- Define
$$t_i = \frac{y_i - \hat{y}_{(i)}}{\hat{\sigma}_{(i)} \left(1 + x_i^T \left(X_{(i)}^T X_{(i)}\right)^{-1} x_i\right)^{1/2}},$$
where $X_{(i)}$ represents the design matrix deleting i-th obesrvation.
  - It can be proved $t_i = \frac{\hat{\epsilon}_i}{\hat{\sigma}_{(i)}\sqrt{1-h_{ii}}} = r_i\sqrt{\frac{(n-p-1)}{n-p-r_i^2}}$. (Easy-to-compute. Proof see Theorem 10.1 in Lee and Seber.)
  - If i-th case is not outlier, model is correct, and $\epsilon \sim N(0, \sigma^2 I_n)$, $t_i \sim t_{(n-1)-p}$, where $n-1$ is the sample size.
- Test outliers
  - Practically, $|t_i| > 3$ can imply possible outliers.
  - If we want a level α test,
    - $P(\text{all tests accept}) = 1 - P(\text{at least one rejects}) \geqslant 1 - \sum_i P(\text{test } i \text{ rejects }) = 1 - n\alpha$.
    - Each test should use level α/n. (Bonferroni correction.)

2.3 Influential point

- An influential point is one whose removal from the dataset would cause a large change in the fit.
  - An influential point may or may not be an outlier,
  - and may or may not have large leverage,
  - but it will tend to have at least one of these two properties.
- Measure of influence: Cook's distance statistic Cook (1977)
$$D_i = \frac{\left(\hat{y} - \hat{y}_{(i)}\right)^T \left(\hat{y} - \hat{y}_{(i)}\right)}{p\hat{\sigma}^2}$$
  - It can also be computed as $D_i = \frac{1}{p} r_i^2 \frac{h_{ii}}{1-h_{ii}}$, where r_i represents i-th standardized residual.

```
cook <- cooks.distance(lmod)
summary(lmod)$coefficients
```

```
##                  Estimate  Std. Error      t value     Pr(>|t|)
## (Intercept)  7.068220709 19.15419782  0.369016796 7.153508e-01
## Area        -0.023938338  0.02242235 -1.067610554 2.963180e-01
## Elevation    0.319464761  0.05366280  5.953187968 3.823409e-06
## Scruz       -0.240524230  0.21540225 -1.116628222 2.752082e-01
## Nearest      0.009143961  1.05413595  0.008674366 9.931506e-01
## Adjacent    -0.074804832  0.01770019 -4.226216850 2.970655e-04
```

```
summary(lmod)$r.squared
```

```
## [1] 0.7658469
```

```
lmodi <- lm(Species ~ Area + Elevation + Scruz + Nearest + Adjacent,
            gala, subset = (cook<max(cook)))
summary(lmodi)$coefficients
```

```
##                Estimate  Std. Error    t value     Pr(>|t|)
## (Intercept) 22.58614473 13.40191356  1.6852925 1.054542e-01
## Area         0.29574351  0.06186188  4.7807068 8.042013e-05
## Elevation    0.14039023  0.04970484  2.8244782 9.613092e-03
## Scruz       -0.09010457  0.14979821 -0.6015063 5.533860e-01
## Nearest     -0.25518223  0.72167754 -0.3535959 7.268624e-01
## Adjacent    -0.06503051  0.01222732 -5.3184596 2.124483e-05
```

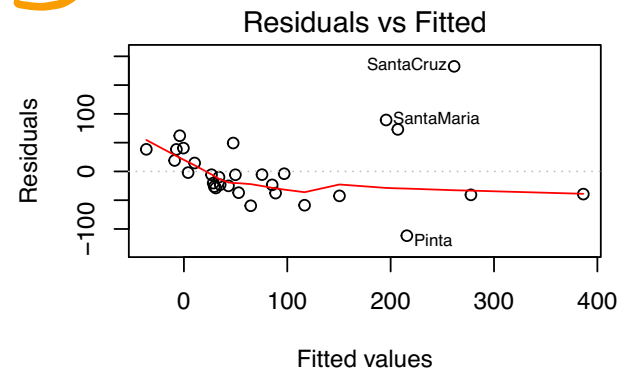
```
summary(lmodi)$r.squared
```

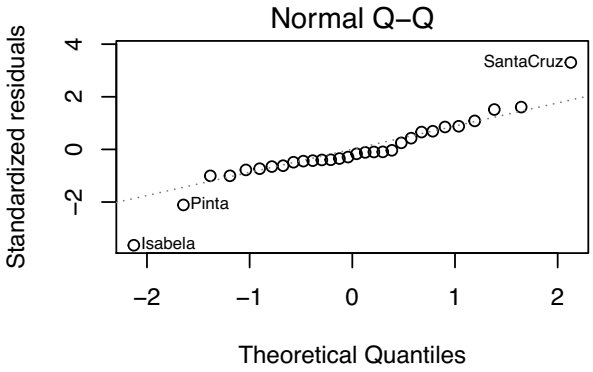
```
## [1] 0.8714011
```

- p-value of covariate Area changes significantly.

