

3D/2D Projected Shape Sensitivity Analysis of Total Joint Arthroplasty Implants

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Abstract

Recent advancements in computer vision and machine learning have facilitated fully autonomous kinematics measurement of Total Knee Arthroplasty (TKA) implant components. However, such performance has yet to be effectively replicated for Reverse Total Shoulder Arthroplasty (rTSA) implants. This study examines the correlation between 3D implant orientation and the sensitivity of their 2D projections, employing the Invariant Angular Radial Transform Descriptor (IARTD). It was observed that minor rotational differences along near-symmetric dimensions, such as the humeral internal-external rotation and the tibial internal/external axis, resulted in negligible shape alterations. Moreover, axes that posed challenges in registration were associated with reduced sensitivity of the descriptor. The findings indicate that symmetrical geometries and orientations inherently limit the amount of silhouette information that can be extracted. Nevertheless, the inclusion of bony anatomical landmarks as optimization constraints emerges as a promising approach to mitigate ambiguity and enhance registration accuracy.

Keywords: Computer vision, TKA, rTSA, Shape Descriptor, Angular Radial Transform

TODO (I left you some room to write as well)

- Figures (please add more if you think we need them, I left you some room)
 - Figure showing different rTSA implant/fluoro views
 - Figure showing each of the rotation axes
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- Verify that there are enough citations and justifications for last two discussion paragraphs
- Last grammar run through (Academic passive, no first person, no jarring sentences)
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1. Introduction

Understanding the in-vivo kinematics of total joint replacement has been essential in implant design, post-operative assessment, and predicting wear and failure patterns for nearly three decades [1–3]. Recent advancements in computer vision and machine learning have enabled these analyses for total knee arthroplasty (TKA) in a fully autonomous and clinically practical setting, utilizing single-plane fluoroscopy [4, 5]. However, using only a single camera inherently limits the optimization due to loss of depth perception and the introduction of ambiguous projected shapes during optimization [6–9]. The observed limitation, predominantly impacting mediolaterally symmetric tibial implants, led to a phenomenon termed “symmetry traps.” In such instances, two distinct three-dimensional orientations of the implant produce indistinguishable two-dimensional projected geometries. A machine learning algorithm was developed to address these symmetry traps. This algorithm was trained to recognize accurate anatomic orientations and correct images caught in optimization minima [10]. However, this approach required the symmetric implant to optimize into one of the two potential local minima, each corresponding to a distinct “symmetry trap.”

The application of the same optimization routine and cost function [5, 6] to reverse total shoulder arthroplasty (rTSA) resulted in significantly lower performance compared to its application in total knee arthroplasty implants. This suboptimal performance manifested in two primary ways. First, there was a consistent error along the internal/external rotation axis. This axis, characterized by near-rotational symmetry, frequently has features occluded by the glenosphere implant in frontal-plane fluoroscopy. Second, the optimization led to a distal shift of the implant. This shift meant that while the local minima correctly registered the humeral stem, they failed to do so for the humeral cup.

This pattern of failure prompted a deeper exploration into the perceptual psychology of shape [11, 12], underscoring the significance of high curvature as a salient feature in binary shapes. Additionally, binary distance metric studies [13, 14] emphasized the need to align the cost-function metric with the problem, considering the data’s underlying structure. In response to these new findings and to address the challenges identified with rTSA kinematics optimization, we developed two novel cost functions.

Menger’s discrete curvature algorithm [15] was applied to the projected implant’s contour to isolate keypoints of the highest curvature. Subsequently, a distance map (DM_k) for each keypoint was calculated, enabling rapid computation of Euclidean distances to these points. The high-curvature distance maps were then integrated into a *Modified Asymmetric Surface Distance*, focusing solely on the identified keypoints (Eq. 1). In this process, the minimum values from the element-wise multiplication of each distance map (DM_k) with the projection estimate (p) were averaged. An error value of 0 would indicate a direct overlap of the estimated projection with all keypoints, confirming accurate registration in high-curvature regions. However, this approach, when applied to humeral implants, mirrored previously encountered errors, resulting in poor performance.

