

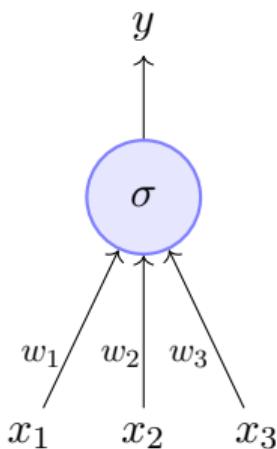
# CS7015 (Deep Learning) : Lecture 2

McCulloch Pitts Neuron, Thresholding Logic, Perceptrons, Perceptron Learning Algorithm and Convergence, Multilayer Perceptrons (MLPs), Representation Power of MLPs

Mitesh M. Khapra

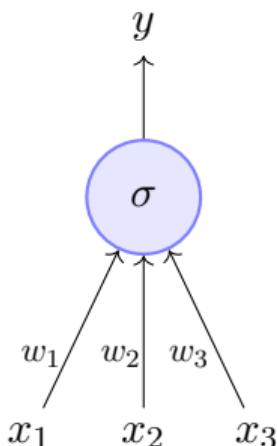
Department of Computer Science and Engineering  
Indian Institute of Technology Madras

# Module 2.1: Biological Neurons



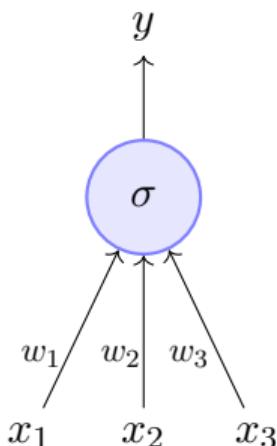
Artificial Neuron

- The most fundamental unit of a deep neural network is called an *artificial neuron*



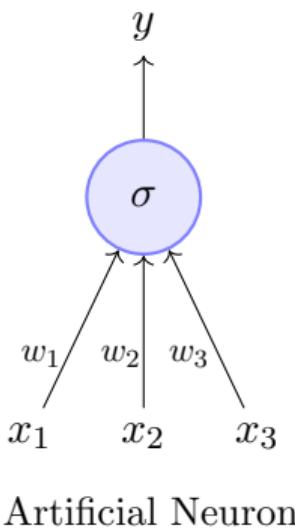
Artificial Neuron

- The most fundamental unit of a deep neural network is called an *artificial neuron*
- Why is it called a neuron ? Where does the inspiration come from ?

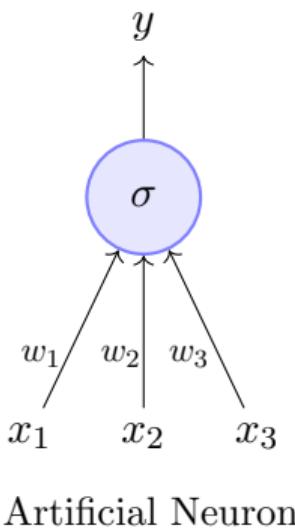


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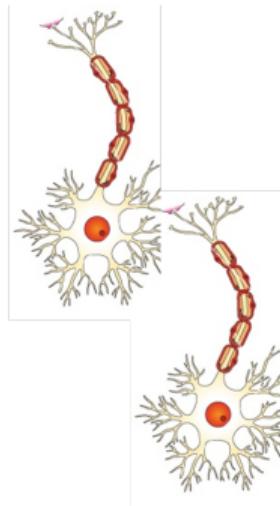


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- *biological neurons = neural cells = neural processing units*



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- Why is it called a neuron ? Where does the inspiration come from ?
- The inspiration comes from biology (more specifically, from the *brain*)
- *biological neurons = neural cells = neural processing units*
- We will first see what a biological neuron looks like ...

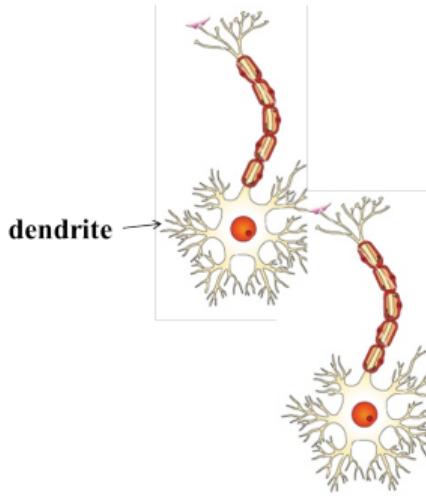


## Biological Neurons\*

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\*Image adapted from

<https://cdn.vectorstock.com/i/composite/12,25/neuron-cell-vector-81225.jpg>



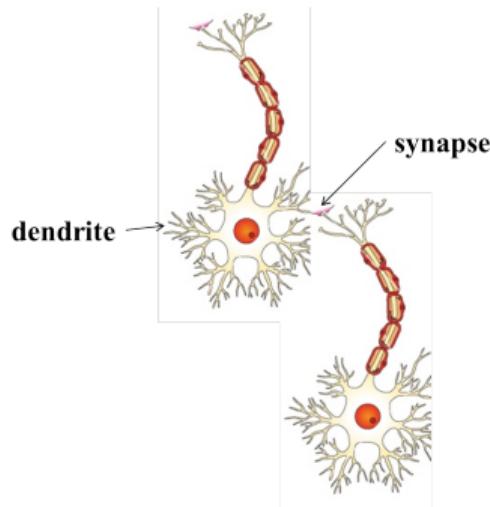
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Biological Neurons\*

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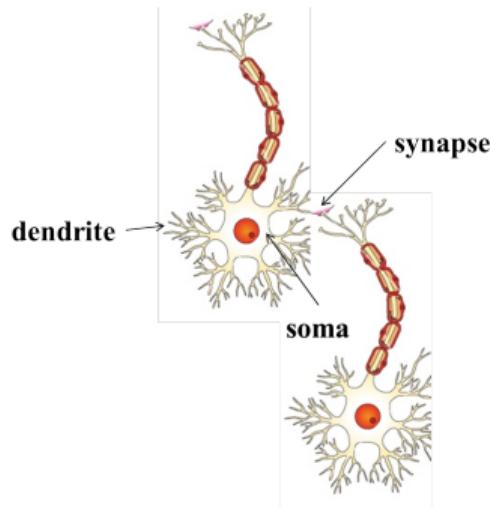
Biological Neurons\*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons

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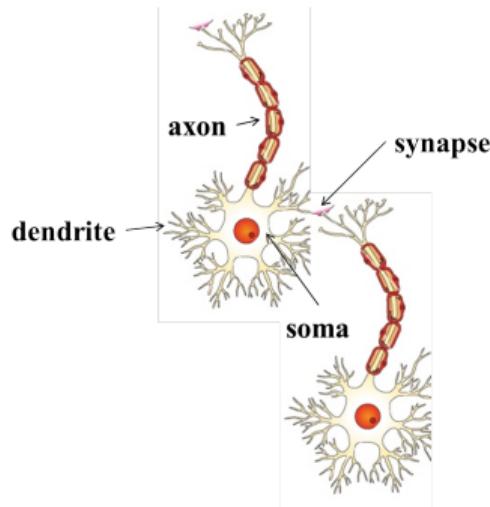
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Biological Neurons\*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons
- **soma:** processes the information
- **axon:** transmits the output of this neuron

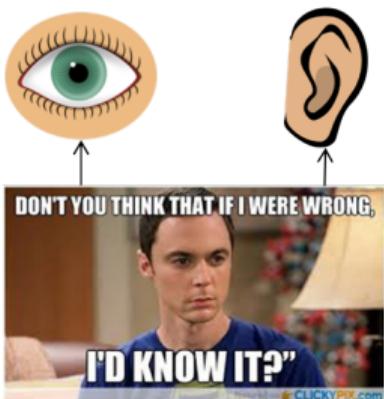
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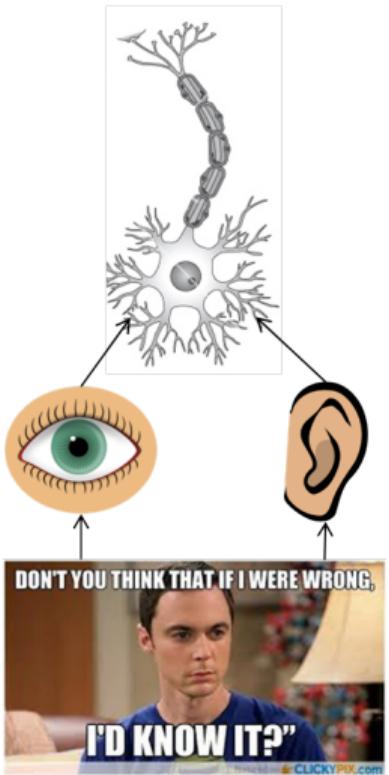
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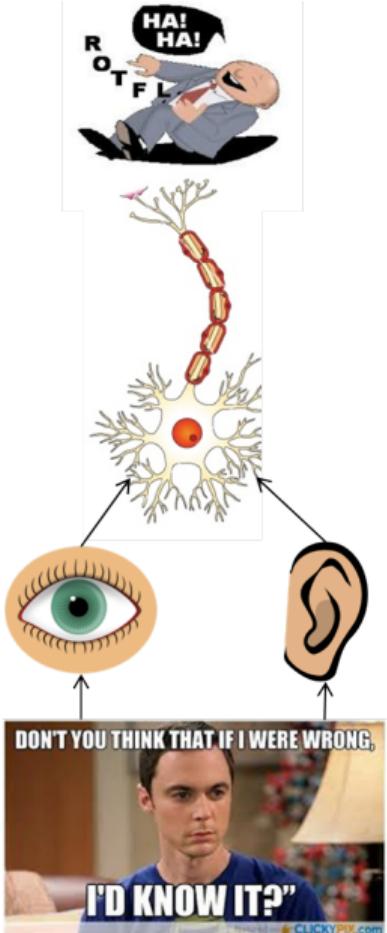
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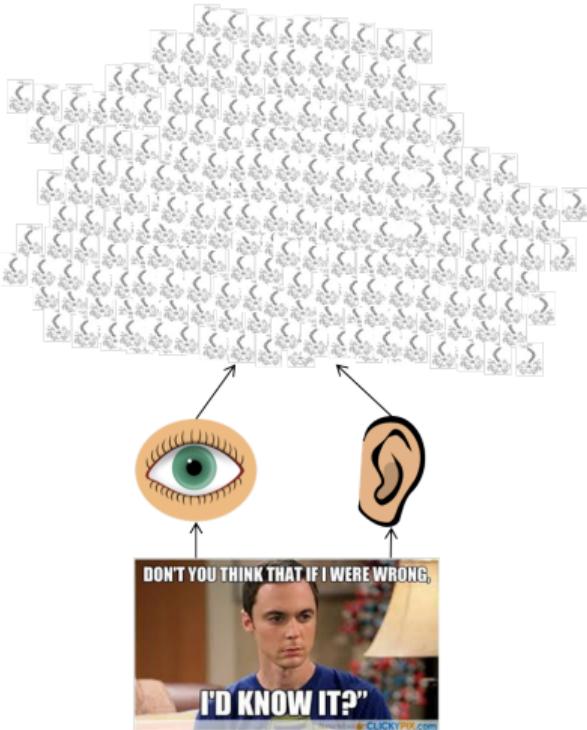
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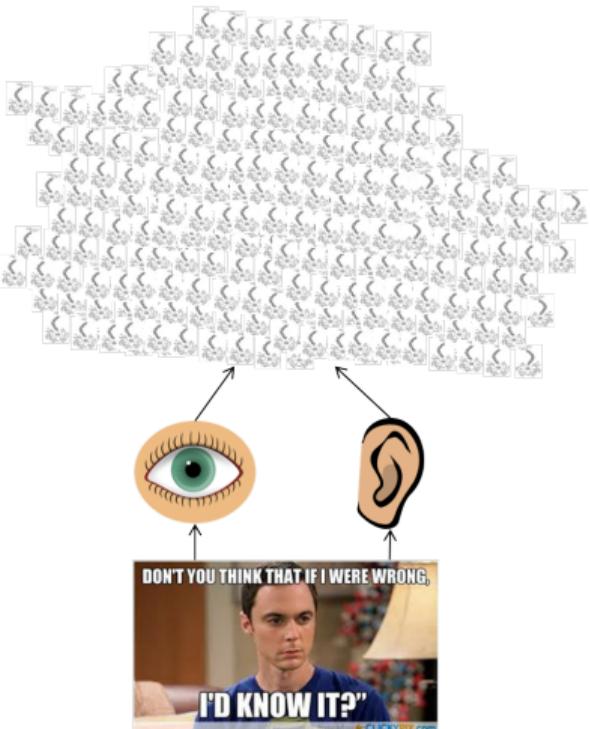


- Let us see a very cartoonish illustration of how a neuron works
- Our sense organs interact with the outside world
- They relay information to the neurons
- The neurons (may) get activated and produces a response (laughter in this case)

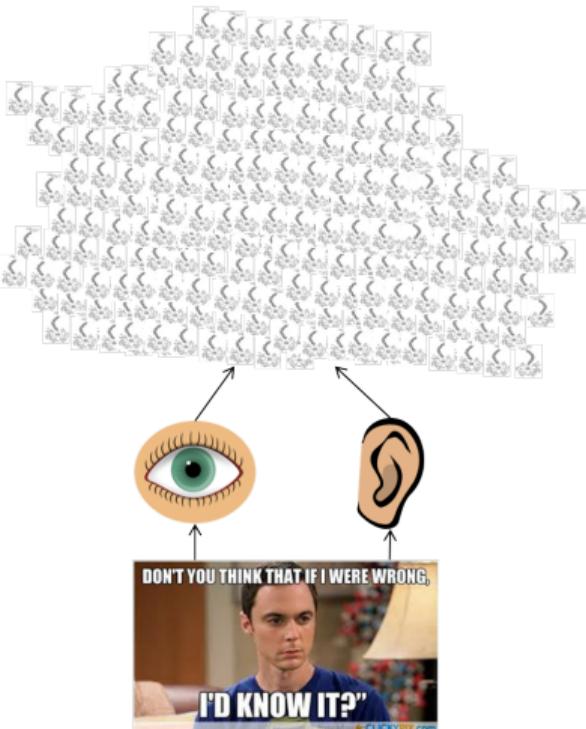
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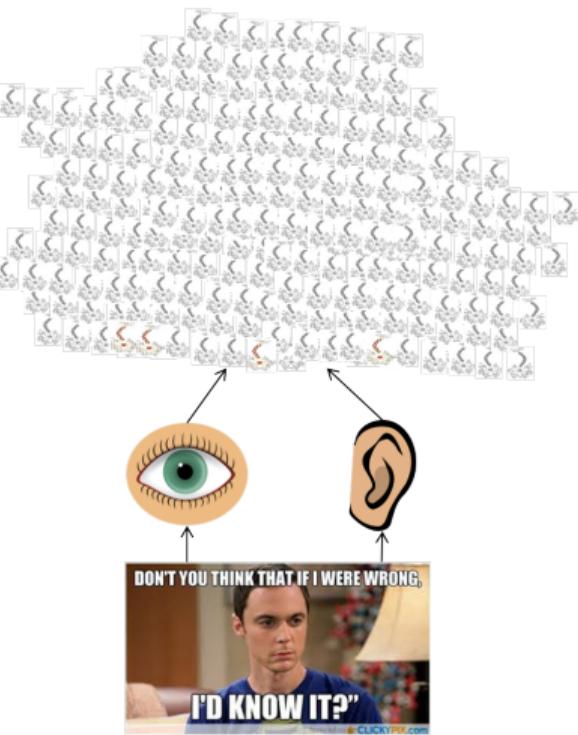


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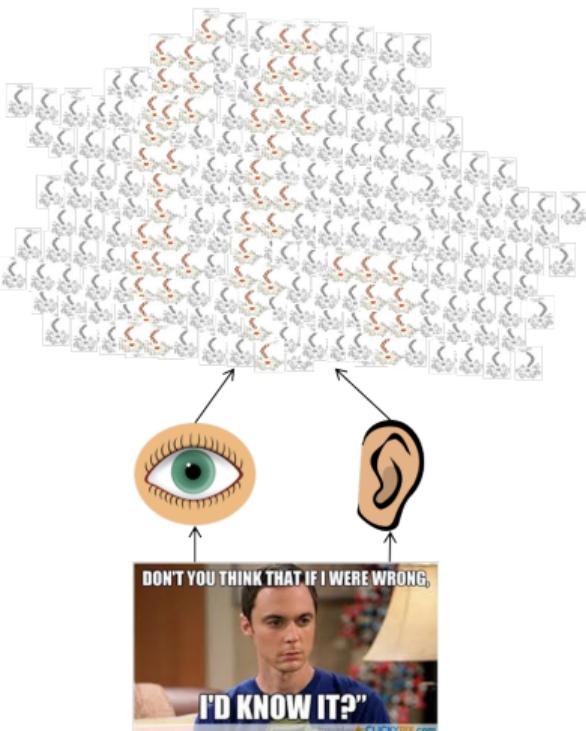


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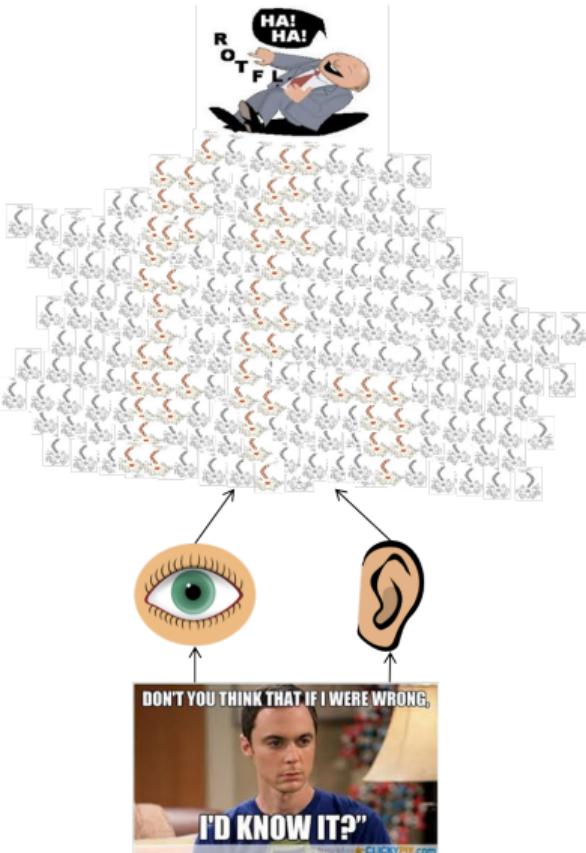




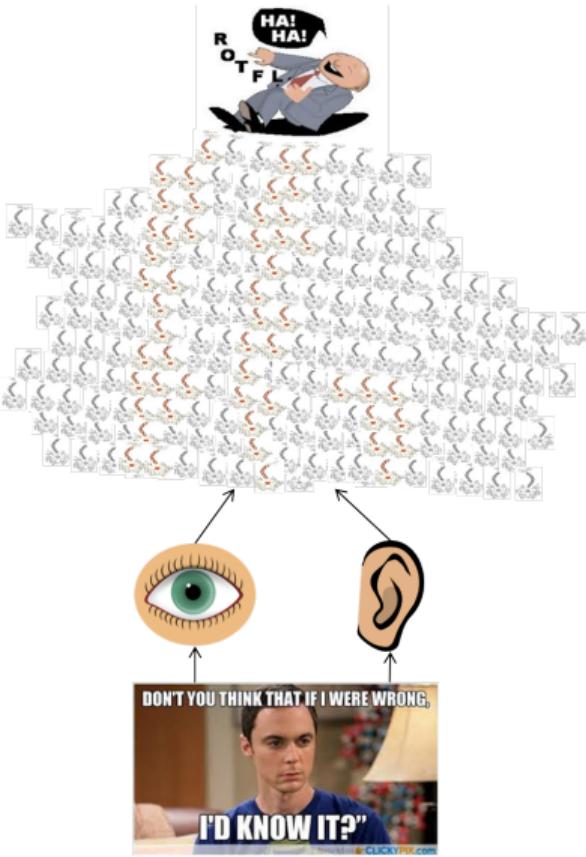
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- The sense organs relay information to the lowest layer of neurons
- Some of these neurons may fire (in red) in response to this information and in turn relay information to other neurons they are connected to



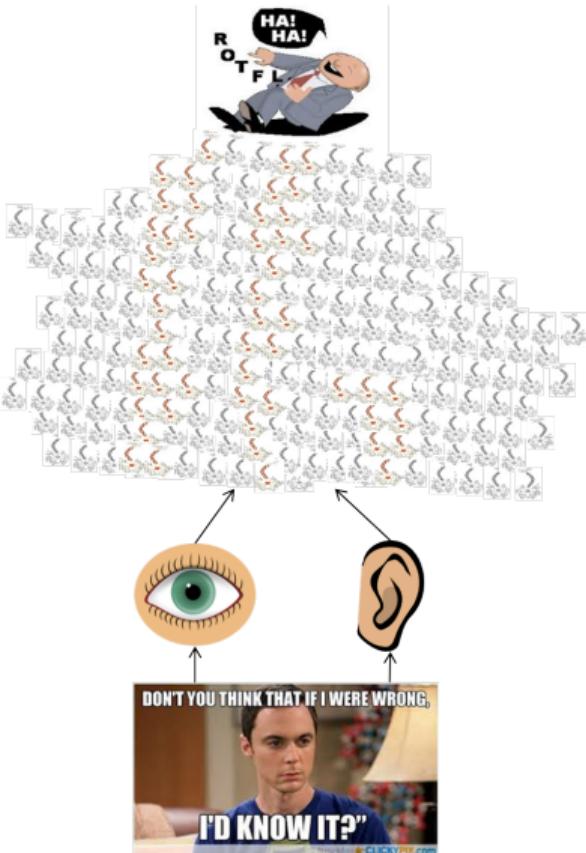
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- These neurons may also fire (again, in red) and the process continues



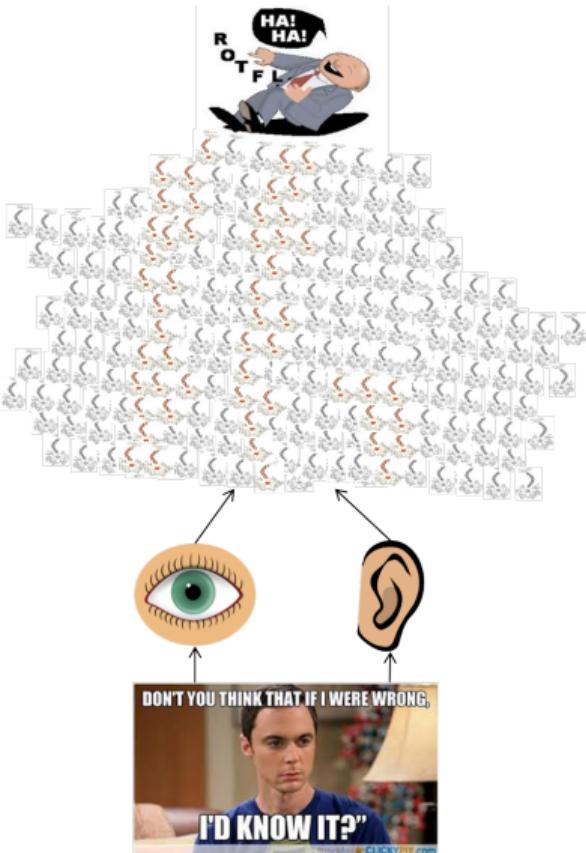
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- An average human brain has around  $10^{11}$  (100 billion) neurons!

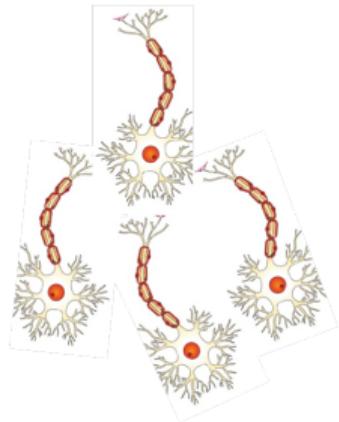


- This massively parallel network also ensures that there is division of work



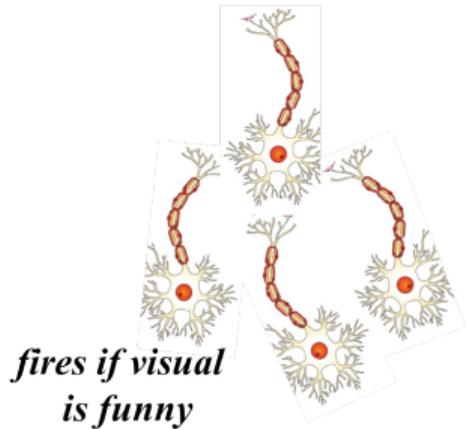
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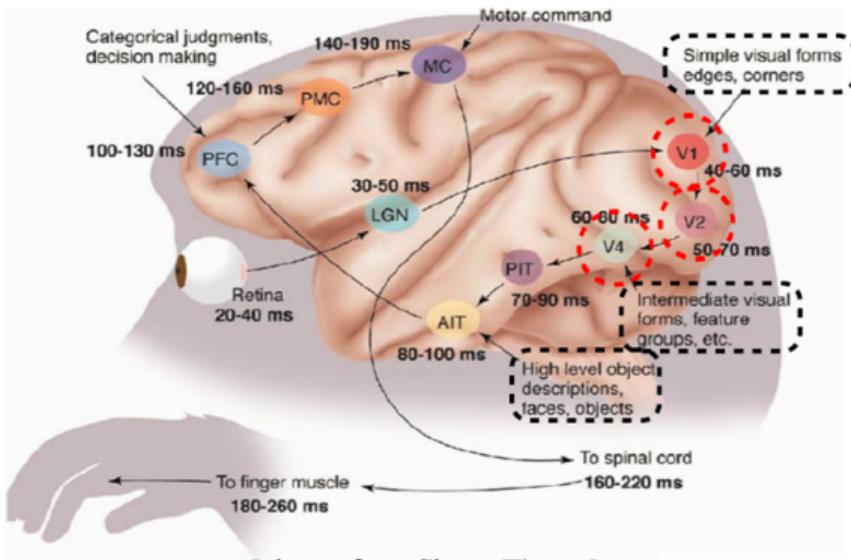
*fires if at least  
2 of the 3 inputs fired*



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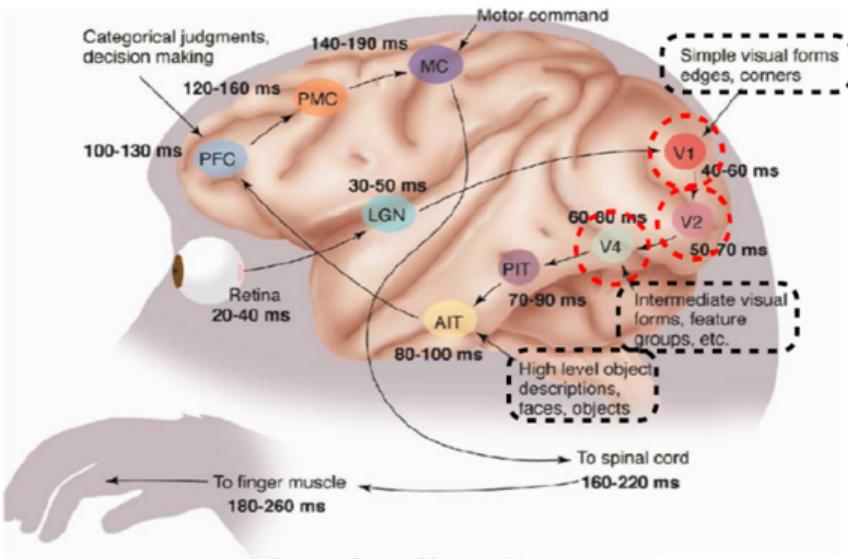
A simplified illustration

- The neurons in the brain are arranged in a hierarchy



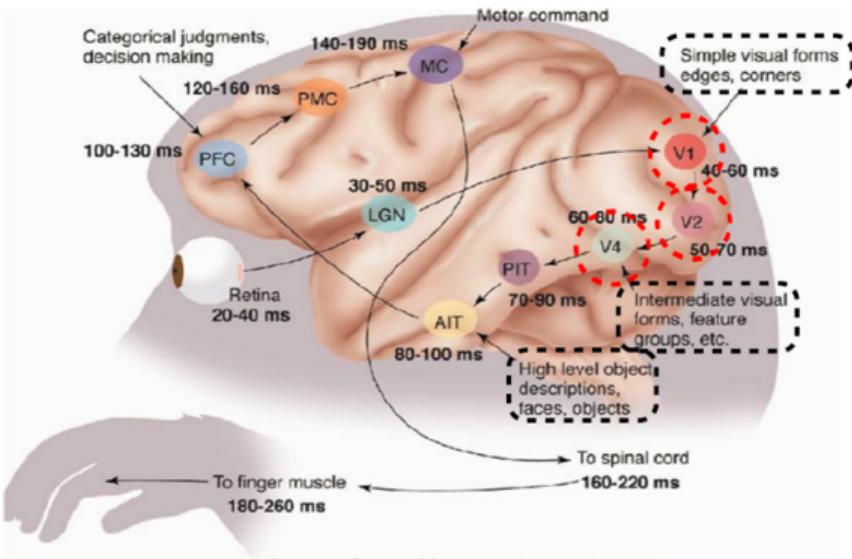
[picture from Simon Thorpe]

- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information



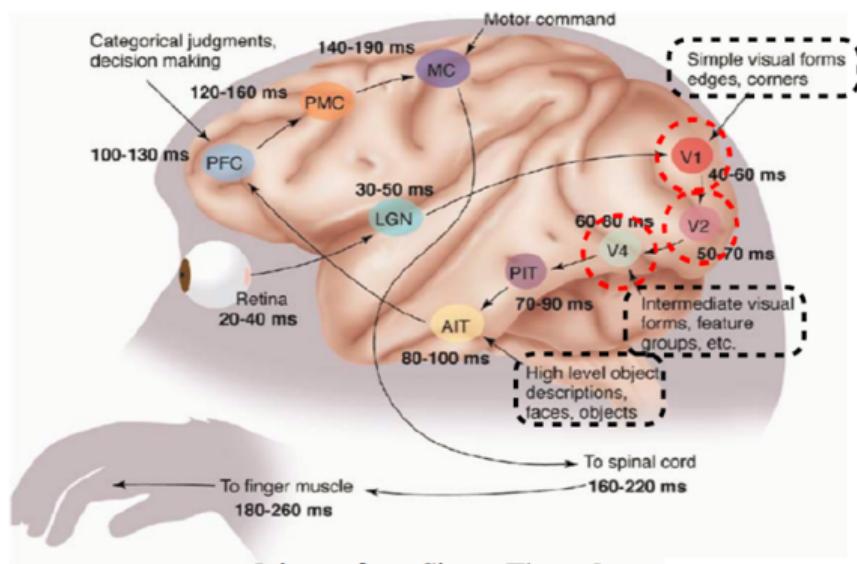
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- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information
- Starting from the retina, the information is relayed to several layers (follow the arrows)
- We observe that the layers  $V1$ ,  $V2$  to  $AIT$  form a hierarchy (from identifying simple visual forms to high level objects)

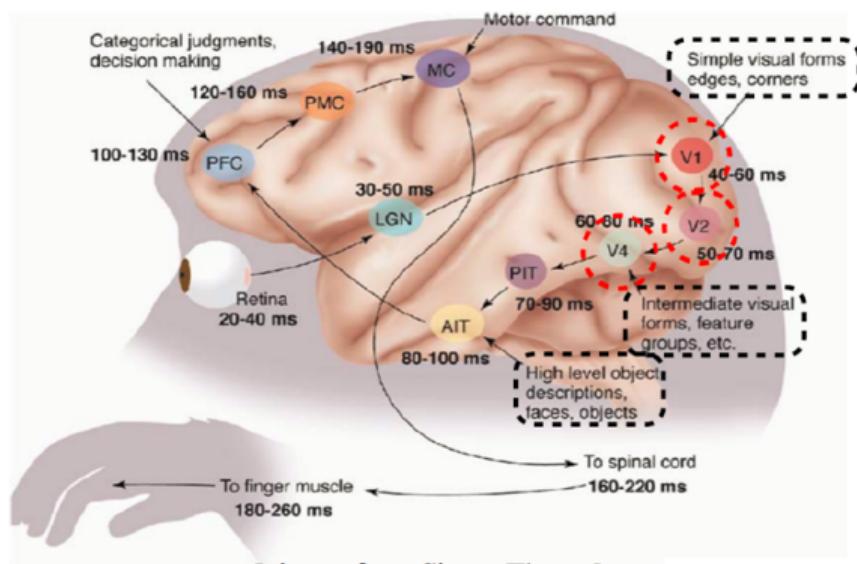


## Layer 1: detect edges & corners

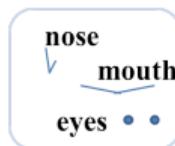
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## Sample illustration of hierarchical processing\*

<sup>\*</sup>Idea borrowed from Hugo Larochelle's lecture slides



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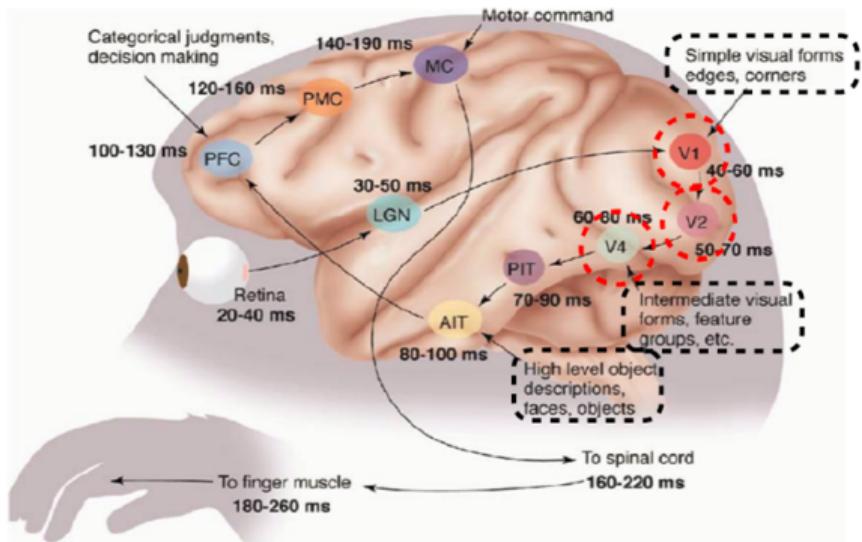


### **Layer 2: form feature groups**

[picture from Simon Thorpe]

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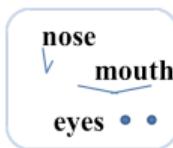
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[picture from Simon Thorpe]



**Layer 1: detect edges & corners**



**Layer 2: form feature groups**



**Layer 3: detect high level objects, faces, etc.**

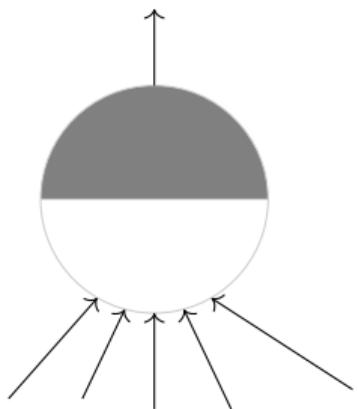
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## Disclaimer

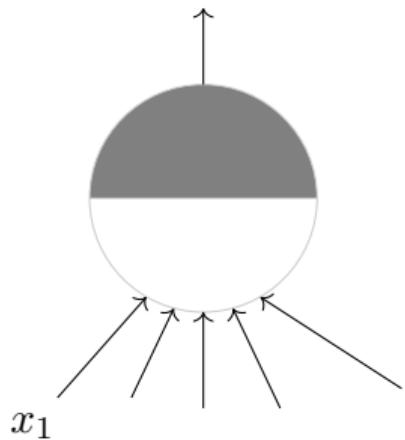
- I understand very little about how the brain works!
- What you saw so far is an overly simplified explanation of how the brain works!
- But this explanation suffices for the purpose of this course!

