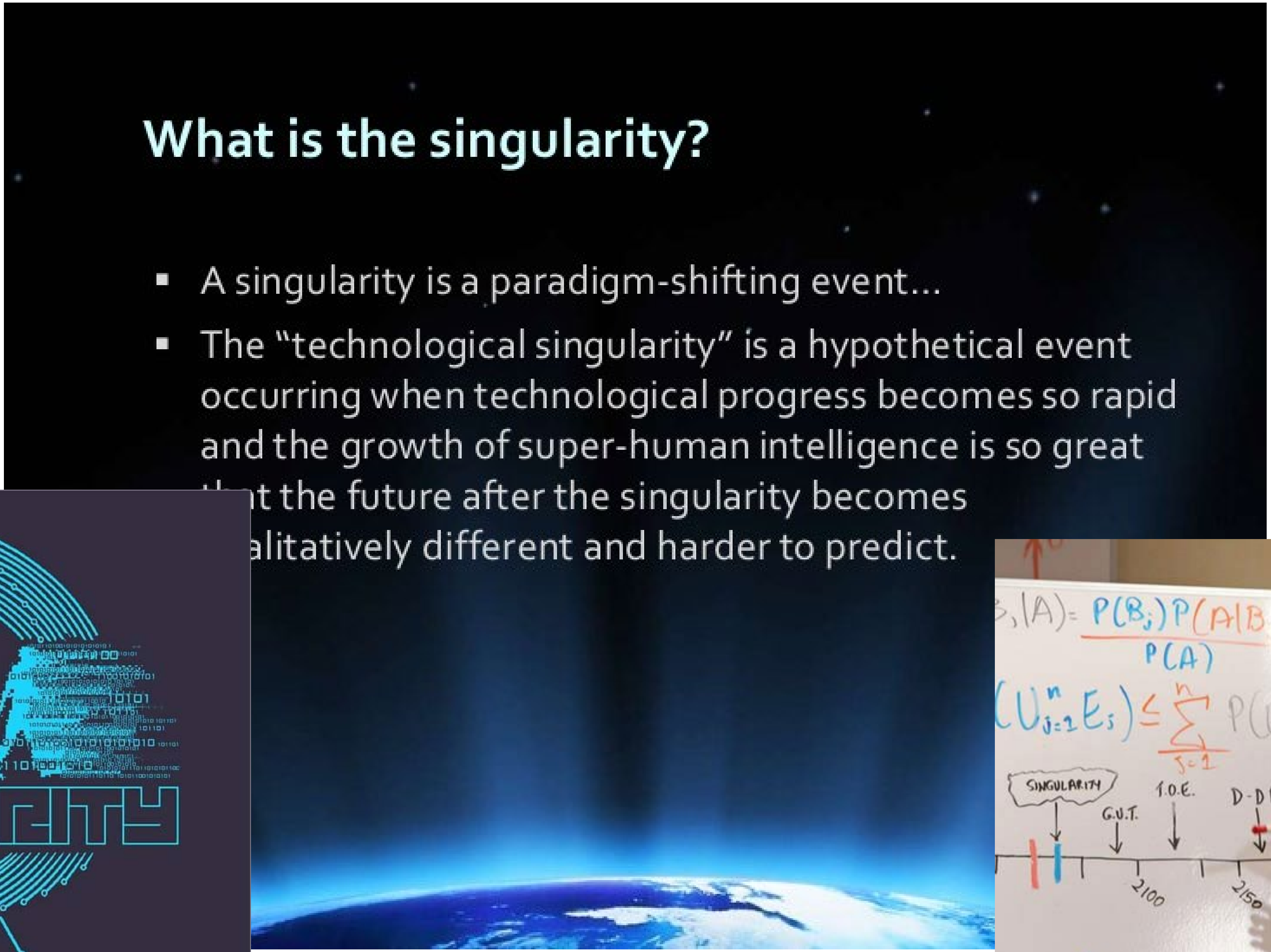
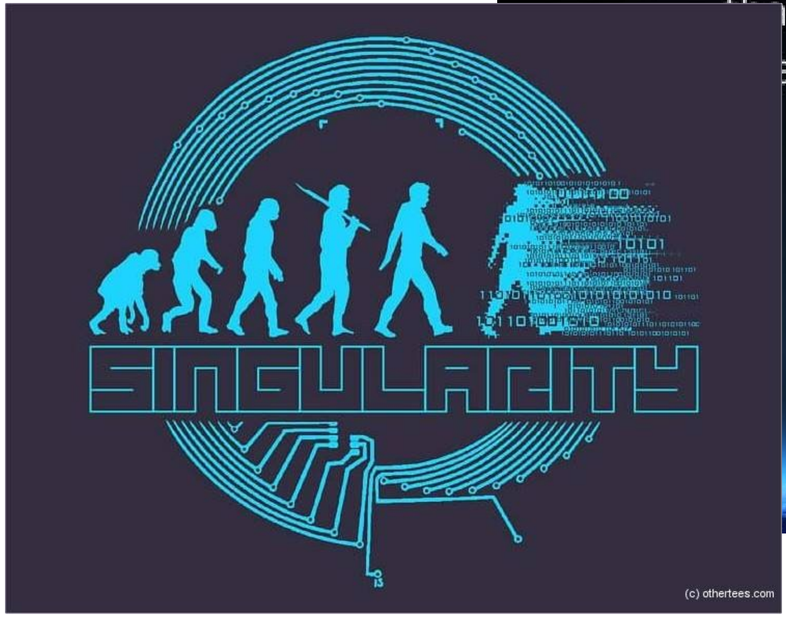
