VECTOR SPACES

Instead of dealing with individual vectors, we look into "spaces" of vectors.

Example: R^n

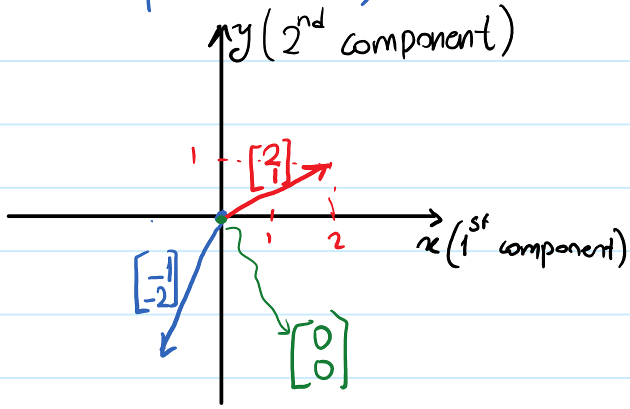
- One of the most important vector space.
- R^n consists of all column vectors with n real components.

R^n: $n \rightarrow$ There are n components.
R ↳ The components are <u>real</u> numbers

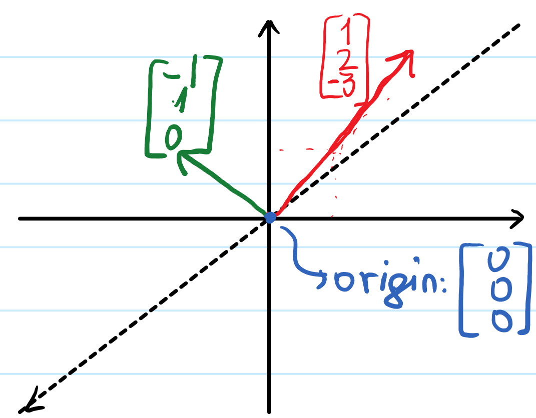
<u>Ex:</u> $\begin{bmatrix}1\\2\end{bmatrix} \in R^2$, $\begin{bmatrix}0\\1\end{bmatrix} \in R^2$

$\begin{bmatrix}3\\4\\5\end{bmatrix} \in R^3$, $\quad 0 \in R^1 \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \in R^4$

<u>Ex:</u> (R^2 representation)



<u>Ex:</u> (R^3 representation)



Note: R^2 represents <u>all</u> 2-dimensional vectors.
R^n consists of <u>all</u> n-dimensional vectors.

!!! We can <u>add</u> any vectors in R^n, and we can multiply any vector in R^n by any scalar c, and we get a result in R^n.

Ex (vector space with only zero vector):

Z is zero-dimensional vector space that consists of zero-vector.

Closedness under addition and multiplication

For a set of vectors to be a vector space both vector addition and multiplication by a scalar should produce results that are part of this set:

If V is a vector space and $v, u \in V$ then any linear combination of v and u is also in V:

$$u, v \in V \Rightarrow w = cu + dv \in V.$$

Ex: R^2: If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2$ then

$$w = u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in R^2 \leftarrow \text{If we add real numbers, we get real numbers.}$$

$$cu = \begin{bmatrix} cu \\ cv \end{bmatrix} \in R^2 \leftarrow \text{If we multiply real numbers by a scalar, we get real numbers.}$$

Subspace

A subspace of a vector space is a set of vectors that satisfies closedness under addition and multiplication:

If v and w are two vectors in the subspace and c is scalar, then

- $v + w$ is in the subspace
- cv is in the subspace.

In other words; any linear combination of two vectors in the subspace produces a vector in the subspace.

Example:

R^3 is a subspace of R^3

A line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a subspace of R^2

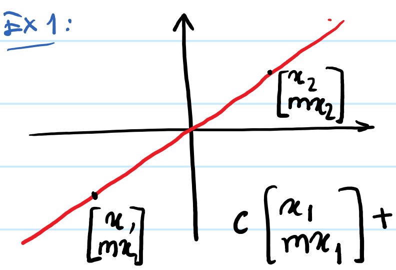
8 rules for vector addition and scalar multiplication:

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. There is a unique zero-vector such that $x + 0 = x$ for all x
4. For each x, there is a unique vector $-x$ such that $x + (-x) = 0$
5. $1x = x$
6. $(c_1 c_2)x = c_1(c_2 x)$
7. $c(x+y) = cx + cy$
8. $(c_1 + c_2)x = c_1 x + c_2 x$

A vector space is <u>a set of vectors</u> which you can <u>add</u> and <u>multiply by a scalar</u>. Addition and scalar multiplication that you define should obey these 8 rules.

