Discrete Structures for Computer Science

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Lecture #13: Proof by Induction

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Based on materials developed by Dr. Adam Lee

We've learned a lot of proof methods…

Basic proof methods

- Direct proof, contradiction, contraposition, cases, …

Proof of quantified statements

- Existential statements (i.e., ∃x P(x))
  - Finding a single example suffices

- Universal statements (i.e., ∀x P(x)) can be harder to prove
  - $$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$
  - $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Bottom line: We need new tools!

Mathematical induction lets us prove universally quantified statements!

Goal: Prove ∀x∈**N** P(x).

Intuition: If P(0) is true, then P(1) is true. If P(1) is true, then P(2) is true…

Procedure:

1. Prove P(0)
2. Show that P(k) → P(k+1) for any arbitrary k
3. Conclude that P(x) is true ∀x∈**N**

$$\frac{\begin{array}{l} P(0) \\ P(k) \rightarrow P(k+1) \end{array}}{\therefore \forall x \in \mathbf{N}\ P(x)}$$

Analogy: Climbing a ladder

Proving P(0):

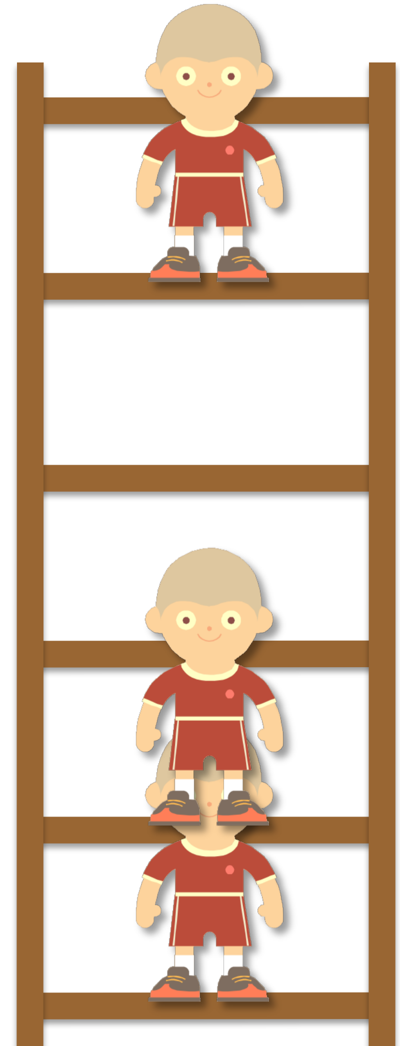
- You can get on the first rung of the ladder

Proving $P(k) \rightarrow P(k+1)$***:***

- If you are on the k^{th} step, you can get to the $(k+1)^{st}$ step

$\therefore \forall x\, P(x)$

- You can get to any step on the ladder



Analogy: Playing with dominoes

Proving P(0):

- The first domino falls

Proving $P(k) \rightarrow P(k+1)$:

- If the k^{th} domino falls, then the $(k+1)^{st}$ domino will fall

$\therefore \forall x\, P(x)$

- All dominoes will fall!

All of your proofs should have the same overall structure

$P(x) \equiv$ *Define the property that you are trying to prove*

Base case: *Prove the "first step onto the ladder." Typically, but not always, this means proving P(0) or P(1).*

Inductive Hypothesis: *Assume that P(k) is true for an arbitrary k*

Inductive step: *Show that $P(k) \rightarrow P(k + 1)$. That is, prove that once you're on one step, you can get to the next step. This is where many proofs will differ from one another.*

Conclusion: *Since you've proven the base case and $P(k) \rightarrow P(k + 1)$, the claim is true!* ❑

Prove that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

P(n) ≡ $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

Base case: P(1): 1(1+1)/2 = 1 ✔

I.H.: Assume that P(k) holds for an arbitrary integer k

Inductive step: We will now show that P(k) → P(k+1)

- 1+2+…+k = k(k+1)/2 by I.H.
- 1+2+…+k+(k+1) = k(k+1)/2 + (k+1) k+1 to both sides
- 1+2+…+k+(k+1) = k(k+1)/2 + 2(k+1)/2
- 1+2+…+k+(k+1) = (k^2 + 3k + 2)/2
- 1+2+…+k+(k+1) = (k+1)(k+2)/2 factoring

Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

Induction cannot give us a formula to prove, but can allow us to verify conjectures

Mathematical induction is not a tool for discovering new theorems, but rather a powerful way to prove them

Example: Make a conjecture about the sum of the first n odd positive numbers, then prove it.

- $1 = 1$
- $1 + 3 = 4$
- $1 + 3 + 5 = 9$
- $1 + 3 + 5 + 7 = 16$
- $1 + 3 + 5 + 7 + 9 = 25$

The sequence 1, 4, 9, 16, 25, … appears to be the sequence $\{n^2\}$

Conjecture: The sum of the first n odd positive integers is n^2

Prove that the sum of the first n positive odd integers is n^2

$P(n) \equiv$ The sum of the first n positive odd numbers is n^2

Base case: $P(1)$: $1 = 1$ ✔

I.H.: Assume that $P(k)$ holds for an arbitrary integer k

Inductive step: We will now show that $P(k) \rightarrow P(k+1)$

- $1+3+\ldots+(2k-1) = k^2$ by I.H.
- $1+3+\ldots+(2k-1)+(2k+1) = k^2+2k+1$ 2k+1 to both sides
- $1+3+\ldots+(2k-1)+(2k+1) = (k+1)^2$ factoring

Note: The k^{th} odd integer is 2k-1, the $(k+1)^{st}$ odd integer is 2k+1

Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

Prove that the sum $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n

P(n) ≡ $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Base case: P(0): $2^0 = 1$ ✔

I.H.: Assume that P(k) holds for an arbitrary integer k

Inductive step: We will now show that P(k) → P(k+1)

by I.H.