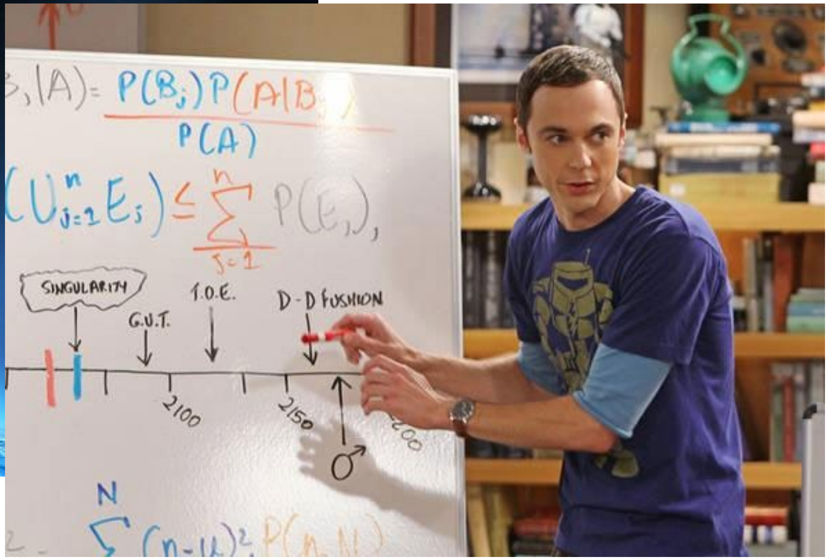
DS 5020
Introduction to Linear Algebra, Statistics, and Probability

