Discrete Structures for Computer Science

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Lecture #10: Sequences and Summations



Today's Topics

Sequences and Summations

- Specifying and recognizing sequences
- Summation notation
- Closed forms of summations
- Cardinality of infinite sets

Sequences are ordered lists of elements

Definition: A sequence is a function from a subset of the set of integers to a set S. We use the notation a_n to denote the image of the integer n. a_n is called a term of the sequence.

Examples:

- 1, 3, 5, 7, 9, 11
- 1, 1/2, 1/3, 1/4, 1/5, ...

A sequence with 6 terms

An infinite sequence

Note: The second example can be described as the sequence $\{a_n\}$ where $a_n = 1/n$



What makes sequences so special?

Question: Aren't sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is ordered, a set is not.

Question: How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

Some special sequences

Geometric progressions are sequences of the form $\{ar^n\}$ where a and r are real numbers

Examples:

- 1, 1/2, 1/4, 1/8, 1/16, ...
- 1, -1, 1, -1, 1, -1, ...

Arithmetic progressions are sequences of the form $\{a + nd\}$ where a and d are real numbers.

Examples:

- 2, 4, 6, 8, 10, ...
- -10, -15, -20, -25, ...

Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

- Are there runs of the same value?
- 2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
- Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
- 4. Are terms obtained by combining previous terms in a certain way?
- 5. Are there cycles amongst terms?

What are the formulas for these sequences?

Problem 1: 1, 5, 9, 13, 17, ...

Problem 2: 1, 3, 9, 27, 81, ...

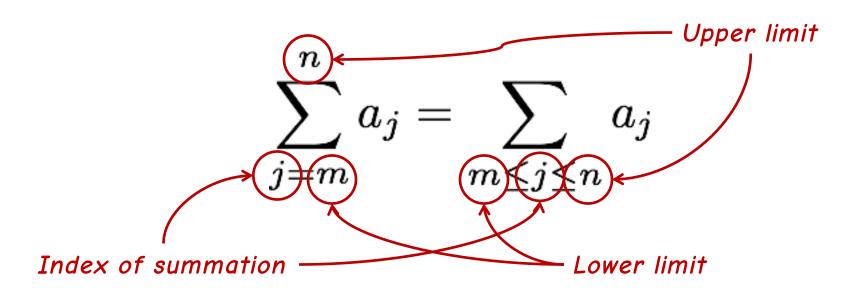
Problem 3: 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, ...

Problem 4: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

This is called the Fibonacci sequence.

Sometimes we want to find the sum of the terms in a sequence

Summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + ... + a_n$



Example:
$$\sum_{1 \le i \le 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$$

The usual laws of arithmetic still apply

$$\sum_{j=1}^{n} (ax_j + by_j - cz_j) = \sum_{j=1}^{n} x_j + b \sum_{j=1}^{n} y_j - c \sum_{j=1}^{n} z_j$$

Constant factors can be pulled out of the summation

A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations

Example:

- $4\sum_{1 \le j \le 3} j + \sum_{1 \le j \le 3} j^2 =$

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Example sums

Example: Express the sum of the first 50 terms of the sequence $1/n^2$ for n = 1, 2, 3, ...

Answer:
$$\sum_{j=1}^{50} \frac{1}{j^2}$$

Example: What is the value of $\sum_{k=4}^{8} (-1)^k$

Answer:
$$\sum_{k=4}^{8} (-1)^k =$$

$$=$$

We can also compute the summation of the elements of some set

Example: Compute
$$\sum_{s \in \{0,2,4,6\}} (s+2)$$

Answer:
$$(0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20$$

Example: Let
$$f(x) = x^3 + 1$$
. Compute $\sum_{s \in \{1,3,5,7\}} f(s)$

Answer:
$$f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500$$

Sometimes it is helpful to shift the index of a summation

This is particularly useful when combining two or more summations. For example:

$$S = \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k-1)$$

$$= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j+1)-1)$$
Need to add 1 to each j
$$= \sum_{j=1}^{10} (j^2 + 2(j+1)-1)$$

$$= \sum_{j=1}^{10} (j^2 + 2j+1)$$

$$= \sum_{j=1}^{10} (j+1)^2$$

Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

Example: Compute
$$\sum_{j=1}^4 \sum_{k=1}^3 (jk)$$

Solution: $\sum_{j=1}^4 \sum_{k=1}^3 (jk) = \sum_{j=1}^4 (j+2j+3j)$
Simplify if possible

$$= \sum_{j=1}^{4} 6j$$

Expand inner sum

$$= 6 + 12 + 18 + 24 = 60$$

Expand outer sum

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In-class exercises

Problem 1: What are the formulas for the following sequences?