2^{k+1} to both sides

associative law

def'n of ×

def'n of exp.

Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

Why does mathematical induction work?

This follows from the well ordering axiom

- i.e., Every set of positive integers has a least element

We can prove that mathematical induction is valid using a proof by contradiction.

- Assume that P(1) holds and $P(k) \rightarrow P(k+1)$, but $\neg\forall x\, P(x)$
- This means that the set $S = \{x \mid \neg P(x)\}$ is nonempty
- By well ordering, S has a least element m with $\neg P(m)$
- Since m is the least element of S, P(m-1) is true
- By $P(k) \rightarrow P(k+1)$, $P(m-1) \rightarrow P(m)$
- Since we have $P(m) \wedge \neg P(m)$ this is a contradiction!

Result: Mathematical induction is a valid proof method

In-class exercises

Problem 1: Prove that $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1}$ if $r \neq 1$

Problem 2: Prove that $\sum_{j=1}^{n} (3j-2) = \frac{n(3n-1)}{2}$

Hint: Be sure to

1. Define P(x)
2. Prove the base case
3. Make an inductive hypothesis
4. Carry out the inductive step
5. Draw the final conclusion

Prove the formula for the sum of the first n positive squares

P(n) $\equiv \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	
Base case: P(1): $1^2 = \frac{1(1+1)(2+1)}{6}$ ✔	
I.H.: Assume that P(k) holds for an arbitrary integer k	
Inductive step: We will now show that P(k) → P(k+1)	
■ $1+4+9+\ldots+k^2 = k(k+1)(2k+1)/6$	by I.H.
■ $1+4+9+\ldots+(k+1)^2 = k(k+1)(2k+1)/6 + (k+1)^2$	$(k+1)^2$ to both sides
■ $= k(k+1)(2k+1)/6 + 6(k+1)^2/6$	common denom.
■ $= (k+1)(2k^2+k+6k+6)/6 = (k+1)(2k^2+7k+6)/6$	factor k+1, mult.
■ $= (k+1)(k+2)(2k+3)/6$	factor
■ $= (k+1)((k+1)+1)(2(k+1)+1)/6$, ∴P(k+1)	proved for k+1
Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑	

Induction can also be used to prove properties other than summations!

$\geq$ $\leq$

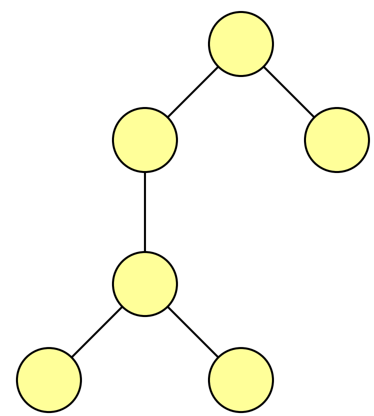
Inequalities

$|$ $\equiv$ $\varphi(p)$

Divisibility and results from number theory

$\subseteq$ $\in$ $\cup$

Set theory



Algorithms and data structures

Prove that $2^n < n!$ for every positive integer $n \geq 4$

Prelude: The expression n! is called the factorial of n.

Definition: $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$

Examples:

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$
- $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

Note how quickly the factorial of n "grows"

Prove that $2^n < n!$ for every positive integer $n \geq 4$

$P(n) \equiv 2^n < n!$

Base case: $P(4)$: $2^4 < 4!$ ✔

I.H.: Assume that $P(k)$ holds for an arbitrary integer k

Inductive step: We will now show that $P(k) \rightarrow P(k+1)$

by I.H.

multiply by 2

def'n of exp.

since $2 < (k+1)$

def'n of factorial

Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

Prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer

$P(n) \equiv 3 \mid (n^3 - n)$

Base case: $P(1): 3 \mid 0$ ✔

I.H.: Assume that $P(k)$ holds for an arbitrary integer k

Inductive step: We will now show that $P(k) \rightarrow P(k+1)$

- $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$
- $= k^3 + 3k^2 + 2k$
- $= (k^3 - k) + (3k^2 + 3k)$
- $= (k^3 - k) + 3(k^2 + k)$
- Note that $3 \mid (k^3 - k)$ by the I.H. and $3 \mid 3(k^2 + k)$ by definition, so $3 \mid [(k+1)^3 - (k+1)]$

Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

In-class exercises

Problem 3: Prove that $n^3 + 2n$ is divisible by 3 for any positive integer n

Problem 4: Prove that $6^n - 1$ is divisible by 5 for any positive integer n

Hint: Be sure to

1. Define P(x)
2. Prove the base case
3. Make an inductive hypothesis
4. Carry out the inductive step
5. Draw the final conclusion

Prove that if S is a finite set with n elements, then S has 2^n subsets.

$P(n) \equiv$ Set S with cardinality n has 2^n subsets
Base case: $P(0)$: $\emptyset$ has $2^0 = 1$ subsets (i.e., $\emptyset \subseteq \emptyset$) ✔
I.H.: Assume that $P(k)$ holds for an arbitrary integer k
Inductive step: We will now show that $P(k) \rightarrow P(k+1)$
Conclusion: Since we have proved the base case and the inductive case, the claim holds by mathematical induction ❑

Final Thoughts

- Mathematical induction lets us prove universally quantified statements using this inference rule:

$$\begin{array}{l} P(0) \\ P(k) \rightarrow P(k+1) \\ \hline \therefore \forall x \in \mathbf{N}\ P(x) \end{array}$$

- Induction is useful for proving:
  - Summations
  - Inequalities
  - Claims about countable sets
  - Theorems from number theory
  - …

- Next time: Strong induction and recursive definitions (Sections 5.2 & 5.3)