IIT Madras
BSc Degree

PDSA - Week 8

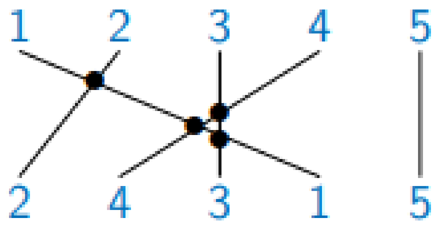
Divide and conquer

- Break up a problem into disjoint subproblems
- Combine these subproblem solutions efficiently
- **Examples**
  - **Merge sort**
    - Split into left and right half and sort each half separately
    - Merge the sorted halves
  - **Quicksort**
    - Rearrange into lower and upper partitions, sort each partition separately
    - Place pivot between sorted lower and upper partitions

Divide and conquer example

Counting inversions

- Compare your profile with other customers
- Identify people who share your likes and dislikes
- No inversions – rankings identical
- Every pair inverted – maximally dissimilar
- Number of inversions ranges from 0 to n(n – 1) / 2



- An inversion is a pair (i, j), i < j, where j appears before i
- Recurrence: $T(n) = 2T(n/2) + n = O(nlogn)$

Implementation

```python
def mergeAndCount(A,B):
    (m,n) = (len(A),len(B))
    (C,i,j,k,count) = ([],0,0,0,0)
    while k < m+n:
        if i == m:
            C.append(B[j])
            j += 1
            k += 1
        elif j == n:
            C.append(A[i])
            i += 1
            k += 1
        elif A[i] < B[j]:
            C.append(A[i])
            i += 1
            k += 1
        else:
            C.append(B[j])
            j += 1
            k += 1
            count += m-i
    return(C,count)

def inversionCount(A):
    n = len(A)
    if n <= 1:
        return(A,0)
    (L,countL) = inversionCount(A[:n//2])
    (R,countR) = inversionCount(A[n//2:])
    (B,countB) = mergeAndCount(L,R)
    return(B,countL + countR + countB)
L = [2,4,3,1,5]
print(inversionCount(L)[1])
```

Output

```
4 # 4 is the number of inversions
```

Closest pair of points

- Several objects on screen
- Basic step: find closest pair of objects
- n objects — naive algorithm is n^2
  - For each pair of objects, compute their distance
  - Report minimum distance across all pairs
- There is a clever algorithm that takes time $O(nlogn)$ using divide and conquer
- Given n points $p1, p2, \ldots, pn$ find the closest pair
  - Assume no two points have same x or y coordinate
  - Split the points into two halves by vertical line
  - Recursively compute closest pair in each half
  - Compare shortest distance in each half to shortest distance across the dividing line
- Recurrence: $Tn = 2Tn/2 + O(n)$
- Overall: $O(nlogn)$

Pseudocode

```
def ClosestPair(Px,Py):
    if len(Px) <= 3:
        compute pairwise distances
        return closest pair and distance
    Construct (Qx,Qy), (Rx,Ry)
    (q1,q2,dQ) = ClosestPair(Qx,Qy)
    (r1,r2,dR) = ClosestPair(Rx,Ry)
    Construct Sy from Qy,Ry
    Scan Sy, find (s1,s2,dS)
    return (q1,q2,dQ), (r1,r2,QR), (s1,s2,dS)
    #depending on which of dQ, dR, dS is minimum
```