- a. 3, 6, 9, 12, 15, ...
- b. 1/3, 2/3, 4/3, 8/3, ...
- c. 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, ...

Problem 2: Compute the following summations:

a.
$$\sum_{k=1}^{5} (k+1)$$

b.
$$\sum_{k=0}^{8} (2^{k+1} - 2^k)$$

Computing the sum of a geometric series by hand is time consuming...

Would you really want to calculate $\sum_{j=0}^{2} (6 \times 2^{j})$ by hand?

Fortunately, we have a closed-form solution for computing the sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1\\ (n+1)a & \text{if } r = 1 \end{cases}$$

So,
$$\sum_{j=0}^{20} (6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906$$



Proof of geometric series closed form

There are other closed form summations that you should know

Sum	Closed Form

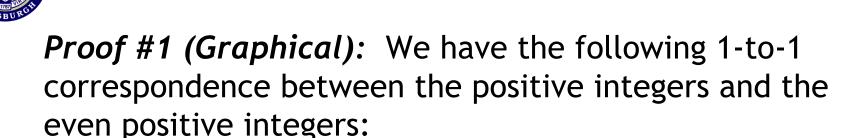
We can use the notion of sequences to analyze the cardinality of infinite sets

Definition: Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence (a bijection) from A to B.

Definition: A finite set or a set that has the same cardinality as the natural numbers (or the positive integers) is called countable. A set that is not countable is called uncountable.

Implication: Any sequence $\{a_n\}$ ranging over the natural numbers is countable.

Show that the set of even positive integers is countable



So, the even positive integers are countable. \Box

Proof #2: We can define the even positive integers as the sequence $\{2k\}$ for all $k \in \mathbb{Z}^+$, so it has the same cardinality as \mathbb{Z}^+ , and is thus countable. \square

Is the set of all rational numbers countable?

Perhaps surprisingly, yes!

This yields the sequence 1/1, 1/2, 2/1, 3/1, 1/3, ..., so the set of rational numbers is countable. \Box

Is the set of real numbers countable?

No, it is not. We can prove this using a proof method called diagonalization, invented by Georg Cantor.

Proof: Assume that the set of real numbers is countable. Then the subset of real numbers between 0 and 1 is also countable, by definition. This implies that the real numbers can be listed in some order, say, r1, r2, r3

Let the decimal representation these numbers be:

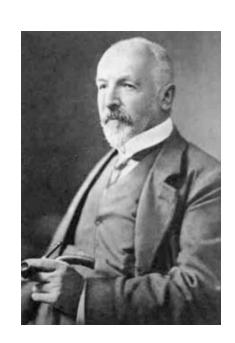
$$r1 = 0.d_{11}d_{12}d_{13}d_{14}...$$

$$r2 = 0.d_{21}d_{22}d_{23}d_{24}...$$

$$r3 = 0.d_{31}d_{32}d_{33}d_{34}...$$

•••

Where $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\} \ \forall i,j$



Proof (continued)

Now, form a new decimal number $r=0.d_1d_2d_3...$ where $d_i=0$ if $d_{ii}=1$, and $d_i=1$ otherwise.

Example:

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r_1 = 0.123456...
r_2 = 0.234524...
r_3 = 0.631234...
...
r = 0.010...
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Note that the i^{th} decimal place of r differs from the i^{th} decimal place of each r_i , by construction. Thus r is not included in the list of all real numbers between 0 and 1. This is a contradiction of the assumption that all real numbers between 0 and 1 could be listed. Thus, not all real numbers can be listed, and \mathbf{R} is uncountable. \square

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Final thoughts

- Sequences allow us to represent (potentially infinite) ordered lists of elements
- Summation notation is a compact representation for adding together the elements of a sequence
- We can use sequences to help us compare the cardinality of infinite sets
- Next time:
 - Integers and division (Section 4.1)