**MATH 250 (1,2), Fall 2010, (Dr. Z.) , Practice Problems for Exam 2**

**1.** (a) Find an $LU$ decomposition of the matrix

$$\begin{bmatrix} -1 & 2 & 1 & -1 & 3 \\ 1 & -4 & 0 & 5 & -5 \\ -2 & 6 & -1 & -5 & 7 \\ -1 & -4 & 4 & 11 & -2 \end{bmatrix}$$

(b) Use the results of part (a) to solve the system

$$-x_1 + 2x_2 + x_3 - x_4 + 3x_5 = 7$$

$$x_1 - 4x_2 \quad + 5x_4 - 5x_5 = -7$$

$$-2x_1 + 6x_2 - x_3 - 5x_4 + 7x_5 = 6$$

$$-x_1 - 4x_2 + 4x_3 + 11x_4 - 2x_5 = 11$$

**2.** Let $A$ be a $4 \times 4$ matrix with row vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$- that is with

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

and determinant equal to 5. Find

(a) The determinant of the matrix

$$\begin{bmatrix} 3\mathbf{a} + \mathbf{b} + \mathbf{d} \\ \mathbf{b} \\ \mathbf{c} + \mathbf{d} \\ \mathbf{d} \end{bmatrix} \quad .$$

(b) The determinant of the matrix $AC^2AC$ where

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 11 \\ 0 & 0 & 1 & -17 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

**3.** Compute the determinant by using elementary row operations (no credit for other methods)

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & -1 \\ -8 & 10 & -20 \end{bmatrix}$$

**4.** (a) Suppose that $A$ and $B$ are $4 \times 5$ matrices and that $B$ is obtained from $A$ by the elementary row operations given below. In each case give an elementary matrix $E$ such that $B = EA$
(i) $\mathbf{r_1} \leftrightarrow \mathbf{r_4}$

(ii) $\mathbf{r_3} + 3\mathbf{r_2} \rightarrow \mathbf{r}_3$

(b) Give the inverses of the elementary matrices found in (i) and (ii).

**5.** For what values of $d$ is the given matrix **not** invertible.

$$\begin{bmatrix} -d & 1 & 1 \\ d & -2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

**6.** A certain $3 \times 3$ matrix has reduced row echelon form

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

Find explicitly a non-trivial linear relation on the columns of $A$, that is a relation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3} = \mathbf{0}$, with $c_1, c_2, c_3$ not all zero.

**7.** Explain why the following set is a subspace of $R^5$ and find a basis for it.

$$\left\{ \begin{bmatrix} r+s+2t \\ r-s \\ 3r+2s+5t \\ -2r+3s+t \\ r-s-t \end{bmatrix} \in R^5 : r, s, \ and \ t \ are \ scalars \right\}$$

**8.** Let $\mathbf{v}$ be a non-zero vector in $R^2$, and let $A = \mathbf{v}\mathbf{v}^T$ ($A$ is a a $2 \times 2$ natrix.)

(a) Show that $\mathbf{v}$ is an eigenvector of $A$. What is the eigenvalue?

(b) What is the rank of $A$? What is the other eigenvalue of A?

**9.** Explain why the following sets in $R^4$ are **not** subspaces
(a)

$$\left\{ \begin{bmatrix} s \\ 2s \\ 3s \\ 5s \end{bmatrix} \in R^4 : s \geq 0 \right\}$$

(b)

$$\left\{ \begin{bmatrix} 1+t \\ 2+t \\ 3+t \\ 10-t \end{bmatrix} \in R^4 : t\, is\ a\ scalar \right\}$$

**10.** The matrix

$$A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & -6 & 9 \\ -5 & -1 & 5 \end{bmatrix}$$

has characteristic polynomial $-(t+2)^2(t-5)$.
(a) Find the eigenvalues of $A$ and the multiplicities of each.

(b) For each eigenvalue found above, give a basis for the corresponding eigenspace.

**11.** Let

$$V = \left\{ \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in R^3 : \mu_1 = 0 \right\}$$

$$W = \left\{ \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in R^3 : \mu_2 = 0 \right\}$$

(a) Prove (using the definition of subspace) that $V$ is a subspace of $R^3$ and that $W$ is a subspace of $R^3$.

(b) Show that $V \cup W$ is **not** a subspace of $R^3$.

**12.**
(a)Let $A$ and $B$ be $n \times n$ matrices such that $B$ is invertible. Prove that $\det(B^{-1}AB) = \det A$.
(b)An $n \times n$ matrix $Q$ is called orthogonal if $Q^T Q = I_n$. Prove that if $Q$ is orthogonal, then $\det Q = \pm 1$.

**13.** Explain why

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} , \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} , \begin{bmatrix} 5 \\ -11 \\ 4 \end{bmatrix} \right\}$$

is not a basis for $R^3$.

**14.** Let $A$ be the matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} .$$

Show that $\det A = (b-a)(c-a)(c-b)$.

