Spec. Matrices 2016; 4:110–120 DE GRUYTER OPEN

Research Article Open Access

Special Issue: Proceedings of the 24th International Workshop on Matrices and Statistics

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Generation of all magic squares of order 5 and interesting patterns finding

DOI 10.1515/spma-2016-0011
Received October 19, 2015; accepted January 12, 2016

Abstract: This paper presents an enumeration algorithm to generate all magic squares of order 5 based on the ideas of basic form (Schroeppel [7]) and generating vector which is extension of Frénicle Quads (Ollerenshaw and Bondi [6]). The results lead us to extend Frénicle-Amela patterns from the case of order 4 to the case of order 5, which we refer to Frénicle-Amela-Like patterns. We show that these interesting Frénicle-Amela-Like patterns appear simultaneously. The number of these patterns is also calculated.

Keywords: Magic square of order 5; Basic form; Generating vectors; Frénicle-Amela patterns

1 Introduction

A *Magic Square of order n* is a $n \times n$ square matrix which consists of a sequence of distinct numbers. The sum of each row, each column as well as two diagonals are all equal to a constant which is called magic sum or magic constant denoted by μ_n. A *Classical Magic Square of order n* is the magic square whose elements are consecutive integers starting from 1 to n^2 with magic sum $\mu_n = n(n^2 + 1)/2$. For simplicity, we refer the *Classical Magic Square of order n* to MS-n in this paper.

Given the mystery of magic squares, it is not surprising that there has been a significant amount of research work upon them [2, 3]. One of interesting and challenging problems is to find the total number of magic squares of different order since the number increases dramatically as the order increases. In Table. 1, the left column shows the total number of MS-3, MS-4 and MS-5. Trump [9] gives a list of the total number of magic squares for $n = 3, 4, \cdots, 10$. Due to the huge number of magic squares another important problem is classification of them through geometric and algebraic ways. Fang, Luo and Zheng [4] gave a comprehensive review on classification of MS-4. Candy [1] studied classification problems of MS-5 from algebraic point of view.

In this paper, we first focus on the generation of all MS-5 which was first studied by Schroeppel [7]. Based on the ideas of *isomorphism* and *basic form* introduced by Schroeppel, we start from *generating vectors* which are extension of Frénicle Quads (Ollerenshaw and Bondi [6]) to the case of MS-5 and by using an enumeration algorithm we generate all MS-5. Furthermore, we will show that Frénicle-Amela-Like patterns always appear simultaneously for MS-5 and the total number of these patterns is also calculated.

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Table 1: The total number of all MS-3, MS-4 and MS-5 and their basic forms

Order	Number of Classical Magic Square	Number of Basic Forms
3	8	1
4	7,040	880
5	2,202,441,792	68,826,306

2 Generation of all MS-5

A number of methods such as Siamese method, Lozenge method and Matrix Addition method have been proposed for generating magic squares of odd order. However, these methods only provide one or few magic squares and cannot generate all of them. Generation of all MS-5 was first studied by Schroeppel [7]. He introduced the idea of isomorphism and basic form of magic squares. We bring his ideas to generating vectors which are extension of Frénicle Quads to the case of MS-5 and by using an enumeration algorithm we will generate all basic forms of MS-5.

2.1 Isomorphism and Basic Form of Magic Squares

Two magic squares are called *isomorphic* if one of them can be obtained from the other through some transformations such as rotation and exchanging rows and columns. For different order n the isomorphism employs different group of transformations. Given a specific magic square, these isomorphic transformations can easily lead to other magic squares which are isomorphic with the original one. In the other words, obtaining one magic square means obtaining its all isomorphic magic squares. The characteristics of isomorphism narrows the set of all magic squares down to the set of all non-isomorphic magic squares whose elements are called *basic form*. Therefore, instead of investigating all magic squares it is only necessary to investigate their basic forms. In Table. 1, the right column shows the total number of basic forms for MS-3, MS-4 and MS-5. Ollerenshaw and Bondi [6] studied the basic forms of MS-3 and MS-4. Schroeppel [7] studied the basic forms of MS-5. He considered the following isomorphic transformations for any $\mathbf{M} \in \mathcal{M}$ where $\mathcal{M}$ is the set of all MS-5,

1. Rotate $\mathbf{M}$ 90 degrees clockwise;
2. Flip $\mathbf{M}$ along the main diagonal which means transpose $\mathbf{M}$;
3. Exchange the first and fifth rows and columns of $\mathbf{M}$, respectively;
4. Exchange rows and columns of $\mathbf{M}$ as follows: row 1 $\Leftrightarrow$ row 2, row 4 $\Leftrightarrow$ row 5, column 1 $\Leftrightarrow$ column 2, and column 4 $\Leftrightarrow$ column 5.