Lecture 6: Eigenvalues and Eigenvectors

SINGULARITY







Why are we discussing about Singularity?

A closer look at the determinant definitions

$$\det(A) = 0 \longrightarrow \text{Singular}$$
$$\det(A) \neq 0 \longrightarrow \text{Non-Singular}$$

Why did we use the term Singular here?

EIGENVALUES AND EIGENVECTORS

A square matrix;

$$Ax = \lambda x$$

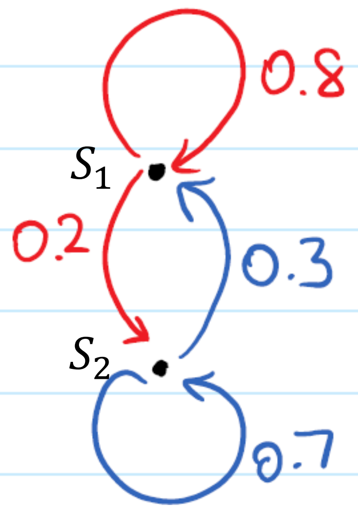
eigenvalue

eigenvector

- A Steady State system

Eigenvalues and eigenvectors can be used to describe "dynamic systems", i.e. how does a system state x change over time.

Ex: $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$



$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.3 \\ 2.7 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$Ax = \lambda x \qquad (A - \lambda I)x = 0 \qquad x$ is in the nullspace of $A - \lambda I$

$$\begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix} x = 0 \qquad \det(A - \lambda I) = 0$$

$$(0.8-\lambda)(0.7-\lambda) - 0.2 \times 0.3 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 0.5$$

$$\lambda_1 = 1 \Rightarrow \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.5 \Rightarrow \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \qquad (0.5-\lambda)^2 - 0.5^2 = 0 \qquad \lambda_1 = 0 \quad \lambda_2 = 1$$

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (0-\lambda)^2 - 1 = 0 \qquad \lambda_1 = -1 \quad \lambda_2 = 1.$$

Note: (Powers)

If $Ax = \lambda x$, then

$$A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$$

$$A^3x = A(\lambda^2 x) = \lambda^2(Ax) = \lambda^3 x$$

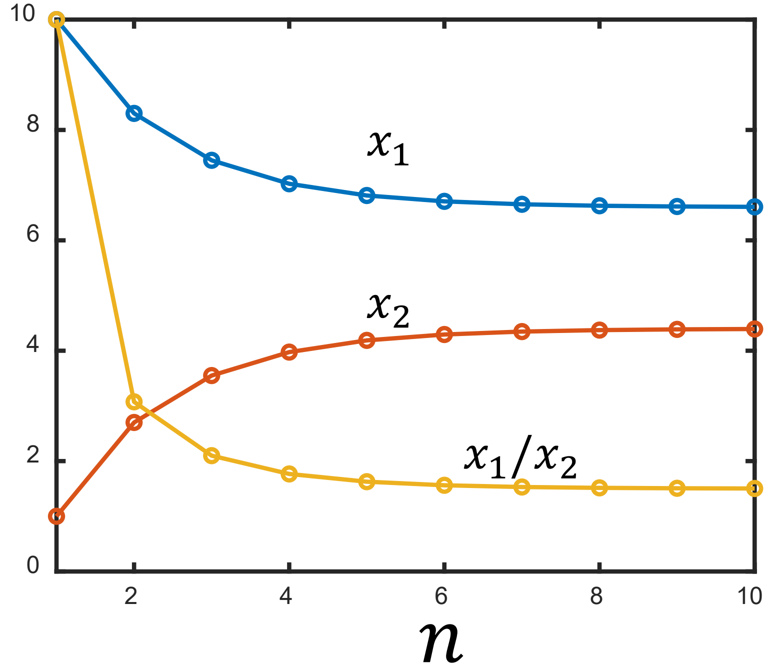
$$\vdots$$

$$A^n x = \lambda^n x$$

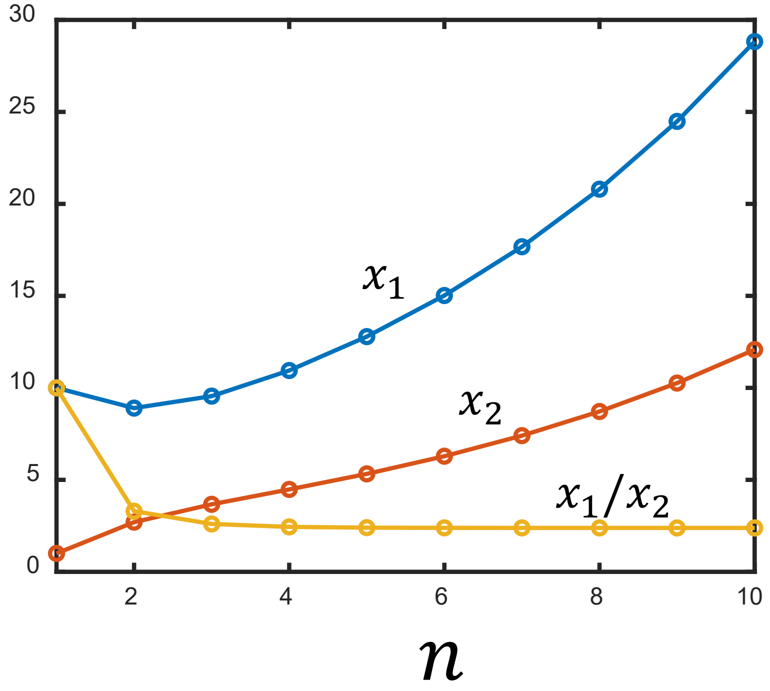
When powers of matrix A are considered ($A^2, A^3, \ldots$) the eigenvectors do not change. The eigenvalues change to that power of the eigenvalues of A.

