Market Equilibrium in Exchange Economies with Some Families of Concave Utility Functions

Bruno Codenotti[1] and Kasturi Varadarajan[2]

[1] Toyota Technological Institute at Chicago, Chicago IL 60637.
Email: bcodenotti@tti-c.org.
[2] Department of Computer Science, The University of Iowa,
Iowa City IA 52242. Email: kvaradar@cs.uiowa.edu.

Abstract. We consider the problem of computing equilibrium prices for exchange economies. For widely used families of utility functions, which include some functions that do not satisfy weak gross substitutability, we translate the equilibrium conditions into a convex feasibility problem. This enables us to obtain new, polynomial time algorithms for computing the equilibrium. As a byproduct of our work, we also prove the uniqueness of the equilibrium in an interesting setting where such a result was not known.

1 Introduction

Theoretical computer scientists have, in the last few years, become aware of the importance of the computation of *market equilibria*, originally a central concept of Theoretical Economics [2–4]. Informally, in an *exchange economy* there is a collection of goods, initially distributed among "actors" who trade them. The preference of each of these independent actors for each bundle of goods is expressed by a "utility function". Each actor tries to maximize her utility, but is constrained by her budget (as determined by the value that the "market" – consisting of herself and all the other actors – gives to the goods she has).

An equilibrium is a set of prices at which there are allocations of goods to traders such that two conditions are simultaneously satisfied: each trader's allocation maximizes her utility, subject to the constraints induced by her budget, and the market clears.

Note that in this formalism there are no restrictions on utility functions other than those imposed by the fact that they represent rational preferences satisfying standard assumptions: they may express very complicated dependencies on all goods held by the actors.

An early triumph of Mathematical Economics was the 1954 result by Arrow and Debreu [2] that, even in a more general situation which includes the production of goods, subject to very mild and natural restrictions, there is always an equilibrium. The proof relies on very general fixpoint theorems. The mathematical nature of such general fixpoint problems does not lead to efficient algorithms

⋆ The first author is on leave from IIT-CNR, Pisa, Italy.

– indeed the computation of fixpoints is suspected to be intractable[3]. In general equilibria are not only not unique, but the set of equilibrium points may be disconnected. Yet many real markets do work, and economists have struggled to capture *realistic* restrictions on markets, where the equilibrium problem exhibits some structure, like uniqueness or convexity. The general approach has been to impose restrictions either at the level of individuals (by restricting the utility functions considered and/or by making assumptions on the initial endowments) or at the level of the *aggregate market* (by assuming that the composition of the individual actions is particularly well behaved).

Two well studied conditions are *gross substitutability* – GS (see [29], p. 611) and the *weak axiom of revealed preferences* – WARP (see [29], Section 2.F). Although restrictive, these conditions are useful and model some realistic scenarios.

A utility function satisfies GS (resp., weak GS – WGS) if increasing the prices of some of the goods while keeping the other prices and the income fixed causes the increase (resp., does not cause the decrease) in demand for the goods whose price is fixed.

Roughly speaking, WARP means that the aggregate behavior of the market fulfills a fundamental property satisfied by the choices made by any rational individual trader.

It is well known that GS implies that the equilibrium prices are unique up to scaling ([37], p. 395), and that WGS and WARP both imply that the set of equilibrium prices is convex ([29], p. 608). When the set of equilibria is convex, it is enough to add a non-degeneracy assumption (which is almost always satisfied) to get the uniqueness of the equilibrium up to scaling [12].

CES utility functions. The most popular family of utility functions is given by CES (constant elasticity of substitution) functions, which have been introduced in [35]. We refer the reader to the book by Shoven and Whalley [34] for a sense of their pervasiveness in applied general equilibrium. A CES function ranks the trader's preferences over bundles of goods $(x_1, \ldots, x_n)$ according to the value of $u(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} c_i x_i^{\rho}\right)^{\frac{1}{\rho}}$.

The success of CES functions is due to the useful combination of their mathematical tractability with their expressive power, which allows for a realistic modeling of a wide range of consumers' preferences. Indeed, one can model markets with very different characteristics, in terms of preference towards variety, substitutability versus complementarity, and multiplicity of price equilibria, by changing the values of ρ and of the utility parameters c_i.

CES functions have been thoroughly analyzed in [1], where it has also been shown how to derive, in the limit, their special cases, i.e., linear, Cobb-Douglas, and Leontief functions (see [1], p. 231). Let $\sigma = \frac{1}{1-\rho}$. The parameter σ is called the *elasticity of substitution*. For $\sigma \to \infty$, CES take the linear form, and the goods are perfect substitutes, so that there is no preference for variety. For $\sigma > 1$,

[3] More precisely, suitable computational versions of Brouwer's and Kakutani's fixpoint theorems are complete for the class PPAD [32]. A polynomial time algorithm for the above problems would imply the polynomial time solvability of a number of important problems, not known to be in P.

the goods are partial substitutes, and different values of σ in this range allow us to express different levels of preference for variety. For $\sigma \to 1$, CES become Cobb-Douglas functions, and express a perfect balance between substitution and complementarity effects. Indeed it is not difficult to show that a trader with a Cobb-Douglas utility spends a fixed fraction of her income on each good.

For $\sigma < 1$, CES functions model markets with significant complementarity effects between goods. This feature reaches its extreme (*perfect complementarity*) as $\sigma \to 0$, i.e., when CES takes the form of Leontief functions. In the latter case, the *shape* of the optimal bundle demanded by the consumer does not depend at all on the prices of the goods, but is fully determined by the parameters defining the utility function.

Whenever the relative incomes of the traders are independent of the prices, CES functions give rise to a market which satisfies WARP. This happens for instance in the Fisher model, a very special case of the exchange model. On the other hand, CES functions satisfy WGS if and only if $\rho \geq 0$, whereas, if $\rho < -1$, they allow for multiple disconnected equilibria.

In summary, CES functions are important because (i) economists use them extensively; (ii) they model markets exposing a variety of different phenomena; (iii) they include, as special cases, utility functions previously studied by computer scientists; (iv) they generate a demand function which facilitates the study of the presence or absence of certain kinds of structure. For these reasons they seem to be the prime candidates for a thorough exploration of the market equilibrium problem.

Our Results. In order to present our results in context, we briefly review recent results on the computation of market equilibria (see [9] for a more complete review.) The main goal in this area of research is to provide polynomial time algorithms for the computation of market equilibria. In a series of papers which started with linear utility functions, more and more general utility functions were considered [13, 24, 23, 11, 10, 20, 21, 7]. Most of the corresponding market settings fall into the framework of one of the two conditions above (WGS or WARP).

The technical tool used in some of these results is to reformulate the problem in terms of mathematical programming in a way that a polynomial time algorithm (or approximation scheme – in general the equilibrium point is not a vector of rationals) can be obtained by known optimization techniques. In particular, *convex programming* has been proven to be a particularly useful tool [23, 11, 10].

