

# Introduction to (Mathematical) Optimization

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- ▶ **system**: mathematical model
- ▶ **change**: change to input variables (parameters)
- ▶ **outcome**: a measure of performance of the model, objective function

# The Raptor Problem

See other slides

## Optimization in one variable

$$\text{minimize } f(x) \in C^2 : \mathbf{R} \rightarrow \mathbf{R}$$

- ▶  $x$  is real number variable
- ▶  $f(x)$  is the objective function, we typically want this to be twice continuously differentiable. This means both the first and second derivative are continuous in  $x$
- ▶ We are looking for a point  $x^*$  such that  $f'(x) = 0$  and  $f''(x) > 0$ . Note that this is a *local* optimizer



## Optimization in one variable

## Key words

- ▶ *domain*: space for input variable  $x$
- ▶ *range*: space for output  $f(x)$
- ▶ *critical point*:  $f'(x) = 0$
- ▶ *local minimizer*:  $f'(x) = 0$  and  $f''(x) > 0$
- ▶ *local maximizer*:  $f'(x) = 0$  and  $f''(x) < 0$
- ▶ *saddle point*:  $f'(x) = 0$  and  $f''(x) = 0$
- ▶ *global minimizer*:  $x^*$  such that  $f(x^*) \leq f(x)$  for all  $x$  in domain

## Optimization in two variables

$$\text{minimize } f(x) \in C^2 : \mathbf{R}^2 \rightarrow \mathbf{R}$$

- ▶  $x$  is a 2-dimensional vector of real variables
- ▶  $f(x)$  is the objective function
  - ▶ First derivative or gradient of  $f$  is written  $\nabla f(x)$
  - ▶ Second derivative or Hessian of  $f$  is written  $\nabla^2 f(x)$
- ▶ We are looking for a point  $x^*$  such that  $\nabla f(x) = 0$  and  $\nabla^2 f(x) \succeq 0$ . Note that this is a *local* optimizer

## The gradient $\nabla f(x)$

Vector of variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Gradient of  $f$ :

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

## The Hessian $\nabla^2 f(x)$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

## Surface plots and contours

- ▶ show local minimizer/maximizer critical point
- ▶ show saddle point

# Higher dimensions

- ▶ can't easily visualize
- ▶ need analysis

# Algorithms

- ▶ Basic loop
- ▶ Descent condition
- ▶ Gradient descent
- ▶ Newton's method



## Show example on rosenbrock function

- ▶ gradient descent vs. newton's method

## Two very important optimization problems

- ▶ linear least squares
- ▶ non-linear least squares

# Constraints

- ▶ basic idea of constraints
- ▶ work through example from multivariate calculus
- ▶ introduce idea of multipliers
- ▶ equality constraints
- ▶ inequality constraints
- ▶ linear constraints
- ▶ nonlinear constraints

## Penalty and barrier methods

- ▶ introduce a penalty into the objective to penalize constraint violation

# Linear programming

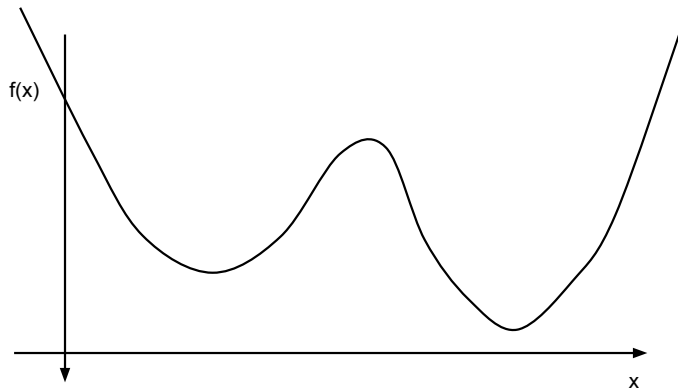
# Discrete variables

- ▶ Mixed integer programming
- ▶ Scheduling problems

## What's next

- ▶ Nonlinear programming
- ▶ Convex modelling
- ▶ Study of algorithms
- ▶ Modeling languages
- ▶ Automated differentiation

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