Algorithms Column: The Computation of Market Equilibria

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November 10, 2004

1 Column editor's note

This issue's column is written by guest columnists, Bruno Codenotti, Sriram Pemmaraju and Kasturi Varadarajan. I am delighted that they agreed to write this timely column on the topic related to the computation of market equilibria that has received much attention recently. Their column introduces the reader to several recent results and provides references for further readings.
Samir Khuller

2 Introduction

The market equilibrium problem has a long and distinguished history. In 1874, Walras published the famous "Elements of Pure Economics", where he describes a model for the state of an economic system in terms of demand and supply, and expresses the *supply equals demand* equilibrium conditions [62]. In 1936, Wald gave the first proof of the existence of an equilibrium for the Walrasian system, albeit under severe restrictions [61]. In 1954, Nobel laureates Arrow and Debreu proved the existence of an equilibrium under milder assumptions [3]. This existence result, along with the two fundamental theorems of welfare are the pillars of modern equilibrium theory. The *First Fundamental Theorem of Welfare* showed the Pareto optimality of allocations at equilibrium prices, thereby formally expressing Adam Smith's "invisible hand" property of markets and providing important social justification for the theory of equilibrium.

We describe a model of the so-called *exchange economy*, an important special case of the model considered by Arrow and Debreu [3]. The exchange economy does not include

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the production of goods, whereas the more general model considered by Arrow and Debreu, henceforth the *Arrow-Debreu model*, does include the production of goods.

Let us consider m economic agents who represent traders of n goods. Let $\mathbf{R}_+^n$ denote the subset of $\mathbf{R}^n$ with all nonnegative coordinates. The j-th coordinate in $\mathbf{R}^n$ will stand for good j. Each trader i has a concave utility function $u_i : \mathbf{R}_+^n \to \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}_+^n$. At given prices $\pi \in \mathbf{R}_+^n$, trader i will sell her endowment, and get the bundle of goods $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}_+^n$ which maximizes $u_i(x)$ subject to the budget constraint[3] $\pi \cdot x \leq \pi \cdot w_i$.

An equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}_+^n$ at which, for each trader i, there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}_+^n$ of goods such that the following two conditions hold:

1. For each trader i, the vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}_+^n$.

2. For each good j, $\sum_i \bar{x}_{ij} \leq \sum_i w_{ij}$.

Starting in the 60's, economists have attempted to use the Arrow-Debreu model and the general equilibrium theory to realistically model actual economies, with the goal of evaluating alternate policy options [55]. Examples of policy options whose analysis is amenable to this kind of modeling include tax-reform (such as the one debated in the U.S. during 1984-86) and simultaneous tariff reductions in several countries (which is often the result of trade negotiations in the GATT). These models are typically large enough that they cannot be analyzed in ways useful to policy makers, other than through numerical techniques. This modeling and analysis work received a crucial stimulus from the research of Scarf and coauthors [50, 51, 52, 54, 53] who devised algorithms for computing equilibria. Other computational techniques that followed Scarf include the homotopy methods [20], and the global Newton's method [29, 56, 57]. While these algorithms are fairly general, their running times are not polynomially bounded. Rutherford [49] mentions research in the 90's in which attempts were made to solve huge models of energy-economy interactions and international trade, whose nonlinear programming subproblems involved thousands of rows, columns, and non-zeros. Markets of such sizes cannot be reliably solved without efficient algorithms, thus motivating a computational investigation of the market equilibrium problem.

Over the last few years, the problem of computing market equilibria has received much attention within the theoretical computer science community. In a short span of time, exciting developments have led to polynomial time algorithms for computing equilibria in several restricted, but often meaningful, market models. In this article, we attempt to summarize these recent developments, relate them to past work, and suggest some open problems.