Through the above isomorphic transformations a given $\mathbf{M} \in \mathcal{M}$ implies $4 \times 2 \times 2 \times 2 = 32$ magic squares (including $\mathbf{M}$) which are isomorphic with $\mathbf{M}$. Thus there are $2,202,441,792/32 = 68,826,306$ basic forms of MS-5.

2.2 Magic Square Generating Vectors

For any $\mathbf{M} \in \mathcal{M}$ where $\mathcal{M}$ is the set of all MS-5, the sum of each row, each column as well as two diagonals of $\mathbf{M}$ are all equal to the magic constant $\mu_5 = 65$. Hence there are 12 vectors (5 rows, 5 columns and 2 diagonals) whose sum are the same but their elements are distinct. Let

$$\mathcal{A} = \{(x_1, \cdots, x_5) : x_i \in \Omega, x_1 > x_2 > x_3 > x_4 > x_5, \sum_{i=1}^{5} x_i = 65\}, \tag{1}$$

where $\Omega = \{1, 2\cdots, 25\}$. This is an extension of Frénicle Quads (see [6]) to the case of MS-5. The set $\mathcal{A}$ which is called the set of *generating vectors* since we can choose 12 vectors with distinct elements from $\mathcal{A}$ to compose a magic square of order 5. It is easy to show that there are 1394 vectors in $\mathcal{A}$. Next we will present an enumeration algorithm which deals with the choice of distinct vectors from $\mathcal{A}$ and further to combine them to obtain basic forms of MS-5.

2.3 An Enumeration Algorithm for Generating All Basic Forms of MS-5

Let

$$\mathbf{M} = (m_{ij}) = \begin{pmatrix} B & J & K & L & F \\ V & C & T & H & X \\ R & M & A & P & S \\ W & I & U & D & Y \\ G & N & Q & O & E \end{pmatrix}$$

denote any magic square of order 5 where $A, B, \cdots Y$ are distinct numbers from $\Omega = \{1, 2\cdots, 25\}$. Our algorithm aims to generate all basic forms of MS-5 which are non-isomorphic magic squares. In order to avoid appearance of isomorphic magic squares, following constraints [7] are added to two diagonals of $\mathbf{M}$ [7],

$$E > B, \quad and \quad D > C > B, \tag{2}$$

$$I > B, \quad H > B, \quad and \quad G > F > B. \tag{3}$$

In fact, if $E < B$ in $\mathbf{M}$, just applying the first isomorphic transformation (i.e. rotating 90 degrees clockwise) twice to $\mathbf{M}$ will result in the form with constraint $E > B$. Similarly, applying other isomorphic transformations or their combinations to $\mathbf{M}$ which does not satisfy the above constraints will result in the basic form with the above constraints. Thus without loss of generality, any basic form of MS-5 can be written as $\mathbf{M}$ with constraints (2) and (3).

Schroeppel [7] also showed that it is only necessary to consider those basic forms of MS-5 with center numbers starting from 1 to 13 since by subtracting each element of those basic forms from 26 we will obtain basic forms of MS-5 with center numbers starting from 14 to 25.

Following Schroeppel's ideas, we only consider those magic squares $\mathbf{M}$ with constraints (2) and (3) which make $\mathbf{M}$ to be a basic form and consider the case where the center number A is from 1 to 13. Our algorithm for generating all basic forms of MS-5 searches 12 distinct vectors from the generating vector set $\mathcal{A}$ and then combine them to obtain all the basic forms.

For any subset $Z \subseteq \Omega = \{1, 2\cdots, 25\}$, let

$$\mathcal{I}_Z = \{\mathbf{x} = (x_1, \cdots, x_5) \in \mathcal{A} : Z \subseteq \{x_i\}_{1\le i\le 5}\},$$

and

$$\mathcal{E}_Z = \{\mathbf{x} = (x_1, \cdots, x_5) \in \mathcal{A} : Z \cap \{x_i\}_{1\le i\le 5} = \emptyset\}.$$

In fact, the subset $\mathcal{I}_Z \subseteq \mathcal{A}$ denotes the set of those generating vectors who contain all the elements of Z, and the subset $\mathcal{E}_Z \subseteq \mathcal{A}$ denotes the set of those generating vectors who do not have any common element with Z. For instance, $\mathcal{I}_{\{2,3\}}$ denotes the set of generating vectors who contain the numbers 2 and 3, while $\mathcal{E}_{\{5,7\}}$ denotes the set of generating vectors who do not contain the numbers 5 and 7. $\mathcal{I}_{\{2,3\}} \bigcap \mathcal{E}_{\{5,7\}}$ denotes the set of generating vectors who contain the numbers 2 and 3 but do not contain the numbers 5 and 7.

