Simple Two Stage Transforms Designed for Optimisation of Shape in Forming Processes

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ABSTRACT: The problem of shape parametrisation suitable for solving optimisation problems in material forming is addressed. The article concentrates on direct approaches where a reference domain is used in which the geometry layout is independent on design parameters. A class of composed two stage transforms is proposed for mapping object configuration from the reference to physical space. The first transform maps a simple domain to an intermediate configuration in the reference space and includes all dependency on parameters. The second transform, which is independent of design parameters, produces the final configuration in the physical space. All parameter dependency is in this way incorporated in a function which is defined on a simple domain and is therefore relatively simple to construct. Due to explicit definition of the transform the derivatives can be readily obtained for sensitivity analysis and the approach is computationally efficient. Evaluation of the inverse transform is limited merely to inversion of the parameter independent part and evaluation of direct terms of the parameter dependent part of the transform.

Key words: shape optimisation, material forming, shape parametrisation, domain transforms

1 INTRODUCTION

Optimisation of shape is an important part of the design of forming sequences and tooling systems. The presented article treats shape parametrisation, which essentially means the definition of the layout of object material points as a function of optimisation parameters. This topic is considered in the context of finite element simulation [1,2], where the task of shape parametrisation is definition of the layout of the initial finite element mesh according to optimisation parameters and eventually the sensitivities of this layout with respect to individual parameters.

There are many different approaches to shape parametrisation. The aim of the first part of this article is therefore to describe some general ideas and to state the definitions that will be used later in the text. The central part introduces the idea of two stage transforms and describes how these transforms are used in shape parametrisation. The presented material mainly refers to two dimensional case, although the approach can be extended to three dimensions. Analogies with three dimensional case are indicated where appropriate. In the final part the approach is discussed from a practical point of view.

2 BASIC CONCEPTS OF SHAPE PARAMETRISATION

Shape parametrisation can be based on two principal approaches. In the first one the continuous geometrical model is parametrised and the finite element mesh is derived from this model at each set of the design parameters. In the second approach, the finite element mesh directly depends on parameters. Optimal geometry of continuous model can be reconstructed on the basis of the optimal finite element mesh.

We look at the problem from the analyst's viewpoint and are only concerned with how to construct analysis input (essentially the finite element mesh) at a given set of design parameters. We can state the task of shape parametrisation as evaluation of the quantities

$$\mathbf{x}_i^{(0)}(\mathbf{p}) \qquad (1)$$

and in case of sensitivity analysis

$$\partial \mathbf{x}_i^{(0)}(\mathbf{p}) / \partial p_j \,, \tag{2}$$

where $\mathbf{x}_i^{(0)}$ is the initial position of the i-th mesh node. In spite of that, we will often use continuous geometrical models for practical reasons. Among the others, this will enable us to adopt some concepts which are well known in computer graphics and CAD [3,4] (Figure 1).

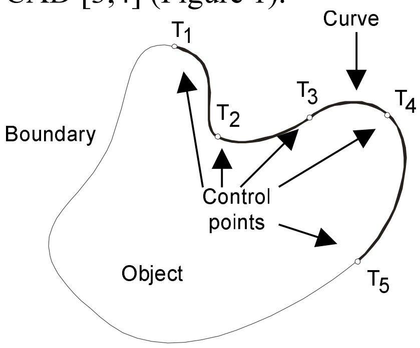

Figure 1: Definition of object boundary by a finite number of control points, which define a continuous curve or surface – a concept widely used in computer graphics and CAD.

Complexity of shape parametrisation is especially affected by the dimension of the problem, complexity of involved shapes and whether remeshing is applied or not.

In the presented approach shape parametrisation is facilitated by introducing the reference domain on which the geometry layout is defined. The layout in the reference coordinate system is independent of design parameters. Material points are mapped to physical domain by a parameter dependent transform, which defines the parametrisation:

$$\mathbf{x}^0 = \left(x^0, y^0\right) = \mathbf{F}(\xi, \eta, \mathbf{p}) \tag{3}$$

ξ and η are coordinates of a material point in the reference domain (a two dimensional case is asumed) and will be used for indexing material points. x and y are coordinates of a material point in the physical domain. Transform **F** must be bijective for any set of parameters **p** that can be encountered during the optimisation process.

There usually exists some initial design which we want to use as a starting point of the optimisation procedure. The geometry corresponding to this design is defined in the physical space, therefore the best way to obtain the layout in the reference domain is to transform material points of this initial layout to the reference domain by inverse of **F** at parameters $\mathbf{p}_0$. It depends on further details of parametrisation what parameters $\mathbf{p}_0$ are; this question will be referred to in the next section.