A subspace

Ex 1:



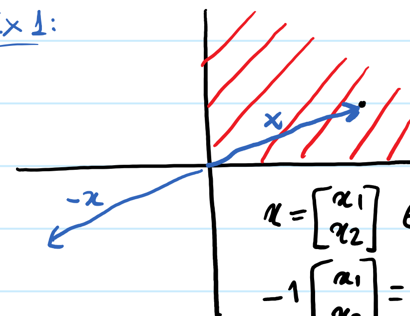
$$c\begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} + d\begin{bmatrix} x_2 \\ mx_2 \end{bmatrix} = \begin{bmatrix} c(x_1+x_2) \\ m\,c(x_1+x_2) \end{bmatrix} \in \text{line}.$$

Ex.2: Z with zero-vector only.

Any linear combination of zero-vectors is a zero-vector: $c0 + d0 = 0$.

Not a subspace

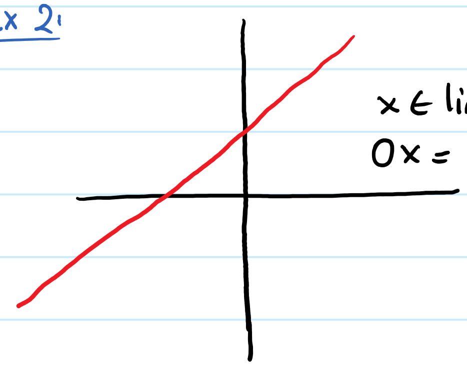
Ex 1:



$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in$ positive orthant $\Rightarrow$

$-1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \notin$ PO.

Ex 2:



$x \in$ line

$0x = 0 \notin$ line.

★ All subspaces of vector spaces contain zero-vector.

Example: M with all 2x2 matrices is a vector space. We can add and scalar-multiply 2x2 matrices and we get 2x2 matrices.

Inside M:

- Upper triangular matrices, and
- Diagonal matrices are two examples of subspaces.

Column space of a matrix

If $A_{m\times n} \in \mathbb{R}^{m\times n}$ is a matrix, the column space of A consists of all the linear combinations of its columns. The column space is a subspace of $\mathbb{R}^m$.

Example: $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\underbrace{\text{Column space of } A}_{C(A)}$ consists of $c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c=0,\ d=0 \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in C(A)$

$c=1,\ d=0 \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in C(A)$

In fact, any $\begin{bmatrix} x \\ y \end{bmatrix} \in C(A)$ $x, y \in \mathbb{R}$

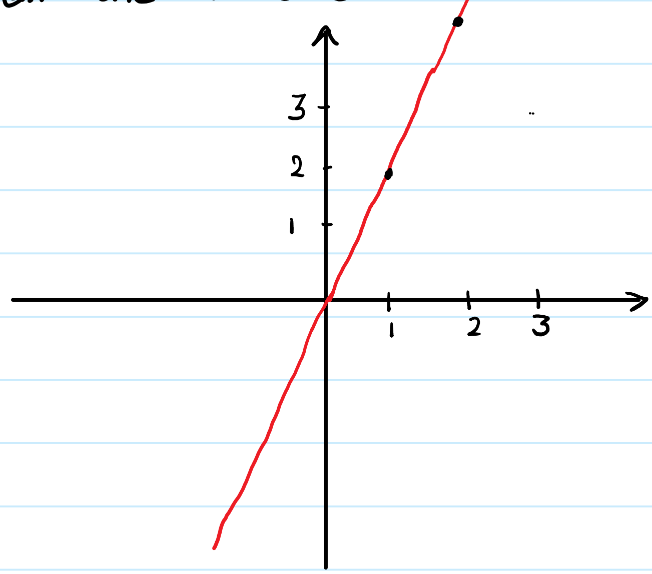
$\Rightarrow C(A)$ is $\mathbb{R}^2$ (the whole plane) in this case.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $C(A)$: all vectors $c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} c+2d \\ 2(c+2d) \end{bmatrix} \Rightarrow$ $\begin{bmatrix} 3 \\ 6 \end{bmatrix} \in C(A)$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \in C(A)$