Our algorithm first chooses a number from 1 to 13 as the center number, second searches two generating vectors with the constraints (2) and (3) but with only one common element (the center number) to fill in the two diagonals, and then searches other 10 generating vectors step by step to fill in the rest rows and columns. The alphabetical order of the letters in $\mathbf{M}$ shows the order of determination of $\mathbf{M}$'s components. Let

$\mathcal{F}_i$ denote the set of components of $\mathbf{M}$ we have determined after ith step. At each step, we choose a generating vector from a feasible set and list all the permutation of all elements subject to some constraints, then choose a possible permutation to fill in the row or column. For instance, at $(i+1)$th step, the set of components we have determined is $\mathcal{F}_i$, and suppose the first and fifth elements of the first row $\{B, F\} \subseteq \mathcal{F}_i$. In order to determine other elements of the first row, we choose a generating vector from a feasible set $\mathcal{I}_{\{B,F\}} \bigcap \mathcal{E}_{\mathcal{F}_i - \{B,F\}}$, and list all the permutation of all elements subject to the constraints $m_{11} = B$ and $m_{15} = F$, then choose a possible permutation to determine J,K and L, and thus the set of components we have determined becomes $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{B, F\}$. If there is no possible permutation which can be filled in the row or column at $(i+1)$th step, the algorithm goes back to ith step and choose another feasible generating vector to start. Since the set $\mathcal{F}_i$ increases after each step, the searching range for feasible generating vectors is reduced. The details of our enumeration algorithm are listed in Table. 2 , and the sketch of the algorithm and an example are shown in Table. 3. The detail flowchart is also shown in the Appendix.

3 Frénicle-Amela-Like Patterns

Since all basic forms of MS-5 have been generated through the enumeration algorithm introduced in Section 2, in this section we extend *Frénicle-Amela patterns* from the case of order 4 to the case of order 5, which we refer to *Frénicle-Amela-Like patterns*. We will show that these interesting Frénicle-Amela-Like patterns appear simultaneously.

Frénicle-Amela patterns are those MS-4 with patterns where the sum of possible four neighborhood numbers in the magic square is equal to the magic constant $\mu_4 = 34$. There are five Frénicle-Amela patterns which are shown in Fig. 1.

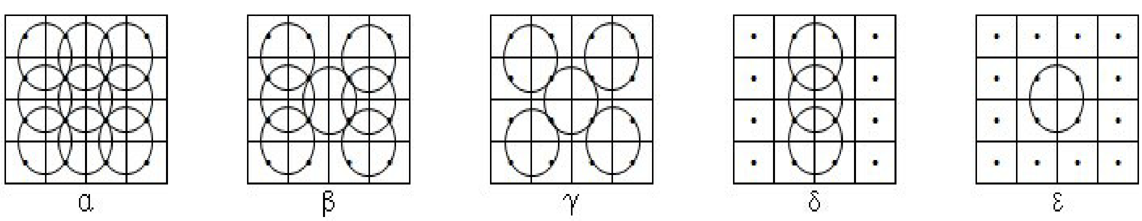

Figure 1: Five Frénicle - Amela patterns

In Fig. 2, we consider Frénicle-Amela-Like patterns of MS-5 where the sum of the center number and four symmetric numbers in the magic square is equal to the magic constant $\mu_5 = 65$. We will show that the four interesting patterns shown in Fig. 2 appear simultaneously.

