Math 240: Discrete Structures I

Assignment 2: Solutions

1. *Proof Techniques*

   (a) **Give a direct proof that: If x is an integer then $x^3 - x$ is divisible by 3.**

   Proof. We have

$$\begin{aligned} x^3 - x &= x(x^2 - 1) \\ &= x(x-1)(x+1) \\ &= (x-1)x(x+1) \end{aligned}$$

   But $x - 1, x$ and $x + 1$ are three consecutive integers. Thus the number 3 must divide exactly one of them. □

   Alternate Proof. Division with remainder gives $x = 3q+r$ for some positive q and r between 0 and 2. Thus, we have

$$\begin{aligned} x^3 - x &= (3q + r)^3 - (3q + r) \\ &= (3q + r)(9q^2 + 6qr + r^2) - 3q - r \\ &= 27q^3 + 18q^2r + 3qr^2 + 9q^2r + 6qr^2 + r^3 - 3q - r \\ &= 3(9q^3 + 9q^2r + 3qr^2 - q) + r^3 - r\ \ . \end{aligned}$$

   Since q and r are integers, the term multiplied by 3 can be ignored. Thus, the expression is divisible by 3 if and only if $r^3 - r$ is divisible by 3. But, we know that r is either 0, 1 or 2. So $r^3 - r$ is $0 - 0 = 0$, $1 - 1 = 0$ or $8 - 2 = 6$, which are all divisible by 3, as desired. □

   (b) **Give a contrapositive proof that: Let x and y be real numbers. If the product $x \cdot y$ is irrational then either x or y is an irrational number.**

   Proof. We will show the contrapositive, namely, that if both x and y are rational, then $x \cdot y$ is rational. Since x and y are rational, we have $x = \frac{a}{b}$ and $y = \frac{c}{d}$ where a, b, c, d are integers. It follows that $x \cdot y = \frac{ac}{bd}$, which is clearly a quotient of two integers, as desired. □

2. *Proofs by Contracition.*
   Prove by contradiction that:

   (a) **There are no integers x and y such that $6x + 14y = 1$.**

   Proof. Suppose not, then there exist integers x and y such that $6x + 14y = 1$. Thus $3x + 7y = \frac{1}{2}$. But x and y are integers so this is impossible. $\square$

   (b) **Let x, y, z be integers. If $x^2 + y^2 = z^2$ then either x or y is even.**

   Proof. Suppose not, then there exist integers x, y and z such that x and y are odd, and $x^2 + y^2 = z^2$. Since both x and y are odd, x^2 and y^2 are odd, and therefore z^2 must be even. Because z^2 is even, we know, by the theorem seen in class, that z must be even. So, we have that $z = 2k$ for some integer k, and $z^2 = 4k^2$. It follows that $4|z^2$. However, $x = 2n + 1$ and $y = 2m + 1$ for some value of n and m, so $x^2 = 4n^2 + 4n + 1$ and $y^2 = 4m^2 + 4m + 1$, from which we get

   $$z^2 = 4n^2 + 4m^2 + 4n + 4m + 2 \ ,$$

   which is clearly not divisible by 4, amounting to a contradiction. $\square$

3. *Proofs by Induction.*

(a) **Prove by induction that, for any $n \in \mathbb{N}$, $n \geq 2$,**

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \left(1 - \frac{1}{4^2}\right) \cdot \cdots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

Proof. We begin by showing the base case $n = 2$, is satisfied. In fact, we have $(1 - \frac{1}{2^2}) = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$, as claimed. It remains to show the induction step. Suppose the property holds for $n = k$, then we show it must hold for $n = k + 1$. In fact, we have

$$\begin{aligned}
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) &= \left(1 - \frac{1}{(k+1)^2}\right) \cdot \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) \\
&= \left(1 - \frac{1}{(k+1)^2}\right) \cdot \frac{k+1}{2k} && \text{by Ind. Hyp.} \\
&= \frac{(k+1)^2 - 1}{(k+1)^2} \cdot \frac{k+1}{2k} \\
&= \frac{(k+1)^2 - 1}{2k(k+1)} \\
&= \frac{k^2 + 2k}{2k(k+1)} \\
&= \frac{k+2}{2(k+1)} \ ,
\end{aligned}$$

as desired. □

(b) **Prove by induction that, for any sets $A_1, A_2, \ldots, A_n$, De Morgan's Law generalises to**