Our contribution is to present new formulations that, for a large class of utility functions, well studied in the economic literature, express the equilibrium as the solution of a convex programming problem. With our techniques we can deal with additively separable concave (ASC) functions of the form $u(x) = \sum_j \alpha_j x_j^{\rho_j}$, where $\alpha_j \geq 0$, and $0 < \rho_j < 1$, and CES functions with $\rho \geq -1$. Note in particular that our results apply to a range of CES functions, those with $-1 \leq \rho < 0$, for which the market satisfies neither WARP nor GS.

Besides the algorithmic contribution of the technique, our formalization allows us to conclude that for this class of functions equilibria are not disconnected, and are thus essentially unique. This was not known by economists. Indeed it turns out that an exchange economy with traders endowed with CES utility functions such that $-1 \leq \rho < 0$ is not covered by any of the known conditions that ensure that there are no multiple disconnected equilibria, such as the *Super Cobb-Douglas Property* of Mas-Colell [27], and thus our result also provides an original contribution to the theory of equilibrium. Combined with a result by Gjerstad presented in Section 2.3, our work leads to a characterization of the CES exchange economies whose set of equilibria is connected.

Our formulation is quite different from Jain's and Nenakov-Primak's formulation [23, 31]. Jain's approach, which works for utility functions $u(x_1, \ldots, x_n)$ for which $\log \frac{\sum_j x_j \partial_j u(x)}{\partial_k u(x)}$ is concave, does not apply to CES functions with $-1 \leq \rho < 0$. The general approach we take is to write the equilibrium conditions as a nonconvex program in the price and allocation variables that involves both equalities and inequalities. We then show that some of the equalities can be relaxed into inequalities. We exploit certain necessary and sufficient conditions that must be satisfied by the market demand function to get rid of the variables representing the allocations of each trader. Finally, we make some variable substitutions which end up making all the inequalities convex.

One of the consequences of our approach is that in some cases the number of variables and constraints in our convex programs is smaller by an order of magnitude in comparison to the programs of Jain and Nenakov-Primak. Our approach is flexible enough to admit a generalization to economies with constant returns-to-scale production [36], where the first convex programs and new uniqueness results are obtained in a model that generalizes exchange and incorporates "no-free-lunch" technologies.

In the sequel we present a more precise set of definitions, as well as a sketch of previous results.

The Model. We now describe the exchange market model. Let us consider m economic agents who represent traders of n goods. Let $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ where the coordinates are nonnegative. The j-th coordinate in $\mathbf{R}^n$ will stand for the good j. Each trader i has a concave utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$. At given prices $\pi \in \mathbf{R}^n_+$, each trader will sell her endowment, and get the bundle of goods $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ which maximizes $u_i(x)$ subject to the budget constraint[4] $\pi \cdot x \leq \pi \cdot w_i$.

An equilibrium is a nonnegative vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ of goods for each trader i such that the following two conditions hold:

1. The vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$.
2. For each good j, $\sum_i \bar{x}_{ij} \leq \sum_i w_{ij}$.

The already mentioned result of Arrow and Debreu [2] implies that, under some quite mild assumptions, such an equilibrium exists. The above described

[4] Given two vectors x and y, we use the notation $x \cdot y$ to denote their inner product.

market model is usually called an *exchange economy*. A simplified model, known as Fisher's model (see [6]), arises when the economic agents are buyers, endowed with fixed incomes, competing for goods, which are available in fixed quantities.

Note that Fisher's model can be seen as a special case of an exchange economy, obtained by assuming that the initial endowments are *proportional*, i.e., $w_i = \delta_i w$, $\delta_i > 0$, so that the relative incomes of the traders are independent of the prices.

Related Results. Devanur et al. have developed a polynomial time algorithm for Fisher's model with linear utilities which is based on a number of max flow computations [13]. A polynomial time algorithm for this setting also follows from a characterization of Eisenberg and Gale [18, 19]. Unaware of an extension of [18, 19] by Eisenberg [17], Codenotti and Varadarajan have introduced a polynomial time algorithm for Fisher's model with Leontief utilities, and have shown how to extend it to homogeneous of degree one utility functions [11]. Jain, Vazirani and Ye [25] have presented some extensions to include economies of scale in production. Chen et al. [7] have proposed an algorithm for Fisher's model with logarithmic utility functions. Their algorithm runs in polynomial time when either the number of goods or the number of buyers is bounded by a constant.

The result of [13] has inspired the definition of a new model, the *spending constraint model* [14], to which the technique used in [13] can still be applied.

The computation of equilibrium prices for Fisher's model has been the main ingredient in iterative poly-time approximation schemes which compute an approximate equilibrium for the exchange model with linear utilities [24, 15]. For the same setting, Garg and Kapoor have introduced an auction based poly-time approximation scheme [20]. This method has been extended in [21] to markets where traders have separable utility functions and the individual excess demands satisfy WGS.

For the exchange model, Eaves introduced a formulation based on linear programming tailored to handle a special case of CES functions, i.e., the Cobb-Douglas utility functions [16].

Unaware of the work by Nenakov and Primak [31], which is written in Russian, Jain [23] introduced a convex program that characterizes the equilibria for the linear exchange model. His convex program turns out to be the same as the one in [31]. This convex program can also be applied to characterize the equilibria for several non-linear utilities, including the CES functions with $\rho > 0$. Ye [38] has suggested how to solve such convex programs using very efficient interior point and related methods.

Classical results [3–5] characterize the equilibria in markets which satisfy WGS as a convex set defined by an infinite number of linear inequalities. Based on the proofs of these characterizations, and a related characterization due to Primak [33], Codenotti, Pemmaraju, and Varadarajan [10] obtained poly-time algorithms for computing equilibria in such markets. Finally, in [8] the authors showed polynomial time approximation schemes based on tatonnement, as well as extensions of the results in [10] to some Fisher and production economies.

Organization of this abstract. In Section 2 we first show that equilibrium prices and allocations for an exchange economy, where the traders are endowed with CES functions with $-1 \leq \rho < 0$, can be computed by solving a feasibility problem, defined in terms of explicitly given convex constraints. We then give a related, but different, convex formulation for CES functions satisfying WGS. We finally point out the difficulty of generalizing our techniques to the entire range of CES functions. In Section 3 we present a convex formulation capturing equilibria for markets where the traders have a family of ASC utility functions. In Section 4 we provide some concluding remarks and mention further work. Most of the proofs are in the Appendix.

2 Exchange Economies with CES Utility Functions

We derive polynomial time algorithms which follow from a characterization of the equilibria in terms of convex feasibility. Since an equilibrium price vector that is rational may not exist in general, we have to settle for an *approximate equilibrium* [24], at which the allocations to the traders almost optimize their utility and the market almost clears. Our algorithm will be polynomial not only in the input parameters but also in the number of bits used in the standard encoding of the rational number representing the approximation parameter. (We postpone a detailed discussion of this to the final version.) Whenever the solution can be irrational, such an algorithm is considered equivalent to an exact algorithm.

