The spiked matrix model
with generative priors

Benjamin Aubin, **Bruno Loureiro,** Antoine Maillaird,
Florent Krzakala, Lenka Zdeborová

IPhT cea saclay

ENS ÉCOLE NORMALE SUPÉRIEURE

Introduction

Motivation

How to **efficiently** exploit the structure of data in signal reconstruction?

Introduction

Motivation

How to **efficiently** exploit the structure of data in signal reconstruction?

$$\mathbf{v} = G(\mathbf{z})$$

$\mathbf{v} \in \mathbb{R}^p$: signal, $\mathbf{z} \in \mathbb{R}^k$: latent variable

G: generative model

Motivation

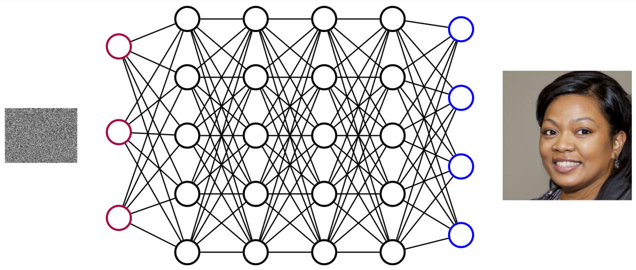
How to **efficiently** exploit the structure of data in signal reconstruction?

$$\mathbf{v} = G(\mathbf{z})$$

$\mathbf{v} \in \mathbb{R}^p$: signal, $\mathbf{z} \in \mathbb{R}^k$: latent variable

G: generative model

Example: GANs (Goodfellow *et al.* 2014)



The model

Spiked matrix estimation

We observe a matrix $Y \in \mathbb{R}^{p\times p}$ generated by a ground truth vector $\mathbf{v}^\star$:

$$Y = \frac{1}{\sqrt{p}}\mathbf{v}^\star(\mathbf{v}^\star)^\intercal + \sqrt{\Delta}\xi \tag{1}$$

$$\text{with} \quad \mathbf{v}^\star \in \mathbb{R}^p \sim P_v, \quad \xi_{ij} \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_p), \quad p \to \infty \quad \text{and} \quad \Delta = \Theta(1)$$

Objective: estimate $\mathbf{v}^\star$.

Sparse PCA

- $k \ll p$: non-zero components $\rho = \frac{k}{p} = \Theta(1)$
- Sparse prior

$$\mathbf{v}^\star \sim (1-\rho)\delta(\mathbf{v}^\star) + \rho\mathcal{N}(0, I_p)$$

- Critical noise for $\Delta_c = 1$ (BBP transition)
- Algorithmic hard phase for small ρ

Generative prior

- $k \ll p$: latent variables, $\alpha = \frac{p}{k} = \Theta(1)$
- Generative prior

$$v^\star = \varphi^{(L)}\left(\frac{1}{\sqrt{k}}W^{(L)} \dots \varphi^{(1)}\left(\frac{1}{\sqrt{k}}W^{(1)}\mathbf{z}^\star\right)\right)$$

- Critical noise $\Delta_c(\alpha) = \dots$?
- Algorithmic hard phase $\dots$?

Results

Main result 1: Analysis of the optimal Bayesian estimator

We prove a rigorous formula describing the performance of the optimal Bayesian estimator for arbitrary prior P_v:

$$MMSE_v(\Delta) = \rho_v - \text{arginf}_{q_v}\, i_{\text{RS}}(\Delta, q_v)$$

where

$$i_{\text{RS}}(\Delta, q_v) = \frac{(\rho_v - q_v)^2}{4\Delta} + \lim_{p\to\infty} \frac{I\left(\mathbf{v}; \mathbf{v} + \sqrt{\frac{\Delta}{q_v}}\boldsymbol{\xi}\right)}{p}$$

with $\boldsymbol{\xi} \sim \mathcal{N}(0, I_p)$, $I(Y; \mathbf{v}^\star) = D_{\text{KL}}(P_{(v^\star, Y)} | P_{v^\star} P_Y)$ and $\rho_v = \lim_{p\to\infty} \mathbb{E}_{P_v}[\mathbf{v}^\intercal \mathbf{v}]/p$.

