Northeastern University

DS5020 - Introduction to Linear Algebra and Probability for Data Science

Assignment 1 Solutions

S1. {10 pts} Suppose $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\-1\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}1\\7\end{bmatrix}$.

a. Find the linear combination $3\mathbf{u} + 2\mathbf{v}$.

$$3u+2v= 3\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} + \begin{bmatrix}4\\-2\end{bmatrix} = \begin{bmatrix}7\\4\end{bmatrix}$$

b. Find the linear combination $\mathbf{u} - \mathbf{v}$.

$$u-v= \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1-2\\2+1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$

c. Find two scalars c and d that satisfy $c\mathbf{u} + d\mathbf{v} = \mathbf{w}$.

$$cu+dv = c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}c+2d\\2c-d\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix} = w \Rightarrow \begin{array}{r} 2c-d=7 \\ -\ 2c+4d=2 \\ \hline -5d=5 \\ d=-1 \end{array} \Rightarrow \boxed{\begin{array}{l} d=-1 \\ c=3 \end{array}}$$

d. Describe geometrically (line, plane etc.) all linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

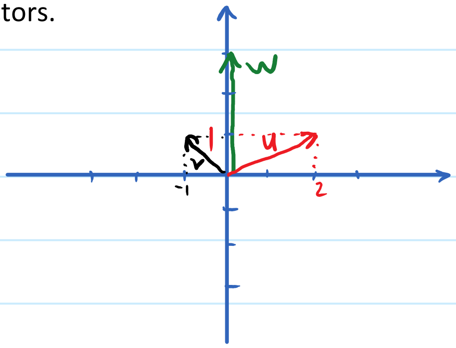
$cu+dv \quad c,d \in \mathbb{R}$ represents all the points in $\underline{\text{a plane}}$.

To show this, assume we would like to get the point $\begin{bmatrix}b_1\\b_2\end{bmatrix}$:

$$cu+dv = \begin{bmatrix}c+2d\\2c-d\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \Rightarrow -5d = b_2 - 2b_1, \quad d = \frac{2b_1 - b_2}{5}, \quad c = \frac{b_1 + 2b_2}{5}$$

S2. {10 pts} Suppose $\mathbf{u} = \begin{bmatrix}2\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-1\\1\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}0\\3\end{bmatrix}$.

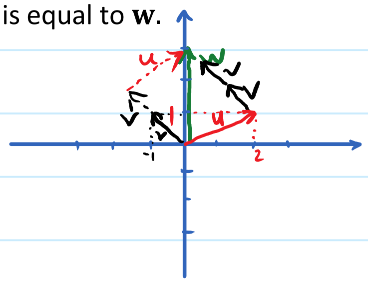
a. Draw vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ on a plane where each axis represents one component of the vectors.



b. Find two scalars c and d that satisfy $c\mathbf{u} + d\mathbf{v} = \mathbf{w}$.

$$cu+dv= c\begin{bmatrix}2\\1\end{bmatrix} + d\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2c-d\\c+d\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix} \quad \begin{array}{r} c+d=3 \\ -\ (2c-d)\times 1/2 = 0\times 1/2 \\ \hline 3d/2 = 3 \end{array} \quad \boxed{d=2,\ c=1}$$

c. Draw $\mathbf{u} + 2\mathbf{v}$ on the same plane you drew on part **a**. Show that this linear combination of $\mathbf{u}$ and $\mathbf{v}$ is equal to $\mathbf{w}$.



$$u+2v= \begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}0\\3\end{bmatrix} = w$$

S3. {20 pts} Suppose $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

a. Find the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$.

$$u.v = u_1 v_1 + u_2 v_2 = 2 \times 1 + -1 \times 2 = 0$$

b. Are vectors $\mathbf{u}$ and $\mathbf{v}$ perpendicular? What is the angle between them?