## Module 2.2: McCulloch Pitts Neuron

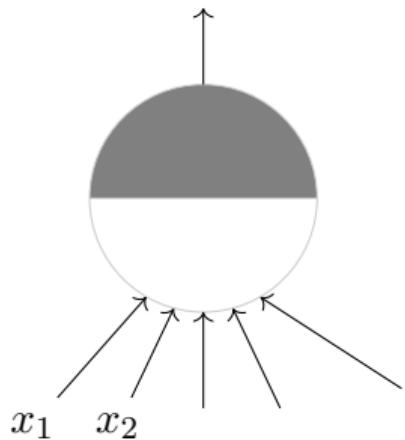


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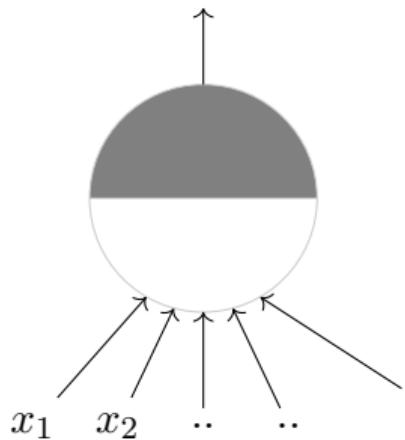
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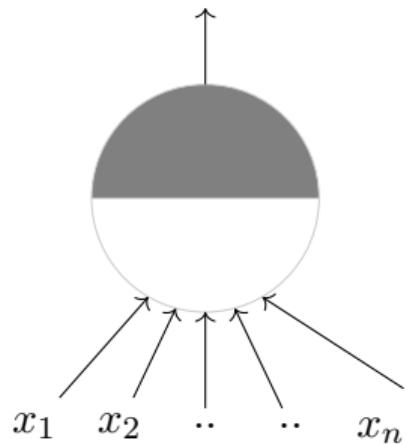
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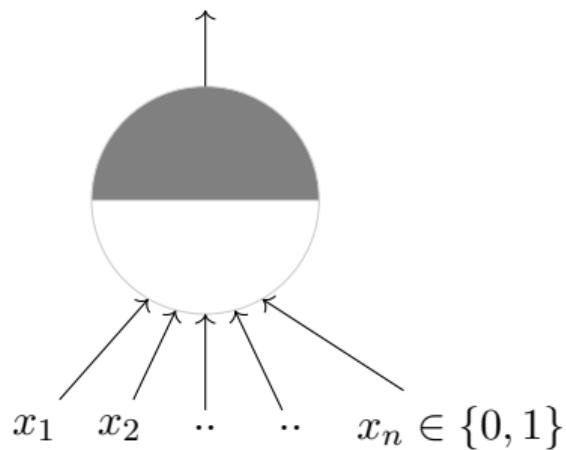
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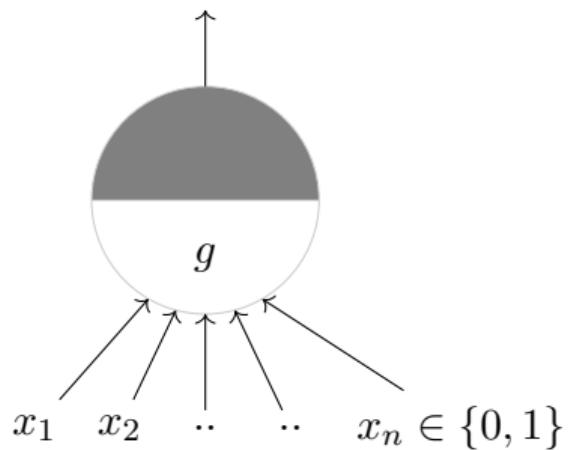
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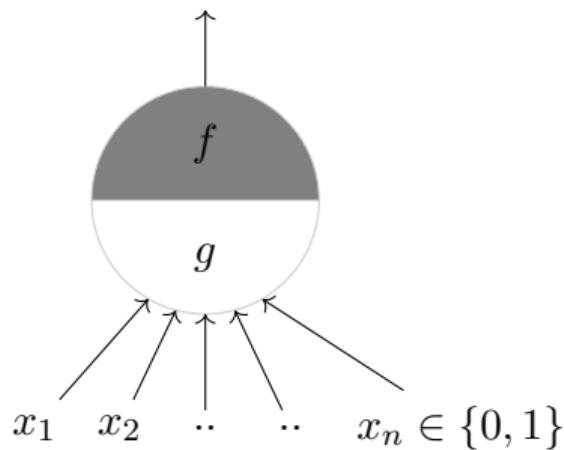


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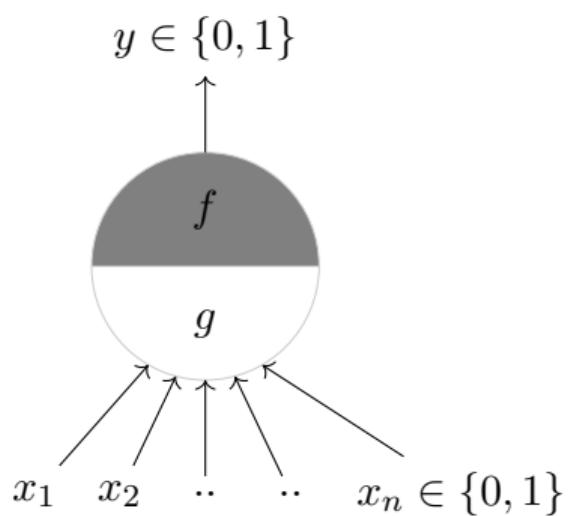


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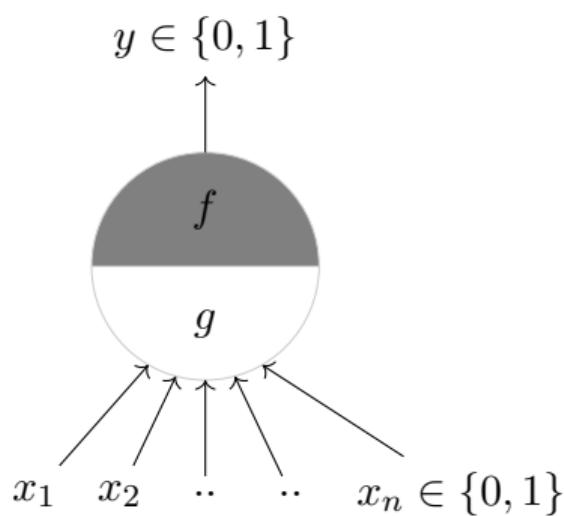




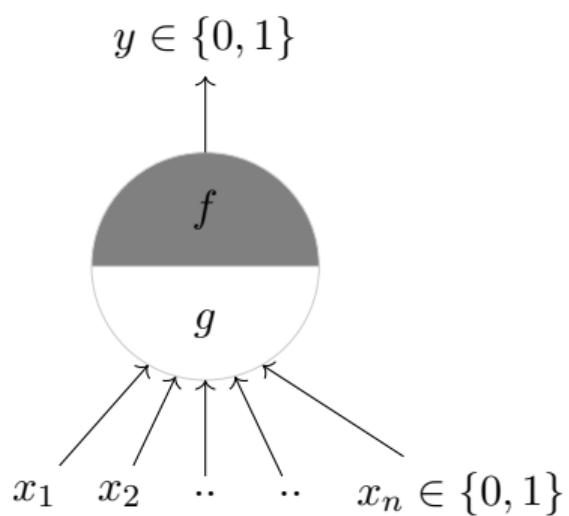
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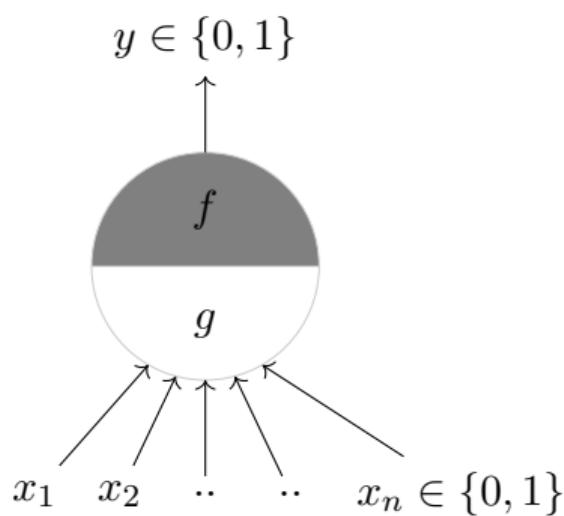
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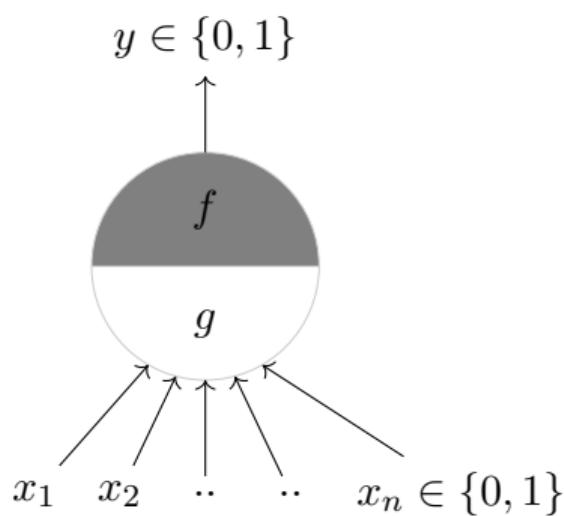


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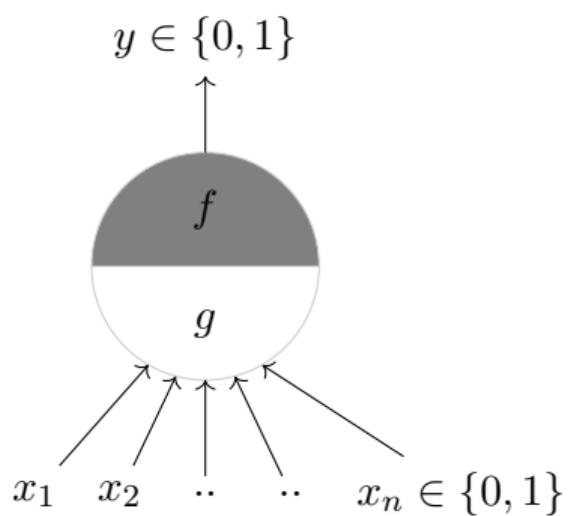
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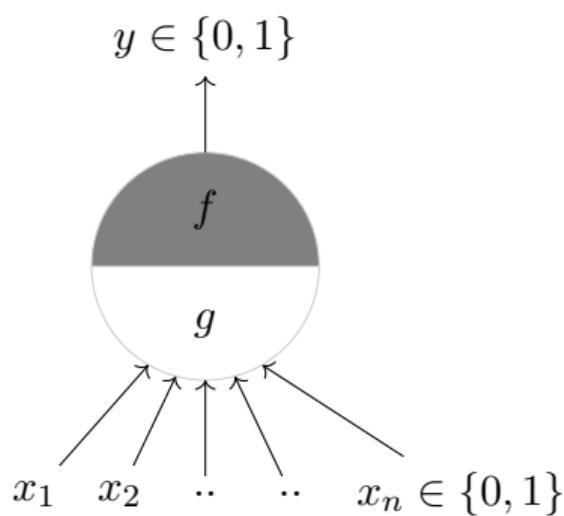
$$y = f(g(\mathbf{x})) = 1 \quad \text{if} \quad g(\mathbf{x}) \geq \theta$$



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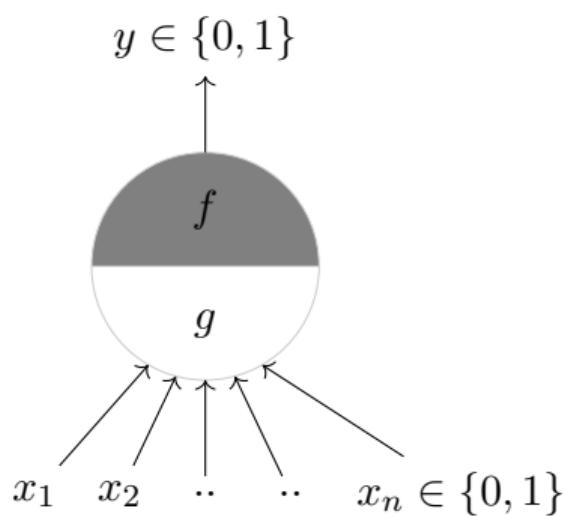


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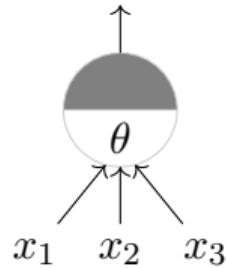
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- $\theta$  is called the thresholding parameter
- This is called Thresholding Logic

Let us implement some boolean functions using this McCulloch Pitts (MP) neuron

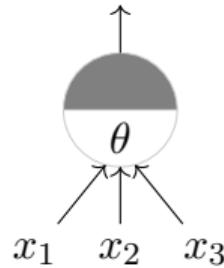
...

$$y \in \{0, 1\}$$



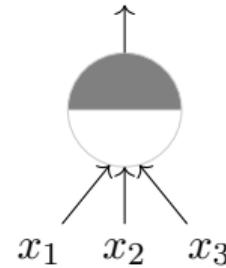
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



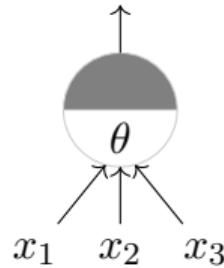
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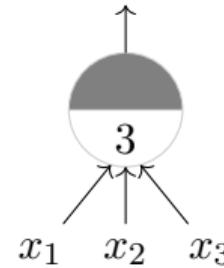
AND function

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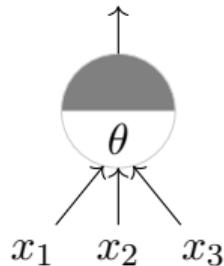
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



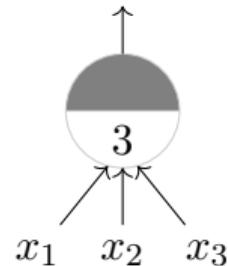
AND function

$$y \in \{0, 1\}$$



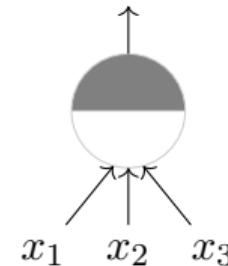
A McCulloch Pitts unit

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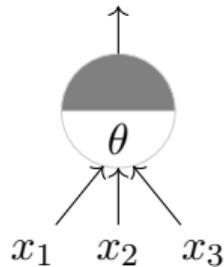
AND function

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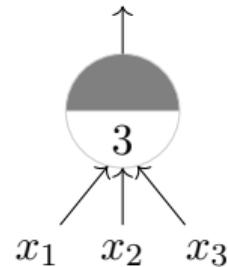
OR function

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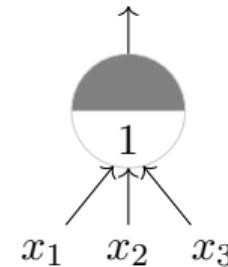
A McCulloch Pitts unit

$$y \in \{0, 1\}$$

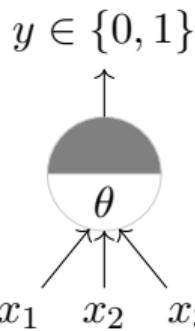


AND function

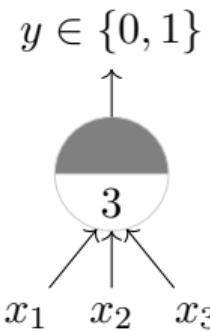
$$y \in \{0, 1\}$$



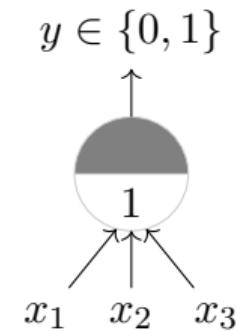
OR function



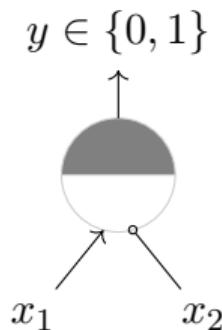
A McCulloch Pitts unit



AND function



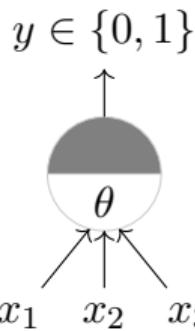
OR function



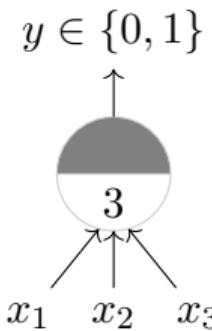
$x_1 \text{ AND } !x_2^*$

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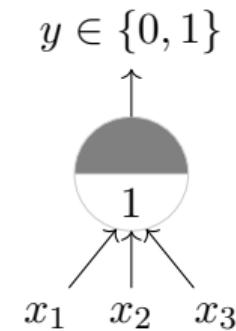
\*circle at the end indicates inhibitory input: if any inhibitory input is 1 the output will be 0



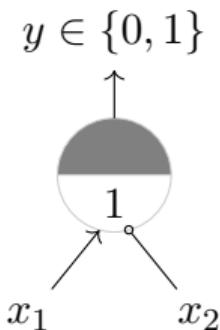
A McCulloch Pitts unit



AND function

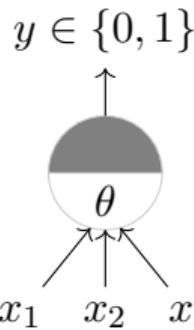


OR function

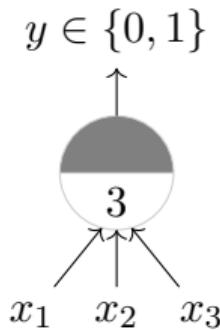


$x_1 \text{ AND } !x_2^*$

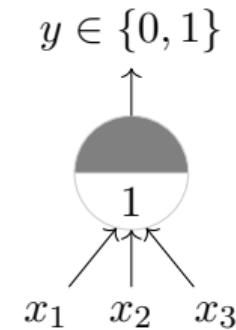
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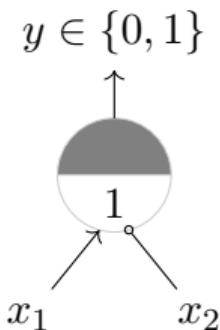
A McCulloch Pitts unit



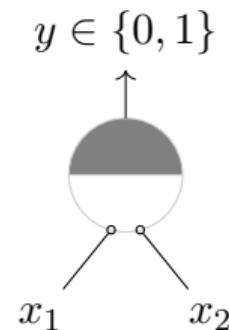
AND function



OR function

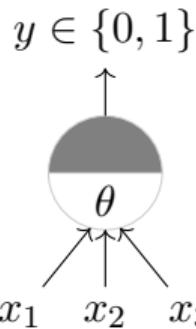


$x_1 \text{ AND } !x_2^*$

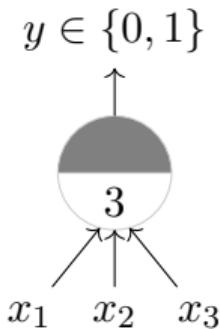


NOR function

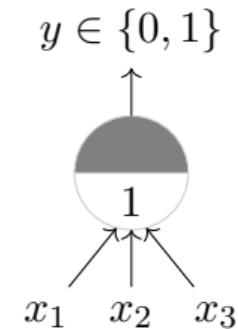
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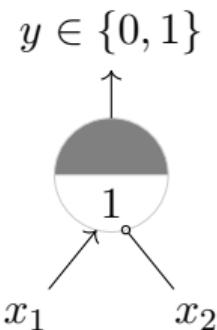
A McCulloch Pitts unit



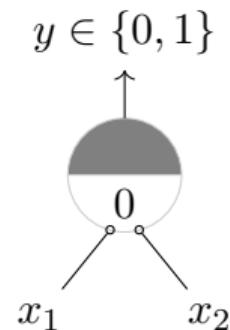
AND function



OR function

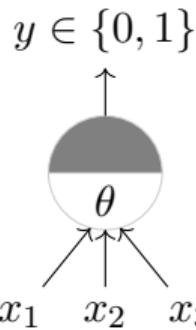


$x_1 \text{ AND } !x_2^*$

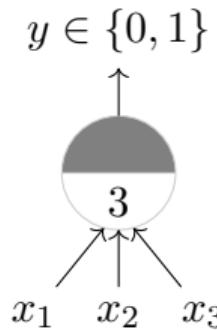


NOR function

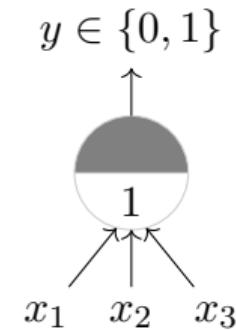
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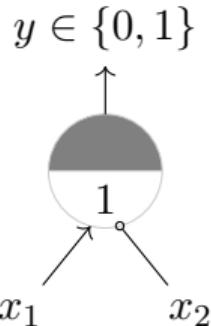
A McCulloch Pitts unit



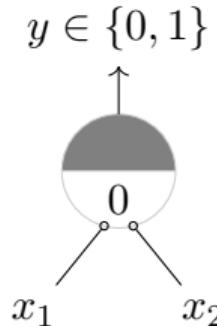
AND function



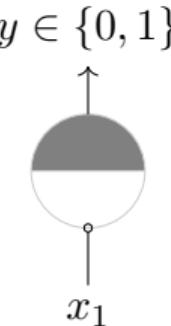
OR function



$x_1$  AND  $\neg x_2^*$

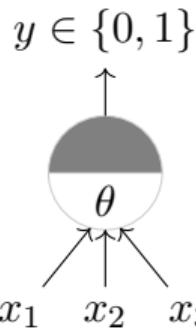


NOR function

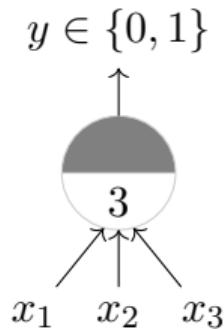


NOT function

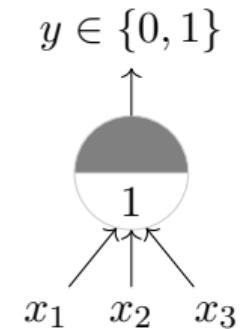
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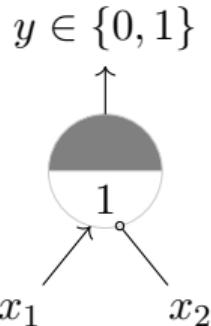
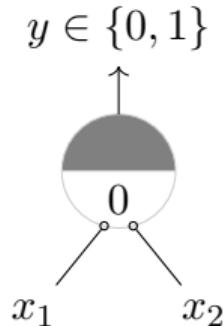
A McCulloch Pitts unit



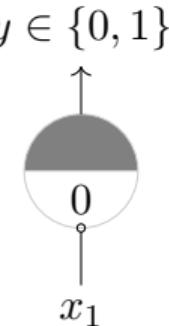
AND function



OR function

 $x_1$  AND  $\neg x_2$ \*

NOR function



NOT function

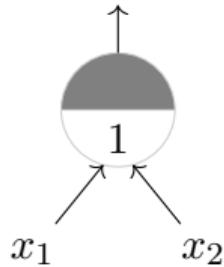
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- Can any boolean function be represented using a McCulloch Pitts unit ?

- Can any boolean function be represented using a McCulloch Pitts unit ?
- Before answering this question let us first see the geometric interpretation of a MP unit ...