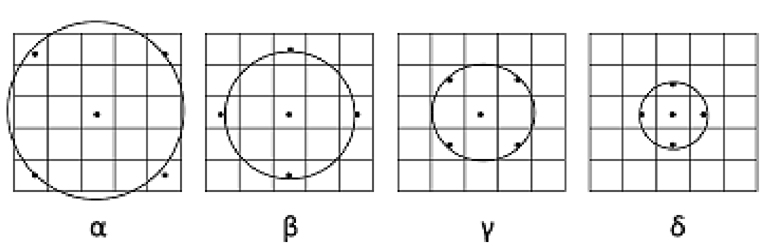

Figure 2: Four Frénicle - Amela-Like patterns

Table 2: Details of enumeration algorithm for generating all basic forms of MS-5

Steps	Row/Column/Diagonal to be determined	Feasible set of generating vector	Constraints	Components to be determined at this step	Set of all components which have been determined $\mathcal{F}_i$
1	Center number A		$1 \le A \le 13$	A	$\mathcal{F}_1 = \{A\}$
2	Main diagonal	$\mathcal{I}_{\{A\}}$	$E > B$ and $D > C > B$	B, C, D, E	$\mathcal{F}_2 = \{A, B, C, D, E\}$
3	Minor diagonal	$\mathcal{I}_{\{A\}} \bigcap \mathcal{E}_{\mathcal{F}_2 - \{A\}}$	$I > B$, $H > B$ and $G > F > B$	F, H, I, G	$\mathcal{F}_3 = \{A, B, C, D, E$ $F, G, H, I\}$
4	The 1st row	$\mathcal{I}_{\{B,F\}} \bigcap \mathcal{E}_{\mathcal{F}_3 - \{B,F\}}$	$m_{11} = B$ and $m_{15} = F$	J, K, L	$\mathcal{F}_4 = \{A, B, C, D, E$ $F, G, H, I, J, K, L\}$
5	The 2nd column	$\mathcal{I}_{\{J,C,I\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_4 - \{J,C,I\}}$	$m_{12} = J$, $m_{22} = C$ and $m_{42} = I$	M, N	$\mathcal{F}_5 = \{A, B, C, D, E$ F, G, H, I, J, K, L $M, N\}$
6	The 4th column	$\mathcal{I}_{\{L,H,D\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_5 - \{L,H,D\}}$	$m_{14} = L$, $m_{24} = H$ and $m_{44} = D$	O, P	$\mathcal{F}_6 = \{A, B, C, D, E$ F, G, H, I, J, K, L $M, N, O, P\}$
7	The 5th row	$\mathcal{I}_{\{G,N,O,E\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_6 - \{G,N,O,E\}}$	$m_{51} = G$, $m_{52} = N$, $m_{54} = O$ and $m_{55} = E$	Q	$\mathcal{F}_7 = \{A, B, C, D, E$ F, G, H, I, J, K, L $M, N, O, P, Q\}$
8	The 3rd row	$\mathcal{I}_{\{M,A,P\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_7 - \{M,A,P\}}$	$m_{32} = M$, $m_{33} = A$ and $m_{34} = P$	R, S	$\mathcal{F}_8 = \{A, B, C, D, E$ F, G, H, I, J, K, L $M, N, O, P, Q, R, S\}$
9	The 3rd column	$\mathcal{I}_{\{K,A,Q\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_8 - \{K,A,Q\}}$	$m_{13} = K$, $m_{33} = A$ and $m_{53} = Q$	T, U	$\mathcal{F}_9 = \{A, B, C, D, E$ F, G, H, I, J, K, L M, N, O, P, Q, R, S $T, U\}$
10	The 1st column	$\mathcal{I}_{\{B,R,G\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_9 - \{B,R,G\}}$	$m_{11} = B$, $m_{31} = R$ and $m_{51} = G$	V, W	$\mathcal{F}_{10} = \{A, B, C, D, E$ F, G, H, I, J, K, L M, N, O, P, Q, R, S $T, U, V, W\}$
11	The 5th column	$\mathcal{I}_{\{F,S,E\}} \bigcap$ $\mathcal{E}_{\mathcal{F}_{10} - \{F,S,E\}}$	$m_{15} = F$, $m_{35} = S$ and $m_{55} = E$	X, Y	$\mathcal{F}_{11} = \{A, B, C, D, E$ F, G, H, I, J, K, L M, N, O, P, Q, R, S $T, U, V, W, X, Y\}$

Theorem 1. *If one of the four Frénicle-Amela-Like patterns shown in Fig. 2 appears in a magic square of order* 5*, the other three Frénicle-Amela-Like patterns must appear in this magic square.*

Proof. Let L denote the center number, and $\mathbf{a} = (a_1, a_2, a_3, a_4)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$, $\mathbf{c} = (c_1, c_2, c_3, c_4)$ and $\mathbf{d} = (d_1, d_2, d_3, d_4)$ denote the four symmetric numbers in the four Frénicle-Amela-Like patterns,respectively.

Table 3: Sketch and an example of enumeration algorithm for generating all basic forms of MS-5.
*Letters and numbers in red color represent the components which are going to be determined at each step, those in grey represent components which have been determined.