Powers of A

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}^n \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$A = \begin{bmatrix} 0.8 & 0.9 \\ 0.2 & 0.7 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}^n \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Example:

$$A^{100}x = ? \qquad A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^{100}x = A^{100}\left(0.8 \times \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} + 0.2 \times \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

$$= 0.8 \; A^{100} \underbrace{\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}}_{\text{eigenvectors}} + 0.2 \; A^{100} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{eigenvectors.}}$$

Recall from previous example:

$$A\begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 0.8 \; 1^{100} \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} + 0.2 \underbrace{\left(\frac{1}{2}\right)^{100}}_{\sim 0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.8 \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Finding eigenvalues and eigenvectors of a matrix.

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0.$

* Determinant of $A - \lambda I$ should be zero for λ to be an eigenvalue.

  $\det(A - \lambda I) = 0$

* For each λ we find from $\det(A - \lambda I) = 0$, solve $Ax = \lambda x$ to find an eigenvector x.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

1. $\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(2-\lambda) - 2 = 0$

   $\lambda^2 - 3\lambda = 0 \quad \lambda(\lambda - 3) = 0 \quad \lambda_1 = 0 \quad \lambda_2 = 3$

   a. $\lambda_1 = 0 \Rightarrow \quad Ax = \lambda x$

   $Ax = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \quad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   All nullspace of A is eigenvectors for $\lambda_1 = 0$.

   b. $\lambda_2 = 3 \Rightarrow Ax = 3x$

   $(A - 3I)x = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \quad x = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

Repeat: To find eigenvalues and eigenvectors of $A_{n\times n}$ matrix:

1. Find λ's such that $\det(A-\lambda I)=0$. This gives a polynomial of degree n, whose roots are eigenvalues λ's.
(Note1: An eigenvalue might repeat multiple times)
(Note2: Sometimes eigenvalues turn out to be <u>complex</u> numbers).

2. The eigenvalues found in part 1. make $A-\lambda I$ singular.
For each λ, solve $(A-\lambda I)x=0$ to find an eigenvector x.

Exercise Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

$$\det(A - \lambda I) = 0,$$

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}$$

$$\begin{aligned}
\det(A - \lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\
&= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (-5-\lambda)6) \\
&= (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\
&= (1-\lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\
&= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\
&= 16 + 12\lambda - \lambda^3.
\end{aligned}$$

$$\lambda^3 - 12\lambda - 16 = 0$$

$$\lambda = 4, -2, -2$$

$$A-4I=\begin{pmatrix}-3&-3&3\\3&-9&3\\6&-6&0\end{pmatrix}$$

$$\begin{array}{cc}\begin{pmatrix}-3&-3&3&0\\3&-9&3&0\\6&-6&0&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}\xrightarrow{R1\to-1/3\times R3}\begin{array}{cc}\begin{pmatrix}1&1&-1&0\\3&-9&3&0\\6&-6&0&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\xrightarrow{\substack{R2\to R2-3\times R1\\R3\to R3-6\times R1}}\begin{array}{cc}\begin{pmatrix}1&1&-1&0\\0&-12&6&0\\0&-12&6&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\xrightarrow{R2\to-1/12\times R2}\begin{array}{cc}\begin{pmatrix}1&1&-1&0\\0&1&-1/2&0\\0&-12&6&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\xrightarrow{R3\to R3+12\times R2}\begin{array}{cc}\begin{pmatrix}1&1&-1&0\\0&1&-1/2&0\\0&0&0&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\xrightarrow{R1\to R1-R2}\begin{array}{cc}\begin{pmatrix}1&0&-1/2&0\\0&1&-1/2&0\\0&0&0&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\begin{aligned}x_1-1/2x_3&=0\\x_2-1/2x_3&=0\end{aligned}$$

$$\mathbf{x}=\begin{pmatrix}x_1=\frac{x_3}{2}\\x_2=\frac{x_3}{2}\\x_3\end{pmatrix}=x_3\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{pmatrix}$$

$$A+2I=\begin{pmatrix}3&-3&3\\3&-3&3\\6&-6&6\end{pmatrix}$$

$$\begin{array}{cc}\begin{pmatrix}3&-3&3&0\\3&-3&3&0\\6&-6&6&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}\xrightarrow{R1\to1/3\times R3}\begin{array}{cc}\begin{pmatrix}1&-1&1&0\\3&-3&3&0\\6&-6&6&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$\xrightarrow{\substack{R2\to R2-3\times R1\\R3\to R3-6\times R1}}\begin{array}{cc}\begin{pmatrix}1&1&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}&\begin{matrix}\text{R1}\\\text{R2}\\\text{R3}\end{matrix}\end{array}$$

