Matrices

A matrix is a rectangular collection of numbers or variables.

Example:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A is a matrix with $m=3$ rows and $n=2$ columns.

A is a 3×2 matrix.

\# of rows by \# columns

We will use matrices to represent (and later understand and solve) systems of linear equations.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{bmatrix} \qquad \begin{aligned} \text{row } 1 &= [1 \quad 4] \\ \text{row } 2 &= [2 \quad 5] \\ \text{row } 3 &= [3 \quad 6] \end{aligned}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = [\text{column } 1 \quad \text{column } 2] \qquad \text{column } 1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{column } 2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \\ A(3,1) & A(3,2) \end{bmatrix}, \quad A(i,j) = a_{ij}$$

The element on the i^{th} row and j^{th} column

Linear equations in matrix form

Assume $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Linear combination of these two vectors would be $x_1 u + x_2 v = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

where x_1 and x_2 are scalars. Two rewrite this in <u>matrix form</u>, the vectors u and v go into the columns of a matrix A, which "multiplies" vector

$$x_1 u + x_2 v = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

Matrix times vector (Linear combination of columns)

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

Two ways to see matrix A times vector x:

1. A linear combination of columns:
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{column 1}} + x_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{column 2}}$$

or

2. Row at a time:
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{row 1} \cdot x \\ \text{row 2} \cdot x \end{bmatrix} = \begin{bmatrix} 1x_1 + 1x_2 \\ 1x_1 + -1x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

dot product

NOTE: At first numbers x_1 and x_2 were multiplying vectors. Now a matrix A is multiplying the vector x that contains these numbers.

Example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} + \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \\ 22 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2\times4 + 1\times5 \\ 1\times4 + 3\times5 \\ 3\times4 + 2\times5 \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \\ 22 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 0x_2 + 0x_3 \\ -1x_1 + x_2 + 0x_3 \\ 0x_1 + 0x_2 + x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 \end{bmatrix}$$

Identity matrix: 1s on the diagonal, 0s elsewhere.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

System of linear equations

Example: Find two numbers whose sum is 3 and difference is 1.

Solution:

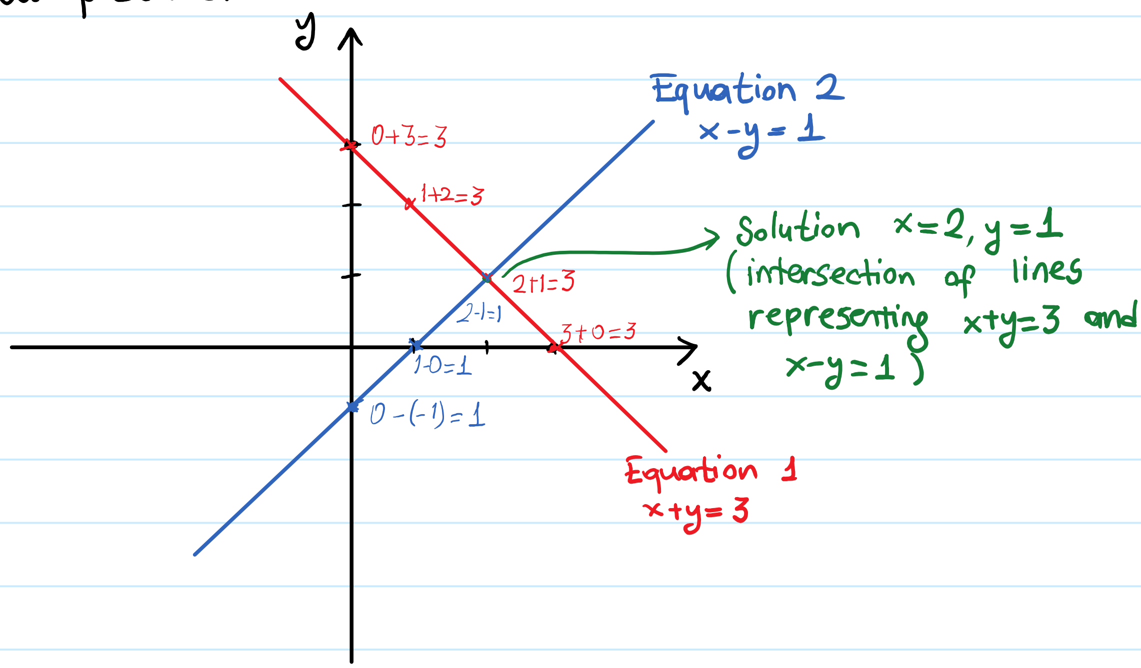
Number 1 : x
Number 2 : y
} Definitions

$$x + y = 3$$
$$x - y = 1$$

} System of linear equations.
2 unknowns (x and y)
2 equations (sum is 3 and difference is 1)

3 different representations of the same problem:

1. Row picture:



Row picture: Find the point where the lines intersect.

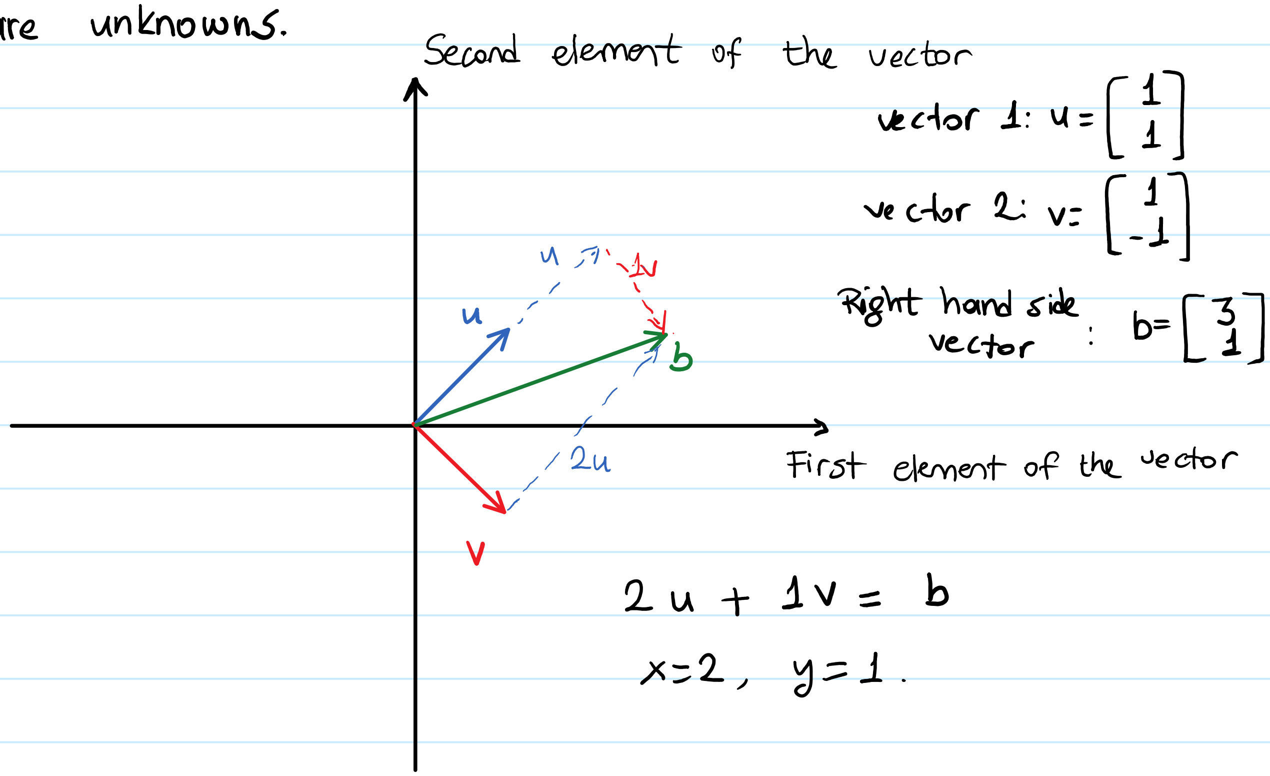
Note: When we deal with more unknowns and equations, each linear equation will represent a plane and solution will be the point where these planes meet.

