College of Computer & Information Science
Northeastern University
Fall 2017

CS5800
Algorithms
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Sample Solution to Quiz 3

1. (3 points) Minimum Spanning Trees

Let G be a connected undirected graph with a positive weight on each edge. Assume all weights are distinct. Let T be a minimum spanning tree of G. Let (u, v) be an arbitrary edge of T. Let C be an arbitrary cut in G such that u and v are on opposite sides of the cut. Prove or disprove: The edge (u, v) is the minimum weight edge crossing the cut C.

Answer: False. Consider the graph with three vertices u, v, and x, and two edges (u, v) and (v, x) with weights 2 and 1, respectively. This graph is a tree, so it is an MST of itself. The cut C that separates v from $\{u, x\}$ contains both edges, and clearly (u, v) is not the minumum-weight edge in the cut.

2. (3 + 4 = 7 points) Pairing in a dance class

A group of n men and n women are attending a dance class. The instructor wants to pair each man with a woman in such a way that the sum of the absolute value of the height differences between partners is minimized. Assume all heights are distinct.

(a) Consider the case where there are two men and two women. Prove that there is an optimal pairing in which the tallest man is paired with the tallest woman.

Answer:

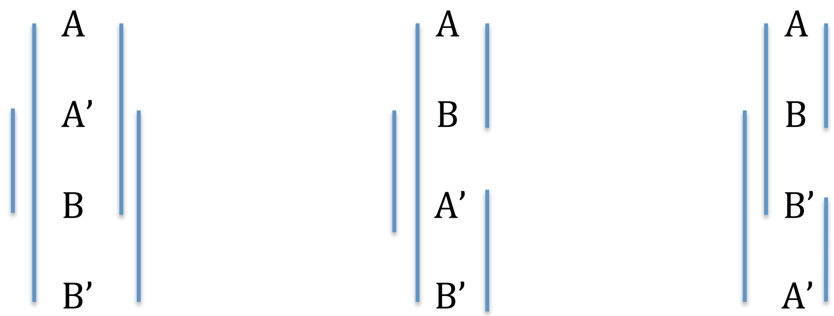
Claim 1 *Let A be a tallest man and let B be a tallest woman. There is an optimal pairing in which A is paired with B.*

Proof: The proof is by contradiction. Consider an optimal pairing. Suppose A is paired with B' shorter than B, and B with A' shorter than A. Let the heights of A, A', B, and B' be $h(A)$, $h(A')$, $h(B)$, and $h(B')$. We show that

$$|h(A) - h(B)| + |h(A') - h(B')| \leq |h(A) - h(B')| + |h(A') - h(B)|,$$

contradicting the assumption that the given pairing is optimal. We consider three cases, illustrated in the figure below.

1. $h(A) \geq h(A') \geq h(B) \geq h(B')$: In this case, both sides are equal to $h(A) - h(A') + 2(h(A') - h(B)) + h(B) - h(B')$.
2. $h(A) \geq h(B) \geq h(A') \geq h(B')$. In this case, the right-hand side exceeds the left-hand side by $2(h(B) - h(A'))$.
3. $h(A) \geq h(B) \geq h(B') \geq h(A')$. In this case, the right-hand side exceeds the left-hand side by $2(h(B) - h(B'))$.



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(b) Based on the result of part (a), give a polynomial-time greedy algorithm for computing an optimal pairing for n men and n women. You need not prove the correctness of your algorithm. State the worst-case running time of your algorithm.

Answer: Now, the algorithm is very simple.

1. Sort the n men and the n women in decreasing order of their heights.
2. Pair the ith man with the ith woman, in order.

Running time equals the time for sorting plus the time for pairing. Sorting can be done in $O(n \log n)$ time, and pairing in $O(n)$ time. So total time is $O(n \log n)$.