Notes for Week 8/12/2020 - 8/19/2020

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1 Fixed graph regression

Suppose we observe a graph $G = \bigl([n], W\bigr)$, and responses

$$Y_i = \theta_{0,i} + \varepsilon_i \tag{1}$$

with signal vector $\theta_0 = (\theta_{0,1}, \dots, \theta_{0,n}) \in \mathbb{R}^n$, and noise vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \sim N(0, I_{n \times n})$. Letting L denote the Laplacian matrix of graph G, the *Laplacian smoothing* estimator $\widetilde{\theta}(G) \in \mathbb{R}^n$ is given by

$$\widetilde{\theta}(G) := \operatorname*{argmin}_{\theta \in \mathbb{R}^n} \Biggl\{ \sum_{i=1}^{n} (Y_i - \theta_i)^2 + \rho \cdot \theta^T L_G^s \theta \Biggr\} = (\rho L_G^s + I)^{-1} Y. \tag{2}$$

Now suppose we wish to test

$$\mathbf{H}_0 : \theta_0 = 0 \;\; \text{vs.} \;\; \mathbf{H}_a : \theta_0 \neq 0 \tag{3}$$

A natural candidate is the quadratic form

$$\widetilde{T}(G) := \frac{1}{n} \sum_{i=1}^{n} \bigl(\widetilde{\theta}_i(G)\bigr)^2 \tag{4}$$

In Lemma 1, we provide concentration bounds on the test statistic $\widetilde{T}(G)$, under both the null and alternative hypotheses. These statements are a function of the spectral decomposition $L_G = \sum_{k=1}^{n} \lambda_k(G) \cdot v_k(G) v_k(G)^T$.

Lemma 1. *Define the threshold $\widetilde{t}_b$ to be*

$$\widetilde{t}_b := \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\bigl(\rho \lambda_k^s + 1\bigr)^2} + \frac{2b}{n} \sqrt{\sum_{k=1}^{n} \frac{1}{\bigl(\rho \lambda_k^s + 1\bigr)^4}}$$

Then,

- ***Type I error.***

$$\mathbb{P}_0\Bigl(\widetilde{T}(G) > \widetilde{t}_b\Bigr) \leq \frac{1}{b^2} \tag{5}$$

- ***Type II error.*** *If*

$$\frac{1}{n} \|\theta_0\|_2^2 \geq \frac{2\rho}{n} \bigl(\theta_0^T L^s \theta_0\bigr) + \frac{4b}{n} \left(\sum_{k=1}^{n} \frac{1}{(\rho \lambda_k^s + 1)^4} \right)^{1/2} \tag{6}$$

then

$$\mathbb{P}_{\theta_0}\Bigl(\widetilde{T}(G) \leq \widetilde{t}_b\Bigr) \leq \frac{4}{b^2} + \frac{8}{b} \left(\sum_{k=1}^{n} \frac{1}{(\rho \lambda_k^s + 1)^4} \right)^{-1/2} \tag{7}$$

1.1 Notation

- When the graph G is obvious from context, we may drop the notational dependence on G and simply write the Laplacian of G as L, and the kth eigenvalue of the Laplacian as λ_k.

2 Proofs

2.1 Proof of Lemma 1

Let $\widetilde{S} := (\rho L^s + I)^{-1}$. The matrix $\widetilde{S} \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite, and our test statistic $\widetilde{T}(G) = \frac{1}{n} Y^T \widetilde{S}^2 Y$. The desired result thus follows from Lemma 2. To see that the conditions of this Lemma are satisfied, we first note that since

$$\lambda_k(S) = \frac{1}{(\rho \lambda_k^s + 1)}$$

and $\rho, \lambda_k > 0$, it is evident that $\lambda_{\max}(\widetilde{S}) \leq 1$. Then, by assumption (6)

$$\theta_0^T \widetilde{S}^2 \theta_0 = \|\theta_0\|_2^2 - \theta_0^T (I - \widetilde{S}^2) \theta_0 \geq 2\rho(\theta_0^T L^s \theta_0) + \theta_0^T (I - \widetilde{S}^2)\theta_0 + 4b \left(\sum_{k=1}^{n} \frac{1}{(\rho \lambda_k^s + 1)^4} \right)^{-1/2},$$

and along with the following calculations,

$$\begin{aligned}
\theta_0^T \left(I - \widetilde{S}^2 \right) \theta_0 &\overset{(i)}{=} \theta_0^T L^{s/2} L^{-s/2} \left(I - \widetilde{S}^2 \right) L^{-s/2} L^{s/2} \theta_0 \\
&\leq \theta_0^T L^s \theta_0 \cdot \lambda_{\max} \left(L^{-s/2} \left(I - \widetilde{S}^2 \right) L^{-s/2} \right) \\
&\overset{(ii)}{=} \theta_0^T L^s \theta_0 \cdot \max_k \left\{ \frac{1}{\lambda_k^s} \left(1 - \frac{1}{(\rho \lambda_k^s + 1)^2} \right) \right\} \\
&\overset{(iii)}{\leq} \theta_0^T L^s \theta_0 \cdot 2\rho,
\end{aligned}$$

we have that

$$\theta_0^T \widetilde{S}^2 \theta_0 \geq 2b \left(\sum_{k=1}^{n} \frac{1}{(\rho \lambda_k^s + 1)^4} \right)^{-1/2}.$$

In other words condition (11) in Lemma 2 is met, and applying that Lemma completes the proof.