$\begin{bmatrix} 3 \\ 3 \end{bmatrix} \notin C(A)$

In this case $C(A)$ is a subspace of $\mathbb{R}^2$ and consists of all the vectors on a line (that goes through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$).



$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, There are <u>many</u> solutions.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, There are <u>no</u> solutions.

IDEA: $Ax = b$ is solvable if and only if b is in the column space of A.

Column space contains all the vectors Ax (since Ax is a linear combination of columns of A). So $Ax=b$ is solvable when $b \in C(A)$.

Example:

$$Ax = \begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{cases} \text{case 1: } \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \text{ there is a solution. } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \\ \text{case2: } \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \text{ there is no solution.} \end{cases}$$

Generalization: Instead of columns of a matrix, we might be interested in all linear combinations of a set of vectors S.

S: a set of vectors

SS: all combinations of vectors in S (a subspace).

If a subspace SS is defined as all linear combinations of a set of vectors S, we say S spans SS.

$$SS = \text{span}(S).$$

Ex: Column vectors of a matrix spans the column space.

NULL SPACE OF A

Definition: The null space $N(A)$ consists of all solutions to $Ax=0$. These vectors x are in R^n.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Find all x such that $Ax=0$.

$$Ax = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1+2x_2 \\ 3(x_1+2x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1+2x_2=0 \\ \text{If } x_2=c, x_1=-2c \end{array}$$

$\Rightarrow$ The null space of A is $c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, where c is any real number.

Two important notes:

1. All solutions to $Ax=0$ form a subspace.
   - If $Ax=0$ and $Ay=0$, then $A(x+y)=Ax+Ay=0$ (closed under summation)
   - If $Ax=0$ then $Acx=cAx=0$, c scalar. (closed under scalar multiplication).

2. For $A \in R^{m \times n}$ ($m \times n$ matrix)
   - The column space consists of all linear combinations of its columns, and is in R^m: $C(A)$ is a subspace of R^m.
   - The null space consists of all solutions (x vectors) to $Ax=0$, and is in R^n: $N(A)$ is a subspace of R^n.

Special solutions

Specify "special solutions" by assigning a value to "free" variables

Ex:

$$Ax=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} x = 0 \Rightarrow x = c\begin{bmatrix} -2 \\ 1 \end{bmatrix}. \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ is a special solution}$$

Since we assigned 1 to 2^{nd} component of x (the "free" one)

Finding the null space $N(A)$ of a matrix:

IDEA: Perform elimination until you get all pivots equal to 1 and everything below and above pivots equal to 0. This form is called Reduced Row echelon form R.

- The columns with pivots are called "pivot columns"
- The columns with no pivots are called "free columns"

We find special solutions (later called the "basis" of null space) by assigning 1's to each free variable one by one.

Example:

Find $N(A)$ $\quad A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 6 & 3 \end{bmatrix}$

We first perform row operations to get to reduced row echelon form.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 6 & 3 \end{bmatrix} \xrightarrow[E_{(21)(31)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}]{\substack{row2 = row2 - 2row1 \\ row3 = row3 - 2row1}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & \boxed{-3} & 0 & -3 \\ 0 & -6 & 0 & -6 \end{bmatrix} \xrightarrow[E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}]{row3 = row3 - 2row2}$$

1^{st} pivot. $\quad 2^{nd}$ pivot.

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{row2 = \frac{row2}{-3}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{12} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{row1 = row1 - 2row2}$$

$$\begin{array}{cccc} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \\ \end{array}$$

pivot column 1 — pivot column 2 — free column 1 (no pivots) — free column 2 (no pivots)

$Ax = 0$

$E(Ax) = E0$

$Rx = 0$

(Row operations did not change null space)

$Rx = 0$: $\quad \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$: $\quad \begin{array}{l} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_4 = 0 \end{array}$

pivot — pivot — free — free

(x_1 and x_2 are pivot variables)
(x_3 and x_4 are free variables)

Special solution 1: assign 1 to free variable x_3, 0's to other free variable x_4.