We start by characterizing the demand function of traders with CES utility functions. Consider a setting where trader i has an initial endowment $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$ of goods, and the CES utility function $u_i(x_{i1}, \ldots, x_{in}) = \left(\sum_{j=1}^n \alpha_{ij} x_{ij}^{\rho_i}\right)^{\frac{1}{\rho_i}}$, where $\rho_i < 1$, $\alpha_{ij} \geq 0$. We assume that each good j is desired by some trader, that is, $\alpha_{ij} > 0$ for some i.

We first present a Lemma which characterizes the *demand function* of trader i at given prices ϕ, i.e., the bundle of goods which maximizes her utility function $u_i(x)$ subject to the budget constraint $\pi \cdot x \leq \pi \cdot w_i$.

Lemma 1. *Suppose trader i has an initial endowment of goods $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbf{R}^n_+$ such that not all w_{ij} are 0, and a CES utility function $u_i(x_{i1}, \ldots, x_{in}) = \left(\sum_{j=1}^n \alpha_{ij} x_{ij}^{\rho_i}\right)^{\frac{1}{\rho_i}}$, where $\rho_i < 1$, $\alpha_{ij} \geq 0$ and not all the α_{ij} are 0. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ be a set of* positive *prices for the goods. Then $\bar{x}_i = (\bar{x}_{i1}, \bar{x}_{i2}, \ldots, \bar{x}_{in})$ is the corresponding demand of trader i if and only if (a) $\pi \cdot \bar{x}_i = \pi \cdot w_i$, (b) $\bar{x}_{ij} = 0$ if $\alpha_{ij} = 0$, and (c) for any k,j such that $\alpha_{ik} > 0$ and $\alpha_{ij} > 0$ we have $\frac{\pi_k \bar{x}_{ik}^{1-\rho_i}}{\alpha_{ik}} = \frac{\pi_j \bar{x}_{ij}^{1-\rho_i}}{\alpha_{ij}}$.*

Lemma 1 is folklore. For the convenience of the reader we present a proof in the Appendix. We now show a simple Corollary of Lemma 1 which will be used in Section 2.1.

Corollary 1. *Under the assumptions of Lemma 1, $\bar{x}_i = (\bar{x}_{i1}, \bar{x}_{i2}, \ldots, \bar{x}_{in})$ is the demand of trader i at prices $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ if and only if* *(a′)* $\pi \cdot \bar{x}_i = \pi \cdot w_i$,

(b′) There exists $\beta_i > 0$ *such that for all* k*, we have* $\bar{x}_{ik} = \frac{\beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{1}{1-\rho_i}}}$.

Proof. Since $\pi \cdot w_i > 0$, conditions (a) and (b) of Lemma 1 tell us that, for each trader i, $x_{ik} > 0$ for some k such that $\alpha_{ik} > 0$; condition (c) tells us that $x_{ik} > 0$ for all k such that $\alpha_{ik} > 0$. Moreover, there exists β_i such that $\beta_i = \frac{\pi_k x_{ik}^{1-\rho_i}}{\alpha_{ik}}$ for all k such that $\alpha_{ik} > 0$. Since we must have $\beta_i > 0$, we can express x_{ik} as a function of β_i, α_{ik}, and π_k, and obtain $x_{ik} = \frac{\beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{1}{1-\rho_i}}}$.

Note that this equality holds even for i and k such that $\alpha_{ik} = 0$ (condition (b) of Lemma 1).

2.1 CES functions not satisfying WGS

We show that there is a convex formulation capturing the positive price equilibria[5] for markets where the traders have CES utility functions from the range $-1 \le \rho_i < 0$. This fact, along with the presence of disconnected equilibria when $\rho_i < -1$ (see Section 2.3), and given that CES functions with $\rho_i \ge 0$ satisfy WGS, settles the question of the elasticities for which a CES exchange economy has multiple disconnected equilibria.

We are now ready to proceed with our derivation of computationally tractable equilibrium conditions. First of all, recall that x_i and π are equilibrium allocations and prices if and only if they satisfy the conditions of Corollary 1 and the conservation of goods. Formally,

$$\sum_{1 \le k \le n} \pi_k x_{ik} = \sum_{1 \le k \le n} \pi_k w_{ik}, \quad \text{for } 1 \le i \le m. \tag{1}$$

$$\exists \beta_i > 0 \text{ such that } \; x_{ik} = \frac{\beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{1}{1-\rho_i}}}, \quad \text{for } 1 \le i \le m, \; 1 \le k \le n. \tag{2}$$

$$\sum_{1 \le i \le m} x_{ik} \le \sum_{1 \le i \le m} w_{ik}, \quad \text{for } 1 \le k \le n. \tag{3}$$

Since for each good k there is a trader i such that $\alpha_{ik} > 0$, we have $u_i(x_{i1}, \ldots, x_{ik}, \ldots, x_{in}) < u_i(x_{i1}, \ldots, x_{ik} + \delta, \ldots, x_{in})$ for any $(x_{i1}, \ldots, x_{ik}, \ldots, x_{in}) \in \mathbf{R}_+^n$ and $\delta > 0$. This

[5] For the exchange economies analyzed in this section, the possibility of some of the prices at equilibrium being zero cannot be ruled out. Such an equilibrium has to be computed by resorting to additional ad hoc but standard methods [16, 23], which we will describe in the full version.

implies that $\pi_k > 0$, for $k = 1, \dots, n$. Also, we have

$$\begin{aligned}\sum_{1\le k\le n} \pi_k \sum_{1\le i\le m} x_{ik} \le \sum_{1\le k\le n} \pi_k \sum_{1\le i\le m} w_{ik} &= \sum_{1\le i\le m}\sum_{1\le k\le n} \pi_k w_{ik} \\ = \sum_{1\le i\le m}\sum_{1\le k\le n} \pi_k x_{ik} &= \sum_{1\le k\le n} \pi_k \sum_{1\le i\le m} x_{ik},\end{aligned}$$

where the first inequality follows from the first condition of an equilibrium, the conservation of goods, and the third relation follows from the fact that x_i is the demand of buyer i at prices π, so that condition (a') of Corollary 1 holds. Thus the first inequality must be an equality, and we have

$$\sum_{1\le i\le m} x_{ik} = \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n. \tag{4}$$

So an equilibrium is a vector of positive prices $\pi = (\pi_1, \dots, \pi_n)$ for which there are allocations $x_i \in \mathbf{R}^n_+$ such that relations 1, 2, and 4 are satisfied.

By an argument similar to the one used above, we now replace equalities 1 and 4 by the following two sets of inequalities:

$$\sum_{1\le k\le n} \pi_k x_{ik} \ge \sum_{1\le k\le n} \pi_k w_{ik}, \quad \text{for } 1 \le i \le m, \tag{5}$$

$$\sum_{1\le i\le m} x_{ik} \le \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n. \tag{6}$$

An equilibrium is a vector of positive prices $\pi = (\pi_1, \dots, \pi_n)$ for which there are allocations $x_i \in \mathbf{R}^n_+$ such that inequalities 5, 6, and equalities 2 are satisfied.