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$$

Proof. The base case here is $n = 2$, that is De Morgan's Law. (It is clear that $\overline{A_1} = \overline{A_1}$, so we could also use $n = 1$ as the base case.) It remains to show the induction step. Suppose the statement holds for $n = k$, we wish to show it holds for $n = k + 1$. We have

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \\
&= \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} && \text{De Morgan} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}} && \text{Ind. Hyp.,}
\end{aligned}$$

as desired. □

4. *Predicate Calculus.*

(a) **What is the negation of the statement**

$$\forall n \in \mathbb{N}\ \ (\textbf{the remainder when } n^2 \textbf{ is divided by 4 is either 0 or 1})$$

Answer: $\exists n \in \mathbb{N}$ (the remainder when n^2 is divided by 4 is either 2 or 3)

(b) **Either the original statement in a) is true or its negation is true. Which one is it?**

The original statement is true. *Proof.* If n is odd, then $n = 2m + 1$ and $n^2 = 4m^2 + 4m + 1$, so the remainder is 1. Otherwise, if n is even, then $n = 2m$ and $n^2 = 4m^2$ so the remainder is 0. □

5. *More Predicate Calculus.*

(a) **What is the negation of the statement**

$$\forall \text{ odd } m \in \mathbb{N}\ \exists n \in \mathbb{N}\ \ (m = n^2 - (n-1)^2)$$

Answer: $\exists$ odd $m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ (m \neq n^2 - (n-1)^2)$.

(b) **Either the original statement in a) is true or its negation is true. Which one is it?**

The original statement is true. *Proof.* We have

$$n^2 - (n-1)^2 \quad = \quad n^2 - n^2 + 2n - 1 \quad = \quad 2n - 1\ ,$$

so for any odd m, we have $m = 2k - 1$ for some $k \geq 1$, and therefore $m = k^2 - (k-1)^2$. □

6. *Axiomatic Systems.*

Consider a geometry with lines and points that satisfy the following axioms:

(A1) ***There is at least one line.***
(A2) ***For any two distinct points, there is exactly one line that goes through them both.***
(A3) ***There is no line that contains every point.***
(A4) ***Any two lines intersect in at least one point.***

Prove that:

(a) **Any two lines intersect in exactly one point.**

Proof. Let L_1 and L_2 be two lines. By (A4), we know that $L_1 \cap L_2$ contains at least one point. We wish to show it contains exactly one point. Suppose not, and let x_1 and x_2 be two points in $L_1 \cap L_2$. By (A2), we know that there is exactly one line through x_1 and x_2, which contradicts the fact that both L_1 and L_2 have this property. $\square$

(b) **There is at least one point.**

Proof. Suppose not, then the set of points is $\emptyset$, the empty set. By (A1), there must be at least one line, call it L. By (A3), the line L cannot contain every point. So there must be some point x that does not lie on L. (Observe that (A3) is trivially false if there are no points!) Thus there is at least one point. $\square$

(c) **No point lies on every line.**

Proof. By (A1), we know that there must be at least one line, and we know that there is a point. So take any arbitrary point x. If x lies on no line, we are done.

If x lies on some line L, then x cannot be the only point, otherwise the line L would contain every point, contradicting (A3).

So let y be some other point. By (A2) we know that there is a line L_1 through x and y. However, by (A3) there must be some point z outside of L_1. Thus, by (A2), we have another line L_2 through x and z. Now $L_2 \neq L_1$ as z is on L_2 but not on L_1.

By (A2), there must also be some line through y and z, call it L_3. Again, $L_3 \neq L_1$ as z is on L_3 but not on L_1.

If x lies on L_3 then $\{x, y\}$ lie on two distinct lines, L_1 and L_3, a contradiction to (A2). Thus we have found a line L_3 that x does not lie on. The choice of x was arbitrary so no point can lie on every line. $\square$