DS5020 - Introduction to Linear Algebra and Probability for Data Science

Assignment 2 Solutions

1. {15 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 4 & 3 & 3 \end{bmatrix}$.

a. Find matrices $\mathbf{E}_{21}$ and $\mathbf{E}_{31}$ that produce zeros in the (2,1) and (3,1) positions of $\mathbf{E}_{21}\mathbf{A}$ and $\mathbf{E}_{31}\mathbf{A}$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 4 & 3 & 3 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{E_{31}} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

b. Find $\mathbf{E} = \mathbf{E}_{31}\mathbf{E}_{21}$ that produces both zeros at once.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

c. Find the result of matrix multiplication $\mathbf{EA}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 2 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

2. {10 pts} Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ d & 3 \end{bmatrix}$, where d is a real number.

a. Find the value d which makes $\mathbf{A}$ singular (non-invertible).

for A to be non-invertible, the columns of A should be linearly dependent: $\frac{2}{1} = \frac{3}{d} \Rightarrow d = \frac{3}{2}$ $\left(\begin{matrix} ac - bd = 0 \\ 3 - 2d = 0, \; d = \frac{3}{2} \end{matrix} \right)$

b. Given that $\mathbf{A}$ is invertible, find $\mathbf{A}^{-1}$. Note that your solution will depend on d.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ d & 3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3-2d & -d & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{-d}{3-2d} & \frac{1}{3-2d} \end{array}\right] \longrightarrow$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 + \frac{2d}{3-2d} & \frac{-2}{3-2d} \\ 0 & 1 & \frac{-d}{3-2d} & \frac{1}{3-2d} \end{array}\right] \Rightarrow A^{-1} = \frac{1}{3-2d} \begin{bmatrix} 3 & -2 \\ -d & 1 \end{bmatrix}$$

3. {15 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

a. Convert matrix $\mathbf{A}$ to upper triangular matrix $\mathbf{U}$ using elimination. Find matrix $\mathbf{E}$ which performs the elimination steps $\mathbf{EA} = \mathbf{U}$.

$$\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{P_{23}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P_{23} E_{21} A = U \Rightarrow E = P_{23} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

b. Find the matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LU}$. *Hint: **L** is the matrix that 'undoes' what **E** does to matrix **A**.*

$$L = E^{-1} = (P_{23} E_{21})^{-1} = E_{21}^{-1} P_{23}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that we only switched the sign of the multiplier for E_{21}^{-1}. And $P_{23}^{-1} = P_{23}$.

c. Find vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad (Ux = c)$$

$$\Rightarrow x_3 = 2, \quad x_2 = 1 - x_3 = 1 - 2 = -1 \quad x_1 = \frac{2-4}{2} = -1 \quad x = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

4. {20 pts} Suppose $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

a. Show that $\mathbf{A}$ is invertible.

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are three non-zero pivots. $\Rightarrow$ A is invertible.

b. Using Gauss-Jordan elimination, find $\mathbf{A}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A^{-1}

c. Find vector $\mathbf{x}$ that gives $\mathbf{Ax} = \mathbf{b}$

$$x = A^{-1} b = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

5. {20 pts} [Based on problem 2.5.30 from text]

a. Prove that $\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$ is invertible if $a \neq 0$ and $a \neq b$ (Find the pivots or $\mathbf{A}^{-1}$).

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & b \\ 0 & (a-b) & 0 \\ 0 & (a-b) & (a-b) \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & b \\ 0 & (a-b) & 0 \\ 0 & 0 & (a-b) \end{bmatrix}$$

Need non-zero pivots for A to be invertible.

Three non-zero pivots if $a \neq 0$ and $a-b \neq 0$.

b. Find three numbers c so that $\mathbf{C} = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$ is not invertible. For $c=0,2,7$, the elimination would produce zero pivots.

$c=0$ (zero-row).

$c=7$ (columns 2 and 3 are same)

$c=2$ (rows 1 and 2 are same).

6. {20 pts} [Based on Problem 2.6.8 from text] This problem shows how the inverses of elimination steps $\mathbf{E}_{ij}^{-1}$ multiply to give $\mathbf{L}$ in $\mathbf{A} = \mathbf{LU}$. We see this best when $\mathbf{A}$ is already a lower triangular matrix with 1's on the diagonal. Then $\mathbf{U} = \mathbf{I}$ and $\mathbf{A} = \mathbf{L}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

a. Find elimination matrices $\mathbf{E}_{21}$, $\mathbf{E}_{31}$ and $\mathbf{E}_{32}$ that produce upper triangular matrix $\mathbf{U} = \mathbf{I}$. Note that this matrices will contain $-a$, $-b$ and $-c$.

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \xrightarrow{E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{bmatrix} \xrightarrow{E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} \xrightarrow{E_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

b. Multiply $\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}$ to get a single matrix $\mathbf{E}$ that produces $\mathbf{EA} = \mathbf{U} = \mathbf{I}$.

$$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b+ca & -c & 1 \end{bmatrix}$$

c. Find the matrix $\mathbf{L} = \mathbf{E}^{-1} = (\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21})^{-1} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1}$ to bring back $\mathbf{A}$ (or $\mathbf{L}$) from $\mathbf{U}$ (or $\mathbf{I}$).

$$L = E^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

Note that multipliers a, b, c get mixed in E but not in L.