Notes for Week 2/28/20 - 3/5/20

Alden Green

March 12, 2020

Let N be an integer, $n = N^d$, and

$$\bar{X} := \left\{ \frac{1}{N}(j_1, \ldots, j_d) : j \in [N]^d \right\}$$

consist of n total evenly spaced grid points on $[0,1]^d$. We observe n samples according to regression model

$$y_j = f(\bar{x}_j) + \varepsilon_j, \ \varepsilon_j \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1) \ \text{ for each } j \in [N]^d, \tag{1}$$

and our goal is to perform the goodness-of-fit testing

$$\mathbb{H}_0 : f = f_0 := 0 \ \text{ vs. } \ \mathbb{H}_a : f \neq f_0.$$

For an arbitrary graph G on n nodes with graph Laplacian $L_G = D_G - A_G$, let $\Lambda(G)$ consist of the eigenvalues $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ of L_G, with $v_k(G) = (v_{k,1}(G), \ldots, v_{k,n}(G)) \in \mathbb{R}^n$ denoting the eigenvector corresponding to the ith eigenvalues $\lambda_k(G)$. The graph Laplacian eigenvector projection test is a linear projection test; letting κ be some integer between 1 and n, we define $\Pi_{k,G}(\theta) := \sum_{k=1}^{\kappa} \langle v_k, \theta \rangle$ to be the projection of θ onto the span of $v_1(G), \ldots, v_k(G)$.

Our test statistic will be

$$T_{\text{spec}}(G) := \|\Pi_{\kappa,G}(Y)\|_n^2 - \frac{\kappa}{n},$$

a simple function of the projection of the data Y onto the span of the first κ eigenvectors of the graph Laplacian $L(G)$; we will suitably choose G based on the assumptions we make on f.

1 Sobolev testing with fixed design.

For now, we confine our attention to the periodic Sobolev space $H^s_{\text{per}}([0,1]^d)$ which consists of all those functions $f \in \mathcal{L}^2([0,1]^d)$ whose tensor product Fourier coefficients decay at a sufficiently fast rate. When $d = 1$ and $x \in [0,1]$, let

$$\begin{aligned} \phi_1(x) &= 1 \\ \phi_{2k}(x) &= \sqrt{2}\cos(2k\pi x), \quad k \in \mathbb{N} \\ \phi_{2k+1}(x) &= \sqrt{2}\sin(2k\pi x), \quad k \in \mathbb{N} \end{aligned}$$

be the trigonometric basis over $[0,1]$. The tensor product Fourier basis is defined on $[0,1]^d$ as

$$\phi_k(x) = \prod_{i=1}^{d} \phi_{k_i}(x_i),$$

the Sobolev space $H^s_{\mathrm{per}}([0,1]^d)$ as

$$H^{\mathrm{per}}([0,1]^d) = \left\{ \sum_{k \in \mathbb{N}^d} \theta_k \phi_k^{(d)} : \sum_{k \in \mathbb{N}^d} \theta_k^2 \, |k|^{2s} < \infty \right\},$$

and the unit ball $H^s_{\mathrm{per}}([0,1]^d, L)$ as

$$H^{\mathrm{per}}([0,1]^d; L) = \left\{ \sum_{k \in \mathbb{N}^d} \theta_k \phi_k^{(d)} : \sum_{k \in \mathbb{N}^d} \theta_k^2 \, |k|^{2s} \leq L \right\},$$

Intuitively, a good choice of graph G for our test statistic $T_{\mathrm{spec}}(G)$ will be one which keeps the approximation error

$$\|f\|_{\mathcal{L}^2}^2 - \mathbb{E}\Big[\|\Pi_{\kappa,G}(Y)\|_n^2\Big] = \|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,G}(f)\|_n^2$$

small when κ is not too large.

1.1 Torus Graph

We will see that the torus graph $\widetilde{G}_d$ fulfills this purpose. The 1-dimensional torus (ring) graph is defined as

$$\widetilde{G}_1 = (\bar{X}; \widetilde{E}), \quad \widetilde{E} = \Big\{ (\bar{x}_i, \bar{x}_j) : i \in [n], j = (i+1) \mod n \Big\};$$

when $d > 1$, we define $\widetilde{G}_d = \widetilde{G}_1 \otimes \cdot \widetilde{G}_1$ as the tensor product of $\widetilde{G}_1$ iterated d times.

Lemma 1. *Let $K := \kappa^{1/d}$. For any $2s > d$ and $f \in H^s_{per}([0,1]^d; L)$,*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,G}(f)\|_n^2 \leq c n^{1/2 - s/d} \kappa^{1/2} \|f\|_2 + \sum_{k \in \mathbb{N}^d - [K]^d} \theta_k^2 \tag{2}$$

for a constant c which depends only on L and d.

It is possible to improve on Lemma 1 under additional assumptions. Specifically, we assume the existence of ε_j satisfying $j\varepsilon_j$ is monotone non-increasing in j, such that for every $f \in H^s_{\mathrm{per}}([0,1]^d)$, $f = \sum_{k \in \mathbb{N}^d} \theta_k \phi_k$,

$$|\theta_k| \leq \prod_{i=1}^{d} \varepsilon_{k_i} \ \text{ for all } k \in \mathbb{N}^d, \quad \sum_{j=1}^{\infty} \varepsilon_j = C, \quad \sum_{j=1}^{\infty} \varepsilon_j^2 \leq C \Bigg(\sum_{k \in \mathbb{N}^d} \theta_k^2 \Bigg)^{1/d} \tag{3}$$

for some constant $C < \infty$.

Lemma 2. *Suppose that (3) holds with respect to $H^s_{per}([0,1]^d)$. Then for every $f \in H^s_{per}([0,1]^d)$,*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,G}(f)\|_n^2 \leq c \|f\|_{\mathcal{L}^2}^2 \left(\frac{K^{1/2}}{N \|f\|_{\mathcal{L}^2}^{1/d}} + \frac{K^{d/2}}{n \|f\|_{\mathcal{L}^2}} \right) + \sum_{k \in \mathbb{N}^d - [K]^d} \theta_k^2 \tag{4}$$

When we choose $\kappa = n^{2d/(4s+d)}$, and additionally $\|f\|_{\mathcal{L}^2} \geq n^{-2s/(4s+d)}$, we note that

$$\frac{K^{1/2}}{N \|f\|_{\mathcal{L}^2}^{1/d}} + \frac{K^{d/2}}{n \|f\|_{\mathcal{L}^2}} = o(1)$$

and therefore the discretization error is asymptotically negligible.

Unfortunately the additional assumption (3) is not mild at all, but instead relatively close to assuming the function is s-Holder. Finally, we state a sharper bound on discretization error than Lemma 1, but that requires less severe assumptions than Lemma 2.

