Confidence Intervel

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Review: Point Estimators

- Most probability distributions are indexed by one or more **parameters**. For example, $N(\mu, \sigma^2)$.
- In hypothesis tests, we have used **point estimators** for parameters. For example, consider an i.i.d. sample $D_1, D_2, \ldots, D_n \sim_{\text{i.i.d}} N(\mu_D, \sigma_D^2)$. Let

$$\bar{D} = \frac{1}{n} \sum_{i=1}^{n} D_n, \quad S_D^2 = \frac{1}{n-1} \sum_{i=1}^{n} (D_n - \bar{D})^2.$$

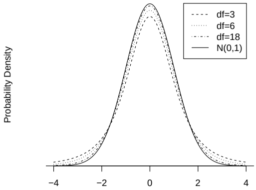
  Then $\bar{D}$ is a point estimator of μ_D and S_D^2 is a point estimator of σ^2.
- We know that $E(\bar{D}) = \mu_D$ and $E(S_D^2) = \sigma_D^2$.
- That is, $\bar{D}$ is an **unbiased estimator** of μ_D and S_D^2 is an unbiased estimator of σ_D^2.

Interval Estimators

- Now we turn to **interval estimators** to give a reasonable interval for parameters.
  - For μ: $[a_1, a_2]$ for some constants a_1, a_2 based on data
  - For σ^2: $[b_1, b_2]$ for some constants b_1, b_2 based on data
- The assumptions are the same as in hypothesis testing, but we do not need a null hypothesis about the parameters (e.g. $\mu_D = \mu_1 - \mu_2 = 0$).

Confidence Interval for μ_D

- Suppose $D_1, D_2, \ldots, D_n$ is an i.i.d. sample from $N(\mu_D, \sigma_D^2)$ and σ_D^2 is unknown.
- Note that
$$\frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} \sim T_{n-1}.$$
- Let $t_{n-1,\alpha/2}$ denote the t critical value such that
$$P(-t_{n-1,\alpha/2} \leq T_{n-1} \leq t_{n-1,\alpha/2}) = 1 - \alpha.$$



Confidence Interval for $\mu_D = \mu_1 - \mu_2$

- Then we have

$$1 - \alpha = P\left(\mu_D \in \left[\bar{D} - t_{n-1,\alpha/2}\frac{S_D}{\sqrt{n}},\ \bar{D} + t_{n-1,\alpha/2}\frac{S_D}{\sqrt{n}}\right]\right)$$

- A $(1 - \alpha)$ CI for μ_D is

$$\mu_D \in \left[\bar{d} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}},\ \bar{d} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}\right]$$

  or

$$\bar{d} \pm t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$$

- The half width of this CI is $t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$.
- The width of this CI is $2 \times t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$.

Confidence Intervals for $\mu_D = \mu_1 - \mu_2$

- A $(1-\alpha)$ CI for μ_D is

$$\mu_D \in \left[\bar{d} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}},\ \bar{d} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}\right]$$

- In the lake clarity 1980 vs. 1990 example, a 95% CI for μ_D is

$$0.497 - 2.080 \times \frac{0.435}{\sqrt{22}} \le \mu_D \le 0.497 + 2.080 \times \frac{0.435}{\sqrt{22}}$$

which is [0.30, 0.69] or 0.497 ± 0.195.

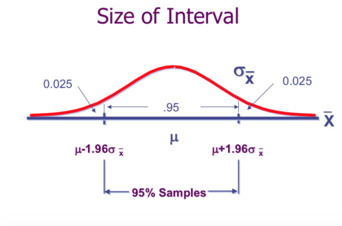
Remarks

- By convention, CIs are two-sided. But one-sided confidence bounds are possible.
- It is not true that $P(0.30 \leq \mu_D \leq 0.69) = 0.95$. Why not? because

  once a sample is observed, there is nothing random.
- The 0.95 probability concerns with the repeated random sampling. It is interpreted as, 95% of the time, the (random) CIs calculated in this way contains (fixed) μ_D.
- For a single case, it is interpreted as ? having 95% confidence that μ_D is between 0.30 m and 0.69 m.
- The interval $[0.30, 0.69]$ (or 0.497 ± 0.195) can be thought of as a plausible range of μ_D.
- What are the assumptions made when we perform a paired T test or construct a corresponding confidence interval for μ?

