Assignment # 4 Solutions.

(1)

a) rank(A) = 2. First 2 rows are LI and third row is their sum.

b)

C(A): dimension = 2 (same as rank(A)).

Basis: {col 1, col 2}

N(A): dimension = 5−2 $(n-r)$ = 3. To find a basis for N(A), we will find the special solutions.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free.

$$S_1 = \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 + 3 = 0. \\ x_2 + 2 = 0 \end{matrix} \Rightarrow S_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 + 4 = 0 \\ x_2 + 3 = 0 \end{matrix} \quad S_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 + 5 = 0 \\ x_2 + 4 = 0 \end{matrix} \quad S_3 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for N(A): $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$C(A^T)$: row 1 and row 2 forms a basis for row space.

Dimension = 2 (same as rank(A))

$N(A^T)$: Dimension = 1. $(m - r = 3 - 2 = 1)$.

Basis for $N(A^T)$: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

c) $\dim(N(A^T)) + \dim(C(A)) = 1 + 2 = 3 = \#$ rows.

$\dim(N(A)) + \dim(C(A^T)) = 3 + 2 = 5 = \#$ columns.

②

a) rank(A) = 2: First two rows are LI. The rest is linearly dependent on the first two rows.

b) $(b-n)\,col1 + (r-b)\,col2 + (n-r)\,col3 =$

$$\begin{bmatrix}(b-n)r\\(b-n)p\\0\\0\\0\\0\\(b-n)p\\(b-n)r\end{bmatrix} + \begin{bmatrix}(r-b)n\\(r-b)p\\0\\0\\0\\0\\(r-b)p\\(r-b)n\end{bmatrix} + \begin{bmatrix}(n-r)b\\(n-r)p\\0\\0\\0\\0\\(n-r)p\\(n-r)b\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}.$$

c) Using a formula similar to part b;

$(k-b)(col\ 2) + (n-k)(col\ 3) + (b-n)(col\ 5) = 0.$

③

a) If $y \in N(A^T)$, then $A^Ty = 0$.

$$y^T(Ax) = (y^TA)x = (A^Ty)^Tx = 0^Tx = 0$$

b) If $x \in N(A)$, then $Ax = 0$.

$$x^T(A^Ty) = (x^TA^T)y = (Ax)^Ty = 0^Ty = 0.$$

④

a) $A^TA = 5$ and $A^Tb = 300$ $\quad A^TA\hat{x} = A^Tb$

$5\hat{x} = 300 \Rightarrow \hat{x} = 60.$

60 is the average of b_i's since A^Tb is their sum, and we devide the sum by the # of readings A^TA.

b) $$e = b - A\cdot 60 = \begin{bmatrix}59\\62\\58\\60\\61\end{bmatrix} - \begin{bmatrix}60\\60\\60\\60\\60\end{bmatrix} = \begin{bmatrix}-1\\2\\-2\\0\\1\end{bmatrix}$$

c) $$e^T(A\hat{x}) = \begin{bmatrix}-1 & 2 & -2 & 0 & 1\end{bmatrix}\begin{bmatrix}60\\60\\60\\60\\60\end{bmatrix} = 0.$$

d) $$\hat{x}_n = \frac{(n-1)\hat{x}_{n-1} + b_n}{n}$$

thus:

$$\hat{x}_6 = \frac{5 \times 60 + 61}{6} = 60.167$$