Lemma 3. *For any $2s > d$ and $f \in H^s_{per}([0,1]^d; L)$,*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,G}(f)\|_n^2 \le cn^{1/2-s/d}\|f\|_2 + \sum_{k \in \mathbb{N}^d - [K]^d} \theta_k^2$$

1.2 Grid graph

In this section we will find it convenient to assume the grid points are symmetric in $[0,1]^d$, so we now treat

$$\bar{X} := \left\{ \frac{1}{2N}(2j_1 - 1, \ldots, 2j_d - 1) : j \in [N]^d \right\}.$$

The grid graph $\bar{G}_d = \bar{G} \otimes \cdots \otimes \bar{G}$, where

$$\bar{G} = (\bar{X}; \bar{E}), \quad \bar{E} = \Big\{ (\bar{x}_i, \bar{x}_j) : i \in [n-1], j = (i+1) \Big\};$$

is the path (i.e. the torus with the edge connecting 1 to n removed.) The eigenvectors of the path can be written as follows:

$$v_{k,i}(\bar{G}) = \cos\Big(\frac{\pi(k-1)(i-1/2)}{n}\Big), \text{for } i = 1, \ldots, n \text{ and } k = 1, \ldots, n-1$$

with the usual consequence for the eigenvectors of the grid:

$$v_{k,i}(\bar{G}_d) = \prod_{j=1}^d v_{k_j,i_j}(\bar{G}), \quad \text{for } i \in [N]^d,\ j \in [N]^d.$$

We extend these eigenvectors to an orthonormal system over $\mathcal{L}^2([0,1]^d)$ as follows. Let

$$\varphi_k(x) = \sqrt{2}\cos\Big(\pi(k-1)x\Big), \quad \text{for } k \in \mathbb{N}, x \in [0,1], \quad \text{and} \quad \varphi_k(x) = \prod_{j=1}^d \varphi_{k_j}(x_j), \quad \text{for } k \in \mathbb{N}^d, x \in [0,1]^d,$$

and note that by shifting the location of the design points $\bar{X}$, we have that $\varphi_k(\bar{x}_i) = \sqrt{n} v_{k,i}(\bar{G}_d)$ for all $k, i \in [N]^d$.

We let the space

$$\widetilde{H}^s([0,1]^d; L) = \Big\{ \sum_{k \in \mathbb{N}^d} \theta_k \varphi_k : \sum_{k \in \mathbb{N}^d} \theta_k^2 a_k^2 < L \Big\}$$

where in the one-dimensional case,

$$a_k = (k-1)^s$$

and for general d, $a_k = \sqrt{\sum_{j=1}^d a_{k_j}^2}$. It is not hard to show that, when s is an integer, $\widetilde{H}^s([0,1]^d)$ consists of those functions in $H^s([0,1]^d)$ which satisfy some Neumann-type boundary conditions. For example when $d = 1$,

$$\widetilde{H}^s([0,1]; L) = \Big\{ f \in H^s([0,1]; L\pi^s) : f^{(\ell)}(0) = f^{(\ell)}(1) = 0, \quad \text{for all odd integers } 0 < \ell < s \Big\} \supseteq H_0^s([0,1]; L'),$$

a fact which we prove in Section 3.1.

For functions $f \in \widetilde{H}^s([0,1]^d; L)$, the error in approximating the continuum norm $\|f\|_{\mathcal{L}^2}$ by the discrete norm of a projection of f onto the eigenvectors of the grid is not too large.

Lemma 4. *Suppose $2s < d$ and $f = \sum_{k \in \mathbb{N}^k} \theta_k \varphi_k$ satisfies $f \in \widetilde{H}^s([0,1]^d; L)$. Then*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,\bar{G}_d}(f)\|_n^2 \leq cn^{1/2-s/d}\|f\|_2 + \sum_{k \in \mathbb{N}^d - [K]^d} \theta_k^2$$

So we've done a lot of interesting work, but we only obtained a sufficient bound on the approximation error when $n^{1/2-s/d} = o(\|f\|_2)$, which is still not good enough for our purposes. In the following Lemma, we state a tighter bound on approximation error when $d = 1$.

Lemma 5. *Suppose $2s > 1$. Then there exists a constant c which depends only on s such that for any $f = \sum_{k=1}^{\infty} \theta_k \varphi_k \in \widetilde{H}^s_{per}([0,1]^d; L)$, the following bound holds:*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,\bar{G}_1}(f)\|_n^2 \leq cL^{1/2}\|f\|_{\mathcal{L}^2} n^{-s} + \sum_{k=\kappa+1}^{\infty} \theta_k^2$$

We extend the results of Lemma 5 to hold for all $d \geq 1$ in Lemma 6. Although Lemma 6 strictly dominates Lemma 5, we keep the latter around since its proof is somewhat easier to follow.

Lemma 6. *Suppose $f = \sum_{k \in \mathbb{N}^d} \theta_k \varphi_k$ satisfies $f \in \widetilde{H}^s([0,1]^d; L)$ for some $2s > d$. Then, there exists a constant c which depends only on s and d such that*

$$\|f\|_{\mathcal{L}^2}^2 - \|\Pi_{\kappa,\bar{G}_d}(f)\|_n^2 \leq cL^{1/2}\|f\|_{\mathcal{L}^2} n^{-s/d} + \sum_{k \in \mathbb{N}^d \setminus [K]^d} \theta_k^2 \tag{5}$$

2 Proofs

2.1 Proof of Lemma 1

Recall that the eigenvectors of the torus graph $v_k := v_k(\widetilde{G}_d)$ satisfy

$$v_k(\bar{x}_i) = \frac{1}{\sqrt{n}} \phi_k(\bar{x}_i)$$

for each $i \in [N]^d$ and $k \in \mathbb{N}^d$. Rewriting $\Pi_{\kappa,G}$ as a function of the tensor product Fourier basis, we obtain

$$\begin{aligned}
\Pi_{\kappa,G}(f) &= \frac{1}{n} \sum_{k \in [K]^d} \Big\{ \sum_{i \in [N]^d} f(\bar{x}_i)\phi_k(\bar{x}_i) \Big\} \phi_k \\
&:= \sum_{k \in [K]^d} \widetilde{\theta}_k \phi_k
\end{aligned}$$

where the right hand side is interpreted as a function over $\bar{X}$. Since ϕ_k are $L_2(\bar{X})$ orthonormal, we obtain

$$\begin{aligned}
\|\Pi_{\kappa,G}(f)\|_n^2 &= \sum_{k \in [K]^d} \widetilde{\theta}_k^2 \\
&= \sum_{k \in [K]^d} \theta_k^2 + \sum_{k \in [K]^d} \widetilde{\theta}_k^2 - \theta_k^2 \\
&= \|f\|_{\mathcal{L}^2([0,1]^d)}^2 - \sum_{k \in \mathbb{N}^d - [K]^d} \theta_k^2 + \sum_{k \in [K]^d} \widetilde{\theta}_k^2 - \theta_k^2
\end{aligned}$$

and rearranging, we have

$$\|f\|^2_{\mathcal{L}^2([0,1]^d)} - \|\Pi_{\kappa,G}(f)\|_n^2 = \sum_{k\in[K]^d} \theta_k^2 - \widetilde{\theta}_k^2 + \sum_{k\in\mathbb{N}^d-[K]^d} \theta_k^2$$