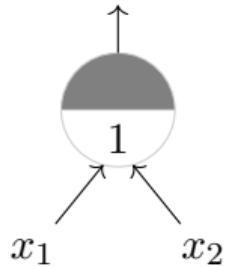
$$y \in \{0, 1\}$$



OR function

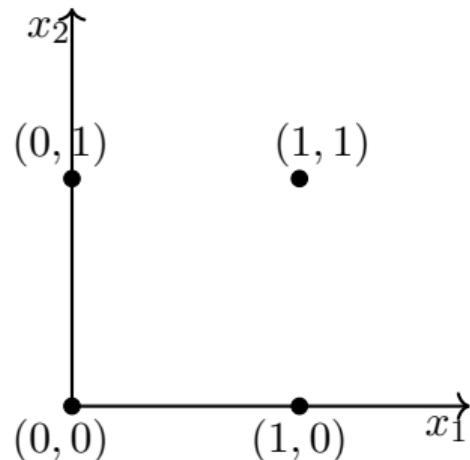
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$

$$y \in \{0, 1\}$$

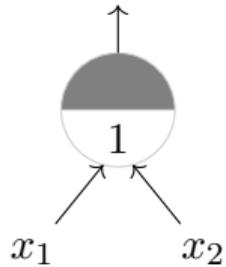


OR function

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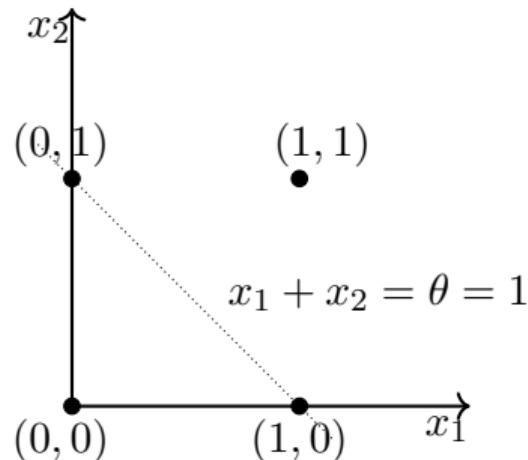


$$y \in \{0, 1\}$$

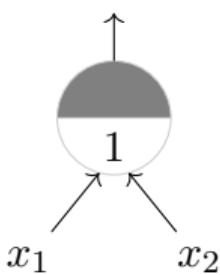


OR function

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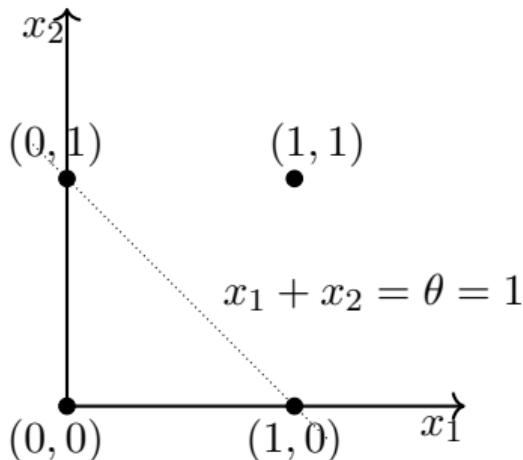
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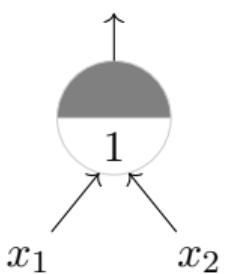
- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves

OR function

$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$



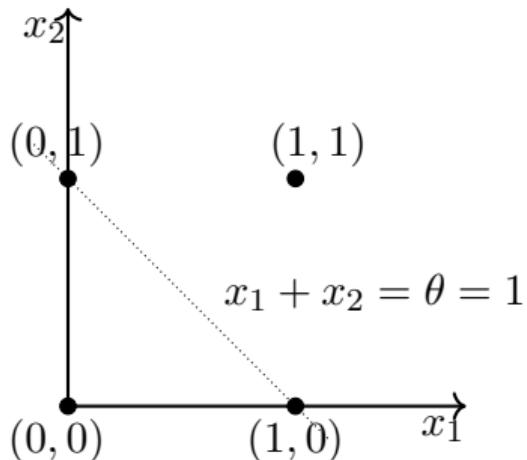
$$y \in \{0, 1\}$$

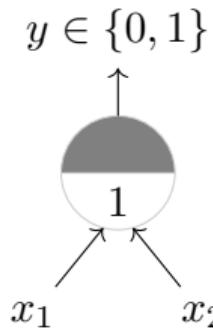


- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line  $\sum_{i=1}^n x_i - \theta = 0$  and points lying below this line

OR function

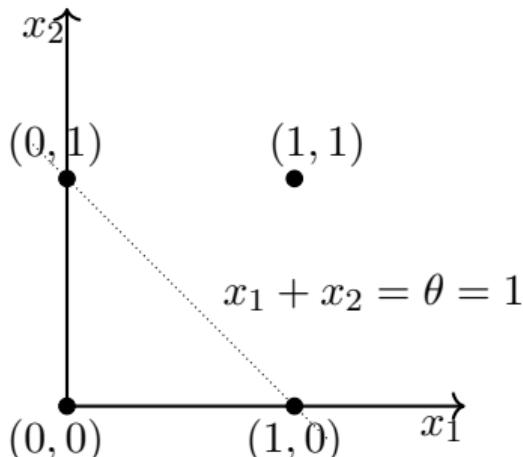
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$



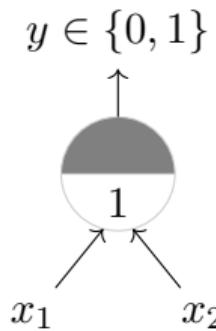


OR function

$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$

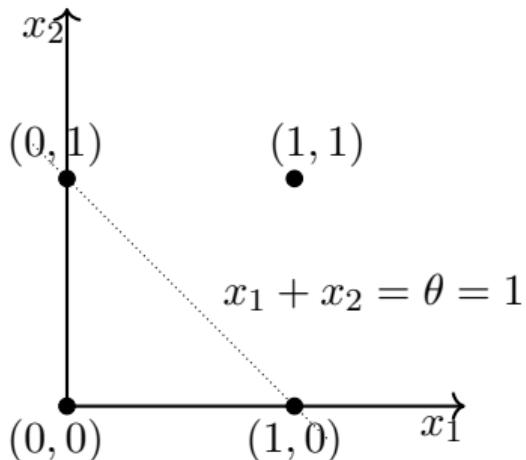


- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line  $\sum_{i=1}^n x_i - \theta = 0$  and points lying below this line
- In other words, all inputs which produce an output 0 will be on one side ( $\sum_{i=1}^n x_i < \theta$ ) of the line and all inputs which produce an output 1 will lie on the other side ( $\sum_{i=1}^n x_i \geq \theta$ ) of this line



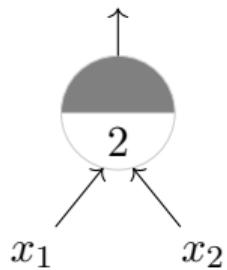
OR function

$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$



- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
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- Let us convince ourselves about this with a few more examples (if it is not already clear from the math)

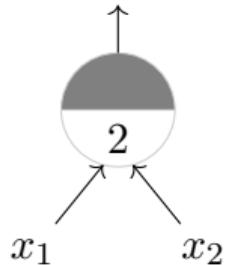
$$y \in \{0, 1\}$$



AND function

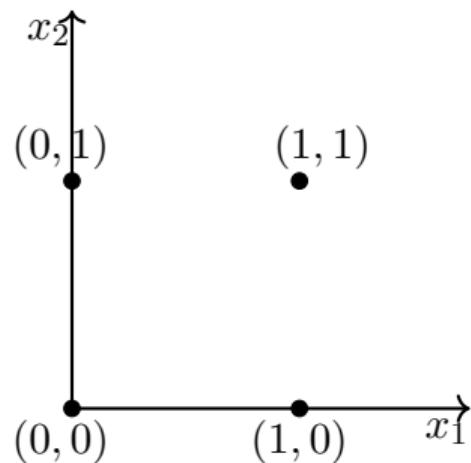
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 2$$

$$y \in \{0, 1\}$$

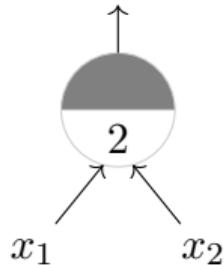


AND function

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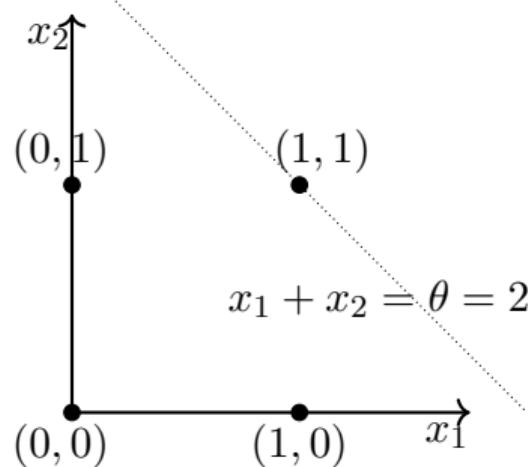


$$y \in \{0, 1\}$$

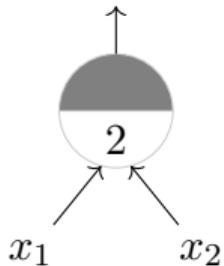


AND function

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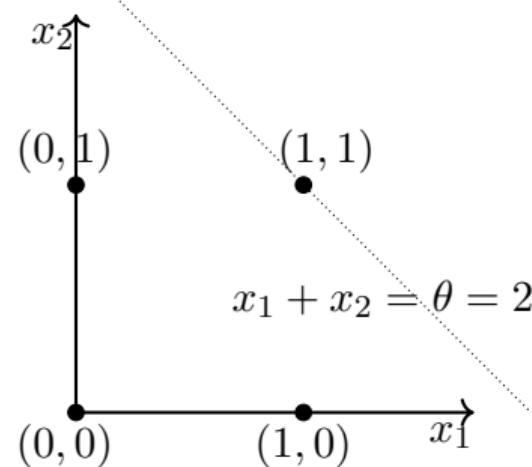


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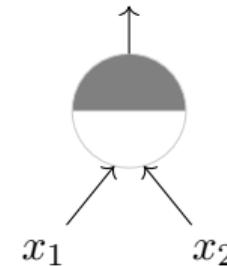


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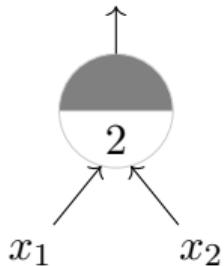


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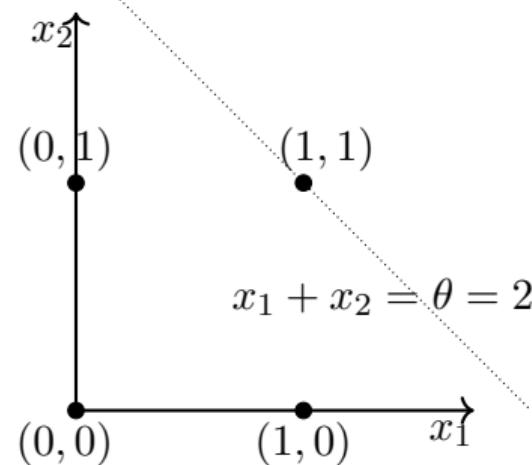
Tautology (always ON)

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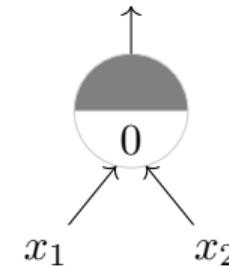


AND function

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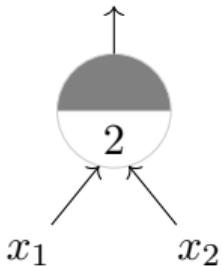


$$y \in \{0, 1\}$$



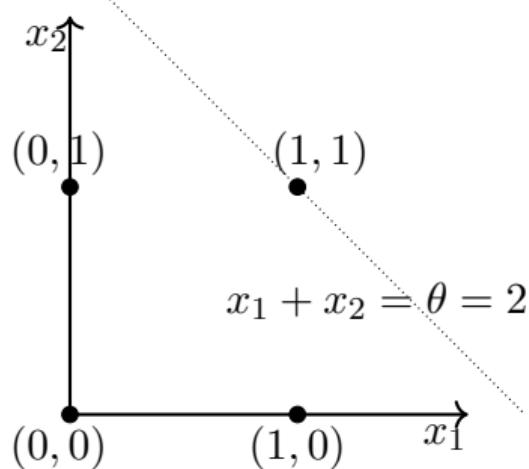
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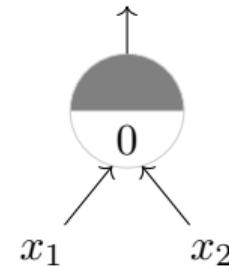


AND function

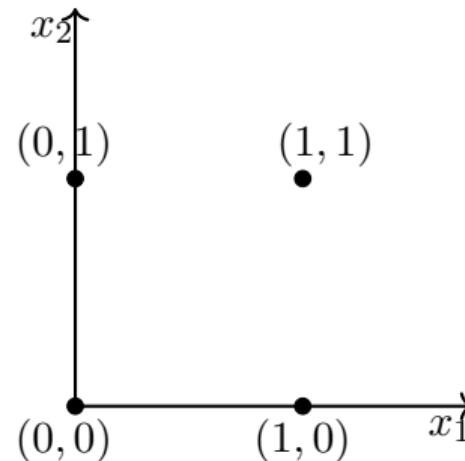
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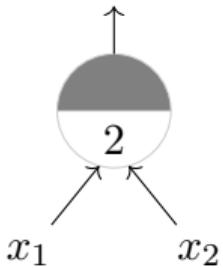
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Tautology (always ON)

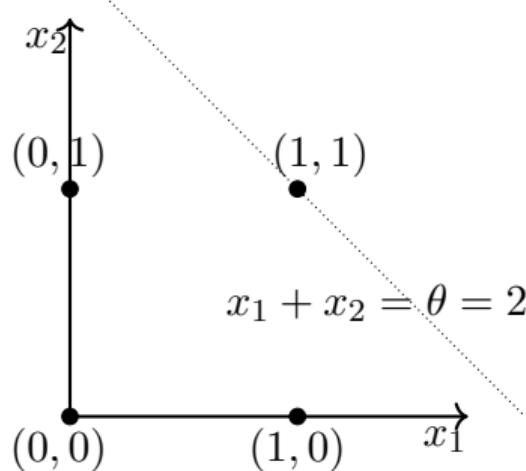


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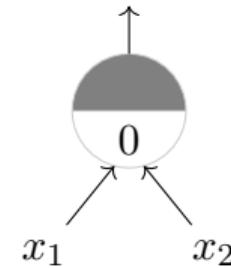


AND function

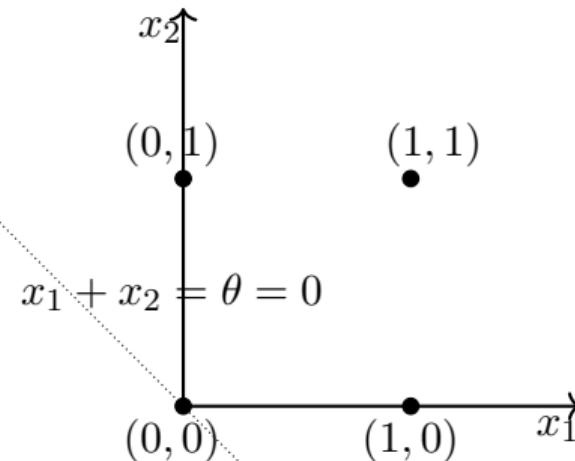
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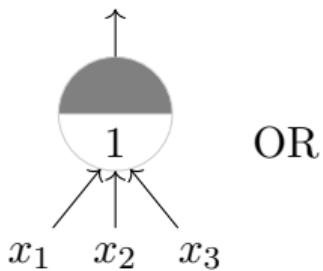
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Tautology (always ON)



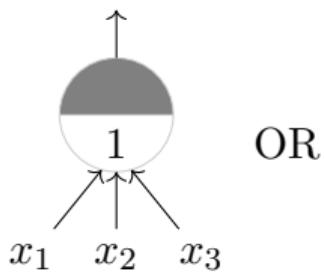
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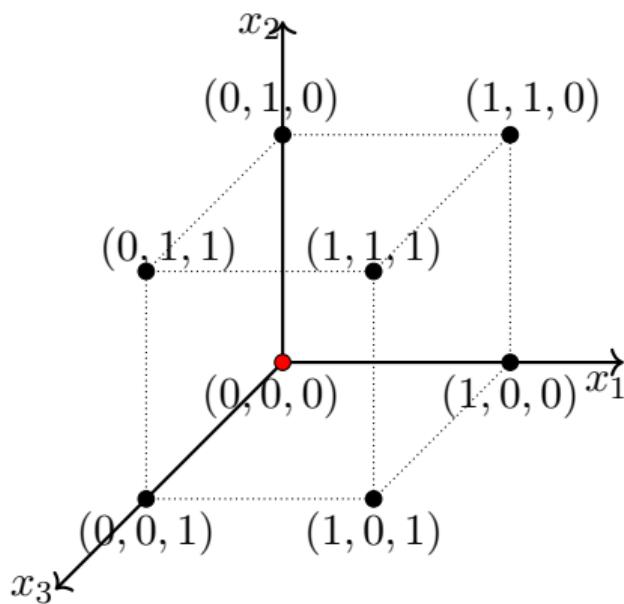
OR

- What if we have more than 2 inputs?

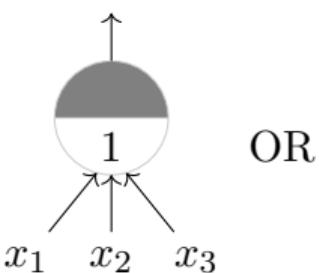
$$y \in \{0, 1\}$$



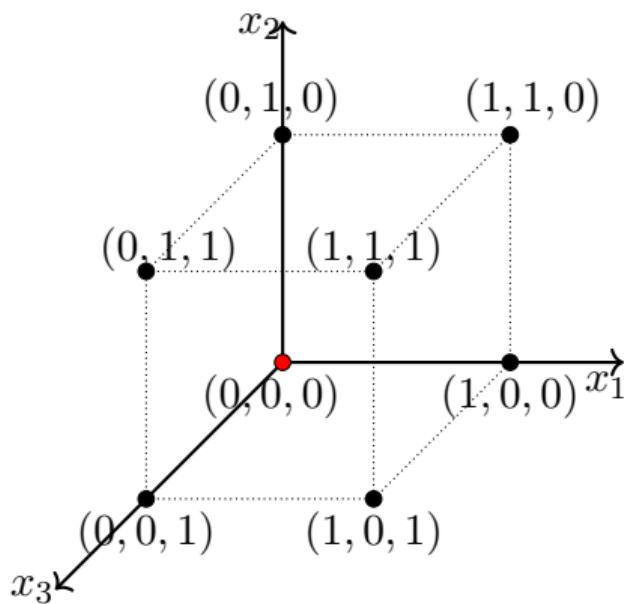
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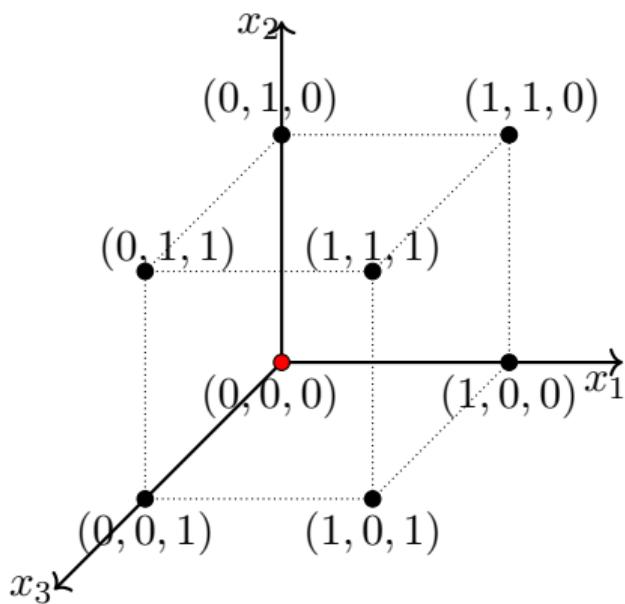
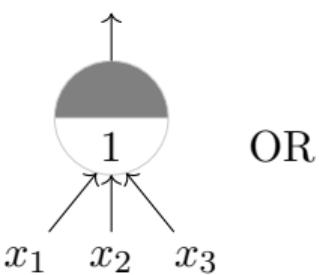
$$y \in \{0, 1\}$$



- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane

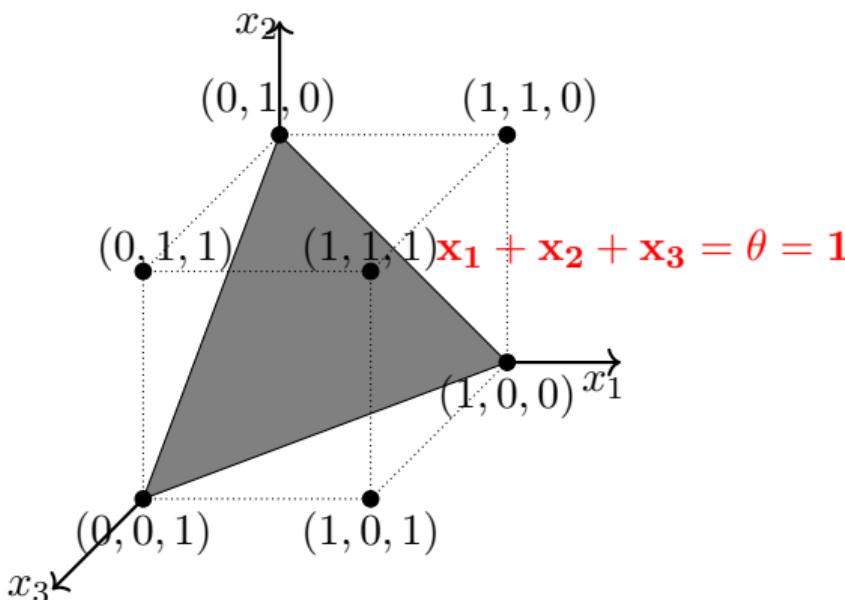
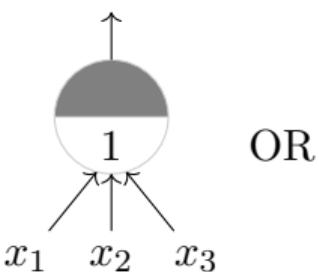


$$y \in \{0, 1\}$$



- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane
- For the OR function, we want a plane such that the point  $(0,0,0)$  lies on one side and the remaining 7 points lie on the other side of the plane

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- For the OR function, we want a plane such that the point  $(0,0,0)$  lies on one side and the remaining 7 points lie on the other side of the plane

## The story so far ...

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- A single McCulloch Pitts Neuron can be used to represent boolean functions which are linearly separable
- Linear separability (for boolean functions) : There exists a line (plane) such that all inputs which produce a 1 lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane)

## Module 2.3: Perceptron

## The story ahead ...

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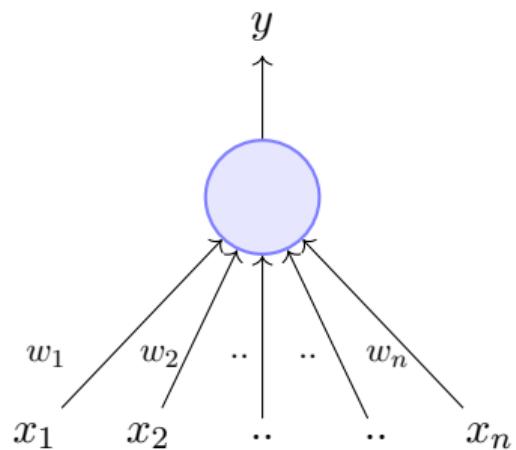
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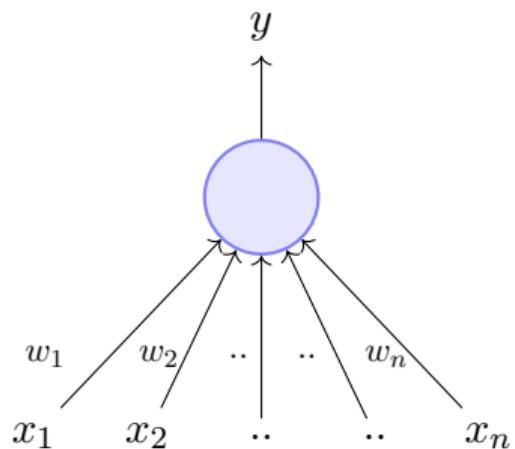
## The story ahead ...

- What about non-boolean (say, real) inputs ?
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- What about functions which are not linearly separable ?

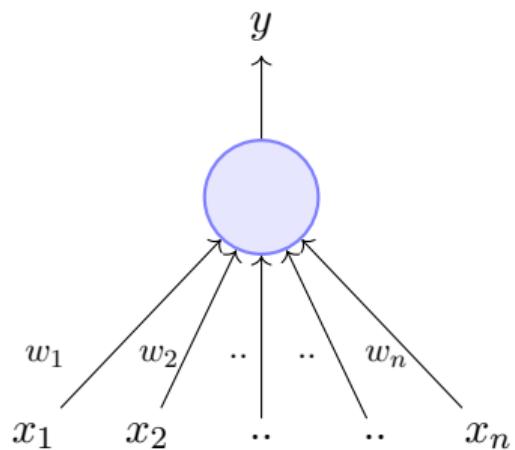
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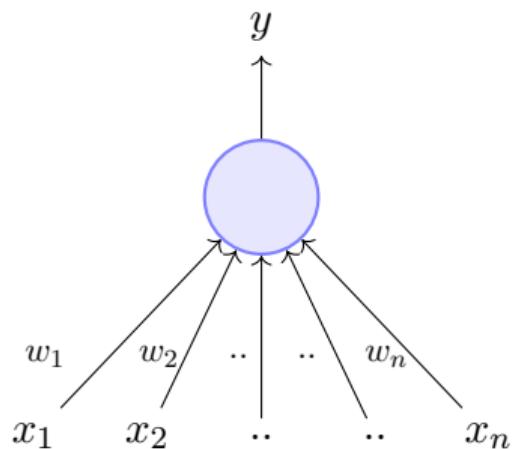
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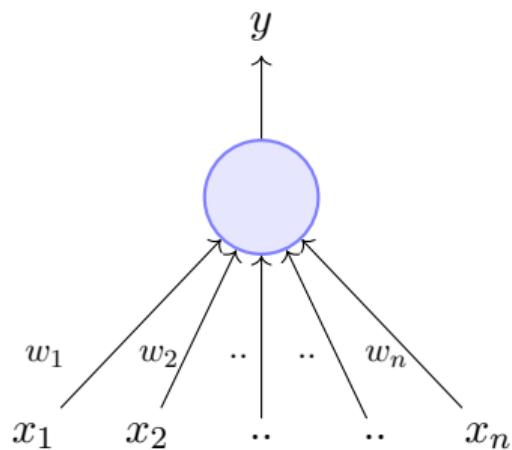
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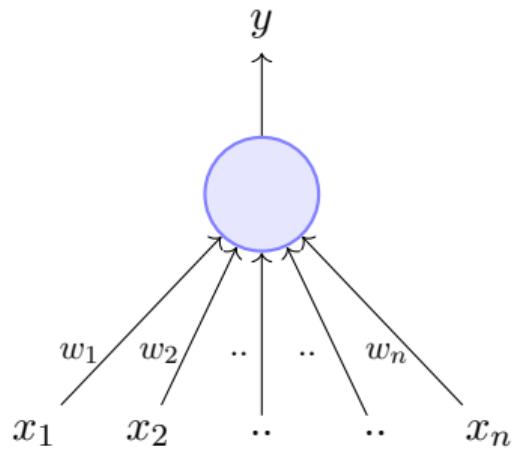
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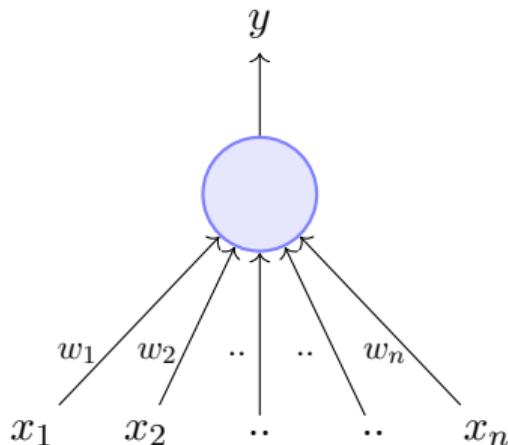


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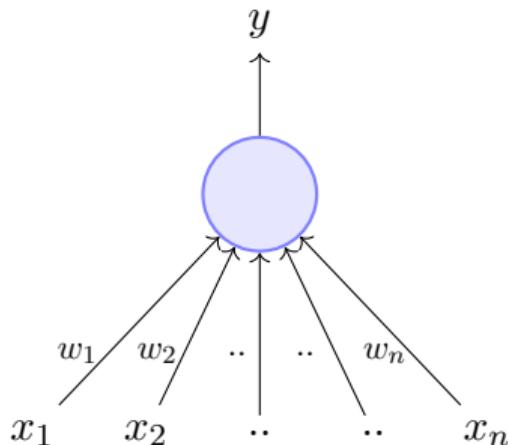


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- Refined and carefully analyzed by Minsky and Papert (1969) - their model is referred to as the **perceptron** model here



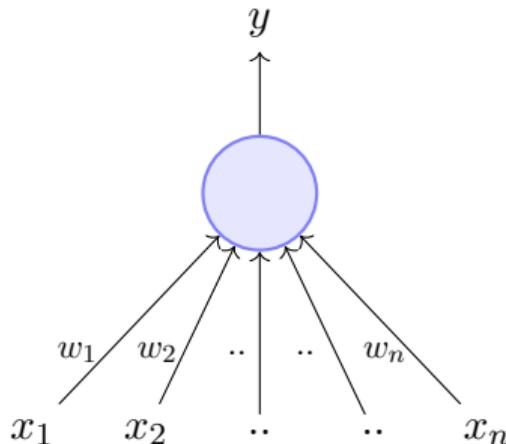


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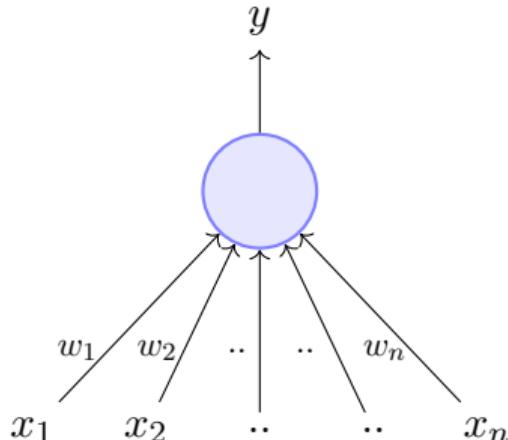
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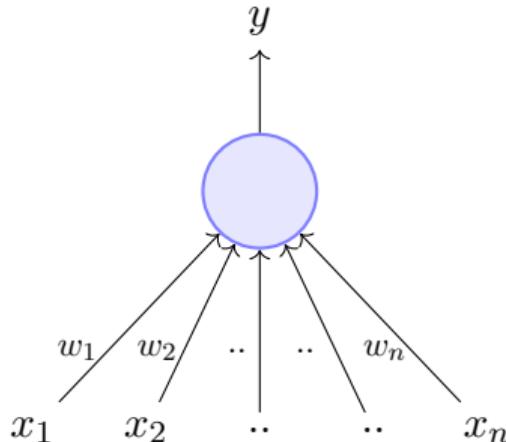


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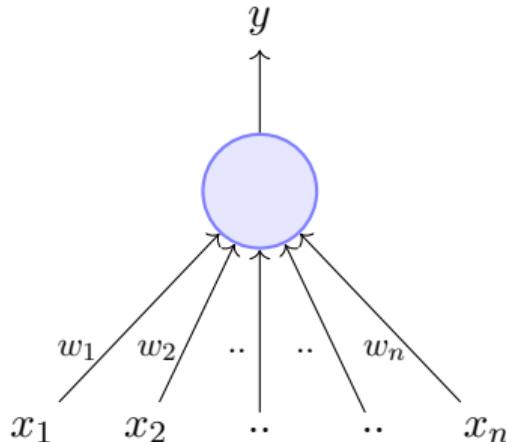
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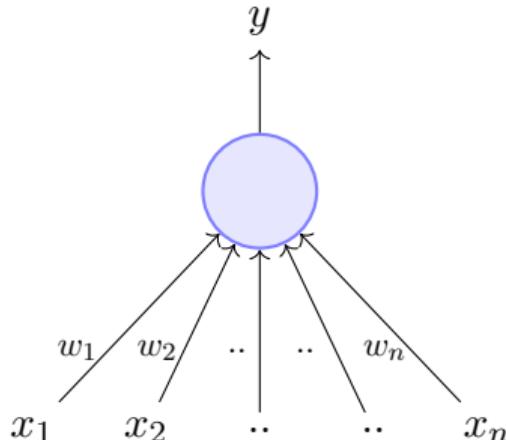
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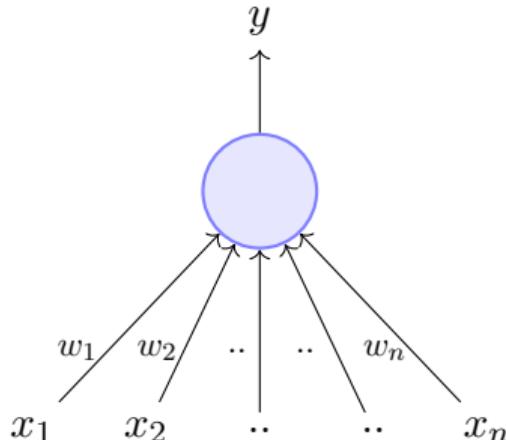
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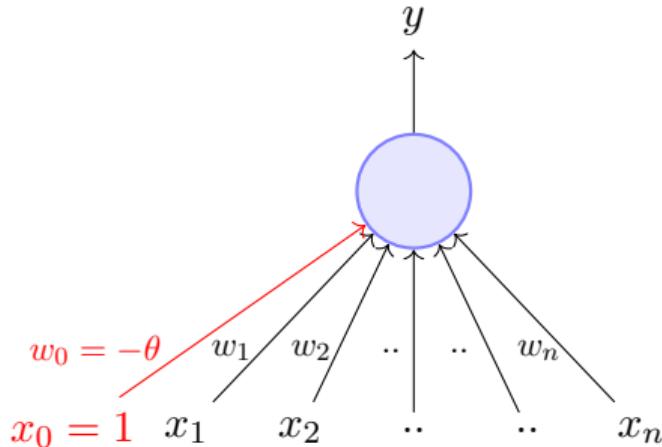
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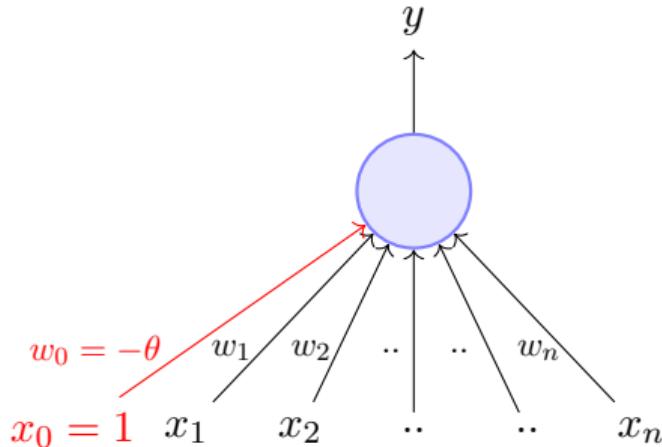
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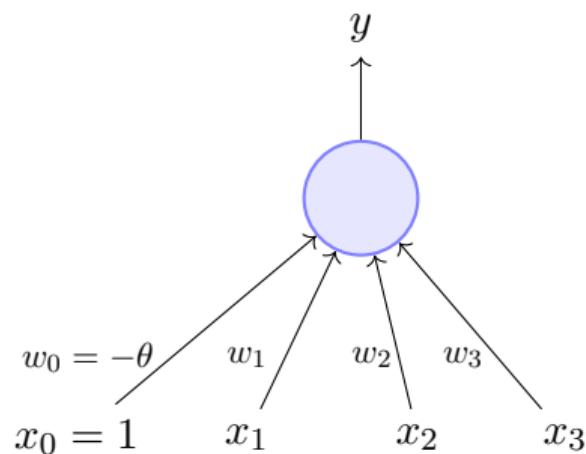
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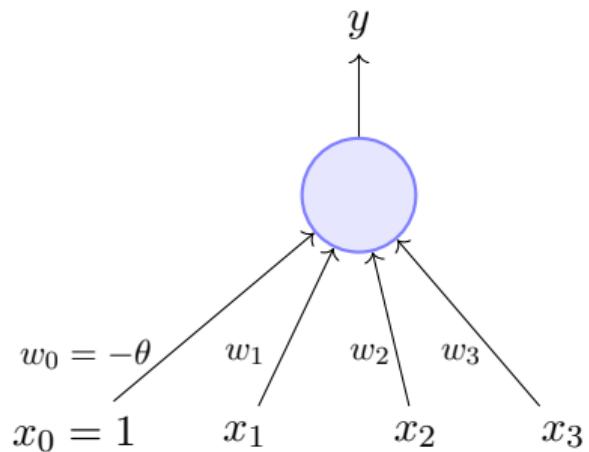
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We will now try to answer the following questions:

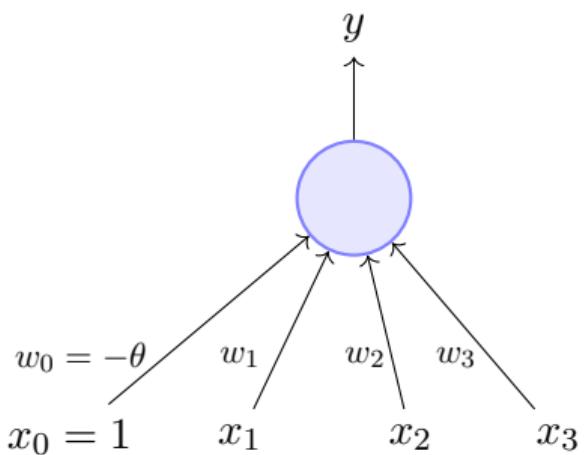
- Why are we trying to implement boolean functions?
- Why do we need weights ?
- Why is  $w_0 = -\theta$  called the bias ?