$$J = \frac{\sum_{k \in \mathbb{K}} (\min_{p \in Proj} (p \cdot DM_k))}{N}$$

where

\mathbb{K} = Set of all high-curvature keypoints (1)

N = Number of high-curvature keypoints

DM_k = Distance map for keypoint k

p = Single point on projection estimate

In response to the Modified Asymmetric Surface Distance cost function’s limitations, there was a shifted focus to refining the Hamming Distance [5, 6], especially due to its uniform error in non-overlapping scenarios between the estimate and target mask. To overcome this limitation, a *Modified Mean Surface Distance* was developed. This involved the element-wise multiplication of the projection estimate ($Proj_{x,y}$) with the global distance map of the target ($DM_{x,y}$) (Eq. 2). Similar to the *Modified Asymmetric Surface Distance*, this approach also led to poor results.

$$J = \frac{\sum_{(x,y) \in Image} Proj_{x,y} DM_{x,y}}{\sum_{(x,y) \in Image} Proj_{x,y}} \quad (2)$$

The current investigation delves into the fundamental shape aspects of each arthroplasty system, with a focus on developing a method for autonomously measuring rTSA kinematics from single-plane images. Central to this is the use of Invariant Shape Descriptors, particularly the Invariant Angular Radial Transform Descriptor (IARTD), which offers a mathematically robust approach to describe object shapes. These descriptors are immune to variations in scale, translation, or orientation [16], and are adept at quantifying the relative “nearness”, “farness”, and “uniqueness” of shapes as vector differences. Such properties are valuable for object categorization [17–19] and kinematics measurement [9], with IARTD’s sensitivity to radial shape differences [20] being particularly beneficial for detailed contour analysis.

The focus of this analysis is on the sensitivity of projected 2D shapes, as depicted by IARTD, to changes in their 3D orientation. This is key to understanding the impact of subtle orientation variations on the projected shape, an aspect integral to shape-based optimization metrics. The ultimate aim is to highlight performance differences in autonomous kinematics measurements between TKA and rTSA implant systems. Additionally, the study seeks to identify areas where imaging methods can be improved to boost the algorithm’s performance.

2. Methods

2.1. Data Collection

For shape sensitivity analysis, representative 3D models of reverse total shoulder arthroplasty (rTSA) humeral and glenosphere implants, as well as total knee arthroplasty (TKA) femoral and tibial implants, were obtained from a manufacturer. The study focused on a single size for each implant type, as the scale of the shapes was normalized using an Invariant Shape Descriptor, rendering multiple sizes unnecessary for this analysis.

2.2. Image Generation

Each implant’s binary silhouette was rendered to a 1024×1024 image plane using an in-house CUDA camera model (CUDA Version 12.1) [21]. The model featured a 1000mm focal length and 0.3mm per pixel resolution. All programming tasks utilized an NVIDIA Quadro P2200 GPU.

2.3. Invariant Angular Radial Transform

The Invariant Angular Radial Transform Descriptor (IARTD) was selected for its radial direction sensitivity, enabling the detection of subtle contour changes in projected shapes [20]. This sensitivity allows us to address minor changes along the contour of the projected shape, which is a desirable property for determining the minor changes in shape with respect to input orientation.

IARTD computation involves aggregating orthogonal basis components across the unit polar disk, forming a complex moment. Each basis function has an order (n) and a repetition (p). The order can be visualized as concentric rings on the polar disk, and the repetition as the count of slices partitioning the unit disk along θ . For these calculations, the image is normalized so that the center is at $(0, 0)$, and the four corners are at $(\pm 1, \pm 1)$.

