Notes for Week 11/8/19 - 11/15/19

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For a function $f : \mathbb{R}^d \to \mathbb{R}$, consider the empirical norm

$$\|f\|_n^2 := \frac{1}{n}\sum_{i=1}^{n}[f(x_i)]^2$$

where $x_1, \ldots, x_n$ are i.i.d. samples from a distribution P, with density p supported on $\mathcal{X} \subset \mathbb{R}^d$. Under what conditions is the empirical norm at least on the order of $\|f\|_{\mathcal{L}^2(\mathcal{X})}^2$?

Lemma 1. *Let $\mathcal{X}$ be a Lipschitz domain over which the density is upper and lower bounded*

$$0 < p_{\min} \leq p(x) \leq p_{\max} < \infty \quad \textit{for all } x \in \mathcal{X},$$

and let $f \in W_d^{s,2}(\mathcal{X})$. Then for any $b \geq 1$, there exists c_1 such that if

$$\|f\|_{\mathcal{L}^2(\mathcal{X})} \geq \begin{cases} c_1 \cdot b \cdot \|f\|_{W_d^{s,2}(\mathcal{X})} \cdot \max\{n^{-1/2}, n^{-s/d}\}, & \text{if } 2s \neq d \\ c_1 \cdot b \cdot \|f\|_{W_d^{s,2}(\mathcal{X})} \cdot n^{-a/2}, & \text{if } 2s = d \text{ for any } 0 < a < 1 \end{cases} \tag{1}$$

then,

$$\mathbb{P}\left[\|f\|_n^2 \geq \frac{1}{b}\mathbb{E}\big[\|f\|_n^2\big]\right] \geq 1 - \frac{5}{b} \tag{2}$$

where c_1 and c_2 are constants which may depend only on s, $\mathcal{X}$, d, $p_{\min}$ and $p_{\max}$.

Proof. To prove (2) we will show

$$\mathbb{E}\big[\|f\|_n^4\big] \leq \left(1 + \frac{1}{b^2}\right) \cdot \left(\mathbb{E}\big[\|f\|_n^2\big]\right)^2$$

whence the claim follows from the Paley-Zygmund inequality (Lemma 3). Since $p \leq p_{\max}$ is uniformly bounded, we can relate $\mathbb{E}\big[\|f\|_n^4\big]$ to the $\mathcal{L}^4$ norm,

$$\mathbb{E}\big[\|f\|_n^4\big] = \frac{(n-1)}{n}\left(\mathbb{E}\Big[\|f\|_n^2\Big]\right)^2 + \frac{\mathbb{E}\Big[\big(f(x_1)\big)^4\Big]}{n} \leq \left(\mathbb{E}\Big[\|f\|_n^2\Big]\right)^2 + p_{\max}^2\frac{\|f\|_{\mathcal{L}^4}^4}{n}.$$

We will use a Sobolev inequality to relate $\|f\|_{\mathcal{L}^4}$ to $\|f\|_{W_d^{s,2}(\mathcal{X})}$. The nature of this inequality depends on the relationship between s and d (see Theorem 6 in Section 5.6.3 of Evans for a formal statement), so from this point on we divide our analysis into three cases: (i) the case where $2s > d$, (ii) the case where $2s < d$, and (iii) the borderline case $2s = d$.

Case 1: $2s > d$**.** When $2s > d$, since $\mathcal{X}$ is a Lipschitz domain the Sobolev inequality establishes that $f \in C^{\gamma}(\overline{\mathcal{X}})$ for some $\gamma > 0$ which depends on s and d, with the accompanying estimate

$$\sup_{x \in \mathcal{X}} |f(x)| \leq \|f\|_{C^{\gamma}(\mathcal{X})} \leq c\|f\|_{W^{s,2}(\mathcal{X})}.$$