The inverse mapping which gives the geometry layout in the reference space can be performed in different ways. If we operate on a fixed mesh, the initial layout is discretised and nodal points of the mesh are transformed, since nodal points form all geometrical data we will possibly need. Another possibility is to transform geometrical entities, which are used in the finite element system pre-processing for modelling object boundaries like it is indicated in Figure 1.

When the reference layout and the transform are defined, the basic quantities (1) and (2) required as analysis input can be evaluated as follows:

$$\begin{aligned} &\mathbf{x}_i^{(0)}(\mathbf{p}) = \mathbf{F}(\xi_i, \eta_i, \mathbf{p}) \\ &d\,\mathbf{x}_i^{(0)}(\mathbf{p}) / d\,p_j = d\,\mathbf{F}(\xi_i, \eta_i, \mathbf{p}) / d\,p_j \end{aligned} \,, \tag{4}$$

The way how these quantities are actually calculated depends on many details. The most important among them is whether a fixed mesh in reference domain is used or not. If yes, then mesh topology is known in advance and evaluation of (4) is straightforward.

If automatic meshing is used, the evaluation is more complicated. We only have images of geometrical entities that define object boundaries in the reference domain. At any set of parameters **p** these entities are transformed to physical domain, in which automatic meshing of geometrical objects is performed. This process produces the mesh topology and initial positions of mesh nodes $\mathbf{x}_i^{(0)}$. If sensitivity analysis is also performed, derivatives of these positions with respect to the design parameters must be evaluated. We must relate the discretised and continuum design and consider $\mathbf{x}_i^{(0)}$ as coordinates of a material point, which is a part of geometrical layout defined in the reference domain. Reference coordinates of this material point are obtained by inverse transform $\mathbf{F}^{-1}$ and the derivative of $\mathbf{x}_i^{(0)}$ with respect to a certain parameter (4) is evaluated in the following way:

$$d\,\mathbf{x}_i^{(0)}(\mathbf{p}) / d\,p_j = d\,\mathbf{F}\left(\mathbf{F}^{-1}\left(\mathbf{x}_i^{(0)}, \mathbf{p}\right), \mathbf{p}\right) / d\,p_j \,. \tag{5}$$

3 PARAMETRISATION OF SHAPE BY USING TWO STAGE TRANSFORMS

The approach proposed in this article is illustrated in Figure 2. The geometry layout must be defined in the reference system. Material points are transformed from reference to physical domain by a two stage transform composed of a parameter dependent transform **g** and parameter independent transform **f**:

$$\begin{aligned} &\mathbf{F}(\xi, \eta, \mathbf{p}) = \mathbf{f}(\mathbf{g}(\xi, \eta, \mathbf{p})); \\ &\mathbf{g}(\xi, \eta, \mathbf{p}) = (\xi', \eta') \end{aligned} \,. \tag{6}$$

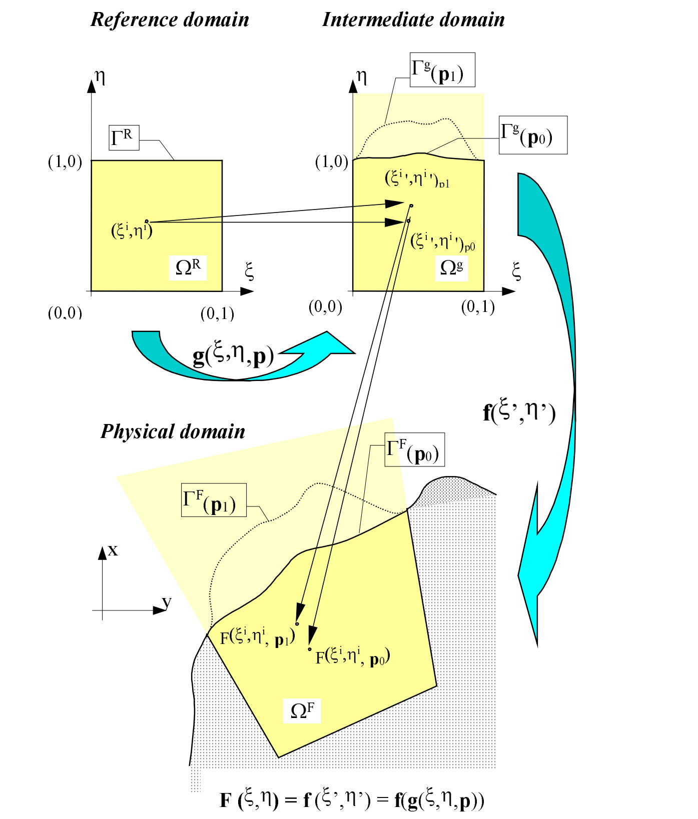

Figure 2: Direct approach shape parametrisation involving two stage transforms.