- We usually do not want estimates to be so sensitive to the presence/deletion of just one observation.

```
par(mfrow=c(2,2))
plot(lmod, which = c(1,2,4, 5), cook.levels = 1) #R codes for multiple diagnostic plots
```

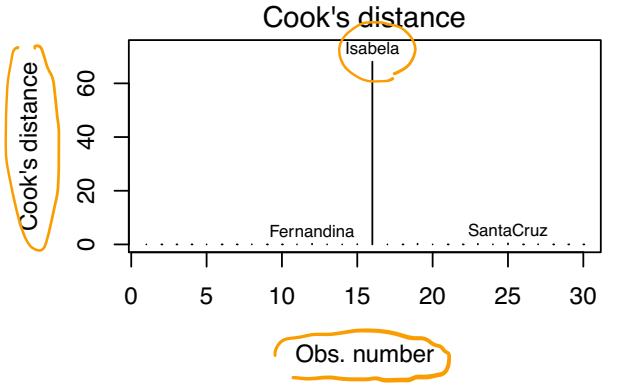


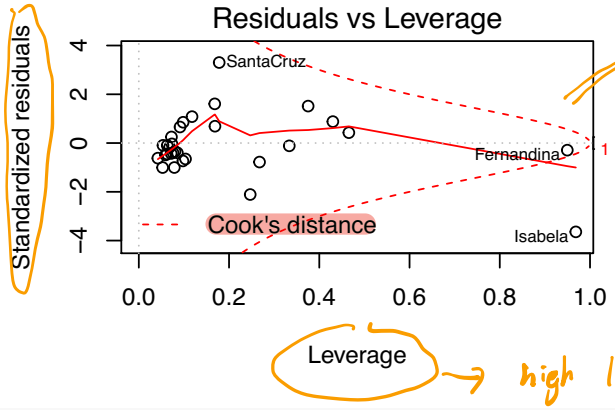


assumptions on error terms

normality

outlier





dashed line

Cook's distance = 1

$D_i = f(r_i, h_{ii})$

high leverage

```
par(mfrow=c(1,1))
```

- As the Cook statistics represent a function of standardized residuals and leverage, we can plot contours (the above plot shows contours with Cook's distance = 1).
- Any point that lies beyond these contours might well be influential and require closer attention.
- A practical guideline: $D_i > 4/n$ can indicate an influential point.

(r_i, h_i)

II. Some remedies of error issues

- We have seen that assumptions of errors can be violated and we must then consider alternatives.
- When the errors are dependent, we can use generalized least squares (GLS).
- When the errors are independent, but not identically distributed, we can use weighted least squares (WLS), which is a special case of GLS.

II.1 Generalized Least Squares

- We have assumed $\text{var}(\epsilon) = \sigma^2 I$.

If Σ is known:

- Suppose instead $\text{var}(\epsilon) = \sigma^2 \Sigma$.
  - σ^2 is unknown but Σ is known.
  - that is, we know the correlation and relative variance between the errors,
  - but we do not know the absolute scale of the variation.
- Write $\Sigma = SS^\top$. We can transform the regression model as

$$Y = X\beta + \varepsilon$$
$$S^{-1}Y = S^{-1}X\beta + S^{-1}\varepsilon \Rightarrow \tilde{Y} = \tilde{X}\beta + \tilde{\varepsilon}$$

- Then OLS can be conducted to the transformed variables $\tilde{Y}$ and $\tilde{X}$.
- Then

$$\hat{\beta} = (\tilde{X}^\top \tilde{X})^{-1}\tilde{X}^\top \tilde{Y} = \left(X^T\Sigma^{-1}X\right)^{-1} X^T\Sigma^{-1}Y$$
$$\text{var}(\hat{\beta}) = \sigma^2(\tilde{X}^\top \tilde{X})^{-1} = \sigma^2(X^\top \Sigma^{-1} X)^{-1}.$$

- Also diagnostics should be applied to the tranformed residuals $S^{-1}\hat{\varepsilon}$, which should be approximately i.i.d.