This column is organized as follows. In Section 3 we provide appropriate definitions and introduce relevant economic properties. In Section 4 we overview the early literature on the

[3] Given two vectors x and y, $x \cdot y$ denotes their inner product.

computation of market equilibria, as it developed starting in the 60's. In Section 5 we cover some of the most recent computational achievements. In Section 6 we list some of the more interesting open problems and challenges in this area.

3 Background: Definitions, Models, Properties

The Fisher model. A special case of the general exchange model defined above occurs when the initial endowments are *proportional*, i.e., when $w_i = \delta_i w$, $\delta_i > 0$, so that the relative incomes of the traders are independent of the prices. This special case is equivalent to the *Fisher model*, which is a market of n goods desired by m utility maximizing buyers with fixed incomes. In the standard account of the Fisher model, each buyer has a concave utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$ and an endowment $e_i > 0$ of *money*. There is a seller with an amount $q_j > 0$ of good j. An equilibrium in the Fisher setting is a nonnegative vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}^n_+$ of goods for each buyer i such that the following two conditions hold:

1. The vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq e_i$ and $x \in \mathbf{R}^n_+$.
2. For each good j, $\sum_i \bar{x}_{ij} = q_j$.

Unless otherwise stated, we treat the Fisher model as a special case of the general exchange model.

Demand and excess demand. For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* of trader i at prices π.[4] Then $X_k(\pi) = \sum_i x_{ik}(\pi)$ denotes the *market (or aggregate) demand* of good k at prices π, and $Z_k(\pi) = X_k(\pi) - \sum_i w_{ik}$ the *market excess demand* of good k at prices π. The vectors $X(\pi) = (X_1(\pi), \ldots, X_n(\pi))$ and $Z(\pi) = (Z_1(\pi), \ldots, Z_n(\pi))$ are called *market demand* (or aggregate demand) and *market excess demand*, respectively. The market is said to satisfy *positive homogeneity* if for any price vector π and any $\lambda > 0$, we have $Z(\pi) = Z(\lambda\pi)$. It is said to satisfy *Walras' Law* if for any price π, we have $\pi \cdot Z(\pi) = 0$. Both positive homogeneity and Walras' law are considered standard and extremely mild assumptions in the theory of equilibrium. Positive homogenity follows simply from the agent's utility maximizing behavior whereas Walras' Law follows from a very weak assumption on agent preferences called *local nonsatiation*. In terms of the aggregate excess demand function, the equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ such that $Z(\pi)$ is well-defined and $Z_j(\pi) \leq 0$, for each j.

[4]Unless otherwise stated, we assume that there is at most one such vector, i.e., the demand is a single-valued function. This is the case for utility functions that satisfy strict quasi-concavity. In general, the demand is a set-valued function.

GS and WARP. Two properties play a significant role in the theory of equilibrium and in related computational results: *gross substitutability* (GS) and the *weak axiom of revealed preferences* (WARP).

The market excess demand is said to satisfy gross substitutability (resp., weak gross substitutability - WGS) if for any two sets of prices π and π' such that $0 < \pi_j \le \pi'_j$, for each j, and $\pi_j < \pi'_j$ for some j, we have that $\pi_k = \pi'_k$ for any good k implies $Z_k(\pi) < Z_k(\pi')$ (resp., $Z_k(\pi) \le Z_k(\pi')$). That is, increasing the price of some of the goods while keeping some others fixed can only cause an increase in the demand for the goods whose price is fixed.

The market excess demand is said to satisfy WARP if for any two sets of prices π and π' such that $Z(\pi) \ne Z(\pi')$ either $\pi \cdot Z(\pi') > 0$ or $(\pi') \cdot Z(\pi) > 0$. This means that if the demands at prices π and π' are different, then either $Z(\pi')$ is not within budget when the price is π or $Z(\pi)$ is not within budget when the price is π'.

It is well known that GS implies that the equilibrium prices are unique up to scaling ([60], p. 395) and that WARP implies that the set of equilibrium prices is convex ([37], p. 608). It is also well known that the market excess demand satisfies weak GS when each individual excess demand does. In contrast, WARP is typically satisfied by the individual excess demand, but it is not in general satisfied by the aggregate excess demand.

