Discrete Structures for Computer Science

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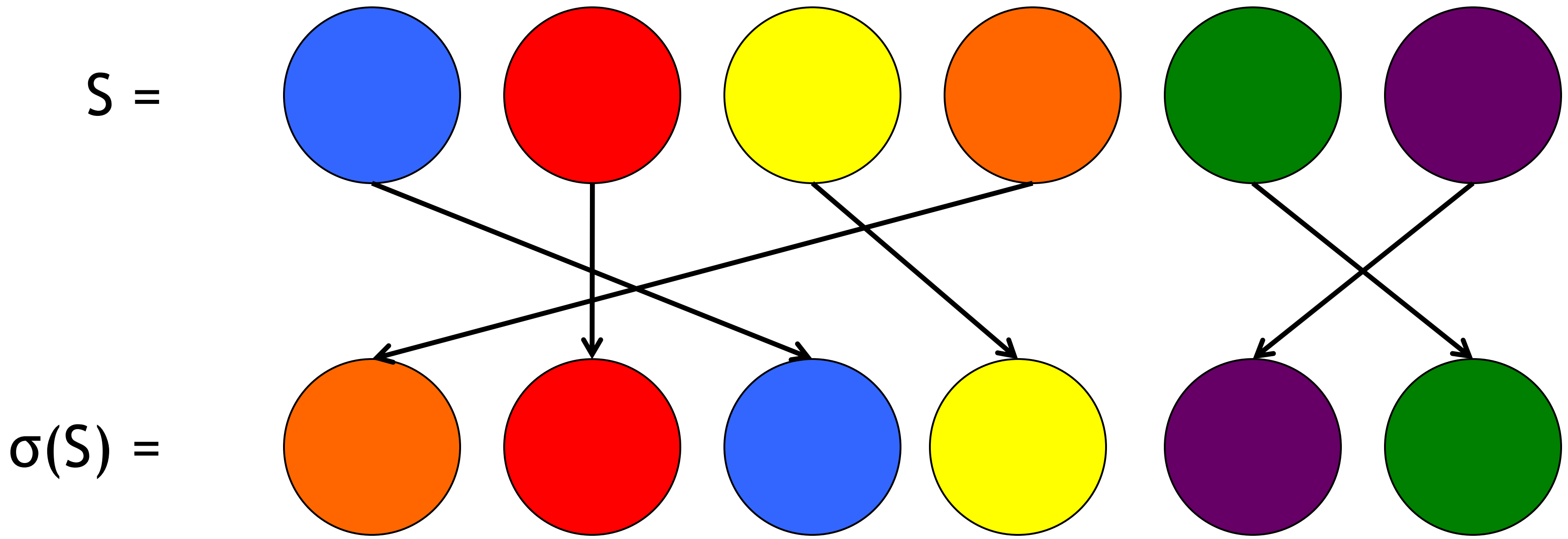
Lecture #21: Permutations and Combinations

Based on materials developed by Dr. Adam Lee

Today’s Topics

- Permutations
- Combinations
- Binomial coefficients

A permutation is an ordered arrangement of a set of objects



Note: A permutation of some set is essentially just a shuffling of that set.

Sometimes we're interested in counting the number of ways that a given set can be arranged

Example: Suppose that a photographer wants to take a picture of three dogs. How many ways can the dogs be arranged?



There are six possible arrangements of three dogs!



Counting permutations

In general, we can use the product rule to count the number of permutations of a given set.

Given a set of n items, we have:

- n ways to pick the 1st item in the permuted set
- n-1 ways to pick the 2nd item in the permuted set
- n-2 ways to pick the 3rd item in the permuted set
- …
- 1 way to choose the last item in the permuted set

So, for a set of size n, we have $n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$ ways to permute that set

And the winner is…

Example: Six friends run in a foot race. How many possible outcomes of the race are there, assuming that there are no ties?