If Σ is unknown:

- We need to estimate Σ. Can be done through R function `gls`.
- Recall the temperature data that we investigated where serial correlation was observed.

```
data(globwarm,package="faraway")
lmod <- lm(nhtemp ~ wusa + jasper + westgreen + chesapeake
           + tornetrask + urals + mongolia + tasman, globwarm)
sumary(lmod)
```

```
##               Estimate Std. Error t value  Pr(>|t|)
## (Intercept) -0.2425552  0.0270115 -8.9797 1.972e-15
## wusa         0.0773844  0.0429266  1.8027 0.0736475
## jasper      -0.2287948  0.0781074 -2.9292 0.0039859
## westgreen    0.0095839  0.0418405  0.2291 0.8191679
## chesapeake  -0.0321117  0.0340522 -0.9430 0.3473462
## tornetrask   0.0926676  0.0450530  2.0569 0.0416114
## urals        0.1853691  0.0914285  2.0275 0.0445674
## mongolia     0.0419725  0.0457935  0.9166 0.3609955
## tasman       0.1154529  0.0301110  3.8342 0.0001919
##
## n = 145, p = 9, Residual SE = 0.17577, R-Squared = 0.48
```

```
cor(residuals(lmod)[-1],residuals(lmod)[-length(residuals(lmod))])
```

```
## [1] 0.583339
```

$(\epsilon_i, \epsilon_{i+1})$

- nlme packageof Pinheiro and Bates (2000) contains a GLS fitting function.
  - restricted maximum likelihood (ReML) (Solve MLE of coefficients and variance parameters simultaneously)
- Consider a autoregressive model $\varepsilon_{i+1} = \phi\varepsilon_i + \delta_i$ where $\delta_i \sim N(0, \tau^2)$.

```
require(nlme)
glmod <- gls(nhtemp ~ wusa + jasper + westgreen + chesapeake
             + tornetrask + urals + mongolia + tasman,
             correlation=corAR1(form=~year), data=na.omit(globwarm))
# na.omit(globwarm) drops missing values
summary(glmod)
```

time series $(\epsilon_1 \epsilon_2 \cdots \epsilon_n)$

ϵ_i : conditionary ϵ_{i-1} . it is independent with the remaining ϵ's

$\epsilon_i = \phi\epsilon_{i-1} + \delta_{i-1}$

fixed parameter

i over year

```
## Generalized least squares fit by REML
##   Model: nhtemp ~ wusa + jasper + westgreen + chesapeake + tornetrask +      urals +
##   Data: na.omit(globwarm)
```

The covariance matrix is determined by ϕ and σ^2

In particular, $cov(\epsilon) = \sigma^2 \times \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \rho \\ \rho^{n-1} & \cdots & \rho & \cdots & 1 \end{pmatrix}$

```
##         AIC       BIC   logLik
##   -108.2074 -76.16822 65.10371
##
## Correlation Structure: AR(1)
##  Formula: ~year
##  Parameter estimate(s):
##       Phi
## 0.7109922
##
## Coefficients:
##                    Value  Std.Error   t-value p-value
## (Intercept) -0.23010624 0.06702406 -3.433188  0.0008
## wusa         0.06673819 0.09877211  0.675678  0.5004
## jasper      -0.20244335 0.18802773 -1.076668  0.2835
## westgreen   -0.00440299 0.08985321 -0.049002  0.9610
## chesapeake  -0.00735289 0.07349791 -0.100042  0.9205
## tornetrask   0.03835169 0.09482515  0.404446  0.6865
## urals        0.24142199 0.22871028  1.055580  0.2930
## mongolia     0.05694978 0.10489786  0.542907  0.5881
## tasman       0.12034918 0.07456983  1.613913  0.1089
##
## Correlation:
##             (Intr) wusa   jasper wstgrn chespk trntrs urals  mongol
## wusa        -0.517
## jasper      -0.058 -0.299
## westgreen    0.330 -0.533  0.121
## chesapeake   0.090 -0.314  0.230  0.147
## tornetrask -0.430  0.499 -0.197 -0.328 -0.441
## urals       -0.110 -0.142 -0.265  0.075 -0.064 -0.346
## mongolia     0.459 -0.437 -0.205  0.217  0.449 -0.343 -0.371
## tasman       0.037 -0.322  0.065  0.134  0.116 -0.434  0.416 -0.017
##
## Standardized residuals:
```

Under model $\epsilon_{i+1} = \phi \epsilon_i + \delta_{i+1}$

$$\rho = \text{corr}(\epsilon_i, \epsilon_{i+1}) = \text{corr}(\epsilon_i, \phi\epsilon_i + \delta_{i+1}) = \phi \times \text{corr}(\epsilon_i, \epsilon_i) \quad (\text{By } \epsilon_i \perp \delta_{i+1}) = \phi$$

$$\text{var}(\epsilon_{i+1}) = \text{var}(\phi\epsilon_i + \delta_{i+1}) = \phi^2 \text{var}(\epsilon_i) + \text{var}(\delta_{i+1}) = \phi^2 \text{var}(\epsilon_i) + \tau^2$$

$$\Rightarrow \quad \sigma^2 = \phi^2 \sigma^2 + \tau^2 \Rightarrow \sigma^2 = \frac{\tau^2}{1-\phi^2}$$

Therefore, with $\hat{\sigma}^2$ and $\hat{\phi}^2$, we can obtain $\hat{\tau}^2$.