Implementation

```
import math

# Returns eucledian disatnce between points p and q
def distance(p, q):
  return math.sqrt(math.pow(p[0] - q[0],2) + math.pow(p[1] - q[1],2))

def minDistanceRec(Px, Py):
  s = len(Px)
  # Given number of points cannot be less than 2.
  # If only 2 or 3 points are left return the minimum distance accordingly.
  if (s == 2):
    return distance(Px[0],Px[1])
  elif (s == 3):
    return min(distance(Px[0],Px[1]), distance(Px[1],Px[2]),
distance(Px[2],Px[0]))

  # For more than 3 points divide the poitns by point around median of x
coordinates
  m = s//2
```

```python
  Qx = Px[:m]
  Rx = Px[m:]
  xR = Rx[0][0]    # minimum x value in Rx

  # Construct Qy and Ry in O(n) rather from Py
  Qy=[]
  Ry=[]
  for p in Py:
    if(p[0] < xR):
      Qy.append(p)
    else:
      Ry.append(p)

  # Extract Sy using delta
  delta = min(minDistanceRec(Qx, Qy), minDistanceRec(Rx, Ry))
  Sy = []
  for p in Py:
    if abs(p[0]-xR) <= delta:
      Sy.append(p)

  #print(xR,delta,Sy)
  sizeS = len(Sy)
  if sizeS > 1:
      minS = distance(Sy[0], Sy[1])
      for i in range(1, sizeS-1):
          for j in range(i, min(i+15, sizeS)-1):
              minS = min(minS, distance(Sy[i], Sy[j+1]))
      return min(delta, minS)
  else:
      return delta

def minDistance(Points):
  Px = sorted(Points)
  Py = Points
  Py.sort(key=lambda x: x[-1])
  #print(Px,Py)
  return round(minDistanceRec(Px, Py), 2)



pts =  [(2, 15), (40, 5), (20, 1), (21, 14), (1,4), (3, 11)]
mul = 0
if (len(pts) > 100): mul = 0
result = minDistance(pts)
for i in range(mul):
  minDistance(pts)
print(result)
```

Output

```
4.12
```

Integer multiplication

- Traditional method: $O(n^2)$
- Naïve divide and conquer strategy: $T(n) = 4T(n/2) + n = O(n^2)$
- Karatsuba's algorithm: $T(n) = 3Tn/2 + n \approx On\ log\ 3)$

Implementation

```
# here 10 represent base of input numbers x and y
def Fast_Multiply(x,y,n):
    if n == 1:
        return x * y
    else:
        m = n/2
        xh = x//10**m
        xl = x % (10**m)
        yh = y//10**m
        yl = y % (10**m)
        a = xh + xl
        b = yh + yl
        p = Fast_Multiply(xh, yh, m)
        q = Fast_Multiply(xl, yl, m)
        r = Fast_Multiply(a, b, m)
        return p*(10**n) + (r - q - p) * (10**(n/2)) + q
print(Fast_Multiply(3456,8902,4))
```

Output

```
30765312.0
```

Quick select and Fast select

- Find the k_{th} smallest value in a sequence of length k
- Sort in descending order and look at position $k - O(nlogn)$
- For any fixed k, find maximum for k times - $O(kn)$
- $k = n/2$ (median) - $O(n^2)$
- Median of medians -$O(n)$
- Selection becomes $O(n)$ in Fast select algorithm
- Quicksort becomes $O(nlogn)$ using MoM

Implementation

```
def quickselect(L,l,r,k): # k-th smallest in L[l:r]
  if (k < 1) or (k > r-l):
    return(None)

  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper + 1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
```

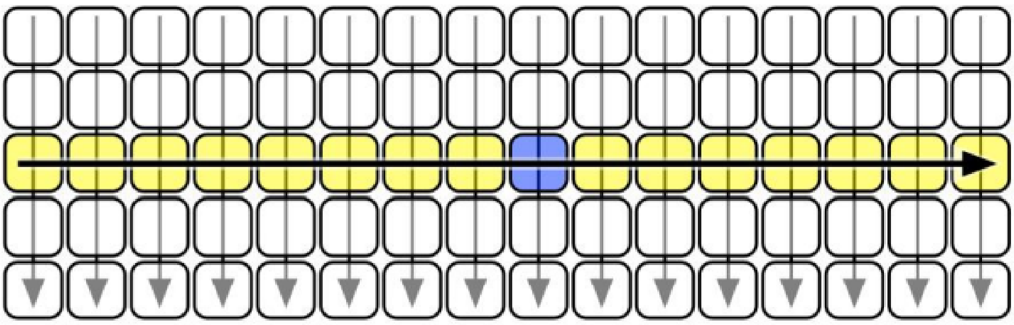
```
      (lower,upper) = (lower+1,upper+1)
  (L[l],L[lower-1]) = (L[lower-1],L[l]) # Move pivot
  lower = lower - 1

  # Recursive calls
  lowerlen = lower - l
  if k <= lowerlen:
    return(quickselect(L,l,lower,k))
  elif k == (lowerlen + 1):
    return(L[lower])
  else:
    return(quickselect(L,lower+1,r,k-(lowerlen+1)))
print(quickselect([5,3,7,2,1],0,5,2))
```

