Recitation Class 1

Q1 – For each statement below, either *formally* prove it or give a counterexample.

(A) $f(n) = O\big(g(n)\big) \Rightarrow \log f(n) = O(\log g(n))$

(B) $f(n) = O\big(g(n)\big) \Rightarrow g(n) = \Omega(f(n))$

Answer:

(A) True with assumption that $g(n) = \Omega(1)$. Start with

$$f(n) = O(g(n)) \Rightarrow \exists n > n_0 \; and \; c > 0, f(n) \le c(g(n))$$

by taking logarithm on both sides, we have

$$\Rightarrow \exists n > n_0 \; and \; c > 0, \log\big(f(n)\big) \le \log(c) + \log(g(n))$$

let $c' = Sup\left\{\frac{\log(c)}{\log(g(n))}\right\} > 0$

$$\Rightarrow \exists n > n_0, \log\big(f(n)\big) \le c' \log\big(g(n)\big) + \log(g(n))$$

$$\Rightarrow \exists n > n_0, \log\big(f(n)\big) \le (c' + 1) \log(g(n))$$

let $c'' = c' + 1$

$$\log\big(f(n)\big) \le c'' \log\big(g(n)\big) \Rightarrow \log f(n) = O(\log g(n))$$

(B) True. $f(n) = O\big(g(n)\big) \Rightarrow \exists n > n_0 \; and \; c > 0, \; f(n) \le c\big(g(n)\big)$

$$\Rightarrow \exists n > n_0 \text{ and } c > 0, \; \frac{1}{c} \text{ f(n)} \le \text{(g(n))}$$

let $\varepsilon = \frac{1}{c}$

$$\Rightarrow \exists n > n_0 \; and \; \varepsilon > 0, \; \varepsilon f(n) \le (g(n))$$

$$\Rightarrow f(n) = \Omega\,(g(n))$$

Q2 – Prove the following using induction $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$.

Answer:

Let P(n) be $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$.

Base case: P(1) is correct ➔ $\sum_{i=1}^{1} i(2^i) = 1(2) = 2 = 2 + 0$

Inductive step: Assume P(k) holds, which is $\sum_{i=1}^{k} i(2^i) = 2 + (k-1)2^{k+1}$. In the following we show that P(k) implies P(k+1).

We start with LHS and rewrite it in this way:

$$\sum_{i=1}^{k+1} i(2^i) = \sum_{i=1}^{k} i(2^i) + (k+1)(2^{k+1})$$

(using induction hypothesis for n = k) $= 2 + (k-1)2^{k+1} + (k+1)(2^{k+1})$

$$= 2 + (k - 1 + k + 1)\, 2^{k+1}$$

$$= 2 + (k)\, 2^{k+2}$$

This completes the proof.

Q3 – Sort the following functions in the order of their asymptotic growth $\log n$, $n^{\frac{1}{3}}$, $n/\log n$.

Answer: The correct order is $\log n \ll n^{\frac{1}{3}} \ll n/\log n$. We just need to show the order between consecutive pairs are correct.

$$\lim_{n\to\infty} \frac{\log n}{n^{1/3}} = \text{by taking derivative } \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{\frac{1}{3}n^{-\frac{2}{3}}} = \lim_{n\to\infty} \frac{c}{n^{\frac{1}{3}}} = 0 \Rightarrow \log n = o(n^{\frac{1}{3}})$$

$$\lim_{n\to\infty} \frac{n^{1/3}}{\frac{n}{\log n}} = \lim_{n\to\infty} \frac{\log n}{n^{2/3}} \text{ (similar to previous one) } = 0 \Rightarrow n^{1/3} = o(\frac{n}{\log n})$$

Q4 – In a summer internship program organized by tech companies, there are n tech companies that each of them has one open internship position for one graduate student for summer. There are n graduate students who are qualified to participate in this program. To assign students to tech companies, each of companies and students give a complete preference list of each other. Among these companies, k of them are located in Bay area and k of students live in the same area. Companies in bay area prefer these k students more than rest of them, also these k students prefer to do their internship in the same they live. Prove that in any stable matching, every k student who lives in the bay area is assigned to a company in the same area.

Answer: (Use contradiction) Consider an arbitrary perfect stable matching that some the k bay area companies are matched to some non-bay area student. Then there is a company in bay area C_B and a student S_B in bay area which are matched to a non-bay area student and company respectively. Argue that (C_B, S_B) are unstable pair; Each prefer each other rather than their current matched company and student. This is contradictory with the assumption that the matching is stable.