Discrete Structures for Computer Science

William Garrison
bill@cs.pitt.edu
6311 Sennott Square

Lecture #27: Relations and Representations

University of Pittsburgh

Based on materials developed by Dr. Adam Lee

Binary relations establish a relationship between elements of two sets

Definition: Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation R is a set of ordered pairs (a_i, b_i) where $a_i \in A$ and $b_i \in B$.

Notation: We say that

- $a \, R \, b$ if $(a,b) \in R$
- $a \not R \, b$ if $(a,b) \notin R$

Example: Course Enrollments

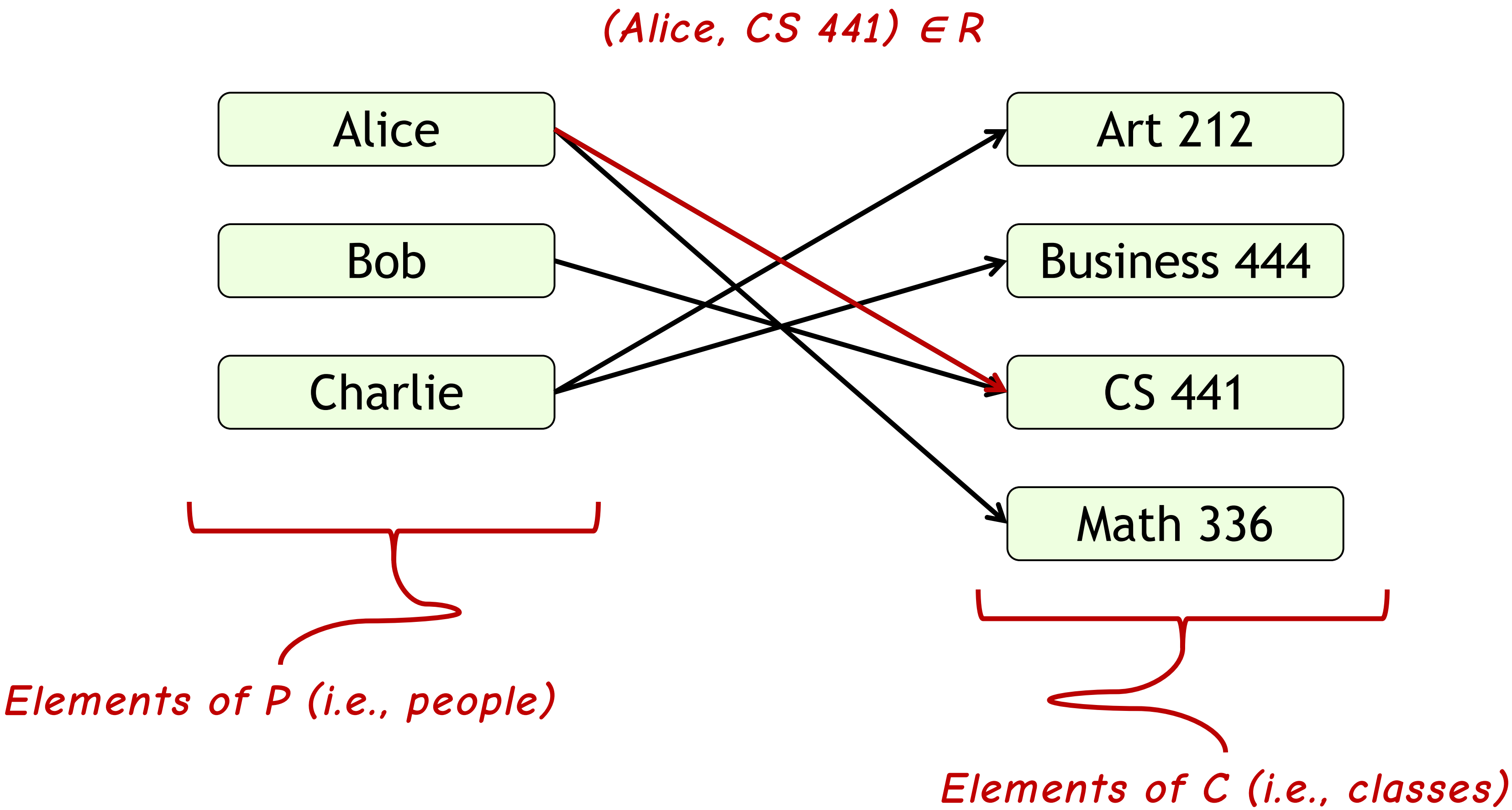
Let's say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Solution:

- Let the set P denote people, so P = {Alice, Bob, Charlie}
- Let the set C denote classes, so C = {CS 441, Math 336, Art 212, Business 444}
- By definition $R \subseteq P \times C$
- From the above statement, we know that
  - (Alice, CS 441) $\in$ R
  - (Bob, CS 441) $\in$ R
  - (Alice, Math 336) $\in$ R
  - (Charlie, Art 212) $\in$ R
  - (Charlie, Business 444) $\in$ R
- So, R = {(Alice, CS 441), (Bob, CS 441), (Alice, Math 336), (Charlie, Art 212), (Charlie, Business 444)}

A relation can also be represented as a graph

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.



A relation can also be represented as a table

Let's say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Name of the relation

(Bob, CS 441) ∈ R

Elements of C (i.e., courses)

R	Art 212	Business 444	CS 441	Math 336
Alice			X	X
Bob			X	
Charlie	X	X		

Elements of P (i.e., people)

Wait, doesn't this mean that relations are the same as functions?

Not quite… Recall the following definition from Lecture #9.

Definition: Let A and B be nonempty sets. A function, f, is an assignment of exactly one element of set B to each element of set A.

This would mean that, e.g., a person only be enrolled in one course!

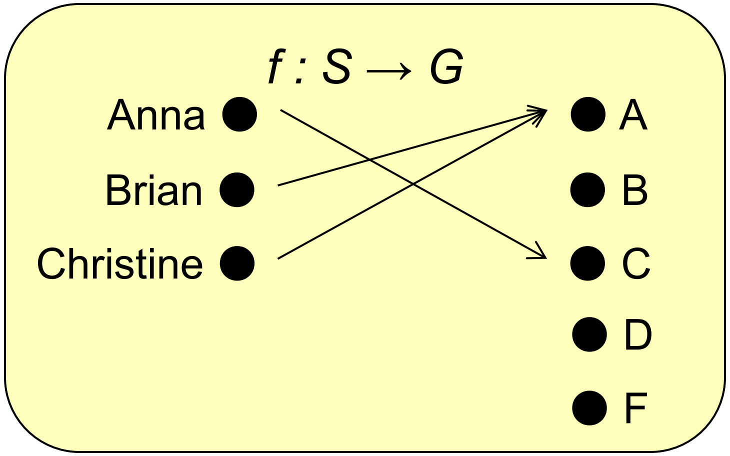
Reconciling this with our definition of a relation, we see that

1. Every function is also a relation
2. Not every relation is a function

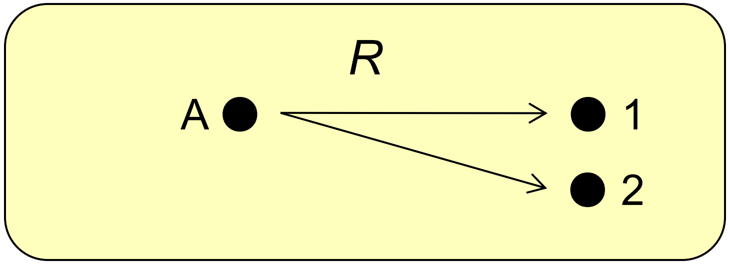
Let's see some quick examples…

Short and sweet…

1. Consider $f : S \rightarrow G$
   - Clearly a function
   - Can also be represented as the relation R = {(Anna, C), (Brian, A), (Christine A)}



1. Consider the set R = {(A, 1), (A, 2)}
   - Clearly a relation
   - Cannot be represented as a function!



We can also define binary relations on a single set

Definition: A relation on the set A is a relation from A to A. That is, a relation on the set A is a subset of $A \times A$.

