Linear regression and dummy variable encoding

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Reading: Chapter 6.1-6.3, 6.6 in RC; Chapter 3 in JF.

Another view of T-test

- Recall the simple linear regression (SLR) model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \quad \varepsilon_i \sim i.i.d.\ N(0, \sigma^2),$$

  for all $i = 1, \ldots, n$.

- Equivalently

$$\boldsymbol{Y} = \boldsymbol{X}\beta + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{MVN}(\mathbf{0}, \sigma^2 \boldsymbol{I}).$$

  where $\boldsymbol{X}_{n\times 2} = \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}$ denote the $n \times 2$ design matrix.

- One-sample test is a special case of SLR.
- Two-sample test is also a special case of SLR.

Equivalence to one-sample test

- Let

$$Y_i = \beta_0 + \varepsilon_i, \quad \varepsilon_i \sim i.i.d.\ N(0, \sigma^2),$$

for all $i = 1, \ldots, n$.

- Equivalently

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{MVN}(\mathbf{0}, \sigma^2 \boldsymbol{I}).$$

where $\boldsymbol{X}_{n\times 1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ denote the $n \times 1$ design matrix, and $\boldsymbol{\beta} = \beta_0$.

- The one-sample mean test is equivalent to

$$H_0 : \beta_0 = \mu \text{ vs. } H_A : \beta_0 \neq \mu$$

Equivalence to two-sample test

- Let
$$Y_i = \beta_0 \mathbb{1}_{i \text{ is in group } 1} + \beta_1 \mathbb{1}_{i \text{ is in group } 2} + \varepsilon_i, \quad \varepsilon_i \sim i.i.d.\ N(0, \sigma^2),$$
for all $i = 1, \ldots, n$.
- Equivalently
$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{MVN}(\mathbf{0}, \sigma^2 \boldsymbol{I}).$$
where $\boldsymbol{X}_{n\times 2} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}$ denote the $n \times 2$ design matrix, and
$\boldsymbol{\beta} = (\beta_0, \beta_1)'$.
- The unpaired two sample mean test is equivalent to
$$H_0 : \beta_0 - \beta_1 = 0 \text{ vs. } H_A : \beta_0 - \beta_1 \neq 0$$

Dummy variable

- The predictors in the linear model can be either continuous (e.g., age, height) or categorical (e.g., gender, group)
- For a categorical predictor that has p categories, define $p-1$ dummy variables:
$$X_{ik} = \begin{cases} 1 & \text{observation } i \text{ is in category } k \\ 0 & \text{otherwise} \end{cases}$$
where $k = 1, \ldots, p-1$.
- Include dummy variables as predictors in the linear model.
- Example. Consider n i.i.d. observations from the following model:
$$Y = \beta_0 + \beta_1 \text{Age} + \beta_2 X + \varepsilon, \quad \text{where } \varepsilon \sim i.i.d.\ N(0, \sigma^2),$$
with $X = 1$ if male, $X = 0$ if female.
- What is the interpretation for β_0, β_1, and β_2?

Example with categorical variables

Consider the effect of education on hourly wages (Y). The education is classified into three categories:

Option in Survey (O)	Meaning (M)
1	College dropout
2	College
3	MS and above

Which model makes more sense?

- $Y = \beta_0 + \beta_1 O + \varepsilon$?
- $Y = \beta_0 + \beta_1 \mathbb{1}_{\text{college}} + \beta_2 \mathbb{1}_{\text{MS and above}} + \varepsilon$?
- $Y = \beta_0 + \beta_1 \mathbb{1}_{\text{college dropout}} + \beta_2 \mathbb{1}_{\text{college}} + \varepsilon$?

(In all cases, assume $\varepsilon \sim i.i.d. N(0, \sigma^2)$)

Example (Cont.)

- To include the eduction as predictor in a regression model, define 2 dummy variables X_1 and X_2:

Option in Survey (O)	Meaning (M)	X_1	X_2
1	College dropout	0	0
2	College	1	0
3	MS and above	0	1

- Baseline (all dummies 0): college dropout;
- $X_1 = 1$, if the highest degree is college, 0 otherwise;
- $X_2 = 1$, if degree with MS and above, 0 otherwise.

Include X_1 and X_2 as dummy variables in a regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \underbrace{\beta_3 X_3 + \ldots + \beta_p X_p}_{\text{other predictors, e.g., age}} + \varepsilon, \quad \varepsilon \sim i.i.d.\ N(0, \sigma^2).$$

Inference on the linear contrast

Recall the study that investigates the effect of education on hourly salary (Y):

Education	X_1	X_2
College dropout	0	0
College	1	0
MS and above	0	1

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon, \quad \text{where} \quad \varepsilon \sim i.i.d.\ N(0, \sigma^2).$$

Suppose we are interested in testing:

- The mean salary for "MS and above" is the same as for "College":
  $H_0 : \beta_1 = \beta_2 \longleftrightarrow H_0 : 0 * \beta_0 + 1 * \beta_1 - 1 * \beta_2 = 0$
- The mean salary for "College" is the same as for "College dropout":
  $H_0 : \beta_1 = 0 \longleftrightarrow H_0 : 0 * \beta_0 + 1 * \beta_1 + 0 * \beta_2 = 0$
- Compared to college dropout, the mean salary increase for "MS and above" is twice as that for "College":
  $\mathrm{H}_0 : \beta_2 = 2\beta_1 \longleftrightarrow H_0 : 0 * \beta_0 + 2 * \beta_1 - 1 * \beta_2 = 0$

Inference on the linear contrast

- All these hypothesis tests could be expressed as a linear contrast:

$$H_0 : c_0\beta_0 + c_1\beta_1 + c_2\beta_2 = 0 \quad \text{v.s.} \quad H_\alpha : c_0\beta_0 + c_1\beta_1 + c_2\beta_2 \neq 0,$$

  for a given vector $\boldsymbol{c} = (c_0, c_1, c_2)$. Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)'$.
- What is the distribution of $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ under the null? Multivariate normal with

$$\mathbb{E}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}) = \boldsymbol{c}'\boldsymbol{\beta}, \quad \text{Var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}) = \underline{\hspace{4cm}} = \sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{c}$$

- In case σ^2 is unknown, plug in the estimator $\hat{\sigma}^2$. (what is the form of $\hat{\sigma}^2$?)

$$\frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\widehat{\text{Var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}})}}} \sim T_{n-3}$$