Each angular radial transform (ART) coefficient is a complex double integral (Eq. 3) over the image in polar coordinates, $f(\rho, \theta)$ multiplied by the ART basis function, $V_{np}(\rho, \theta)$ (Eq. 4).

$$F_{np} = \int_0^{2\pi} \int_0^1 f(\rho, \theta) V_{np}(\rho, \theta) \rho d\rho d\theta \quad (3)$$

$$V_{np}(\rho, \theta) = A_p(\theta) R_n(\rho) \quad (4)$$

The radial basis function includes a complex exponential, $A_p(\theta)$ (Eq. 5), ensuring rotational invariance, and a trigonometric transform, $R_p(\theta)$ (Eq. 6), to establish orthogonality.

$$A_p(\theta) = \frac{1}{2\pi} e^{jp\theta} \quad (5)$$

$$R_n(\rho) = \begin{cases} 1 & n = 0 \\ 2 \cos(\pi n \rho) & n \neq 0 \end{cases} \quad (6)$$

Phase correction is applied to each ART coefficient (Eq. 7, Eq. 8) to adjust for differences in in-plane rotation.

$$\phi'_{np} = \phi_{np} - \phi_{n,1} \quad (7)$$

$$F'_{np} = F_{np} e^{-jp\phi_{n,1}} \quad (8)$$

Subsequently, the final feature vector is formulated by the polar decomposition of each coefficient at every order and repetition (Eq. 9). Values from the first two repetitions are excluded, as they do not provide significant information [20]. The complete IARTD feature vector encompasses values of $n = 0, \dots, 3$ and $p = 0, \dots, 8$ per the original authors' suggestion [20].

$$IARTD = \{|F'_{np}|, \phi'_{np}\} \text{ where } n \geq 0, p \geq 2 \quad (9)$$

2.4. Shape Differences and Sensitivity

In order to quantify the overall change between shapes, a readily interpretable value must be established. To simplify notation, successive rotations are denoted as subscripts, with $R_z R_x R_y$ being represented as $R_{z,x,y}$. If more than 3 rotations are applied successively, the full rotation sequence will be captured as $R_{r_1, r_2, r_3, \dots, r_n}$. Similarly, the application of the IARTD equation to an implant at a specific input orientation $R_{z,x,y}$ is denoted as $IARTD(R_{z,x,y})$. Shape differences were calculated using the central difference equation on the IARTD vector produced from two different orientations. The grid of sampled orientations had extrema of $\pm 30^\circ$ with a step size of 5° for each of the x , y , and z axes. The “differences” along each axis were computed using a positive and negative rotation ($\pm\delta$) of 1 degree. Therefore, for every set of x, y, z rotations, three distinct shape differences are computed, one each for δ_x , δ_y , and δ_z (Eq. 10).

For notational brevity, we will condense the total equation to a single $\Delta S(\delta)$, (representing $\Delta Shape$ for a differential rotation δ).

$$\begin{aligned} \Delta S(\delta)_{z,x,y} &\equiv IARTD(R_{z,x,y,+\delta}) \\ &\quad - IARTD(R_{z,x,y,-\delta}) \\ \forall \delta &\in \{\delta_x, \delta_y, \delta_z\} \end{aligned} \quad (10)$$

Is this approximation averaged over separate unit perturbations in each DoF(1), or is each axis tweaked simultaneously so there are only two IARTDs to diff (2)?
 This is the (1) option. I tried to emphasize this further in the above section by describing > 3 rotations using a successive notation. I also tried to emphasize that this was for each $\delta_{x,y,z}$ in the equation itself.

The disparate scales of IARTD vector elements necessitate their standardization, ensuring a balanced assessment of global behavior without overemphasis on any individual element. Z-scaling provides a practical approach to standardizing each element relative to its distribution. After z-scaling, the Euclidean norm of each $S(\delta)_{z,x,y}$ is calculated to quantify the total shape change for a specific differential rotation (Eq. 11).