Commonly used utility functions. A utility function $u(\cdot)$ is *homogeneous* (of degree one) if it satisfies $u(\alpha x) = \alpha u(x)$, for all $\alpha > 0$. A *linear* utility function has the form $u_i(x) = \sum_j a_{ij} x_{ij}$. The *Cobb-Douglas* utility has the form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, where $\sum_j a_{ij} = 1$. The *Leontief* utility function has the form $u_i(x) = \min_j a_{ij} x_{ij}$. A *CES (constant elasticity of substitution)* utility function has the form $u(x_i) = (\sum_j (a_{ij} x_{ij})^\rho)^{1/\rho}$, where $-\infty < \rho < 1$ but $\rho \ne 0$. In all of these definitions, $a_{ij} \ge 0$. As ρ tends to 1 (resp. 0, $-\infty$), the CES utility function tends to a linear (resp. Cobb-Douglas, Leontief) utility function ([2], page 231). The CES class of functions are popular among economists for their ability to express a wide variety of consumer preferences as well as their mathematical tractability which allows for explicit computation of the associated demand function.

Approximate equilibria. In general, equilibrium prices are vectors of irrationals. Rational solutions exist only in very special cases and therefore most algorithms actually compute an approximate equilibrium as defined below.

Definition 1 *A bundle $x_i \in \mathbf{R}^n_+$ is a μ-*approximate demand*, for $\mu \ge 1$, of trader i at prices π if $u_i(x_i) \ge \frac{1}{\mu} u^*$ and $\pi \cdot x_i \le \mu \pi \cdot w_i$, where $u^* = \max\{u_i(x) | x \in \mathbf{R}^n_+, \pi \cdot x \le \pi \cdot w_i\}$.*

Prices π and bundles x_i's are a strong μ-approximate equilibrium *($\mu \ge 1$) if (1) for each trader i, x_i is the demand of trader i at prices π, and (2) $\sum_i x_{ij} \le \mu \sum_i w_{ij}$ for each good j. Prices π and bundles x_i's are a* weak μ-approximate equilibrium *($\mu \ge 1$) if (1) for each trader i, x_i is a μ-approximate demand of trader i at prices π, and (2) $\sum_i x_{ij} \le \mu \sum_i w_{ij}$ for each good j.*

Definition 2 *An algorithm that computes a* $(1+\varepsilon)$*-approximate equilibrium, for any* $\varepsilon > 0$*, in time that is polynomial in the input parameters and* $1/\varepsilon$ *(resp.,* $\log 1/\varepsilon$*) is called a poly-time approximation scheme (resp., poly-time algorithm).*

4 Computation of Market Equilibria: Early History

This section surveys some of the older techniques (pre-2001) for computing equilibria. Most of this material appears in the literature on mathematical economics and related areas.

General Methods. The market equilibrium problem can be transformed into a fixed point problem – indeed, the proofs of existence of a market equilibrium are based on either Brouwer's or Kakutani's fixed point theorem, depending on the setting (see, e.g., the beautiful monograph [6]).

Starting from the 60's, this intimate connection between the notions of fixed-point and market equilibrium was exploited for computational goals by Scarf and some coauthors, who employed path-following techniques to compute approximate fixed points (or equilibrium prices) [50, 51, 52, 54, 53]. In their simplest form these methods are based upon a decomposition of the *price simplex* into a large number of small regions and on the use of information about the problem instance to construct a path that can be shown to terminate at a fixed point. The worst case running time of these algorithms turns out to be exponential in the number of goods.