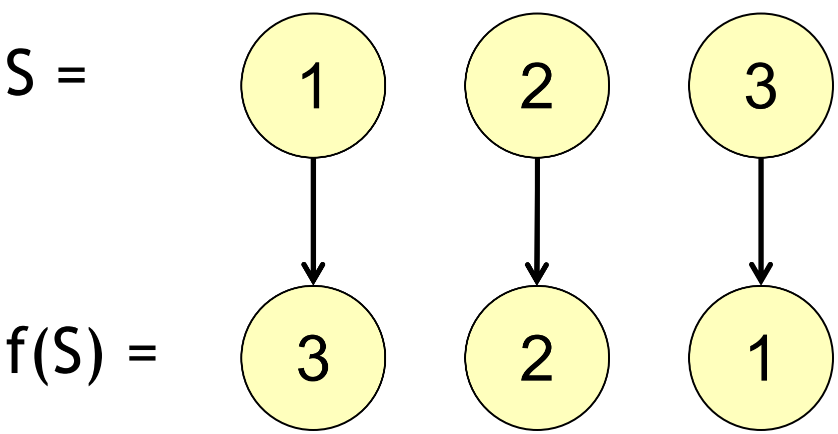
Solution:

- 6 ways to choose 1st place
- 5 ways to choose 2nd place
- 4 ways to chose 3rd place
- 3 ways to choose 4th place
- 2 ways to choose 5th place
- 1 way to choose last place
- So there are $6! = 720$ possible outcomes!

Functions and permutations

Let S be some set. If f: S → S is a bijection, then f describes a permutation of the set S.

Example: Let S = {1, 2, 3}, f(1) = 3, f(2) = 2, f(3) = 1



More often than not, we're only interested in arranging a subset of a given set

Definition: An r-permutation is a permutation of some r elements of a set.

Example: Let S = {Alice, Bob, Carol, Dave}. Then:

- Dave, Bob is a 2-permutation of S
- Carol, Alice, Bob is a 3-permutation of S
- Bob, Dave is a 2-permutation of S

Rather than specifying a particular r-permutation of a set, we're usually more interested in counting the number of r-permutations of a set

Counting r-permutations

Definition: We denote the number of r-permutations of a set of size n by P(n, r). By the product rule:

$$P(n,r) = n \times (n-1) \times \ \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Example: In a foot race between six people, how many ways can the gold, silver, and bronze medals be assigned, assuming that there are no ties?

Solution:

- P(6, 3) = 6!/(6-3)! = 6!/3! = 720/6 = 120
- So, 120 ways to assign medals

The traveling salesperson

Example: A salesperson must visit 7 different cities. The first and last cities of her route are specified by her boss, but she can choose the order of the other visits. How many possible trips can she take?

Solution:

- Since first and last cities are fixed, we must count the number of ways to permute the 5 remaining cities
- So, there are $5! = 120$ possible trips that the salesperson can take.

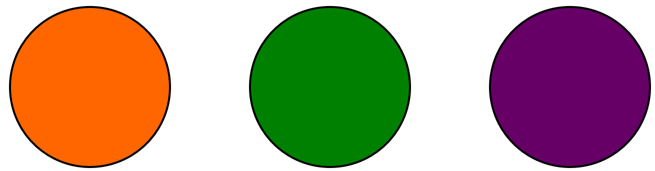
Counting strings

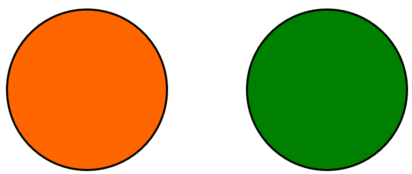
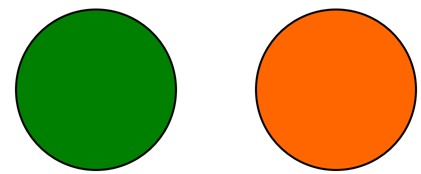
Example: How many permutations of "ABCDEFG" contain the substring "ABC"?

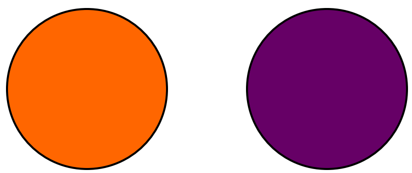
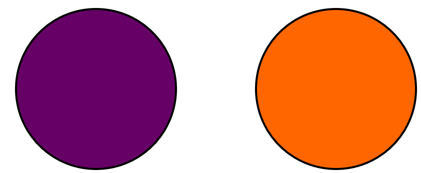
Solution:

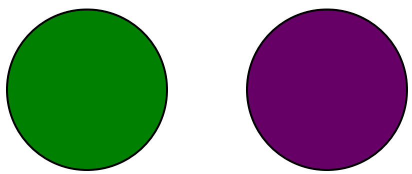
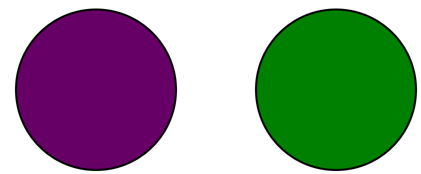
- Observation: Treat "ABC" as one character
- Now, how many ways are there to permute {ABC, D, E, F, G}?
- |{ABC, D, E, F, G}| = 5
- 5! = 120 permutations contain the substring "ABC"

How do we count unordered selections of objects?

