Matrix multiplication revisited

Four ways to see matrix multiplication of $A_{m\times n}$ and $B_{n\times p}$

1.

$$\underset{A}{\begin{bmatrix} \\ \text{— row } i \text{ —} \\ \\ \end{bmatrix}} \underset{B}{\begin{bmatrix} | \\ \text{column } j \\ | \end{bmatrix}} = \begin{array}{c|c} & \\ i & \begin{bmatrix} & \vdots & \\ \cdots & \bigcirc & \cdots \\ & \vdots & \end{bmatrix} \\ & j \end{array}$$

Entry $AB(i,j)$

$AB(i,j) = (\text{row } i \text{ of } A)\cdot(\text{column } j \text{ of } B)$

2.

$$\underset{A}{\begin{bmatrix} \\ \text{— row } i \text{ —} \\ \\ \end{bmatrix}} \underset{B}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} = \underset{AB}{\begin{bmatrix} \\ \text{— row } i \text{ —} \\ \\ \end{bmatrix}}$$

$(\text{row } i \text{ of } A)\, B = \text{row } i \text{ of } AB$

3.

$$\underset{A}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} \underset{B}{\begin{bmatrix} | \\ \text{column } j \\ | \end{bmatrix}} = \underset{AB}{\begin{bmatrix} | \\ \text{column } j \\ | \end{bmatrix}}$$

$A\,(\text{column } j \text{ of } B) = \text{column } j \text{ of } AB$

4.

$$\begin{bmatrix} | & & | \\ \text{column } 1 & \cdots & \text{column } n \\ | & & | \end{bmatrix} \begin{bmatrix} \text{— row } 1 \text{ —} \\ \text{— row } 2 \text{ —} \\ \vdots \\ \text{— row } n \text{ —} \end{bmatrix} = \underset{AB}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}$$

$$AB = \sum_{i=1}^{n} (\text{column } i \text{ of } A)(\text{row } i \text{ of } B)$$

Elimination with row exchanges

Row exchange matrix: P_{ij} exchanges row i with row j. It is the identity matrix with rows i and j reversed.

We will need row exchange matrices to change zeros in the pivot elements with nonzeros.

Example:

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

Example:

$$P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example: (Elimination failure - zero pivots)

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 2 \\ 2 & 7 & 5 \end{bmatrix} \xrightarrow{P_{21}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 2 & 7 & 5 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} \boxed{1} & 2 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

The third pivot is zero. (This tells us that A is not invertible.)

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{(no solution! Elimination gives } 0 = -1)$$

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \quad \text{(infinitely many solutions! Elimination gives } 0 = 0).$$

$$b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \xrightarrow{P_{21}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \xrightarrow{P_{21}} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Augmented matrix $[A \;\; b]$

Elimination does the same row operations on both sides of $Ax=b$. Thus we can include b as another column and perform the elimination on matrix $[A \;\; b]$, called augmented matrix.

Example:

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 8 & 9 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \text{Augmented matrix} = [A \,\vdots\, b] = \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 1 & 3 & 2 & 1 \end{array}\right]$$

$$\underbrace{}_{A} \quad \underbrace{}_{b}$$

Elimination on augmented matrix $[A \;\; b]$:

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{P_{23}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad Ax=b \xrightarrow{\text{Elimination}} Ux=c$$

$$\underbrace{}_{U} \quad \underbrace{}_{c}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \begin{aligned} x_3 &= -1 \\ x_2 &= 0 \\ x_1 &= 3 \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

INVERSE MATRIX

Matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

A^{-1} "undoes" what A does (to a vector or a matrix).

Example:

$Ax = b$. Multiply both sides with A^{-1} (assuming it exists!):

$\underbrace{A^{-1}A}_{I} x = A^{-1}b \Rightarrow x = A^{-1}b$, meaning we can use inverses to solve for unknown vector x.

Note: Sometimes A^{-1} might not exist. If A^{-1} (inverse of matrix A) doesn't exist, we say A is singular.

Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$ should not be zero!

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

A^{-1} and B^{-1} don't exist.

If E_{21} subtracts 2 times row 1 from row 2, then E_{21}^{-1} adds 2 times row 1 to row 2.

How can we tell if A^{-1} exists or not?

1. A^{-1} exists if elimination produces n pivots.

$$A \xrightarrow[\text{(row exchanges allowed)}]{\text{Elimination}} U$$

If U has only non-zero values on its diagonal (pivot positions), then A^{-1} exists.

2. If there is a non-zero vector x such that $Ax = 0$, A is singular.

Non-zero vector: any vector with at least one non-zero value.

Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{x \text{ is non-zero}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is singular (non-invertible).

3. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible as long as $ad - bc$ is not zero.

Example: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is singular.