(There is no need to present $\hat{\tau}^2$ separately)

```
##         Min          Q1         Med          Q3         Max
## -2.31122523 -0.53484054  0.02342908  0.50015642  2.97224724
##
## Residual standard error: 0.204572
## Degrees of freedom: 145 total; 136 residual
```

II.2 Weighted least squares

- Errors are uncorrelated, but have unequal and unknown variances.
- $\Sigma = \text{diag}(1/w_1, \ldots, 1/w_n)$, where w_i are the weights.
- By GLS, $S = \text{diag}(1/\sqrt{w_1}, \ldots, 1/\sqrt{w_n})$.
- Regress $S^{-1}Y$ on $S^{-1}X$, i.e., $\sqrt{w_i}y_i \sim \sqrt{w_i}x_i$.
- Data example:
  - We consider an experiment studying interactions of unstable elementary particles in collision with proton targets (Weisberg et al., 1978).
  - These particles interact via the so-called strong interaction force that holds nuclei together.
  - The experiment was carried out with beam having various values of incident momentum, or equivalently for various values of s, the square of the total energy in the center-of-mass frame of reference system.
  - For each value of s, we observe the scattering cross-section y, measured in millibarns.
  - A theoretical model of the strong interaction force predicts that

$$\text{E}(y \mid s) = \beta_0 + \beta_1 s^{-1/2} + \text{ relatively small terms}$$

    (leading part is a simple linear regression.)
  - At each value of s, a very large number of particles was counted, and as a result the values of $\text{Var}(y \mid s = s_i) = \sigma^2/w_i$ are known almost exactly.

```
library(alr4)
head(physics1,3)
```

```
##      x   y SD
## 1 0.345 284 13
## 2 0.287 288  9
## 3 0.251 304  9
```

- In this case, variances are not constant but are different for each value of s.
- Therefore, WLS should be used.

III. Transformations

- We have seen that transformations of the response and/or predictors can improve the fit and correct violations of model assumptions, such as non-constant error variance.

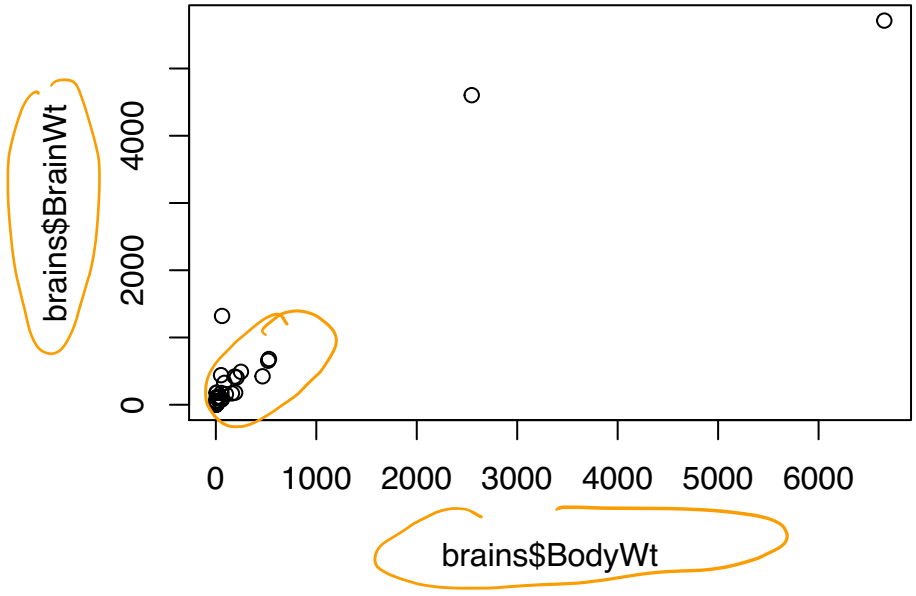
III.1 Transform the responses

- Transforming responses is to achieve a mean function that is linear in the transformed scale: $\mathrm{E}(\tilde{Y} \mid \tilde{X}) \approx \beta_0 + \beta_1 \tilde{X}$.
  - In simple linear regression:
    - let scatterplot have an approximate straight line mean function
  - In multiple linear regression:
    - Harder, but we can consider one predictor case first.
- A transformation family is a collection of transformations indexed by one or a few parameters that the analyst can select.

```
library(alr4)
head(brains,3)
```

```
##             BrainWt BodyWt
## Arctic fox   44.500   3.385
## Owl monkey   15.499   0.480
## Beaver        8.100   1.350
```

```
plot(brains$BodyWt, brains$BrainWt )
```



- Little evidence of linear mean function.