Example: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

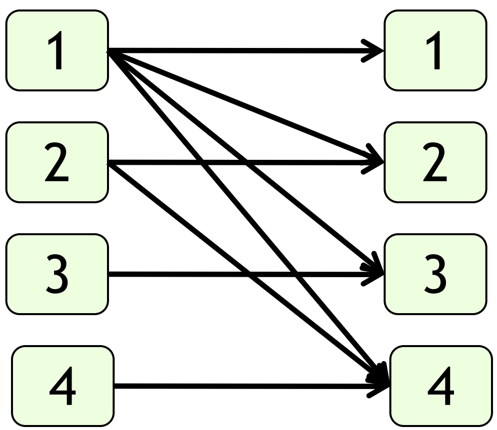
Solution:

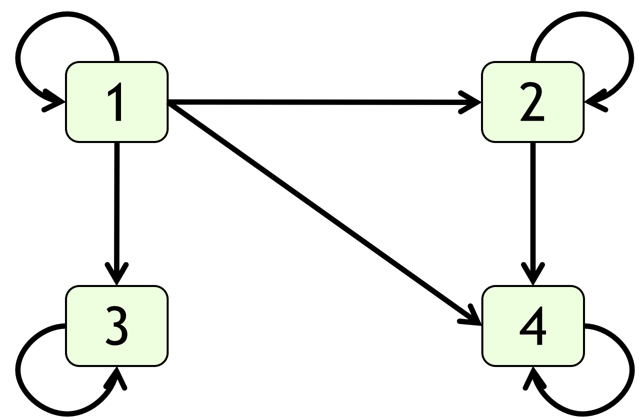
- 1 divides everything
- 2 divides itself and 4
- 3 divides itself
- 4 divides itself

- So, R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Representing the last example as a graph…

Example: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?





Tell me what you know…

Question: Which of the following relations contain each of the pairs (1,1), (1,2), (2,1), (1,-1), and (2,2)?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

These are all relations on an infinite set!

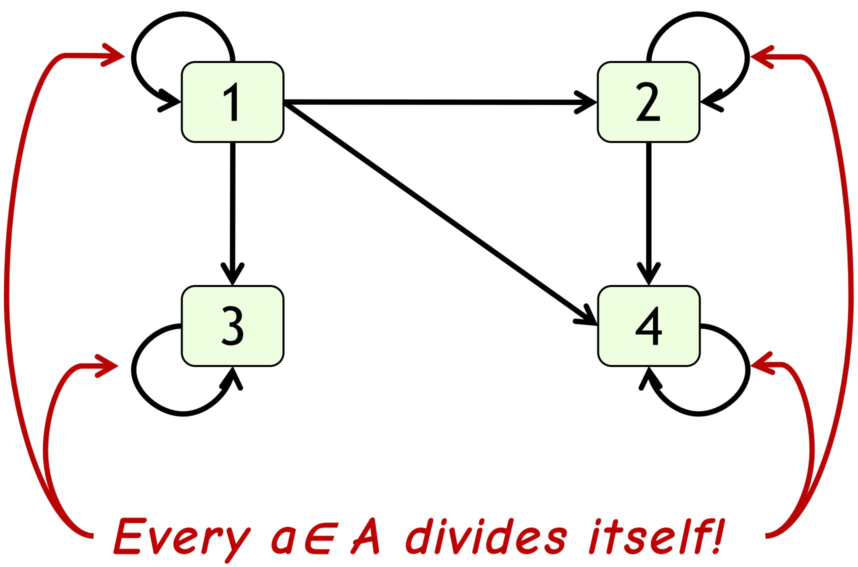
Answer:

	(1,1)	(1,2)	(2,1)	(1,-1)	(2,2)
R_1					
R_2					
R_3					
R_4					
R_5					
R_6					

Properties of Relations

Definition: A relation R on a set A is reflexive if $(a,a) \in R$ for every $a \in A$.

Note: Our "divides" relation on the set $A = \{1,2,3,4\}$ is reflexive.



	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

Properties of Relations

Definition: A relation R on a set A is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for every $a,b \in A$. If R is a relation in which $(a,b) \in R$ and $(b,a) \in R$ implies that $a=b$, we say that R is antisymmetric.