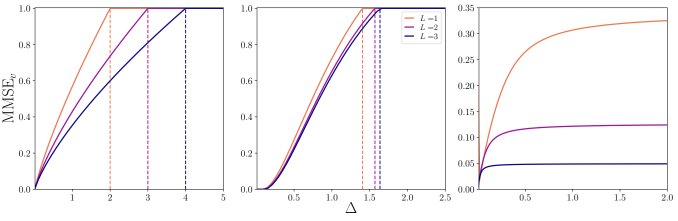
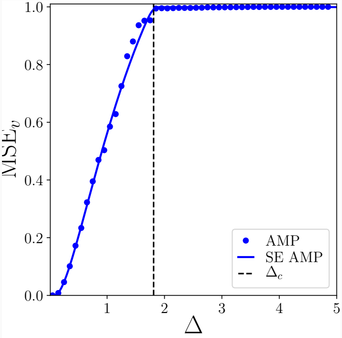


Figure: Linear (left), Sign (middle) and ReLU (right) activation functions.

Results

Main result 2: Approximate Message Passing algorithm

We derive an Approximate Message Passing algorithm (AMP) and analyse its performance. For our class of random multi-layer generative priors with linear, sign and ReLU activations, **AMP achieves the optimal statistical threshold previously derived**.



$$\mathbf{v} = \mathrm{sign}(W\mathbf{z}).$$

Results

Main result 3: L-AMP spectral method

The linearised AMP algorithm yield a simple spectral method also achieving the optimal statistical threshold. In particular, it beats PCA.

Input: Observed matrix $Y \in \mathbb{R}^{p\times p}$, prior P_v on $\mathbf{v} \in \mathbb{R}^p$

Take the leading eigenvector $\hat{\mathbf{v}} \in \mathbb{R}^p$ of $K_p \left[\frac{Y}{\sqrt{p}} - \mathrm{I}_p\right]$ with $K_p = \mathbb{E}_{P_v}\left[\mathbf{v}\mathbf{v}^\intercal\right]$.

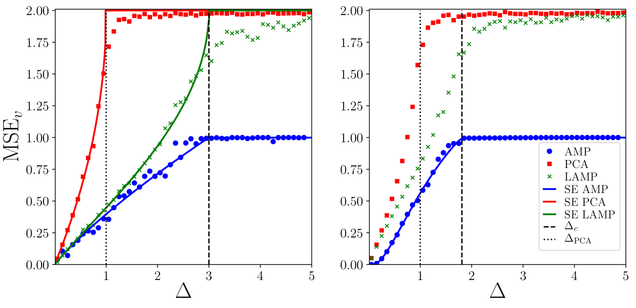
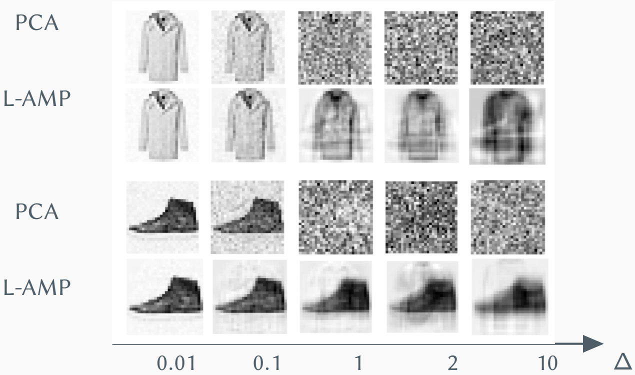


Figure: Linear (left) and Sign activation functions, $L = 1$.

Spectral algorithm L-AMP - Application to FashionMnist

- P_v : underlying distribution of *Fashion Mnist* dataset $\{\mathbf{v}^\mu\}_{\mu=1}^m$.
- Use empirical covariance $K_p = \mathbb{E}[\mathbf{v}\mathbf{v}^\intercal] \simeq \frac{1}{m}\sum_{\mu=1}^m v^\mu (v^\mu)^\intercal$ and apply the L-AMP spectral method.



Thank you!

Thank you for the attention :) !

- Curious to know more? Check our paper with the details of these results (and others!) at arXiv:1905.12385.
- Come check our poster and discuss with us live!
- Not in NeurIPS 2019? Feel free to send me an e-mail: bruno.loureiro@ipht.fr