Box-Cox transformation

- Named after statisticians George Box and Sir David Roxbee Cox (1964)
- Designed for strictly positive responses and chooses the transformation to find the best fit to the data: $y \to g_\lambda(y)$

$$g_\lambda(y) = \begin{cases} \frac{y^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \log y & \lambda = 0 \end{cases}$$

- Choose λ that maximizes

normal distribution likelihood

$$L(\lambda) = -\frac{n}{2}\log\left(\mathrm{RSS}_\lambda/n\right) + (\lambda - 1)\sum \log y_i$$

where RSS_λ is the residual sum of squares when $g_\lambda(y)$ is the response.

⇒ See details in Lee and Seber Section 10.5.2

- A $100(1-\alpha)\%$ confidence interval for λ is (from $H_0 : \lambda = \lambda_0$):

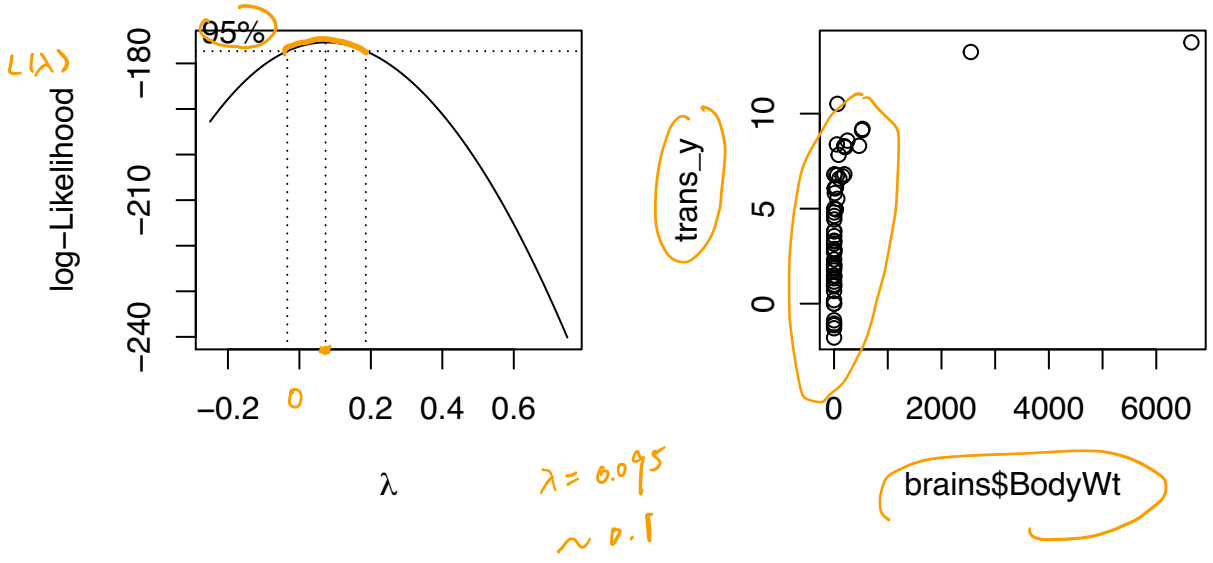
$$\left\{\lambda : \quad L(\lambda) > L(\hat{\lambda}) - \frac{1}{2}\chi_1^{2(1-\alpha)}\right\}$$

df 1 : λ one parameter

```
require(MASS)
par(mfrow=c(1,2))
lmod <- lm( BrainWt ~ BodyWt, brains)
boxcox(lmod, lambda=seq(-0.25,0.75,by=0.05),plotit=T)
lambda_opt <- 0.1
trans_y <- (brains$BrainWt^(lambda_opt)-1)/(lambda_opt)
plot(brains$BodyWt, trans_y )
```

(-0.25, -0.2, -0.15 ... 0.75)

TRUE



L(λ)

λ = 0.095 ~ 0.1

```r
par(mfrow=c(1,1))
```

Remark

- Conclusion can be influenced by outliers.
- If some $y_i < 0$, we can add a constant to all the y_i.