$$x_1+x_2-x_3=0$$

$$\mathbf{x}=\begin{pmatrix}x_1=x_3-x_2\\x_2\\x_3\end{pmatrix}$$

$$\mathbf{x}=\begin{pmatrix}x_3-x_2\\x_2\\x_3\end{pmatrix}=x_3\begin{pmatrix}1\\0\\1\end{pmatrix}+x_2\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Determinant and trace

- The product of eigenvalues is equal to det(A).
- The sum of eigenvalues is equal to sum of the main diagonal elements of A (i.e. the trace of A, trace(A)).

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

product of.

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

sum of.

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad (1-\lambda)(2-\lambda) - 0\times 1 = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 2.$$

$$\det(A) = 2 = \lambda_1 \cdot \lambda_2$$

$$tr(A) = 1+2 = 3 = \lambda_1 + \lambda_2.$$

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad (a-\lambda)(d-\lambda) - bc = 0.$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

Example: (Triangular matrix)

The eigenvalues of a triangular matrix lie on its diagonal.

Examples

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \qquad tr(A) = 0, \sum_i \lambda_i = 0 \qquad det(A) = 16, \prod_i \lambda_i = 16$$

$$\lambda = 4, -2, -2$$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad tr(A) = 1.5, \sum_i \lambda_i = 1.5$$

$$\lambda = 1, 0.5 \qquad det(A) = 0.5, \prod_i \lambda_i = 0.5$$

AB and $A+B$

The eigenvalues of AB are usually <u>not</u> equal to the multiplication of eigenvalues of A and B (unless A and B share the same eigenvectors).

Similarly, eigenvalues of $A+B$ are generall <u>not</u> sum of eigenvalues of A and B.

* If $AB=BA$, then A and B share the same n independent eigenvectors.

DIAGONALIZING A MATRIX

Suppose $A_{n\times n}$ has n linearly independent eigenvectors. Then

$$Ax_1 = \lambda_1 x_1,\quad Ax_2 = \lambda_2,\quad \ldots,\quad Ax_n = \lambda_n x_n$$

$$\Downarrow$$

$$A\underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{X} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & \cdots & \cdots & \lambda_n x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \\ & & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$AX = X\Lambda$$

Since we assumed x_i's are LI, X has inverse and

$$AXX^{-1} = X\Lambda X^{-1} \Rightarrow A = X\Lambda X^{-1}$$

and

$$X^{-1}AX = X^{-1}X\Lambda = \Lambda$$

Diagonalization:

If A has n linearly independent eigenvectors $x_1, x_2, \dots, x_n$, put them in columns of matrix X. Then X^{-1} exists and:

$$A = X\Lambda X^{-1} \leftarrow \text{Eigen-decomposition}$$

$$\Lambda = X^{-1}AX \leftarrow \text{Diagonalization}$$

where Λ is the diagonal matrix containing eigenvalues.

Example: $A = \begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}$ $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $X^{-1}AX = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}}_{\Lambda}$

Note: For a matrix $A_{n\times n}$ to be diagonalizable, it should have n linearly independent eigenvectors.

Powers of A:

If $A = X\Lambda X^{-1}$, then $A^k = X\Lambda^k X^{-1}$

$k=0$: $I = I$

$k=1$: $A = X\Lambda X^{-1}$

$k=-1$: $A^{-1} = X\Lambda^{-1} X^{-1}$

$\boxed{\begin{array}{l} \text{Calculating } u_{k+1} = Au_k: \\ 1.\ A = X\Lambda X^{-1} \\ 2.\ u_k = Au_{k-1} = A^2 u_{k-2} = \dots = A^k u_0 \\ 3.\ A^k = (X\Lambda X^{-1})^k = \underbrace{(X\Lambda X^{-1})(X\Lambda X^{-1})\dots}_{k \text{ times}} = X\Lambda^k X^{-1} \\ 4.\ u_k = A^k u_0 = X\Lambda^k X^{-1} u_0 \end{array}}$

(***) The eigenvectors corresponding to two different eigenvalues are linearly independent.