2. Column picture:

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned} \qquad x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

What is the right linear combination of vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ that gives vector $\begin{bmatrix}3\\1\end{bmatrix}$?

Column picture represents the system of linear equations as a linear combination of vectors where coefficients are unknowns.



$$2u + 1v = b$$

$$x = 2, \quad y = 1.$$

To repeat: The left hand side of the vector equation $\left(x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix}\right)$ is a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. The problem was to find the right coefficients $x=2$ and $y=1$ that produces $b = \begin{bmatrix}3\\1\end{bmatrix}$.

3. Matrix form

$$1x + 1y = 3$$
$$1x - 1y = 1$$

Coefficients

Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Matrix representation of above system of linear equations:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{b}$$

rows of A: $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (coefficients of x and y in equation 1)

$\begin{bmatrix} 1 & -1 \end{bmatrix}$ (coefficients of x and y in equation 2)

!!! Each row of coefficient matrix A represents coefficients of unknowns in the corresponding equation.

columns of A: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (coefficients of x in all equations)

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (coefficients of y in all equations)

Row picture: Each row represents an equation and we would like to find a solution (a vector) that satisfies all the equations.

Column picture: Each column represents the coefficients of a particular unknown. We would like to find the right combination of these columns that produces the column vector on the right hand side.

Revisiting the matrix-vector product:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \text{Dot product of row 1 and } \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{Dot product of row 2 and } \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1x + 1y \\ 1x - 1y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

OR we could see it as a linear combination of columns:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{column 1}} x + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{column 2}} y$$

Example: 3 equations and 3 unknowns

Emma has three types of food containers in her backpack: 1) Energy bar, which has a 3 units of carbs, 1 unit of fat and no protein per package, 2) trail mix, which has 2 units of carbs, 4 units of fat and 1 unit of protein per package, and 3) protein bar, which has no carbs, 1 unit of fat and 3 units of protein.

If Emma wants a daily intake of 8 units of carbs, 9 units of fat and 10 units of protein, how much should she eat from each container in a day?

Lets write this problem as a system of linear equations.

x: Amount of energy bar intake in a day

y: Amount of trail mix intake in a day

z: Amount of protein bar intake in a day

We would like to find the right combination of energy bars, trail mixes, and protein bars that produces a perfect daily diet.

$$\begin{aligned}
\text{Carb intake}: &\quad 3x + 2y + 0z = 8 \\
\text{Fat intake}: &\quad 1x + 4y + 1z = 9 \\
\text{Protein intake}: &\quad 0x + 1y + 3z = 10
\end{aligned}$$

Row picture

(we make sure we take right amount of each nutrition)

$$\underbrace{\begin{bmatrix}3\\1\\0\end{bmatrix}}_{\text{energy bar}} x + \underbrace{\begin{bmatrix}2\\4\\1\end{bmatrix}}_{\text{trail mix}} y + \underbrace{\begin{bmatrix}0\\1\\3\end{bmatrix}}_{\text{protein bar}} z = \begin{bmatrix}8\\9\\10\end{bmatrix}$$

Column picture

(What is the right combination of energ bar, trail mix, and protein bar for a perfect diet?)

$$\underbrace{\begin{bmatrix}3 & 2 & 0\\1 & 4 & 1\\0 & 1 & 3\end{bmatrix}}_{\text{Coefficient matrix } A} \underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\text{vector of unknowns } x} = \underbrace{\begin{bmatrix}8\\9\\10\end{bmatrix}}_{\text{Right hand side vector } b}$$

matrix form

$(A\,x = b)$

Note: while writing the system of linear equations in matrix form, the vectors whose linear combination with unknown coefficients equals right hand side become the columns of the coefficient matrix.