Yes, u & v are perpendicular since their dot product is 0.
The angle between them is 90° (or $\pi/2$).

c. Find the dot product of vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v} \cdot \mathbf{w}$.

$$v.w = v_1 w_1 + v_2 w_2 = 1 \times 2 + 2 \times 4 = 10$$

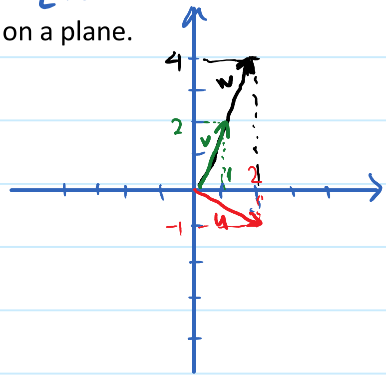
d. Using the cosine formula, find the angle between vectors $\mathbf{v}$ and $\mathbf{w}$.

$$\cos\theta = \frac{|v.w|}{\|v\| \ \|w\|} = \frac{10}{\sqrt{1^2+2^2}\ \sqrt{2^2+4^2}} = \frac{10}{\sqrt{100}} = 1 \Rightarrow \theta = 0$$

e. Find scalar c that satisfies $c\mathbf{v} = \mathbf{w}$.

$$cv = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow c = 2$$

f. Draw vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ on a plane.



S4. {15 pts} Suppose $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

a. Describe the columns and rows of A.

column 1 = $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ row 1 = $\begin{bmatrix} -1 & 2 \end{bmatrix}$

column 2 = $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ row 2 = $\begin{bmatrix} 0 & 1 \end{bmatrix}$

b. Find vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that satisfies $\mathbf{Ax} = \mathbf{b}$.

$$Ax = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = 2 \\ x_1 = 3 \end{matrix} \qquad x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

c. Write down the system of equations that the matrix form $\mathbf{Ax} = \mathbf{b}$ represents.

$$Ax = b \Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} \rightarrow \text{Eqn 1: } -x_1 + 2x_2 = 1 \\ \rightarrow \text{Eqn 2: } x_2 = 2 \end{matrix}$$

S5. {25 pts} Suppose $\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{y} = [1\ 1\ 0]$.

a. Compute **yA**. Describe in words what the result is in terms of rows of **A**.

$$yA = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \end{bmatrix}$$

(Sum of row1 and row2)

b. Compute **Ax**. Describe in words what the result is in terms of columns of **A**.

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \\ 26 \end{bmatrix}$$

(2x column3 + column 2)

c. Compute **EA**. Describe in words what this operation does to rows of **A**.

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Changed row1 to row 2.
Subtracted row 1 from row2, put result in row2.

d. Compute **PA**. Describe in words what this operation does to rows of **A**.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

Exchanged row 2 and row3.
(Permuted the rows).

e. Compute **AP**. Describe in words what this operation does to columns of **A**. Is the result the same as the result in part **d**?

$$AP = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}$$

Exchanged column 2 and column 3.
(Permuted the columns).

Comparing results in part d. and e., we see that $PA \neq AP$

S6. {20 pts} Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 5 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

a. Using elimination, convert matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 5 & 3 \end{bmatrix}$ to upper triangular matrix $\mathbf{U} = \begin{bmatrix} ? & ? & ? \\ 0 & ? & ? \\ 0 & 0 & ? \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 5 & 3 \end{bmatrix} \xrightarrow{E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 5 & 3 \end{bmatrix} \xrightarrow{E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix} \text{Cont's}$$

$$\xrightarrow{E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}}_{U}$$

$$E_{32} E_{31} E_{21} A = U$$

b. Find vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $\mathbf{Ax} = \mathbf{b}$.

We will perform the same elimination steps; this time on augmented matrix $[A \ \ b]$. This performs the same operations on the right hand side of equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 4 & 1 \\ 1 & 5 & 3 & -1 \end{array}\right] \xrightarrow{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 5 & 3 & -1 \end{array}\right] \xrightarrow{E_{31}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 2 & -1 \end{array}\right] \text{Cont's} \rightarrow$$

$$\left[\underbrace{\begin{matrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 5 & 3 \end{matrix}}_{A} \ \middle| \ \underbrace{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}}_{b}\right]$$

$$\xrightarrow{E_{32}} \left[\underbrace{\begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{matrix}}_{U} \ \middle| \ \underbrace{\begin{matrix} 0 \\ 1 \\ -4 \end{matrix}}_{\text{right hand side after elimination}}\right] \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \leftarrow \text{updated equations after elimination}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 1 \\ -4x_3 &= -4 \end{aligned}\right\} \quad \begin{aligned} x_3 &= 1 \\ x_2 &= -1 \\ x_1 &= 1. \end{aligned}$$

$$\Rightarrow \ x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$