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- Suppose, we base our decision on 3 inputs (binary, for simplicity)

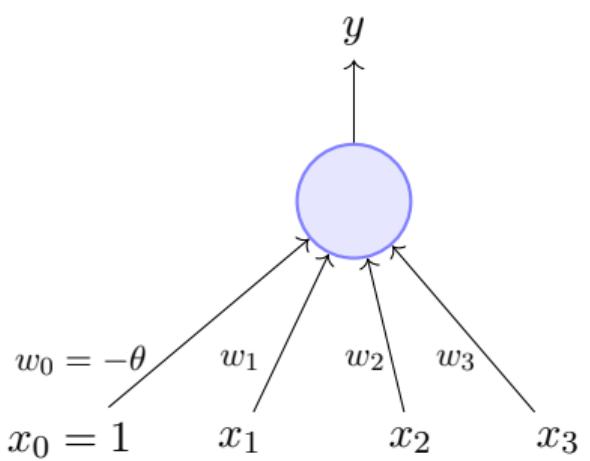


- Consider the task of predicting whether we would like a movie or not
- Suppose, we base our decision on 3 inputs (binary, for simplicity)
- Based on our past viewing experience (**data**), we may give a high weight to *isDirectorNolan* as compared to the other inputs

$x_1 = \text{isActorDamon}$

$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

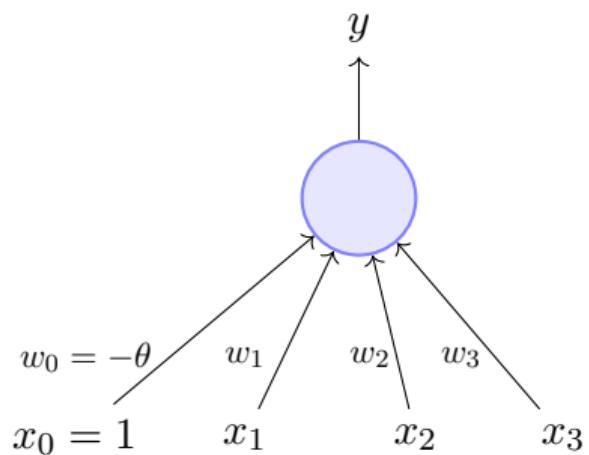


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- Suppose, we base our decision on 3 inputs (binary, for simplicity)
- Based on our past viewing experience (**data**), we may give a high weight to *isDirectorNolan* as compared to the other inputs
- Specifically, even if the actor is not *Matt Damon* and the genre is not *thriller* we would still want to cross the threshold  $\theta$  by assigning a high weight to *isDirectorNolan*

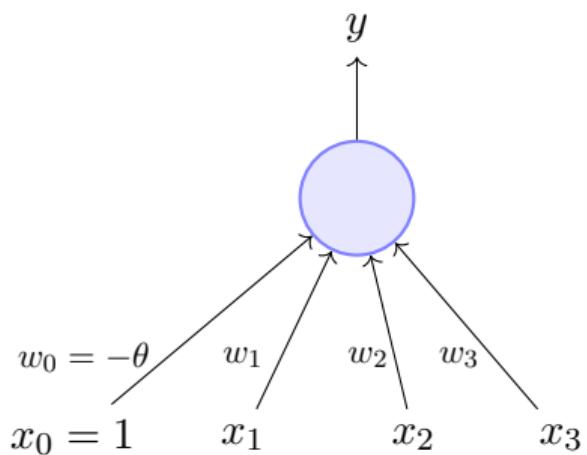


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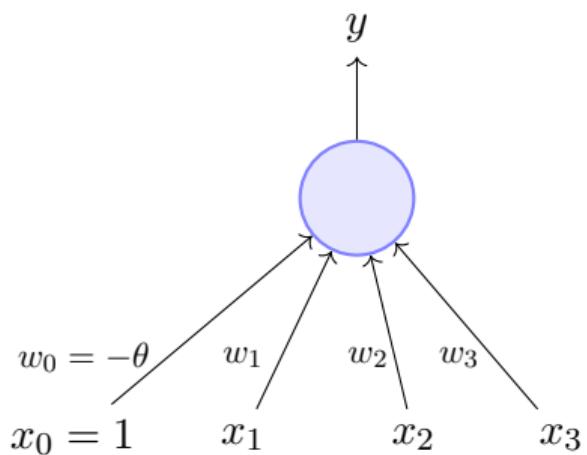


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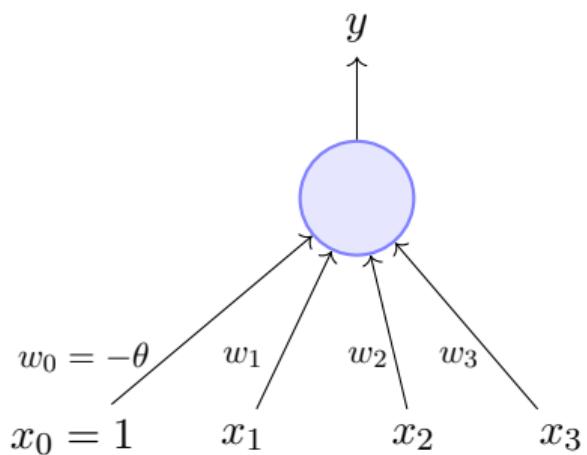


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- The weights ( $w_1, w_2, \dots, w_n$ ) and the bias ( $w_0$ ) will depend on the data (viewer history in this case)

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What kind of functions can be implemented using the perceptron? Any difference from McCulloch Pitts neurons?

# McCulloch Pitts Neuron

(assuming no inhibitory inputs)

$$y = 1 \quad if \sum_{i=0}^n x_i \geq 0$$

$$= 0 \quad if \sum_{i=0}^n x_i < 0$$

# Perceptron

$$y = 1 \quad if \sum_{i=0}^n \textcolor{red}{w_i} * x_i \geq 0$$

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- In other words, a single perceptron can only be used to implement linearly separable functions
- Then what is the difference? The weights (including threshold) can be learned and the inputs can be real valued
- We will first revisit some boolean functions and then see the perceptron learning algorithm (for learning weights)

$x_1$	$x_2$	OR
0	0	

---

$x_1$	$x_2$	OR
0	0	0

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i$

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

---

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
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$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
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0	1	1	

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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- One possible solution to this set of inequalities is  $w_0 = -1, w_1 = 1.1, w_2 = 1.1$  (and various other solutions are possible)

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
1	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$

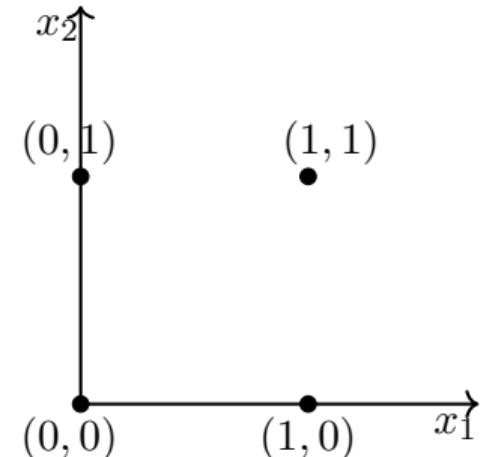
$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

$$w_0 + w_1 \cdot 0 + w_2 \cdot 1 \geq 0 \implies w_2 \geq -w_0$$

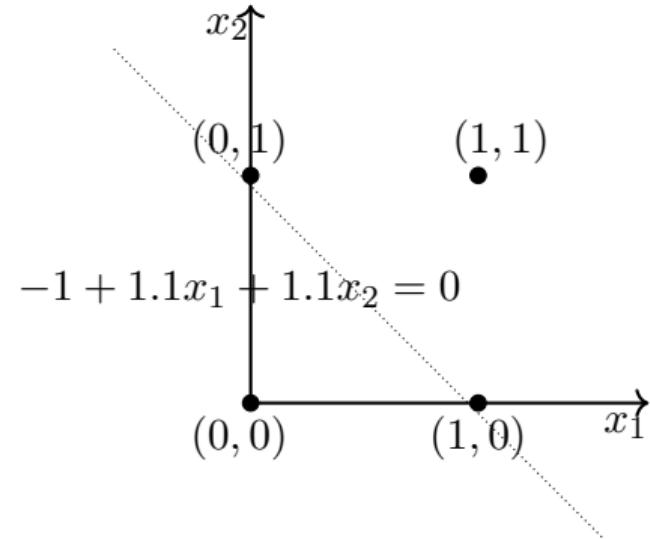
$$w_0 + w_1 \cdot 1 + w_2 \cdot 0 \geq 0 \implies w_1 \geq -w_0$$

$$w_0 + w_1 \cdot 1 + w_2 \cdot 1 \geq 0 \implies w_1 + w_2 \geq -w_0$$

- One possible solution to this set of inequalities is  $w_0 = -1, w_1 = 1, w_2 = 1$  (and various other solutions are possible)



$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
1	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$



$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

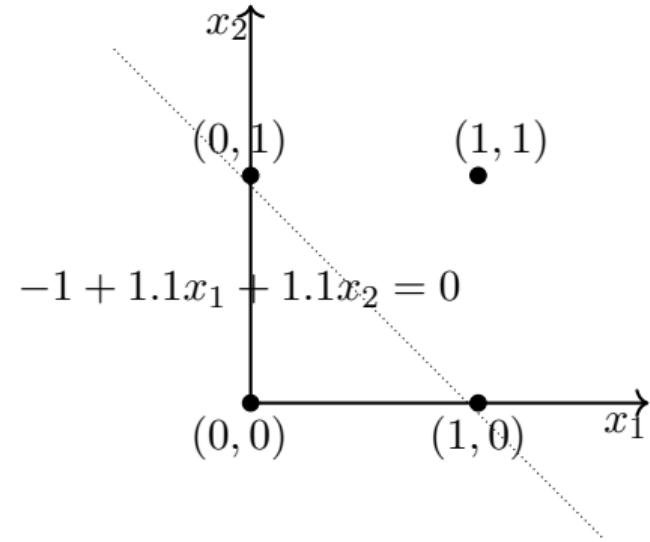
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- One possible solution to this set of inequalities is  $w_0 = -1, w_1 = 1.1, w_2 = 1.1$  (and various other solutions are possible)

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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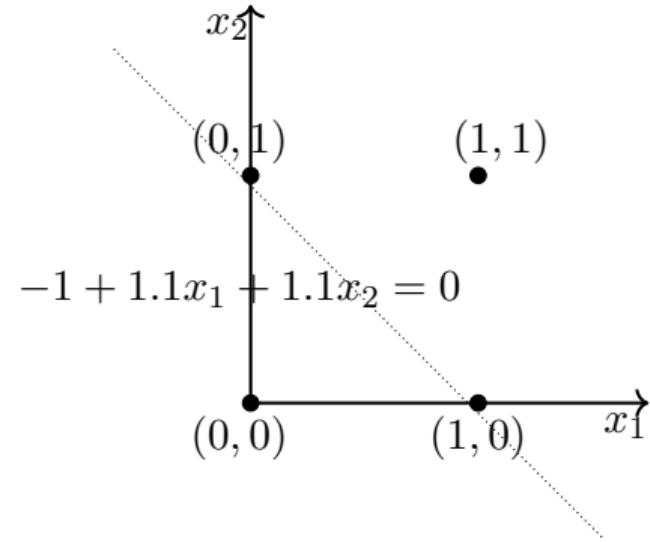
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- Note that we can come up with a similar set of inequalities and find the value of  $\theta$  for a McCulloch Pitts neuron also

$x_1$	$x_2$	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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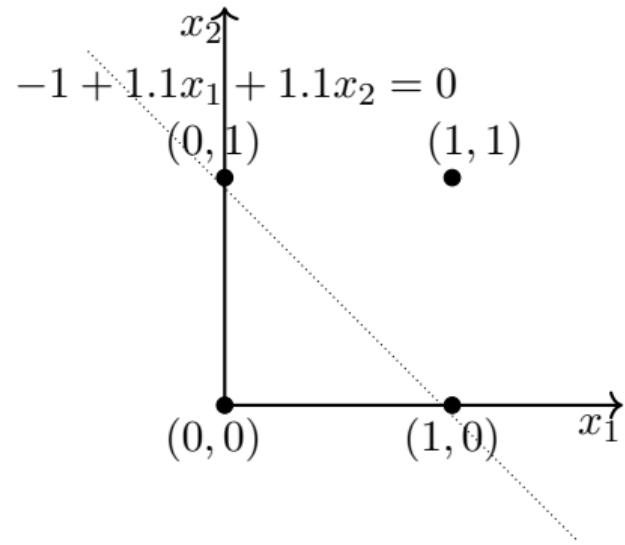
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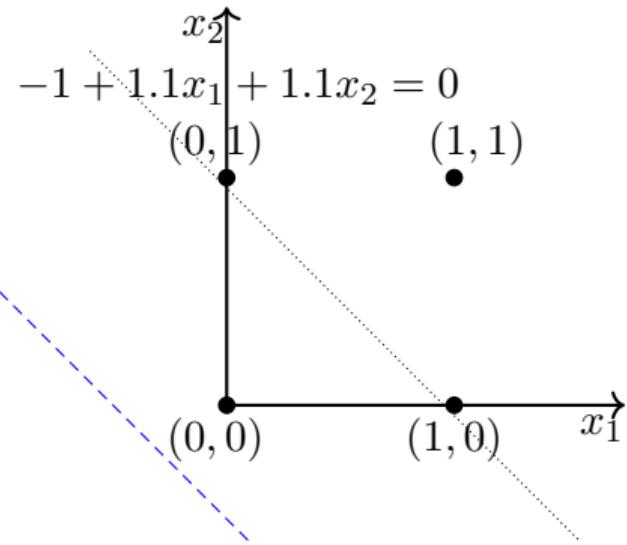
- Note that we can come up with a similar set of inequalities and find the value of  $\theta$  for a McCulloch Pitts neuron also (Try it!)

## Module 2.4: Errors and Error Surfaces

- Let us fix the threshold ( $-w_0 = 1$ ) and try different values of  $w_1, w_2$

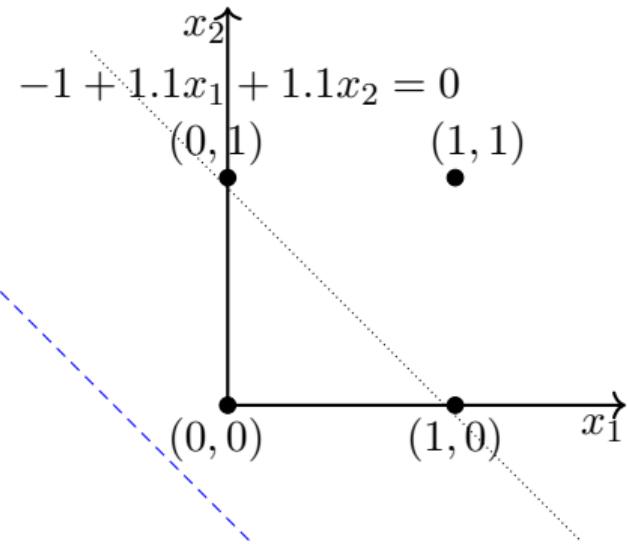


- Let us fix the threshold ( $-w_0 = 1$ ) and try different values of  $w_1, w_2$
- Say,  $w_1 = -1, w_2 = -1$



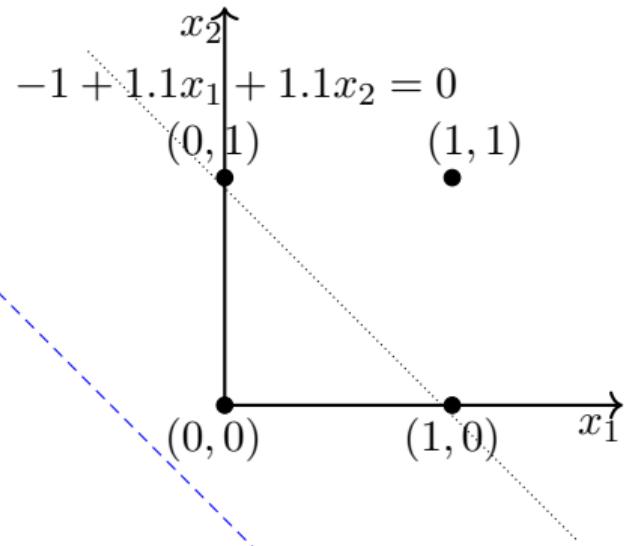
$$-1 + (-1)x_1 + (-1)x_2 = 0$$

- Let us fix the threshold ( $-w_0 = 1$ ) and try different values of  $w_1, w_2$
- Say,  $w_1 = -1, w_2 = -1$
- What is wrong with this line?



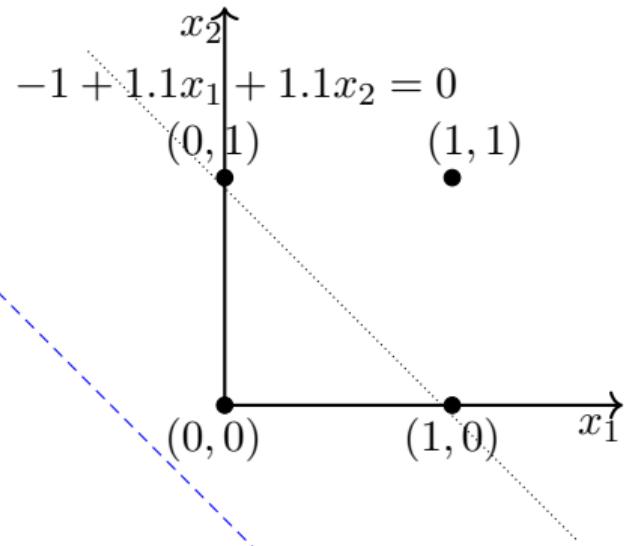
$$-1 + (-1)x_1 + (-1)x_2 = 0$$

- Let us fix the threshold ( $-w_0 = 1$ ) and try different values of  $w_1, w_2$
- Say,  $w_1 = -1, w_2 = -1$
- What is wrong with this line? We make an error on 1 out of the 4 inputs



$$-1 + (-1)x_1 + (-1)x_2 = 0$$

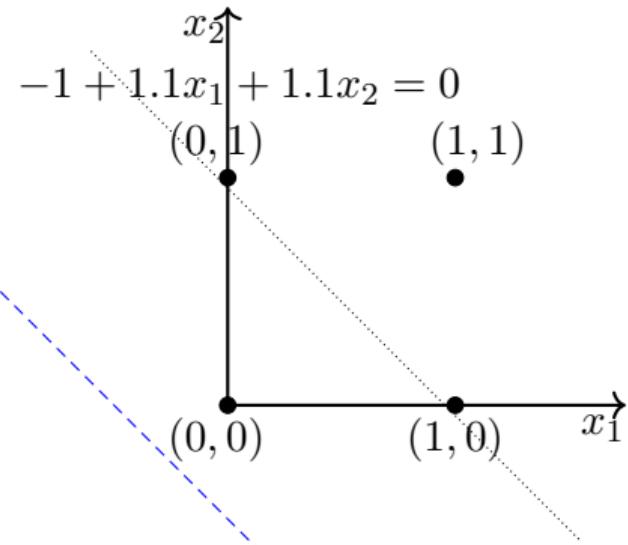
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- Say,  $w_1 = -1, w_2 = -1$
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- Lets try some more values of  $w_1, w_2$  and note how many errors we make



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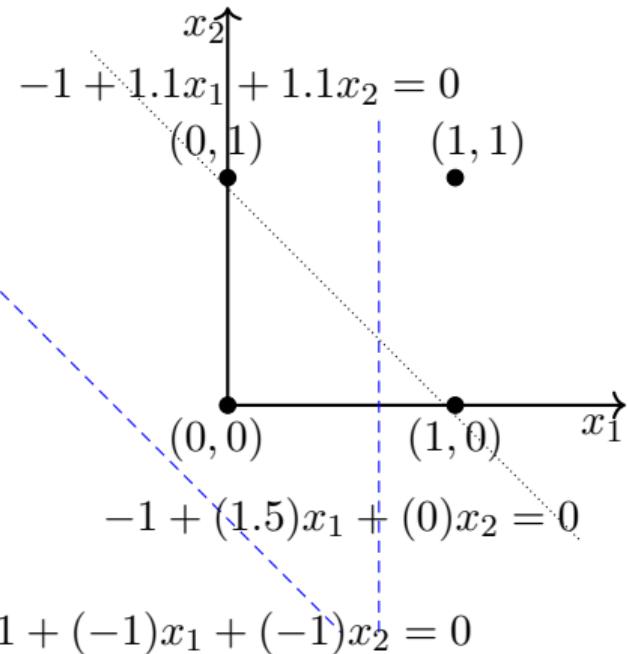
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$w_1$	$w_2$	errors
-1	-1	3



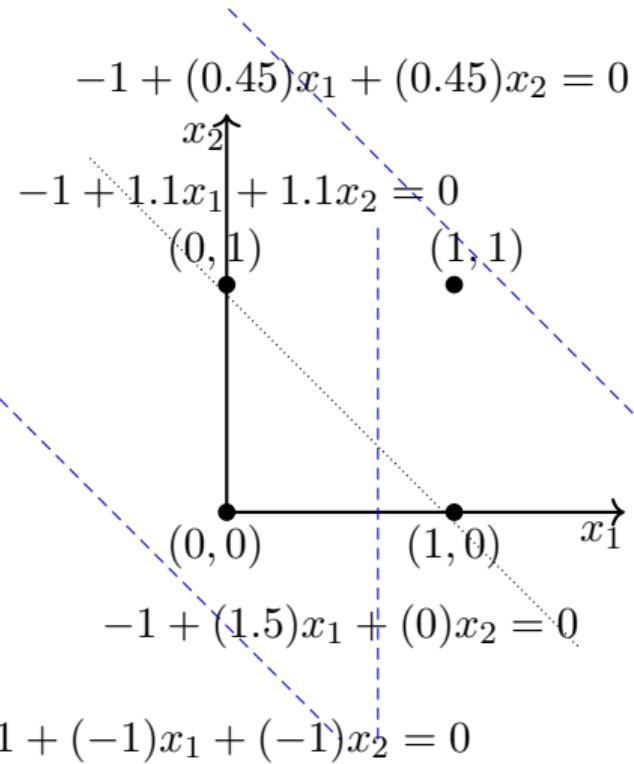
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$w_1$	$w_2$	errors
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1.5	0	1



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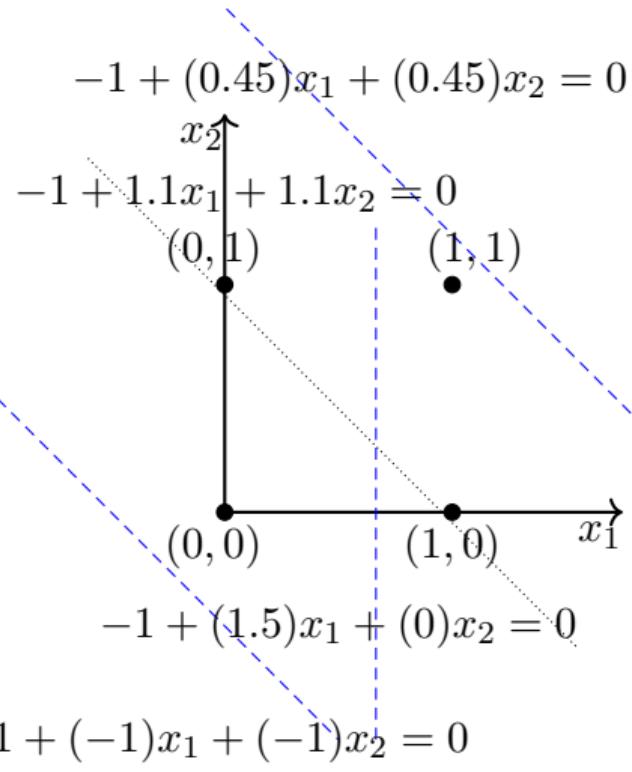
$w_1$	$w_2$	errors
-1	-1	3
1.5	0	1
0.45	0.45	3



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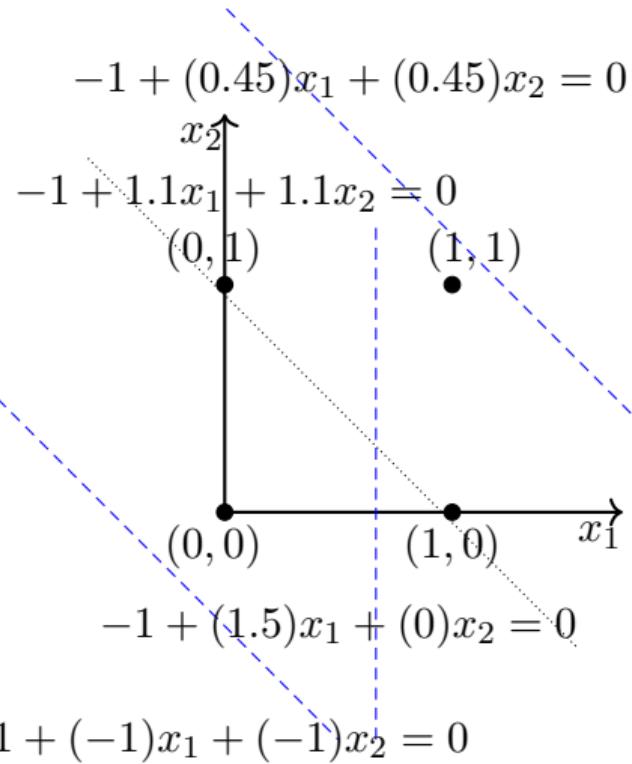
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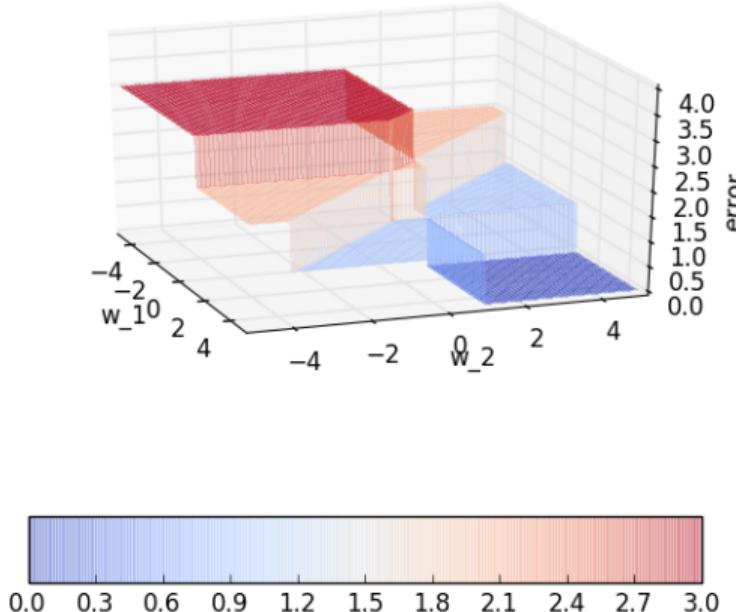
- We are interested in those values of  $w_0, w_1, w_2$  which result in 0 error
- Let us plot the error surface corresponding to different values of  $w_0, w_1, w_2$



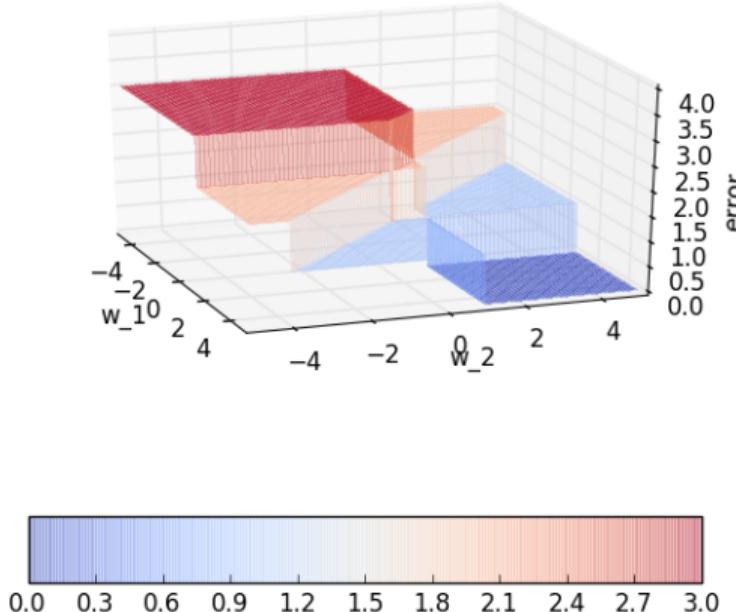
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- For the OR function, an error occurs if  $(x_1, x_2) = (0, 0)$  but  $-w_0 + w_1 * x_1 + w_2 * x_2 \geq 0$  or if  $(x_1, x_2) \neq (0, 0)$  but  $-w_0 + w_1 * x_1 + w_2 * x_2 < 0$



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- We are interested in finding an algorithm which finds the values of  $w_1, w_2$  which minimize this error

## Module 2.5: Perceptron Learning Algorithm

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- Apart from implementing boolean functions (which does not look very interesting) what can a perceptron be used for ?
- Our interest lies in the use of perceptron as a binary classifier. Let us see what this means...