The second term in the above equation corresponds to the second term in (2), and so we focus our attention on the first term, which represents discretization error. Applying the parallelogram law and the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\sum_{k\in[K]^d} \theta_k^2 - \widetilde{\theta}_k^2 &\le \sum_{k\in[K]^d} \max\Big\{(\theta_k + \widetilde{\theta}_k)(\theta_k - \widetilde{\theta}_k), 0\Big\} \\
&\le \left(\sum_{k\in[K]^d} \max\Big\{(\theta_k + \widetilde{\theta}_k)^2, 0\Big\}\right)^{1/2} \left(\sum_{k\in[K]^d} (\theta_k - \widetilde{\theta}_k)^2\right)^{1/2} \\
&\le \sqrt{2}\|f\|_{\mathcal{L}^2([0,1]^d)} \left(\sum_{k\in[K]^d} (\theta_k - \widetilde{\theta}_k)^2\right)^{1/2}
\end{aligned} \tag{6}$$

We focus on the differences $\theta_k - \widetilde{\theta}_k$, and using the same steps as (Tsybakov) Lemma 1.8 we obtain

$$\max_{k\in[K]^d} \left|\theta_k - \widetilde{\theta}_k\right| \le 2L^{1/2} \left(\sum_{j\in\mathbb{N}^d-[N]^d} |j|^{-2s}\right)^{1/2} \tag{7}$$

so long as $K^d \le n-1$. Upper bounding the Riemann sum in (7) by a corresponding integral, and converting to spherical coordinates, we have

$$\begin{aligned}
\sum_{j\in\mathbb{N}^d-[N]^d} |j|^{-2s} &\le d \sum_{j\in\mathbb{N}^d: j_1 \ge N+1} |j|^{-2s} \\
&\le d \int_{\mathbb{R}^d \setminus B(0,N+1)} |x|^{-2s}\, dx \\
&\le c \int_{N+1}^{\infty} r^{-2s} r^{d-1}\, dr \\
&\le cN^{d-2s} = cn^{1-2s/d}
\end{aligned}$$

where the last inequality holds since $2s > d$, and c is a constant which depends only on d. By plugging back in to (7) we obtain

$$\max_{k\in[K]^d} \left|\theta_k - \widetilde{\theta}_k\right| \le cn^{1/2-s/d}$$

and the claim then follows from (6).

2.2 Proof of Lemma 2

As shown in the proof of Lemma 1,

$$\|f\|^2_{\mathcal{L}^2} - \|\Pi_{\kappa,G}(f)\|_n^2 \le \sqrt{2}\|f\|_{\mathcal{L}^2([0,1]^d)} \left(\sum_{k\in[K]^d} (\theta_k - \widetilde{\theta}_k)^2\right)^{1/2} + \sum_{k\in\mathbb{N}^d-[K]^d} \theta_k^2 \tag{8}$$

We rewrite the discrete Fourier coefficients $\widetilde{\theta}_k$ as a weighted sum of continuum Fourier coefficients θ_k,

$$\begin{aligned}
\widetilde{\theta}_k - \theta_k &= \frac{1}{n} \sum_{i \in [N]^d} \Big\{ \sum_{l \in \mathbb{N}^d} \theta_l \phi_l(\bar{x}_i) \Big\} \phi_k(\bar{x}_i) - \theta_k \\
&= \sum_{l \in \mathbb{N}^d} \theta_l \Big\{ \frac{1}{n} \sum_{i \in [N]^d} \phi_l(\bar{x}_i) \phi_k(\bar{x}_i) \Big\} - \theta_k \qquad (9)
\end{aligned}$$

Using the tensor decomposition of ϕ_k, we obtain

$$\begin{aligned}
\frac{1}{n} \sum_{i \in [N]^d} \phi_l(\bar{x}_i) \phi_k(\bar{x}_i) &= \frac{1}{n} \sum_{i \in [N]^d} \Big\{ \prod_{j=1}^{d} \phi_{l_j}(\bar{x}_{ij}) \phi_{k_j}(\bar{x}_{ij}) \Big\} \\
&= \frac{1}{n} \prod_{j=1}^{d} \Big\{ \sum_{i_j=1}^{N} \phi_{l_j}(\bar{x}_{ij}) \phi_{k_j}(\bar{x}_{ij}) \Big\} \\
&= \prod_{j=1}^{d} \Big(\delta_{l_j k_j} + \Delta_{l_j k_j}^{(N)} \Big)
\end{aligned}$$

where the last line follows from Lemma 7 (a restatement of results of (Polyak 90)) and $\Delta_{l_j k_j}$ is as in that Lemma. Inserting this into (6) gives

$$\begin{aligned}
\Big| \widetilde{\theta}_k - \theta_k \Big| &= \Big| \sum_{l \in \mathbb{N}^d} \theta_l \prod_{j=1}^{d} \Big(\delta_{l_j k_j} + \Delta_{l_j k_j}^{(N)} \Big) - \theta_k \Big| \\
&\overset{(i)}{=} \Big| \sum_{l \in \mathbb{N}^d \setminus [N]^d} \theta_l \prod_{j=1}^{d} \Big(\delta_{l_j k_j} + \Delta_{l_j k_j}^{(N)} \Big) \Big| \\
&\overset{(ii)}{\leq} \sum_{l \in \mathbb{N}^d \setminus [N]^d} \prod_{j=1}^{d} \varepsilon_{l_j} \Big(\delta_{l_j k_j} + \Big| \Delta_{l_j k_j}^{(N)} \Big| \Big) \\
&\overset{(iii)}{=} \sum_{b \in \{0,1\}^d : |b| \geq 1} \prod_{j : b_j = 0} \varepsilon_{k_j} \prod_{j : b_j = 1} \Big(\sum_{m=N+1}^{\infty} \varepsilon_m \Big| \Delta_{m k_j}^{(N)} \Big| \Big) \\
&\overset{(iv)}{\leq} c \sum_{b \in \{0,1\}^d : |b| \geq 1} N^{-|b|} \prod_{j : b_j = 0} \varepsilon_{k_j}
\end{aligned}$$

where (i) and (iv) follow from Lemma 7, (ii) follows from the condition (3), and (iii) follows since for each index l_j at most one of $\delta_{l_j k_j}$ or $\Delta_{l_j k_j}^{(N)}$ can be non-zero, and $\delta_{l_j k_j}$ must equal 0 for at least one $j = 1, \ldots, d$.