$\Rightarrow \begin{array}{ll} x_1 + 2 = 0 & x_1 = -2 \\ x_2 + 0 = 0 & x_2 = 0 \end{array} \quad \Rightarrow s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Special Solution 2: Assign 1 to second free variable x_4 and 0's to the other free variable x_3.

$$\Rightarrow \quad \begin{matrix} x_1 + 1 = 0 & x_1 = -1 \\ x_2 + 1 = 0 & x_2 = -1 \end{matrix} \quad \Rightarrow \quad s_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Null space $N(A)$ = all linear combinations of special Solutions:

$$N(A) = cs_1 + ds_2 = c\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Quick check: Is $A(cs_1 + ds_2) = 0$?

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 6 & 3 \end{bmatrix}\begin{bmatrix} -2c-d \\ -d \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1(-2c-d) + 2(-d) + 2(c) + 3(d) \\ 2(-2c-d) + 1(-d) + 4(c) + 3(d) \\ 3(-2c-d) + 0(-d) + 6(c) + 3(d) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \checkmark$$

The reason why we called x_3 and x_4 "free variables" is now clear: We can assign <u>any</u> value we want to these two variables. Only then we are able to find what x_1 and x_2 are, which depend on our x_3 and x_4 choice (through the equations $Ax = 0$). If we select x_3 to be c and x_4 to be d, then $x_1 = -2c-d$, $x_2 = -d$.

Exercise: What if we choose different row operations and pivots:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 0 & 3 & 0 & 3 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{cccc} & & & \\ \multicolumn{4}{c}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{free} & \text{free} & \text{pivot} \end{array}$$

$x_2 = 1 \quad x_3 = 0 \Rightarrow x_1 = 1, \; x_4 = -1$

$x_3 = 1 \quad x_2 = 0 \Rightarrow x_1 = -2, \; x_4 = 0$

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that the special vectors s_1 and s_2 we got are the original s_1 and s_2 we found above, times some scalars. So all their linear combinations will represent the same null space.

Exercise:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 2 & 1 & -3 & 1 & -2 \\ 2 & 0 & -4 & 0 & -2 \end{bmatrix} \qquad \text{Find } N(A)$$

Solution:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 2 & 1 & -3 & 1 & -2 \\ 2 & 0 & -4 & 0 & -2 \end{bmatrix} \xrightarrow[r_4 = r_4 - 2r_1]{\substack{r2 = r2 - r1 \\ r3 = r3 - 2r1}} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & -3 & -3 & -1 & -2 \\ 0 & -4 & -4 & -2 & -2 \end{bmatrix} \xrightarrow[r_4 = r_4 - 4r_2]{r3 = r_3 - 3r_2}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{r_4 = r_4 - r_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_3 = r_3/2]{r_2 = r_2/-1}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_1 = r_1 - r_3]{r_2 = r_2 - r_3} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 = r_1 - 2r_2}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \text{pivot} & \text{pivot} & \text{free} & \text{pivot} & \text{free} \end{array} \end{array}$$

(those columns have no pivots)

In equations:

$$\begin{aligned} x_1 - 2x_3 - x_5 &= 0 \\ x_2 + x_3 + x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

pivots: x_1, x_2, x_4

free variables.

Special solution 1: $x_3 = 1,\ x_5 = 0 \Rightarrow x_4 = 0,\ x_2 = -1,\ x_1 = 2$

Special solution 2: $x_3 = 0,\ x_5 = 1 \Rightarrow x_4 = 1,\ x_2 = -1,\ x_1 = 1$

$$s_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$N(A) = cs_1 + ds_2 = c\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c+d \\ -c-d \\ c \\ d \\ d \end{bmatrix}$$

Free

Example:

$$R = \begin{bmatrix} 1 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & * & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

pivots   Free columns.

Pivot variables: x_1, x_2, x_5

Free variables: x_3, x_4, x_6

3 special solutions with 3 free variables.