We can now use equalities 2 to eliminate the variables x_{ik} from inequalities 5 and 6. We obtain the following two sets of inequalities:

$$\beta_i^{\frac{1}{1-\rho_i}} \sum_{1\le k\le n} \pi_k^{\frac{-\rho_i}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \ge \sum_{1\le k\le n} \pi_k w_{ik}, \quad \text{for } 1 \le i \le m, \tag{7}$$

$$\sum_{1\le i\le m} \left(\frac{\beta_i \alpha_{ik}}{\pi_k}\right)^{\frac{1}{1-\rho_i}} \le \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n. \tag{8}$$

That is, an equilibrium is a vector of positive prices $\pi = (\pi_1, \dots, \pi_n)$ such that there exists $\beta_i > 0$, for each i, such that the relations 7 and 8 are satisfied.

Let $\rho = \min_i \rho_i$. We now introduce some new variables σ_k, related to the prices as $\sigma_k = \pi_k^{\frac{1}{1-\rho}}$, and z_i, related to β_i as $z_i = \beta_i^{\frac{1}{1-\rho}}$. We then show that after these variable substitutions, inequalities 7 and 8 become inequalities that define convex sets, thus giving us a convex program in terms of the β_i and σ_k that characterizes equilibria.

In terms of the σ_k's and z_i's, inequalities 7 become

$$z_i^{\frac{1-\rho}{1-\rho_i}} \sum_{1\le k\le n} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \ge \sum_{1\le k\le n} \sigma_k^{1-\rho} w_{ik}, \quad \text{for } 1 \le i \le m. \tag{9}$$

A convenient way to see that each inequality in 9 defines a convex set is to take the power $\frac{1}{1-\rho}$ of both sides and obtain

$$z_i^{\frac{1}{(1-\rho_i)}} \left(\sum_{1\le k\le n} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \right)^{\frac{1}{1-\rho}} \ge \left(\sum_{1\le k\le n} \sigma_k^{1-\rho} w_{ik} \right)^{\frac{1}{1-\rho}}, \quad \text{for } 1 \le i \le m. \tag{10}$$

The left hand side of this inequality is

$$(z_i)^{\frac{1}{1-\rho_i}} \left[\left(\sum_{1\le k\le n} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \right)^{\frac{1-\rho_i}{-\rho_i(1-\rho)}} \right]^{\frac{-\rho_i}{1-\rho_i}}. \tag{11}$$

The function

$$f_i(\sigma_1, \ldots, \sigma_k) = \left(\sum_{1\le k\le n} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \right)^{\frac{1-\rho_i}{-\rho_i(1-\rho)}}$$

is concave. Since $0 < \frac{-\rho_i(1-\rho)}{1-\rho_i} \le 1$, it is in fact a CES function of the σ_k. Since both $\frac{1}{1-\rho_i}$ and $\frac{-\rho_i}{1-\rho_i}$ are positive, and $\frac{1}{1-\rho_i} + \frac{-\rho_i}{1-\rho_i} = 1$, the left hand side 11, which is $z_i^{\frac{1}{1-\rho_i}} f_i^{\frac{-\rho_i}{1-\rho_i}}$, is a concave function. Since $\rho < 0$, it is easy to check that the right hand side of inequality 10 is a convex function. Therefore the set of inequalities 10 are convex and each inequality in 9 defines a convex set.

We now turn to inequalities 8 and rewrite them as

$$\sum_{1\le i\le m} \frac{\beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{1}{1-\rho_i} - \frac{1}{1-\rho}}} \le \pi_k^{\frac{1}{1-\rho}} \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n, \tag{12}$$

We can now plug in the σ_k's and the z_i's to get the inequalities

$$\sum_{1\le i\le m} \frac{z_i^{\frac{1-\rho}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\sigma_k^{\frac{\rho_i-\rho}{1-\rho_i}}} \le \sigma_k \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n. \tag{13}$$

To see that each inequality in 13 describes a convex set, we finally introduce new variables t_{ik}, and substitute inequalities 13 with

$$\frac{z_i^{\frac{1-\rho}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\sigma_k^{\frac{\rho_i-\rho}{1-\rho_i}}} \leq t_{ik}, \quad \text{for } 1 \leq k \leq n, \quad \text{and } 1 \leq i \leq m. \tag{14}$$

and

$$\sum_{1 \leq i \leq m} t_{ik} \leq \sigma_k \sum_{1 \leq i \leq m} w_{ik}, \quad \text{for } 1 \leq k \leq n. \tag{15}$$

It is easy to see that inequalities 14 and 15 are equivalent to inequalities 13.

Inequalities 14 can be seen to be convex after rewriting them as

$$t_{ik}^{\frac{1-\rho_i}{1-\rho}} \sigma_k^{\frac{\rho_i-\rho}{1-\rho}} \geq \alpha_{ik}^{\frac{1}{1-\rho}} z_i, \quad \text{for } 1 \leq k \leq n, \quad \text{and } 1 \leq i \leq m, \tag{16}$$

since both $\frac{1-\rho_i}{1-\rho}$ and $\frac{\rho_i-\rho}{1-\rho}$ are nonnegative and $\frac{1-\rho_i}{1-\rho} + \frac{\rho_i-\rho}{1-\rho} = 1$. Inequalities 15 are linear. We therefore have the following theorems.

Theorem 1. *Let $\sigma_1, \ldots, \sigma_n$, and $z_1, \ldots, z_m$ be positive real numbers satisfying the set of inequalities 9 and 13 each of which defines a convex set (or equivalently, the set of inequalities 10, 15, and 16, each of which is a convex constraint). Then the vector $\pi = (\pi_1, \ldots, \pi_n)$, given by $\pi_k = \sigma_k^{1-\rho}$, is an equilibrium for the market.*

Note that the allocations that correspond to this equilibrium can also be easily computed.

Theorem 2. *Suppose the vector $\pi = (\pi_1, \ldots, \pi_n)$ is an equilibrium price vector. Let $\sigma_k = \pi_k^{\frac{1}{1-\rho}}$, where $\rho = \min_i \rho_i$. Then $\sigma_1, \ldots, \sigma_n$ is part of a feasible, positive, solution to the system of inequalities 9 and 13 (or equivalently, the system of inequalities 10, 15, and 16.)*

2.2 CES functions satisfying WGS

In this section, we consider exchange economies with CES utilities whose elasticity of substitution is greater than one, and thus satisfy WGS. We show how to formulate the problem of computing an equilibrium as convex feasibility. The overall approach is quite similar to the one used in Section 2.1.

Consider the program CP1 , which consists of finding positive real numbers $\sigma_1, \ldots, \sigma_n$ and $z_1, \ldots, z_m$ satisfying

$$
\begin{aligned}
&\sum_{1\le i\le m} c_{ik} z_i^{t_i} \sigma_k^{1-t_i} \ge q_k \sigma_k \text{ for } 1 \le k \le n \\
&z_i \le \left[\left(\sum_k \sigma_k^{1-\rho} w_{ik}\right)^{\frac{1}{1-\rho}}\right]^{1-\rho_i} \times \left[\left(\sum_k c_{ik} \sigma_k^{-t_i\rho_i}\right)^{\frac{1}{-t_i\rho_i}}\right]^{\rho_i} \quad \text{for } 1 \le i \le m, \\
&\sigma_k > 0 \text{ for } 1 \le k \le n \\
&z_i > 0 \text{ for } 1 \le i \le m\,,
\end{aligned}
$$

where $\rho = \max_i \rho_i$, $t_i = \frac{1-\rho}{1-\rho_i}$, $c_{ik} = \alpha_{ik}^{\frac{1}{1-\rho_i}}$, and $q_k = \sum_i w_{ik}$. Recall that $0 < \rho_i < 1$, and thus $0 < \rho < 1$ and $0 < t_i \le 1$.