Two-sided Confidence Interval

- Based on Z-statistic: $(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$

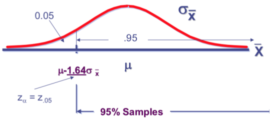


- Based on T-statistic: $(\bar{X} - t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}},\ \bar{X} + t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}})$

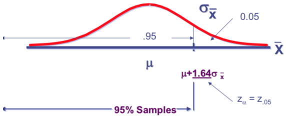
One-sided Confidence Interval

- Based on Z-statistic:
  - Lower interval: $(-\infty,\ \underbrace{\bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}}_{\text{Upper bound}})$
  - Upper interval: $(\underbrace{\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}}}_{\text{Lower bound}},\ \infty)$

Upper Interval



Lower Interval



- Based on T-statistic:
  - Lower interval: $(-\infty,\ \underbrace{\bar{X} + t_{n-1,\alpha} \frac{\hat{\sigma}}{\sqrt{n}}}_{\text{Upper bound}})$
  - Upper interval: $(\underbrace{\bar{X} - t_{n-1,\alpha} \frac{\hat{\sigma}}{\sqrt{n}}}_{\text{Lower bound}},\ \infty)$

Example

Problem: Suppose the mean of an i.i.d. sample of $n = 100$ is $\bar{x} = 50$ with sample standard deviation 10. Set up an upper 95%-CI estimate for the population mean μ.

Answer: Assume the observation $X_i \sim_{\text{i.i.d.}} N(\mu, \sigma^2)$ for all $i = 1, \ldots, 100$. Since σ is unknown, we consider the T-statistic. Note that $t_{99,0.05} = 1.66$ and $\hat{\sigma} = 10$. So the 95%-CI for μ is

$$(\bar{x} - t_{99,0.05} * \frac{\hat{\sigma}}{\sqrt{n}}, \infty) = (50 - 1.66 * \frac{10}{\sqrt{100}},\ \infty) = (48.34,\ \infty).$$

Connection between hypothesis testing and confidence interval

Distribution of Test Statistics

Let $(X_1, X_2, ..., X_n)$ be an i.i.d. sample drawn from a population $N(\mu, \sigma^2)$.

- If μ is unknown, σ is known, then
  - Sample Mean:
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$
  - Sample Variance:
$$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$$
- If both μ and σ are unknown, then

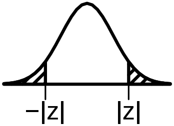
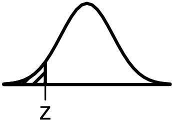
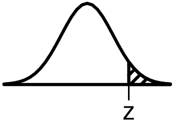
$$T = \frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} \sim T_{n-1}$$

.

Summary: Hypothesis Testing on Population Mean

If σ is known, z-statistics: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

significance level	α	0.1	0.05	0.01
	$1-\alpha$	90%	95%	99%
N(0,1)	$z^*_{\alpha/2}$	1.64	1.96	2.58

	Two-Sided	Lower One-Sided	Upper One-Sided
H_0	$\mu = \mu_0$	$\mu = \mu_0$	$\mu = \mu_0$
H_1	$\mu \neq \mu_0$	$\mu < \mu_0$	$\mu > \mu_0$
Test Statistic	$z = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$		
P-value	$P(\lvert\text{Norm(0,1)}\rvert > \lvert z\rvert)$ 	$P(\text{Norm(0,1)} < z)$ 	$P(\text{Norm(0,1)} > z)$ 
Accept H_0 w/ significance level α	$\lvert z\rvert < z^*_{\alpha/2}$ or equivalently $\lvert\bar{x}-\mu_0\rvert < z^*_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$z > -z^*_{\alpha}$ or equivalently $\bar{x}-\mu_0 > -z^*_{\alpha}\frac{\sigma}{\sqrt{n}}$	$z < z^*_{\alpha}$ or equivalently $\bar{x}-\mu_0 < z^*_{\alpha}\frac{\sigma}{\sqrt{n}}$

Summary: Hypothesis Testing on Population Mean

If σ is unknown, t-statistics: $t = \frac{\bar{x}-\mu_0}{\hat{\sigma}/\sqrt{n}} \sim T_{n-1}$

significance level	α	0.1	0.05	0.01
	$1-\alpha$	90%	95%	99%
T_{n-1}	$t^*_{n-1,\alpha/2}$	qt(α/2, n-1)		

<table>
<tr><th></th><th>Two-Sided</th><th>Lower One-Sided</th><th>Upper One-Sided</th></tr>
<tr><td>H$_0$</td><td>$\mu = \mu_0$</td><td>$\mu = \mu_0$</td><td>$\mu = \mu_0$</td></tr>
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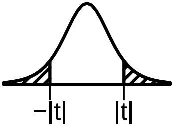
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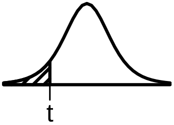
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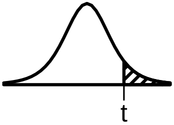
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or equivalently
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</table>