4. A diagonal matrix is invertible as long as no diagonal entries are zero.

Example:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is invertible.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is singular.

5. If A and B are invertible, AB is also invertible.

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Note: The order is reversed!

$$(AB)(B^{-1}A^{-1}) \underset{\uparrow}{=} A(\underbrace{BB^{-1}}_{I})A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

change the paranthesis locations

Rule: Inverses come in reverse order:

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

Example: (Inverse of elimination matrices)

Suppose $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 5 & 1 \end{bmatrix}$ Elimination matrices are:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Exercise: Verify that $E_{32}E_{31}E_{21}A$ gives an upper triangular matrix U.

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = U$$

Multiply both sides with $\boxed{(E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}$

$$\underbrace{E_{21}^{-1}\underbrace{E_{31}^{-1}\underbrace{E_{32}^{-1}E_{32}}_{I}E_{31}}_{I}E_{21}}_{I}A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{=L}U$$

$$A = LU$$

Gauss-Jordan Elimination to Calculate A^{-1}

$$\underbrace{A}_{\text{Known}}\underbrace{A^{-1}}_{\text{Unknown}} = \underbrace{I}_{\text{Known}}$$

Separate A^{-1} and I into columns and rewrite (for simplicity assume A is 3x3):

$$A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

where x_1, x_2, x_3 are column vectors of A^{-1} and e_1, e_2, e_3 are column vectors of I.

Solve three systems of linear equations to find A^{-1}:

1. $Ax_1 = e_1$ 2. $Ax_2 = e_2$ 3. $Ax_3 = e_3$

Gauss-Jordan method solves all these equations together, by using elimination on augmented matrix $[A \ I]$.

Previously augmented matrix had only one extra column b, this time it has identity matrix I.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ Augmented matrix: $[A \ I] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$ with A under the first three columns and I under the last three.

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$E\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} EA & EI \end{bmatrix} = \begin{bmatrix} I & E \end{bmatrix}$$

Since $EA = I$, $E = A^{-1}$. Note that E is the second half of the augmented matrix after elimination to get identity matrix I on the first half.

Idea: If elimination steps E take me from A to I, then the same steps take me from I to A^{-1}.

$$EA = I \Rightarrow EI = E = A^{-1}$$

Gauss-Jordan Elimination:

Apply elimination to augmented matrix $[A \ I]$ until you get I on the left half (in place of A). Then, the resulting matrix on the right half (in place of I) is A^{-1}.

→ After getting an upper triangular matrix in place of A, don't stop! Apply back-substitution to get a diagonal matrix instead.

→ Then, divide each row by a scalar to get I on the left. The matrix on the right is then A^{-1}.

Example:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Quick check: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example:

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

Gauss-Jordan: $A^{-1}[A \;\; I] = [I \;\; A^{-1}]$

* If A is upper triangular, so is A^{-1}.

Example: $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$ True for bigger matrices as well.

Test for invertibility: A should have n pivots.

Factorization $A = LU$

Elimination matrices E_{ij} and their inverses could be used to factorize A into multiplication of a lower and upper triangular matrices.

(For the moment assume there are no row exchanges, we will deal with those later).

Steps:

1. Invert each elimination step $E_{ij}^{-1} = L_{ij}$.
   Note that both E_{ij} and its inverse L_{ij} are lower triangular.
2. Invert all elimination steps using the reverse order rule:
$$(\underbrace{E_{n\,n-1} \cdots E_{32} E_{n1} \cdots E_{21}}_{E})^{-1} = \underbrace{L_{21} L_{31} \cdots L_{n1} L_{32} \cdots L_{n\,n-1}}_{L}$$
3. Noting $EA = U$ and multiplying both sides with $L = E^{-1}$,
   we get $LEA = LU$
$$A = LU.$$ (lower triangular times upper triangular).

Note: The product matrix L is still lower triangular since it was product of lower triangular matrices. It has all 1's on its diagonal.

Example:

$$A=\begin{bmatrix}2&1&0\\0&4&2\\6&3&5\end{bmatrix}\xrightarrow[L_{31}=E_{31}^{-1}]{E_{31}=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\quad L_{31}=\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}}\underbrace{\begin{bmatrix}2&1&0\\0&4&2\\0&0&5\end{bmatrix}}_{U}$$

$$E_{31}A=U \Rightarrow L_{31}E_{31}A=L_{31}U \Rightarrow A=L_{31}U$$

Example:

$$A=\begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix}\xrightarrow[L_{21}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}]{E_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\begin{bmatrix}1&1&1\\0&2&3\\0&4&0\end{bmatrix}\xrightarrow[L_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}]{E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}}\underbrace{\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}}_{U}$$

$E_{32}E_{21}A=U$ (multiply both sides by $(E_{32}E_{21})^{-1}=E_{21}^{-1}E_{32}^{-1}=L_{21}L_{32}$)

$$\underbrace{L_{21}\underbrace{L_{32}E_{32}}_{I}E_{21}}_{I}A=L_{21}L_{32}U=LU$$

$$L=L_{21}L_{32}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}$$

!!! The entries of L are multipliers l_{ij} used in the elimination steps. !!!

