Discrete Structures for Computer Science

William Garrison
bill@cs.pitt.edu
6311 Sennott Square

Lecture #4: Predicates and Quantifiers

University of Pittsburgh

Based on materials developed by Dr. Adam Lee

Topics

- Predicates
- Quantifiers
- Logical equivalences in predicate logic
- Translations using quantifiers

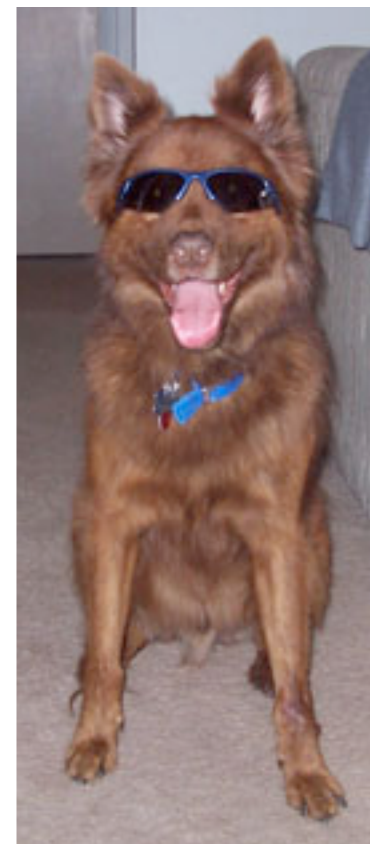
Propositional logic is simple, therefore limited

Propositional logic cannot represent some classes of natural language statements…

Given: All of my dogs like peanut butter

Propositional logic gives us no way to draw the (obvious) conclusion that Kody likes peanut butter!

Given: Kody is one of my dogs



Propositional logic also limits the mathematical truths that we can express and reason about

Consider the following:

- $p_1 \equiv$ 2 has no divisors other than 1 and itself
- $p_2 \equiv$ 3 has no divisors other than 1 and itself
- $p_3 \equiv$ 5 has no divisors other than 1 and itself
- $p_4 \equiv$ 7 has no divisors other than 1 and itself
- $p_5 \equiv$ 11 has no divisors other than 1 and itself
- …

This is an inefficient way to reason about the properties of prime numbers!

General problem: Propositional logic has no way of reasoning about instances of general statements.

Historical Context

The previous examples are called syllogisms

Aristotle used syllogisms in his *Prior Analytics* to deductively infer new facts from existing knowledge

Major premise

All men are mortal

Socrates is a man

Minor premise

$\therefore$ Socrates is mortal

Conclusion

Predicate logic allows us to reason about the properties of individual objects and classes of objects

Predicate logic allows us to use propositional functions during our logical reasoning

$$\overbrace{P(\underbrace{x}_{\text{variable}}) \equiv \underbrace{x^3 > 0}_{\text{predicate}}}$$

Note: A propositional function P(x) has no truth value unless it is evaluated for a given x or set of xs.

Examples

Assume $P(x) \equiv x^3 > 0$. What are the truth values of the following expressions:

- P(0)
- P(23)
- P(-42)

We can express the prime number property using predicate logic:

Predicates can also be defined on more than one variable

Let $P(x, y) \equiv x + y = 42$. What are the truth values of the following expressions:

- $P(45, -3)$
- $P(23, 23)$
- $P(1, 119)$

Let $S(x, y, z) \equiv x + y = z$. What are the truth values of the following expressions:

- $S(1, 1, 2)$
- $S(23, 24, 42)$
- $S(-9, 18, 9)$

Predicates play a central role in program control flow and debugging

If/then statements:

- **if** x > 17 **then** y = 13

Loops:

- **while** y <= 14 **do**

  …

  end while

Debugging in C/C++:

- assert(strlen(passwd) > 0);

This is a predicate!

Quantifiers allow us to make general statements that turn propositional functions into propositions

In English, we use quantifiers on a regular basis:

- All students can ride the bus for free
- Many people like chocolate
- I enjoy some types of tea
- At least one person will sleep through their final exam

Quantifiers require us to define a universe of discourse (also called a domain) in order for the quantification to make sense

- “Many like chocolate” doesn’t make sense!

What are the universes of discourse for the above statements?

Universal quantification allows us to make statements about the entire universe of discourse

Examples:

- All of my dogs like peanut butter
- Every even integer is a multiple of two
- For each positive integer x, 2x > x

Given a propositional function P(x), we express the universal quantification of P(x) as $\forall x\, P(x)$

What is the truth value of $\forall x\, P(x)$?

Examples

All rational numbers are greater than 42

If a natural number is prime, it has no divisors other than 1 and itself

Existential quantifiers allow us to make statements about some objects

Examples:

- Some elephants are scared of mice
- There exist integers a, b, and c such that the equality $a^2 + b^2 = c^2$ is true
- There is at least one person who did better than John on the midterm

Given a propositional function P(x), we express the existential quantification of P(x) as $\exists x\, P(x)$

What is the truth value of $\exists x\, P(x)$?