$$\underbrace{\begin{bmatrix}3 & 2 & 0\\1 & 4 & 1\\0 & 1 & 3\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}8\\9\\10\end{bmatrix}}_{b}$$

Column 1: energy bar vector

Column2: trail mix vector

Column 3: protein bar vector

amount vector (unknown)

perfect daily diet vector

$$Ax = b$$

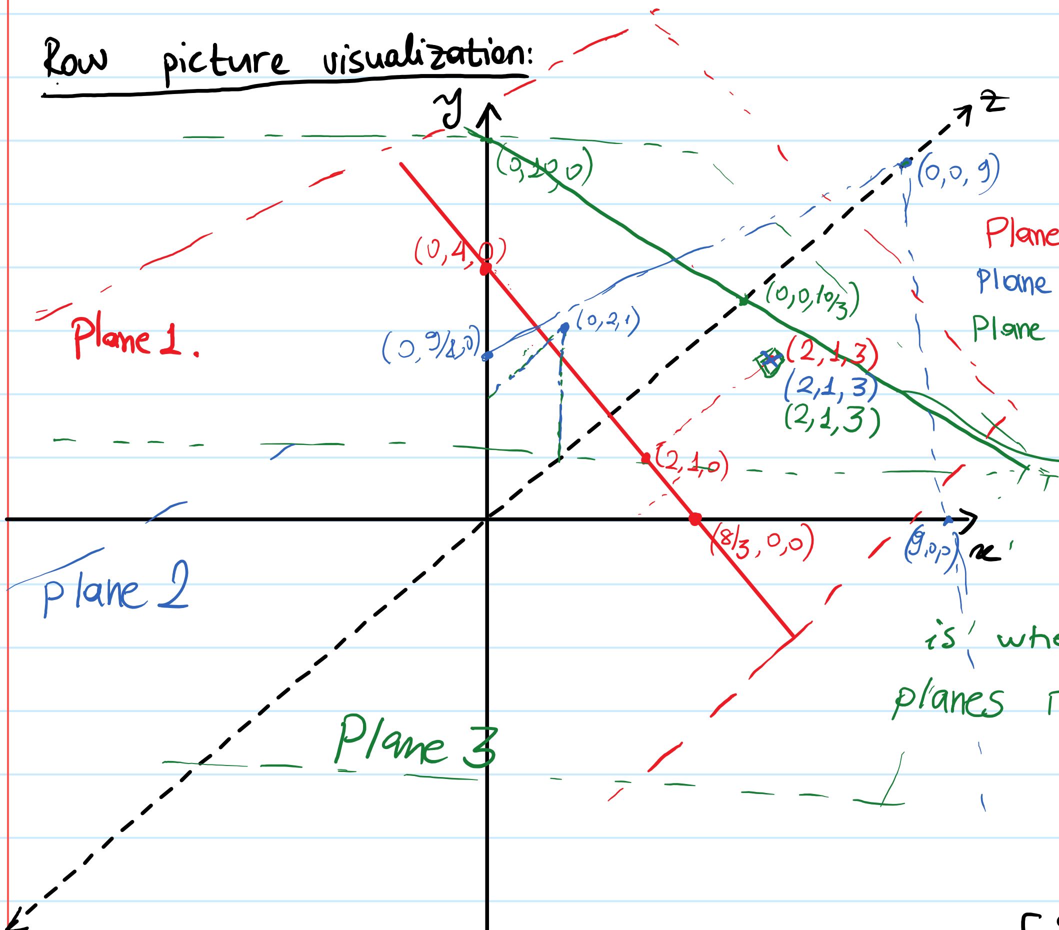
$$Ax = \begin{bmatrix} (\text{row } 1) \cdot x \\ (\text{row } 2) \cdot x \\ (\text{row } 3) \cdot x \end{bmatrix} = \begin{bmatrix} 3x + 2y + 0z \\ 1x + 4y + 1z \\ 0x + 1y + 3z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix} = b$$

dot product

Note: unknown vector x and amount of energy bar x are different!