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$x_1 = \text{isActorDamon}$

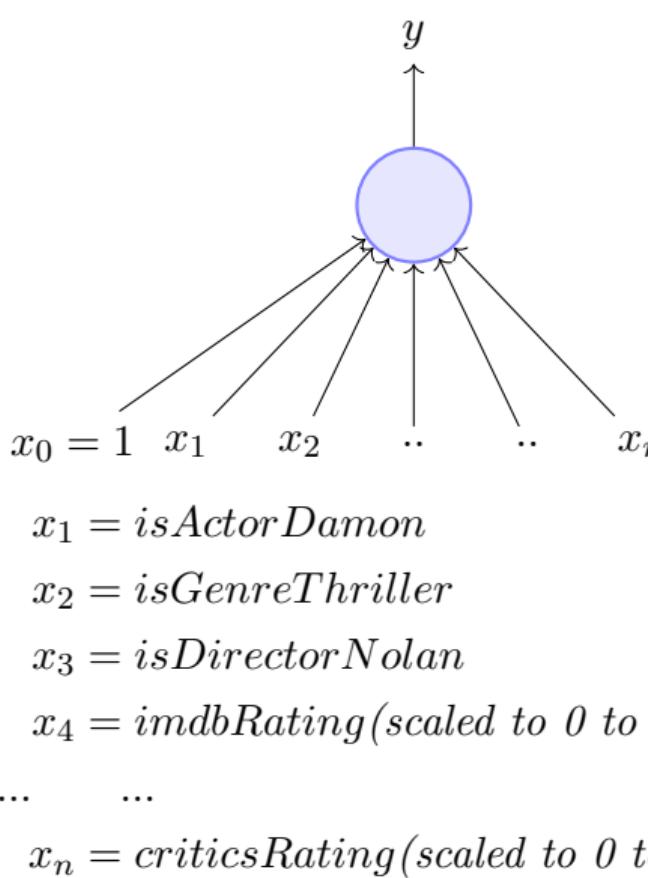
$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

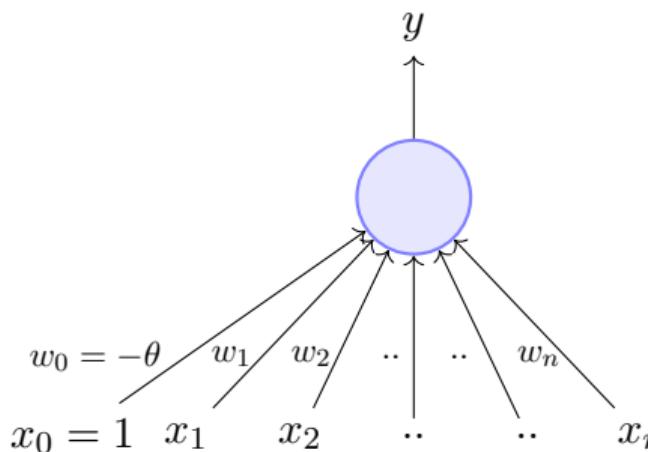
$x_4 = \text{imdbRating}(\text{scaled to 0 to 1})$

... ...

$x_n = \text{criticsRating}(\text{scaled to 0 to 1})$



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- Further, suppose we represent each movie with  $n$  features (some boolean, some real valued)
- We will assume that the data is linearly separable and we want a perceptron to learn how to make this decision
- In other words, we want the perceptron to find the equation of this separating plane (or find the values of  $w_0, w_1, w_2, \dots, w_m$ )

## **Algorithm:** Perceptron Learning Algorithm

---

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---

$P \leftarrow \text{inputs with label 1};$

## **Algorithm:** Perceptron Learning Algorithm

---

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**if**  $\mathbf{x} \in P$  and  $\sum_{i=0}^n w_i * x_i < 0$  **then**