Mathematically:

- Symmetric: $\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$
- Antisymmetric: $\forall a \forall b(((a,b) \in R \land (b,a) \in R) \rightarrow (a = b))$

Examples:

- Symmetric: $R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}$
- Antisymmetric: $R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}$

Symmetric and Antisymmetric Relations

R = {(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)}

	1	2	3	4
1	X	X	X	X
2	X		X	
3	X	X		
4	X			X

Symmetric relation

- Diagonal axis of symmetry
- Not all elements on the axis of symmetry need to be included in the relation

R = {(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)}

	1	2	3	4
1	X	X	X	X
2				X
3			X	
4				X

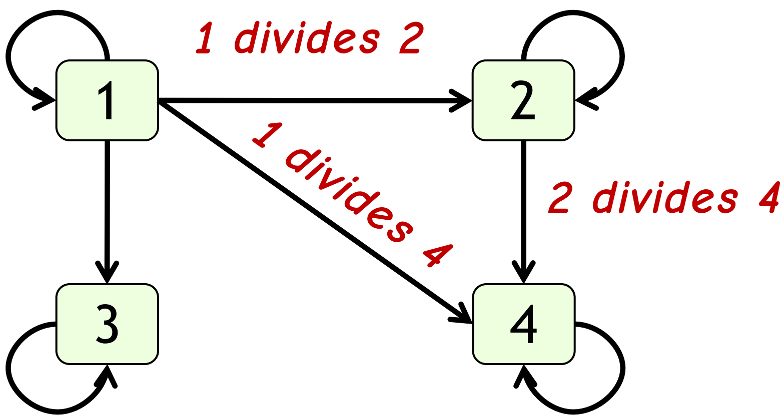
Antisymmetric relation

- No axis of symmetry
- Only symmetry occurs on diagonal
- Not all elements on the diagonal need to be included in the relation

Properties of Relations

Definition: A relation R on a set A is transitive if, whenever (a,b) ∈ R and (b,c) ∈ R, then (a,c) ∈ R for every a,b,c ∈ A.

Note: Our "divides" relation on the set A = {1,2,3,4} is transitive.



Recall that:

$$a|b \wedge b|c \rightarrow a|c$$

More common transitive relations include equality and comparison operators like <, >, ≤, and ≥.

Examples, redux

Question: Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

Answer:

	Reflexive	Symmetric	Antisymmetric	Transitive
R_1				
R_2				
R_3				
R_4				
R_5				
R_6				

Relations can be combined using set operations

Example: Let R be the relation that pairs students with courses that they have taken. Let S be the relation that pairs students with courses that they need to graduate. What do the relations R ∪ S, R ∩ S, and S – R represent?

Solution:

- R ∪ S = All pairs (a,b) where
  - student a has taken course b OR
  - student a needs to take course b to graduate
- R ∩ S = All pairs (a,b) where
  - Student a has taken course b AND
  - Student a needs course b to graduate
- S – R = All pairs (a,b) where
  - Student a needs to take course b to graduate BUT
  - Student a has not yet taken course b

Relations can be combined using functional composition

Definition: Let R be a relation from the set A to the set B, and S be a relation from the set B to the set C. The composite of R and S is the relation of ordered pairs (a, c), where $a \in A$ and $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $R \circ S$.

Example: What is the composite relation of R and S?

R: $\{1,2,3\} \rightarrow \{1,2,3,4\}$

- R = {(1,1),(1,4),(2,3),(3,1),(3,4)}

S: $\{1,2,3,4\} \rightarrow \{0,1,2\}$

- S = {(1,0),(2,0),(3,1),(3,2),(4,1)}

So: $R \circ S$ = {(1,0), (3,0), (1,1), (3,1), (2,1), (2,2)}

In-class exercises

Problem 1: List the ordered pairs of the relation R from A = {0,1,2,3,4} to B = {0,1,2,3} where (a,b) ∈ R iff a + b = 4.

Problem 2: Draw the graph and table representations of the above relation.

Problem 3: Is the above relation reflexive, symmetric, antisymmetric, and/or transitive?

Final Thoughts

Relations allow us to represent and reason about the relationships between sets

Relations are more general than functions

Relations are use all over…

- Mathematical operators
- Bindings between sets of objects
- Etc.

Next time: n-ary relations