We can now upper bound the sum of squared differences between discrete Fourier coefficients and Fourier coefficients,

$$\begin{aligned}
\sum_{k\in[K]^d}(\widetilde{\theta}_k-\theta_k)^2 &\le c\sum_{k\in[K]^d}\Big(\sum_{b\in\{0,1\}^d:|b|\ge 1} N^{-|b|}\prod_{j:b_j=0}\varepsilon_{k_j}\Big)\\
&\le c\sum_{k\in[K]^d}\sum_{b\in\{0,1\}^d:|b|\ge 1} N^{-2|b|}\prod_{j:b_j=0}\varepsilon_{k_j}^2\\
&\le c\sum_{b\in\{0,1\}^d:|b|\ge 1} N^{-2|b|}\sum_{k\in[K]^d}\prod_{j:b_j=0}\varepsilon_{k_j}^2\\
&\overset{(v)}{\le} c\sum_{b\in\{0,1\}^d:|b|\ge 1} N^{-2|b|}\|f\|_{\mathcal{L}^2}^{2(d-|b|)/d}K^{|b|}\\
&\overset{(vi)}{\le} c\|f\|_{\mathcal{L}^2}^2\Big(\frac{K}{N^2\|f\|_{\mathcal{L}^2}^{2/d}}+\frac{K^d}{N^{2d}\|f\|_{\mathcal{L}^2}^2}\Big)
\end{aligned}$$

where (v) follows from condition (3), (vi) since the summand achieves its maximum either when $b=1$ or $b=d$, and in the above expression $c\ge 0$ is a constant which changes from line to line. Plugging back into (6) implies Lemma 2.

2.3 Proof of Lemma 3

As shown in the proof of Lemma 1,

$$\|f\|_{\mathcal{L}^2}^2-\|\Pi_{\kappa,G}(f)\|_n^2\le\sqrt{2}\|f\|_{\mathcal{L}^2([0,1]^d)}\left(\sum_{k\in[K]^d}(\theta_k-\widetilde{\theta}_k)^2\right)^{1/2}+\sum_{k\in\mathbb{N}^d-[K]^d}\theta_k^2 \tag{10}$$

By Lemma 7, we have

$$\begin{aligned}
\left|\widetilde{\theta}_k-\theta_k\right| &= \left|\sum_{l\in\mathbb{N}^d}\theta_l\prod_{j=1}^d\Big(\delta_{l_jk_j}+\Delta_{l_jk_j}^{(N)}\Big)-\theta_k\right|\\
&=\left|\sum_{l\in\mathbb{N}^d\setminus[N]^d}\theta_l\prod_{j=1}^d\Big(\delta_{l_jk_j}+\Delta_{l_jk_j}^{(N)}\Big)\right|\\
&=\sum_{b\in\{0,1\}^d:|b|<N}\ \sum_{l\in I_b(k)}|\theta_l|\prod_{j:b_j=1}^d\left|\Delta_{l_jk_j}^{(N)}\right|
\end{aligned}$$

and therefore, upper bounding l_2 norm by l_1 norm and exchanging sums, we have

$$\begin{aligned}
\Big(\sum_{k\in[K]^d}(\widetilde{\theta}_k-\theta_k)^2\Big)^{1/2} &\le\sum_{k\in[K]^d}\left|\widetilde{\theta}_k-\theta_k\right|\\
&\le\sum_{b\in\{0,1\}^d:|b|<N}\ \sum_{k\in[K]^d}\ \sum_{l\in I_b(k)}|\theta_l|\prod_{j:b_j=1}^d\left|\Delta_{l_jk_j}^{(N)}\right|.
\end{aligned}$$

In Lemma 10, we establish that the following upper bound holds for each $b\in\{0,1\}^d$ such that $|b|<N$:

$$\sum_{k\in[K]^d}\ \sum_{l\in I_b(k)}|\theta_l|\prod_{j:b_j=1}^d\left|\Delta_{l_jk_j}^{(N)}\right|\le 2^d\sum_{l\in\mathbb{N}^d-[N]^d}|\theta_l|$$

and therefore,

$$\Big(\sum_{k\in[K]^d}(\widetilde{\theta}_k-\theta_k)^2\Big)^{1/2} \leq 4^d \sum_{l\in\mathbb{N}^d-[N]^d}|\theta_l|$$
$$\leq 4^d\Big(\sum_{l\in\mathbb{N}^d-[N]^d}|\theta_l|^2\,|l|^{2s}\Big)^{1/2}\Big(\sum_{l\in\mathbb{N}^d-[N]^d}|l|^{-2s}\Big)^{1/2}$$
$$\leq 4^d L^{1/2}\Big(\sum_{l\in\mathbb{N}^d-[N]^d}|l|^{-2s}\Big)^{1/2}.$$

Note that this upper bound (7), which upper bounds the maximum over $k\in[K]^d$ rather than the sum over all $k\in[K]^d$ by the same term, and therefore incurs an additional loss of $K^{d/2}$. Continuing as in the proof of Lemma 2 yields the claimed result.

2.4 Proof of Lemma 4

This proof follows the along the same lines as the proof of Lemma 3, replacing functions ϕ in the Fourier basis by functions φ in the modified Fourier basis when necessary.

2.5 Proof of Lemma 5

Recall that the eigenvectors of the grid graph $v_k := v_k(\bar{G})$ satisfy

$$v_k(\bar{x}_i) = \frac{1}{\sqrt{n}}\varphi_k(\bar{x}_i)$$

for each $i\in[n]$ and $k\in\mathbb{N}$. Therefore we may rewrite $\Pi_{\kappa,\bar{G}}$ as a function of the cosine Fourier basis,

$$\Pi_{\kappa,\bar{G}}(f) = \frac{1}{n}\sum_{k=1}^{\kappa}\Big\{\sum_{i\in[n]}f(\bar{x}_i)\varphi_k(\bar{x}_i)\Big\}\varphi_k$$
$$:= \sum_{k=1}^{\kappa}\widetilde{\theta}_k\varphi_k$$

where we interpret the right hand side as a function over $\bar{X}$. Since (φ_k) is an $L_2(\bar{X})$ orthonormal sequence, we obtain

$$\|\Pi_{\kappa,\bar{G}}(f)\|_n^2 = \sum_{k=1}^{\kappa}\widetilde{\theta}_k^2$$
$$= \sum_{k=1}^{\kappa}\theta_k^2 + \sum_{k=1}^{\kappa}\widetilde{\theta}_k^2 - \theta_k^2$$
$$= \|f\|^2_{\mathcal{L}^2([0,1]^d)} - \sum_{k=\kappa+1}^{\infty}\theta_k^2 + \sum_{k=1}^{\kappa}\widetilde{\theta}_k^2 - \theta_k^2$$

or equivalently

$$\|f\|^2_{\mathcal{L}^2([0,1]^d)} - \|\Pi_{\kappa,\bar{G}}(f)\|_n^2 = \sum_{k=1}^{\kappa}\theta_k^2 - \widetilde{\theta}_k^2 + \sum_{k=\kappa+1}^{\infty}\theta_k^2$$