$$Ax = x\,(\text{column } 1) + y\,(\text{column } 2) + z\,(\text{column } 3) = b$$

Row picture visualization:



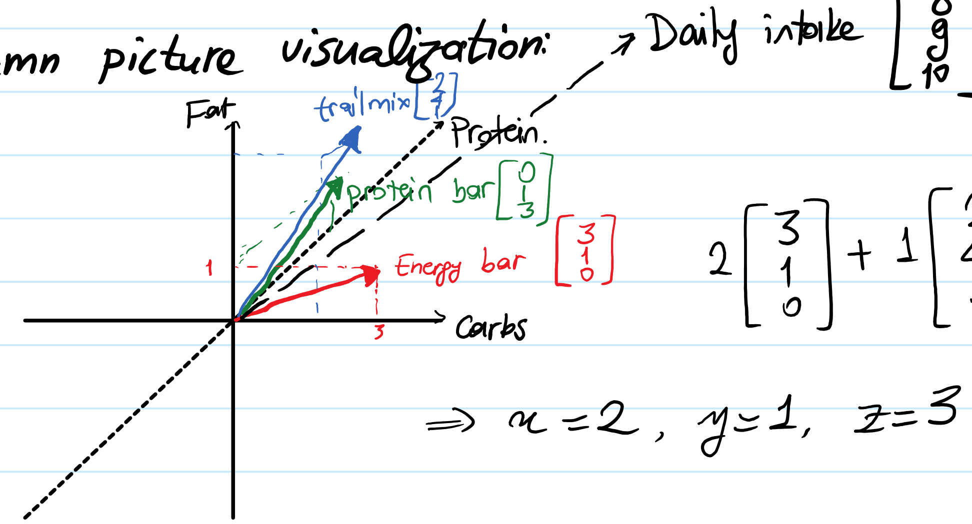
Plane 1: $3x + 2y + 0z = 8$

Plane 2: $1x + 4y + 1z = 9$

Plane 3: $0x + 1y + 3z = 10$

Solution: Point (2,1,3) $(x=2,\ y=1,\ z=3)$ is where all three planes meet.

Column picture visualization:



Daily intake $\begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$

trailmix $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, protein bar $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, Energy bar $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

$$2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$$

$$\Rightarrow x = 2,\ y = 1,\ z = 3.$$

Solving linear equations with elimination

Assume we have a system of linear equations $Ax=b$, with n unknowns and n equations:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note that this is no different than the examples we have seen so far except that the number of equations and unknowns increased from 2 or 3 to n!

Elimination steps:

1. First equation is $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$. Multiply this by the ratio a_{21}/a_{11} and subtract from second equation. Then, x_1 is eliminated from second equation:

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad \leftarrow \text{Equation 2}$$

$$\frac{a_{21}}{a_{11}}\left(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\right) = \frac{a_{21}}{a_{11}} b_1 \quad \leftarrow \text{Equation 1 times } \frac{a_{21}}{a_{11}}$$

$-$ ← subtraction

$$\underbrace{\left(a_{21} - \frac{a_{21}}{\cancel{a_{11}}}\cancel{a_{11}}\right)}_{\text{new } a_{21}} x_1 + \underbrace{\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)}_{\text{new } a_{22}} x_2 + \cdots + \underbrace{\left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)}_{\text{new } a_{2n}} x_n = \underbrace{b_2 - \frac{a_{21}}{a_{11}}b_1}_{\text{new } b_2}$$

Idea: New $a_{21} = a_{21} - \frac{a_{21}}{a_{11}}a_{11} = 0$, so we have eliminated x_1 from second equation. Corner element a_{11} is called "pivot".

2. Similar to step 1, eliminate x_1 from every remaining equation i by multiplying equation 1 by a_{i1}/a_{11} and subtracting from equation i.
3. After step 2, last $n-1$ equations contain $n-1$ unknowns $x_2, \ldots, x_n$. Repeat the same procedure to eliminate x_2 from last $n-2$ equations.

