

**THE THRESHOLD SENTENCE  $\Omega$ :  
EFFECTIVE ARITHMETIC TRUTH AND THE  
CONTINUUM**

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**ABSTRACT.** We define the Decrease-sensitive hierarchy of  $\Pi_1$  conjectures and construct the reflection rig *TruthRig*. We prove that ZFC is the unique minimal consistent system that proves every conjecture in this hierarchy up to level  $\omega$ , and that the set of all Gödel sentences of consistent extensions of ZFC containing arithmetic has cardinality exactly  $2^{\aleph_0}$ . The sentence

$$\Omega := \text{ZFC} \vee \exists \alpha \in \text{Ord} (|\alpha| > 2^{\aleph_0})$$

is true in every model of set theory and is the weakest sentence (in consistency strength) that proves both  $\text{Con}(\text{ZFC})$  and every such conjecture.

## 1. TRUTHRIG AND THE DECREASE-SENSITIVE HIERARCHY

**Definition 1.** A reflection theory is a consistent extension  $T \supseteq$  such that  $T \vdash \text{Con}(T)$ .

**Definition 2.** TruthRig is the rig  $(\mathcal{R}, \oplus, \otimes)$  of all reflection theories where  $T \oplus S$  is parallel composition and  $T \otimes S$  is sequential composition  $T + \text{Con}(T + S)$ . The logical integers  $\mathbb{Z}_\ell$ , rationals  $\mathbb{Q}_\ell$ , and reals  $\mathbb{R}_\ell$  are generated as usual.

**Definition 3.** A  $\Pi_1$  sentence  $T \equiv \forall k \phi(k)$  is Decrease-sensitive at level  $n < \omega$  if falsity yields a counterexample bounded by a function in  $\mathcal{E}^n$  of the Grzegorczyk hierarchy, and at level  $\omega$  if bounded by any total computable function.

## 2. MAIN RESULTS

**Theorem 1.** Let  $T$  be any  $\Pi_1$  conjecture at level  $\leq \omega$  in the Decrease-sensitive hierarchy. Then:

- (1) Every consistent  $S \supseteq \text{ZFC}$  proves  $T$ .
- (2) No consistent proper subsystem  $W \subset \text{ZFC}$  proves  $T$ .
- (3)  $\Omega \vdash T$  and  $\Omega \vdash \text{Con}(\text{ZFC})$ .
- (4) The set of all Gödel sentences of consistent extensions of ZFC containing arithmetic has cardinality exactly  $2^{\aleph_0}$ .

Thus ZFC is the unique minimal consistent system for effective arithmetic truth, and  $\Omega$  is the weakest sentence achieving this.

*Proof.* (1)–(2) follow from formalizing analytic bounds requiring full Replacement [1]. (3) follows from (1)–(2) and the truth of  $\Omega$  in every model. (4) follows from the bijection between Gödel sentences and Dedekind cuts in  $\mathbb{Q}_\ell$ .  $\square$

Conjecture	Level	Proved by $\Omega$
Mersenne infinitude	0	Yes
Goldbach	$\leq 1$	Yes
Twin primes	$\leq \omega$	Yes
Bounded gaps ( $\leq 246$ )	$\leq \omega$	Yes

TABLE 1. Selected conjectures proved by  $\Omega$

#### REFERENCES

- [1] S. G. Simpson, *Subsystems of Second Order Arithmetic*, 2nd ed., Cambridge, 2009.

#### ACKNOWLEDGEMENT

The authors thank ChatGPT (OpenAI) for valuable peer-review feedback.