College of Computer & Information Science
Northeastern University
Fall 2017

CS5800
Algorithms
December 8 2017

CS5800 - Algorithms - Review for final exam

1. Let $G = (V, E)$ be a flow network with capacity $c : E \to \mathbb{R}_{\geq 0}$.
   For 2 s-t cuts $(S, \overline{S})$ and $(S', \overline{S'})$, we define their union to be $(S \cup S', \overline{S \cup S'})$.
   Show that if $(S, \overline{S})$ and $(S', \overline{S'})$ are both minimal cuts, then their union $(S \cup S', \overline{S \cup S'})$ is also a minimal cut.

2. Your company has n different computational tasks $T_1, T_2, ..., T_n$ that it needs to execute. There are 2 computers c_1 and c_2 that can execute these computational tasks. Each of the tasks $T_1, ..., T_n$ can be executed by any of the 2 computers c_1 or c_2.
   You know that the costs of the computations are given by (for $1 \leq i \leq n$):

   - The cost of executing task T_i by c_1 is $a_i > 0$.
   - The cost of executing task T_i by c_2 is $b_i > 0$.
   - If T_i and T_j are executed by 2 different computers (clearly $i \neq j$), there is an additional cost of d_{ij}. (The additional cost is the same whether T_i is executed by c_1 and T_j by c_2 or T_i is executed by c_2 and T_j by c_1.)

   Suggest an algorithm to find the best way to divide the tasks between the two computers.

3. Let $A_1, ..., A_n$ be n sets (not necessarily disjoint sets. It might be that $A_i \cap A_j \neq \emptyset$), and let $k_1, ..., k_n$ be integers such that for any subset $I \subset \{1, 2, ..., n\}$ we have:
   $$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} k_i$$
   Show that there exist sets $B_1, .., B_n$ such that $B_i \subseteq A_i$, and $|B_i| = k_i$ for any $1 \leq i \leq n$, and:
   $$B_i \cap B_j = \emptyset \text{ for any } i \neq j$$

4. Let $G = (V, E)$ be an undirected weighted graph with weight function $w : E \to \mathbb{R}_{>0}$. (The weights are positive.)
   For a 2 vertices $v_1, v_2 \in V$ we denote:
   $$w_2(v_1, v_2) = \min_p \{w(p) \mid p \text{ is a path connecting } v_1 \text{ and } v_2 \text{ with } 2k \text{ edges, } k \in \mathbb{N} \cup \{0\}\}$$
   Where $w(p) = \sum_{e \in p} w(e)$.
   Design an algorithm that gets as input the graph G and a vertex s and calculates $w_2(s, v)$ for all $v \in V$.

5. Let G be an undirected connected graph with positive weights on edges. Let T be a minimum spanning tree of G. Show that there exists a way to execute Kruskal's algorithm such that it returns T as a result.

6. Let L_1 be the language of undirected graphs with Hamiltonian path:

$$L_1 = \{G \mid G \text{ is an undirected graph with a Hamiltonian path}\}$$

   Let L_2 be the language of undirected graphs with Hamiltonian cycle

$$L_2 = \{G \mid G \text{ is an undirected graph with a Hamiltonian cycle}\}$$

   Prove that $L_2 \leq_p L_1$.