4. Repeat until last $n-i$ equations contain only $n-i$ unknowns $x_{i+1}, x_{i+2}, \dots, x_n$ for $i = 1, 2, \dots, n-1$

5. This process might break down if zero appears in the "pivot" elements $a_{11}, \dots, a_{nn}$. Exchanging two equations might solve this.

Revisit of two numbers example:

$$x + y = 3$$
$$x - y = 1$$

unknowns: x, y

$$\begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{1} \\ \underset{a_{21}}{1} & \underset{a_{22}}{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Step 1 (and the only step in this example): Eliminate x from second equation:

multiply first equation by $\frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$ and subtract from equation 2.

$$\begin{array}{r} x - y = 1 \\ - \quad x + y = 3 \\ \hline 0x - 2y = -2 \end{array}$$

Updated system of equations after step 1: $\quad \begin{aligned} x + y &= 3 \\ -2y &= -2 \end{aligned}$

$\Rightarrow y = 1$. Substituting $y=1$ back into the first equation gives $x + 1 = 3$, $x = 2$.

!!! Elimination produces an upper triangular system, this is the goal:

$$\begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{-2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

□: Pivot elements that are on the diagonal.

◸: Only upper right side of coefficient matrix is non-zero.

Revisit of diet example:

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$$

Step 1:

Eliminate x from second equation:

$$1x + 4y + 1z = 9$$
$$- \quad \tfrac{1}{3}\left(3x + 2y + 0z\right) = \tfrac{1}{3}8$$
$$\overline{0x + \tfrac{10}{3}y + 1z = \tfrac{19}{3}}$$

After first step:
$$\begin{bmatrix} 3 & 2 & 0 \\ 0 & \frac{10}{3} & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 19/3 \\ 10 \end{bmatrix}$$

2nd step: Eliminate x from third equation. No need!

3rd step: Eliminate y from third equation:

$$0x + 1y + 3z = 10$$
$$- \quad \tfrac{3}{10}\left(0x + \tfrac{10}{3}y + 1z\right) = \left(19/3\right)\tfrac{3}{10}$$
$$\overline{0x + 0y + \left(3 - \tfrac{3}{10}\right)z = 10 - \tfrac{19}{10}}$$
$$0x + 0y + \frac{27z}{10} = \frac{81}{10}$$

Before elimination:

$Ax = b$

After elimination

$Ux = c$

upper triangular matrix.

After 3rd step:
$$\begin{bmatrix} 3 & 2 & 0 \\ 0 & 10/3 & 1 \\ 0 & 0 & 27/10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 19/3 \\ 81/10 \end{bmatrix}$$

Back substitution after elimination.

$$z = 3$$
$$y = \left(\tfrac{19}{3} - 3\right) \times \tfrac{3}{10} = 1$$
$$x = (8 - 2)/3 = 2.$$

Question: How can we perform elimination using matrices?
Answer: We need to represent elimination steps with matrices, and learn how to multiple matrices.

Elimination in matrix form:

Given $Ax=b$;

Step 1:

Multiple $Ax=b$ by a matrix E_{21} to get $E_{21}Ax=E_{21}b$

We would like E_{21} to eliminate x_1 from second equation.

Step2: Repeat step 1 to eliminate x_i from last $n-i$ equations.

Step 3: If row exchanges are needed, multiply both sides with "row exchange matrix" P_{iJ}.

Matrix multiplication

Definition: If A is a matrix with m rows and n columns, and B is a matrix with n rows and p columns, matrix multiplication AB is defined as:

$$AB(i,J) = \sum_{k=1}^{n} A(i,k)\,B(k,J)$$

entry at i^{th} row and J^{th} column of AB = Dot product of i^{th} row of A and J^{th} column of B.

In words, the value in row i and column J of AB is dot product of row i of A and column J of B.