Column space: $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}$ It is a subspace of R^4.

Null space: It is a subspace of R^6, and consists of all linear combinations of 3 special solutions. (3 special solutions <u>span</u> the null space.)

Rank of a Matrix

Rank of a matrix A is the number of pivots.

$$\text{Rank}(A) = r = \#\text{ pivots}.$$

Equivalent definitions of rank(A):

- Number of non-zero rows after elimination
  (Number of "useful" or "independent" equations in the system)
- Number of "linearly independent" columns of A.

Linear independence of vectors:

Vectors $u_1, u_2, \dots, u_n$ are linearly independent if one can not be written as a linear combination of the others. In other words;

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0 \quad \text{implies} \quad c_1 = 0, c_2 = 0, \dots, c_n = 0.$$

Example: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are not.

In terms of $Ax=0$ and $Rx=0$ after elimination:

Free columns in A are linear combination of its pivot columns. The special solutions tell us what the coefficients are.

Example (previous example revisited):

$$A=\begin{bmatrix}1&2&2&3\\2&1&4&3\\3&0&6&3\end{bmatrix} \qquad R=\begin{bmatrix}1&0&2&1\\0&1&0&1\\0&0&0&0\end{bmatrix} \qquad s_1=\begin{bmatrix}-2\\0\\1\\0\end{bmatrix} \qquad s_2=\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}$$

$$\begin{array}{cccc}\uparrow&\uparrow&\uparrow&\uparrow\\ \text{pivots}&&\text{free}&\end{array}$$

$$As_1=0=-2\begin{bmatrix}1\\2\\3\end{bmatrix}+0\begin{bmatrix}2\\1\\0\end{bmatrix}+1\begin{bmatrix}2\\4\\6\end{bmatrix}+0\begin{bmatrix}3\\3\\3\end{bmatrix}\Rightarrow\begin{pmatrix}\text{col } 3\\ \text{of } A\end{pmatrix}=2\begin{pmatrix}\text{col } 1\\ \text{of } A\end{pmatrix}$$

$$As_2=0=-1\begin{bmatrix}1\\2\\3\end{bmatrix}+-1\begin{bmatrix}2\\1\\0\end{bmatrix}+0\begin{bmatrix}2\\4\\6\end{bmatrix}+1\begin{bmatrix}3\\3\\3\end{bmatrix}\Rightarrow\begin{pmatrix}\text{col } 4\\ \text{of } A\end{pmatrix}=\begin{pmatrix}\text{col } 1\\ \text{of } A\end{pmatrix}+\begin{pmatrix}\text{col } 2\\ \text{of } A\end{pmatrix}$$

Rank One matrix

Matrix that has rank 1. (has 1 pivot). Every column is a scalar times the pivot column.

$$A=\begin{bmatrix}1&2&4\\2&4&8\\3&6&12\end{bmatrix}\xrightarrow{\text{Elimination}}R=\begin{bmatrix}1&2&4\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{array}{ccc}\uparrow&\uparrow&\uparrow\\ \text{pivot}&\text{free}&\text{free}\end{array}$$

A can be written as a column times a row vector:

$$\begin{bmatrix}1&2&4\\2&4&8\\3&6&12\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1&2&4\end{bmatrix}$$

$$s_1=\begin{bmatrix}-2\\1\\0\end{bmatrix} \qquad s_2=\begin{bmatrix}-4\\0\\1\end{bmatrix} \qquad N(A)=cs_1+ds_2=c\begin{bmatrix}-2\\1\\0\end{bmatrix}+d\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

Complete Solution $Ax=b$

We know how to find solutions to $Ax=0$ (null space).
How about $Ax=b$?