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**end**

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## **Algorithm:** Perceptron Learning Algorithm

```
P ← inputs with label 1;  
N ← inputs with label 0;  
Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
    if x ∈ P and  $\sum_{i=0}^n w_i * x_i < 0$  then  
        | w = w + x ;  
    end  
    if x ∈ N and  $\sum_{i=0}^n w_i * x_i \geq 0$  then  
        |  
    end  
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**end**

**if**  $\mathbf{x} \in N$  and  $\sum_{i=0}^n w_i * x_i \geq 0$  **then**  
|    $\mathbf{w} = \mathbf{w} - \mathbf{x}$  ;

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- Why would this work ?
- To understand why this works we will have to get into a bit of Linear Algebra and a bit of geometry...

- Consider two vectors  $\mathbf{w}$  and  $\mathbf{x}$

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$$\mathbf{w} = [w_0, w_1, w_2, \dots, w_n]$$

$$\mathbf{x} = [1, x_1, x_2, \dots, x_n]$$

- Consider two vectors  $\mathbf{w}$  and  $\mathbf{x}$

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$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^n w_i * x_i$$

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- We can thus rewrite the perceptron rule as

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$$y = 1 \quad if \quad \mathbf{w}^T \mathbf{x} \geq 0$$

$$= 0 \quad if \quad \mathbf{w}^T \mathbf{x} < 0$$

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- We are interested in finding the line  $\mathbf{w}^T \mathbf{x} = 0$  which divides the input space into two halves
- Every point ( $\mathbf{x}$ ) on this line satisfies the equation  $\mathbf{w}^T \mathbf{x} = 0$

- We can thus rewrite the perceptron rule as

$$y = 1 \quad if \quad \mathbf{w}^T \mathbf{x} \geq 0$$

$$= 0 \quad if \quad \mathbf{w}^T \mathbf{x} < 0$$

- Consider two vectors  $\mathbf{w}$  and  $\mathbf{x}$

$$\mathbf{w} = [w_0, w_1, w_2, \dots, w_n]$$

$$\mathbf{x} = [1, x_1, x_2, \dots, x_n]$$

$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^n w_i * x_i$$

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- We are interested in finding the line  $\mathbf{w}^T \mathbf{x} = 0$  which divides the input space into two halves
- Every point ( $\mathbf{x}$ ) on this line satisfies the equation  $\mathbf{w}^T \mathbf{x} = 0$
- What can you tell about the angle ( $\alpha$ ) between  $\mathbf{w}$  and any point ( $\mathbf{x}$ ) which lies on this line ?

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- The angle is  $90^\circ$  ( $\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$ )

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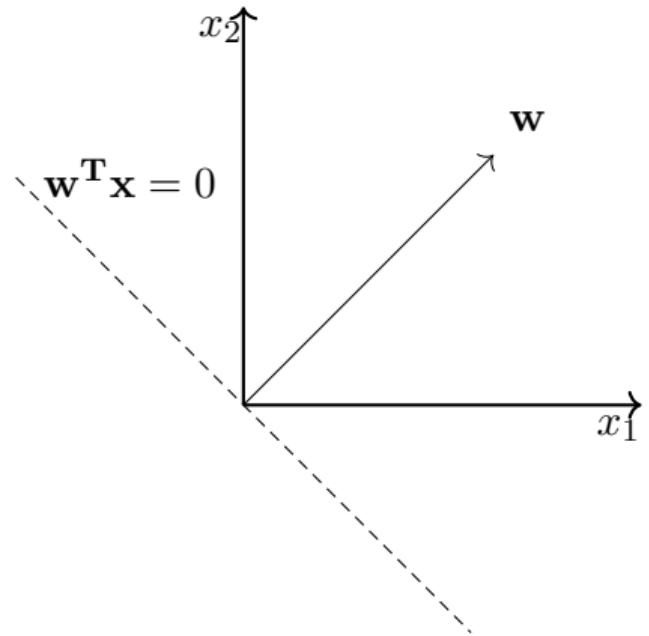
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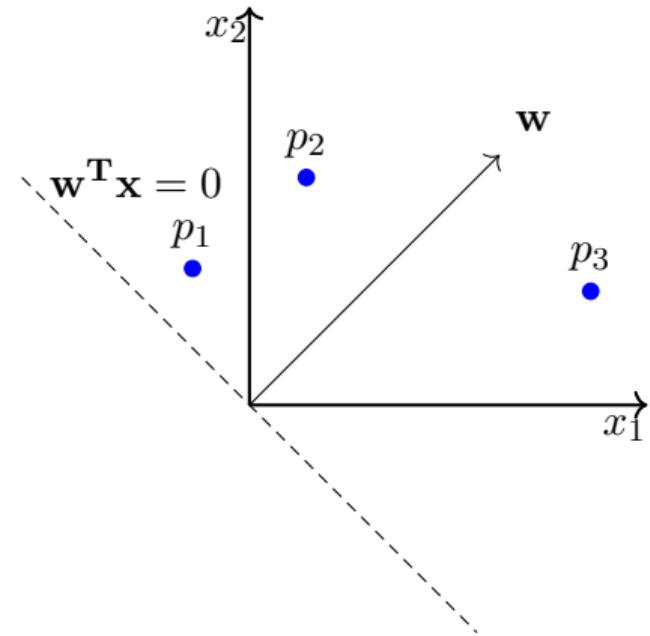
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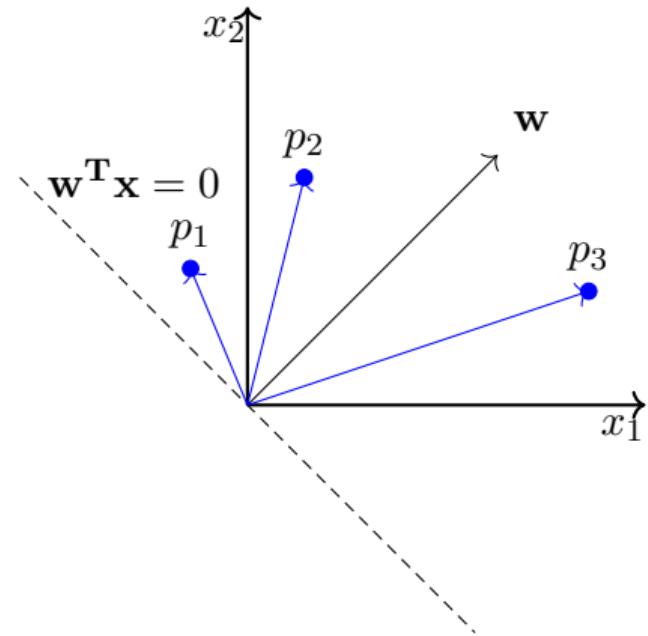
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- The angle is  $90^\circ$  ( $\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$ )
- Since the vector  $\mathbf{w}$  is perpendicular to every point on the line it is actually perpendicular to the line itself



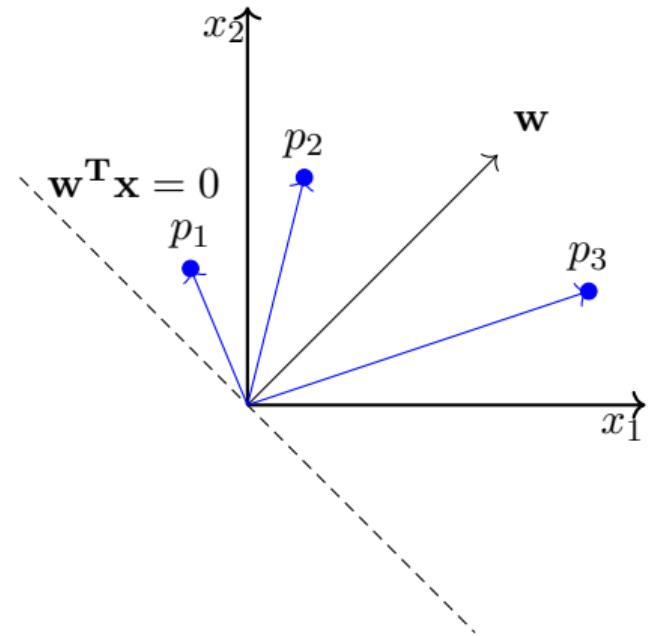
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*,  $\mathbf{w}^T \mathbf{x} \geq 0$ )



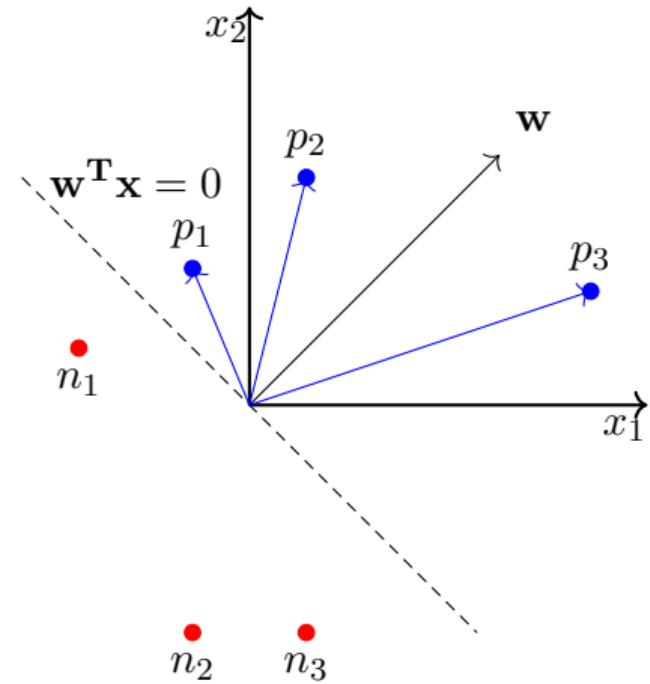
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*,  $\mathbf{w}^T \mathbf{x} \geq 0$ )
- What will be the angle between any such vector and  $\mathbf{w}$  ?



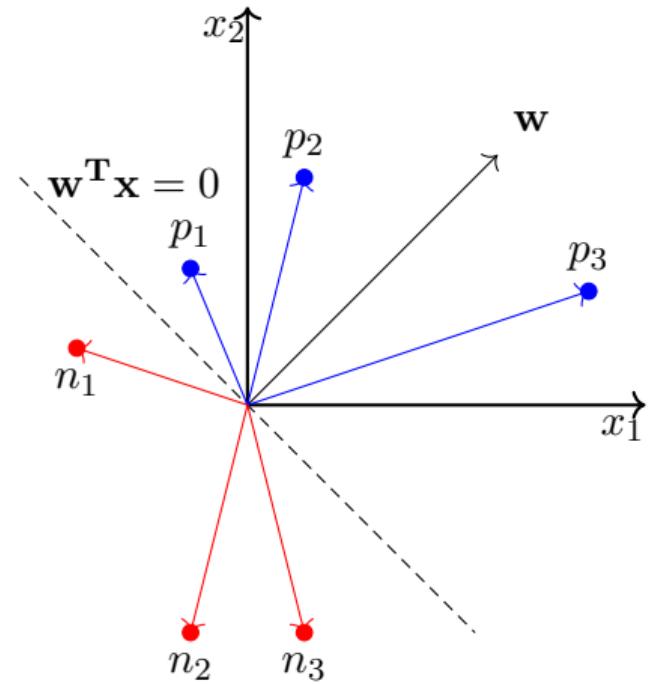
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*,  $\mathbf{w}^T \mathbf{x} \geq 0$ )
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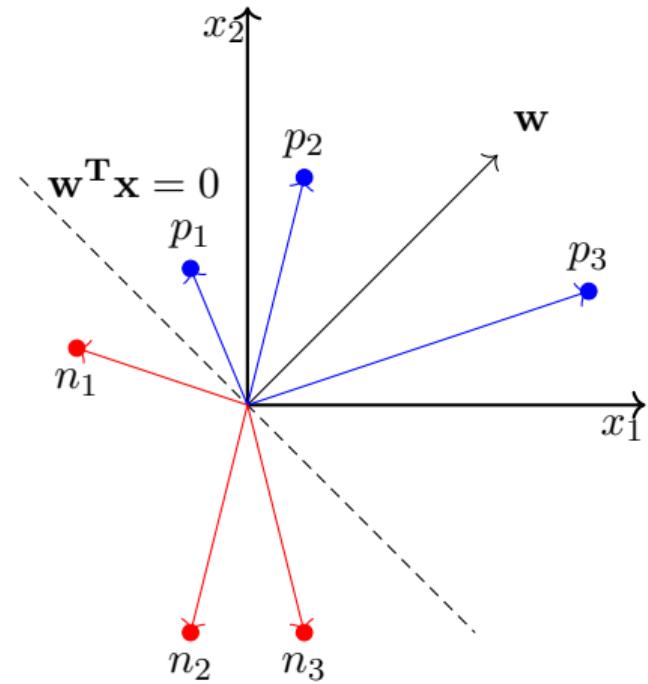
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- What about points (vectors) which lie in the negative half space of this line (*i.e.*,  $\mathbf{w}^T \mathbf{x} < 0$ )



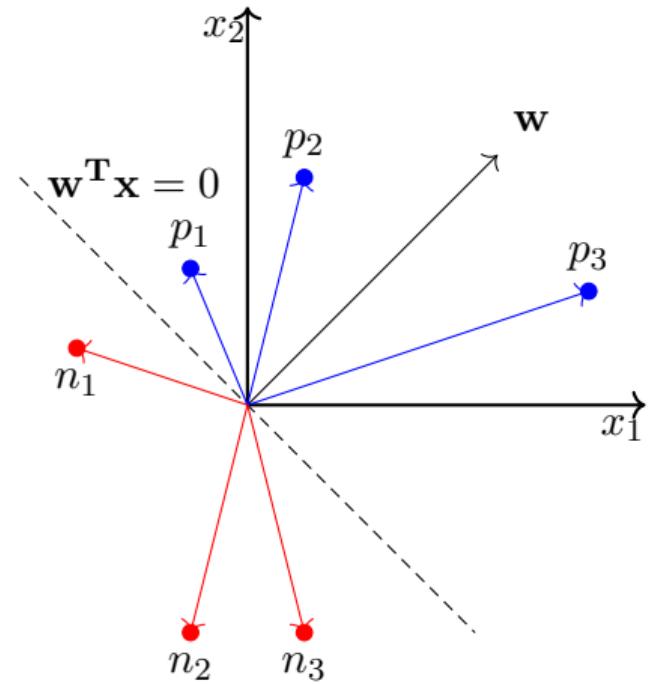
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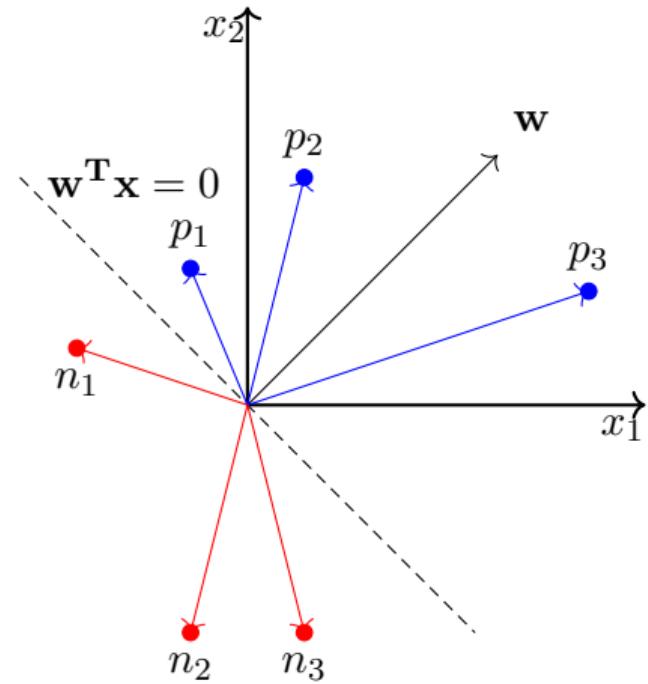
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- Of course, this also follows from the formula ( $\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$ )



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- Keeping this picture in mind let us revisit the algorithm



## **Algorithm:** Perceptron Learning Algorithm

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P ← inputs with label 1;  
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Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
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    end  
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$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$

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$$\cos(\alpha_{new}) > \cos\alpha$$

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$$\cos(\alpha_{new}) > \cos\alpha$$

- Thus  $\alpha_{new}$  will be less than  $\alpha$  and this is exactly what we want

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        | w = w + x ;  
    end  
    if x ∈ N and w.x ≥ 0 then  
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//the algorithm converges when all the  
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$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

- For  $\mathbf{x} \in N$  if  $\mathbf{w} \cdot \mathbf{x} \geq 0$  then it means that the angle ( $\alpha$ ) between this  $\mathbf{x}$  and the current  $\mathbf{w}$  is less than  $90^\circ$  (but we want  $\alpha$  to be greater than  $90^\circ$ )
- What happens to the new angle ( $\alpha_{new}$ ) when  $\mathbf{w}_{new} = \mathbf{w} - \mathbf{x}$

$$\begin{aligned}\cos(\alpha_{new}) &\propto \mathbf{w}_{new}^T \mathbf{x} \\ &\propto (\mathbf{w} - \mathbf{x})^T \mathbf{x}\end{aligned}$$

## **Algorithm:** Perceptron Learning Algorithm

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P ← inputs with label 1;  
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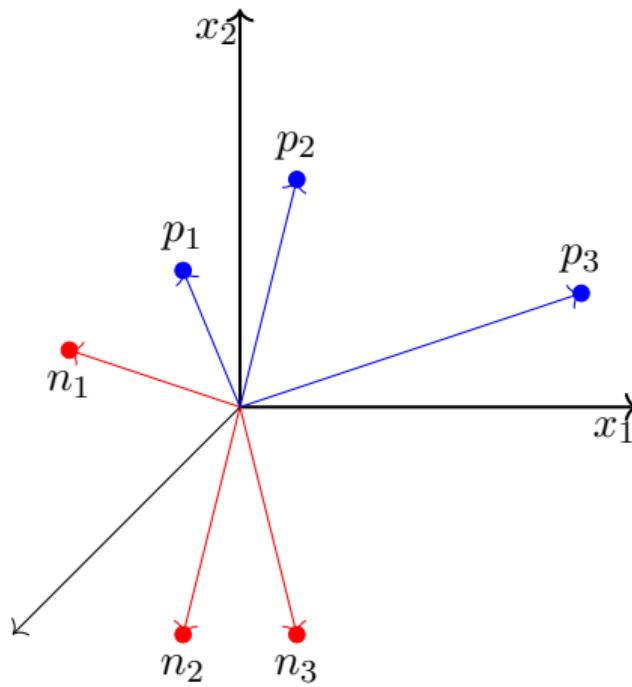
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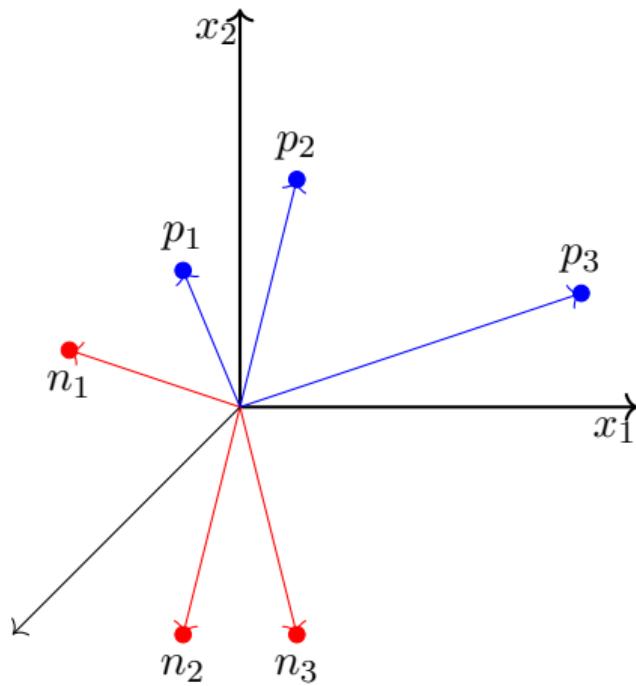
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$$\cos(\alpha_{new}) < \cos\alpha$$

- Thus  $\alpha_{new}$  will be greater than  $\alpha$  and this is exactly what we want

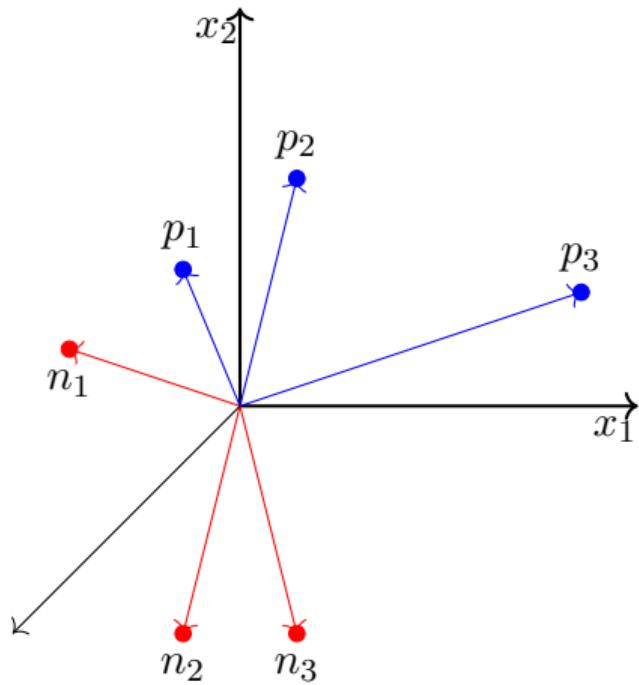
- We will now see this algorithm in action for a toy dataset



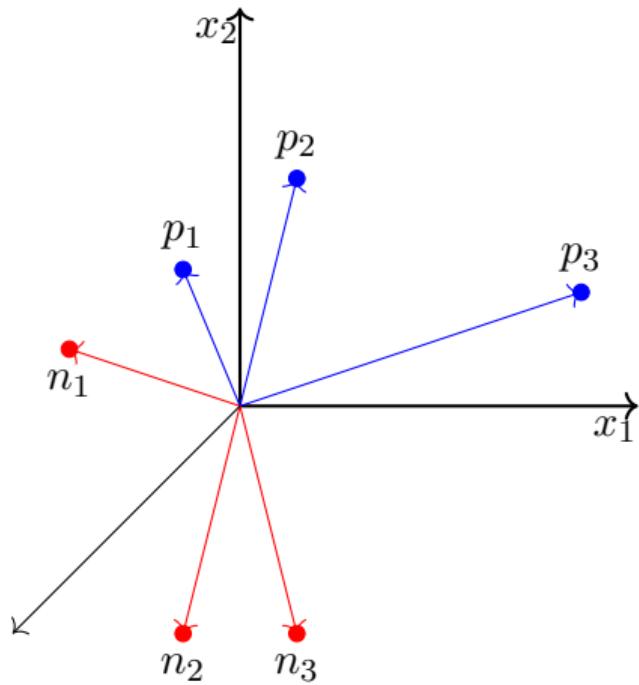
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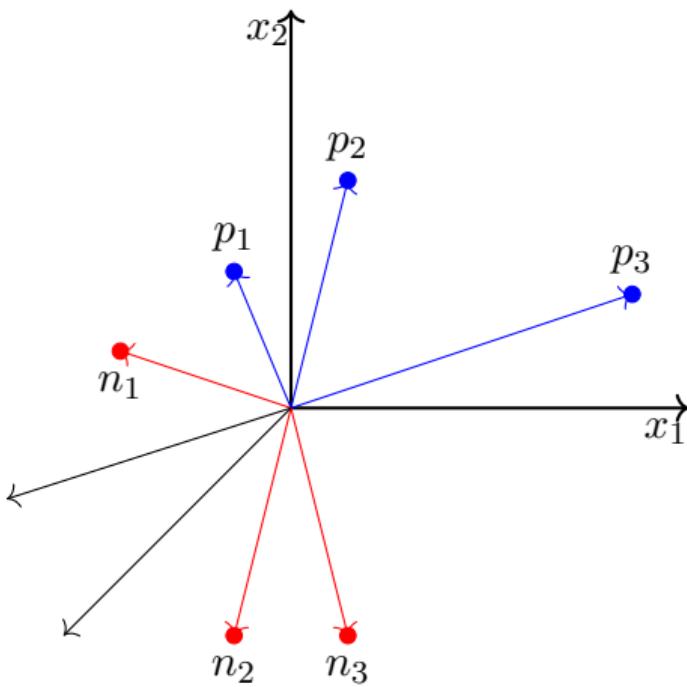
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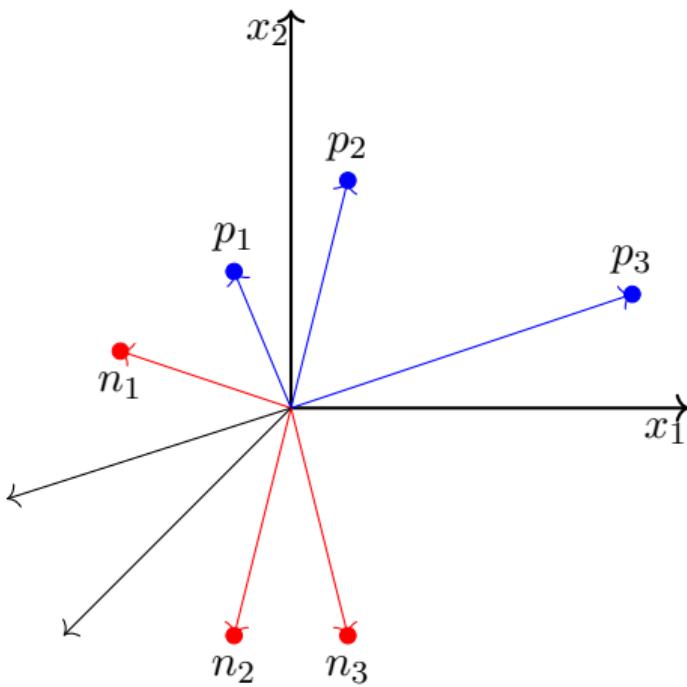
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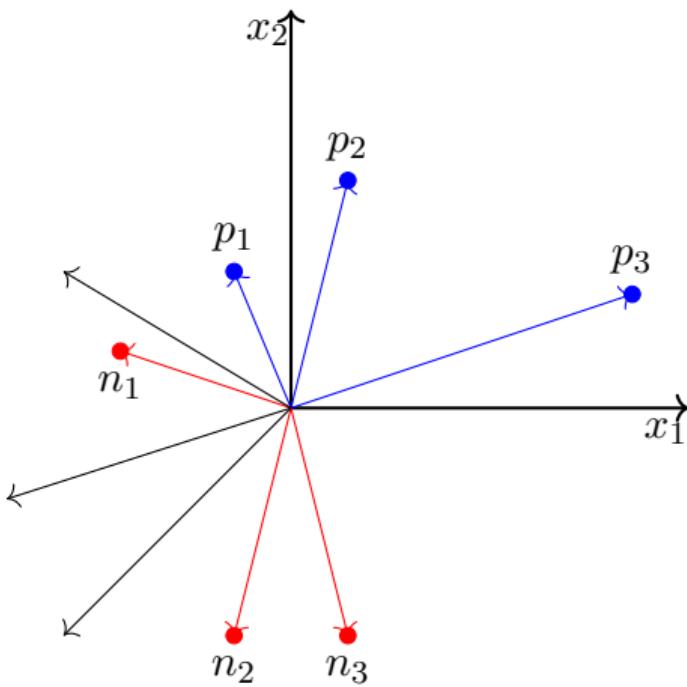
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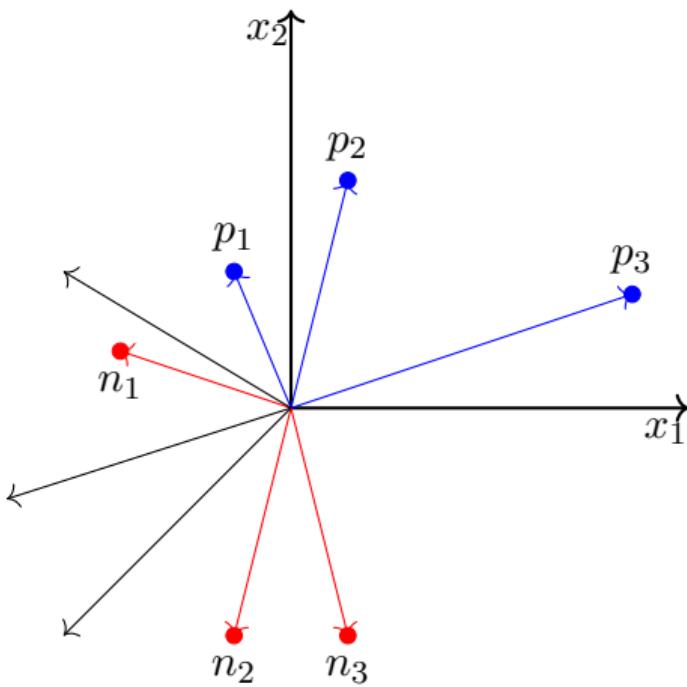
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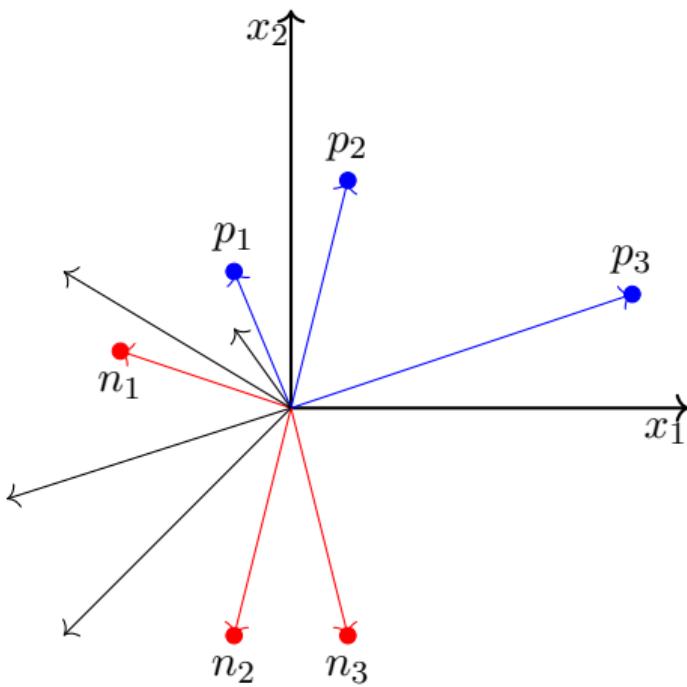
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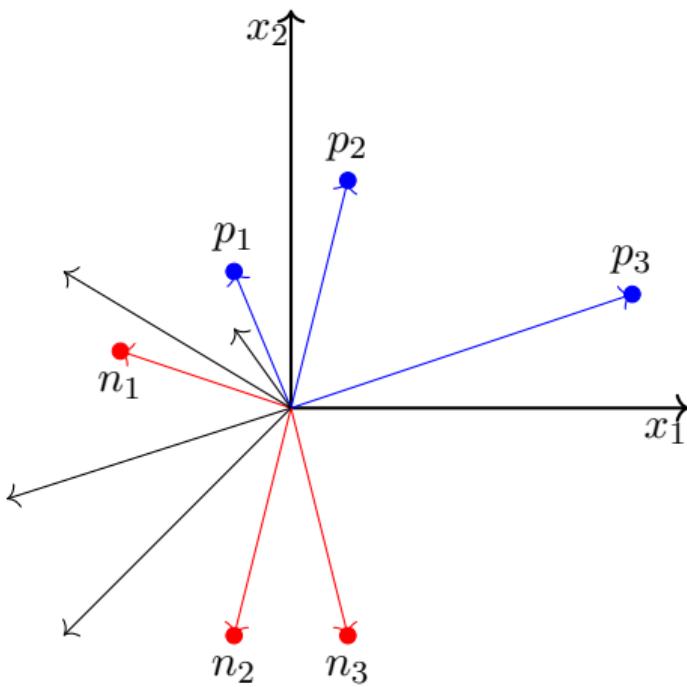
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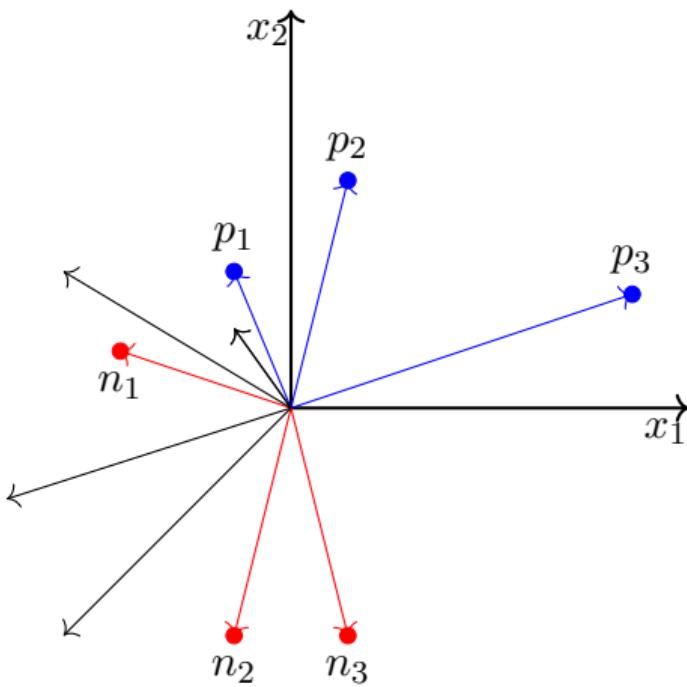
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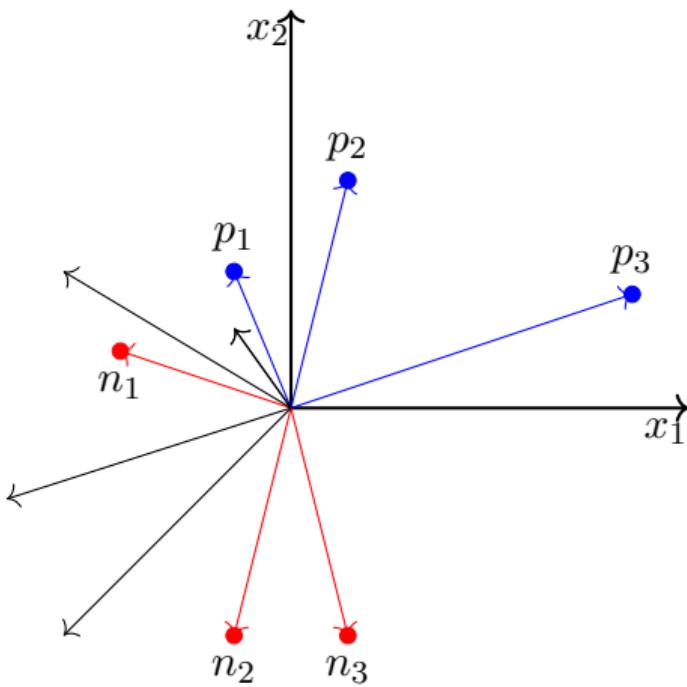
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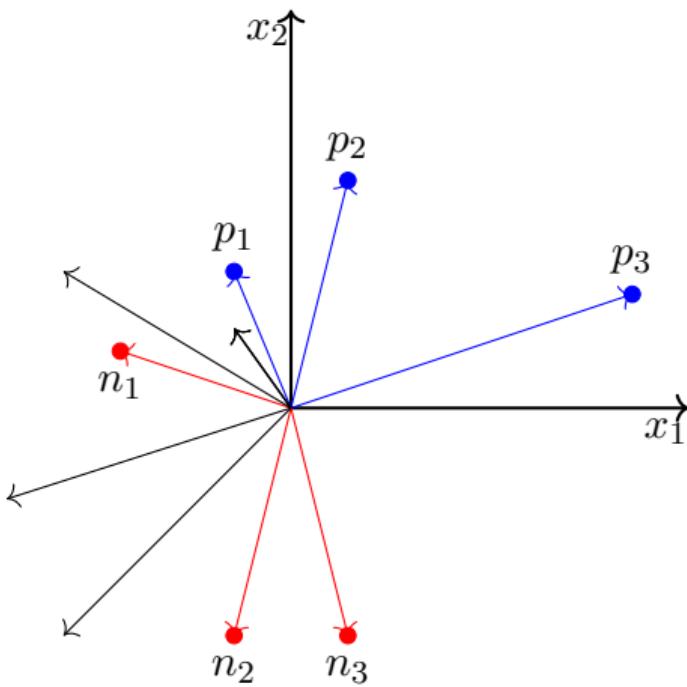
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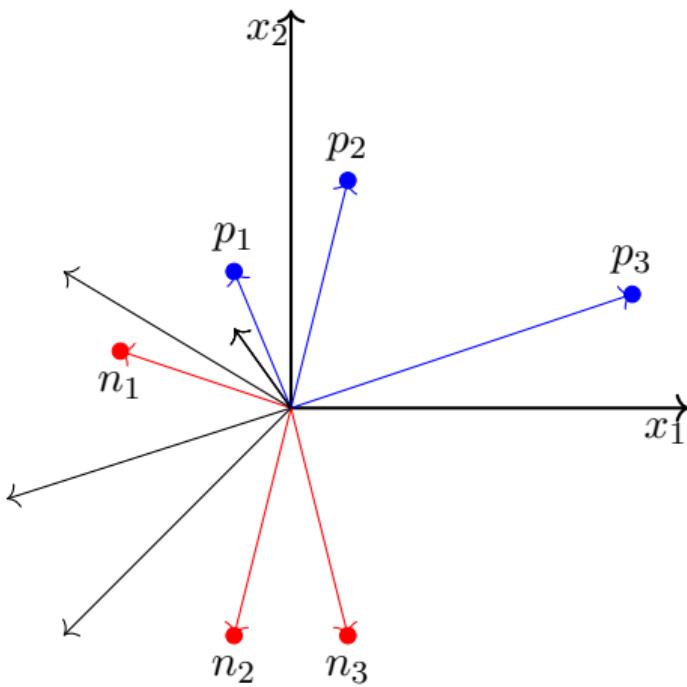
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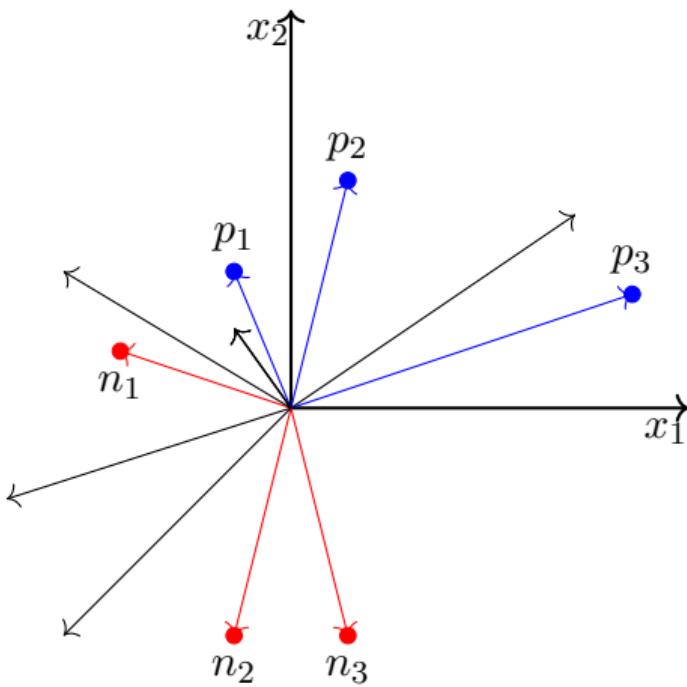
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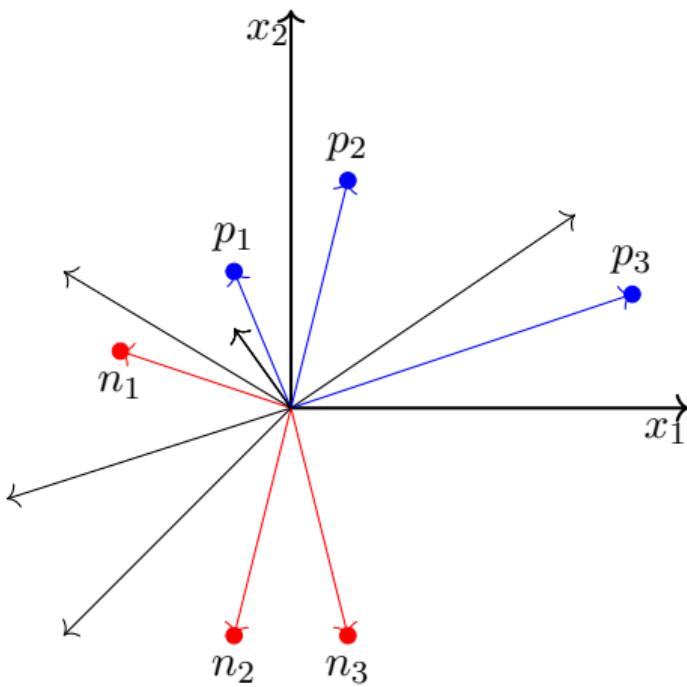
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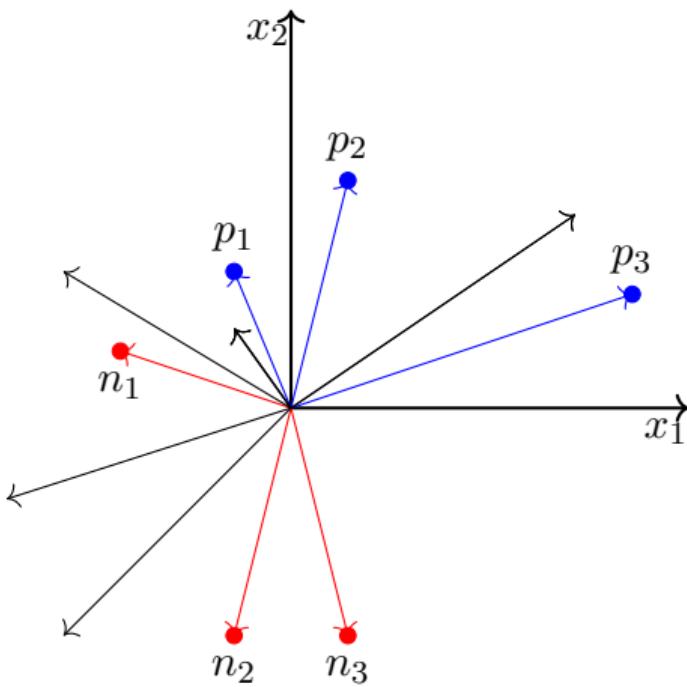
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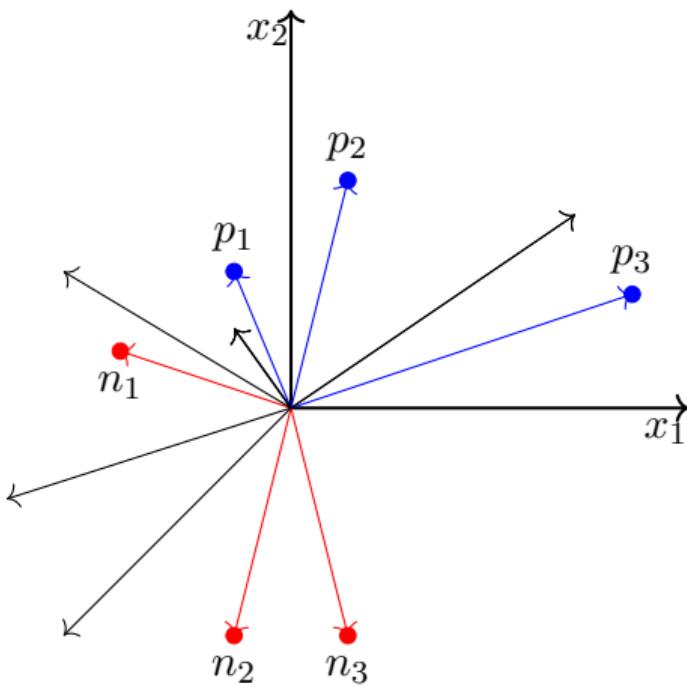
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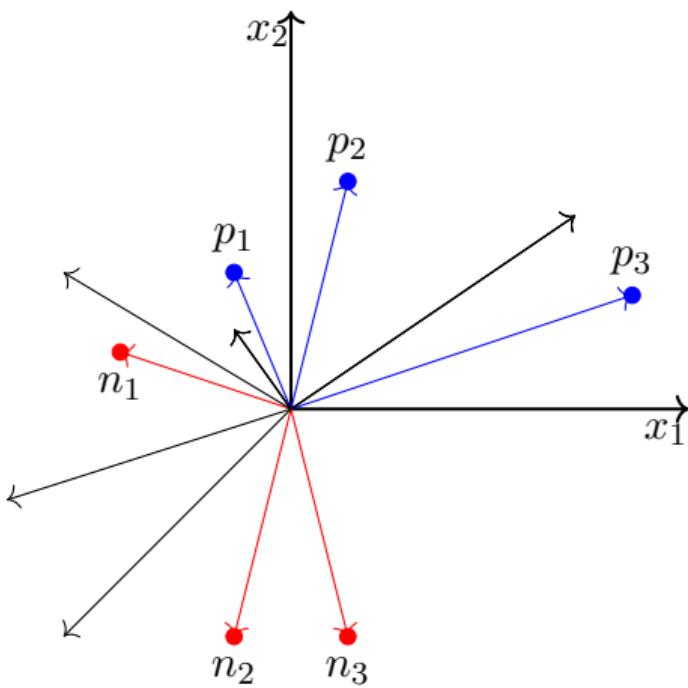
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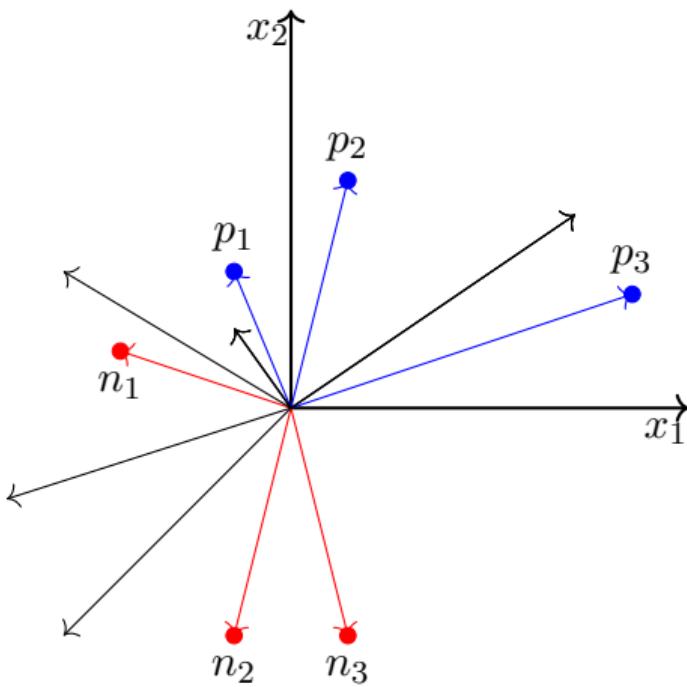
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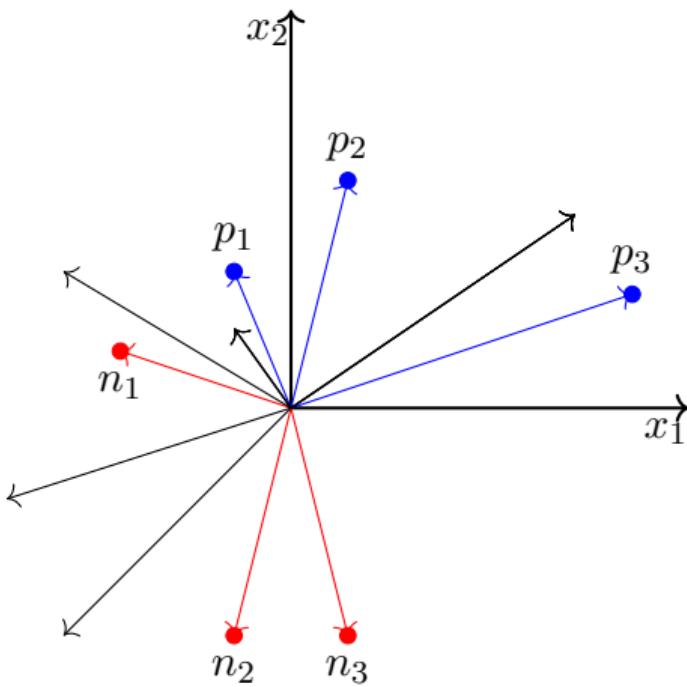
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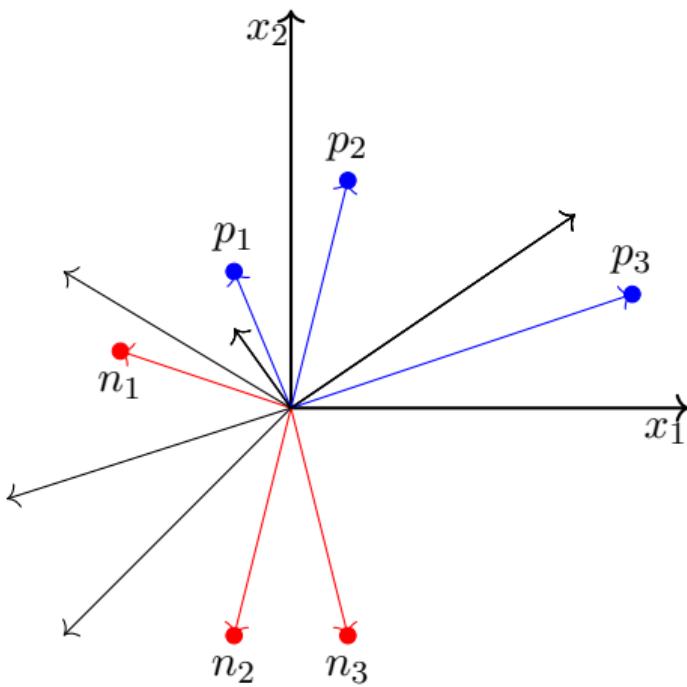
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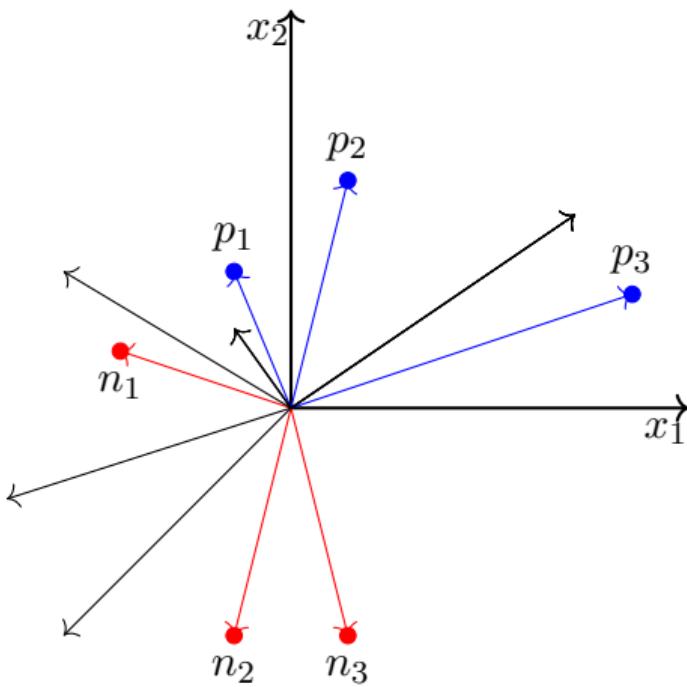
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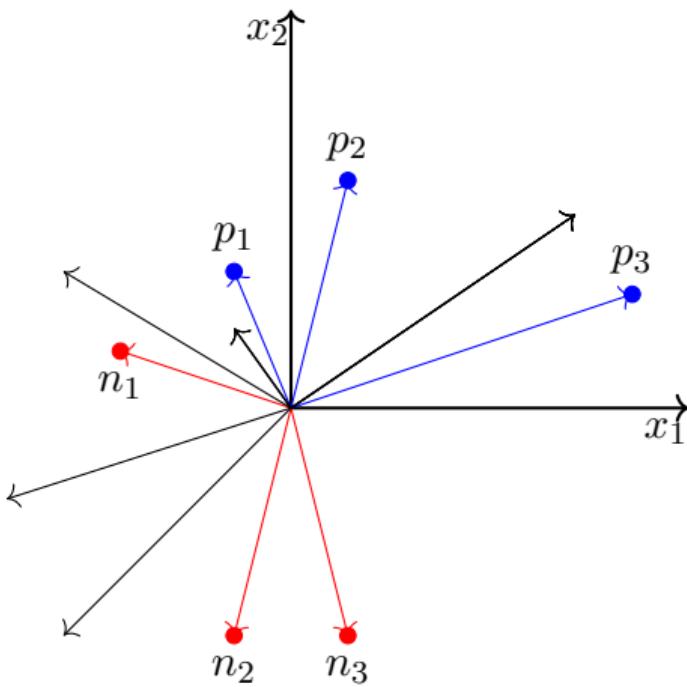
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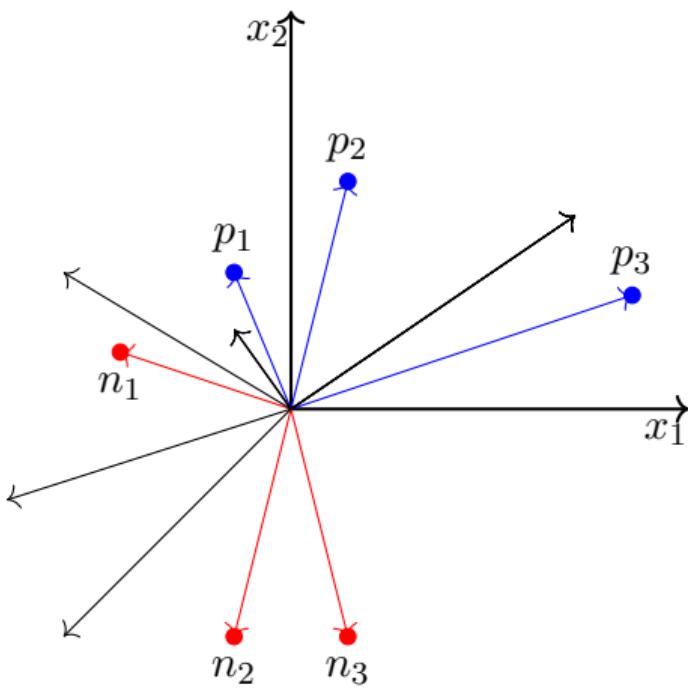
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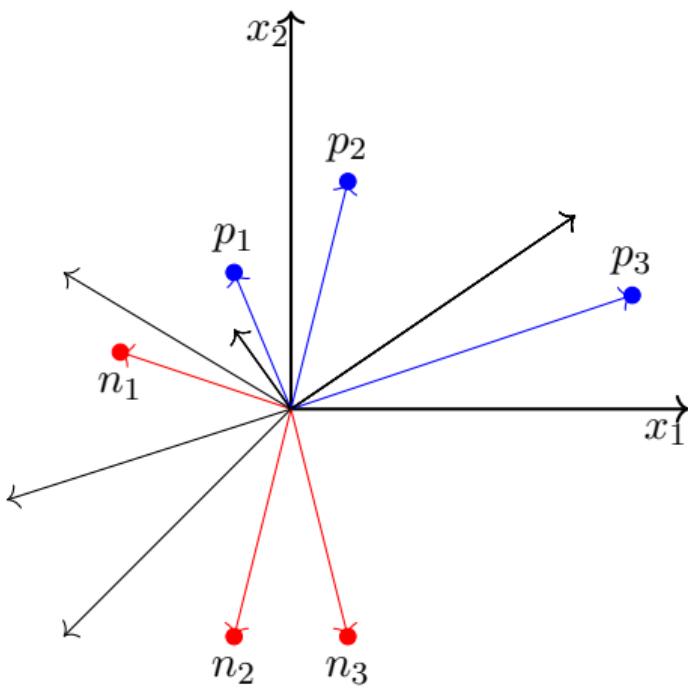
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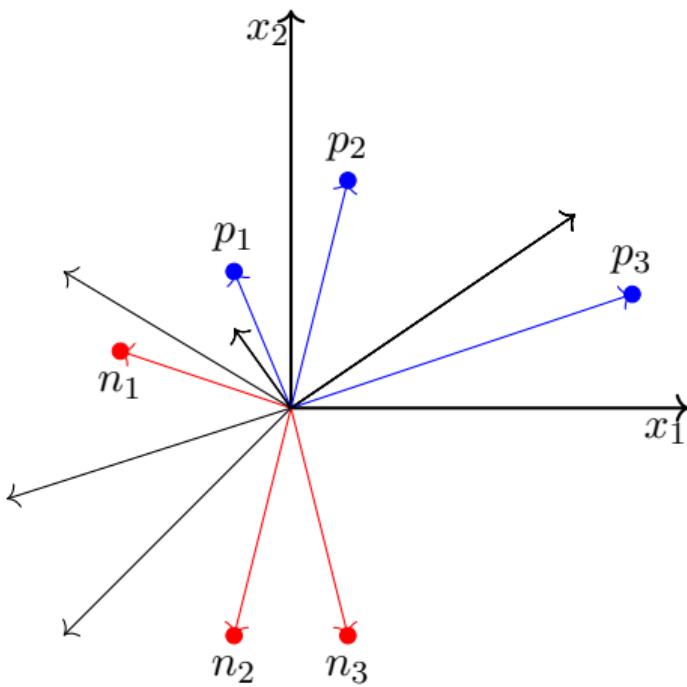