The second term in the above equation is exactly the second term in (2), and so we focus our attention on the first term, which represents discretization error. Applying the parallelogram law and the Cauchy-Schwarz inequality gives

$$\begin{aligned}\sum_{k=1}^{\kappa} \theta_k^2 - \widetilde{\theta}_k^2 &\le \sum_{k=1}^{\kappa} \max\Big\{(\theta_k + \widetilde{\theta}_k)(\theta_k - \widetilde{\theta}_k), 0\Big\} \\ &\le \left(\sum_{k=1}^{\kappa} \max\Big\{(\theta_k + \widetilde{\theta}_k)^2, 0\Big\}\right)^{1/2} \left(\sum_{k=1}^{\kappa} (\theta_k - \widetilde{\theta}_k)^2\right)^{1/2} \\ &\le \sqrt{2}\|f\|_{\mathcal{L}^2([0,1])} \left(\sum_{k=1}^{\kappa} (\theta_k - \widetilde{\theta}_k)^2\right)^{1/2}\end{aligned} \tag{11}$$

It is known that the error $\theta_k - \widetilde{\theta}_k$ is due to the aliasing phenomenon, where high frequency sinusoidal waves agree with low frequency waves at evenly spaced points. This phenomenon has a very precise periodic nature – expressed in Lemma 8–and we use this Lemma to get the upper bound

$$\begin{aligned}\left|\theta_k - \widetilde{\theta}_k\right| &= \left|\sum_{\ell=1}^{\infty} \theta_\ell \frac{1}{n} \sum_{i=1}^{n} \varphi_\ell(\bar{x}_i)\varphi_k(\bar{x}_i)\right| \\ &= \sum_{m=1}^{\infty} |\theta_{2mn+k}| + \sum_{m=0}^{\infty} \left|\theta_{2(m+1)n-(k-1)}\right|.\end{aligned}$$

Since θ belongs to the Sobolev ellipsoid $\Theta(s, L)$, by applying Cauchy-Schwarz we can convert the above two tail sums to bounds which explicitly depend on n. Applying this logic to the first tail sum gives us

$$\begin{aligned}\sum_{m=1}^{\infty} |\theta_{2mn+k}| &\le \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\theta_{2mn+k}\big)^2\right)^{1/2} \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\big)^{-2}\right)^{1/2} \\ &\le \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\theta_{2mn+k}\big)^2\right)^{1/2} \left(\sum_{m=1}^{\infty} (2mn)^{-2s}\right)^{1/2} \\ &\le \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\theta_{2mn+k}\big)^2\right)^{1/2} \left(\left(1 + \int_1^{\infty} x^{-2s}\, dx\right)\right)^{1/2} (2n)^{-s} \\ &= \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\theta_{2mn+k}\big)^2\right)^{1/2} \left(\left(1 + \frac{1}{2s-1}\right)\right)^{1/2} (2n)^{-s}\end{aligned} \tag{12}$$

where we have used the fact that the Riemann sum of a monotone non-increasing function evaluated at right end points is always no greater than the corresponding integral, and the last equality holds since $2s > 1$. By using essentially equivalent reasoning, we obtain a comparable bound on the second tail sum

$$\sum_{m=0}^{\infty} \left|\theta_{2(m+1)n-(k-1)}\right| \le \left(\sum_{m=1}^{\infty} \big(a_{2(m+1)n-(k-1)}\theta_{2(m+1)n-(k-1)}\big)^2\right)^{1/2} \left(\left(2 + \frac{1}{2s-1}\right)\right)^{1/2} (2n)^{-s}$$

Thus,

$$\begin{aligned}\sum_{k=1}^{\kappa} \big(\theta_k - \widetilde{\theta}_k\big)^2 &\le c(s)n^{-2s} \sum_{k=1}^{\kappa} \left(\sum_{m=1}^{\infty} \big(a_{2mn+k}\theta_{2mn+k}\big)^2 + \sum_{m=0}^{\infty} \big(a_{2(m+1)n-(k-1)}\theta_{2(m+1)n-k+1}\big)^2\right) \\ &\le c(s)n^{-2s} \sum_{j=n+1}^{\infty} (\theta_j^2 a_j^2) \\ &\le Lc(s)n^{-2s}.\end{aligned} \tag{13}$$

where the second inequality in the preceding display follows since for any $(m,k) \neq (m',k') \in \mathbb{N} \times [n]$,

$$2mn + k \neq 2m'n + k', \quad 2(m+1)n - k + 1 \neq 2(m'+1)n - k' + 1, \quad \text{and} \quad 2mn + k \neq 2(m'+1)n - k' + 1$$

We plug (13) back in to (11) to obtain the claim of the Lemma.

2.6 Proof of Lemma 6

Recall that the eigenvectors of the grid graph $v_k := v_k(\bar{G}_d)$ satisfy

$$v_k(\bar{x}_i) = \frac{1}{\sqrt{n}} \varphi_k(\bar{x}_i)$$

for each $i \in [N]^d$ and $k \in \mathbb{N}^d$. Therefore we may rewrite $\Pi_{\kappa,\bar{G}_d}$ as a function of the tensor product cosine Fourier basis,

$$\begin{aligned}
\Pi_{\kappa,\bar{G}_d}(f) &= \frac{1}{n} \sum_{k \in [K]^d} \Big\{ \sum_{i \in [N]^d} f(\bar{x}_i)\varphi_k(\bar{x}_i) \Big\} \varphi_k \\
&:= \sum_{k \in [K]^d} \widetilde{\theta}_k \varphi_k
\end{aligned}$$

where we interpret the right hand side as a function over $\bar{X}$. Since (φ_k) is an $L_2(\bar{X})$ orthonormal sequence, we obtain

$$\begin{aligned}
\|\Pi_{\kappa,\bar{G}_d}(f)\|_n^2 &= \sum_{k \in [K]^d} \widetilde{\theta}_k^2 \\
&= \sum_{k \in [K]^d} \theta_k^2 + \sum_{k \in [K]^d} \widetilde{\theta}_k^2 - \theta_k^2 \\
&= \|f\|_{\mathcal{L}^2([0,1]^d)}^2 - \sum_{k \in \mathbb{N}^d \setminus [K]^d} \theta_k^2 + \sum_{k \in [K]^d} \widetilde{\theta}_k^2 - \theta_k^2
\end{aligned}$$

or equivalently

$$\|f\|_{\mathcal{L}^2([0,1]^d)}^2 - \|\Pi_{\kappa,\bar{G}_d}(f)\|_n^2 = \sum_{k \in [K]^d} \theta_k^2 - \widetilde{\theta}_k^2 + \sum_{k \in \mathbb{N}^d \setminus [K]^d} \theta_k^2$$

