Discrete Structures for Computer Science

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Lecture #2: Propositional Logic

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Based on materials developed by Dr. Adam Lee

Today's Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic

Logic is the basis of all mathematical and analytical reasoning

Given a collection of known truths, logic allows us to deduce new truths

Example

Base facts:

- If it is raining, I will not go outside
- If I am inside, Lisa will stay home
- Lisa and I always play video games if we are together during the weekend
- Today is a rainy Saturday

Conclusion: Lisa and I will play video games today

Logic allows us to advance mathematics through an iterative process of conjecture and proof

Propositional logic is a very simple logic

Definition: A proposition is a precise statement that is either true or false, but not both.

Examples:

- 2 + 2 = 4 (true)
- All dogs have 3 legs (false)
- $x^2 < 0$ (false)
- Washington, D.C. is the capital of the USA (true)

Not all statements are propositions

- Charlie is handsome
  - “Handsome” is a subjective term.
- $x^3 < 0$
  - True if $x < 0$, false otherwise.
- Springfield is the capital
  - True in Illinois, false in Massachusetts.

We can use logical connectives to build complex propositions

We will discuss the following logical connectives:

- $\neg$ (not)
- $\wedge$ (conjunction / and)
- $\vee$ (disjunction / or)
- $\oplus$ (exclusive disjunction / xor)
- $\rightarrow$ (implication)
- $\leftrightarrow$ (biconditional)

Negation

The negation of a proposition is true iff the proposition is false

What we know

What we want to know

One row for each possible value of "what we know"

p	$\neg p$

The truth table for negation

(I'll sometimes use ⊤ and ⊥)

Negation Examples

Negate the following propositions

- Today is Monday
- 21 * 2 = 42

What is the truth value of the following propositions

- ¬(9 is a prime number)
- ¬(Pittsburgh is in Pennsylvania)

Conjunction

The conjunction of two propositions is true iff both propositions are true

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

The truth table for conjunction

2^2 = 4 rows since we know both p and q!

Disjunction

The disjunction of two propositions is true if *at least one* proposition is true

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

The truth table for disjunction

Conjunction and disjunction examples

This symbol means "is defined as" or "is equivalent to" (sometimes I'll use $\triangleq$)

Let:

- $p \equiv x^2 \geq 0$
- $q \equiv$ A lion weighs less than a mouse
- $r \equiv 10 < 7$
- $s \equiv$ Pittsburgh is located in Pennsylvania

What are the truth values of these expressions:

- $p \wedge q$
- $p \wedge s$
- $p \vee q$
- $q \vee r$

In-class Exercises

Problem 1: Let $p \equiv 2+2=5$, $q \equiv$ eagles can fly, $r \equiv 1=1$. Determine the value for each of the following:

- $p \wedge q$
- $\neg p \vee q$
- $p \vee (q \wedge r)$
- $(p \vee q) \wedge (\neg r \vee \neg p)$

Exclusive or (XOR)

The exclusive or of two propositions is true if *exactly one* proposition is true

p	q	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

The truth table for exclusive or

Note: Exclusive or is typically used to natural language to identify *choices*. For example “You may have a soup or salad with your entree.”

Implication

The implication $p \rightarrow q$ is false if p is true and q is false, and true otherwise

Terminology

- p is called the hypothesis
- q is called the conclusion

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

The truth table for implication

Implication (cont.)

The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If p then q
- p only if q
- p is sufficient for q
- q whenever p

Implication examples

Let:

- $p \equiv$ Jane gets a 100% on her final
- $q \equiv$ Jane gets an A

What are the truth values of these implications:

- $p \rightarrow q$
- $q \rightarrow p$

Other conditional statements

Given an implication $p \to q$:

- $q \to p$ is its converse
- $\neg q \to \neg p$ is its contrapositive
- $\neg p \to \neg q$ is its inverse

Why might this be useful?

Note: An implication and its contrapositive *always* have the same truth value

Biconditional

The biconditional $p \leftrightarrow q$ is true if and only if p and q assume the same truth value

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

The truth table for the biconditional

Note: The biconditional statement $p \leftrightarrow q$ is often read as "p if and only if q" or "p is a necessary and sufficient condition for q."

Truth tables can also be made for more complex expressions

Example: What is the truth table for $(p \wedge q) \rightarrow \neg r$?

Subexpressions of "what we want to know"

What we want to know

p	q	r			

$2^3 = 8$ rows

Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - $\neg q \vee r$ means $(\neg q) \vee r$, not $\neg(q \vee r)$
2. Conjunction
3. Disjunction
   - $q \wedge r \vee s$ means $(q \wedge r) \vee s$, not $q \wedge (r \vee s)$
4. Implication
   - $q \wedge r \rightarrow s$ means $(q \wedge r) \rightarrow s$, not $q \wedge (r \rightarrow s)$
5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.

In-class Exercises

Problem 2: Show that an implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same value

- **Hint:** Construct two truth tables

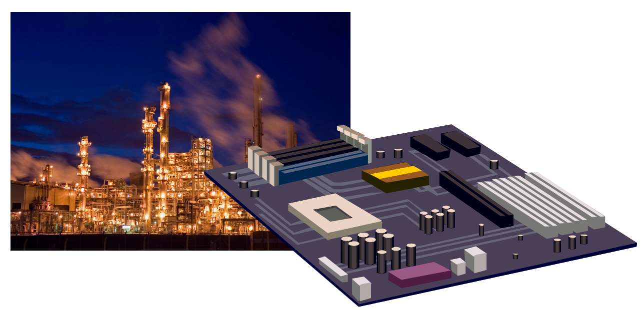
Problem 3: Construct the truth table for the compound proposition $p \wedge (\neg q \vee r) \rightarrow s$

English sentences can often be translated into propositional sentences

But *why* would we do that?

Philosophy and epistemology

Reasoning about law



Verifying complex system specifications

Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Find logical connectives

Let:

Translate fragments

Create logical expression

Example #2

Example: You can have free coffee if you are senior citizen and it is a Tuesday

Let:

Example #3

Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the translation from example 1!

Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - e.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as true and 0 as false, our logic truth tables tell us how to carry out bitwise logical operations

Bitwise logic examples

$$\begin{array}{r} 1010\ 1110 \\ \wedge\ \ 1110\ 1010 \\ \hline \end{array}$$

$$\begin{array}{r} 1010\ 1110 \\ \vee\ \ 1110\ 1010 \\ \hline \end{array}$$

$$\begin{array}{r} 1010\ 1110 \\ \oplus\ \ 1110\ 1010 \\ \hline \end{array}$$

In-class Exercises

Problem 4: Translate the following sentences

- If it is raining and today is Saturday then I will either play video games or watch a movie
- You get a free salad only if you order off of the extended menu and it is a Wednesday

Problem 5: Solve the following bitwise problems

$$\begin{array}{r} 1011\ 1000 \\ \oplus\ 1010\ 0110 \\ \hline \end{array} \qquad \begin{array}{r} 1011\ 1000 \\ \wedge\ 1010\ 0110 \\ \hline \end{array}$$

Final Thoughts

- Propositional logic is a simple logic that allows us to reason about a variety of concepts

- In recitation:
  - More examples and practice problems
  - Be sure to attend!

- Next:
  - Logic puzzles and logical equivalences
  - Please read sections 1.2 and 1.3
    - In general: do the assigned reading!