Discrete Structures for Computer Science

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Lecture #14: Strong Induction

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Based on materials developed by Dr. Adam Lee

Recall that mathematical induction let us prove universally quantified statements

Intuition: If P(0) is true, then P(1) is true. If P(1) is true, then P(2) is true…

Goal: Prove $\forall x \in \mathbf{N}\ P(x)$.

Procedure:

1. Prove $P(0)$
2. Show that $P(k) \rightarrow P(k+1)$ for any arbitrary k
3. Conclude that $P(x)$ is true $\forall x \in \mathbf{N}$

$$\begin{array}{l} P(0) \\ P(k) \rightarrow P(k+1) \\ \hline \therefore \forall x \in \mathbf{N}\ P(x) \end{array}$$

Strong mathematical induction is another flavor of induction

Goal: Prove $\forall x \in \mathbf{N}\ P(x)$.

Procedure:

1. Prove $P(0)$
2. Show that $[P(0) \wedge P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ for any arbitrary k
3. Conclude that $P(x)$ is true $\forall x \in \mathbf{N}$

$$\begin{array}{l} P(0) \\ [P(0) \wedge P(1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1) \\ \hline \therefore \forall x \in \mathbf{N}\ P(x) \end{array}$$

So what's the big deal?

Recall: In mathematical induction, our inductive hypothesis allows us to assume that P(k) is true and use this knowledge to prove P(k+1)

However, in strong induction, we can assume that $P(0) \wedge P(1) \wedge \ldots \wedge P(k)$ is true before trying to prove P(k+1)

For certain types of proofs, this is much easier than trying to prove P(k+1) from P(k) alone.

For example…

Show that if n is an integer greater than 1, then n can be written as the product of primes

$P(n) \equiv$ n can be written as a product of primes

Base case: $P(2)$: $2 = 2^1$ ✔

I.H.: Assume that $P(2) \wedge \ldots \wedge P(k)$ holds for an arbitrary k

Inductive step: We will now show that $[P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

- Two cases to consider: k+1 prime and k+1 composite
- If k+1 is prime, then we're done
- If k+1 is composite, then by definition, $k+1 = ab$
- Since $2 \leq a < k$ and $2 \leq b < k$, a and b can be written as products of primes by the I.H.
- Thus, k+1 can be written as a product of primes

Conclusion: Since we have proved the base case and the inductive case, the claim holds by strong induction ❑

Is strong induction somehow more powerful than mathematical induction?

The ability to assume $P(0) \land P(1) \land \ldots \land P(k)$ true before proving $P(k+1)$ seems more powerful than just assuming $P(k)$ is true

Perhaps surprisingly, mathematical induction, strong induction, and well ordering are all equivalent!

That is, a proof using one of these methods can always be written using the other two methods

This may not be easy, though!

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps

$P(n) \equiv$ n cents of postage can be made using 4- and 5-cent stamps

Base case: P(12): 3 4-cent stamps
P(13): 1 5-cent stamp, 2 4-cent stamps
P(14): 2 5-cent stamps, 1 4-cent stamp
P(15): 3 5-cent stamps ✔

I.H.: Assume that $P(12) \wedge \ldots \wedge P(k)$ holds for an arbitrary integer k

Inductive step: We will now show that $[P(12) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

- By the I.H., we can make k-3 cents worth of postage using only 4-cent and 5-cent stamps
- By adding another 4-cent stamp, we end up with k+1 cents worth of postage

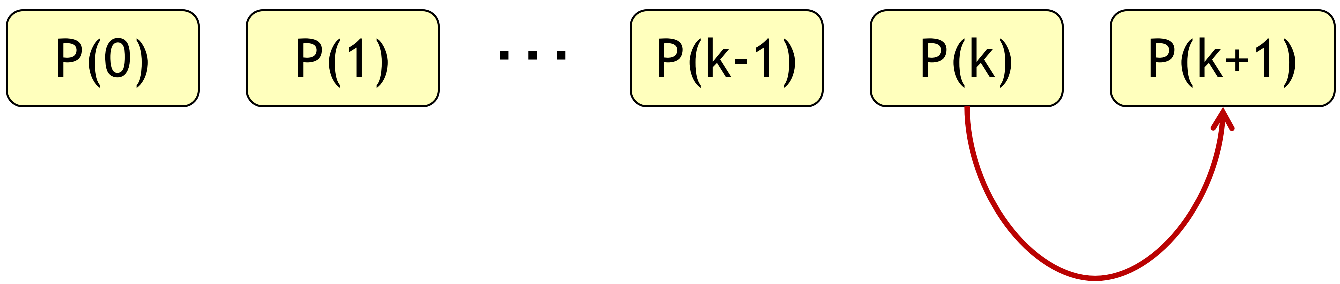
Conclusion: Since we have proved the base case and the inductive case, the claim holds by strong induction ❑