The second term in the above equation is exactly the second term in (5), and so we focus our attention on the first term, which represents discretization error. Applying the parallelogram law and the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\sum_{k=1}^{\kappa} \theta_k^2 - \widetilde{\theta}_k^2 &\leq \sum_{k \in [K]^d} \max\Big\{ (\theta_k + \widetilde{\theta}_k)(\theta_k - \widetilde{\theta}_k), 0 \Big\} \\
&\leq \left(\sum_{k \in [K]^d} \max\Big\{ (\theta_k + \widetilde{\theta}_k)^2, 0 \Big\} \right)^{1/2} \left(\sum_{k \in [K]^d} (\theta_k - \widetilde{\theta}_k)^2 \right)^{1/2} \\
&\leq \sqrt{2} \|f\|_{\mathcal{L}^2([0,1])} \left(\sum_{k \in [K]^d} (\theta_k - \widetilde{\theta}_k)^2 \right)^{1/2}
\end{aligned} \tag{14}$$

It is known that the error $\theta_k - \widetilde{\theta}_k$ is due to the aliasing phenomenon, where high frequency sinusoidal waves agree with low frequency waves at evenly spaced points. This phenomenon has a very precise periodic nature, as expressed in Lemma 8, which allows us to rewrite $\widetilde{\theta}_k$. Note that for every $\ell \in \mathbb{N}^d$, there exists exactly one $m \in \mathbb{N}_0^d$, $o \in [N]^d$ and $b \in \{0,1\}^d$ such that $\ell = 2mN + (1-b)o + b(-(o-1))$ (where for $a, b \in \mathbb{R}^d$ we denote $ab := (a_1 b_1, \ldots, a_d b_d)$). Adopting the convention $\theta_\ell := 0$ if ℓ contains a negative index, we have

$$\begin{aligned}
\widetilde{\theta}_k &= \frac{1}{n} \sum_{i \in [N]^d} f(\bar{x}_i) \varphi_k(\bar{x}_i) \\
&= \frac{1}{n} \sum_{i \in [N]^d} \Big(\sum_{\ell \in \mathbb{N}^d} \theta_\ell \varphi_\ell(\bar{x}_i) \Big) \varphi_k(\bar{x}_i) \\
&= \sum_{\ell \in \mathbb{N}^d} \theta_\ell \Big(\frac{1}{n} \sum_{i \in [N]^d} \varphi_\ell(\bar{x}_i) \varphi_k(\bar{x}_i) \Big) \\
&= \sum_{m \in \mathbb{N}_0^d} \sum_{o \in [N]^d} \sum_{b \in \{0,1\}^d} \theta_{I_N(m,b,o)} \Big(\frac{1}{n} \sum_{i \in [N]^d} \varphi_{I_N(m,b,o)}(\bar{x}_i) \varphi_k(\bar{x}_i) \Big) \\
&= \sum_{m \in \mathbb{N}_0^d} \sum_{b \in \{0,1\}^d} \theta_{I_N(m,b,k)}
\end{aligned}$$

where the last equality follows from Lemma 8. By applying the Cauchy-Schwarz inequality, and noting that $I_N(m,b,k) \geq mN$ (where inequality between two vectors is interpreted entrywise) we get

$$\begin{aligned}
\Big| \widetilde{\theta}_k - \theta_k \Big| &\leq \sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{b \in \{0,1\}^d} \big| \theta_{I_N(m,b,k)} \big| \\
&\leq \Big(\sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{b \in \{0,1\}^d} \Big(\theta_{I_N(m,b,k)} a_{I_N(m,b,k)} \Big)^2 \Big)^{1/2} \Big(\sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{b \in \{0,1\}^d} \Big(a_{I_N(m,b,k)} \Big)^{-2} \Big)^{1/2} \\
&\leq 2^{d/2} N^{-s} \Big(\sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{b \in \{0,1\}^d} \Big(\theta_{I_N(m,b,k)} a_{I_N(m,b,k)} \Big)^2 \Big)^{1/2} \Big(\sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{j=1}^{d} m_j^{-2s} \Big)^{1/2}
\end{aligned}$$

Letting $c(d,s)$ be a finite constant which may change from line to line but can depend only on d and s, and letting $\mathbb{N}_2 = \mathbb{N} \setminus \{1\}$, we can upper bound the second sum in the previous display,

$$\begin{aligned}
\sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \sum_{j=1}^{d} m_j^{-2s} &\overset{(i)}{\leq} d^{2(s+1)} \sum_{m \in \mathbb{N}_0^d \setminus \{0\}} \Big(\sum_{j=1}^{d} m_j^{-2} \Big)^s \\
&\overset{(ii)}{\leq} d^{2(s+1)} \sum_{\widetilde{d}=1}^{d} 2^{\widetilde{d}} \frac{d!}{\widetilde{d}!} \sum_{m \in \mathbb{N}_2^{\widetilde{d}}} \Big(\sum_{j=1}^{d} m_j^{-2} \Big)^s \\
&\overset{(iii)}{\leq} d^{2(s+1)} \sum_{\widetilde{d}=1}^{d} 2^{\widetilde{d}} \frac{d!}{\widetilde{d}!} \int_{[1,\infty)^{\widetilde{d}}} \Big(\sum_{j=1}^{\widetilde{d}} x_j^{-2} \Big)^s dx_1 \ldots dx_{\widetilde{d}} \\
&\leq \sum_{\widetilde{d}=1}^{d} c(d,s) \int_{\sqrt{\widetilde{d}}}^{\infty} r^{-2s+d-1}\, dr \\
&\overset{(iv)}{\leq} c(d,s) < \infty.
\end{aligned}$$

where (i) is an equality when $d = 1$, and otherwise follows from Jensen's inequality when $d \geq 2$ and $s \geq 1$; (ii) follows from specially considering the edge cases where $m_j = 0$ or $m_j = 1$ for $j = \widetilde{d}+1, \ldots, d$; (iii) uses the fact that the Riemann sum of a monotone non-increasing function evaluated at right end points is always no greater than the corresponding integral; and (iv) follows since $2s > d$. We are now in a position to conclude that