Assume: $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ If $\lambda_1 \neq \lambda_2$, then we want to show x_1 and x_2 are LI.

Suppose $cx_1 + dx_2 = 0$

$$A(cx_1 + dx_2) = c\lambda_1 x_1 + d\lambda_2 x_2 = 0$$
$$\lambda_2(cx_1 + dx_2) = c\lambda_2 x_1 + d\lambda_2 x_2 = 0$$
$$c(\lambda_1 - \lambda_2)x_1 = 0 \rightarrow c = 0$$

Similarly $d=0$ and thus x_1 and x_2 are LI.

From (***), we can say that a matrix $A_{n\times n}$ is diagonalizable if it has n different eigenvalues.

A has n different eigenvalues $\Rightarrow$ A is diagonalizable.

$$X^{-1}AX = \Lambda \quad \text{where } X \text{ is eigenvector matrix.}$$

A has repeated eigenvalues $\Rightarrow$ A is not diagonalizable.

☺ Summary (Invertible or diagonalizable):

- A is invertible if $\det A =$ product of eigenvalues $\neq 0$
  ($\lambda = 0 \Rightarrow$ singular, $\lambda \neq 0 \Rightarrow$ invertible).
- A is diagonalizable if n different eigenvalues.

Definition (similar matrices): A and B are similar if they have the same eigenvalues:

$$\left.\begin{aligned} A &= X \Lambda X^{-1} \\ B &= Y \Lambda Y^{-1} \end{aligned}\right\}$$

Eigenvectors could differ but they have the same eigenvalues.
$\Rightarrow$ A and B are "similar".

Q: For Fibonacci numbers defined as $F_k = F_{k-1} + F_{k-2}$, find F_{100} without direct calculation

Define: $u_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$.

Then: $u_k = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}^{k} u_0$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = -\lambda + \lambda^2 - 1 = 0$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$F_{100} = A^{100} u_0 = A^{100} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^{100} \frac{1}{\lambda_1 - \lambda_2} \left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right)$

$= \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^{100} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \underbrace{\lambda_2^{100}}_{\approx 0} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right)$

$\approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{100} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \Rightarrow F_{100} \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{100}$

$$u_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}; u_{k-1} = \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}$$
$$u_k = \begin{bmatrix} F_{k-1} + F_{k-2} \\ F_{k-1} \end{bmatrix}$$
$$u_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}$$
$$u_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_{k-1}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + b \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$
$$a\lambda_1 + b\lambda_2 = 1$$
$$a + b = 0$$

Solve for a and b

$$a = \frac{1}{\lambda_1 - \lambda_2}; b = -\frac{1}{\lambda_1 - \lambda_2}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right)$$

System of Differential Equations

Some preliminary concepts from calculus

$$\left.\begin{array}{l} \dfrac{du}{dt} = u \quad \Rightarrow \quad u = Ce^{t} \\[2ex] \dfrac{du}{dt} = \lambda u \quad \Rightarrow \quad u = Ce^{\lambda t} \end{array}\right\} \quad t=0 \Rightarrow u_0 = C \text{ (initial value).}$$

System of differential equations: $\boxed{\dfrac{du}{dt} = Au}$ (They are linear)

$$\frac{du}{dt} = Au, \quad Ax = \lambda x \quad \Rightarrow \quad \text{choose } u = e^{\lambda t}x$$

$$\left(u = e^{(\text{eigenvalue})t}(\text{eigenvector})\right)$$

$$u = e^{\lambda t}x \quad \Rightarrow \quad \frac{du}{dt} = \lambda e^{\lambda t}x = Ae^{\lambda t}x = Au.$$

Steps to solve system of differential equations

Given $\dfrac{du}{dt} = Au.$

1. Find eigenvalues and eigenvectors (λ_i, x_i) of A.

$Q_i := e^{\lambda_i t} x_i$ are solutions to above system.

3. $u = c_i e^{\lambda_i t} x_i$ (linear combination of solutions in part 2.) is also a solution.

4. If an initial solution $u_0 = u(t=0)$ is given, find c_i's in part 3. from this initial solution.

Example: $\dfrac{du_1}{dt} = u_2$, $\dfrac{du_2}{dt} = u_1$. Assume $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $u(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

$\dfrac{du}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} u \Rightarrow \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_1(t) = e^{\lambda_1 t} x_1 = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$ $\quad u_2(t) = e^{\lambda_2 t} x_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ $\quad u(t) = c u_1(t) + d u_2(t)$

$c = 3$, $d = 1$ from $u(0)$.