Confidence Interval (Variance Known)

Parameter of interest: population mean μ

- When σ^2 known: z-test
  - Statistics

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

  - 95% **Confidence Interval (CI)**

$$\mu \in (\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \quad , \quad \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$$

  - $(1-\alpha)$-**Confidence Interval (CI)**:

$$\mu \in (\bar{x} - z^*_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad , \quad \bar{x} + z^*_{\alpha/2}\frac{\sigma}{\sqrt{n}})$$

  - $z^*_{\alpha/2}$ is called critical value at level $\alpha/2$.

significance level	α	0.1	0.05	0.01
	$1-\alpha$	90%	95%	90%
N(0,1)	$z^*_{\alpha/2}$	1.64	1.96	2.58

Confidence Interval (Variance Unknown)

Parameter of interest: population mean μ

- When σ^2 is unknown: t-test
  - Statistics
    $$t = \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$
  - $(1-\alpha)$-**Confidence Interval (CI)**
    $$\mu \in (\bar{x} - t^*_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \quad , \quad \bar{x} + t^*_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}})$$
  - $t^*_{n-1,\alpha/2}$ is called critical value at level $\alpha/2$.

    In R: qt(...,df=n-1).

Margin of Error & Sample Size & Confidence Level

$$\underbrace{\bar{x}}_{\text{estimate}} \pm \underbrace{z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{margin of error}} \quad \text{or} \quad \underbrace{\bar{x}}_{\text{estimate}} \pm \underbrace{t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}}_{\text{margin of error}}$$

The size of the margin of error can be reduced if

- confidence level is smaller (e.g. 95% $\rightarrow$ 90%);
- sample size n is larger;
- or if σ is smaller

We usually prefer **shorter** Confidence Interval.

Duality of Confidence Intervals and Hypothesis Tests

$$\boxed{\text{In a two sided test, } H_0 : \mu = \mu_0 \text{ is not rejected at level } \alpha}$$

if and only if

$$\boxed{\mu_0 \text{ is in the } (1-\alpha) \text{ CI for } \mu}$$

Proof: In a two sided z-test, $H_0 : \mu = \mu_0$ is not rejected if

$$
\begin{aligned}
|\bar{x} - \mu_0| \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}} &\iff -z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x} - \mu_0 \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \\
&\iff \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}
\end{aligned}
$$

Confidence Interval

In general, $(1-\alpha)$-CI for population parameter θ can be calculated from the test statistic for θ.

- Find the test statistic for θ and its null distribution;
- Find the critical value at level $\alpha/2$ (if two sided test) based on null distribution, say $c^*_{\alpha/2}$;
- Write the $(1-\alpha)$-CI in the form of

$$\text{estimate} \pm \text{margin of error}$$

  where the margin of error usually is the $c^*_{\alpha/2}\times$denominator in test statistics.

Comparison of Two Population Means: Paired T Test

- Parameter of interest: $\mu_1 - \mu_2$
- Data: $D_1 = y_1 - y_2, \ldots, D_n = y_1 - y_n$
- Paired two-sample inference:
  - Hypothesis testing $H_0 : \mu_D = \mu_D^0$

$$T = \frac{\bar{D} - \mu_D^0}{S_D/\sqrt{n}} \sim T_{n-1}, \text{ where } S_D = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2},$$

  - $(1-\alpha)$ CI for $\mu_D = \mu_1 - \mu_2$:

$$\bar{d} \pm t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$$

Comparison of Two Population Means: Independent Two Sample T Test

- Independent two-sample inference assuming $\sigma_1^2 = \sigma_2^2$:
  - Hypothesis testing $H_0 : \mu_1 - \mu_2 = \mu_D^0$

$$T = \frac{\bar{Y}_1 - \bar{Y}_2 - \mu_D^0}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim T_{n_1+n_2-2},$$

    where $S_p^2 = n_1 + n_2 - 2\sum_{i=1}^{n_1}(Y_{1i} - \bar{Y}_1)^2 + \sum_{i=1}^{n_2}(Y_{2i} - \bar{Y}_2)$.
  - $(1-\alpha)$ CI for $\mu_1 - \mu_2$:

$$\bar{y}_1 - \bar{y}_2 \pm t_{n_1+n_2-2,\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$