Discrete Structures for Computer Science

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Lecture #10: Sequences and Summations

Based on materials developed by Dr. Adam Lee

Today's Topics

Sequences and Summations

- Specifying and recognizing sequences
- Summation notation
- Closed forms of summations
- Cardinality of infinite sets

Sequences are ordered lists of elements

Definition: A sequence is a function from a subset of the set of integers to a set S. We use the notation a_n to denote the image of the integer n. a_n is called a term of the sequence.

Examples:

- 1, 3, 5, 7, 9, 11 — A sequence with 6 terms
- 1, 1/2, 1/3, 1/4, 1/5, … — An infinite sequence

Note: The second example can be described as the sequence $\{a_n\}$ where $a_n = 1/n$

What makes sequences so special?

Question: Aren't sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is ordered, a set is not.

Question: How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

Some special sequences

Geometric progressions are sequences of the form $\{ar^n\}$ where a and r are real numbers

Examples:

- 1, 1/2, 1/4, 1/8, 1/16, …
- 1, -1, 1, -1, 1, -1, …

Arithmetic progressions are sequences of the form $\{a + nd\}$ where a and d are real numbers.

Examples:

- 2, 4, 6, 8, 10, …
- -10, -15, -20, -25, …

Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?
2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles amongst terms?

What are the formulas for these sequences?

Problem 1: 1, 5, 9, 13, 17, …

Problem 2: 1, 3, 9, 27, 81, …

Problem 3: 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, …

Problem 4: 1, 1, 2, 3, 5, 8, 13, 21, 34, …

This is called the Fibonacci sequence.

Sometimes we want to find the sum of the terms in a sequence

Summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + \ldots + a_n$

Upper limit

$$\sum_{j=m}^{n} a_j = \sum_{m \le j \le n} a_j$$

Index of summation *Lower limit*

Example: $\sum_{1 \le i \le 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$

The usual laws of arithmetic still apply

$$\sum_{j=1}^{n}(ax_j + by_j - cz_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j - c\sum_{j=1}^{n} z_j$$

Constant factors can be pulled out of the summation

A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations

Example:

- $\Sigma_{1 \le j \le 3} (4j + j^2) =$
- $4\Sigma_{1 \le j \le 3}\, j + \Sigma_{1 \le j \le 3}\, j^2 =$

Example sums

Example: Express the sum of the first 50 terms of the sequence $1/n^2$ for n = 1, 2, 3, …

Answer: $$\sum_{j=1}^{50} \frac{1}{j^2}$$

Example: What is the value of $\sum_{k=4}^{8} (-1)^k$

Answer: $$\sum_{k=4}^{8} (-1)^k =$$

$$=$$

$$=$$

We can also compute the summation of the elements of some set

Example: Compute $\sum_{s \in \{0,2,4,6\}} (s+2)$

Answer: (0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20

Example: Let f(x) = x^3 + 1. Compute $\sum_{s \in \{1,3,5,7\}} f(s)$

Answer: $f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500$

Sometimes it is helpful to shift the index of a summation

This is particularly useful when combining two or more summations. For example:

$$
\begin{aligned}
S &= \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k-1) \\
&= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j+1)-1) \\
&= \sum_{j=1}^{10} (j^2 + 2(j+1) - 1) \\
&= \sum_{j=1}^{10} (j^2 + 2j + 1) \\
&= \sum_{j=1}^{10} (j+1)^2
\end{aligned}
$$

Let j = k - 1

Need to add 1 to each j

Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

Expand inner sum

Example: Compute $\sum_{j=1}^{4}\sum_{k=1}^{3}(jk)$

Solution:
$$\sum_{j=1}^{4}\sum_{k=1}^{3}(jk) = \sum_{j=1}^{4}(j+2j+3j)$$

Simplify if possible

$$= \sum_{j=1}^{4} 6j$$

$$= 6 + 12 + 18 + 24 = 60$$

Expand outer sum

In-class exercises

Problem 1: What are the formulas for the following sequences?

a. 3, 6, 9, 12, 15, …
b. 1/3, 2/3, 4/3, 8/3, …
c. 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

Problem 2: Compute the following summations:

a. $$\sum_{k=1}^{5}(k+1)$$

b. $$\sum_{k=0}^{8}(2^{k+1}-2^{k})$$

Computing the sum of a geometric series by hand is time consuming…

Would you really want to calculate $\sum_{j=0}^{20}(6 \times 2^j)$ by hand?

Fortunately, we have a closed-form solution for computing the sum of a geometric series:

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

So, $\sum_{j=0}^{20}(6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2-1} = 12,582,906$

Why?

Proof of geometric series closed form

There are other closed form summations that you should know

Sum	*Closed Form*

We can use the notion of sequences to analyze the cardinality of infinite sets

Definition: Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence (a bijection) from A to B.

Definition: A finite set or a set that has the same cardinality as the natural numbers (or the positive integers) is called countable. A set that is not countable is called uncountable.

Implication: Any sequence $\{a_n\}$ ranging over the natural numbers is countable.

Show that the set of even positive integers is countable

Proof #1 (Graphical): We have the following 1-to-1 correspondence between the positive integers and the even positive integers:

So, the even positive integers are countable. ❑

Proof #2: We can define the even positive integers as the sequence $\{2k\}$ for all $k \in \mathbf{Z}^+$, so it has the same cardinality as $\mathbf{Z}^+$, and is thus countable. ❑

Is the set of all rational numbers countable?

Perhaps surprisingly, yes!

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

This yields the sequence 1/1, 1/2, 2/1, 3/1, 1/3, …, so the set of rational numbers is countable. ❑

Is the set of real numbers countable?

No, it is not. We can prove this using a proof method called diagonalization, invented by Georg Cantor.

Proof: Assume that the set of real numbers is countable. Then the subset of real numbers between 0 and 1 is also countable, by definition. This implies that the real numbers can be listed in some order, say, *r1*, *r2*, *r3* ….

Let the decimal representation these numbers be:

$r1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$

$r2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$

$r3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$

…

Where $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}\ \forall i,j$

Proof (continued)

Now, form a new decimal number $r=0.d_1d_2d_3\ldots$ where $d_i = 0$ if $d_{ii} = 1$, and $d_i=1$ otherwise.

Example:

$r_1 = 0.123456\ldots$

$r_2 = 0.234524\ldots$

$r_3 = 0.631234\ldots$

…

$r = 0.010\ldots$

Note that the i^{th} decimal place of r differs from the i^{th} decimal place of each r_i, by construction. Thus r is not included in the list of all real numbers between 0 and 1. This is a contradiction of the assumption that all real numbers between 0 and 1 could be listed. Thus, not all real numbers can be listed, and **R** is uncountable. ❑

Final thoughts

- Sequences allow us to represent (potentially infinite) ordered lists of elements
- Summation notation is a compact representation for adding together the elements of a sequence
- We can use sequences to help us compare the cardinality of infinite sets
- Next time:
  - Integers and division (Section 4.1)