$$\begin{aligned}\sum_{k\in[K]^d} (\theta_k - \widetilde{\theta}_k)^2 &\le c(d,s)N^{-2s} \sum_{k\in[K]^d} \sum_{m\in\mathbb{N}_0^d\setminus\{0\}} \sum_{b\in\{0,1\}^d} \Big(\theta_{I_N(m,b,k)} a_{I_N(m,b,k)}\Big)^2 \\ &\le c(d,s)N^{-2s} \sum_{k\in\mathbb{N}^d\setminus[N]^d} \theta_k^2 a_k^2 \\ &\le Lc(d,s)N^{-2s} \qquad (15)\end{aligned}$$

where the second inequality in the preceding display follows since for any $(m,k,b) \neq (m',k',b')$, $I_N(m,b,k) \neq I_N(m',b',k')$. We plug (15) back in to (14) to obtain the claim of the Lemma.

3 Supporting Theory

Lemma 7 (Restatement and Extension of Lemma 1 of (Polyak 90))**.**

$$\frac{1}{n}\sum_{i=1}^n \phi_k(\bar{x}_i)\phi_l(\bar{x}_i) = \delta_{kl} + \Delta_{kl}^{(n)}$$

where $\Delta_{kl}^{(n)} = 0$ if $k,l \le n$ or k and l are of different parities, and otherwise

$$|\Delta_{kl}| \le \sum_{m=1}^{\infty} \delta_{|[k/2]-[l/2]|=mn} + \delta_{|[k/2]+[l/2]|=mn}$$

The same statement holds true when replacing ϕ_k by φ_k and ϕ_l by φ_l, and using symmetric grid points $\bar{X}_i$.

The following Lemma is similar to (7), but tweaked to be more useful for our setting. It applies to the elements in the cosine basis of period 2, rather than elements in the Fourier basis of period 1, and is stated slightly differently.

Lemma 8. *Let $k \in [n]$ and let $\ell \in \mathbb{N}$. There exists a unique $m \in \mathbb{N}_0$, $j \in [n]$ and $b \in \{0,1\}$ such that $\ell = 2mn + (1-b)j - b(j-1)$. Then*

$$\frac{1}{n}\sum_{i=1}^n \varphi_k(\bar{x}_i)\varphi_\ell(\bar{x}_i) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise.}\end{cases}$$

Proof. Note that for $k = 1,\dots,n$, and $i = 1,\dots,n$,

$$v_{k,i} = \frac{1}{\sqrt{n}}\varphi_k(\bar{x}_i).$$

Additionally, it is easy to check that

$$\varphi_\ell(\bar{x}_i) \propto \begin{cases} \varphi_j(\bar{x}_i), & \text{if } j \in [n] \\ \varphi_{-(j-1)}(\bar{x}_i), & \text{if } j \in [-(n-1)] \\ \varphi_1(\bar{x}_i), & \text{if } j = 0.\end{cases}$$

The claim follows since (v_k) is orthonormal in empirical norm. $\square$

Lemma 9. *If $(\theta_k) \in \ell^2(\mathbb{N})$ satisfies (3), then additionally*

$$\max_{1 \le k \le n} \left(\sum_{l=n+1}^{\infty} \theta_l \Delta_{kl} \right)^2 \le \frac{C'}{n^2}$$

for some C' which depends only on C.

For $b \in \{0,1\}^d$, let $I_b(k) = \left\{ l \in \mathbb{N}^d : l_i b_i = k_i b_i \text{ for each } i = 1, \dots, d \right\}$.

Lemma 10. *For any $\theta : \mathbb{N}^d \to \mathbb{R}^+$, any $K \le N$, and any $b \in \{0,1\}^d$ such that $|b| < N$,*

$$\left| \sum_{k \in [K]^d} \sum_{l \in I_b(k)} \theta_l \prod_{j : b_j = 0} \Delta^{(N)}_{l_j k_j} \right| \le 2^d \sum_{\mathbb{N}^d - [N]^d} |\theta_l| \tag{16}$$

Proof. Without loss of generality, suppose $b_1 = \dots = b_{|b|} = 1$, and $b_{|b|+1} = \dots = b_d = 0$. We can upper bound the left hand side of (16) as

$$\left| \sum_{k \in [K]^d} \sum_{l \in I_b(k)} \theta_l \prod_{j : b_j = 0} \Delta^{(N)}_{l_j k_j} \right| \le \sum_{k_1, \dots, k_{|b|}=1}^{K} \sum_{k_{|b|+1}, \dots, k_d = 1}^{K} \sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \prod_{j=|b|+1}^{d} \left| \Delta^{(N)}_{l_j k_j} \right|$$

and it is therefore sufficient to show that

$$\sum_{k_{|b|+1}, \dots, k_d = 1}^{K} \sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \prod_{j=|b|+1}^{d} \left| \Delta^{(N)}_{l_j k_j} \right| \le \sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \tag{17}$$

Using the upper bound on Δ_{kl} given by Lemma 10, for a given $k_{|b|+1}, \dots, k_d$ we have

$$\begin{aligned}
&\sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \prod_{j=|b|+1}^{d} \left| \Delta^{(N)}_{l_j k_j} \right| \\
&\qquad \le \sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \sum_{m_{|b|+1}, \dots, m_d = 1}^{\infty} \prod_{j=|b|+1}^{d} \delta_{|[k_j/2] - [l_j/2]| = m_j N} + \delta_{|[k_j/2] + [l_j/2]| = m_j N} \\
&\qquad = \sum_{m_{|b|+1}, \dots, m_d = 1}^{\infty} \sum_{l_{|b|+1}, \dots, l_d = N+1}^{\infty} \left| \theta_{k_1 \dots k_{|b|} l_{|b|+1} \dots l_d} \right| \prod_{j=|b|+1}^{d} \delta_{|[k_j/2] - [l_j/2]| = m_j N} + \delta_{|[k_j/2] + [l_j/2]| = m_j N} \\
&\qquad = \sum_{m_{|b|+1}, \dots, m_d = 1}^{\infty} \sum_{l \in L(m,k)} |\theta_l| \,.
\end{aligned} \tag{18}$$