Why $A=LU$ (Read the book for a more detailed explanation)?

Row 3 of U = Row 3 of A $- l_{31}$(Row 1 of U) $- l_{32}$(Row 2 of U)

Row 3 of A = l_{31}(Row 1 of U) + l_{32}(Row 2 of U) + 1(Row 3 of U)

$$=\begin{bmatrix}l_{31}&l_{32}&1\end{bmatrix}U$$

Similarly;

Row 2 of A = l_{21}(Row 1 of U) + 1(Row 2 of U) + 0(Row 3 of U)

Row 1 of A = 1(Row 1 of U) + 0(Row 2 of U) + 0(Row 3 of U)

$$\Rightarrow \begin{bmatrix}\text{Row 1 of } A\\\text{Row 2 of } A\\\text{Row 3 of } A\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}U \Rightarrow A=LU$$

A = LDU

Sometimes $A = LU$ is written as $A = LDU$ where D is the diagonal matrix containing the pivots.

Example:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}$$

Diagonal matrix of pivots.

Diagonal entries are 1.

Example:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 2 & 6 & 11 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 6 & 7 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\Rightarrow A = LU = \begin{bmatrix} & L_{31} & \end{bmatrix}\begin{bmatrix} & L_{32} & \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

diagonal contains only 1's.

Where to use $A = LU$?

$Ax = b \qquad LUx = b \qquad Ux = L^{-1}b \qquad x = U^{-1}(L^{-1}b)$

Factor A into L and U — Forward elimination — back substitution

Cost of elimination:

Elimination step for 1^{st} col: n^2 multiplication & n^2 subtraction

2^{n} col: $(n-1)^2$ multiplication & $(n-1)^2$ subtraction.

$\vdots$

$+ \quad 1^2$ multiplication & 1^2 subtraction

$\sim \frac{1}{3}n^3$ multiplication & $\frac{1}{3}n^3$ subtractions. (From A to U)

On the right hand side:

$(n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$ from b to c

$1 + 2 + \dots + n = \frac{n(n+1)}{2}$ from c to x

$\Bigg\}$ n^2 multiplications & n^2 subtractions.

Transpose and Permutations

Transpose of A: A^T.

The columns of A^T are rows of A.

Definition: A^T (transpose of A)

$$A^T(i,j) = A(j,i)$$

If A is $m \times n$, then A^T is $n \times m$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Rules and observations:

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ Prove! Also $(Ax)^T = x^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$ Prove!

- Inner (dot) product: $x \cdot y = x^T y$ ← a scalar
- Outer (rank one) product: xy^T ← a $n \times n$ matrix

- $(Ax)^T y = x^T (A^T y)$

- Symmetric matrix:

  S is symmetric if $S^T = S$ Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

  S^{-1} is also symmetric.

  If $S = S^T$ factors into LDU with no row exchanges, then $U = L^T$.

  $$\boxed{S = LDL^T}$$

★ Creating a symmetric matrix:

Given A, A^TA and AA^T is symmetric.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad AA^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 8 \\ 3 & 8 & 10 \end{bmatrix}$$

Permutation matrix

A permutation matrix has rows of I in any order.

$P \rightarrow$ permutation matrix = row-shuffled identity matrix I

P has exactly one 1 in every column and row. Rest is 0.

Example:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{32} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Total of $n!$ (n factorial, e.g. $5! = 120$) $n \times n$ permutation matrices.

$PA = LU$ (Factorization with row exchanges)

In $A = LU$ factorization before, we assumed there were no row exchanges.

$PA = LU$ is generalized version of $A = LU$. It allows for row exchanges.

Example:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{P} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U}$$

Time Complexity: Let Ω be the notation for rate of growth. Most computer scientists use Ω to talk about time or space complexity of an algorithm. To solve $Ax = b$ using the LU factorization requires Ωn^3 operations or more precisely: $\frac{2}{3}n^3 - \frac{1}{2}n^2 + 2n^2 = \frac{2}{3}n^3 + \frac{3}{2}n^2$, where $\frac{2}{3}n^3 - \frac{1}{2}n^2$ is the time it takes to find LU and $2n^2$ is the time it takes to solve for x. While Gaussian Elimination is also Ωn^3, the real savings for the LU factorization comes when there is repeated use of the same matrix A. If you consider this savings, finding other solutions will only be Ωn^2 when using LU factorization.