Assignment #3 Solutions

(1)
$$\underbrace{\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

a)
$$A^T y = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 700 + 120 + 6000 \\ 35000 + 3000 + 150000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$$

Cost of a truck: \$6,820

$\Rightarrow$ Cost of a plane: \$188,000

b) Cost of 3 trucks and 5 planes:

$$\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 6,820 \\ 188000 \end{bmatrix} = 3 \times 6,820 + 5 \times 188000 = 960460$$

(2)

a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_2 \end{array}\right]$$

zero row

If $b_2 = b_3$, above system is solvable. (many solutions)

If $b_2 \neq b_3$, above system has no solution.

b)
$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 3 & b_2 \\ 3 & 4 & b_3 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 3b_1 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

zero row.

$\Rightarrow$ If $b_3 - b_2 - b_1 = 0$ $(b_3 = b_1 + b_2)$, then above system is solvable.

If $b_3 - b_2 - b_1 \neq 0$, then there is no solution.

③ a)

Free columns.

$$\begin{bmatrix} 1 & 2 & 2 & 5 & 6 \\ 1 & 2 & 3 & 6 & 8 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 5 & 6 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 5 & 6 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

$$S_1: \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 2 + 2x_3 = 0 \\ x_3 = 0 \end{array} \Rightarrow s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$S_2: \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 2x_3 + 5 = 0 \\ x_3 + 1 = 0 \end{array} \Rightarrow S_2 = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$S_3: \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 2x_3 + 6 = 0 \\ x_3 + 2 = 0 \end{array} \Rightarrow S_3 = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

b) Null space contains all linear combinations of special solutions.

$$N(A) = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + e\begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \qquad c, d, e \in \mathbb{R}$$

c) Rank (A) = # pivots = 2.

(4)

a)

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix}$$

rank(A) = 3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$ rank(B) = 2.

b) Pivot columns 1, 3, 4 of A we found in part a are linearly independent.

Columns 1, 2 of matrix B are LI.

c) Since rank(A) = 3, we need 3 linearly independent columns of A (which will span C(A) and thus will form a basis).

Columns 1, 3, 4 of A are a basis for C(A).

- They are LI.
- They span C(A): Any vector in C(A) could be written as a linear combination of these 3 colums.

(5)

a) $$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(A)=2 and there is non-zero vectors in the nullspace.

There are infinitely many solutions. R^{-1} doesn't exist, R is singular.

b) $C(A) = c.\,col1 + d.\,col2 = c\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

C(A) doesn't fill entire R^3. Example: Vector $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ is not in C(A).

c) $v = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

d) $N(A) = c\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ is one and ~~the~~ the only special solution to $Rc = 0$.

⑥

a) $\begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 3 & 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 4/3 & -1/3 & 1 & 5/3 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & 7/3 & 7/3 & 9/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & 7 & 7 & 9 \end{bmatrix}$

rank(A) = 3   Pivot columns   free columns.

$S_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ -x_2 + 2x_3 + 1 &= 0 \\ 7x_3 + 7 &= 0 \end{aligned} \Rightarrow S_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$S_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} 1 + 3x_1 + 2x_2 + x_3 &= 0 \\ -x_2 + 2x_3 + 1 &= 0 \\ 7x_3 + 9 &= 0 \end{aligned} \Rightarrow S_2 = \begin{bmatrix} 8/7 \\ -11/7 \\ -9/7 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow N(A) = cS_1 + dS_2 = c \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 8/7 \\ -11/7 \\ -9/7 \\ 0 \\ 1 \end{bmatrix}$

b) Columns 1, 2, and 3 are LI. and rank(A) = 3.

$\Rightarrow C(A) = c \text{ col}1 + d \text{ Col}2 + e \text{ col}3 \qquad c, d, e \in \mathbb{R}.$

Yes it fills the entire $\mathbb{R}^3$.

$\begin{bmatrix} 3 & 2 & 1 \\ 3 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix} x = b$ has a unique solution for any $b \in \mathbb{R}^3$

So any $b \in \mathbb{R}^3$ can be represented as a linear combination of columns of A and thus is in $C(A)$.