The sets of inequalities defining CP1 are convex constraints. For the first set of inequalities, we have that the function on the left hand side is a concave function, and the one on the right is a linear function. For the second set of inequalities, the right hand side is a function of the form $f^{1-\rho_i} g^{\rho_i}$, where the f and g are concave functions (in fact CES functions). Thus the right hand side is a concave function, and the second set of inequalities define convex constraints. Thus CP1 is a convex feasibility problem. Note that it is homogeneous in the σ_j's and the z_i's. So we can solve it by replacing the constraints that these variables be positive by the constraints that they be at least 1.

Theorem 3. *Let $\hat{\sigma}$ and $\hat{z}$ be a solution to CP1 . Let $x_i \in \mathbf{R}_+^n$ be the vector whose k-th component is $x_{ik} = \frac{\hat{z_i}^{t_i} c_{ik}}{\hat{\sigma}_k^{t_i}}$, and $\pi \in \mathbf{R}_+^n$ be the vector whose k-th component is $\pi_k = \hat{\sigma}_k^{1-\rho}$. Then π and x_i, for $i = 1, \ldots, m$, are equilibrium prices and allocations.*

Theorem 4. *Let π and x_i, for $i = 1, \ldots, m$, be equilibrium prices and allocations. Let $\sigma_k = \pi_k^{\frac{1}{1-\rho}}$, and for each i, pick some k such that $\alpha_{ik} > 0$ and set $z_i = \frac{x_{ik}^{\frac{1-\rho_i}{1-\rho}} \pi_k^{\frac{1}{1-\rho}}}{\alpha_{ik}^{\frac{1}{1-\rho}}}$. Then the z_i's and the σ_k's satisfy CP1 .*

2.3 CES exchange economies with multiple disconnected equilibria

In [22], Gjerstad gives the following example of a market with two traders and two goods that has multiple disconnected equilibria. The first trader has an initial bundle $w_1 = (1, 0)$ and the CES utility function $u_1(x, y) = ((ax)^\rho + y^\rho)^{1/\rho}$, where $a > 0$. The second trader has an initial bundle $w_2 = (0, 1)$ and the CES utility function $u_2(x, y) = ((x/a)^\rho + y^\rho)^{1/\rho}$. He shows that for each $\rho < -1$ there is a sufficiently small value of a for which

1. The vector $(1/2, 1/2)$ is an equilibrium price.
2. The vector $(p, 1-p)$ is an equilibrium price for some $p < 1/2$, and the vector $(q, 1-q)$ is not an equilibrium price for any $p < q < 1/2$.

This example therefore does not admit a convex programming formulation in terms of some "relative" of the prices (such as the one given in Sections 2.1 and 2.2 in terms of the σ_k) that captures *all* the price equilibria. Such a formulation implies that if $(p_1, 1-p_1)$ is a price equilibrium and $(p_2, 1-p_2)$ is a price equilibrium for some $p_1 < p_2$, then $(p_3, 1-p_3)$ is also a price equilibrium for every $p_1 < p_3 < p_2$.

This suggests that it may not be possible to extend convex programming techniques to encompass markets where some buyers have a CES utility function with the elasticity $\sigma < 1/2$ (which corresponds to $\rho < -1$ in the example above).

3 Exchange Economies with ASC Utility Functions

We now show how to extend the results of Section 2.2 to economies with some additively separable utility functions.

Consider an exchange economy with n goods and m traders, where the i-th trader has the separable and additive utility function $u_i(x_i) = \sum_j \alpha_{ij} x_{ij}^{\rho_{ij}}$, where $\alpha_{ij} \geq 0$, and $0 < \rho_{ij} < 1$. We assume that, for each trader i, there is some j such that $\alpha_{ij}\rho_{ij} > 0$, and that, for each good j, there is some i such that $\alpha_{ij}\rho_{ij} > 0$. Let $\rho = \max_{i,j} \rho_{ij}$. We consider the program CP2 which consists of finding positive real numbers $\sigma_1, \ldots, \sigma_n$ and $z_1, \ldots, z_m$ satisfying

$$\sum_{1 \leq i \leq m} \rho_{ij}^{\frac{1}{1-\rho_{ij}}} \alpha_{ij}^{\frac{1}{1-\rho_{ij}}} z_i^{\frac{1-\rho}{1-\rho_{ij}}} \sigma_j^{\frac{\rho-\rho_{ij}}{1-\rho_{ij}}} \geq \sigma_j \sum_{1 \leq i \leq m} w_{ij} \text{ for } 1 \leq j \leq n$$

$$z_i \leq \left[\left(\sum_j \sigma_j^{1-\rho} w_{ij} \right)^{\frac{1}{1-\rho}} \right]^{1-\rho} \times \left[\left[\sum_j \rho_{ij}^{\frac{1}{1-\rho_{ij}}} \alpha_{ij}^{\frac{1}{1-\rho_{ij}}} \left(\sigma_j^{\frac{(1-\rho)\rho_{ij}}{\rho(1-\rho_{ij})}} z_i^{\frac{\rho-\rho_{ij}}{\rho(1-\rho_{ij})}} \right)^{-\rho} \right]^{-\frac{1}{\rho}} \right]^{\rho} \text{ for } 1 \leq i \leq m\,.$$

Note that the sets of inequalities defining CP2 are convex constraints. For lack of space, the details about the characterization of this Section are in the Appendix.

4 Conclusions

The existence of multiple disconnected equilibria had troubled the economists several decades ago. It pointed out the inadequacy, and, sometimes, the inapplicability of tools from general equilibrium theory. As computer scientists, we are now facing a similar challenge.

Previous work showed the computational tractability of markets that satisfy well understood properties like gross substitutability and the weak axiom of revealed preferences. We think that the unrestricted problem is likely to be intractable. The challenge is to explore the unknown territory between these classes, by trying to characterize other tractable classes, develop new techniques to compute their equilibria, and understand why and how these techniques work. This paper is a modest first step in this program.

We see several directions which we believe are amenable to analysis along these lines. We expect progress in the investigation of some exchange markets where (1) multiple equilibria do exist, but (2) the market demand is structured and need not be tightly coupled with general fixed point computations, as it occurs in the general case [32]. Features (1) and (2) are present, perhaps, when the traders have CES functions with elasticity of substitution smaller than $\frac{1}{2}$.

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Appendix

Proof of Lemma 1.