Reduced row echelon form

Using the elimination steps (until we get R with 1's for pivots and 0's below and above pivots) on augmented matrix:

$$[A \;\; b] \xrightarrow{\text{Elimination}} [R \;\; d]$$
$$Ax=b \qquad\qquad Rx=d.$$

Example:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\text{Elimination cnt's}} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

First check: Zero rows : $0=0$ ✓

Now we write solution as $x = x_p + x_n$

solutions that come from null space.

a particular solution

Particular solution

If we set free variables x_3 & x_4 to zero:

$$x_1 = d_1 = 1 \qquad\qquad \Rightarrow \quad x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$
$$x_2 = d_2 = 2$$

Null space solutions:

$$s_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad x_n = c\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad Ax = b \qquad Ax_p = b \qquad Ax_n = 0$$

Sum

$$A(x_p + x_n) = b$$

Complete Solution: $x = x_p + x_n$

$$= \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\text{One particular solution.}} + \underbrace{c\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{null space vectors}}$$

Adding vectors from null space gives other solutions.

* If null space has only zero-vector (meaning we have <u>no</u> free variables), then $Ax=b$ has a unique solution x_p.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

What is the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ that will make $Ax=b$ solvable?

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ 2 & 3 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & 1 & b_3-2b_1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2b_1-b_2 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_3-b_1-b_2 \end{array}\right]$$

Condition: $b_3 - b_1 - b_2 = 0$.

Combining what we learned: (full column rank)

If a matrix $A_{m\times n}$ is full column rank $(\text{rank}(A)=r=n)$:

* Its reduced row echelon form will look like: $R=\begin{bmatrix} I \\ 0 \end{bmatrix}$

* All columns will have pivots, no free variables.
* Null space has only zero-vector. No special solutions.
* If $Ax=b$ has a solution, it is the only solution.
  So $Ax=b$ either has exactly one solution, or no solution.

* Note: Full column rank means all columns are linearly independent: $Ax=0$ has only one solution: $x=0$. (Null space consists of $x=0$).

Combining what we learned (full row rank).

If $A_{m\times n}$ is full row rank $(\text{rank}(A)=r=m)$

* Elimination doesn't produce zero rows. (All rows have pivots)
* $Ax=b$ has a solution for every right side b.
* Column space $C(A)=R^m$
* There are $n-r=n-m$ special solutions in the null space.
  (Later we will say null space has dimension $n-r$.

Linear Independence, Basis and Dimension

Linear Independence:

$u_1, u_2, \dots, u_n$ are linearly independent (LI) if the only linear combination that gives the zero vector is:

$$0u_1 + 0u_2 + \dots + 0u_n .$$

If there is a combination that gives zero, when some coefficients are not zero, the vectors are linearly dependent (LD).

Examples:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ are LI.}$$

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix} \text{ are LD} : \underbrace{-2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}2\\2\end{bmatrix}}_{\text{non-zero coefficients}} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \text{ are LI.}$$

$$\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}, \begin{bmatrix}e\\f\end{bmatrix} \text{ are LD. (exercise)}$$

Generalization: Every $n+1$ or more vectors in R^n are LD.

Example:

$$\begin{bmatrix}a & d & g & k\\ b & e & h & l\\ c & f & j & m\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$ will have special solutions.

4 vectors in R^3 are LD, regardless of what they are.

Example: Columns of $A = \begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 3\\ 1 & 1 & 2\end{bmatrix}$ are LD: Col 3 = col 2 + col 1.

Meaning: there are nonzero solutions to $Ax=0$. These solutions are all part of null space of A, $N(A)$.

Example: Full column rank matrix $A_{m\times n}$:
Columns are LI by definition and $\text{rank}(A) = r = n$.
Only $x = 0$ is in the null space.

Other way to see linear dependence: One vector is a combination of others.

Vectors that span a subspace

Definition: A set of vectors "span" a space if their linear combinations fill that space.

Example:

Vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ span R^2.

Vectors $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix}$ also span R^2.

Vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ span the subspace $\begin{bmatrix}a\\b\\0\end{bmatrix}$ in R^3.

Vector $\begin{bmatrix}1\\0\end{bmatrix}$ span a line $\begin{bmatrix}a\\0\end{bmatrix}$ in R^2.

Example: Columns of A span the column space of A.
Rows of A span the row space of A.

Note: Row space of A is the column space of A^T.

$$A = \begin{bmatrix}1 & 0\\2 & 1\\3 & 1\end{bmatrix} \Rightarrow C(A) = c\begin{bmatrix}1\\2\\3\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} \qquad c, d \in R$$

$C(A)$ is a subspace of R^3

Even if the two columns of A are LI, they aren't enough to fill the entire R^3.