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- Randomly pick a point (say,  $n_2$ ), no correction needed  $\because \mathbf{w} \cdot \mathbf{x} < 0$  (you can check the angle visually)



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- We now run the algorithm by randomly going over the points
- The algorithm has converged

## Module 2.6: Proof of Convergence

- Now that we have some faith and intuition about why the algorithm works, we will see a more formal proof of convergence ...

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**Proof:** On the next slide

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- Let  $w^*$  be the normalized solution vector (we know one exists as the data is linearly separable)

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## Observations:

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- We do not make a correction at every time-step

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- As  $k$  (number of corrections) increases  $\cos\beta$  can become arbitrarily large
- But since  $\cos\beta \leq 1$ ,  $k$  must be bounded by a maximum number
- Thus, there can only be a finite number of corrections ( $k$ ) to  $w$  and the algorithm will converge!

Coming back to our questions ...

- What about non-boolean (say, real) inputs?
- Do we always need to hand code the threshold?
- Are all inputs equal? What if we want to assign more weight (importance) to some inputs?
- What about functions which are not linearly separable ?

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- Are all inputs equal? What if we want to assign more weight (importance) to some inputs? A perceptron allows weights to be assigned to inputs
- What about functions which are not linearly separable ? Not possible with a single perceptron but we will see how to handle this ..

## Module 2.7: Linearly Separable Boolean Functions

- So what do we do about functions which are not linearly separable ?

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- Let us see one such simple boolean function first ?

$x_1$	$x_2$	XOR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
1	1	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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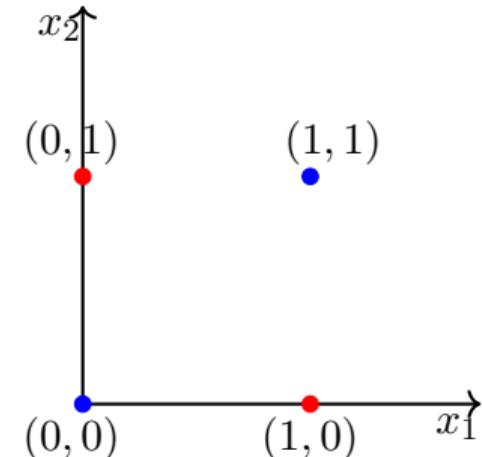
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- The fourth condition contradicts conditions 2 and 3

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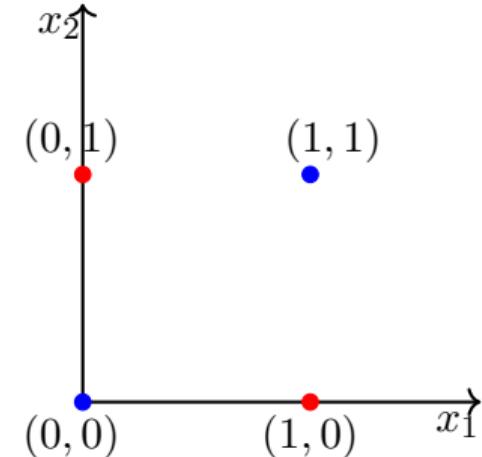
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- Hence we cannot have a solution to this set of inequalities

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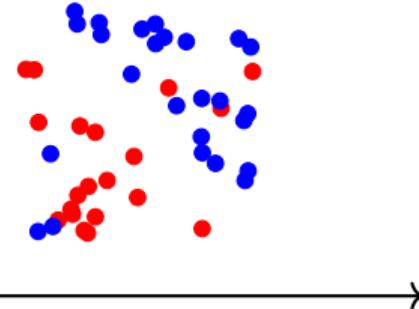
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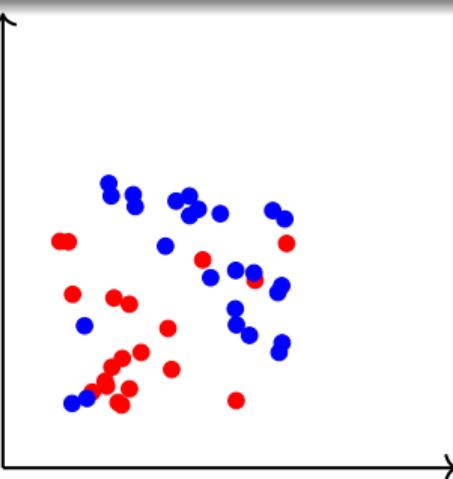
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- Hence we cannot have a solution to this set of inequalities

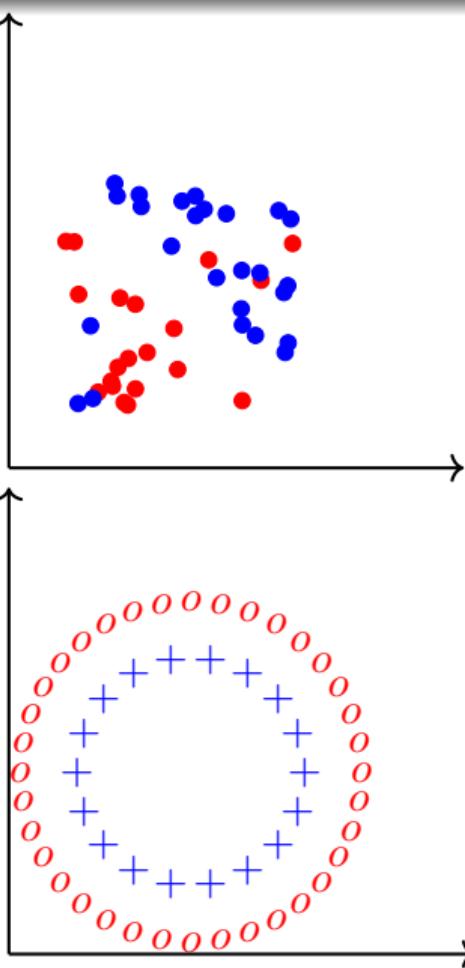
- And indeed you can see that it is impossible to draw a line which separates the red points from the blue points



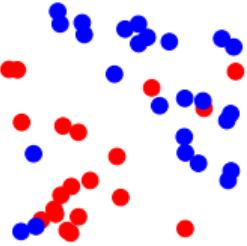
- Most real world data is not linearly separable and will always contain some outliers



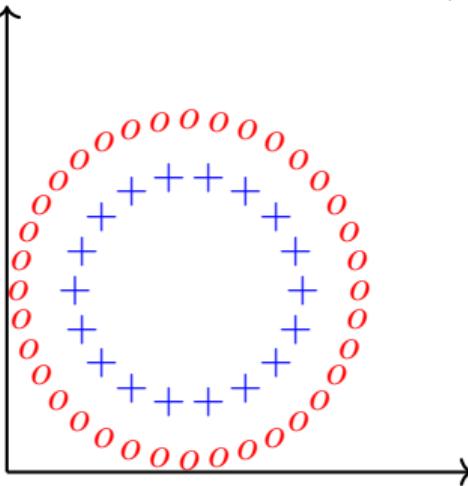
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- In fact, sometimes there may not be any outliers but still the data may not be linearly separable



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- We need computational units (models) which can deal with such data



- Most real world data is not linearly separable and will always contain some outliers
- In fact, sometimes there may not be any outliers but still the data may not be linearly separable
- We need computational units (models) which can deal with such data
- While a single perceptron cannot deal with such data, we will show that a network of perceptrons can indeed deal with such data



- Before seeing how a network of perceptrons can deal with linearly inseparable data, we will discuss boolean functions in some more detail ...

- How many boolean functions can you design from 2 inputs ?

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- Let us begin with some easy ones which you already know ..

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---

$x_1$	$x_2$
0	0
0	1
1	0
1	1

---

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$
0	0	0
0	1	0
1	0	0
1	1	0

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_{16}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	0	1

- How many boolean functions can you design from 2 inputs ?
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$x_1$	$x_2$	$f_1$	$f_2$	$f_{16}$
0	0	0	0	1
0	1	0	0	1
1	0	0	0	1
1	1	0	1	1

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_8$	$f_{16}$
0	0	0	0	0	1
0	1	0	0	1	1
1	0	0	0	1	1
1	1	0	1	1	1

- How many boolean functions can you design from 2 inputs ?
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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_8$	$f_{16}$
0	0	0	0	0	0	1
0	1	0	0	0	1	1
1	0	0	0	1	1	1
1	1	0	1	0	1	1

- How many boolean functions can you design from 2 inputs ?
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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_8$	$f_{16}$
0	0	0	0	0	0	0	1
0	1	0	0	0	0	1	1
1	0	0	0	1	1	1	1
1	1	0	1	0	1	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_8$	$f_{16}$
0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	1	1	1
1	0	0	0	1	1	0	1	1
1	1	0	1	0	1	0	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_8$	$f_{16}$
0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	1	1	1	1	1
1	0	0	0	1	1	0	0	1	1	1
1	1	0	1	0	1	0	1	0	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1
0	1	0	0	0	0	1	1	1	1	0	1
1	0	0	0	1	1	0	0	1	1	0	1
1	1	0	1	0	1	0	1	0	1	0	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	1
1	0	0	0	1	1	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	0	1	0	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	1

- How many boolean functions can you design from 2 inputs ?
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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

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$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for  $n$  inputs ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for  $n$  inputs ?  $2^{2^n}$

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for  $n$  inputs ?  $2^{2^n}$
- How many of these  $2^{2^n}$  functions are not linearly separable ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

$x_1$	$x_2$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for  $n$  inputs ?  $2^{2^n}$
- How many of these  $2^{2^n}$  functions are not linearly separable ? For the time being, it suffices to know that at least some of these may not be linearly inseparable (I encourage you to figure out the exact answer :-))

## Module 2.8: Representation Power of a Network of Perceptrons

- We will now see how to implement **any** boolean function using a network of perceptrons ...

- For this discussion, we will assume True = +1 and False = -1

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- We consider 2 inputs and 4 perceptrons



- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



$x_1$

$x_2$

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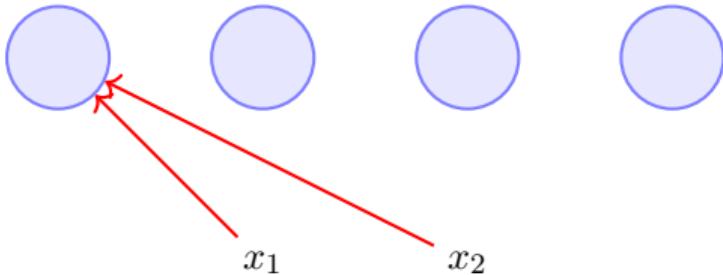


$x_1$

$x_2$

red edge indicates  $w = -1$   
blue edge indicates  $w = +1$

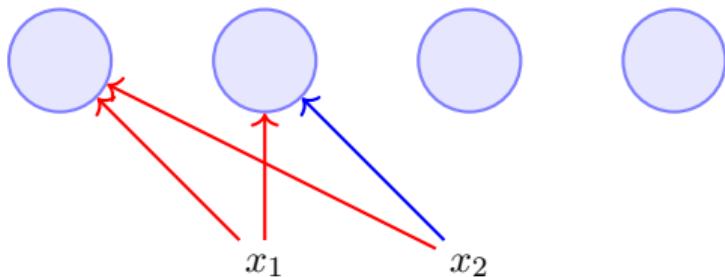
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red edge indicates  $w = -1$

blue edge indicates  $w = +1$

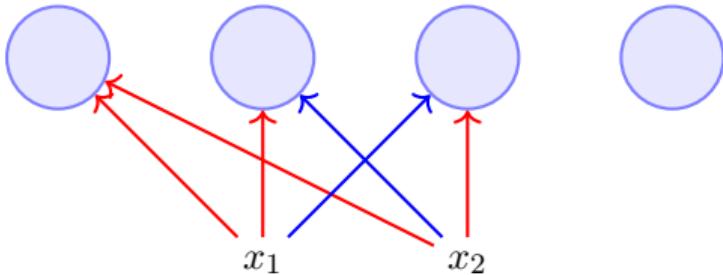
- For this discussion, we will assume True = +1 and False = -1
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red edge indicates  $w = -1$

blue edge indicates  $w = +1$

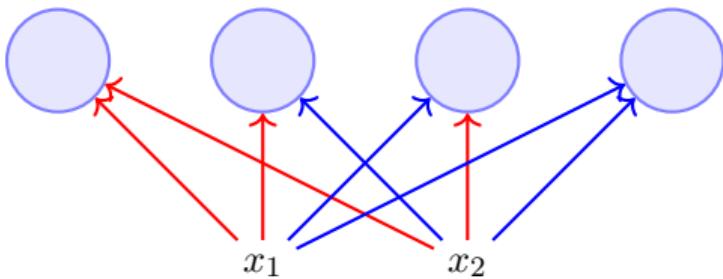
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates  $w = -1$

blue edge indicates  $w = +1$

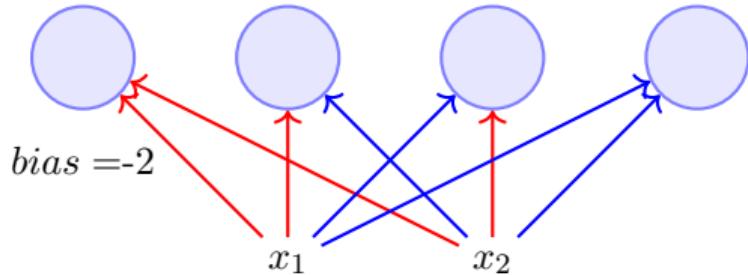
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates  $w = -1$

blue edge indicates  $w = +1$

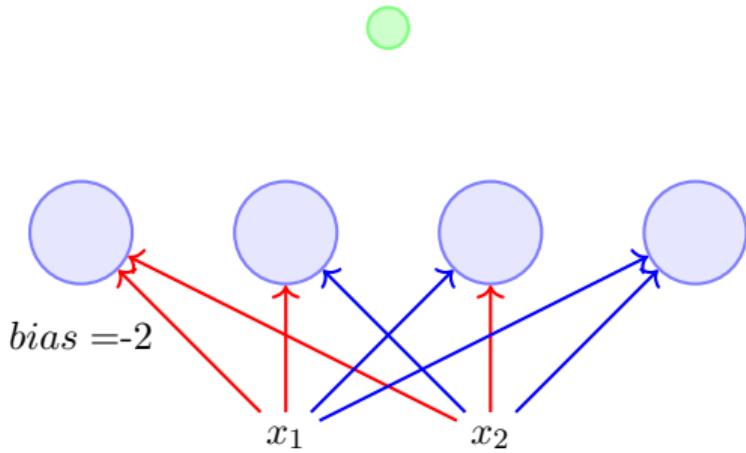
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias ( $w_0$ ) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is  $\geq 2$ )



red edge indicates  $w = -1$

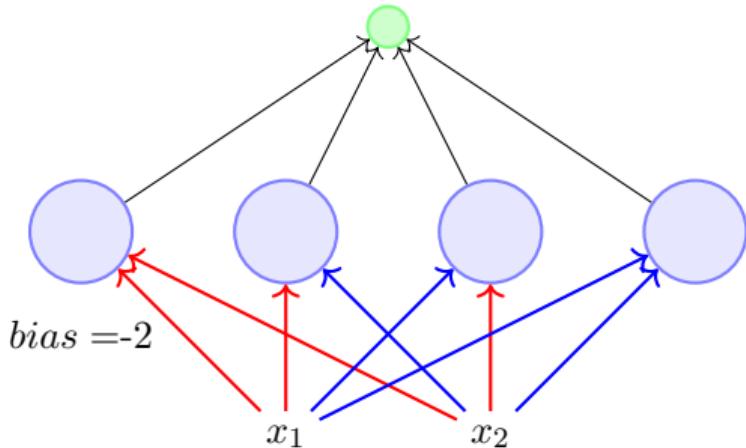
blue edge indicates  $w = +1$

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- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias ( $w_0$ ) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is  $\geq 2$ )
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



red edge indicates  $w = -1$   
 blue edge indicates  $w = +1$

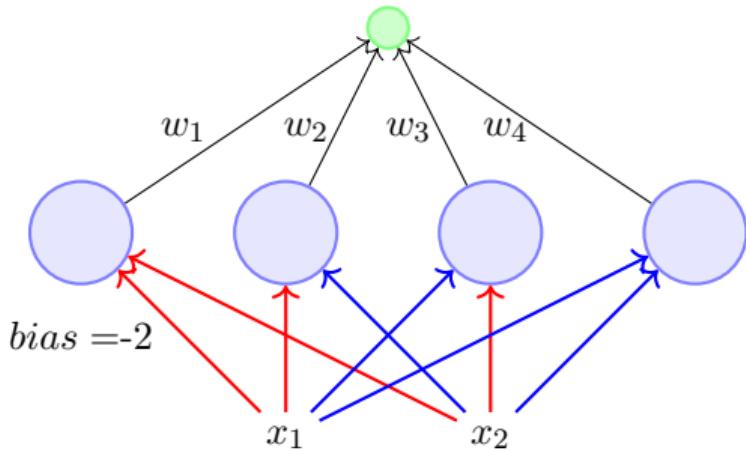
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias ( $w_0$ ) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is  $\geq 2$ )