The final step takes advantage of two factors: first, that the in-plane rotations are the first in the Euler sequence (z -axis), and second, that this type of rotation does not affect the in-plane shape. For each x and y input rotation, an average is computed from values where x and y remain constant while z varies (Eq. 12). The final values are obtained from this equation, denoted by \mathbb{S} . Individual plots will be created for $\mathbb{S}_{x,y}$, corresponding to each differential rotation in x , y , and z , and for each of the four implant types. An analysis of these plots will be conducted to assess the performance of JTML optimization and to explore areas where optimization faces significant challenges.

$$\|S(\delta)_{z,x,y}\|_2 \quad (11)$$

$$\mathbb{S}(\delta)_{x,y} = \frac{\sum_z \|S(\delta)_{z,x,y}\|_2}{N} \quad (12)$$

Table 1: Average projected-shape sensitivity values for each of the implant models.

Implant Type	Average $\mathbb{S}(\delta_x)$	Average $\mathbb{S}(\delta_y)$	Average $\mathbb{S}(\delta_z)$
Humeral	8.83	4.82	7.08
Glenosphere	6.37	6.22	4.86
Femoral	6.88	8.68	4.93
Tibial	9.0	5.52	3.72

3. Results

The humeral implant exhibited the lowest mean $\mathbb{S}(\delta_y)$ across all implant types (Fig. 1) (Table 1). This rotation represents the final rotation in our Euler rotation sequence (Z-X-Y) and captures the internal/external rotation of the humeral implant. Additionally, the surface plotted by the humeral shape sensitivity for all $\delta_{x,y,z}$ is much smoother than any of the other plots, demonstrating the relative lack of shape difference for a wide range of input orientations. Several plots showed regions with relatively low sensitivity. Specifically, the glenosphere's δ_y sensitivity along the $y = 0$ axis (Fig. 2) and the tibial implant's δ_y sensitivity along the $x = 0$ axis (Fig. 4). The femoral implant had the highest average sensitivity ($\frac{\mathbb{S}(\delta_x) + \mathbb{S}(\delta_y) + \mathbb{S}(\delta_z)}{3}$) among all implant types.

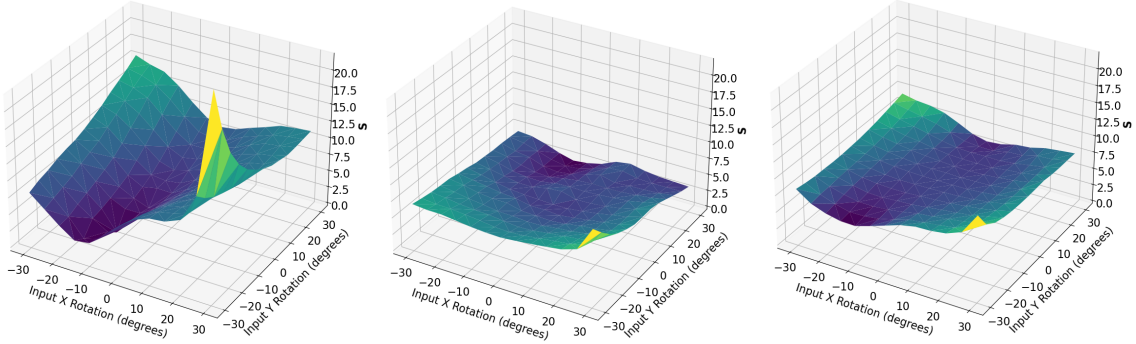


Figure 1: The \mathbb{S} plot for a humeral implant for δ rotations along the x, y, and z axis, respectively.