Basis

Vectors $u_1, \dots, u_n$ form a basis for a vector space if

1. they are linearly independent
2. they span the vector space

Example:

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for R^2:

- they are LI
- they span R^2: we can write any vector in R^2 as a linear combination of u and v.

!!! There is one and only one way to write a vector b as a linear combination of the basis vectors.

This means:

$Ax = b$ $\quad \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ & & & \end{bmatrix} x = b$ has a unique solution. (for every b in space spanned by columns)

basis vectors in columns of A

Example: Pivot columns of A is a basis for $C(A)$.

- They are linearly independent
- Every vector $C(A)$ is a linear combination of these column vectors.

Pivot rows of R is a basis for row space of A (or R).

If A is $n \times n$ and all columns are linearly independent, then columns span the column space, which is R^n. And A is also invertible; there is a unique solution to $Ax = b$.

Important: Basis is not unique. We can define multiple bases for the same space.

Example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for R^2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also a basis for R^2.

Dimension of a vector space:

Dimension of a vector space is the number of vectors in one of its bases.

All bases of a vector space contain the same number of vectors; this number is the dimension of that space.

Examples: R^2 is 2-dimensional space.

$c\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a 1-dimensional space. Every basis of it contains exactly one vector.

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a basis for $c\begin{bmatrix}1\\0\\0\end{bmatrix}$, So is $\begin{bmatrix}-1\\0\\0\end{bmatrix}$.

Example: Column space of a rank-one matrix $A = uv^T$ is 1-dimensional. Every basis of $C(A)$ contains only one vector (which is a scalar times u).

Matrix spaces and their dimensions:

Think of all the matrices of size 3×2. This is a matrix space: we can add matrices with size 3×2 and scalar-multiply each with real numbers:

$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$ A basis for this space would be:

$$\left\{ \begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix} \right\}$$

The dimension of this matrix space is $3 \times 2 = 6$.

Example:

The dimension of all the $n \times n$ upper triangular matrices is:

$\frac{n \times (n+1)}{2}$ (# max non-zero entries in an upper triangular matrix)

Four subspaces related to matrix A

1. Column space of A: $C(A)$ (All linear combinations of its columns)
2. Null space of A: $N(A)$ (All solutions to $Ax=0$)
3. Row space of A: $C(A^T)$ (all linear combinations of columns of A^T)
4. Left null space: $N(A^T)$ (all solutions to $A^Ty=0$).

If A: $m \times n$, then

$C(A)$ is a subspace of R^m
$N(A)$ is a subspace of R^n
$C(A^T)$ is a subspace of R^n
$N(A^T)$ is a subspace of R^m.

!!! The row space and column space of A have the same dimension r, which is the rank of A.

Then,

$N(A)$ has dimension $n-r$.
$N(A^T)$ has dimension $m-r$. !!!

SUPER IMPORTANT!

Example:

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Row space has dimension 2.
   (Two "independent" equations)
2. Column space has dimension 2.
   (Two pivot columns)
3. Null space has dimension $n-r = 5-2 = 3$.
   (Three special solutions that span $N(A)$)
   (Three free variable columns).
4. Null space of R^T has dimension $m-r = 3-2 = 1$
   (One zero row).

Fundamental theorem of linear algebra:

Part 1:

Column and row spaces have dimension r.

Null spaces have dimension $\underset{N(A)}{\underset{\uparrow}{n-r}}$ and $\underset{N(A^T)}{\underset{\uparrow}{m-r}}$.

Part 2: (chapter 4).

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$$

$$A = LR = \begin{bmatrix} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \end{bmatrix}$$

$$= (\text{col } 1)(r_1) + (\text{col } 2)(r_2)$$

A has rank: $1 + 1 = 2$.

Again: The number of LI columns of A is equal to the number of LI rows of A, which is the rank of A, which is also the dimension of its column space, which is also the dimension of its row space.

$$\# \text{ LI columns} = \# \text{ LI rows} = \text{rank}(A) = \dim(C(A)) = \dim(C(A^T))$$

rank of A.

dimension of $C(A)$

dimension of $C(A^T)$