Proof. This lemma is folklore: it follows from well-known techniques. For completeness we give a proof here. Let us first show that if $\bar{x}_i$ is a utility maximizing bundle, it satisfies (a), (b), and (c). The condition (a) follows from local non-satiation of the CES functions, because if part of the budget is not spent the utility can always be increased by spending it on any good j such that $\alpha_{ij} > 0$. The condition (b) follows because all the prices are positive, and so the buyer will not spend any part of her budget on a good that does not add to her utility. For (c), we write the concave maximization program for maximizing buyer i's utility:

$$\begin{aligned}
&\text{Maximize } u_i(x_i)\\
&\text{Subject to } \pi \cdot x_i \leq \pi \cdot w_i\\
&\qquad\qquad x_{ij} \geq 0 \text{ for } 1 \leq j \leq n
\end{aligned}$$

We have that $\bar{x}_i$ maximizes this program. We must have $\bar{x}_{ij} > 0$ if $\alpha_{ij} > 0$. Otherwise, the partial derivative $\frac{\partial u_i}{\partial x_{ij}}$ is infinite at $\bar{x}_i$ and this easily implies that $\bar{x}_i$ does not maximize this program. Now the Kuhn-Tucker stationary-point necessary optimality theorem ([26], page 105) says that there exists $\lambda \geq 0$ such that

1. $\frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}} \leq \lambda \pi_j$ for $1 \leq j \leq n$.
2. $\frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}} = \lambda \pi_j$ if $\bar{x}_{ij} > 0$.
3. $\lambda = 0$ if $\pi \cdot \bar{x}_i < \pi \cdot w_i$.

From the second condition, we see that $\lambda > 0$. Using the second condition once again and eliminating the Lagrange multiplier λ, we obtain for any k and j such that $\alpha_{ik} > 0$ and $\alpha_{ij} > 0$ the equality
$\frac{\pi_j}{\pi_k} = \frac{\frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}}}{\frac{\partial u_i(\bar{x}_i)}{\partial x_{ik}}}$,
which states the well known fact that, at the optimal solution for trader i, the *marginal rate of substitution* between goods j and k (which is the ratio on the right hand side) must be equal to their price ratios.

The computation of the partial derivatives then gives

$$\frac{\pi_j}{\pi_k} = \frac{\frac{1}{\rho_i} \cdot \left(\sum_{j=1}^n \alpha_{ij} \bar{x}_{ij}^{\rho_i}\right)^{\frac{1}{\rho_i}-1} \cdot \alpha_{ij} \cdot \rho_i \cdot \bar{x}_{ij}^{\rho_i - 1}}{\frac{1}{\rho_i} \cdot \left(\sum_{j=1}^n \alpha_{ij} \bar{x}_{ij}^{\rho_i}\right)^{\frac{1}{\rho_i}-1} \cdot \alpha_{ik} \cdot \rho_i \cdot \bar{x}_{ik}^{\rho_i - 1}},$$

i.e.,

$$\frac{\pi_j}{\pi_k} = \frac{\alpha_{ij} \cdot \bar{x}_{ij}^{\rho_i - 1}}{\alpha_{ik} \cdot \bar{x}_{ik}^{\rho_i - 1}}.$$

We can rewrite this equation as

$$\frac{\pi_j \bar{x}_{ij}^{1-\rho_i}}{\alpha_{ij}} = \frac{\pi_k \bar{x}_{ik}^{1-\rho_i}}{\alpha_{ik}},$$

which completes the proof in one direction.

For the other direction, suppose we have a $\bar{x}_i \in \mathbf{R}_+^n$ that satisfies (a), (b), and (c). From (a) and (b), we have that $\bar{x}_{ij} > 0$ for some j such that $\alpha_{ij} > 0$, so (c) then implies that $\bar{x}_{ij} > 0$ for all j such that $\alpha_{ij} > 0$. Now by calculations similar to the one above, we get from (c) that for any k and j such that $\alpha_{ik} > 0$ and $\alpha_{ij} > 0$:
$\frac{\pi_j}{\pi_k} = \frac{\frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}}}{\frac{\partial u_i(\bar{x}_i)}{\partial x_{ik}}}$.

Let $\lambda = \pi_j / \frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}}$ for all j such that $\alpha_{ij} > 0$. Clearly λ is well-defined and $\lambda > 0$. We also have that $\frac{\partial u_i(\bar{x}_i)}{\partial x_{ij}} = 0$, for all j such that $\alpha_{ij} = 0$.

Thus we see that $\bar{x}_i$ and λ satisfy the three Kuhn-Tucker conditions above. Since the utility maximization program of buyer i is convex, we can now use the Kuhn-Tucker sufficient optimality theorem ([26], page 94) to conclude that $\bar{x}_i$ is the optimal solution to this program and is therefore the demand of trader i at prices π.

Proof of Theorem 3

Proof. From the first set of inequalities in CP1 we get

$$\sum_{1 \le i \le m} \left(\frac{c_{ik} \hat{z}_i^{t_i}}{\hat{\sigma}_k^{t_i}} \right) \hat{\sigma}_k \ge q_k \hat{\sigma}_k \text{ for } 1 \le k \le n.$$

Canceling $\hat{\sigma}_k$ out, we get

$$\sum_{1 \le i \le m} \left(\frac{c_{ik} \hat{z}_i^{t_i}}{\hat{\sigma}_k^{t_i}} \right) \ge q_k \text{ for } 1 \le k \le n.$$

From the expressions $x_{ik} = \frac{\hat{z}_i^{t_i} c_{ik}}{\hat{\sigma}_k^{t_i}}$ and $q_k = \sum_i w_{ik}$, we get

$$\sum_{1 \le i \le m} x_{ik} \ge \sum_{1 \le i \le m} w_{ik} \text{ for } 1 \le k \le n\,. \tag{17}$$

The second set of inequalities of CP1 implies that

$$\hat{z}_i \le \left[\left(\sum_k \hat{\sigma}_k^{1-\rho} w_{ik} \right)^{\frac{1}{1-\rho}} \right]^{1-\rho_i} \cdot \frac{1}{\left[\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}} \right)^{\frac{1-\rho_i}{(1-\rho)\rho_i}} \right]^{\rho_i}} \text{ for } 1 \le i \le m.$$

This simplifies to

$$\hat{z}_i \le \left(\sum_k \hat{\sigma}_k^{1-\rho} w_{ik}\right)^{\frac{1-\rho_i}{1-\rho}} \cdot \frac{1}{\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right)^{\frac{1-\rho_i}{1-\rho}}}.$$

Raising both sides to the power $\frac{1-\rho}{1-\rho_i}$, we obtain

$$\hat{z}_i^{\frac{1-\rho}{1-\rho_i}} \le \left(\sum_k \hat{\sigma}_k^{1-\rho} w_{ik}\right) \cdot \frac{1}{\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right)}.$$

Rearranging, we get

$$\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}} \hat{z}_i^{\frac{1-\rho}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right) \le \sum_k \hat{\sigma}_k^{1-\rho} w_{ik}.$$

This is rewritten as

$$\left(\sum_k \left(\frac{\alpha_{ik}^{\frac{1}{1-\rho_i}} \hat{z}_i^{\frac{1-\rho}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{1-\rho}{1-\rho_i}}}\right) \hat{\sigma}_k^{1-\rho}\right) \le \sum_k \hat{\sigma}_k^{1-\rho} w_{ik}.$$