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps

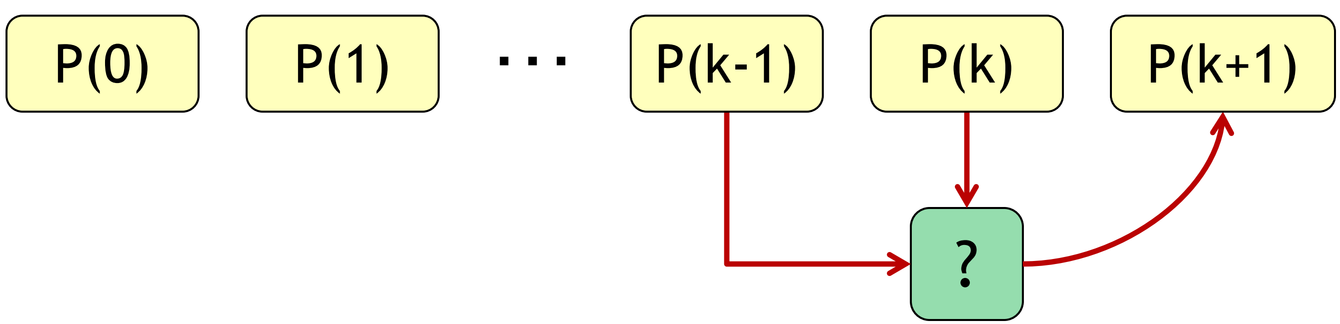
$P(n) \equiv$ n cents of postage can be made using 4- and 5-cent stamps
Base case: P(12): 3 4-cent stamps ✔
I.H.: Assume that P(k) holds for an arbitrary integer k
Inductive step: We will now show that $P(k) \rightarrow P(k+1)$ ■ We can k cents using 4- and 5-cent stamps, by the I.H. ■ If at least one 4-cent stamp was used, remove that stamp and add in a 5-cent stamp, thereby making k+1 cents of postage ■ If no 4-cent stamps were used, then only 5-cent stamps were used. ■ Since k > 12, at least 3 5-cent stamps were used. ■ Replace 3 5-cent stamps with 4 4-cent stamps, thereby making k+1 cents of postage
Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

So when should we use strong induction?

If it is straightforward to prove P(k+1) from P(k) alone, use mathematical induction



If it would be easier to prove P(k+1) using one or more P(j) for 0 ≤ j < k, use strong induction



In-class exercises

Problem 1: Use strong induction to prove that any whole dollar amount greater than or equal to \$4 can be formed using only \$2 and \$5 bills.

In-class exercises

Problem 2: Suppose you have a box of Maltesers containing n candies, and you want to split it into n piles of 1 candy each by repeatedly splitting a pile into two smaller piles. Each time you split a pile, you multiply the number of candies in each of the two smaller piles you form, and add it to a running total. For example, if you split a pile of 13 candies into two piles of size 4 and 9, you would add $4 \times 9 = 36$ to the total.

Show that, no matter how you split the piles, the sum of the products computed at each step will equal $\frac{n(n-1)}{2}$

Final Thoughts

- Strong induction lets us prove universally quantified statements using this inference rule:

$$\begin{array}{l} P(0) \\ [P(0) \land P(1) \land \ldots \land P(k)\,] \rightarrow P(k+1) \\ \hline \therefore \forall x \in \mathbf{N}\ P(x) \end{array}$$

- Although sometimes more convenient than mathematical induction, strong induction is no more powerful