In the above lines, the sums are always over l_j such that l_j and k_j have the same parity, since otherwise $\Delta^{(N)}_{l_j k_j} = 0$ by Lemma 7, and $L(m,k) := \cap_{j=1}^{d} S_j(m,k) \cap \cap_{j=1}^{d} R_j(l)$ for

$$S_j(m,k) = \begin{cases} \left\{ l_j \in \mathbb{N} : l_j = k_j \right\}, & \text{for } j = 1, \dots, |b| \\ \left\{ l_j \in \mathbb{N} \setminus [N] : [l_j/2] - [k_j/2] = m_j N \right\} \cup \left\{ l_j \in \mathbb{N} \setminus N : [l_j/2] + [k_j/2] = m_j N \right\}, & \text{for } j = |b|+1, \dots, d. \end{cases}$$

$$R_j(k) = \left\{ l_j \in \mathbb{N} : l_j \mod 2 = k_j \mod 2 \right\}$$

The key point is that for any distinct $k_j, k'_j \in [K]$ the sets $S_j(m,k)$ and $S_j(m,k')$ are disjoint unless (a) $[k_j/2] = [k'_j/2]$, in which case $R_j(k)$ and $R_j(k')$ are disjoint since k_j and k'_j are of different parity, or (b)

$[k_j/2] = N - [k_j/2]$ and k_j, k_j' are of the same parity. Therefore, for any $k \in [K]^d$ and $m \in \mathbb{N}^d$, there are at most $2^{d-|b|} < 2^d$ choices of k' such that $L(m,k)$ and $L(m,k')$ are not disjoint. Moreover, for any $m \in \mathbb{N}^d$ the inclusion $l_j \in S_j(m,k)$ implies that $|l_j| \geq N+1$. As a result,

$$\bigcup_{k_{|b|+1},\ldots,k_d=1}^{N} \left\{ \bigcup_{m \in \mathbb{N}^d} L(m,k) \right\} \subseteq \bigcup_{l_{|b|+1},\ldots,l_d=N}^{\infty} (k_1,\ldots,k_{|b|},l_{|b|+1},\ldots,l_d)$$

Plugging (18) back into the left hand side of (17), we have

$$\begin{aligned}\sum_{k_{|b|+1},\ldots,k_d=1}^{K} \sum_{l_{|b|+1},\ldots,l_d=N+1}^{\infty} \left|\theta_{k_1\ldots k_{|b|}l_{|b|+1}\ldots l_d}\right| \prod_{j=|b|+1}^{d} \left|\Delta_{l_j k_j}^{(N)}\right| &\leq \sum_{k_{|b|+1},\ldots,k_d=1}^{K} \sum_{m_{|b|+1},\ldots,m_d=1}^{\infty} \sum_{l \in L(m,k)} |\theta_l| \\ &\leq 2^d \sum_{l_{|b|+1},\ldots,l_d=N+1}^{\infty} \left|\theta_{k_1\ldots k_{|b|}l_{|b|+1}\ldots l_d}\right|\end{aligned}$$

and the proof (17)–and therefore the proof the Lemma–is complete. ☐

3.1 Equivalence between Sobolev classes

We will show that when $s \geq 1$,

$$\widetilde{H}^s([0,1];L) = \left\{ f \in H^s([0,1];L') : f^{(\ell)}(0) = f^{(\ell)}(1) = 0, \quad \text{for each } \ell < s \text{ odd.} \right\}$$

First, we show the $\subseteq$ direction. Let $f = \sum_{k=1}^{\infty} \theta_k \varphi_k$ satisfy $\sum_{k=1}^{\infty} \theta_k^2 a_k^2 \leq L^2$. Put

$$f_\diamond := \sqrt{2} \sum_{k=1}^{\infty} \theta_k \phi_k$$

where the functions $\phi_k = \cos(\pi[k-1]x)$ are orthonormal in $\mathcal{L}^2([-1,1])$. Since $\sum_{k=1}^{\infty} (\sqrt{2}\theta_k)^2 a_k^2 \leq 2L^2$, we know $f_\diamond \in H^s_{\mathrm{per}}([-1,1];2L)$. Moreover since $f_\diamond$ is an even function, we have that

$$\sum_{\ell=0}^{s} \int_0^1 \left(f_\diamond^{(\ell)}(x)\right)^2 dx = \frac{1}{2} \sum_{\ell=0}^{s} \int_{-1}^1 \left(f_\diamond^{(\ell)}(x)\right)^2 dx \leq \frac{1}{\pi^{2s}} L^2$$

where the latter inequality follows from Tsybakov. As $f(x) = f_\diamond(x)$ for all $x \in [0,1]$, we have established that $f \in H^s([0,1], L')$ for $L' = L/\pi^s$. As for the boundary cases $x = 0$ and $x = 1$, putting

$$f_N := \sum_{k=1}^{N} \theta_k \varphi_k$$

we observe the following facts: first that $f_N^{(\ell)}(0) = f_N^{(\ell)}(1) = 0$ for all $N \in \mathbb{N}$ and ℓ odd, and second that $f_N^{(\ell)}$ converges uniformly for all $0 \leq \ell < s$. Therefore we have $f^{(\ell)}(x) = \lim_{N\to\infty} f_N^{(\ell)}(x)$ for all $x \in [0,1]$ and $0 \leq \ell < s$, and in particular $f^{(\ell)}(0) = f^{(\ell)}(1) = 0$ for all $0 < \ell < s$ odd.

Now for the $\supseteq$ direction. Let $f \in H^s([0,1];L)$ satisfy $f^{(\ell)}(0)$ for all $\ell = 1,\ldots,s-1$ odd. Put

$$f_\diamond = f(|x|) \;\; \text{for } x \in [-1,1]$$

and note the following facts: first that $f_\diamond^{(\ell)}(0)$ is well-defined for each $\ell = 0,\ldots,s-1$, and second that $f_\diamond^{(s-1)}$ is absolutely continuous on $[-1,1]$. As a result, $f_\diamond$ is s-times weakly differentiable (and $s-1$ times classically differentiable), and by construction satisfies

$$\|f_\diamond\|_{H^s([-1,1])} = \sqrt{2}\|f\|_{H^s([0,1])} \leq 2L$$

Since clearly $f_\diamond^{(\ell)}(-1) = f_\diamond^{(\ell)}(1)$ for each $\ell = 0, \ldots, s-1$, we have that $f_\diamond \in H^s_{\mathrm{per}}([-1,1]; \sqrt{2}L)$. Since $f_\diamond$ is even, it may be expressed as $f_\diamond = \sum_{k=1}^\infty \theta_k \phi_k$ where $\sum_{k=1}^\infty \theta_k^2 a_k^2 \le 2L^2/\pi^{2s}$. Since $f(x) = f_\diamond(x)$ for $x \in [0,1]$, this implies

$$f(x) = \sum_{k=1}^\infty \theta_k \phi_k(x) = \sum_{k=1}^\infty \frac{1}{\sqrt{2}} \theta_k \varphi_k(x)$$

and the claim is proved.