Notes for Week 3/19/20 - 3/26/20

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Let $G_0 = ([\mathcal{M}], E_0)$ be a graph on $\mathcal{M} \geq 1$ vertices. Let $N_1, \ldots, N_{\mathcal{M}}$ each be positive integers. The (Alden product) graph H_0 on $n = \sum N_m$ nodes is defined as

$$H_0 = \left(\bigcup_{m=1}^{\mathcal{M}} \bigcup_{i=1}^{N_m} (m,i), \quad F_0\right), \quad \text{where } (\ell, i) \sim (m,j) \ \text{ in } H_0 \text{ if } \ell \sim m \text{ in } G_0.$$

Let $\lambda_k(H_0)$ denote the kth smallest eigenvalue of the Laplacian L_{H_0}, and let

$$N_{\min} = \min_{m \in [\mathcal{M}]} N_m, \quad N_{\max} = \min_{m \in [\mathcal{M}]} N_m$$

We wish to prove the following Lemma

Lemma 1. *For $k \in [n]$,*

$$\lambda_k(H_0) \geq \frac{\deg_{\min}(G_0) N_{\min}^3}{\deg_{\max}(G_0) N_{\max}^2} \cdot \begin{cases} \lambda_k(G_0), & \textit{if } k \in [\mathcal{M}] \\ 1, & \textit{otherwise.} \end{cases} \tag{1}$$

Our proof will involve a series of matrix and graph comparisons. For a graph G on M vertices, let N_G be the normalized Laplacian matrix, with associated eigenvalues $\lambda_1(N_G) \leq \cdots \leq \lambda_M(N_G)$. Let $\sigma_1, \ldots, \sigma_{\mathcal{M}}$ be contractions from $[N_{\max}]$ to $[N_1], \ldots, [N_{\mathcal{M}}]$, respectively, to be defined later. The mappings σ_m and the graph H_0 jointly induce a weighted graph

$$\bar{H}_0 = \Big([\mathcal{M}] \times [N_{\max}], W_0\Big), \quad \text{where } W_0[(m,j),(m',j')] = \frac{\mathbf{1}\{(m, \sigma_m(j)) \sim (m', \sigma_{m'}(j')) \text{ in } H\}}{|\{\ell : \sigma_m(\ell) = \sigma_m(j)\}| \cdot |\{\ell' : \sigma_{m'}(\ell') = j'\}|} \tag{2}$$

(TODO): Define $\bar{G}_0$ here.

We shall proceed according to the following steps:

1. For each $k \in [n]$,
$$\lambda_k(H_0) \geq \deg_{\min}(H_0) \cdot \lambda_k(N_{H_0}) \tag{3}$$

2. For each $k \in \big[n\big]$,
$$\lambda_k(N_{H_0}) \geq \lambda_k(N_{\overline{H}_0}) \tag{4}$$

3. For each $k \in [\mathcal{M} N_{\max}]$,
$$\lambda_k(N_{\overline{H}_0}) \geq \frac{N_{\min}^2}{N_{\max}^2} \cdot \lambda_k(N_{\bar{G}_0}) \tag{5}$$

4. For each $k \in [\mathcal{M}]$,
$$\widetilde{\lambda}_k(\bar{G}_0) = \widetilde{\lambda}_k(G_0) \tag{6}$$
Otherwise if $k > \mathcal{M}$, $\widetilde{\lambda}_k(\bar{G}_0) = 1$.
5. For each $k \in [\mathcal{M}]$,
$$\widetilde{\lambda}_k(G_0) \geq \frac{\lambda_k(G_0)}{\deg_{\max}(G)}.$$

(TODO): Fill in proof.

Step 1: Moving to Normalized Laplacian. For simplicity, in this section we will deal with an arbitrary $G = ([n], E)$, and show
$$\lambda_k(G) \geq \deg_{\min}(G) \cdot \lambda_k(N_G)$$
for all $k \in [n]$. If $\deg_{\min}(G) = 0$, the statement is trivially obvious, and so we will suppose without loss of generality that $\deg_{\min}(G) > 0$.