III.2 Transform the predictors

Broken Stick Regression

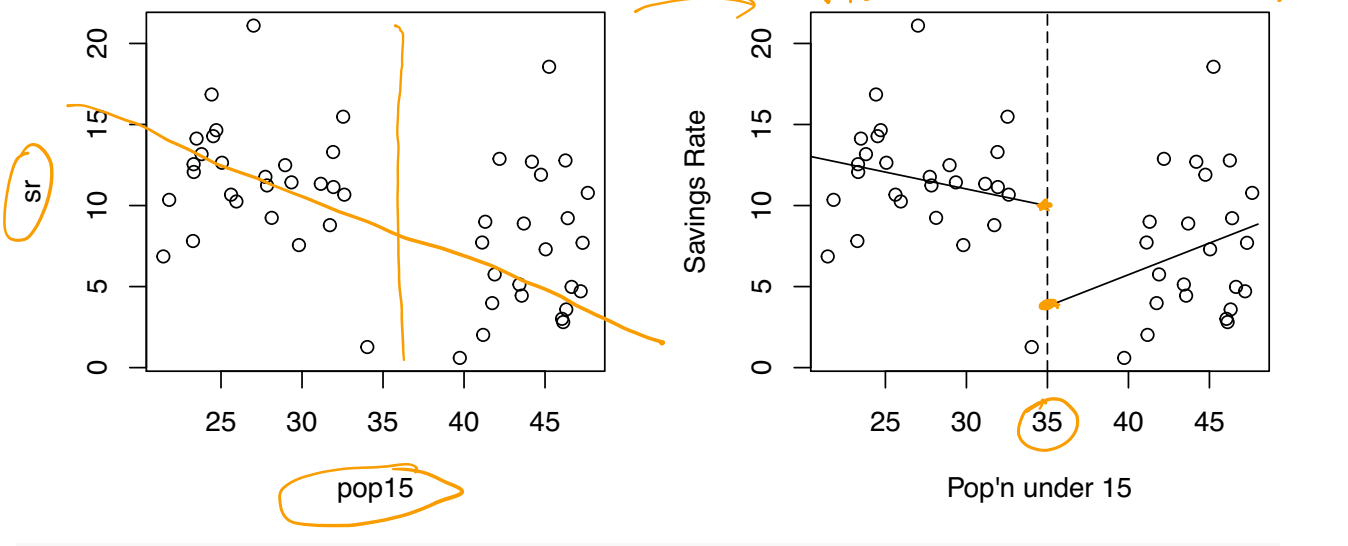
- Sometimes we have reason to believe that different linear regression models apply in different regions of the data. (predictors)
- Analyzing a data on savings rates in 50 countries (averaged over the period 1960- 1970.)
  - sr: savings rate - personal saving divided by disposable income
  - pop15: percent population under age of 15

```
par(mfrow=c(1,2))
plot(sr ~ pop15,savings)

lmod1 <- lm(sr ~ pop15, savings, subset=(pop15 < 35))
lmod2 <- lm(sr ~ pop15, savings, subset=(pop15 > 35))
plot(sr ~ pop15,savings,xlab="Pop'n under 15",  ylab="Savings Rate")

abline(v=35,lty=5)
segments(20,lmod1$coef[1]+lmod1$coef[2]*20,35,  lmod1$coef[1]+lmod1$coef[2]*35)
segments(48,lmod2$coef[1]+lmod2$coef[2]*48,35,  lmod2$coef[1]+lmod2$coef[2]*35)
```

Divide dataset into 2 parts




```
lhs <- function(x) ifelse(x < 35,35-x,0)
rhs <- function(x) ifelse(x < 35,0,x-35)
lmod <- lm(sr ~ lhs(pop15) + rhs(pop15), savings)
x <- seq(20,48,by=1)
```

```
par(mfrow=c(1,1))
```

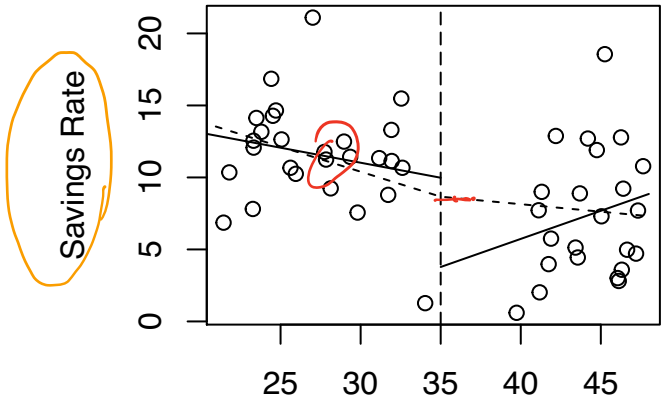
- Subsetted regression fit is that the two parts of the fit do not meet at the join.
- If we believe the fit should be continuous as the predictor varies, we should consider the broken stick regression fit.
- Define

$$B_l(x) = \begin{cases} c - x & \text{if } x < c \\ 0 & \text{otherwise} \end{cases} \qquad B_r(x) = \begin{cases} x - c & \text{if } x > c \\ 0 & \text{otherwise} \end{cases}$$

  where c marks the division between two groups.
- Fit a model of the form:

$$y = \beta_0 + \beta_1 B_l(x) + \beta_2 B_r(x) + \varepsilon$$

```
plot(sr ~ pop15,savings,xlab="Pop'n under 15",  ylab="Savings Rate")
abline(v=35,lty=5)
segments(20,lmod1$coef[1]+lmod1$coef[2]*20,35,  lmod1$coef[1]+lmod1$coef[2]*35)
segments(48,lmod2$coef[1]+lmod2$coef[2]*48,35,  lmod2$coef[1]+lmod2$coef[2]*35)
lhs <- function(x) ifelse(x < 35,35-x,0)
rhs <- function(x) ifelse(x < 35,0,x-35)
lmod <- lm(sr ~ lhs(pop15) + rhs(pop15), savings)
x <- seq(20,48,by=1)
py <- lmod$coef[1] + lmod$coef[2]*lhs(x) + lmod$coef[3]*rhs(x)
lines(x,py,lty=2)
```