Therefore,

$$\begin{aligned}
\|f\|_{\mathcal{L}^4}^4 &= \int_{\mathcal{X}} [f(x)]^4 \, dx \\
&\leq \left(\sup_{x \in \mathcal{X}} |f(x)|\right)^2 \cdot \int_{\mathcal{X}} [f(x)]^2 \, dx \\
&\leq c\|f\|_{W^{s,2}(\mathcal{X})}^2 \cdot \|f\|_{\mathcal{L}^2(\mathcal{X})}^2.
\end{aligned}$$

Since by assumption

$$\|f\|_{\mathcal{L}^2(\mathcal{X})}^2 \geq c_1^2 \cdot b^2 \cdot \|f\|_{W_d^{s,2}(\mathcal{X})}^2 \cdot \frac{1}{n},$$

we have

$$p_{\max}^2 \frac{\|f\|_{\mathcal{L}^4(\mathcal{X})}^4}{n} \leq c\|f\|_{W^{s,2}(\mathcal{X})}^2 \cdot \frac{\|f\|_{\mathcal{L}^2(\mathcal{X})}^4}{n\|f\|_{\mathcal{L}^2(\mathcal{X})}^2} \leq c\frac{\|f\|_{\mathcal{L}^2(\mathcal{X})}^4}{c_1^2 b^2} \leq \frac{\mathbb{E}\big[\|f\|_n^2\big]}{b^2},$$

where the last inequality follows by taking c_1 sufficiently large.

Case 2: $2s < d$**.** When $2s < d$, since $\mathcal{X}$ is a Lipschitz domain the Sobolev inequality establishes that $f \in \mathcal{L}^q(\mathcal{X})$ for $q = 2d/(d-2s)$, and moreover that

$$\|f\|_{\mathcal{L}^q(\mathcal{X})} \leq c\|f\|_{W^{s,2}(\mathcal{X})}.$$

Since $4 = 2\theta + (1-\theta)q$ for $\theta = 2 - d/(2s)$, Lyapunov's inequality implies

$$\|f\|_{\mathcal{L}^4(\mathcal{X})}^4 \leq \|f\|_{\mathcal{L}^2}^{2\theta} \cdot \|f\|_{\mathcal{L}^q(\mathcal{X})}^{(1-\theta)q} \leq c\|f\|_{\mathcal{L}^2(\mathcal{X})}^4 \cdot \left(\frac{\|f\|_{W^{s,2}(\mathcal{X})}}{\|f\|_{\mathcal{L}^2(\mathcal{X})}}\right)^{d/s}.$$

By assumption, $\|f\|_{\mathcal{L}^2(\mathcal{X})} \geq c_1 b\|f\|_{W^{s,2}(\mathcal{X})} n^{-s/d}$, and therefore

$$p_{\max}^2 \frac{\|f\|_{\mathcal{L}^4(\mathcal{X})}^4}{n} \leq c\|f\|_{\mathcal{L}^2(\mathcal{X})}^4 \left(\frac{\|f\|_{W^{s,2}(\mathcal{X})}}{n^{s/d}\|f\|_{\mathcal{L}^2(\mathcal{X})}}\right)^{d/s} \leq \frac{c\|f\|_{\mathcal{L}^2(\mathcal{X})}^4}{c_1 b^{d/s}} \leq \frac{\|f\|_{\mathcal{L}^2(\mathcal{X})}^4}{b^2}.$$

where the last inequality follows when c_1 is sufficiently large, and keeping in mind that $d/s > 2$ and $b \geq 1$.

Case 3: $2s = d$**.** Assume f satisfies (3) for a given $0 < a < 1$. When $2s = d$, since $\mathcal{X}$ is a Lipschitz domain we have that $f \in L^q(\mathcal{X})$ for any $q < \infty$, with the accompanying estimate

$$\|f\|_{\mathcal{L}^q(\mathcal{X})} \leq c\|f\|_{W^{s,2}(\mathcal{X})}.$$

In particular the above holds for $q = 2/(1-a)$ when $1/2 < a < 1$, and for any $q > 4$ when $0 < a < 1/2$. Using Lyapunov's inequality as in the previous case then implies the desired result. ☐

1 Old Stuff

Lemma 2. *Suppose* $\|f\|_\infty \leq 1$ *and* $\mathbb{E}[f^2(X)] \geq \frac{1}{n}$. *Then,*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)^2 \geq \frac{1}{2}\mathbb{E}[f^2(X)]\right) \geq \frac{1}{8}.$$

The proof of Lemma 2 relies on (a variant of) the Paley-Zygmund Inequality.

