(1)

a) $A^TA = \begin{bmatrix} 3 & 4 & 0 \\ -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 4 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & -25 \\ -25 & 26 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 3 & 4 & 0 \\ -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ -23 \end{bmatrix}$

$A^TA\hat{x} = A^Tb \Rightarrow \hat{x} = (A^TA)^{-1} A^Tb = \frac{1}{25} \begin{bmatrix} 26 & 25 \\ 25 & 25 \end{bmatrix} \begin{bmatrix} 25 \\ -23 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$e = b - A\hat{x} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

$e^TA\hat{x} = \begin{bmatrix} -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = 0.$

b) $q_1 = \frac{\text{col } 1}{\|\text{col} 1\|} = \frac{(\text{col } 1)}{\sqrt{3^2+4^2}} \Rightarrow q_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$

$p = (\text{col } 2)^T q_1 \cdot q_1 = \left(\begin{bmatrix} -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \right) q_1 = -5q_1 = \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix}$

$e = \text{col} 2 - p = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad q_2 = \frac{e}{\|e\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 3/5 & 0 \\ 4/5 & 0 \\ 0 & 1 \end{bmatrix} \quad Q^TQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

c) $q_1^T c_1 = \begin{bmatrix} 3/5 & 4/5 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 5$

$$A = \overbrace{\begin{bmatrix} q_1 & q_2 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} q_1^T c_1 & q_1^T c_2 \\ 0 & q_2^T c_2 \end{bmatrix}}^{R}$$

$q_2^T c_1 = \begin{bmatrix} 3/5 & 4/5 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = -5$

$q_2^T c_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = 1 \Rightarrow R = \begin{bmatrix} 5 & -5 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 4 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 0 \\ 4/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 \\ 0 & 1 \end{bmatrix}$$

d)

$A^T A \hat{x} = A^T b$

$(QR)^T QR\hat{x} = (QR)^T b$

$R^T Q^T Q R \hat{x} = R^T Q^T b$

$(R^T)^{-1}(R^T R \hat{x}) = R^{T^{-1}}(R^T Q b)$

$R\hat{x} = Qb$

e) $\begin{bmatrix} 5 & -5 \\ 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\hat{x} = \frac{1}{5} \begin{bmatrix} 1 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Same as $\hat{x}$ in part a.

(2) a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$ $\quad A^T b = \begin{bmatrix} 742 \\ 1912 \end{bmatrix}$

$(A^T A)^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T b = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{bmatrix} \begin{bmatrix} 742 \\ 1912 \end{bmatrix} = \begin{bmatrix} 157 \\ 11.4 \end{bmatrix}$

b) $e = b - A\hat{x} = \begin{bmatrix} 163 \\ 186 \\ 195 \\ 198 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 157 \\ 11.4 \end{bmatrix} = \begin{bmatrix} -5.4 \\ 6.2 \\ 3.8 \\ -4.6 \end{bmatrix}$

c) $A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.63 & 0.86 & 0.95 & 0.98 \end{bmatrix} \begin{bmatrix} 1 & 0.63 \\ 1 & 0.86 \\ 1 & 0.95 \\ 1 & 0.98 \end{bmatrix} = \begin{bmatrix} 4 & 3.42 \\ 3.42 & 2.9994 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.63 & 0.86 & 0.95 & 0.98 \end{bmatrix} \begin{bmatrix} 163 \\ 186 \\ 195 \\ 198 \end{bmatrix} = \begin{bmatrix} 742 \\ 641.94 \end{bmatrix}$

$\Rightarrow \hat{x} = (A^TA)^{-1} A^Tb = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

d) $e = b - A\hat{x} = \begin{bmatrix} 163 \\ 186 \\ 195 \\ 198 \end{bmatrix} - \begin{bmatrix} 163 \\ 186 \\ 195 \\ 198 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Since $e=0$, the least squares solution is the exact solution to $Ax=b$. That is, there exists a solution to $\begin{bmatrix} 1 & 0.63 \\ 1 & 0.86 \\ 1 & 0.95 \\ 1 & 0.98 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 163 \\ 186 \\ 195 \\ 198 \end{bmatrix}$, which is $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

(3)

a) $\det(T-\lambda I) = 0 \Rightarrow \begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$

$\lambda^2 - 1.7\lambda + 0.7 = 0 \qquad (\lambda-1)(\lambda-0.7) = 0$

$\lambda_1 = 1\ ,\ \lambda_2 = 0.7.$

$\begin{bmatrix} 0.8-1 & 0.1 \\ 0.2 & 0.9-1 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} x = 0 \quad \Rightarrow \quad x = \begin{bmatrix} +1 \\ +2 \end{bmatrix}$ is an eigenvector.