Plugging in $x_{ik} = \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}} \hat{z}_i^{\frac{1-\rho}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{1-\rho}{1-\rho_i}}}$ and $\pi_k = \hat{\sigma}_k^{1-\rho}$, we get

$$\sum_{1 \le k \le n} \pi_k x_{ik} \le \sum_{1 \le k \le n} \pi_k w_{ik} \text{ for } 1 \le i \le m. \tag{18}$$

Observe now that

$$\begin{aligned}\sum_{1 \le k \le n} \pi_k \sum_{1 \le i \le m} x_{ik} \ge \sum_{1 \le k \le n} \pi_k \sum_{1 \le i \le m} w_{ik} &= \sum_{1 \le i \le m} \sum_{1 \le k \le n} \pi_k w_{ik} \\ \ge \sum_{1 \le i \le m} \sum_{1 \le k \le n} \pi_k x_{ik} &= \sum_{1 \le k \le n} \pi_k \sum_{1 \le i \le m} x_{ik}\end{aligned}$$

where the first inequality follows from (17) and the second from (18). Note that the two inequalities in this sequence must be equalities. Given that all the $\pi_k > 0$, this readily implies that

$$\sum_{1\le i\le m} x_{ik} = \sum_{1\le i\le m} w_{ik} \text{ for } 1 \le k \le n, \tag{19}$$

and

$$\sum_{1\le k\le n} \pi_k x_{ik} = \sum_{1\le k\le n} \pi_k w_{ik} \text{ for } 1 \le i \le m. \tag{20}$$

Now, setting $\beta_i = \hat{z}_i^{1-\rho}$ and $\hat{\sigma}_k^{1-\rho} = \pi_k$ in $x_{ik} = \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}} \hat{z}_i^{\frac{1-\rho}{1-\rho_i}}}{\hat{\sigma}_k^{\frac{1-\rho}{1-\rho_i}}}$, we get $x_{ik} = \left(\frac{\beta_i \alpha_{ik}}{\pi_k}\right)^{\frac{1}{1-\rho_i}}$. This means that

$$\frac{x_{ik}^{1-\rho_i}\pi_k}{\alpha_{ik}} = \beta_i \tag{21}$$

if $\alpha_{ik} > 0$, and $x_{ik} = 0$ if $\alpha_{ik} = 0$. The latter fact along with Equations 20 and 21 show that conditions (a') and (b') of Corollary 1 are fulfilled for each trader i, so x_i is her demand at prices π. Equation 19 gives the conservation of goods. So we have an equilibrium.

Proof of Theorem 4

Proof. Since the first part of the proof is the same as the derivation of Section 2.1, we can start from relations 1, 2, and 4.

We now replace 1 and 4 with inequalities, which are reversed as compared with those from Section 2.1. We can then use equality 2 to obtain the two sets of inequalities:

$$\sum_i \frac{\beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{1}{1-\rho_i}}} \ge \sum_i w_{ik}, \;\; 1 \le k \le n, \tag{22}$$

$$\beta_i^{\frac{1}{1-\rho_i}} \sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{\rho_i}{1-\rho_i}}} \le \sum_k \pi_k w_{ik}, \;\; 1 \le i \le m. \tag{23}$$

From 22 and 23, we now derive the convex inequalities of CP1 .

If we now multiply both sides of 22 by $\pi_k^{\frac{1}{1-\rho}}$, we obtain

$$\sum_i \beta_i^{\frac{1}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \pi_k^{\frac{1}{1-\rho} - \frac{1}{1-\rho_i}} \ge \pi_k^{\frac{1}{1-\rho}} \sum_i w_{ik}.$$

If we now plug in the z_i's and the σ_k's, we finally obtain

$$\sum_i z_i^{\frac{1-\rho}{1-\rho_i}} \alpha_{ik}^{\frac{1}{1-\rho_i}} \sigma_k^{1-\frac{1-\rho}{1-\rho_i}} \geq \sigma_k \sum_i w_{ik},$$

which means that the z_i's and the σ_k's satisfy the first set of inequalities in CP1.

We can write inequalities 23 as

$$\beta_i^{\frac{1}{1-\rho_i}} \leq \left(\sum_k \pi_k w_{ik}\right) \cdot \frac{1}{\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\pi_k^{\frac{\rho_i}{1-\rho_i}}}\right)}.$$

If we now plug in the z_i's and the σ_k's, we obtain

$$z_i^{\frac{1-\rho}{1-\rho_i}} \leq \left(\sum_k \sigma_k^{1-\rho} w_{ik}\right) \cdot \frac{1}{\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\sigma_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right)}.$$

Raising both sides to the power $\frac{1-\rho_i}{1-\rho}$, we get

$$z_i \leq \left(\sum_k \sigma_k^{1-\rho} w_{ik}\right)^{\frac{1-\rho_i}{1-\rho}} \cdot \frac{1}{\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\sigma_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right)^{\frac{1-\rho_i}{1-\rho}}}.$$

The last inequality can finally be rewritten as

$$z_i \leq \left[\left(\sum_k \sigma_k^{1-\rho} w_{ik}\right)^{\frac{1}{1-\rho}}\right]^{1-\rho_i} \cdot \frac{1}{\left[\left(\sum_k \frac{\alpha_{ik}^{\frac{1}{1-\rho_i}}}{\sigma_k^{\frac{(1-\rho)\rho_i}{1-\rho_i}}}\right)^{\frac{1-\rho_i}{(1-\rho)\rho_i}}\right]^{\rho_i}},$$

which means that the second set of inequalities in CP1 are satisfied.

Exchange Economies with ASC Utility Functions

We present here the results of Section 3.

Consider an exchange economy with n goods and m traders, where the i-th trader has the separable and additive utility function $u_i(x_i) = \sum_j \alpha_{ij} x_{ij}^{\rho_{ij}}$, where $\alpha_{ij} \geq 0$, and $0 < \rho_{ij} < 1$. We assume that, for each trader i, there is some j such that $\alpha_{ij}\rho_{ij} > 0$, and that, for each good j, there is some i such that $\alpha_{ij}\rho_{ij} > 0$. Let $\rho = \max_{i,j} \rho_{ij}$. We consider the program CP2 which consists of finding positive real numbers $\sigma_1, \ldots, \sigma_n$ and $z_1, \ldots, z_m$ satisfying

$$\sum_{1 \leq i \leq m} \rho_{ij}^{\frac{1}{1-\rho_{ij}}} \alpha_{ij}^{\frac{1}{1-\rho_{ij}}} z_i^{\frac{1-\rho}{1-\rho_{ij}}} \sigma_j^{\frac{\rho-\rho_{ij}}{1-\rho_{ij}}} \geq \sigma_j \sum_{1 \leq i \leq m} w_{ij} \text{ for } 1 \leq j \leq n$$

$$z_i \leq \left[\left(\sum_j \sigma_j^{1-\rho} w_{ij} \right)^{\frac{1}{1-\rho}} \right]^{1-\rho} \times \left[\left[\sum_j \rho_{ij}^{\frac{1}{1-\rho_{ij}}} \alpha_{ij}^{\frac{1}{1-\rho_{ij}}} \left(\sigma_j^{\frac{(1-\rho)\rho_{ij}}{\rho(1-\rho_{ij})}} z_i^{\frac{\rho-\rho_{ij}}{\rho(1-\rho_{ij})}} \right)^{-\rho} \right]^{-\frac{1}{\rho}} \right]^{\rho} \text{ for } 1 \leq i \leq m\,.$$

