Sample Solution to Quiz 1

1. **(2+2+1 = 5 points) Stable Matching**

   For the 3 men $m1, m2, m3$ and the 3 women $w1, w2, w3$ the following lists of preferences are given:

man	first	second	third
$m1$	$w1$	$w2$	$w3$
$m2$	$w2$	$w1$	$w3$
$m3$	$w1$	$w3$	$w2$

woman	first	second	third
$w1$	$m2$	$m1$	$m3$
$w2$	$m3$	$m2$	$m1$
$w3$	$m1$	$m2$	$m3$

   (a) List the stable matching returned by the Gale-Shapley algorithm for the above lists of preferences assuming the **men** propose to the **women**.

   Answer:

   $\{(m1, w1), (m2, w2), (m3, w3)\}$

   (b) List the stable matching returned by the Gale-Shapley algorithm for the above lists of preferences assuming the **women** propose to the **men**.

   Answer:

   $\{(m2, w1), (m3, w2), (m1, w3)\}$

   (c) Does there exist a man that got a better result in part (a) than in part (b)? If so, indicate which man.

   Answer:

   All of the men got a better result in part (a) than in part (b).

   - $m1$ prefers $w1$, his partner in part (a) to $w3$, his partner in part(b).
   - $m2$ prefers $w2$, his partner in part (a) to $w1$, his partner in part(b).
   - $m3$ prefers $w3$, his partner in part (a) to $w2$, his partner in part(b).

2. **(5 points) Asymptotic Notation**

   Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

   Answer:

   In order to prove that $max(f(n), g(n)) = \Theta(f(n) + g(n))$, we need to prove that $\exists$ constants $c_1, c_2 > 0$ and integer $n_0 \geq 0$, such that:

$$c_1(f(n) + g(n)) \leq max(f(n), g(n)) \leq c_2(f(n) + g(n)), \forall n \geq n_0 \tag{1}$$

   We will prove each inequality in turn.

   Part 1:

   By definition of maximum,

$$f(n) \leq max(f(n), g(n)), \forall n \tag{2}$$

$$g(n) \leq max(f(n), g(n)), \forall n \tag{3}$$

   Adding (2) and (3) we get:

$$f(n) + g(n) \leq 2(max(f(n), g(n))), \forall n$$

   or

$$\frac{1}{2}(f(n) + g(n)) \leq max(f(n), g(n)), \forall n$$

   This proves the left hand side inequality of (1) above with $c_1 = \frac{1}{2}$ and $n \geq 0$.

   Part 2:

   Functions $f(n)$ and $g(n)$ are given to be asymptotically non-negative. This implies that $\exists\, n_0 \geq 0$ such that both $f(n)$ and $g(n)$ are $\geq 0, \forall n \geq n_0$

   By the definition of maximum,

$$max(f(n), g(n)) = f(n) \text{ } or \text{ } g(n), \forall n \tag{4}$$

   and specifically for $n \geq n_0$.

   Also,

$$f(n) \leq f(n) + g(n), \ \forall n \geq n_0 \tag{5}$$

   and

$$g(n) \leq f(n) + g(n), \ \forall n \geq n_0 \tag{6}$$

   From (4), (5) and (6), we can infer that:

$$max(f(n), g(n)) \leq f(n) + g(n), \ \forall n \geq n_0$$

   This proves the right hand side inequality of (1) above with $c_2 = 1$ and $n \geq n_0$.

   Combining the results from Parts 1 and 2 above, we have:

$$\frac{1}{2}((f(n) + g(n)) \leq max(f(n), g(n)) \leq (f(n) + g(n)), \ \forall n \geq n_0$$

   which proves $max(f(n), g(n)) = \Theta(f(n) + g(n))$.