DS5020 - Introduction to Linear Algebra and Probability for Data Science

Assignment 2

1. {15 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 4 & 3 & 3 \end{bmatrix}$.

- **a.** Find matrices $\mathbf{E}_{21}$ and $\mathbf{E}_{31}$ that produce zeros in the (2,1) and (3,1) positions of $\mathbf{E}_{21}\mathbf{A}$ and $\mathbf{E}_{31}\mathbf{A}$.
- **b.** Find $\mathbf{E} = \mathbf{E}_{31}\mathbf{E}_{21}$ that produces both zeros at once.
- **c.** Find the result of matrix multiplication $\mathbf{EA}$.

2. {10 pts} Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ d & 3 \end{bmatrix}$, where d is a real number.

- **a.** Find the value d which makes $\mathbf{A}$ singular (non-invertible).
- **b.** Given that $\mathbf{A}$ is invertible, find $\mathbf{A}^{-1}$. Note that your solution will depend on d.

3. {15 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

- **a.** Convert matrix $\mathbf{A}$ to upper triangular matrix $\mathbf{U}$ using elimination. Find matrix $\mathbf{E}$ which performs the elimination steps $\mathbf{EA} = \mathbf{U}$.
- **b.** Find the matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LU}$. *Hint: **L** is the matrix that 'undoes' what **E** does to matrix **A**.*
- **c.** Find vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

4. {20 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

- **a.** Show that $\mathbf{A}$ is invertible.
- **b.** Using Gauss-Jordan elimination, find $\mathbf{A}^{-1}$.
- **c.** Find vector $\mathbf{x}$ that gives $\mathbf{Ax} = \mathbf{b}$.

5. {20 pts} [Based on problem 2.5.30 from text]

- **a.** Prove that $\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$ is invertible if $a \neq 0$ and $a \neq b$ (Find the pivots or $\mathbf{A}^{-1}$).
- **b.** Find three numbers c so that $\mathbf{C} = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$ is not invertible.

6. {20 pts} [Based on Problem 2.6.8 from text] This problem shows how the inverses of elimination steps ${\mathbf{E}_{ij}}^{-1}$ multiply to give $\mathbf{L}$ in $\mathbf{A} = \mathbf{LU}$. We see this best when $\mathbf{A}$ is already a lower triangular matrix with 1's on the diagonal. Then $\mathbf{U} = \mathbf{I}$ and $\mathbf{A} = \mathbf{L}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

- **a.** Find elimination matrices $\mathbf{E}_{21}$, $\mathbf{E}_{31}$ and $\mathbf{E}_{32}$ that produce upper triangular matrix $\mathbf{U} = \mathbf{I}$. Note that this matrices will contain $-a$, $-b$ and $-c$.
- **b.** Multiply $\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}$ to get a single matrix $\mathbf{E}$ that produces $\mathbf{EA} = \mathbf{U} = \mathbf{I}$.
- **c.** Find the matrix $\mathbf{L} = \mathbf{E}^{-1} = (\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21})^{-1} = {\mathbf{E}_{21}}^{-1}{\mathbf{E}_{31}}^{-1}{\mathbf{E}_{32}}^{-1}$ to bring back $\mathbf{A}$ (or $\mathbf{L}$) from $\mathbf{U}$ (or $\mathbf{I}$).

Key observation to make: The multipliers a , b and c are mixed up in $\mathbf{E}$ but perfect in $\mathbf{L}$.