Examples

The inequality $x + 1 < x$ holds for at least one integer

For some integers, the equality $a^2 + b^2 = c^2$ is true

We can restrict the domain of quantification

The square of every natural number less than 4 is no more than 9

- *Domain:* natural numbers
- *Statement:* $\forall x{<}4\ (x^2 \leq 9)$
- *Truth value:* true

This is equivalent to writing

$\forall x\ [(x < 4) \rightarrow (x^2 \leq 9)]$

Some integers between 0 and 6 are prime

- *Domain:* Integers
- *Propositional function:* $P(x) \equiv$ "x is prime"
- *Statement:* $\exists 0{\leq}x{\leq}6\ P(x)$
- *Truth value:* true

This is equivalent to writing

$\exists x\ [(0 \leq x \leq 6) \wedge P(x)]$

Precedence of quantifiers

The universal and existential quantifiers have the highest precedence of all logical operators

For example:

- $\forall x\ P(x) \wedge Q(x)$ actually means $(\forall x\ P(x)) \wedge Q(x)$
- $\exists x\ P(x) \rightarrow Q(x)$ actually means $(\exists x\ P(x)) \rightarrow Q(x)$

For the most part, we will use parentheses to disambiguate these types of statements

But you are still responsible for understanding precedence!

In-class exercises

Problem 1: Assume $M(x) \equiv$ "x is a Monday" and $D(x,y) \equiv 2x = y$. What are the truth values of the following statements?

- M("May 13, 2019")
- $D(2, 5)$

Problem 2: Let $P(x) \equiv$ "x is prime" where the domain of x is the integers. Let $T(x,y) \equiv$ "$x = 3y$" where the domain of x and y is all natural numbers. What are the truth values of the following statements?

- $\forall x\, P(x)$
- $\exists x,y\; T(x,y)$

We can extend the notion of logical equivalence to expressions containing predicates or quantifiers

Definition: Two statements involving predicates and quantifiers are logically equivalent iff they take on the same truth value *regardless* of which predicates are substituted into these statements and which domains of discourse are used.

Prove: $\exists x\, [P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

Prove: $\exists x\ [P(x) \vee Q(x)] \equiv \exists x\ P(x) \vee \exists x\ Q(x)$

We also have DeMorgan's laws for quantifiers

Negation over universal quantifier: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

Negation over existential quantifier: $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$

These are **very** useful logical equivalences, so let's prove one of them…

Prove: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

Translations from English

To translate English sentences into logical expressions:

1. Rewrite the sentence to make it easier to translate
2. Determine the appropriate quantifiers to use
3. Look for words that indicate logical operators
4. Formalize sentence fragments
5. Put it all together

Example: At least one person in this classroom is named Bill and has lived in Pittsburgh for 8 years

Existential quantifier

Rewrite: There exists at least one person who is in this classroom, is named Bill, and has lived in Pittsburgh for 8 years

Conjunction

Formalize:

- $C(x) \equiv$ "x is in this classroom"
- $N(x) \equiv$ "x is named Bill"
- $P(x) \equiv$ "x has lived in Pittsburgh for 8 years"

Final expression: $\exists x\ [C(x) \wedge N(x) \wedge P(x)]$

Example: If a student is taking CS441, then they have taken high school algebra

Universal quantifier

Rewrite: For all students, if a student is in CS 441, then they have taken high school algebra

Implication

Formalize:

- $C(x) \equiv$ "x is taking CS441"
- $H(x) \equiv$ "x has taken high school algebra"

Final expression: $\forall x\,[C(x) \rightarrow H(x)]$

Negate the previous example

$\neg\forall x\ [C(x) \rightarrow H(x)]$

Translate back into English:

- There is a student taking CS441 that has not taken high school algebra!

Example: Jane enjoys drinking some types of tea

Rewrite: There exist some types of tea that Jane enjoys drinking

Formalize:

- $T(x) \equiv$ "x is a type of tea"
- $D(x) \equiv$ "Jane enjoys drinking x"

Final expression: $\exists x\ [T(x) \wedge D(x)]$

Negate the previous example:

$$\neg\exists x\ [T(x) \wedge D(x)] \equiv$$

$$\equiv$$

$$\equiv$$

In-class exercises

Problem 3: Translate the following sentences into logical expressions.

a) Some cows have black spots
b) At least one student likes to watch football or ice hockey
c) Any adult citizen of the US can register to vote if he or she is not a convicted felon

Problem 4: Negate the translated expressions from problem 3. Translate these back into English.

Final Thoughts

- The simplicity of propositional logic makes it unsuitable for solving certain types of problems

- Predicate logic makes use of
  - Propositional functions to describe properties of objects
  - The universal quantifier to assert properties of all objects within a given domain
  - The existential quantifier to assert properties of some objects within a given domain

- Predicate logic can be used to reason about relationships between objects and classes of objects

- Next lecture:
  - Applications of predicate logic and nested quantifiers
  - Please read section 1.5