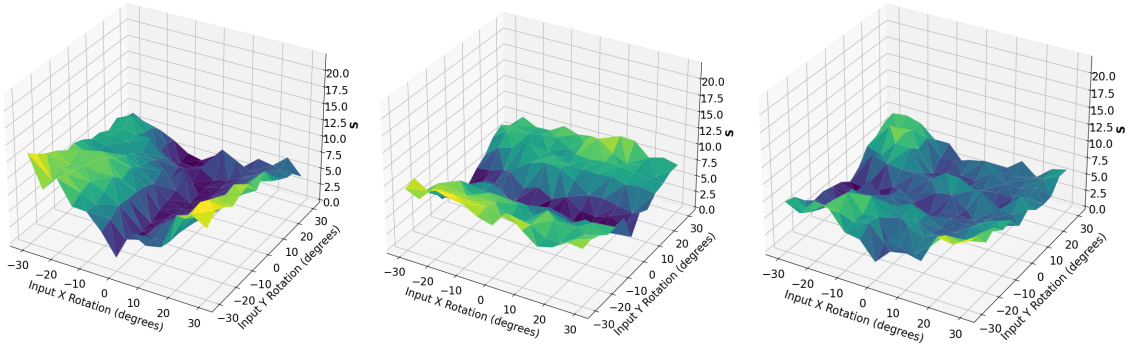


Figure 2: The \mathbb{S} plot for a glenosphere implant for δ rotations along the x, y, and z axis, respectively.

4. Discussion

The findings correspond closely with initial expectations regarding the sensitivity measurement of projected shapes relative to 3D object orientation and are consistent with areas challenging for JTML optimization. Specifically, the humeral implant showed a generally smooth and minimal shape sensitivity profile,

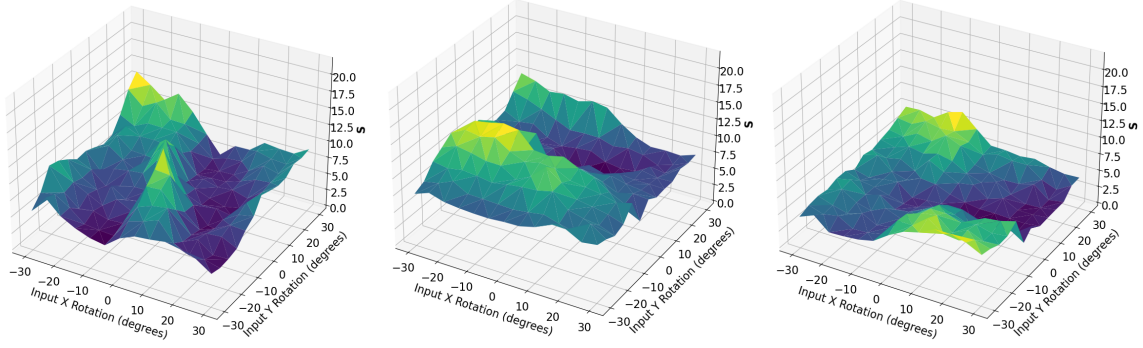


Figure 3: The \mathbb{S} plot for a femoral implant for δ rotations along the x, y, and z axis, respectively.

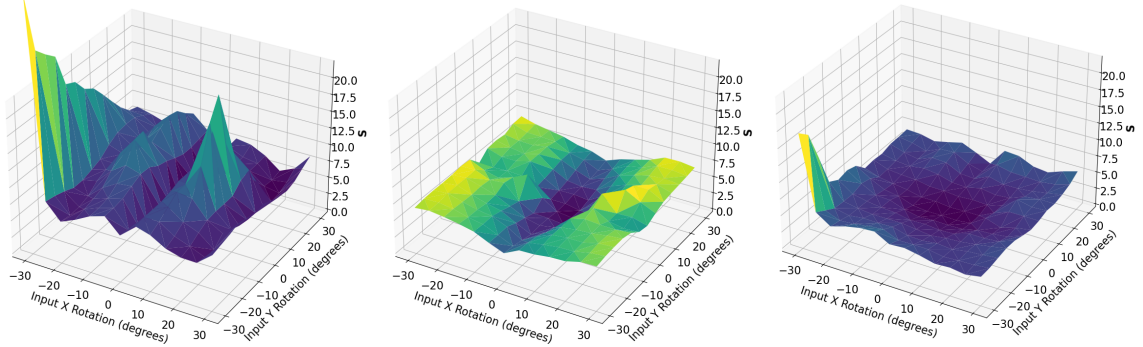


Figure 4: The \mathbb{S} plot for a tibial implant for δ rotations along the x, y, and z axis, respectively.

particularly for δ_y rotations (Table 1). Along this axis, the humeral implant is the most cylindrical, meaning we would not expect to see a significant change in the shape descriptor with minor δ_y rotations. Furthermore, it is noteworthy that this axis presented the most significant difficulties in JTML optimization.