Kuhn [35] showed the connection between Scarf's result [50] and Sperner's lemma, and proposed a technique for the subdivision of the simplex, which yields a simple algorithm for the simplicial approximation of fixed points. This line of work culminated in the homotopy methods [20].[5]

These algorithms do not achieve polynomial running times, and their performance on large size applications has been superseded by more efficient iterative schemes, based for example on the global Newton's method [29, 56, 57]. These last approaches enjoy global convergence, and, despite not being guaranteed to run in polynomial time, seem to be amenable for tackling real world applications (see [23], p. 670, and references therein).

Markets satisfying WGS. In the late fifties, several researchers considered the question of the stability of market equilibrium under a price adjustment mechanism known as *tâtonnement*, that raises (resp. lowers) the prices of goods whose aggregate excess demand is positive (resp. negative). It was shown that if the aggregate excess demand of the market satisfies gross substitutability, then the dynamics of a differential form of the tâtonnement process leads the market to an equilibrium starting from any price vector [1]. The notion of tâtonnement inspired several algorithms (see, e.g., [8, 14]).

In analyzing the stability of the tâtonnement process, Arrow, Block, and Hurwicz [1] proved the following Lemma.

[5]For an excellent presentation of these methods, we refer the reader to the monograph by Todd [58].

Lemma 3 **[Separation Lemma]** *If an equilibrium price vector $\hat{\pi}$ satisfies $\hat{\pi}_j > 0$, for each good j, if the market satisfies gross substitutability, positive homogeneity, and Walras' law, then for any non-equilibrium price vector π such that $\pi_j > 0$ for each j, we have $\hat{\pi} \cdot Z(\pi) > 0$.*

The Separation Lemma says that, under GS, WARP holds for any pair of price vectors, provided that one of them is an equilibrium price vector and the other is not. The lemma generalizes to the case where there is only weak GS [4, 5] and immediately implies that the set of equilibrium prices form a convex set. It also gives, for any positive price vector π that is not an equilibrium price vector, a *separating hyperplane* [28], that is, a hyperplane that separates π from the convex set of equilibrium prices. Indeed we have $\sum_j \hat{\pi}_j Z_j(\pi) > 0$, but $\sum_j \pi_j Z_j(\pi) = 0$, by Walras' law. To compute this separating hyperplane, one needs to compute the demands $Z_j(\pi)$ at the prices π.

Polterovich and Spivak [45] extended the characterization of the Separation Lemma to scenarios where the demand is a set-valued function of the prices, which includes in particular the exchange model with linear utilities. This extension says that for any equilibrium price $\hat{\pi}$, and non-equilibrium price π, and any vector $z \in \mathbf{R}^n$ that is chosen from the set of aggregate excess demands of the market at π, we have $\hat{\pi} \cdot z > 0$.

Cutting plane methods that exploit characterizations like that of the Separation Lemma and its extension in [45] have been studied (see the paper by Primak [47] and the references therein). Primak [46] (see also [47]) has also shown that the Polterovich-Spivak characterization holds when traders have linear utilities and production sets are limited to the positive orthant $\mathbf{R}^n_+$. This result generalizes to the case where each trader has a utility function that generates, for any initial endowment, an individual excess demand function that satisfies WGS. Newman and Primak [41] have applied the Ellipsoid Algorithm to a setting with linear utilities and production sets restricted to the positive orthant. This setting is a generalization of the linear exchange model, but as discussed in [11], their work does not guarantee an approximate equilibrium in polynomial time.

Explicit Convex Programs. Nenakov and Primak [40] wrote the equilibrium conditions for the exchange model with linear utilities as a finite convex feasibility problem, and for the Arrow-Debreu model with linear utilities and a restricted form of production as an infinite set of convex inequalities. Suppose that the linear utility function of the i'th trader is $\sum_j a_{ij} x_{ij}$. Their program for the exchange model amounts to finding ψ_j's and nonnegative x_{ij}'s such that:

$$\sum_k a_{ik} x_{ik} \geq a_{ij} \sum_k w_{ik} e^{\psi_k - \psi_j}, \text{ for each } 1 \leq i \leq m, 1 \leq j \leq n.$$

$$\sum_i x_i = \sum_i w_i.$$

They showed that any solution to this program corresponds to an equilibrium obtained by setting $\pi_j = e^{\psi_j}$. They also proved the converse, i.e., that any equilibrium corresponds to a solution to this program. They showed that their approach also works when the traders have Cobb-Douglas utility functions.