**15.** Determine the dimensions of (a) $Col\, A$ (b) $Null\, A$ (c) $Row\, A$ and (d) $Null\, A^T$, if

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & -2 & 2 & -2 \\ 2 & 3 & 0 & 3 \end{bmatrix} \quad .$$

**16.** Show that each set is not a subspace of the appropriate $R^n$.

(a)

$$\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2 : u_1 u_2 = 0 \right\} \quad .$$

(b)

$$\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2 : u_1^2 + u_2^2 \leq 1 \right\} \quad .$$

**17.** Classify each statement as true or false and give a brief justification of your answer.

(a)There are some subspaces of $R^n$ that do not contain $\mathbf{0}$.

(b) A vector $\mathbf{v}$ is in $Col\, A$ if and only if $A\mathbf{v} = \mathbf{0}$.

(c) A vector $\mathbf{v}$ is in $Row\, A$ if and only if $A^T\mathbf{x} = \mathbf{v}$ is consistent.

(d) Every subspace of $R^n$ has a basis.

**18.** Classify each statement as true or false and give a brief justification of your answer.

(a) If $A\mathbf{x} = \mathbf{0}$ has a unique solution then the nullspace of $A$ is non-empty.

(b) If $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ belongs to a subspace $W$ of $R^n$ then $5\mathbf{u_1} + 3\mathbf{u_2} - \mathbf{u_3}$ also belongs to $W$.

(c) A square matrix is invertible if and only of $\det A \neq 0$ .

(d) If $A$ is a $5 \times 2$ matrix, then the nullspace of $A$ is never $\{\mathbf{0}\}$.

**19.** Show that

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\} \quad ,$$

is a basis for the subspace

$$V = \left\{ \begin{bmatrix} 4t \\ s+t \\ -3s+t \end{bmatrix} \in R^3 : s \textit{ and } t \textit{ are scalars} \right\} \quad .$$

**20.** Prove that if $\lambda$ is an eigenvalue of the matrix $A$, then $\lambda^5$ is an eigenvalue of the matrix $A^5$.

**21**. Let

$$A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & -2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

(i) Find a basis for the column space of $A$

(ii) Find a basis for the null space of $A$.

**22.** Compute the determinant of

$$\begin{bmatrix} 1 & 1 & x \\ -1 & 7 & y \\ 2 & 8 & z \end{bmatrix}$$

by cofactor expansion along the third column.

**23.** Use Cramer's rule (no credit for other methods!) to solve the following system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 \quad\quad + x_3 = 3$$

$$x_1 + x_2 - x_3 = 1 \quad .$$

**24.** Let

$$A = \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} :$$

Compute:

(a) $\det(ABA^3B^8)$

(b) $\det(AB^{-3}A^4B^{-10})$

**25.** Find a basis for the following subspace

$$Span\left\{\begin{bmatrix}4\\6\\-10\end{bmatrix} , \begin{bmatrix}8\\-12\\20\end{bmatrix} , \begin{bmatrix}1\\0\\-2\end{bmatrix} , \begin{bmatrix}0\\8\\-4\end{bmatrix} , \begin{bmatrix}21\\6\\0\end{bmatrix}\right\}$$

**26.** Explain why

$$\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix} , \begin{bmatrix}2\\-3\\1\end{bmatrix}\right\}$$

is not a generating set for $R^3$.

**27.** Find the eigenvalues of the following matrix, and determine a basis for each eigenspace.

$$A = \begin{bmatrix}1 & 0 & 0 & 0\\ 9 & -2 & -3 & 3\\ -6 & 0 & 1 & -3\\ -6 & 0 & 0 & -2\end{bmatrix}$$

**28.** Show that

$$\begin{bmatrix}4 & -5\\ 3 & -2\end{bmatrix}$$

has no real eigenvalues.

**29.** Classify each statement as true or false and give a brief justification of your answer.

(a) If $A$ is an $m \times n$ matrix then $dim\, Null\, A + dim\, Col\, A = m$

(b) The rank of a matrix $A$ is equal to the rank of $A^T$.

(c) If $A$ is a $10 \times 10$ matrix and $rank\ A = 7$ then 0 is a root of the characteristic polynomial of $A$.

(d) If $\mathbf{v}$ is an eigenvector of a matrix, then there is a unique eigenvalue of the matrix corresponding to $\mathbf{v}$.

**30.** Let $A$ be a certain $3\times 3$ matrix, and you know that $A^4 = O$, where $O$ is the zero-matrix. Show that the only eigenvalue of $A$ is 0.

**31.** Classify each statement as true or false and gibe a brief justification of your answer.

(a) If $\mathbf{v}$ is an eigenvector of matrix $A$ then $c\mathbf{v}$ is also an eigenvector for any scalar $c$.

(b) A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if the equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a non-zero solution.

(c) The rank of any matrix equals the dimension of its row space.

(d) If $\mathbf{v}$ is an eigenvector of matrix $A$ then $c\mathbf{v}$ is also an eigenvector for any non-zero scalar $c$.

**32.** Determine wheter the matrix is invertible, and if it is, find its inverse:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$