Lemma 3. *Let* f *satisfy the following moment inequality for some* $b \geq 1$*:*

$$\mathbb{E}\big[\|f\|_n^4\big] \leq \left(1 + \frac{1}{b^2}\right) \cdot \left(\mathbb{E}\big[\|f\|_n^2\big]\right)^2. \tag{3}$$

Then,

$$\mathbb{P}\left[\|f\|_n^2 \geq \frac{1}{b}\mathbb{E}\big[\|f\|_n^2\big]\right] \geq 1 - \frac{5}{b}. \tag{4}$$

Proof. Let Z be a non-negative random variable such that $\mathbb{E}(Z^q) < \infty$. The Paley-Zygmund inequality says that for all $0 \leq \lambda \leq 1$,

$$\mathbb{P}(Z > \lambda\mathbb{E}(Z^p)) \geq \left[(1 - \lambda^p)\frac{\mathbb{E}(Z^p)}{(\mathbb{E}(Z^q))^{p/q}}\right]^{\frac{q}{q-p}} \tag{5}$$

Applying (5) with $Z = \|f\|_n^2$, $p = 1$, $q = 2$ and $\lambda = \frac{1}{b}$, by assumption (3) we have

$$\mathbb{P}\Big(\|f\|_n^2 > \frac{1}{b}\mathbb{E}[\|f\|_n^2]\Big) \geq \Big(1 - \frac{1}{b}\Big)^2 \cdot \frac{\left(\mathbb{E}[\|f\|_n^2]\right)^2}{\mathbb{E}[\|f\|_n^4]} \geq \frac{\left(1 - \frac{2}{b}\right)}{\left(1 + \frac{1}{b^2}\right)} \geq 1 - \frac{5}{b}.$$

$\square$

Proof of Lemma 2: To apply Lemma 3, we need an upper bound on $\mathbb{E}((\frac{1}{n}\sum_{i=1}^n f(x_i)^2)^2)$.

$$\begin{aligned}
\mathbb{E}\left((\frac{1}{n}\sum_{i=1}^n f(x_i)^2)^2\right) &= \frac{1}{n^2}\left(n(n-1)\mathbb{E}(f^2(X))^2 + n\mathbb{E}(f^4(X))\right) \\
&\leq \frac{1}{n^2}\left(n(n-1)\mathbb{E}(f^2(X))^2 + n\mathbb{E}(f^2(X))\right) \\
&= \frac{1}{n^2}\left(\mathbb{E}(f^2(X))\left(n(n-1)\mathbb{E}(f^2(X)) + n\right)\right) \\
&\leq \frac{1}{n^2}\left(\mathbb{E}(f^2(X))\left(n(n-1)\mathbb{E}(f^2(X)) + n^2\mathbb{E}(f^2(X))\right)\right) \\
&\leq 2\left(\mathbb{E}(f^2(X))\right)^2
\end{aligned}$$

Applying Lemma 3 with $p = 1$, $q = 2$, $Z = \frac{1}{n}\sum_{i=1}^n f(x_i)^2$, and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)^2 \geq \frac{1}{2}\mathbb{E}[f^2(X)]\right) &\geq \left[\frac{1}{2}\frac{\mathbb{E}(f^2(X))}{\mathbb{E}\left((\frac{1}{n}\sum_{i=1}^n f(x_i)^2)^2\right)^{1/2}}\right]^2 \\
&\geq \frac{1}{4}\left[\frac{1}{\sqrt{2}}\right]^2 = \frac{1}{8}.
\end{aligned}$$