Problem Set 1 (due Sunday, January 27, 11:59 PM)

Instructions:

- The assignment is due at the time and date specified. Late assignments will be accepted, up until Monday, January 28th, 11:59 PM. *Note, however, that you can use at most 3 late days for your problem set and programming assignment submissions throughout the course of the term.*
- *We encourage you to attempt and work out all of the problems on your own. You are permitted to study with friends and discuss the problems;* however, you must write up you own solutions, in your own words.
- Please refrain from searching online or asking your peers or other students for solutions. The best way to learn the material is to attempt the problem yourself, and if you are stuck, identify where and why you are stuck and seek help to overcome the associated hurdles. If you do collaborate with any of the other students on any problem, please *list all your collaborators in your submission for each problem.*
- *We require that all homework submissions be neat, organized, and typeset. You may use plain text or a word processor like Microsoft Word or LaTeX for your submissions. If you need to draw any diagrams, however, you may draw them with your hand.*
- *Submit your solution to the problem set as a single PDF file. Multiple submissions are allowed and the last one before the assignment deadline will be graded.*

1. **(4 points) True or False?**

   Consider an instance of the Stable Matching Problem in which there are n employers and n job applicants. Each employer has exactly one job offering, and each applicant applies to all n available jobs. Suppose there exists an employer e and an applicant a such that e is ranked first on the preference list of a and a is ranked first on the preference list of e. Then in every stable matching S for this instance, the pair (e, a) belongs to S.
   Decide whether you think the statement above is true or false. If it is true, provide a proof. If it is false, give a counterexample.

2. **(6 points) Stable Matching with Unequal Sets**

   In this problem, you will consider the familiar problem of matching residents to hospitals, but with some of the constraints relaxed. There are m hospitals, each with a certain number of available positions for hiring residents. There are n medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there are more students graduating than there were slots available in the m hospitals.

We are interested in assigning each student to at most one hospital, in such a way that all available positions in all hospitals are filled. (Bear in mind that since we are assuming a surplus of students, there will be some students who do not get assigned to any hospital.

We say that an assignment of students to hospitals is *stable* if neither of the following situations arises.

- First type of instability: There are students s and s', and a hospital h, so that:
  - s is assigned to h, and
  - s' is assigned to no hospital, and
  - h prefers s' to s.
- Second type of instability: There are students s and s', and hospitals h and h' so that:
  - s is assigned to h, and
  - s' is assigned to h', and
  - h prefers s' to s, and
  - s' prefers h to h'.

Show that there is always a stable assignment of students to hospitals, and give an algorithm to find one.

3. **(4 points) Proof by Cases**

   Suppose that

   $$x^2 + y^2 + z^2 = w^2$$

   where x, y, z, w are nonnegative integers. Let P be the assertion that w is even, and let R be the assertion that all three of x, y, z are even. Prove by cases that:

   $$P \iff R$$

   Hint: An odd number equals $2m + 1$ for some integer m, so its square equals $4m^2 + 4m + 1$.

4. **(3 points) Asymptotic Growth Rate**

   Compare the following pair of functions in terms of asymptotic growth rate. Say whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, and/or $f(n) = \Theta(g(n))$. Justify your answer.

   - $f(n) = n2^n \qquad g(n) = 3^n$
   - $f(n) = n(\log_3 n)^5 \qquad g(n) = n^{1.2}$
   - $f(n) = 2^{\sqrt{\log n}} \qquad g(n) = n^{\log n}$

5. **(3 points) Asymptotic Functions**

   Let $f(n)$, $g(n)$, $r(n)$ and $s(n)$ be asymptotically positive functions. Prove or disprove the following. Justify your answer.
   $f(n) = \Theta(r(n))$ and $g(n) = \Theta(s(n))$ imply that $f(n)g(n) = \Theta(r(n)s(n))$