Similar intuitive outcomes are observed with the glenosphere implant, which exhibited the lowest average $\mathbb{S}(\delta)$ among all the implant types. This implant primarily consists of an articulation surface closely approximating a spherical shape. Given that the projection of a sphere (a circle) remains constant regardless of the sphere’s orientation, the closer a shape is to a spherical form, the lower its overall shape sensitivity is expected to be.

The observed shape sensitivity of the tibial implant with respect to δ_y rotation aligns with the concept of symmetry traps. There is a consistently low shape sensitivity along the line where $x = 0$. This axis, associated with internal/external rotation, is the same one that contributes to symmetry traps, where two different 3D orientations result in an identical projected shape. In terms of this analysis, the ΔS value would be 0 for these two orientations of the tibial implant.

This study sheds light on an important aspect of Joint Track Machine Learning, particularly the use of Euler angles in the DIRECT-JTA optimization routine. Currently, the optimization does not involve independently varying all angles within a body-centered reference frame, as this approach is not conducive to hyperbox creation [6, 22]. Instead, optimization is performed over a range of ordered rotations, projected through the sequence $R_z R_x R_y$. The challenges the humeral implant encounters in aligning the y -axis illustrate that this ordered sequence, especially with a symmetric final axis, can hinder the convergence process.

Beyond the inherent shape sensitivities, such optimization limitations motivate exploring alternatives to Euler angles. Performing registration optimization directly on the Special Orthogonal group $SO(3)$ poses an intriguing direction. $SO(3)$ encapsulates all possible 3D rotations in a mathematically convenient structure (A *Lie Group*, which is both a manifold and a group) [23, 24]. By optimizing on this manifold instead of Euler angle parametrizations, issues with gimbal lock and cascading rotation effects can be avoided. However, most manifold optimization is specifically gear toward derivative-based optimization, which is not the currently

supported by the DIRECT-JTA algorithm that Joint Track Machine Learning incorporates [5, 6, 22]. This limitation motivates either a restructuring of the current optimization methods to incorporate derivative information, or to explore the potentials of derivative-free (black-box) optimization [25] over manifolds.

Historical manual registration software heavily relied on bony landmarks within images to disentangle challenging poses. As landmarks, the tibial tuberosity, fibula, and bicipital groove were vital in identifying the specific orientation of implants. To accurately assess implant kinematics in highly symmetric unicompartmental knee arthroplasty, surface meshes of femoral and tibial components were incorporated into the registration process [26]. These bone models were particularly beneficial in addressing implants’ internal/external rotation along semi-symmetric axes. Additionally, the use of densely selected keypoints in the precise measurement of preoperative shoulder kinematics through biplane fluoroscopy has been documented [27], with keypoint registration achieved via a modified Perspective-N-Points optimization routine. Furthermore, intensity-based metrics have played a crucial role in the accurate determination of non-operative tibiofemoral kinematics [28]. Rather than minimizing the Euclidean distance of projected model keypoints, these methods leverage digitally reconstructed radiographs and directly compare the intensity values of the x-ray image and the projection estimate. Thus, utilizing bony landmarks for measuring implant kinematics presents a robust and reliable approach to measuring accurate joint kinematics in clinical environments.

5. Conclusion

This study demonstrates intrinsic differences between implant types regarding projected 2D shape sensitivity. Measurement difficulties aligned with low sensitivity along problematic axes—humeral internal rotation and tibial symmetry traps. Fundamentally, minor orientation changes yielded negligible 2D variability for near-symmetrical geometries. While inherent shape constraints limit data extractable solely from fluoroscopic silhouettes, incorporating additional image information like bone offers promise. Despite unavoidable ambiguity along select dimensions, boosting descriptor sensitivity and employing precise anatomical constraints could enable robust clinical tracking. Overall, relating optimization performance to shape response underscores routes toward accurate autonomous kinematic analysis.

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