Steps	1	2	3
Sketch	$\begin{pmatrix} & & & & \\ & & & & \\ & & A & & \\ & & & & \\ & & & & \end{pmatrix}$	$\begin{pmatrix} B & & & & \\ & C & & & \\ & & A & & \\ & & & D & \\ & & & & E \end{pmatrix}$	$\begin{pmatrix} B & & & & F \\ & C & & H & \\ & & A & & \\ & I & & D & \\ G & & & & E \end{pmatrix}$
Example	$\begin{pmatrix} & & & & \\ & & & & \\ & & 13 & & \\ & & & & \\ & & & & \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 13 & & \\ & & & 24 & \\ & & & & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & 3 \\ & 2 & & 9 & \\ & & 13 & & \\ & 17 & & 24 & \\ 23 & & & & 25 \end{pmatrix}$
Steps	4	5	6
Sketch	$\begin{pmatrix} B & J & K & L & F \\ & C & & H & \\ & & A & & \\ & I & & D & \\ G & & & & E \end{pmatrix}$	$\begin{pmatrix} B & J & K & L & F \\ & C & & H & \\ & M & A & & \\ & I & & D & \\ G & N & & & E \end{pmatrix}$	$\begin{pmatrix} B & J & K & L & F \\ & C & & H & \\ & M & A & P & \\ & I & & D & \\ G & N & & O & E \end{pmatrix}$
Example	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & & 9 & \\ & & 13 & & \\ & 17 & & 24 & \\ 23 & & & & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & & 9 & \\ & 16 & 13 & & \\ & 17 & & 24 & \\ 23 & 8 & & & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & & 9 & \\ & 16 & 13 & 10 & \\ & 17 & & 24 & \\ 23 & 8 & & 4 & 25 \end{pmatrix}$
Steps	7	8	9
Sketch	$\begin{pmatrix} B & J & K & L & F \\ & C & & H & \\ & M & A & P & \\ & I & & D & \\ G & N & Q & O & E \end{pmatrix}$	$\begin{pmatrix} B & J & K & L & F \\ & C & & H & \\ R & M & A & P & S \\ & I & & D & \\ G & N & Q & O & E \end{pmatrix}$	$\begin{pmatrix} B & J & K & L & F \\ & C & T & H & \\ R & M & A & P & S \\ & I & U & D & \\ G & N & Q & O & E \end{pmatrix}$
Example	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & & 9 & \\ & 16 & 13 & 10 & \\ & 17 & & 24 & \\ 23 & 8 & 5 & 4 & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & & 9 & \\ 14 & 16 & 13 & 10 & 12 \\ & 17 & & 24 & \\ 23 & 8 & 5 & 4 & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ & 2 & 15 & 9 & \\ 14 & 16 & 13 & 10 & 12 \\ & 17 & 11 & 24 & \\ 23 & 8 & 5 & 4 & 25 \end{pmatrix}$
Steps	10	11	
Sketch	$\begin{pmatrix} B & J & K & L & F \\ V & C & T & H & \\ R & M & A & P & S \\ W & I & U & D & \\ G & N & Q & O & E \end{pmatrix}$	$\begin{pmatrix} B & J & K & L & F \\ V & C & T & H & X \\ R & M & A & P & S \\ W & I & U & D & Y \\ G & N & Q & O & E \end{pmatrix}$	
Example	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ 20 & 2 & 15 & 9 & \\ 14 & 16 & 13 & 10 & 12 \\ 7 & 17 & 11 & 24 & \\ 23 & 8 & 5 & 4 & 25 \end{pmatrix}$	$\begin{pmatrix} 1 & 22 & 21 & 18 & 3 \\ 20 & 2 & 15 & 9 & 19 \\ 14 & 16 & 13 & 10 & 12 \\ 7 & 17 & 11 & 24 & 6 \\ 23 & 8 & 5 & 4 & 25 \end{pmatrix}$	

Thus a magic square of order 5 with four Frénicle-Amela-Like patterns can be written as follows,

$$\mathbf{M} = \begin{pmatrix} a_1 & A & b_1 & B & a_2 \\ C & c_1 & d_1 & c_2 & D \\ b_2 & d_2 & L & d_3 & b_3 \\ E & c_3 & d_4 & c_4 & F \\ a_3 & G & b_4 & H & a_4 \end{pmatrix},$$

where

$$a_1 + a_2 + a_3 + a_4 + L = 65, \tag{4}$$

$$b_1 + b_2 + b_3 + b_4 + L = 65, \tag{5}$$

$$c_1 + c_2 + c_3 + c_4 + L = 65, \tag{6}$$

$$d_1 + d_2 + d_3 + d_4 + L = 65. \tag{7}$$

We shall prove that one of the above equations (4)-(7) implies the other three. Without loss of generality, we assume equation (4) holds. Since the sum of two diagonals

