Northeastern University

DS5020 - Introduction to Linear Algebra and Probability for Data Science

Midterm - Spring 2018

1) **{10 pts}** Let V be a vector space of all 2×2 matrices with real entries and W be a subset of all matrices X satisfying both of the two condition $X = X^T$ and $tr(X) = 0$.
   (a) Show that W is a subspace of V.
   (b) Find a basis and the dimension of W.

2) **{10 pts}** Let

$$A = \begin{bmatrix} 5 & -4 & 1 & 8 \\ 1 & -2 & 1 & 2 \\ 3 & 0 & -1 & 4 \end{bmatrix}$$

   (a) Find the basis for row space and column space of A
   (b) Find the basis of left and right null space of A.
   (c) Let $\boldsymbol{b} = \begin{bmatrix} c \\ 8 \\ 6 \end{bmatrix}$. For what value or values of c will $Ax = b$ will be consistent?

3) **{10 pts}**Find the least squares solution of
   a. Intersection of the three lines.

$$\begin{Bmatrix} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{Bmatrix}$$

   b. Intersection of the three planes.

$$\begin{Bmatrix} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{Bmatrix}$$

4) **{10 pts}** Let A be a $n \times n$ real skew symmetric matrix ($A^T = -A$).

   a. Prove that the matrices $I - A$ and $I + A$ are non-singular.

   b. Prove that $B = (I - A)(I + A)^{-1}$ is an orthonormal matrix.

   c. Solve the equations $\begin{Bmatrix} 2x + 3y - z = 1 \\ 4x + y - 3z = 11 \\ 3x - 2y + 5z = 21 \end{Bmatrix}$ using Cramer's rule.

5) **{10 pts}** Consider the system of differential equations

$$\frac{dx_1(t)}{dt} = 2x_1(t) - x_2(t) - x_3(t)$$
$$\frac{dx_2(t)}{dt} = -x_1(t) + 2x_2(t) - x_3(t)$$
$$\frac{dx_1(t)}{dt} = -x_1(t) - x_2(t) + 2x_3(t)$$

   a. Express the system in a matrix form.

   b. Find the general solution of the system.

   c. Find the solution of the system with the initial value $x_1 = 0, x_2 = 1, x_3 = 5$.

6) **{10 pts}** Find SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

   a. Find eigenvalues and eigenvectors of $A^T A$.

   b. Form the diagonal matrix Σ using singular values.

   c. Find the eigenvectors of AA^T.

   d. Form the orthonormal vectors U and V using a, b and c above. If needed complete the matrices to squares by adding orthonormal vectors from null spaces A^T and A.

   e. Verify $A = U\Sigma V^T$.