Output

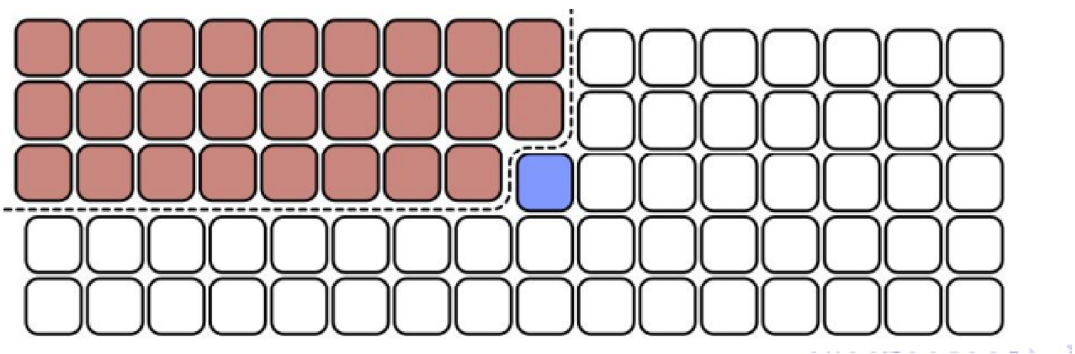
```
2
```

Median of Medians(MoM)

- Divide L into blocks of 5
- Find the median of each block (brute force)
- Let M be the list of block medians
- Recursively apply the process to M
- We can visualize the blocks as follows



- Each block of 5 is arranged in ascending order, top to bottom
- Block medians are the middle row
- The median of block medians lies between 3len(L)/10 and 7len(L)/10



Implementation

```
def MoM(L): # Median of medians
  if len(L) <= 5:
    L.sort()
    return(L[len(L)//2])
  # Construct list of block medians
  M = []
  for i in range(0,len(L),5):
    X = L[i:i+5]
    X.sort()
    M.append(X[len(X)//2])
  return(MoM(M))
print(MoM([4,3,5,6,2,1,8,9,7,10,13,15,18,17,11]))
```

Output

```
8
```

Fast select using MOM

Implementation

```
def MoM(L): # Median of medians
  if len(L) <= 5:
    L.sort()
    return(L[len(L)//2])
  # Construct list of block medians
  M = []
  for i in range(0,len(L),5):
    X = L[i:i+5]
    X.sort()
    M.append(X[len(X)//2])
  return(MoM(M))



def fastselect(L,l,r,k): # k-th smallest in L[l:r]
  if (k < 1) or (k > r-l):
    return(None)

  # Find MoM pivot and move to L[l]
  pivot = MoM(L[l:r])
  pivotpos = min([i for i in range(l,r) if L[i] == pivot])
  (L[l],L[pivotpos]) = (L[pivotpos],L[l])

  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper + 1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
```

