Northeastern University

DS5020 - Introduction to Linear Algebra and Probability for Data Science

Assignment 3

1. {10 pts} [Based on problems 2.7.30/31 from text] Producing x_1 trucks and x_2 planes needs $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. The unit costs $y_1,\ y_2,\ y_3$ are \$700 per ton of steel, \$3 per pound of rubber, and \$3000 per month of labor. Writing this in matrix form $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b},$$

where $\mathbf{b}$ represents the amount of steel, rubber, and labor to produce x_1 trucks and x_2 planes.

a. What are the values of one truck and one plane? Note that those are the components of $\mathbf{A}^{\mathrm{T}}\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

b. Find the cost of producing 3 trucks and 5 planes.

2. {10 pts} For which right sides (find conditions on $b_1,\ b_2,\ b_3$) are below systems of equations solvable?

a.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

3. {20 pts} Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 5 & 6 \\ 1 & 2 & 3 & 6 & 8 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

a. Find the special solutions to $\mathbf{Ax} = \mathbf{0}$.

b. Describe the null space in terms of special solutions you found in part **a.**

c. Find the rank r of $\mathbf{A}$, rank($\mathbf{A}$).

4. {20 pts} Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

a. What are the ranks of $\mathbf{A}$ and $\mathbf{B}$?

b. Find 3 columns of A that are linearly independent. Similarly find 2 columns of B that are linearly independent.

c. Find a basis for the column space of $\mathbf{A}$, $\mathbf{C}(\mathbf{A})$.

5. {20 pts} Suppose we have a circuitry where we measure voltages at three locations, group them in voltage vector $\mathbf{v} \in R^3$, and would like to make inferences about current levels through three wires, grouped in current vector $\mathbf{s} \in R^3$. Our goal is to make sure the circuitry operates within the safety limits. The relationship between $\mathbf{v}$ and $\mathbf{s}$ is described by a 3x3 resistance matrix $\mathbf{R}$, where $\mathbf{Rc} = \mathbf{v}$.

$$\mathbf{R} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

a. Given measurement voltage vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, can we uniquely determine the current vector $\mathbf{c}$? What does this tell us about the inverse of $\mathbf{R}$, $\mathbf{R}^{-1}$?

b. Find the column space of $\mathbf{R}$. Does it fill the entire R^3 vector space?

c. If your answer to part **b.** is no, then find a voltage vector $\mathbf{v}$ which can not be expressed as a linear combination of columns of $\mathbf{R}$ (that is $\mathbf{Rc} = \mathbf{v}$ has no solution).

d. Find the null space of $\mathbf{R}$, that is all the current vectors $\mathbf{c}$ such that $\mathbf{Rc} = \mathbf{0}$.

Interpretation of null space in above scenario: Increasing the currents by any amount vector $\mathbf{a}$ *from null space does not change the voltage vector* $\mathbf{v}$. *Thus voltage vector would be a bad metric for safety if null space contains non-zero vectors: we can increase currents as much as we want without affecting the voltage readings.*

6. {20 pts} Suppose Netflix uses a similarity matrix $\mathbf{A}$ and a rating vector $\mathbf{r}$ of five old movies to score three new releases: $\mathbf{Ar} = \mathbf{s}$. Netflix will use the score vector $\mathbf{s}$ in deciding how much it is willing to pay for streaming each of the new releases.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 3 & 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix}, \text{ and } \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}.$$

a. Find the null space of $\mathbf{A}$. That is, all the rating vectors $\mathbf{r}$ that lead to a zero score vector $\mathbf{s} = \mathbf{0}$.

b. Find the column space of $\mathbf{A}$. Does it fill the entire R^3? A possible way to check this would be to check if a basis for column space of $\mathbf{A}$ is also a basis for R^3.