Note that the sets of inequalities defining CP2 are convex constraints. In the first set of inequalities, the left hand side is a concave function, and the right hand side is a linear function. For the second set of inequalities, note that the second factor on the right hand side has the form B^ρ, where $B = \left(\frac{1}{\sum_k \frac{c_k}{B_k^\rho}} \right)^{\frac{1}{\rho}}$. Furthermore, note that $\frac{(1-\rho)\rho_{ik}}{\rho(1-\rho_{ik})} + \frac{\rho-\rho_{ik}}{\rho(1-\rho_{ik})} = 1$. Therefore B_k is a Cobb-Douglas function, which implies that B is a concave nested CES function. Thus the right hand side is a concave function, which shows that these inequalities are convex.

The following two Theorems show that the convex program CP2 captures the equilibrium conditions.

Theorem 5. *Let σ and z be a solution to CP2 . Let $\pi_j = \sigma_j^{1-\rho}$, and $\beta_i = z_i^{1-\rho}$. Let $x_i \in \mathbf{R}_+^n$ be the vector whose j-th component is $x_{ij} = \left(\frac{\rho_{ij}\beta_i\alpha_{ij}}{\pi_j} \right)^{\frac{1}{1-\rho_{ij}}}$. Then π and x_i's are equilibrium prices and allocations.*

Proof. The proof is quite similar to that of Theorem 3 and is omitted.

Theorem 6. *Let π and x_i's be equilibrium prices and allocations. For each i, pick some j such that $\alpha_{ij} > 0$, and let $\beta_i = \left(\frac{\pi_j x_{ij}^{1-\rho_{ij}}}{\rho_{ij}\alpha_{ij}} \right)$. Let $\sigma_j = \pi_j^{\frac{1}{1-\rho}}$, and $z_i = \beta_i^{\frac{1}{1-\rho}}$. Then σ and z are a solution to CP2 .*

Proof. Equilibrium prices and allocations satisfy the sets of equalities 1 and 4 from Section 2.1. These conditions can be equivalently rewritten as inequalities:

$$\sum_{1 \leq i \leq m} x_{ik} \geq \sum_{1 \leq i \leq m} w_{ik}\,, \quad \text{for } 1 \leq k \leq n, \tag{24}$$

and

$$\sum_{1 \leq k \leq n} \pi_k x_{ik} \leq \sum_{1 \leq k \leq n} \pi_k w_{ik}\,, \quad \text{for } 1 \leq i \leq m. \tag{25}$$

The solution to the consumer's optimization problem for ASC functions (which consists of a simple extension of Lemma 1) implies that there exists $\beta_i > 0$ such that

$$\begin{cases} \frac{\pi_j x_{ij}^{1-\rho_{ij}}}{\rho_{ij}\alpha_{ij}} = \beta_i \text{ if } \alpha_{ij} > 0 \\ x_{ij} = 0 \qquad\quad \text{if } \alpha_{ij} = 0 \end{cases} \tag{26}$$

Conditions 26 are equivalent to saying that there is a $\beta_i > 0$ such that

$$x_{ij} = \left(\frac{\rho_{ij}\alpha_{ij}\beta_i}{\pi_j}\right)^{\frac{1}{1-\rho_{ij}}}, \tag{27}$$

which is always well defined since $\pi_j > 0$, for all j.

Now the sets of relations 24, 25, and 27, have a solution if and only if the inequalities 28, 29, and 30 below do:

$$\sum_{1\le i\le m} \left(\frac{\rho_{ik}\alpha_{ik}\beta_i}{\pi_k}\right)^{\frac{1}{1-\rho_{ik}}} \ge \sum_{1\le i\le m} w_{ik}, \quad \text{for } 1 \le k \le n, \tag{28}$$

$$\sum_{1\le k\le n} \pi_k \left(\frac{\rho_{ik}\alpha_{ik}\beta_i}{\pi_k}\right)^{\frac{1}{1-\rho_{ik}}} \le \sum_{1\le k\le n} \pi_k w_{ik}\,, \quad \text{for } 1 \le i \le m, \tag{29}$$

$$\pi_k > 0, \quad \text{for } 1 \le k \le n,\ \beta_i > 0\,, \quad \text{for } 1 \le i \le m. \tag{30}$$

Let us now define $\rho = \max \rho_{ij}$, and let us make the substitutions $z_i = \beta_i^{\frac{1}{1-\rho}}$, and $\sigma_k = \pi_k^{\frac{1}{1-\rho}}$. First of all, note that conditions 30 translate into $\sigma_k > 0$, and $z_i > 0$.

After plugging in the σ_k's and the z_i's, inequalities 28 become

$$\sum_{1\le i\le m} \rho_{ik}^{\frac{1}{1-\rho_{ik}}} \alpha_{ik}^{\frac{1}{1-\rho_{ik}}} z_i^{\frac{1-\rho}{1-\rho_{ik}}} \sigma_k^{\frac{\rho-\rho_{ik}}{1-\rho_{ik}}} \ge \sigma_k \sum_{1\le i\le m} w_{ik} \text{ for } 1 \le k \le n, \tag{31}$$

which is the first set of convex constraints in CP2 .

Similarly, inequalities 29 become

$$z_i \sum_{1\le k\le n} \frac{\rho_{ik}^{\frac{1}{1-\rho_{ik}}} \alpha_{ik}^{\frac{1}{1-\rho_{ik}}}}{\sigma_k^{\frac{(1-\rho)\rho_{ik}}{1-\rho_{ik}}} z_i^{\frac{\rho-\rho_{ik}}{1-\rho_{ik}}}} \le \sum_{1\le k\le n} \sigma_k^{1-\rho} w_{ik} \text{ for } 1 \le i \le m, \tag{32}$$

which can be conveniently rewritten in the form

$$z_i \leq \left[\left(\sum_{1 \leq k \leq n} \sigma_k^{1-\rho} w_{ik} \right)^{\frac{1}{1-\rho}} \right]^{1-\rho} \times \left[\left(\frac{1}{g(i)} \right)^{\frac{1}{\rho}} \right]^{\rho} \text{ for } 1 \leq i \leq m, \tag{33}$$

where

$$g(i) = \sum_k \frac{\rho_{ik}^{1/(1-\rho_{ik})} \alpha_{ik}^{1/(1-\rho_{ik})}}{(\sigma_k^{((1-\rho)\rho_{ik})/(\rho(1-\rho_{ik}))} z_i^{(\rho-\rho_{ik})/(\rho(1-\rho_{ik}))})^{\rho}}$$

which is the second set of constraints in CP2 .