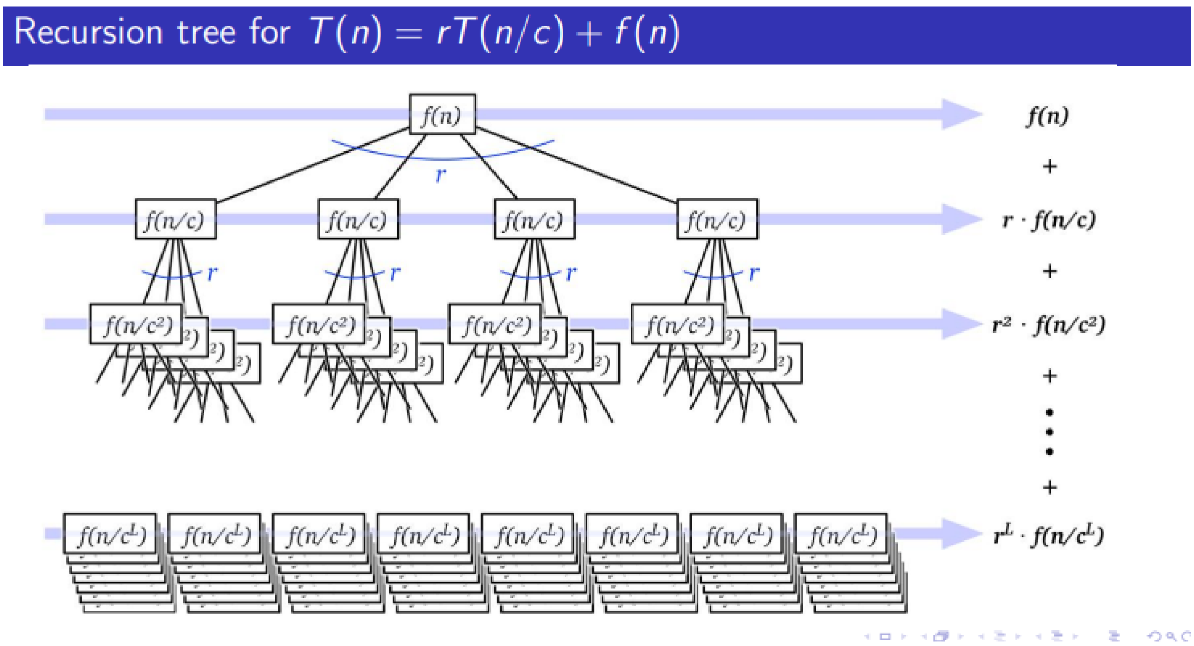
```python
        (lower,upper) = (lower+1,upper+1)
    (L[l],L[lower-1]) = (L[lower-1],L[l]) # Move pivot
    lower = lower - 1

    # Recursive calls
    lowerlen = lower - l
    if k <= lowerlen:
      return(fastselect(L,l,lower,k))
    elif k == (lowerlen + 1):
      return(L[lower])
    else:
      return(fastselect(L,lower+1,r,k-(lowerlen+1)))
print(fastselect([4,3,5,6,2,1,8,9,7,10,13,15,18,17,11],0,15,4))
```

Output

```
4
```

Recursion trees

- **Recursion tree**-Rooted tree with one node for each recursive subproblem
- **Value** of each node is time spent on that subproblem excluding recursive calls
- Concretely, on an input of size n
  - $f(n)$ is the time spent on non-recursive work
  - r is the number of recursive calls
  - Each recursive call works on a subproblem of size n/c
- Resulting recurrence: $T(n) = rT(n/c) + f(n)$
- Root of recursion tree for $T(n)$ has value $f(n)$
- Root has r children, each (recursively) the root of a tree for $T(n/c)$
- Each node at level d has value $f(n/c^d)$
  - Assume, for simplicity, that n was a power of c



- Leaves correspond to the base case $T(1)$
    - Safe to assume $T(1) = 1$, asymptotic complexity ignores constants
- Level i has r^i nodes, each with value $f(n/c^i)$
- Tree has L levels, $L = \log_c n$
- Total cost is $T(n) = \sum_{i=0}^{L} r^i \cdot f(n/c^i)$
- Number of leaves is r^L
    - Last term in the level by level sum is $r^L \cdot f(1) = r^{\log_c n} \cdot 1 = n^{\log_c r}$
    - Recall that $a^{\log_b c} = c^{\log_b a}$

- Tree has $\log_c n$ levels, last level has cost is $n^{\log_c r}$
- Total cost is $T(n) = \sum_{i=0}^{L} r^i \cdot f(n/c^i)$
- Think of the total cost as a series. Three common cases
- Decreasing Each term is a constant factor smaller than previous term
    - Root dominates the sum, $T(n) = O(f(n))$
- Equal All terms in the series are equal
    - $T(n) = O(f(n) \cdot L) = O(f(n) \log n)$ — $\log_c n$ is asymptotically same as $\log n$
- Increasing Series grows exponentially, each term a constant factor larger than previous term
    - Leaves dominate the sum, $T(n) = O(n^{\log_c r})$