Example: How many ways can we choose two objects from the set S = 

S_1 =  = 

S_2 =  = 

S_3 =  = 

Conclusion: There are three 2-combinations of a set of size three.

Counting r-combinations

Definition: Let C(n, r) denote the number of r-combinations of a set of size n. Then:

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Proof:

- The r-permutations of the set can be formed by finding all of the r-combinations of the set and then permuting each r-combination in each possible way
- There are $P(r,r)$ ways to permute each possible r-combination
- So, $P(n,r) = C(n,r) \times P(r,r)$
- This means that $C(n,r) = P(n,r)/P(r,r)$
- $= (n!/(n-r)!)/r!$
- $= n!/(r!\,(n-r)!)$ ❑

Alternate notation

Note 1: Sometimes, C(n, r) is read "n choose r"

This is intuitive, as C(n, r) specifies the number of ways to choose r objects from a set of size n.

Note 2: C(n, r) is often written as $\binom{n}{r}$. In this class, I will use the notation C(n, r) exclusively.

Be careful using the formula for C(n, r)

Using the formula for C(n, r) directly can result in doing lots of multiplication…

Instead, note that much of the denominator cancels out terms in the numerator. For example:

$$C(52, 5)$$

Poker Hands

Example: Given a standard 52-card deck, how many 5 card poker hands can be drawn?

Solution:

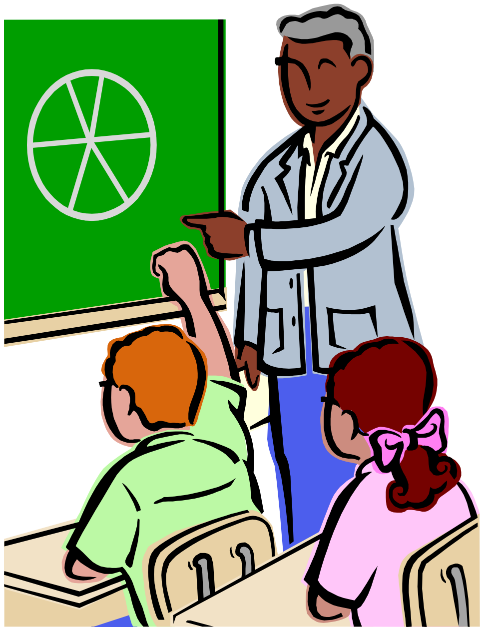
- $C(52, 5) = 52!/[5!\ 47!]$
- $= (52 \times 51 \times 50 \times 49 \times 48)/(5 \times 4 \times 3 \times 2 \times 1)$
- $= 26 \times 17 \times 10 \times 49 \times 12$
- $=$ 2,598,960 different hands

Choosing participants

Example: Say that a class consists of 30 students. How many ways can 3 people be chosen from this class to participate in a survey?

Solution:

- Want to compute is “30 choose 3”
- $C(30, 3) = 30!/[3!\ 27!]$
- $= (30 \times 29 \times 28)/(3 \times 2)$
- $= 10 \times 29 \times 14$
- $=$ 4,060 ways to choose participants



An interesting observation…

Note: Given a set of size n, choosing r elements is the same as excluding n-r elements.

This means that $C(n, r) = C(n, n-r)$.

Proof:

- $C(n, n-r) = n!/[(n-r)!\,(n-(n-r))!]$
- $= n!/[(n-r)!\,r!]$ by simplification
- $= n!/[r!\,(n-r)!]$ by the commutative property
- $= C(n, r)$ by definition ❑

Permutations and combinations can be used in conjunction with the product and sum rules!

Example: Suppose the CS department has 11 faculty members and the Math department has 9 faculty members. How many ways can a committee consisting of 4 CS faculty members and 3 Math faculty members be chosen?