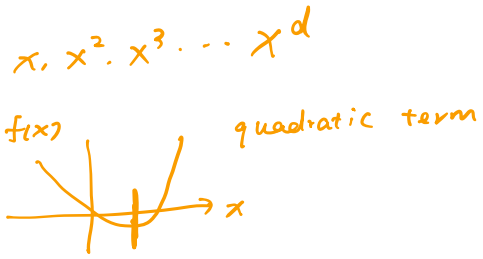


Polynomial regression

- Generalizing $X\beta$ part by adding polynomial terms:
  - e.g.,

$$y = \beta_0 + \beta_1 x + \cdots + \beta_d x^d + \varepsilon$$

    allows for a more flexible relationship.



- Adding a quadratic term:
  - e.g., there may be a best temperature for baking bread – a hotter or colder temperature may result in a less tasty outcome.
  - If you believe a predictor behaves in this manner, it makes sense to add a quadratic term.
- Again look at `savings` dataset
  - dpi: per-capita disposable income in dollars
  - ddpi: percent growth rate of dpi
- Choosing d:
  - Keep adding terms until the added term is not statistically significant.
  - Start with a large d and eliminate non-statistically significant terms starting with the highest order term.

```
summary(lm(sr ~ ddpi,savings))$coefficients #linear of ddpi is significant
```

```
##             Estimate Std. Error  t value     Pr(>|t|)
## (Intercept) 7.883021  1.0110011 7.797243 4.464697e-10
## ddpi        0.475830  0.2146166 2.217117 3.138509e-02
```

```
summary(lm(sr ~ ddpi+I(ddpi^2),savings))$coefficients #quadratic of ddpi is significant
```

```
##                Estimate Std. Error   t value     Pr(>|t|)
## (Intercept)  5.13038069 1.43471517  3.575888 0.0008211413
## ddpi         1.75751897 0.53772368  3.268443 0.0020258542
## I(ddpi^2)   -0.09298521 0.03612318 -2.574115 0.0132617330
```

```
summary(lm(sr ~ ddpi+I(ddpi^2)+I(ddpi^3),savings))$coefficients  #cubit of ddpi is NOT
```

```
##                  Estimate Std. Error      t value   Pr(>|t|)
## (Intercept)  5.145360e+00 2.19860644  2.340282237 0.02366212
```

```
## ddpi          1.746017e+00 1.38045499  1.264812459 0.21230898
## I(ddpi^2)    -9.096724e-02 0.22559835 -0.403226554 0.68864973
## I(ddpi^3)    -8.496955e-05 0.00937393 -0.009064453 0.99280691
```

- You have to refit the model each time a term is removed which is inconvenient and numerically unstable.
- Orthogonal polynomials get around this problem by defining:

$$z_1 = a_1 + b_1 x$$

$$z_2 = a_2 + b_2 x + c_2 x^2$$

$$z_3 = a_3 + b_3 x + c_3 x^2 + d_3 x^3$$

  where coefficients are chosen so that $z_i^\top z_j = 0$ when $i \neq j$.
- In r, `poly()` function constructs orthogonal polynomials:

```
lmod <- lm(sr ~ poly(ddpi,4),savings)
summary(lmod)$coefficients
```

```
##                    Estimate Std. Error      t value     Pr(>|t|)
## (Intercept)      9.67100000   0.584602 16.542879686 9.477039e-21
## poly(ddpi, 4)1   9.55899338   4.133760  2.312420904 2.538538e-02
## poly(ddpi, 4)2 -10.49987612   4.133760 -2.540030321 1.460646e-02
## poly(ddpi, 4)3  -0.03737382   4.133760 -0.009041119 9.928263e-01
## poly(ddpi, 4)4   3.61196847   4.133760  0.873773113 3.868811e-01
```

- You can also define polynomials in more than one variable. These are sometimes called response surface models.
- A second degree model would be like:

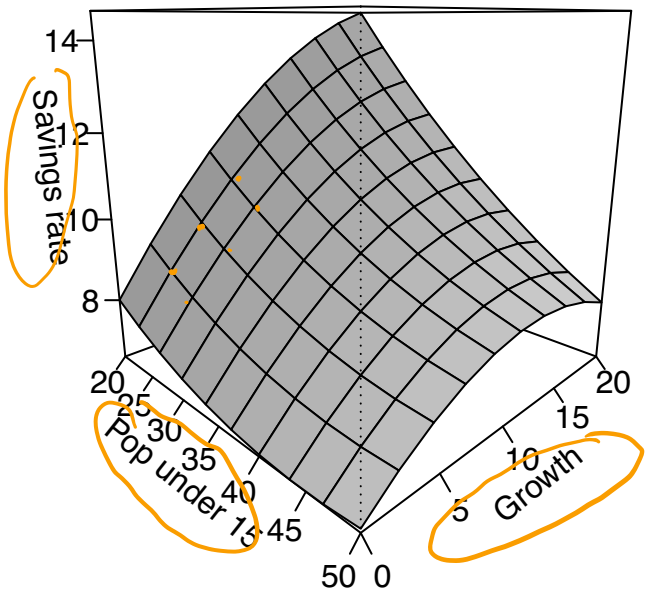
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2$$

```
lmod <- lm(sr ~ polym(pop15,ddpi,degree=2),savings)
```

- We can construct a perspective plot of the fitted surface.

```
pop15r <- seq(20, 50, len=10) #choose grid vlaues
ddpir <- seq(0, 20, len=10) #choose grid vlaues
pgrid <- expand.grid(pop15=pop15r, ddpi=ddpir) #construct 2-dim grids
pv <- predict(lmod, pgrid)
```

```r
persp(pop15r, ddpir, matrix(pv, 10, 10), theta=45,
      xlab="Pop under 15", ylab="Growth", zlab = "Savings rate",
      ticktype="detailed", shade = 0.25)
```



Predictors → polynomials

orthogonal polynomials (numerical)

Splines (local) ⇒ non-parametric statistics

IV. Measurement errors

- Errors in both responses Y and covariates X.
- Suppose that what we observe is (Y, X).
- But the true relationship is

$$Y = Z\beta + \epsilon \qquad \text{with} \qquad X = Z + \tau,$$

  so τ denotes errors of measuring true Z.
- Suppose we use observed data to calculate least squares estimates

$$\begin{aligned}\hat{\beta} &= (X'X)^{-1} X'Y \\ &= (Z'Z + Z'\tau + \tau'Z + \tau'\tau)^{-1} (Z'Y + \tau'Y) \\ &= \left(\frac{Z'Z}{n} + \frac{Z'\tau}{n} + \frac{\tau'Z}{n} + \frac{\tau'\tau}{n}\right)^{-1} \left(\frac{Z'Y}{n} + \frac{\tau'Z\beta}{n} + \frac{\tau'\epsilon}{n}\right)\end{aligned}$$

  where we plug in $Y = Z\beta + \epsilon$ and $X = Z + \tau$.
- In many cases, it can be assumed that
  - the covariate measurement error is uncorrelated with the covariate levels $Z'\tau/n \to 0$
  - covariate measurement error and observation error are uncorrelated $\tau'\epsilon/n \to 0$
- Then

$$\begin{aligned}\hat{\beta} \to & \left(\frac{Z'Z}{n} + \frac{\tau'\tau}{n}\right)^{-1} Z'Y/n = (M_z + M_\tau)^{-1} \frac{Z'Y}{n} \\ = & (M_z + M_\tau)^{-1} \left(\frac{Z'Z}{n}\beta + \frac{Z'}{n}\epsilon\right) \to (M_z + M_\tau)^{-1} M_z\beta,\end{aligned}$$

  where $M_z = Z'Z/n$ and $M_\tau = \tau'\tau/n$.
- The limiting bias is

$$\hat{\beta} - \beta = \left(\left(I + M_z^{-1}M_\tau\right)^{-1} - I\right)\beta.$$

SIMEX (SIMulation-EXtrapolation)

- If the variances and covariances among the measurement errors can be considered known.
- Regress Y on $X + \lambda E$
  - E is simulated noise having the same variance as the assumed measurement error.
- Denote the coefficient vector of this fit as $\hat{\beta}_\lambda$.
- Repeat this for several values of $\lambda \geq 0$, leading to a set of $\hat{\beta}_\lambda$ vectors.
  - Ideally, $\hat{\beta}_{-1}$ would approximate the coefficient estimates under no measurement error.

– By fitting a line or smooth curve to the $\hat{\beta}_\lambda$ values (separately for each component of β), it becomes possible to extrapolate back to $\hat{\beta}_{-1}$.

```
require(simex)
set.seed(123)
lmod <- lm(dist ~ speed, cars, x=TRUE) #stopping distances vs speed
#Suppose the predictor, speed, was measured with
#a known error standard deviation, say 0.5.
simout <- simex(lmod,"speed",0.5, B=1000)
simout
```

```
##
## Naive model:
## lm(formula = dist ~ speed, data = cars, x = TRUE)
##
## SIMEX-Variables: speed
## Number of Simulations: 1000
##
## Coefficients:
## (Intercept)        speed
##      -18.01         3.96
```

```
par(mfrow=c(1,2))
plot(simout)
```