(In the previous derivation: in (i) when we write $L^{-s/2}$ we are referring to the pseudoinverse of L, since the Laplacian always has at least one eigenvalue equal to 0; in (ii) the maximum is over all indices k such that the eigenvalue λ_k is strictly positive; and (iii) follows from the basic algebraic identity $1 - 1/(1 + \rho x)^2 \leq 2\rho x$ for any $x, \rho > 0$.

3 Technical Lemmas

3.1 Type I and Type II error of quadratic forms.

Let $S \in \mathbb{R}^{n \times n}$ be a square symmetric matrix. The quadratic form

$$T = Y^T S^2 Y \tag{8}$$

can be used as a test statistic to distinguish $\mathbf{H}_0$ from $\mathbf{H}_a$. In Lemma 2, we establish conditions under which a test based on T has small Type I and Type II error.

Lemma 2. *Define the threshold* t_b *to be*

$$t_b := \sum_{k=1}^{n} \big(\lambda_k(S)\big)^2 + 2b\sqrt{\sum_{k=1}^{n} \big(\lambda_k(S)\big)^4} \tag{9}$$

Suppose the eigenvalues $0 \leq \lambda_{\min}(S) \leq \lambda_{\max}(S) \leq 1$. *Then,*

- ***Type I error.***

$$\mathbb{P}_0\big(T > t_b\big) \leq \frac{1}{b^2} \tag{10}$$

- ***Type II error.*** *Assuming that*

$$\theta_0^T S^2 \theta_0 \geq 4b\sqrt{\sum_{k=1}^{n} \big(\lambda_k(S)\big)^4} \tag{11}$$

  then

$$\mathbb{P}_{\theta_0}\big(T \leq t_b\big) \leq \frac{4}{b^2} + \frac{8}{b}\left(\sum_{k=1}^{n} \big(\lambda_k(S)\big)^4\right)^{-1/2} \tag{12}$$

Proof of Lemma 2. We compute the mean and variance of T as a function of θ_0, then apply Chebyshev's inequality.

Mean. Writing $Y = \theta_0 + \varepsilon$, we make use of the eigendecomposition $S = \sum_{k=1}^{n} \lambda_k(S) \cdot v_k(S) v_k(S)^T$—where in this case we fix the eigenvectors $v_1(S), \dots, v_n(S)$ to be unit-norm—and obtain

$$\begin{aligned}
T &= \theta_0^T S^2 \theta_0 + 2\theta_0^T S^2 \varepsilon + \varepsilon^T S^2 \varepsilon \\
&= \theta_0^T S^2 \theta_0 + 2\theta_0^T S^2 \varepsilon + \sum_{k=1}^{n} \big(\lambda_k(S)\big)^2 (\varepsilon^T v_k(S))^2 \\
&= \theta_0^T S^2 \theta_0 + 2\theta_0^T S^2 \varepsilon + \sum_{k=1}^{n} \big(\lambda_k(S)\big)^2 Z_k^2
\end{aligned} \tag{13}$$

where in the last line $Z_k = (\varepsilon^T v_k(S))$, and $Z = (Z_1, \dots, Z_k) \sim N(0, I)$ follows from the rotational invariance of the Gaussian distribution. Thus

$$\mathbb{E}_{\theta_0}[T] = \theta_0^T S^2 \theta_0 + \sum_{k=1}^{n} \big(\lambda_k(S)\big)^2. \tag{14}$$

Variance. Starting from (13) and recalling the basic fact $\mathrm{Var}(Z_k^2) = 2$, we derive

$$\mathrm{Var}_{\theta_0}[T] \leq 8\theta_0^T S^4 \theta_0 + 4\sum_{k=1}^{n} \big(\lambda_k(S)\big)^4 \leq 8\theta_0^T S^2 \theta_0 + 4\sum_{k=1}^{n} \big(\lambda_k(S)\big)^4 \tag{15}$$

where the second inequality follows since by assumption $\lambda_{\max}(S) \leq 1$.

Bounding Type I and Type II error. The bound (10) follows directly from Chebyshev's inequality, along with our above calculations on the mean and variance of T.

The bound (12) also follows from Chebyshev's inequality, as can be seen by the following manipulations,

$$\begin{aligned}
\mathbb{P}_{\theta_0}\big(T \le t_b\big) &= \mathbb{P}_{\theta_0}\big(T - \mathbb{E}_{\theta_0}[T] \le t_b - \mathbb{E}_{\theta_0}[T]\big) \\
&\overset{(i)}{\le} \mathbb{P}_{\theta_0}\big(|T - \mathbb{E}_{\theta_0}[T]| \ge |t_b - \mathbb{E}_{\theta_0}[T]|\big) \\
&\overset{(ii)}{\le} 4\frac{\mathrm{Var}_{\theta_0}[T]}{(\theta_0^T S^2 \theta_0)^2} \\
&\overset{(iii)}{\le} \frac{32}{\theta_0^T S^2 \theta_0} + \frac{4}{b^2} \\
&\overset{(iv)}{\le} \frac{8}{b}\left(\sum_{k=1}^{n}\big(\lambda_k(S)\big)^4\right)^{-1/2} + \frac{4}{b^2}
\end{aligned}$$

In the previous expression, (i) and (ii) follow since assumption (11) and equation (14) together imply $\mathbb{E}_{\theta_0}(T) - \frac{1}{2}\theta_0^T S^2 \theta_0 \ge t_b$, (iii) follows from assumption (11) and the inequality (15), and (iv) follows assumption (11). □