Let $L = D - A$, where $A = A(G)$ is the adjacency matrix of G, $D := D(G)$ is the degree matrix associated with G, and L is therefore the Laplacian of G. Let $N = D^{-1/2} L D^{-1/2}$ be the normalized Laplacian of G. Using the Courant-Fischer min max theorem, and letting $D^{-1/2}V = \{D^{-1/2}v : v \in V\}$ for any subspace $V \subset \mathbb{R}^n$, we have
$$\begin{aligned}
\lambda_k(G) &= \min_V \Big\{ \max_{v \in V} \frac{v^T L v}{v^T v} \Big\} \\
&= \min_V \Big\{ \max_{u \in D^{-1/2}V} \frac{u^T N u}{u^T D^{-1} u} \Big\} \\
&\geq \deg_{\min}(G) \min_V \Big\{ \max_{u \in D^{-1/2}V} \frac{u^T N u}{u^T u} \Big\}
\end{aligned}$$
where the minimum is always over all k dimensional subspaces of $\mathbb{R}^n$ and the second line follows upon substituting $u = D^{1/2}v$. Since every vertex has non-zero degree, both $D^{1/2}$ and $D^{-1/2}$ are full rank matrices, and $\dim(D^{-1/2}V) = \dim(V)$ for all subspaces V. Hence, we have
$$\min_V \Big\{ \max_{u \in D^{-1/2}V} \frac{u^T N u}{u^T u} \Big\} = \min_U \Big\{ \max_{u \in U} \frac{u^T N u}{u^T u} \Big\} = \lambda_k(N_G).$$
Choosing $G = H_0$, we obtain (3).

Step 2: Moving to a weighted graph. By definition, the graph H_0 is a contraction of $\bar{H}_0$. Moreover, for any vertices (m, j) and (m', j') which are contracted together, $m = m'$ and $\sigma_m(j) = \sigma_m(j')$, so the two vertices have all the same edge weights in $\bar{H}_0$. The eigenvalue inequality (4) then follows from Lemma 2.

Step 3: Moving to unweighted tensor product graph. We first lower bound $N_{\overline{H}_0}$ by an unweighted tensor product graph $N_{\bar{G}_0}$, where $\bar{G}_0$ is defined as
$$\bar{G}_0 = \big([\mathcal{M}] \times [N_{\max}], E(\bar{G}_0)\big), \quad \text{where} \;\; (m, j) \sim (m', j') \text{ in } \bar{G}_0 \text{ if } m \sim m' \text{ in } G_0.$$
The graph $\bar{H}_0$ has the same adjacency structure as $\bar{G}_0$, but edges with weights between 0 and 1. To have our lower bound be sufficiently large, we would like to make the minimum of these weights large. We achieve by this ensuring the maps σ_m do not map too many vertices in $[\mathcal{M}] \times [N_{\max}]$ to the same vertex in $V(H_0)$. Formally, we let
$$\sigma_m(i) := i \mod N_m$$

Then, for any $(m, i) \in [\mathcal{M}] \times [N_{\max}]$, clearly $|\{\ell : \sigma_m(\ell) = \sigma_m(j)\}| \le N_{\max}/N_{\min}$. Recalling the definition of the weight matrix W_0 in (2), by Lemma 3, we have that

$$\lambda_k(N_{\overline{H}_0}) \ge \frac{N_{\min}^2}{N_{\max}^2} \lambda_k(N_{\bar{G}_0}).$$

Step 4: Completing the proof. We have that $\bar{G}_0 = G_0 \otimes K_{N_{\max}}$, and we may therefore characterize the spectrum $\widetilde{\Lambda}(\bar{G}_0)$ by $\widetilde{\Lambda}(G_0)$ and $\widetilde{\Lambda}(K_{N_{\max}})$ using Lemma 4. The latter spectrum is simply

$$\lambda_1(N_{K_{N_{\max}}}) = 0, \lambda_2(N_{K_{N_{\max}}}), \ldots, \lambda_N(N_{K_{N_{\max}}}) = 1,$$

and therefore by Lemma 4

$$\widetilde{\lambda}_k(\bar{G}_0) = \begin{cases} \widetilde{\lambda}_k(G_0), & \text{for } k = 1, \ldots, \mathcal{M} \\ 1, & \text{for } k = \mathcal{M} + 1, \ldots, \mathcal{M} \cdot N_{\max}. \end{cases} \tag{7}$$

1 Additional Theory

1.1 Contractions

Let $H = ([n+m], W_H)$ be an arbitrary weighted graph on $n+m$ vertices. Any mapping $\sigma : [n+m] \to [n]$ induces a graph $G = ([n], W_G)$ with weights

$$W_G[k, k'] = \sum_{\ell : \sigma(\ell) = k} \sum_{\ell' : \sigma(\ell') = k'} W_H[\ell, \ell']$$

which we call the contraction of H induced by σ.