Example:

$$\underbrace{\begin{bmatrix} 1 & -3 & 4 & 0 \\ 2 & 1 & 2 & 4 \\ 3 & 2 & -1 & 0 \end{bmatrix}}_{A:\ 3\times 4} \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \\ -2 & 2 \end{bmatrix}}_{B:\ 4\times 2} = \overbrace{\begin{bmatrix} * & * \\ * & * \\ 6 & * \end{bmatrix}}^{C}$$

$AB = C$

$$\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix} = 6$$

Rules and observations:

1. $A_{m\times n} \, B_{n\times p} = C_{m\times p}$ sizes of the matrices.

Matrix A with n columns multiply matrix B with n rows.

2. $C_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$

dot product.

3. $(AB)C = A(BC) \qquad (AB)x = A(Bx)$

note: we do not touch the order, just change the paranthesis

4. It is usually not true that $AB = BA$

5. $\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$ (multiplication by blocks)

6. Two other ways to see matrix multiplication:

- Column j of AB is equal to A times column j of B.
- Row i of AB is equal to row A times B.

7. Another way to see matrix multiplication

$$AB = \underbrace{\begin{bmatrix} | & | & & | \\ \text{column}1 & \text{column}2 & \cdots & \text{column } n \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} - \text{row}1 - \\ - \text{row}2 - \\ \vdots \\ - \text{row } n - \end{bmatrix}}_{B} = \sum_{i=1}^{n} \underbrace{(\text{column } i)}_{n\times 1} \underbrace{(\text{row } i)}_{1\times n}$$

This way, we wrote AB as a sum of n matrices.

Ex:

$$\underbrace{\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}_{C}$$

$C_{11} = 3\times 1 + 1\times 0 + -1\times 1 = 2$

$C_{12} = 3\times 1 + 1\times 2 + -1\times 0 = 5$

$C_{21} = 2\times 1 + 0\times 0 + 1\times 1 = 3$

$C_{22} = 2\times 1 + 0\times 2 + 1\times 0 = 2$

$$\Rightarrow C = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$$

* Row 1 of C = Row 1 of A times B:

$$\begin{bmatrix} 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

* Column 2 of C = A times column 2 of B:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} =$$

* $AB = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$

Laws of matrix multiplication

1. Matrix sum:

$$A + B = B + A \quad \text{commutative}$$
$$c(A + B) = cA + cB \quad \text{distributive}$$
$$A + (B+C) = (A+B) + C$$

2. Matrix product

$$AB \neq BA \quad (\text{commutative law is usually broken})$$
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$
$$A(BC) = (AB)C \quad \text{associative law for } ABC.$$

3. If A is square $\underbrace{AAA \dots A}_{p \text{ times}} = A^p$

4. $$\left[\begin{array}{c|c} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} B_{11} \\ \hline B_{21} \end{array}\right] = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

cuts should match.

Inverse matrix

Square matrix A is invertible if there exist a matrix A^{-1} for which $A^{-1}A = I$.

Elimination as a matrix product

If we multiply a matrix by a row vector from left, we get a row vector. We can use this to represent elimination steps as matrix multiplication.

Step 1:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 10/3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

matrix E_{21} performs the following on matrix A:

Doesn't change first and third rows. multiplies first row by $\frac{a_{21}}{a_{11}} = \frac{1}{3}$ and subtracts that from row 2.

Step 2: Do nothing since x is already eliminated from third row.

Step 3: Eliminate y from third equation:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{10} & 1 \end{bmatrix}}_{E_{32}} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 10/3 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 0 & 10/3 & 1 \\ 0 & 0 & 27/10 \end{bmatrix}}_{U}$$

Overall we have:

upper triangular.

$$E_{32}(E_{21}A) = U \quad \text{or}$$

$$(E_{32}E_{21})A = U$$

since matrix multiplication is associative.

Example: (Breakdown of elimination)

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 7 \end{aligned}$$

There is no solution to these system of equations.