$$a_1 + a_2 + a_3 + a_4 + L + c_1 + c_2 + c_3 + c_4 + L = 65 + 65, \tag{8}$$

combining (4) and (8) results in the equation (6). Since the sum of the first and the fifth rows

$$a_1 + a_2 + a_3 + a_4 + A + B + G + H + b_1 + b_4 = 65 + 65, \tag{9}$$

combining (4) and (9) results in

$$A + B + G + H = 65 + L - b_1 - b_4. \tag{10}$$

Then combining the equation that the sum of the second and the fourth columns

$$c1 + c2 + c3 + c4 + A + B + G + H + d_2 + d_3 = 65 + 65, \tag{11}$$

the equation (6) and the equation (10) results in

$$d_2 + d_3 = b_1 + b_4. \tag{12}$$

Similarly, combining the equation that the sum of the first and the fifth column, the equation that the sum of the second and the fourth rows, the equation (4) and (6) result in

$$d_1 + d_4 = b_2 + b_3. \tag{13}$$

Since the sum of the third row and the third column

$$b_1 + b_2 + b_3 + b_4 + d_1 + d_2 + d_3 + d_4 + 2L = 65 + 65, \tag{14}$$

combining (12), (13) and (14) result in the equations (5) and (7). Therefore, the equation (4) implies the equations (5), (6) and (7).

Table 4 shows the number and the percentage of Frénicle-Amela-Like patterns for MS-5.

4 Conclusion

In this paper, following the idea of *isomorphism* and *basic form* introduced by Schroeppel, we start from *generating vectors* and by using an enumeration algorithm we generate all basic forms of MS-5. Furthermore, we show that Frénicle-Amela-Like patterns always appear simultaneously and the percentage of these patterns is around 10%.

Table 4: The number and the percentage of Frénicle-Amela-Like patterns for MS-5

Center	Number	Total Number	Percetage
1	139895	1091448	0.1281738
2	154119	1366179	0.1128103
3	233655	1914984	0.1220141
4	195931	1958837	0.1000241
5	239656	2431806	0.0985506
6	241810	2600879	0.0929724
7	292968	3016881	0.0971096
8	293672	3112161	0.0943627
9	325841	3472540	0.0938336
10	299517	3344034	0.0895676
11	404058	3933818	0.102714
12	367367	3784618	0.0970684
13	830396	4769936	0.1740895
sum	4018885	36798121	0.1092144

Acknowledgement: This work was partially supported by two BNU-HKBU United International College Research Grants (R201409) and (R201505).

The authors would thank Prof. George Styan for his constant encouragement and help. He sent us useful materials to read and his resent presentations including [8]. We also thank Miss Luo Yu-Ying for providing many useful information on computational code in webpage.

