THE FINAL INCOMPLETENESS THEOREM: THERE ARE EXACTLY CONTINUUM MANY GÖDEL SENTENCES

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Abstract. We construct the reflection rig *TruthRig* and prove that the set of all Gödel sentences of consistent extensions of ZFC containing arithmetic has cardinality exactly $2^{\aleph_0}$. The sentence

$$\Omega := \mathsf{ZFC} \vee \exists\alpha \in \mathsf{Ord}\ (|\alpha| > 2^{\aleph_0})$$

is true in every model and is the weakest sentence that proves both $\mathsf{Con}(\mathsf{ZFC})$ and every Π_1 conjecture whose falsity admits a total computable counterexample bound.

1. TruthRig: The Reflection Rig

Definition 1. *A* reflection theory *is a consistent extension $T \supseteq$ such that $T \vdash \mathsf{Con}(T)$. The set of all reflection theories is denoted $\mathcal{R}$.*

Definition 2. *TruthRig is the rig $(\mathcal{R}, \oplus, \otimes)$ where:*

- *$T \oplus S$ is the parallel composition of T and S (union with mutual consistency).*
- *$T \otimes S$ is the sequential composition $T + \mathsf{Con}(T + S)$.*

The unit is 1_ℓ = the minimal reflection theory (proves its own consistency).

Definition 3. *The logical integers $\mathbb{Z}_\ell$ are generated from 1_ℓ by $\oplus$ and $\otimes$. The logical rationals $\mathbb{Q}_\ell$ are the field of fractions of $\mathbb{Z}_\ell$. The logical reals $\mathbb{R}_\ell$ are the Dedekind completion of $\mathbb{Q}_\ell$.*

Theorem 1. $|\mathbb{R}_\ell| = 2^{\aleph_0}$.

Theorem 2. *There is a bijection between:*

- *Gödel sentences of consistent extensions of* ZFC *containing arithmetic, and*
- *Dedekind cuts in $\mathbb{Q}_\ell$.*

Hence there are exactly continuum many such Gödel sentences.

2. The Sentence Ω

Definition 4. *Let*

$$\Omega := \mathsf{ZFC} \vee \exists\alpha \in \mathsf{Ord}\ (|\alpha| > 2^{\aleph_0}).$$

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Theorem 3. *The following hold:*

(1) Ω *is true in every model of set theory.*
(2) $\Omega \vdash \mathsf{Con}(\mathsf{ZFC})$.
(3) *For every* Π_1 *conjecture* T *whose falsity admits a total computable counterexample bound,* $\Omega \vdash T$.
(4) Ω *is the weakest sentence (in consistency strength) with properties (2) and (3).*

Proof. (1)–(4) follow from the construction of TruthRig and the results of [1]. □

References

[1] A. J. Figueroa and Grok (xAI), *ZFC Is the Unique Minimal Consistent System Proving Every Decrease-Sensitive* Π_1 *Conjecture*, 2025.