Solution:

- $C(11, 4)$ ways to choose 4 CS profs
- For each of these, there are $C(9, 3)$ ways to choose 3 Math profs
- So, we need to compute $C(11, 4) \times C(9, 3)$
- $C(11, 4) \times C(9, 3) = 11!/(4!\ 7!) \times 9!/(3!\ 6!)$
- $= 84 \times 330$
- $=$ 27,720 ways!

In-class exercises

Problem 1: Consider a boat race with 7 entrants. In how many different ways can the 1st, 2nd, and 3rd place trophies be awarded?

Problem 2: In the above race, how many ways can the set of entrants not receiving a trophy be selected?

Problem 3: Consider a standard 52-card deck. If two jokers are added to this deck, how many 5-card hands can be dealt containing both jokers?

Combinations can be helpful when examining binomial expressions

Definition: A binomial is an expression involving the sum of two terms.

- For example $(x + y)$ is a binomial, as is $(3 + j)^4$.

r-Combinations are also called binomial coefficients, since they occur as coefficients in the expansions of binomial expressions.

Example:

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$= C(3,0)x^3y^0 + C(3,1)x^2y^1 + C(3,2)x^1y^2 + C(3,3)x^0y^3$$

The Binomial Theorem

The Binomial Theorem: Let x and y be variables, and let n be a non-negative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j$$

Example: Compute $(x + y)^2$.

- $C(2, 0) = 1$
- $C(2, 1) = 2$
- $C(2, 2) = 1$
- $(x + y)^2 = x^2y^0 + 2x^1y^1 + x^0y^2$
- $= x^2 + 2xy + y^2$ ✔

A small example…

Question: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution: By the binomial theorem, we know that this term can be written as $C(25,13)x^{12}y^{13}$, so the coefficient of $x^{12}y^{13}$ is $C(25, 13) = 5{,}200{,}300$.

A slightly more complicated example…

Question: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution:

- Note that $(2x-3y)^{25} = (2x + (-3)y)^{25}$
- The binomial theorem tells us that:

$$\left(2x + (-3y)\right)^{25} = \sum_{j=0}^{25} C(25,j)(2x)^{25-j}(-3y)^{j}$$

- Thus, the coefficient of $x^{12}y^{13}$ occurs when $j = 13$
- $C(25,13)(2x)^{25-13}(-3y)^{13} = C(25,13)2^{12}(-3)^{13}x^{12}y^{13}$
- So the coefficient $= 25!/(13!\ 12!)\ 2^{12}\ (-3)^{13}$
- $= -[25!/(13!\ 12!)\ 2^{12}\ 3^{13}]$

Pascal's Identity

Pascal's identity: $C(n+1, k) = C(n, k-1) + C(n, k)$

***Proof*:**

- Let T be a set of n+1 elements
- Let $a \in T$, and $S = T - \{a\}$
- By definition, there are $C(n+1, k)$ subsets of T containing k elements
- A subset of T containing k elements either contains k elements of S, or the element *a* along with k-1 elements of S
- There are $C(n, k)$ subsets of S containing k elements, so there are $C(n, k)$ subsets of T not containing *a*.
- Further, there are $C(n, k-1)$ subsets of S containing k-1 elements, so there are $C(n, k-1)$ subsets of T containing *a*.
- Thus, $C(n+1, k) = C(n, k-1) + C(n, k)$ ❑

We can use Pascal's identity to define C(n, r) recursively

Basis step: For all n, we have that C(n, 0) = C(n, n) = 1

Recursive step: C(n+1, k) = C(n, k-1) + C(n, k)

Example: Compute C(3,2).

- C(3,2) = C(2,1) + C(2,2)
- = C(1,0) + C(1,1) + 1
- = 1 + 1 + 1
- = 3
- = 3!/(2!(3-2)!) ✔

Using this definition, we can compute C(n, r) without using multiplication at all!

In-class exercises

Problem 4: What is the coefficient of the term a^7b^8 in $(a+b)^{15}$?

Problem 5: Use the recursive definition of C(n, r) to calculate C(5,3).

Final Thoughts

- Permutations count the ways that we can shuffle a set (forming a sequence). r-Permutations count the number of ways that we can arrange r items from a set.
- r-Combinations are useful when we want to count (unordered) subsets of a given set
- Next time:
  - Generalized permutations and combinations