The first transform maps a simple domain to an intermediate one in the reference coordinate system:

$$\begin{aligned}&\Omega^g = \{(\xi,\eta); 0 \le \xi \le 1 \wedge \eta \ge 0\}\\ &\mathbf{g} : \Omega^g \to \Omega^g\end{aligned} . \tag{7}$$

$\mathbf{g}$ is defined through a parameter dependent function $s(\xi,\mathbf{p})$ and is actually a simple stretch in the direction of the η axis by the value of s at a given set of parameters and a certain ξ:

$$(\xi',\eta') = \mathbf{g}(\xi,\eta,\mathbf{p}) = (\xi, s(\xi,\mathbf{p})\cdot\eta) . \tag{8}$$

$s(\xi,\mathbf{p})$ can be any parametrised explicit function of ξ for which

$$(s(0,\mathbf{p}) = s(1,\mathbf{p}) = 1;\; s(\xi,\mathbf{p}) > 0)\,\forall \mathbf{p}, \forall \xi \in [0,1] . \tag{9}$$

Lagrange polynomials, Bezier curves and cubic splines [3,5] have been applied in this work. Parameters $\mathbf{p}$ are in these cases η coordinates of control points which define the curve. Optionally parameters can also include ξ coordinates of control points. More often control points are equidistantly distributed along the ξ axis or fixed ξ coordinates are provided in some other way.

Transform $\mathbf{f}$ maps the domain $\Omega^\mathbf{g}$ to the physical space independently on the design parameters (Figure 3). It was defined in such a way that image of $\Omega^\mathbf{g}$ is limited by straight lines on the three sides inside the parametrised object. $\mathbf{f}$ must therefore map lines ξ=0, η=0 and η=1 into straight lines. In two dimensions the following transform can be used:

$$\begin{aligned}&\mathbf{f}(\xi,\eta) = \mathbf{r}_1 + (-\mathbf{r}_1 + \mathbf{r}_2)\xi + (-\mathbf{r}_1 + \mathbf{r}_4)\eta +\\ &(\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4)\xi\eta\end{aligned} , \tag{10}$$

where points $\mathbf{r}_1$ through $\mathbf{r}_4$ are images of points ρ_1 through ρ_4, whose meaning is shown in Figure 3.

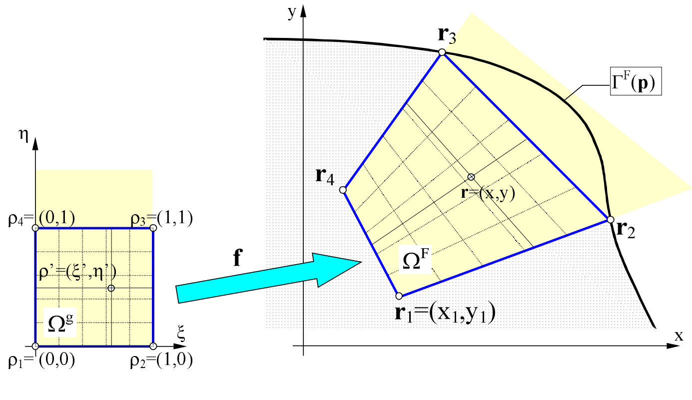

Figure 3: Parameter independent transform **f** that maps from the reference to physical space.

The two stage transform maps the domain $\Omega^\mathbf{R}$ to a simple domain $\Omega^\mathbf{F}$ in the physical space. The part of this domain limited by $\Gamma^\mathbf{F}(\mathbf{p})$ represents a part of the object whose geometry is parametrised. Only this part of object geometry is parameter dependent, while other parts are not affected by changing the design parameters. Parametrisation follows the following steps: First initial geometry is defined in the physical space. Then a part of the geometry, which will be parameter dependent, is defined by specifying the points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$, as it is shown in Figure 3. Inverse images of these points in the reference domain are ρ_1 through ρ_4, and the transform $\mathbf{f}$ is now defined. Function $s(\xi,\mathbf{p})$ is defined next. In this way $\mathbf{g}$ and therefore the complete transform $\mathbf{F}$ is defined. A set of initial parameters $\mathbf{p}_0$ is chosen. The parameter dependent part of the initial geometry is then transformed to the reference coordinate system by inverse transform $F^{-1}(\mathbf{p}_0)$. In the case of fixed mesh the mesh nodes within $\Omega^\mathbf{F}$ are transformed. Geometric entities that define the parameter dependent segment of object boundary are transformed when automatic meshing is applied. In this way the reference layout of the parameter dependent part of the object is defined and either (4) or (5) can be applied to obtain the information related to shape parametrisation.