$\Rightarrow \lambda_1 = 1,\ x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (or any multiplier times x_1).

$\begin{bmatrix} 0.8-0.7 & 0.1 \\ 0.2 & 0.9-0.7 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector.

$\Rightarrow \lambda_2 = 0.7,\ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

b) $P_1 = AP_0 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix}$

c) $P_{100} = A^{100} P_0 = A^{100}\left(\frac{2}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

$= \frac{2}{3} \underbrace{A^{100}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\lambda^{100} x_1} + \frac{1}{3} \underbrace{A^{100}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\lambda^{100} x_2}$

$= \frac{2}{3}\, 1^{100} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5} \underbrace{(0.7)^{100}}_{0} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.66 \\ 1.33 \end{bmatrix}$

$\Rightarrow P_{100} = \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix}.$

(4)

(a) det $T-\lambda I=0 \Rightarrow \begin{vmatrix} \alpha-\lambda & 1-\beta \\ 1-\alpha & \beta-\lambda \end{vmatrix} = 0$

$(\alpha-\lambda)(\beta-\lambda) - (1-\beta)(1-\alpha) = 0$

$\lambda^2 - (\alpha+\beta)\lambda + (\alpha+\beta-1) = 0 \Rightarrow (\lambda-1)(\lambda-\beta-\alpha+1) = 0.$

$\lambda_1 = 1 \qquad \lambda_2 = \alpha+\beta-1.$

(b) $\lambda_1 = 1 \Rightarrow \begin{bmatrix} \alpha-1 & 1-\beta \\ 1-\alpha & \beta-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \alpha-1 & 1-\beta \\ 0 & \underset{\text{free}}{0} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 1 \Rightarrow x_1 = \dfrac{1-\beta}{1-\alpha}.$

Indeed: $\begin{bmatrix} \alpha-1 & 1-\beta \\ 1-\alpha & \beta-1 \end{bmatrix}\begin{bmatrix} \frac{1-\beta}{1-\alpha} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} \frac{1-\beta}{1-\alpha} \\ 1 \end{bmatrix}$ is an eigenvector.

$\lambda_2 = \alpha+\beta-1 \Rightarrow \begin{bmatrix} 1-\beta & 1-\beta \\ 1-\alpha & 1-\cancel{\beta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector.

(c) $cv_1 + dv_2 = 0 \Rightarrow \begin{aligned} \frac{c(1-\beta)}{1-\alpha} + -d &= 0 \\ c + d &= 0 \end{aligned}$

$c\left(\dfrac{1-\beta}{1-\alpha} + 1\right) = 0 \Rightarrow c = 0$ and $d = 0$.

Since the only linear combination of v_1 and v_2 that gives zero is all coefficients zero, v_1 & v_2 are LI.

(d) $P_\infty = A^{\infty} P_0 = A^{\infty}(c v_1 + d v_2) = c A^{\infty} v_1 + d A^{\infty} v_2$

$$= c\, 1^{\infty} v_1 + d \underbrace{(\alpha+\beta-1)^{\infty}}_{\sim 0} v_2$$

$$= c v_1 + 0 = c v_1 .$$

(5) a) $\det A - \lambda I = 0$

$$(1-\lambda)(3-\lambda) - 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda-5)(\lambda+1) = 0$$

$\lambda_1 = 5 \quad \lambda_2 = -1.$

b) $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \lambda_1 = 5,\ x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda_2 = -1,\ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

c) Yes. A is digonalizable since it has distinct eigenvalues.

d) $Ax_1 = \lambda_1 x_1 \qquad Ax_2 = \lambda_2 x_2 \Rightarrow$

$$A\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\Rightarrow A\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} A \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}}_{X^{-1}}\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}_{X}$$

e) $A^{100} = (X\Lambda \underbrace{X^{-1})(X}_{I}\Lambda X^{-1}) \cdots (X\Lambda X^{-1}) = X\Lambda\Lambda \ldots \Lambda X^{-1} = X\Lambda^{100}X^{-1}.$

(Noting $A = X\Lambda X^{-1}$)