Eaves [19] showed that the equilibrium for exchange markets with Cobb-Douglas utilities can be written as the solution to a linear program.

The Fisher Case. Translating work done in a different context by Eisenberg and Gale [22], Gale showed that the equilibrium for a Fisher market with linear utilities can be obtained by solving a convex program ([25], pp. 281-287). This approach was generalized to a great extent by Eisenberg [21], who proved that it works for homogeneous utility functions. Eisenberg first solves the following convex program whose variables x_{ij} are nonnegative:

$$\text{Maximize} \quad \sum_i e_i \log u_i(x_i)$$

$$\text{Subject to} \quad \sum_i x_{ij} \leq q_j \text{ for each } j.$$

The solution to this program gives the allocations of the goods at equilibrium. Notice that the program does not have variables corresponding to prices – these are obtained by applying the Kuhn-Tucker optimality conditions to the solution of the program. Polterovich [44] gives an even more general result, which allows production on the side of the seller instead of a fixed quantity of each good.

Eisenberg [21] actually showed a result that is more fundamental than his characterization of equilibrium – he describes a fictitious consumer with her own concave utility function and money such that the aggregate demand function of the original market is the same as the individual demand function of the fictitious consumer.

Chipman [9] showed the existence of a single fictitious representative consumer even in the exchange model when all traders have the same homogeneous utility function (identical tastes) but possibly different initial endowments. Consequently, in both these scenarios the excess demand function satisfies WARP ([60], p. 133 and pp. 399-400), and thus the set of equilibrium prices is convex.

Furthermore, when there are producers instead of the seller in the Fisher setting, we have an economy with one fictitious producer and one fictitious consumer. (It can be shown that the producers can be collapsed into one fictitious producer.) The Pareto-optimality of any equilibrium implies that it must maximize the utility of the fictitious consumer subject to the production constraints. Thus we have a derivation from more fundamental principles of the fact that any equilibrium allocation must be a solution to Eisenberg's program [21] (when we have a seller) or to Polterovich's program [44] (when we have production). Note however that we have not argued the converse, i.e., that any solution to these programs yields an equilibrium allocation. We refer the reader to [37] (Chapter 15.C) for a discussion of this issue.

5 Computation of Market Equilibria: Recent Work

At STOC 2001 Papadimitriou delivered an inspiring lecture which highlighted several points of contacts between Game Theory, Economic Theory, and Theoretical Computer Science

[43]. Since then, starting with the work of Deng, Papadimitriou, and Safra [15], there has been a flurry of research devoted to the problem of computing market equilibria. In a short span of time, theoretical computer scientists have developed polynomial time algorithms for several restricted versions of the market equilibrium problem. In this section we survey these exciting developments. It is noteworthy that a fair share of this work has been done without awareness of some of the results from the mathematics and economics communities, which we have sketched in Section 4. This may be due partly to the existence of a vast body of literature on the theory of market equilibrium and due partly to the fact that some of the results relevant to equilibrium computation have appeared in Russian journals.

The Fisher Setting. In 2002, Devanur, Papadimitriou, Saberi and Vazirani [16] introduced a polynomial time algorithm for the *linear version* of the Fisher model, i.e., for the special case where the buyers have linear utility functions. Their approach is based on a primal-dual scheme and boils down to a number of max-flow computations. The result of [16] has inspired the definition of a new model, the *spending constraint model* [17], to which the technique used in [16] can still be applied.