Example: (2^{nd} order)

$$\frac{d^2y}{dt^2} + y = 0 \Rightarrow \frac{d^2y}{dt^2} = -y. \quad \text{Assume } u = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du}{dt} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{d(y')}{dt} \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ y' \end{bmatrix}}_{u} \qquad \frac{du}{dt} = Au.$$

Find $\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 1 = 0 \quad \lambda_1 = i \quad \lambda_2 = -i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\Rightarrow u(t) = c\, e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + d\, e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c+d \\ ci - di \end{bmatrix} \Rightarrow \left.\begin{array}{l} c+d = 1 \\ c-d = 0 \end{array}\right\} \begin{array}{l} c = 1/2 \\ d = 1/2 \end{array}$$

and $u(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$.

Example 3

$$\frac{dx_1(t)}{dt} = 2x_1(t) - x_2(t) - x_3(t)$$
$$\frac{dx_2(t)}{dt} = -x_1(t) + 2x_2(t) - x_3(t)$$
$$\frac{dx_1(t)}{dt} = -x_1(t) - x_2(t) + 2x_3(t)$$

a. Express the system in a matrix form.
b. Find the general solution of the system.
c. Find the solution of the system with the initial value $x_1 = 0, x_2 = 1, x_3 = 5$.

Solution

(a)

$$\frac{d}{dt}\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}$$

(a)

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad |A-\lambda I| = 0.$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)^2-1\right] - (-1)\left[(-1)(2-\lambda)-1\right] + (-1)\left[1-(-1)\cdot(2-\lambda)\right]$$

$$= (2-\lambda)(4-4\lambda+\lambda^2-1) + (-2+\lambda-1) - (1+2-\lambda)$$

$$= (2-\lambda)(\lambda^2-4\lambda+3) + 2(\lambda-3)$$

$$= (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-3)$$

$$= (\lambda-3)\left[(2-\lambda)(\lambda-1)+2\right]$$

$$= (\lambda-3)(2\lambda\cancel{-2}-\lambda^2+\lambda\cancel{+2})$$

$$= (\lambda-3)(3\lambda-\lambda^2)$$

$$= \lambda(\lambda-3)(3-\lambda) = 0$$

$\therefore\ \lambda_1 = 0,\ \lambda_2 = 3$

$\lambda_1=0$:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{pivot} & \text{pivot} & \text{free} \end{array} \end{array} \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

Set $x_3=1$

$x_1=1$, $x_2=1$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_2=3$:

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{pivot} & \text{free} & \text{free} \end{array} \end{array} \Rightarrow x_1+x_2+x_3=0$$

Set $x_2=1$, $x_3=0$

$$\therefore x_1=-1 \Rightarrow s_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Set $x_2=0$, $x_3=1$

$$\therefore x_1=-1 \Rightarrow s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore X_2 = h\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

General Solution

$$X(t) = e^{\lambda t} X = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix} + c\begin{bmatrix} -e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$$

c. Find the solution of the system with the initial value $x_1 = 0, x_2 = 1, x_3 = 5$.

$$a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{cases} a - b - c = 0 \\ a + b = 1 \\ a + c = 5 \end{cases} \Rightarrow \begin{cases} 2a - c = 1 \\ a + c = 5 \end{cases} \Rightarrow \begin{matrix} a = 2 \\ b = -1 \\ c = 3 \end{matrix}$$

Symmetric Matrices

$S = S^T$ and $S = LDL^T$

What happens to eigenvalues and eigenvectors when S symmetric:

$$Sx = \lambda x$$

If S is symmetric: $S = (X \Lambda X^{-1})$

$$S^T = (X^{-1})^T \Lambda X^T$$

$$X^{-1} \stackrel{?}{=} X^T \quad \text{so} \quad X^T X = I$$

- A symmetric matrix has only real eigenvalues.
- The eigenvectors can be chosen "orthonormal":

$$X = Q.$$

$\Rightarrow$ **Spectral theorem:**
If S is symmetric:

$$\boxed{S = Q \Lambda Q^{-1} = Q \Lambda Q^T}$$

Example:

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad \det(S-\lambda I)=0 \Rightarrow (1-\lambda)(3-\lambda)-4=0$$

$$\lambda^2 - 4\lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16+4}}{2}$$

$$= 2 \pm \sqrt{5}$$

$$\lambda_1 = 2+\sqrt{5} \Rightarrow \begin{bmatrix} -1-\sqrt{5} & 2 \\ 2 & 1-\sqrt{5} \end{bmatrix} x_1 = 0 \qquad x_1 = \begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix} \qquad q_1 = \frac{x_1}{\|x_1\|}$$

$$\lambda_2 = 2-\sqrt{5} \Rightarrow \begin{bmatrix} -1+\sqrt{5} & 2 \\ 2 & 1+\sqrt{5} \end{bmatrix} x_2 = 0 \qquad x_2 = \begin{bmatrix} 1 \\ -\frac{(\sqrt{5}-1)}{2} \end{bmatrix} \qquad q_2 = \frac{x_2}{\|x_2\|}$$

Note:

1. λ_1 and λ_2 are real-valued
2. x_1 and x_2 are orthogonal. So $Q=[q_1\ q_2]$ is orthonormal.

Relation to projection matrices

If S symmetric;

$$S = Q\Lambda Q^T = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 q_1 & \lambda_2 q_2 & \cdots & \lambda_n q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \, q_i q_i^T$$

Projection Matrix

A symmetric matrix projects a vector into its eigenspace

Pivots vs Eigenvalues

There is no connection except:

- $\det(A)$ = product of pivots = product of eigenvalues
- for symmetric matrices, # of positive pivots = # positive eigenvalues.
  - → Special case: all pivots positive ⇒ all eigenvalues positive.

Positive Definite Matrices

Symmetric S is <u>positive definite</u> (PD) if $x^T S x > 0$ for all $x \neq 0$.

Another definition: Symmetric S is positive definite if all eigenvalues are positive.

$(1-\lambda)(5-\lambda)-4=0$
$\lambda^2-6\lambda+1=0$

Example:

$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is PD. $\lambda_1>0, \ \lambda_2>0$.

$S=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ if $a>0$ and $ac-b^2=\det A>0 \Rightarrow$ S is positive definite.

(*) Symmetric S is positive definite if and only if all the pivots are positive.

$\boxed{\begin{array}{l}\text{Positive pivots} \Rightarrow \text{positive eigenvalues} \\ \text{positive eigenvalues} \Rightarrow \text{positive pivots.}\end{array}}$

Example.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix}$$

all positive $\Rightarrow$ positive definite.

If S and T are PD then

$$x^T(S+T)x = x^TSx + x^TTx > 0 \text{ so } S+T \text{ is PD too.}$$

**

If S symmetric, one of the conditions below mean all the others are satisfied too. (They are equivalent).

1. All n pivots are positive.
2. All n upper left determinants are positive.
3. All n eigenvalues are positive
4. $x^TSx > 0$ for all $x \neq 0$
5. $S = A^TA$ for a matrix A with <u>independent columns</u>.

Example:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

pivots $= 2, 3/2, 4/3$ all positive ✓.

so S is PD.

Or, upper left determinants: $2, 3, 4$ all positive. ✓

Or, all eigenvalues > 0 (not shown here). ✓

Or: $S = A^TA$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, noting columns of A are LI. ✓

Either of the checks above mean the rest is satisfied.

Positive semidefinite matrices:

A symmetric S is PSD if $x^T S x \geq 0$ for all x.
Or, all eigenvalues of S are non-negative.

Example:

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \lambda_1 = 5, \quad \lambda_2 = 0.$$

Summary:

Check lookup table on pg 363 of the text to see a summary of various special matrices and their eigenvalues and eigenvectors.

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors $\boldsymbol{x}_i$.

Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\|\lambda\| = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $A^{\mathrm{T}} = A$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $B = M^{-1}AM$	$\lambda(B) = \lambda(A)$	$\boldsymbol{x}(B) = M^{-1}\boldsymbol{x}(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\|\lambda\| < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi i k/n}$	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $J = M^{-1}AM$	diagonal of J	each block gives $\boldsymbol{x} = (0, .., 1, .., 0)$
Rectangular: $A = U\Sigma V^{\mathrm{T}}$	$\operatorname{rank}(A) = \operatorname{rank}(\Sigma)$	eigenvectors of $A^{\mathrm{T}}A, AA^{\mathrm{T}}$ in V, U