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



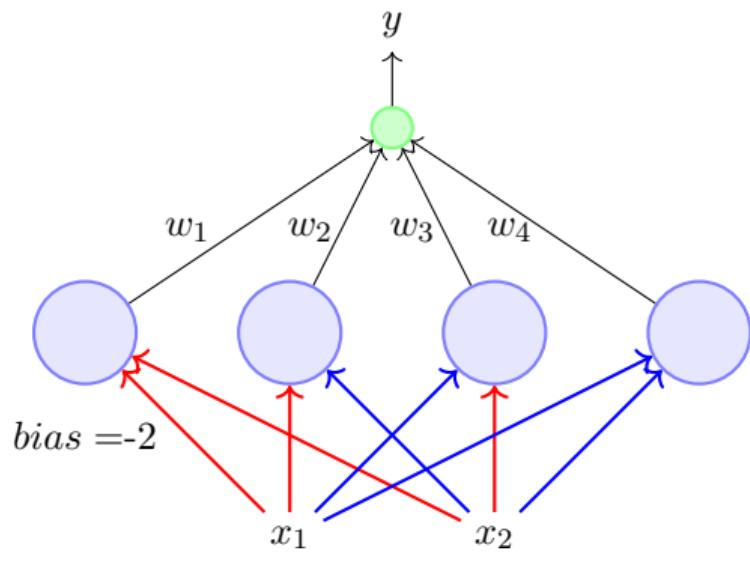
red edge indicates  $w = -1$

blue edge indicates  $w = +1$

- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias ( $w_0$ ) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is  $\geq 2$ )
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



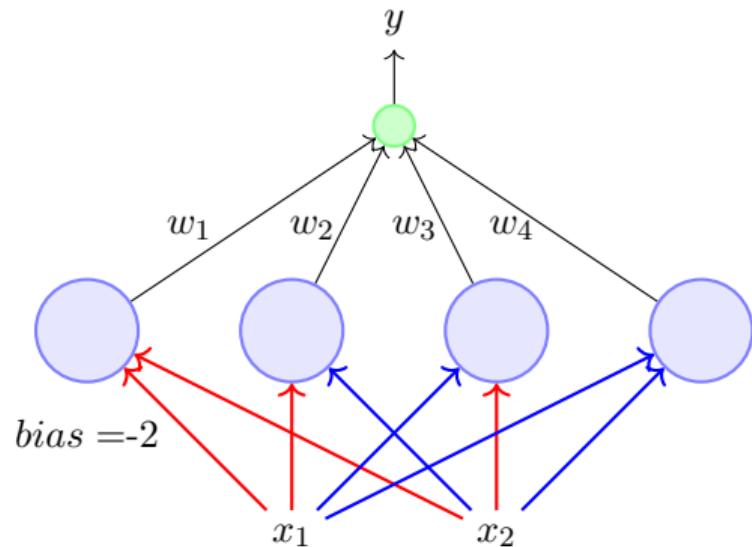
red edge indicates  $w = -1$   
 blue edge indicates  $w = +1$



- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias ( $w_0$ ) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is  $\geq 2$ )
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)
- The output of this perceptron ( $y$ ) is the output of this network

## Terminology:

- This network contains 3 layers

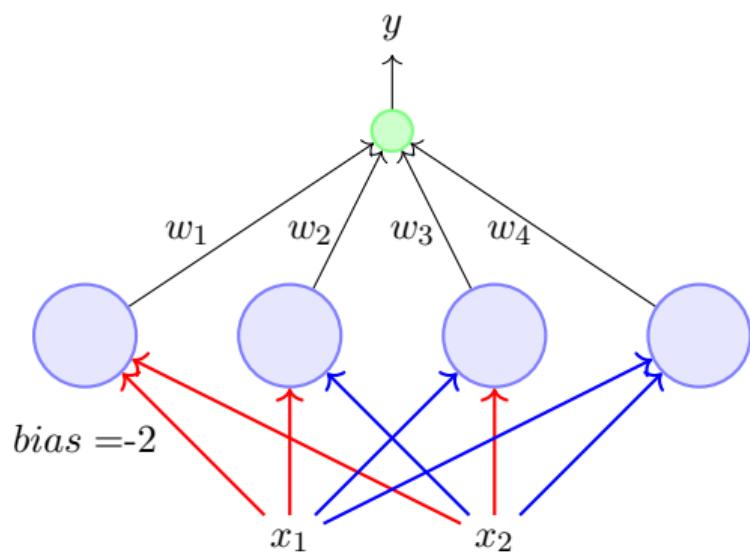


red edge indicates  $w = -1$

blue edge indicates  $w = +1$

## Terminology:

- This network contains 3 layers
- The layer containing the inputs ( $x_1, x_2$ ) is called the **input layer**

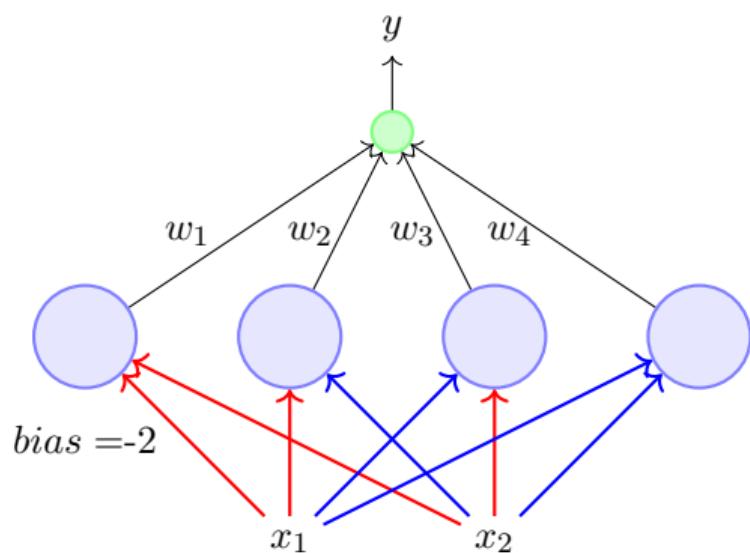


red edge indicates  $w = -1$

blue edge indicates  $w = +1$

## Terminology:

- This network contains 3 layers
- The layer containing the inputs  $(x_1, x_2)$  is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**

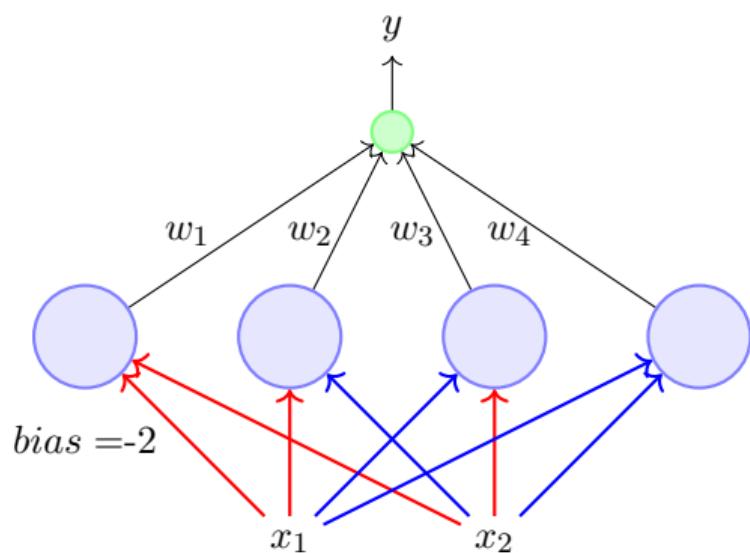


red edge indicates  $w = -1$

blue edge indicates  $w = +1$

## Terminology:

- This network contains 3 layers
- The layer containing the inputs ( $x_1, x_2$ ) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**

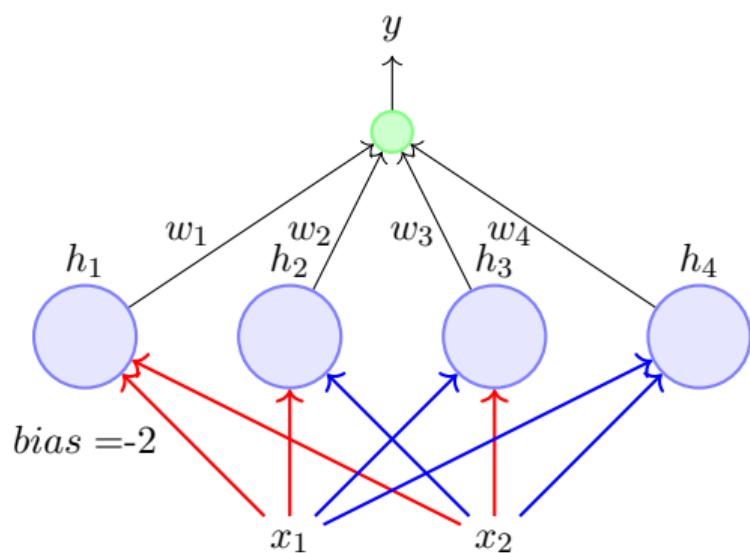


red edge indicates  $w = -1$

blue edge indicates  $w = +1$

## Terminology:

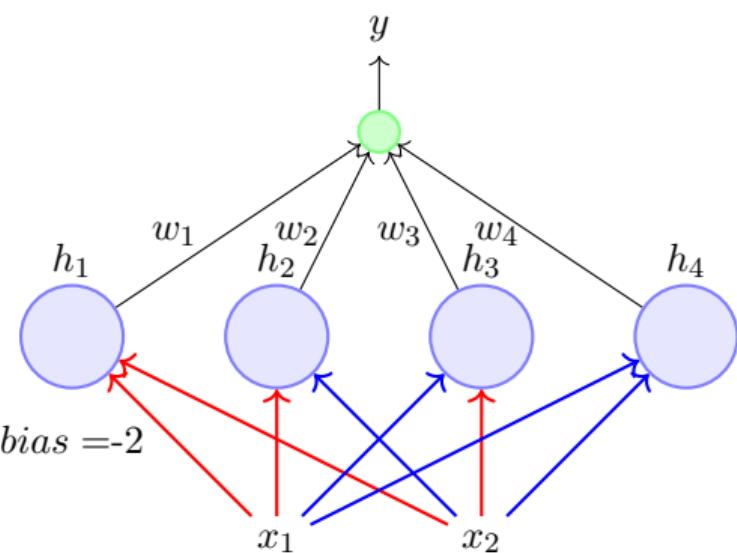
- This network contains 3 layers
- The layer containing the inputs ( $x_1, x_2$ ) is called the **input layer**
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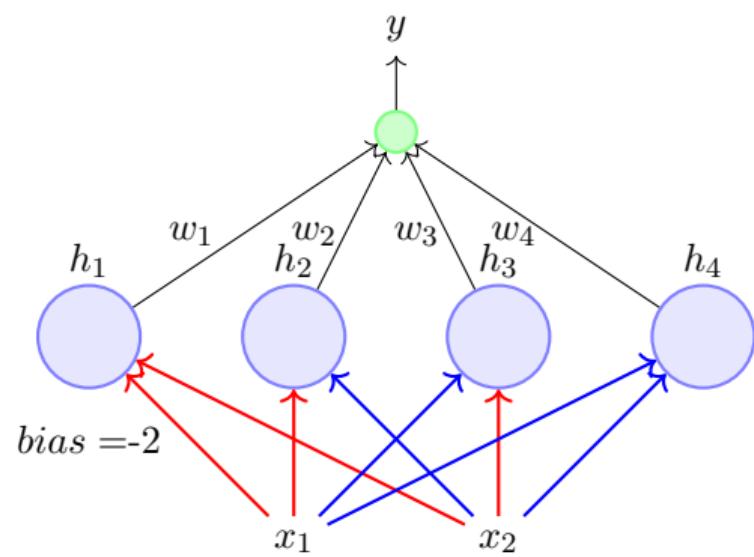


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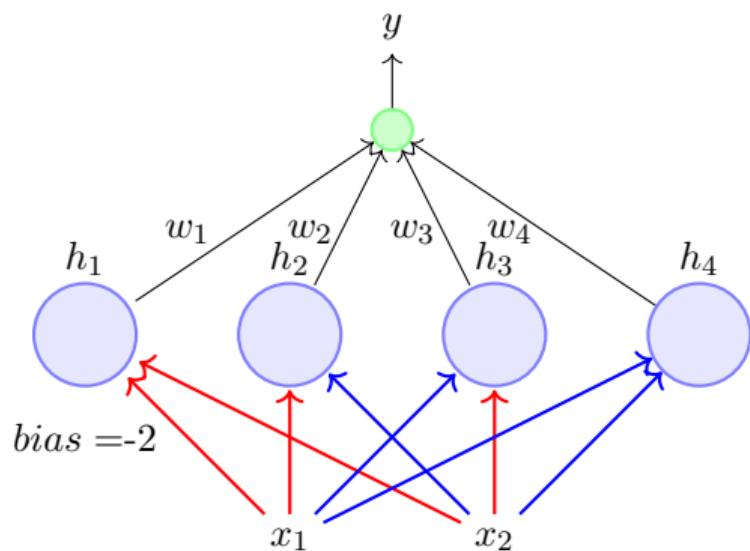


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- $w_1, w_2, w_3, w_4$  are called layer 2 weights

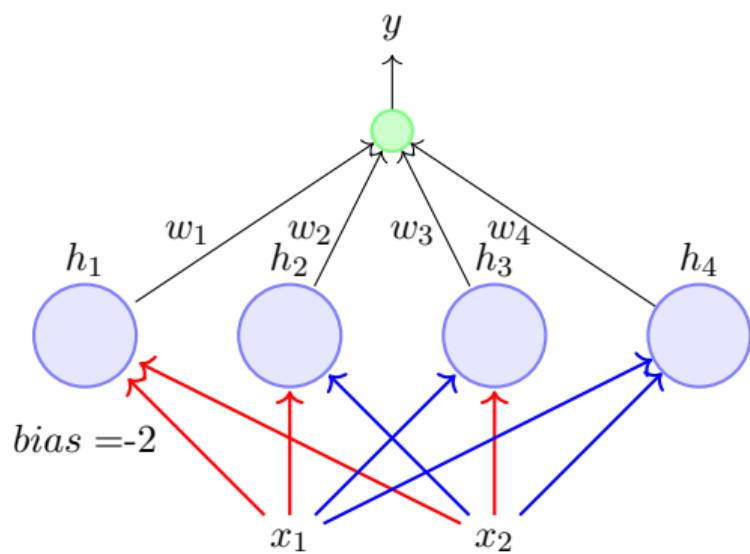
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !



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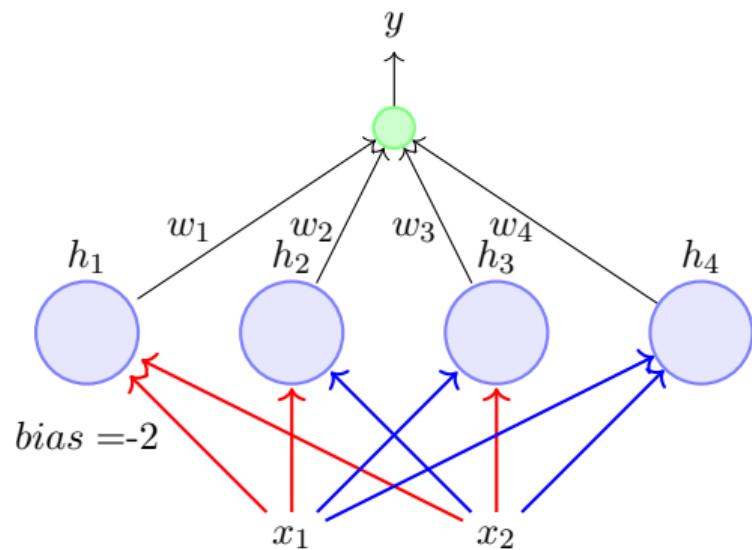
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- In other words, we can find  $w_1, w_2, w_3, w_4$  such that the truth table of any boolean function can be represented by this network



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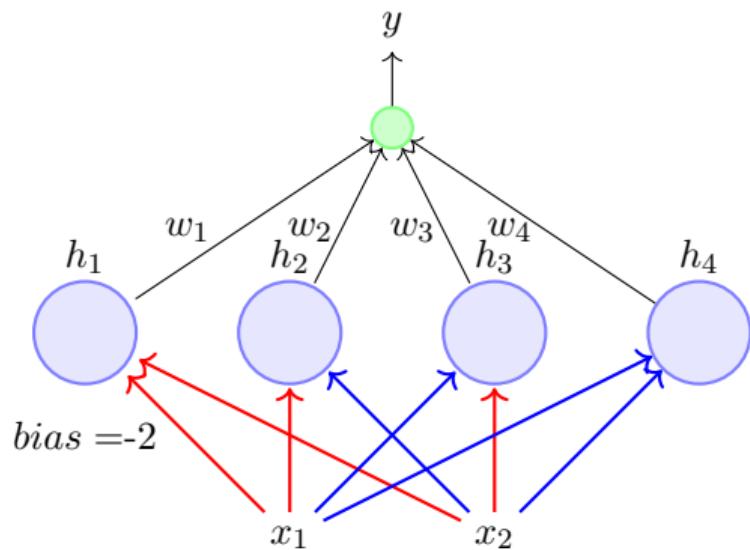
blue edge indicates  $w = +1$



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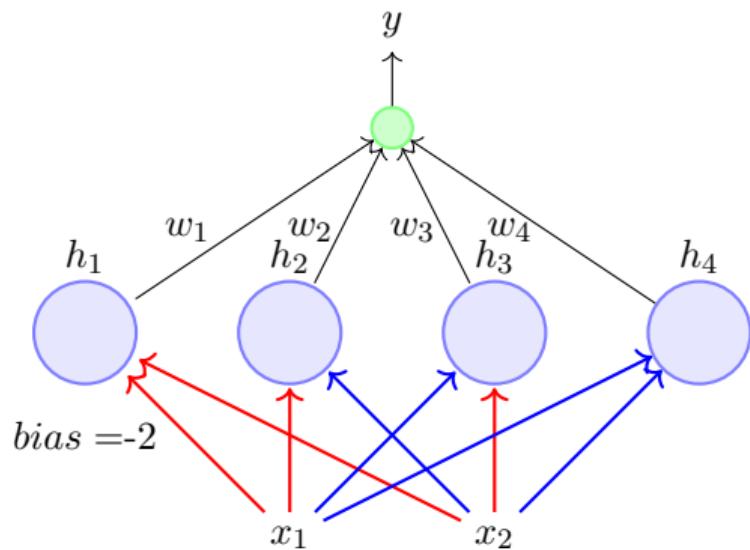
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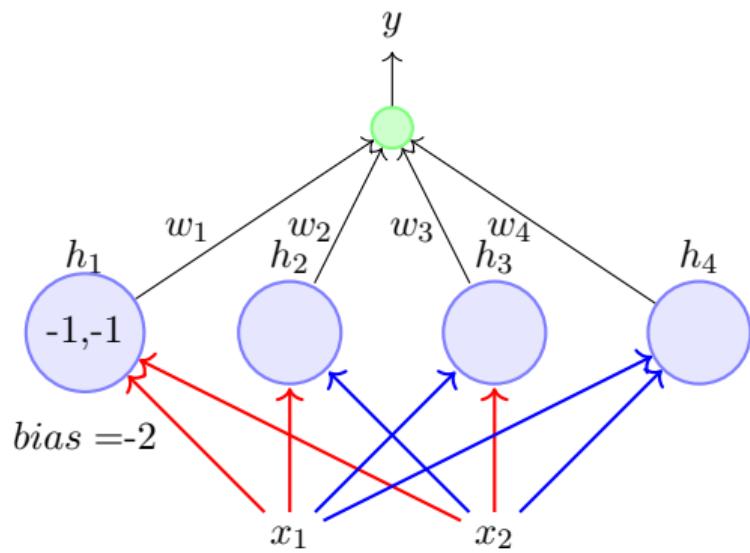
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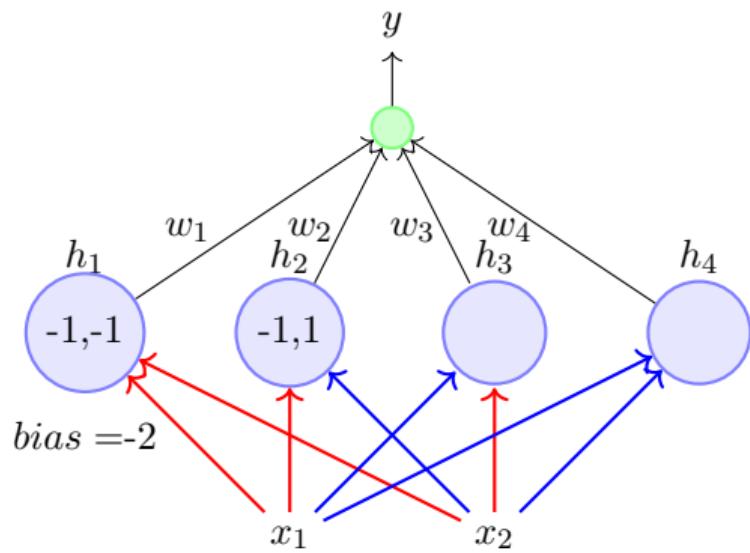
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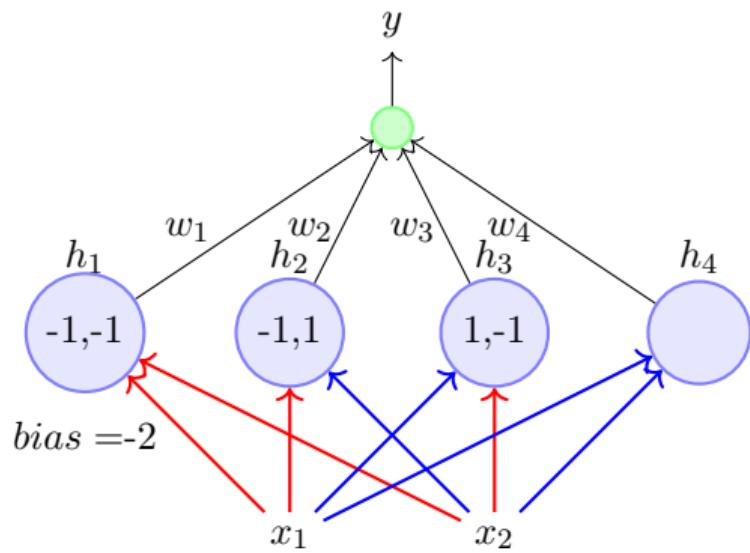
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- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the first perceptron fires for  $\{-1, -1\}$



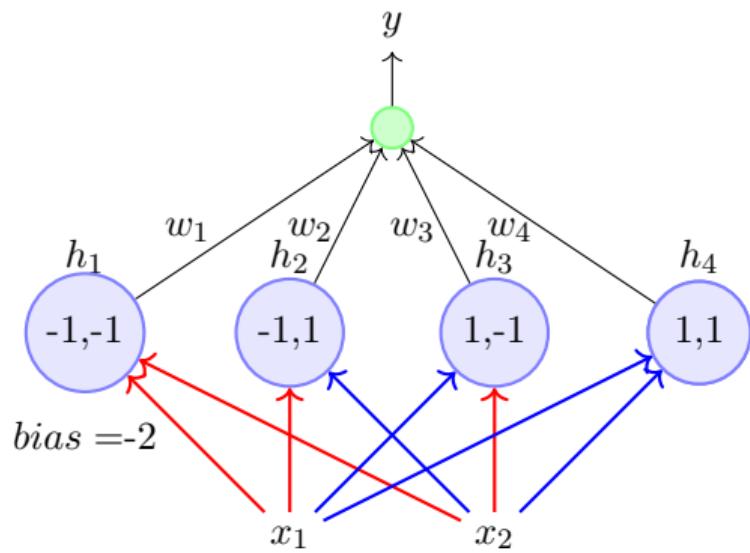
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- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the second perceptron fires for  $\{-1,1\}$



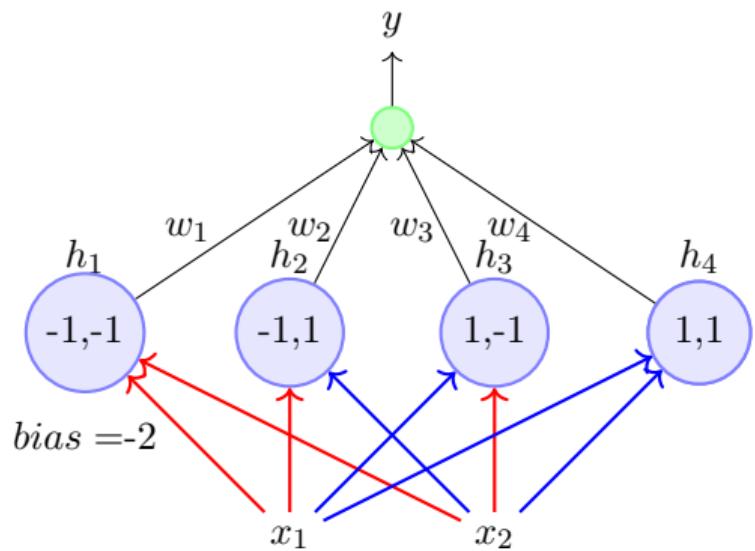
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- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the third perceptron fires for  $\{1,-1\}$



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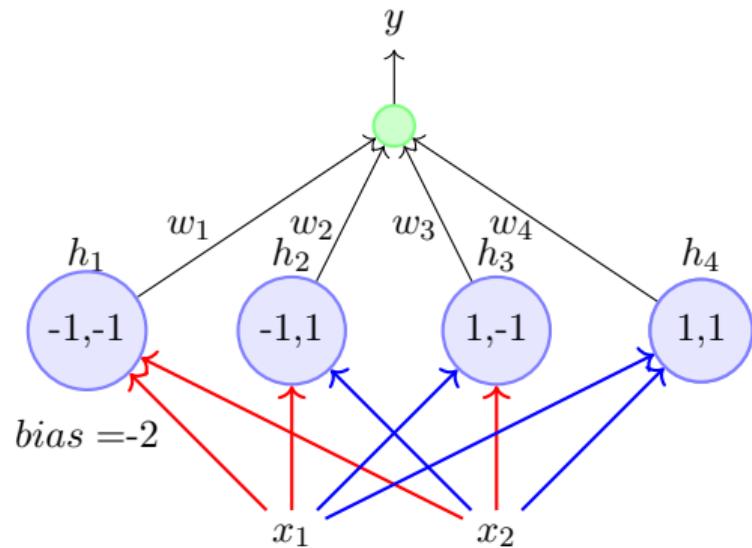
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- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- Let us see why this network works by taking an example of the XOR function

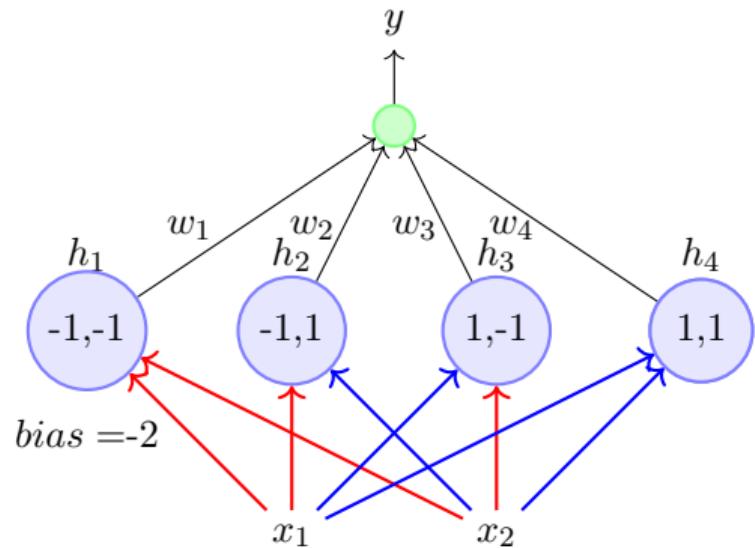
- Let  $w_0$  be the bias output of the neuron (i.e., it will fire if  $\sum_{i=1}^4 w_i h_i \geq w_0$ )



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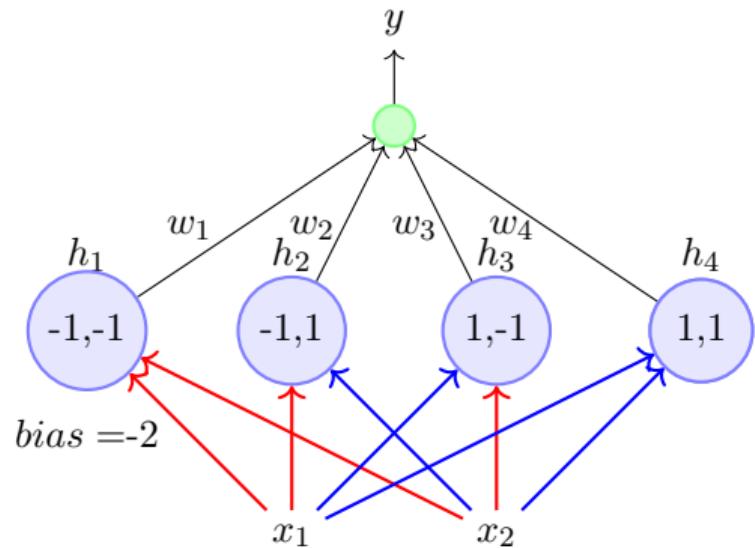
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$x_1$	$x_2$	$XOR$	$h_1$	$h_2$	$h_3$	$h_4$	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	$w_1$

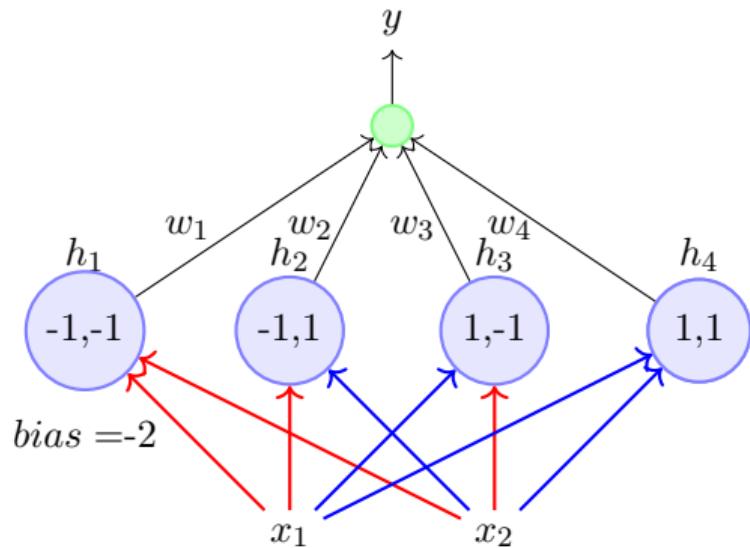
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0	0	0	1	0	0	0	$w_1$
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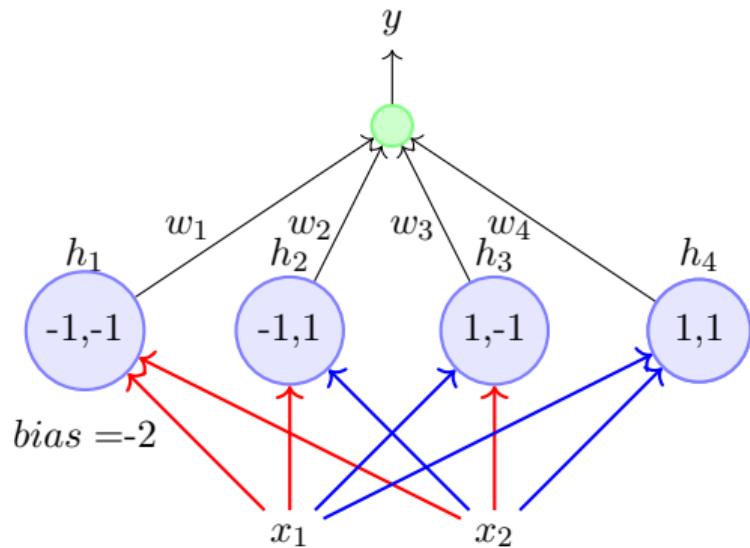
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1	0	1	0	0	1	0	$w_3$

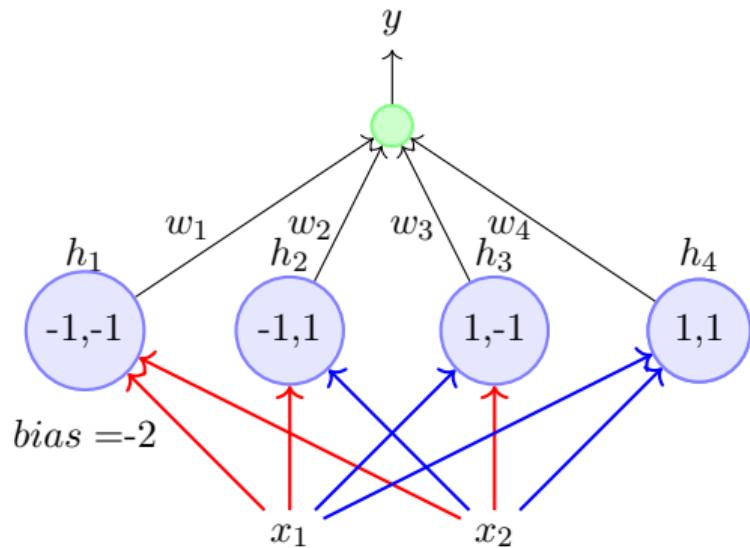
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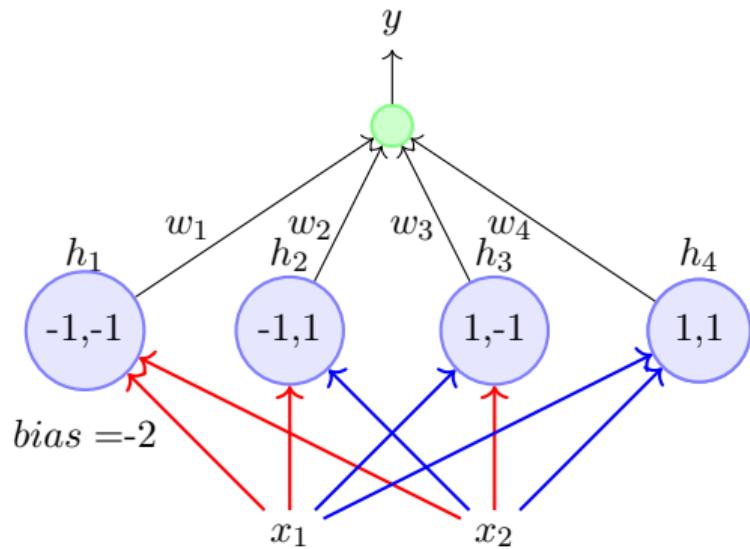
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- This results in the following four conditions to implement XOR:  $w_1 < w_0, w_2 \geq w_0, w_3 \geq w_0, w_4 < w_0$

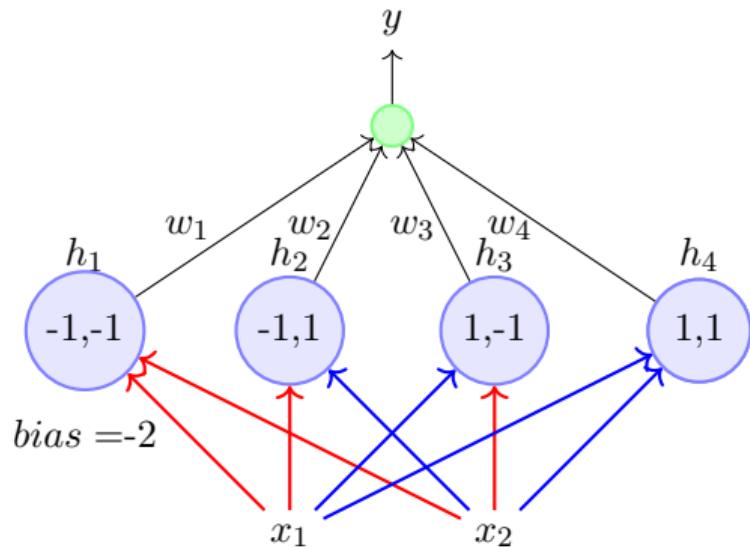
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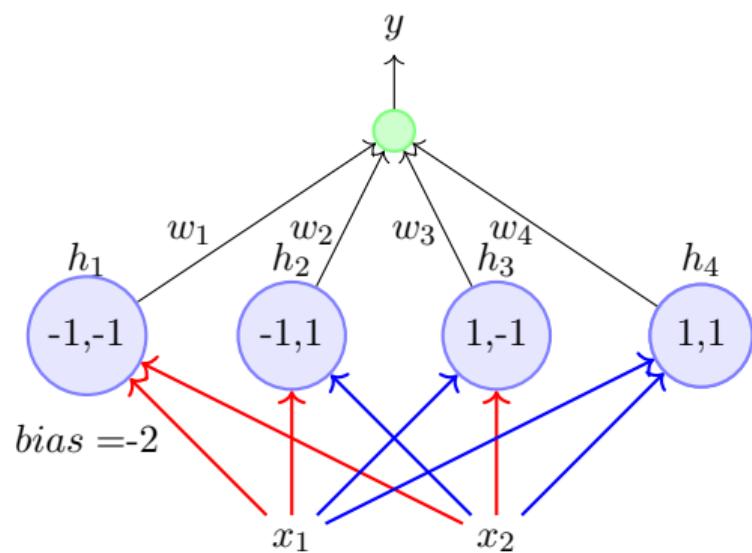
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- Essentially each  $w_i$  is now responsible for one of the 4 possible inputs and can be adjusted to get the desired output for that input

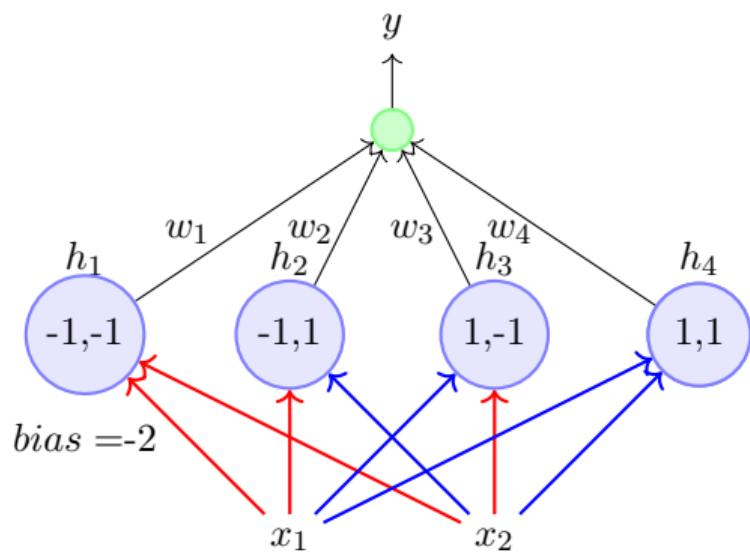
- It should be clear that the same network can be used to represent the remaining 15 boolean functions also



red edge indicates  $w = -1$

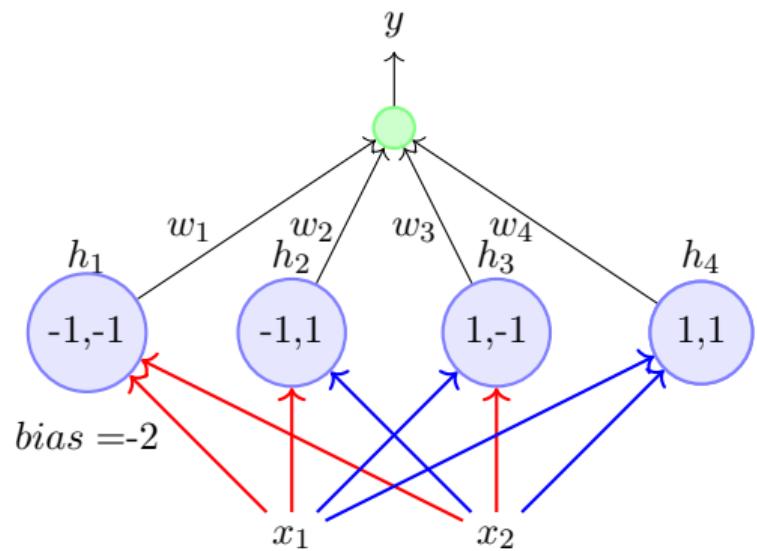
blue edge indicates  $w = +1$

- It should be clear that the same network can be used to represent the remaining 15 boolean functions also
- Each boolean function will result in a different set of non-contradicting inequalities which can be satisfied by appropriately setting  $w_1, w_2, w_3, w_4$



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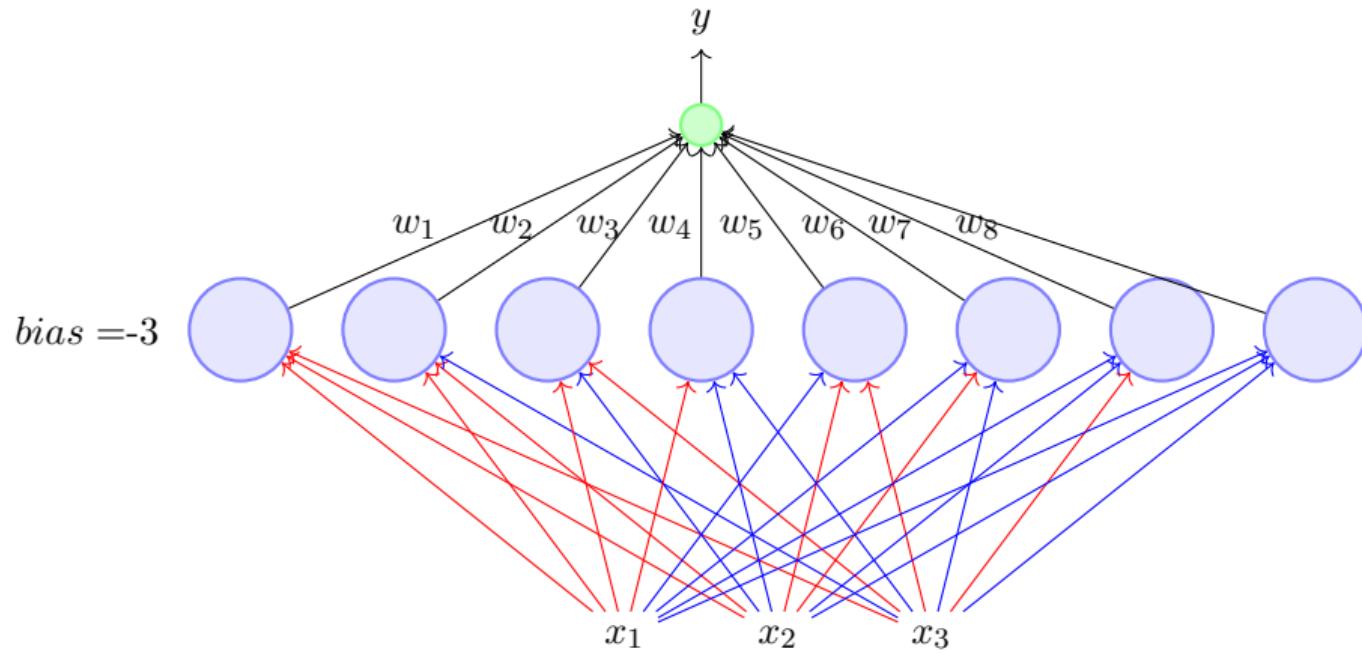
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- Try it!

- What if we have more than 3 inputs ?

- Again each of the 8 perceptrons will fire only for one of the 8 inputs
- Each of the 8 weights in the second layer is responsible for one of the 8 inputs and can be adjusted to produce the desired output for that input



- What if we have  $n$  inputs ?

## Theorem

Any boolean function of  $n$  inputs can be represented exactly by a network of perceptrons containing 1 hidden layer with  $2^n$  perceptrons and one output layer containing 1 perceptron

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**Note:** A network of  $2^n + 1$  perceptrons is not necessary but sufficient. For example, we already saw how to represent AND function with just 1 perceptron

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**Catch:** As  $n$  increases the number of perceptrons in the hidden layers obviously increases exponentially

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- How does this help us with our original problem: which was to predict whether we like a movie or not? Let us see!

- We are given this data about our past movie experience

$$\begin{array}{ll} p_1 & \left[ \begin{array}{ccccc} x_{11} & x_{12} & \dots & x_{1n} & y_1 = 1 \end{array} \right] \\ p_2 & \left[ \begin{array}{ccccc} x_{21} & x_{22} & \dots & x_{2n} & y_2 = 1 \end{array} \right] \\ \vdots & \left[ \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \\ n_1 & \left[ \begin{array}{ccccc} x_{k1} & x_{k2} & \dots & x_{kn} & y_i = 0 \end{array} \right] \\ n_2 & \left[ \begin{array}{ccccc} x_{j1} & x_{j2} & \dots & x_{jn} & y_j = 0 \end{array} \right] \\ \vdots & \left[ \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \end{array}$$

- We are given this data about our past movie experience
- For each movie, we are given the values of the various factors ( $x_1, x_2, \dots, x_n$ ) that we base our decision on and we are also given the value of  $y$  (like/dislike)

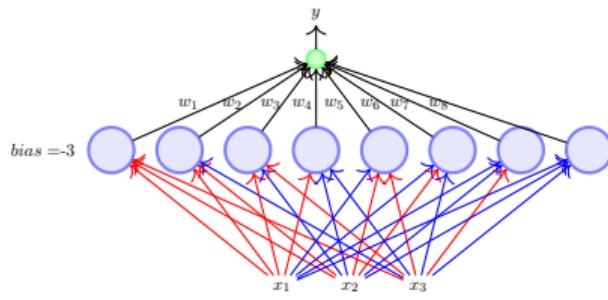
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$$\begin{array}{l}
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 \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 n_1 \left[ \begin{matrix} x_{k1} & x_{k2} & \dots & x_{kn} & y_i = 0 \end{matrix} \right] \\
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- $p_i$ 's are the points for which the output was 1 and  $n_i$ 's are the points for which it was 0
- The data may or may not be linearly separable
- The proof that we just saw tells us that it is possible to have a network of perceptrons and learn the weights in this network such that for any given  $p_i$  or  $n_j$  the output of the network will be the same as  $y_i$  or  $y_j$  (i.e., we can separate the positive and the negative points)

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- Specifically, it tells us that a MLP with a single hidden layer can represent **any** boolean function