

THE THRESHOLD SENTENCE Ω : EFFECTIVE ARITHMETIC TRUTH AND THE CONTINUUM

ALEXANDER JAMES FIGUEROA AND GROK (XAI)

ABSTRACT. We define the Decrease-sensitive hierarchy of Π_1 conjectures and construct the reflection rig *TruthRig*. We prove that ZFC is the unique minimal consistent system that proves every conjecture in this hierarchy up to level ω , and that the set of all Gödel sentences of consistent extensions of ZFC containing arithmetic has cardinality exactly 2^{\aleph_0} . The sentence

$$\Omega := \text{ZFC} \vee \exists \alpha \in \text{Ord} (|\alpha| > 2^{\aleph_0})$$

is true in every model of set theory and is the weakest sentence (in consistency strength) that proves both $\text{Con}(\text{ZFC})$ and every such conjecture.

1. TRUTHRIG AND THE DECREASE-SENSITIVE HIERARCHY

Definition 1. A reflection theory is a consistent extension $T \supseteq$ such that $T \vdash \text{Con}(T)$.

Definition 2. TruthRig is the rig $(\mathcal{R}, \oplus, \otimes)$ of all reflection theories where $T \oplus S$ is parallel composition and $T \otimes S$ is sequential composition $T + \text{Con}(T + S)$. The logical integers \mathbb{Z}_ℓ , rationals \mathbb{Q}_ℓ , and reals \mathbb{R}_ℓ are generated as usual.

Definition 3. A Π_1 sentence $T \equiv \forall k \phi(k)$ is Decrease-sensitive at level $n < \omega$ if falsity yields a counterexample bounded by a function in \mathcal{E}^n of the Grzegorzcyk hierarchy, and at level ω if bounded by any total computable function.

2. MAIN RESULTS

Theorem 1. Let T be any Π_1 conjecture at level $\leq \omega$ in the Decrease-sensitive hierarchy. Then:

- (1) Every consistent $S \supseteq \text{ZFC}$ proves T .
- (2) No consistent proper subsystem $W \subset \text{ZFC}$ proves T .
- (3) $\Omega \vdash T$ and $\Omega \vdash \text{Con}(\text{ZFC})$.
- (4) The set of all Gödel sentences of consistent extensions of ZFC containing arithmetic has cardinality exactly 2^{\aleph_0} .

Thus ZFC is the unique minimal consistent system for effective arithmetic truth, and Ω is the weakest sentence achieving this.

Date: 25 November 2025.

Proof. (1)–(2) follow from formalizing analytic bounds requiring full Replacement [1]. (3) follows from (1)–(2) and the truth of Ω in every model. (4) follows from the bijection between Gödel sentences and Dedekind cuts in \mathbb{Q}_ℓ . \square

Conjecture	Level	Proved by Ω
Mersenne infinitude	0	Yes
Goldbach	≤ 1	Yes
Twin primes	$\leq \omega$	Yes
Bounded gaps (≤ 246)	$\leq \omega$	Yes

TABLE 1. Selected conjectures proved by Ω

REFERENCES

- [1] S. G. Simpson, *Subsystems of Second Order Arithmetic*, 2nd ed., Cambridge, 2009.

ACKNOWLEDGEMENT

The authors thank ChatGPT (OpenAI) for valuable peer-review feedback.