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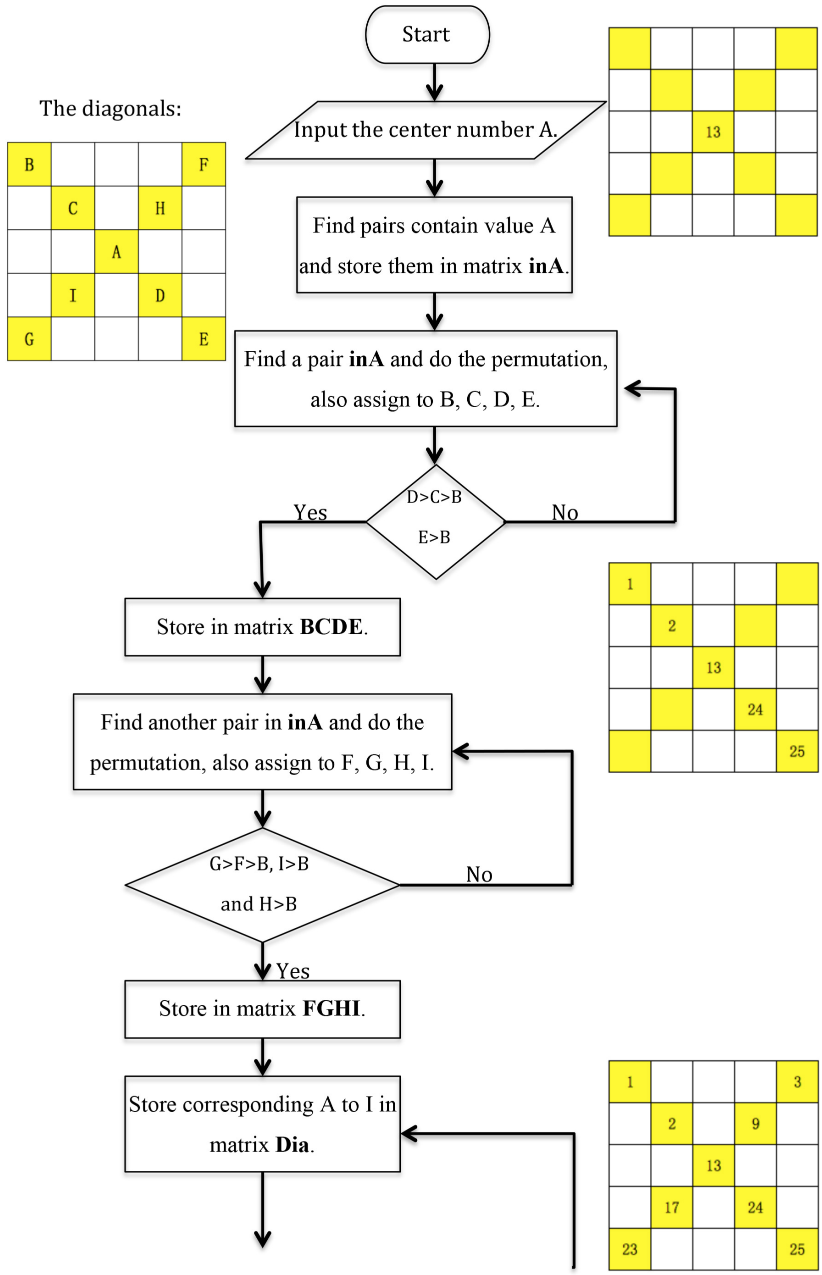
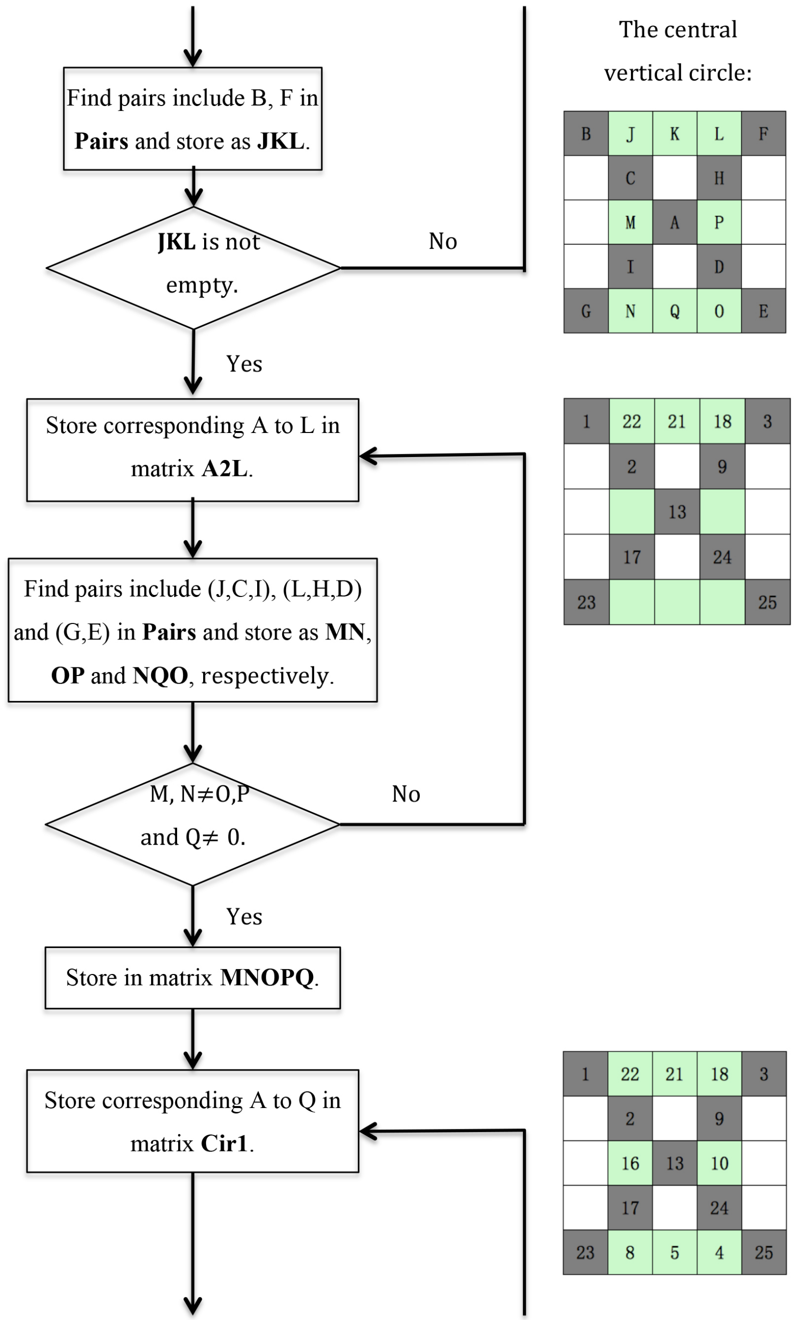
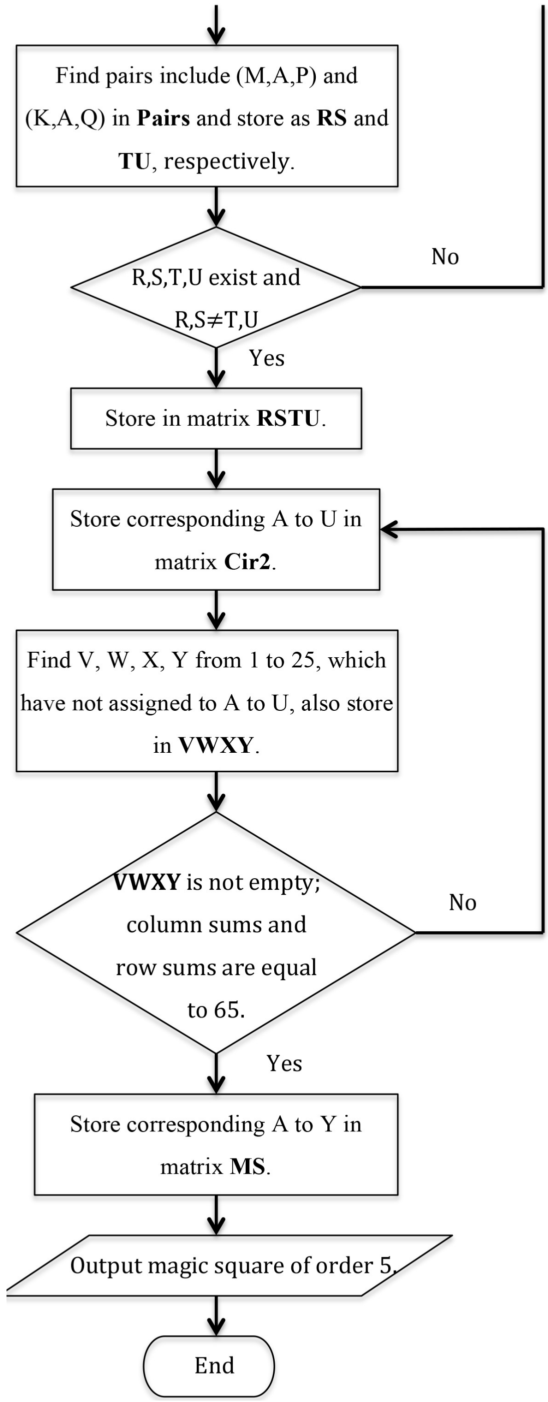
5 Appendix

Flowchart and Example




Find pairs include B, F in Pairs and store as JKL.
JKL is not empty.
No
Yes
Store corresponding A to L in matrix A2L.
Find pairs include (J,C,I), (L,H,D) and (G,E) in Pairs and store as MN, OP and NQO, respectively.
M, N≠0,P and Q≠ 0.
No
Yes
Store in matrix MNOPQ.
Store corresponding A to Q in matrix Cir1.
The central vertical circle:
B J K L F
C H
M A P
I D
G N Q O E
1 22 21 18 3
2 9
13
17 24
23 25
1 22 21 18 3
2 9
16 13 10
17 24
23 8 5 4 25



The central horizontal circle:

B	J	K	L	F
V	C	T	H	X
R	M	A	P	S
W	I	U	D	Y
G	N	Q	O	E

1	22	21	18	3
	2	15	9	
14	16	13	10	12
	17	11	24	
23	8	5	4	25

1	22	21	18	3
20	2	15	9	19
14	16	13	10	12
7	17	11	24	6
23	8	5	4	25