

## Sample Solution to Quiz 1

### Problem 1. (5 points) Stable Matching

Decide whether you think the following is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

**True or False?** For every instance of the stable matching problem with  $n \geq 2$  men and  $n$  women, in every stable matching, there is a pair  $(m, w)$  such that either  $w$  is ranked last on the preference list of  $m$  or  $m$  is ranked last on the preference list of  $w$ .

**Answer:** False. Consider the following instance with 2 men  $M_1, M_2$  and 2 women  $W_1, W_2$ . Suppose the preferences are the following.

$M_1 : W_1, W_2$   
 $M_2 : W_2, W_1$   
 $W_1 : M_1, M_2$   
 $W_2 : M_2, M_1$

The only stable matching is  $(M_1, W_1), (M_2, W_2)$ . Every person gets their first preference.

**Problem 2. (5 points) Asymptotic Notation**

Let  $f(n)$ ,  $g(n)$ , and  $h(n)$  be positive monotonically increasing functions such that  $f(n) = O(g(n))$  and  $g(n) = O(h(n))$ . Prove or disprove:  $f(n) \cdot g(n) = O(h(n)^2)$ .

**Answer:** True.

Since  $f(n) = O(g(n))$ , there exists constants  $c_1$  and  $n_1 > 0$  such that  $f(n) \leq c_1 g(n)$  for all  $n \geq n_1$ .

Since  $g(n) = O(h(n))$ , there exists constants  $c_2$  and  $n_2 > 0$  such that  $g(n) \leq c_2 h(n)$  for all  $n \geq n_2$ .

Therefore, if  $n_3 = \max\{n_1, n_2\}$ , then for  $n \geq n_3$ , we have:

$$\begin{aligned} f(n) \cdot g(n) &\leq c_1 g(n) \cdot c_2 h(n) \\ &\leq c_1 c_2^2 h(n)^2. \end{aligned}$$

So, for  $c_3 = c_1 c_2^2$ , we obtain that there exists constants  $c_3$  and  $n_3 > 0$  such that  $f(n) \cdot g(n) \leq c_3 h(n)^2$  for all  $n \geq n_3$ .