Northeastern University

DS5020 - Introduction to Linear Algebra and Probability for Data Science

# Assignment 4

**1. {20 pts}** Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 & 9 \end{bmatrix}.$$

**a.** Determine the rank $r$ of $\mathbf{A}$, $r = rank(\mathbf{A})$.

**b.** Find bases and dimensions for the four subspaces associated with $\mathbf{A}$. *Note: These four subspaces are: 1. column space, 2. null space, 3. row space, 4. left null space.*

**c.** Confirm that the dimensions of left null space and column space add up to 3 (the number of rows). Similarly confirm that the dimensions of null space and row space add up to 5 (the number of columns).

**2. {20 pts}** [Based on problem 3.5.27 from text] Suppose we have the following 8 by 8 matrix $\mathbf{A}$ that represents the points of each piece on the chess board. Assume that the numbers $r, n, b, q, k, p$ are all different.

$$\mathbf{A} = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \text{four zero rows} & & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix} \tag{1}$$

**a.** Find the rank of $\mathbf{A}$, $rank(\mathbf{A})$.

**b.** Calculate the following linear combination of the first three columns : $(b-n)(column\,1) + (r-b)(column\,2) + (n-r)(column\,3)$.

**c.** Show that columns 2,3 and 5 of matrix $\mathbf{A}$ are linearly dependent. *Hint: Find a non-zero linear combination of them that leads to a zero-vector, as we did in part **b** for the first three columns.*

**3. {20 pts}**

**a.** Show that every $\mathbf{y}$ in $N(\mathbf{A}^{\mathrm{T}})$ (left null space of $\mathbf{A}$) is perpendicular to every $\mathbf{Ax}$ in the column space. Start from $\mathbf{A}^{\mathrm{T}}\mathbf{y} = \mathbf{0}$.

**b.** Show that every $\mathbf{x}$ in null space $N(\mathbf{A})$ of $\mathbf{A}$ is perpendicular to the every $\mathbf{y}$ in the row space. Start from $\mathbf{Ax} = \mathbf{0}$.

**4. {40 pts}** [ Based on problems 4.3.12/15/16 in text]

Your doctor takes 5 readings of your heart rate: $60, 62, 57, 60$, and $61$ beats/minute. Your doctor tries to get a good estimate of your heart rate $x$ (the only unknown) by taking multiple readings. The corresponding system of equations is :

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 59 \\ 62 \\ 58 \\ 60 \\ 61 \end{bmatrix}}_{\mathbf{b}} \tag{2}$$

Note that the readings are not the same so there is no solution to above system of equations. We will solve it (5 equations, 1 unknown) by least squares.

**a.** Solve $\mathbf{A}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ to show that $\hat{\mathbf{x}}$ is the mean (the average) of the readings ($b_i$'s).

**b.** Find the error vector $\mathbf{e} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$.

**c.** Show that $\mathbf{e}^{\mathrm{T}}(\mathbf{A}\hat{\mathbf{x}}) = 0$. That is, the error vector $\mathbf{e}$ is perpendicular to the projection $\mathbf{A}\hat{\mathbf{x}}$ of $\mathbf{b}$ onto the column space of $\mathbf{A}$.

**d.** If we know the average $\hat{\mathbf{x}}$ of previous 5 readings, how can we quickly find the average with one more reading $b_6 = 61$ beats/minute? Find a recursive formula to calculate the average of $n$ numbers $\hat{x}_n = average(b_1, b_2, \cdots, b_n)$ from the average of first $n-1$ numbers $\hat{x}_{n-1} = average(b_1, b_2, \cdots, b_{n-1})$ and the last number $b_n$.