Points outside the region limited by $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ are not affected by changing the design parameters. For these points we take

$$\mathbf{x}(\mathbf{p}) = \mathbf{x}(\mathbf{p}_0),\; d\,\mathbf{x}(\mathbf{p})/d\,p_i = 0 , \tag{11}$$

where $\mathbf{x}(\mathbf{p}_0)$ represents physical coordinates of the material point in the initial layout.

For points inside the parametrised region coordinates and their derivatives are evaluated according to (3) and (5). Because of the condition (9) derivatives of material point coordinates with respect to parameters are zero on line segments ($\mathbf{r}_1$,$\mathbf{r}_2$), ($\mathbf{r}_1$,$\mathbf{r}_4$) and ($\mathbf{r}_4$,$\mathbf{r}_3$) (see Figure 3), which ensures continuity of these derivatives over the whole object.

4 CONCLUSINOS AND PRACTICAL CONSIDERATIONS

By using the described approach only a part of object geometry with limited complexity can be parametrised. This problem can be overcome by decomposition of the object to several simple subdomains where the described two stage transform is applied only to the domains with parametrised boundaries (Figure 4). In this case it can be beneficial if positions of boundary points in which two neighbouring domains meet are also parametrised. This increases the variety of possible shapes that can be obtained.

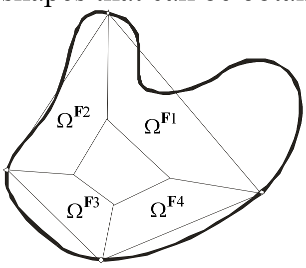

Figure 4: Division of an object to several domains parametrised by two stage transfoerm.

The described approach has several advantages. The transform is quick because it is defined explicitly. All parameter dependence is included in the definition of $s(\xi,\mathbf{p})$, which is an explicit function defined on an interval [0,1]. In three dimensions the equivalent $\mathbf{s}(\xi,\eta,\mathbf{p})$ is defined over a simple unit square domain. This offers a big advantage when constructing $\mathbf{F}$ because design of surfaces over arbitrarily oriented and shaped domains is avoided.

In two dimensions analytical expression for inverse transform exists. This is especially advantageous when automatic meshing is applied. The second stage transform $\mathbf{f}$ has also been constructed in three dimensions. Unfortunately this transform could not be inverted analytically. Its inverse must be calculated numerically. This includes solution of a single non-linear equation and does normally not cause serious efficiency problems. In the case of fixed mesh in the reference domain this is not a serious problem, because inverse images of mesh nodes must be evaluated only once in the entire optimisation process.

Another problematic issue is uniqueness of the parameter independent part of the transform $\mathbf{f}$. In two dimensions, uniqueness of this transform is guaranteed inside the image of a unit cube

$$\{(\xi,\eta);\, 0 \le \xi \le 1 \wedge 0 \le \xi \le 1\}\,, \tag{12}$$

if points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ define a convex configuration. The equivalent condition in three dimensions is more complex, but can be easily satisfied as well. The real problem is uniqueness of the transform above the unit square, i.e. for η>1. If the angle between ($\mathbf{r}_1$,$\mathbf{r}_2$,) and ($\mathbf{r}_4$,$\mathbf{r}_3$,) is less than 0, then the transform is unique only below the inverse image of intersection of these lines.

One way of dealing with this uniqueness problem is to choose the parameter dependent domains in the physical space in such a way that the angle condition is satisfied in each domain (e.g. like in Figure 4). There are however situations where it is not possible to satisfy the condition without affecting the set of shapes that can be achieved. In such situation it is better to constraint the values of $s(\xi,\mathbf{p})$ in such a way that material points which belong to the parametrised body never fall outside the region where $\mathbf{f}$ is unique, after they are mapped by $\mathbf{g}$.

The problem of uniqueness is much more dificult in three dimensions. Although the ways of dealing with the problem are similar than in two dimensions, the complete methodology has not been designed yet and will be a subject of further research.

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