Unaware of Eisenberg's result [21], Codenotti and Varadarajan [12] introduced a convex programming framework for Fisher markets with Leontief utility functions, and described its generalization to homogeneous utility functions. They also showed that, unlike in the linear case, the equilibrium prices can be irrational, so that the only choice is to shoot for approximation algorithms.

Jain, Vazirani and Ye [32] have presented some extensions to include economies of scale in production.

Chen et al. [7] have proposed an algorithm for the problem when the buyers have logarithmic utility functions. Their algorithm runs in polynomial time when either the number of goods or the number of buyers is bounded by a constant.

The Exchange Model: Iterative Schemes. For the exchange model where traders have linear utilities, Jain, Mahdian, and Saberi [31] presented a poly-time approximation scheme that returns a strong $(1+\varepsilon)$-approximate equilibrium. Devanur and Vazirani [18] presented a strongly polynomial time approximation scheme for the same problem. The algorithm of [31] works by iteratively solving Fisher instances of the problem. Their approach is in fact applicable whenever WGS holds, provided that the Fisher instances can be solved efficiently.

Garg and Kapoor [26] presented a simple poly-time approximation scheme, which has an auction interpretation, for the same problem. Garg, Kapoor, and Vazirani [27] extended this method to markets where traders have separable utility functions and the individual excess demands satisfy WGS. Their algorithm is a version of tâtonnement, as it searches the price space using steps that only increase the price of a good that has positive excess demand. Both these algorithms compute a weak approximation.

Explicit Convex Programs. Unaware of the work by Nenakov and Primak [40], which is written in Russian, Jain [30] introduced a convex program that characterizes the equilibria for the linear exchange model. His convex program turns out to be the same as the one in [40]. This convex program can also be applied to characterize the equilibria for several non-linear utilities, including the CES functions with $\rho > 0$. For linear utilities, a rational normalized equilibrium exists. Exploiting this property, Jain proposed a sophisticated variant of the Ellipsoid Algorithm that makes it possible to use the convex formulation to compute the equilibrium exactly in polynomial time.

For CES functions with $\rho > 0$, Codenotti and Varadarajan [13] have provided a different convex feasibility formulation of the equilibrium conditions. Their technique generalizes to some other utility functions, including CES utility functions with $-1 \leq \rho < 0$, which do not satisfy WGS.

The general approach of these two papers (and that of [40]) is to first write down the equilibrium conditions as a non-convex feasibility program. The program is then made convex by replacing some equalities with inequalities and performing appropriate variable substitutions. It is shown that these manipulations preserve the set of feasible solutions.

One of the advantages of explicit convex programs is that they are amenable to methods that are efficient in practice. Ye [63] showed how interior point methods can be applied efficiently to solve Jain's convex program [30].

Algorithms Exploiting the Separation Lemma. Codenotti, Pemmaraju and Varadarajan [11] built upon the proofs of the Separation Lemma (Lemma 3) and related characterizations to present a poly-time algorithm for computing a weak $(1+\varepsilon)$-approximate equilibrium when the aggregate excess demand function is single valued, satisfies WGS, and an oracle for computing the aggregate excess demand to within a specified tolerance is available. Whenever the demand does not change too rapidly as a function of the prices, they compute a strong approximate equilibrium.

The separation oracle provided by the Separation Lemma is not sufficient to compute an approximate equilibrium by the Ellipsoid Algorithm. A stronger separation oracle is needed: given a price π that is not a $(1+\varepsilon)$-approximate equilibrium, compute a hyperplane that separates π from all points within distance $\delta > 0$ from the set of equilibria, where δ is reasonably large in terms of the input size. In [11] the authors construct such an oracle by analyzing the proof of the Separation Lemma and the proof of a related characterization due to Primak [47].

Their approach makes it possible to compute approximate equilibria in polynomial time under several different scenarios. For example, it is possible to compute in polynomial time a weak $(1+\varepsilon)$-approximate equilibrium in exchange markets where traders have either linear, Cobb-Douglas, or CES utility functions with $\rho > 0$.[6]

Their work also leads to a poly-time approximation scheme (forthcoming) based on tâtonnement that computes an approximate equilibrium when the aggregate excess demand satisfies WGS.

[6]With linear utility functions, the demand is not single valued. However, this situation can be handled by approximating a linear utility function by a CES utility function.

6 The road ahead: open problems and future work

We conclude this column with a list of topics that we deem worthy of future investigation.

1. From the Sonnenschein-Mantel-Debreu (SMD) Theorem ([37], Section 17.E) it is clear that in general the set of equilibrium prices need not even be connected. This is true even for the Fisher setting with non-homogeneous utility functions [34] and the exchange model with homogeneous utility functions [36]. The situation with production is even worse – multiple disconnected equilibria are the norm (see [33]) unless there are severe restrictions on the consumer side (such as fixed income [44]) or on the side of production (such as the ones in [40, 46, 47]). Current poly-time schemes do not handle these general settings, and it seems that the search for efficient algorithms will have to reach beyond convex programming.

2. Papadimitriou [42] has analyzed the computational complexity of a problem closely related to that of computing an equilibrium for a market given in terms of an excess demand oracle. Based on results of Uzawa [59], which establish the equivalence of this problem to that of computing fixed points, he has shown the problem to be complete for the complexity class **PPAD**. Perhaps it is possible, using the SMD Theorem, to conclude the hardness of the market equilibrium problem itself when it is given in terms of an excess demand oracle. Can we prove similar results for the computational complexity of the market equilibrium problem given in terms of the initial endowments and utility functions of the traders?

3. Though the scope of the approach seems restricted, it is worth seeing how far the convex-feasibility approach will go. In the exchange setting for example, there are significant special markets which do not satisfy weak gross substitutability, but seem to possess sufficient structure. Examples are CES functions with $\rho < 0$ and *nested CES functions*. Some of these special cases might be efficiently solvable. Preliminary results have been obtained in [13], where a convex formulation has been proposed for CES functions with $-1 \leq \rho < 0$, which do not satisfy WGS.

   Other instances which might be computationally tractable arise when the uniqueness of equilibrium is known to hold under non-degeneracy assumptions [33].

4. The exchange economies with proportional endowments are easier to deal with than the general case, whenever the utility functions are homogeneous. This fact has been used to derive iterative schemes for the approximation of the equilibrium in the exchange model, using a solver for the Fisher model as a black box. One variant of this approach converges in polynomial time for linear utilities [31], and can be extended to handle utilities satisfying WGS. Preliminary experimental results [10] suggest that simpler schemes might also work, and that there could be convergence in other cases, too. Note that this iterative approach has been adopted in an algorithm widely used in practice [49]. However, analysis showing convergence of such iterative schemes does not yet exist.

A local version of the Separation Lemma works for production models, whenever the consumer side of the economy satisfies GS [48]. It seems worthwhile to investigate if this result has algorithmic applications.

5. The existence of equilibrium prices has been proven under mild assumptions on the utility functions and the initial endowments [3]. However, if these assumptions are dropped and we ask if a given market has an equilibrium price vector, then we are in an unknown territory. More precisely, efficient characterizations of existence of equilibrium are not known, even for the exchange model. In the special cases of linear and Cobb-Douglas utility functions, Eaves and Gale have provided such characterizations [19, 24]. In [38] the sufficient conditions have been generalized to handle more general families of utility functions, and the existential problem has been related to the strong connectivity of a certain graph obtained from the initial endowments and the parameters describing the utility functions. However, it is not clear if there is an efficient algorithm, that, given a description of an exchange market, reports if the market has an equilibrium or not. Some very preliminary results are shown in [39].

Acknowledgement: We thank Janos Simon and Varsha Dani for their comments on drafts of this article. We thank Pavel Pudlak for sending us copies of [40] and [46] and Ashot Vardanyan for translating these papers.

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