Lemma 2. *Suppose that the graph H and contraction σ satisfy the following property: for all ℓ, ℓ' such that $\sigma(\ell) = \sigma(\ell')$, $W_H[\ell, \cdot] = W_H[\ell', \cdot]$. Then,*

$$\lambda_k(N_G) \ge \lambda_k(N_H), \quad \textit{for all } k \in [n].$$

Proof. The following two facts are key consequences of the assumption that $W_H[\ell, \cdot] = W_H[\ell', \cdot]$ for all $\sigma(\ell) = \sigma(\ell')$. First, for any i and $i' \in [n+m]$,

$$\begin{aligned} W_G[\sigma(i), \sigma(i')] &= \sum_{\ell : \sigma(\ell) = \sigma(i)} \sum_{\ell' : \sigma(\ell') = \sigma(i')} W_H[\ell, \ell'] \\ &= \sum_{\ell : \sigma(\ell) = \sigma(i)} \sum_{\ell' : \sigma(\ell') = \sigma(i')} W_H[i, i'] = W_H[i, i'] \cdot N_\sigma(i) N_\sigma(i'). \end{aligned}$$

Second, for any $\ell \in [n+m]$,

$$\begin{aligned} \deg_H(\ell) &= \sum_{i=1}^{n+m} W_H[i, \ell] \\ &= \sum_{i=1}^{n+m} \frac{W_G\big[\sigma(i), \sigma(\ell)\big]}{N_\sigma(i) N_{\sigma(\ell)}} \\ &= \sum_{j=1}^{n} \frac{W_G\big[j, \ell\big]}{N_\sigma(\ell)} \\ &= \frac{\deg_G(\sigma(\ell))}{N_\sigma(\ell)}. \end{aligned}$$

Now, let v be an eigenvector of N_G, meaning there exists $\lambda > 0$ such that

$$N_G v = \lambda v$$

Define $u : [n+m] \to \mathbb{R}$ to be the vector

$$u(\ell) = \frac{v(\sigma(\ell))}{\sqrt{N_\sigma(\ell)}}, \quad \text{where } N_\sigma(\ell) = \Big|\{\ell' : \sigma(\ell') = \sigma(\ell)\}\Big|$$

The following manipulations show that u is an eigenvector of N_H, with eigenvalue λ.

$$\begin{aligned}
\Big(N_H u\Big)(\ell) &= \sum_{i=1}^{n+m} \Big\{ \frac{u(\ell)}{\deg_H(\ell)} - \frac{u(i)}{\sqrt{\deg_H(\ell)\deg_H(i)}} \Big\} W_H[\ell, i] \\
&= \sum_{i=1}^{n+m} \Big\{ \frac{v\big(\sigma(\ell)\big)}{\sqrt{N_\sigma(\ell)}\deg_H(\ell)} - \frac{v\big(\sigma(i)\big)}{\sqrt{N_\sigma(i)\deg_H(\ell)\deg_H(i)}} \Big\} W_H[\ell, i] \\
&= \sum_{i=1}^{n+m} \Big\{ \frac{\sqrt{N_\sigma(\ell)} v\big(\sigma(\ell)\big)}{\deg_G\big(\sigma(\ell)\big)} - \frac{\sqrt{N_\sigma(\ell)} v\big(\sigma(i)\big)}{\sqrt{\deg_G\big(\sigma(\ell)\big)\deg_G\big(\sigma(i)\big)}} \Big\} W_H[\ell, i] \\
&= \sum_{i=1}^{n+m} \Big\{ \frac{\sqrt{N_\sigma(\ell)} v\big(\sigma(\ell)\big)}{\deg_G\big(\sigma(\ell)\big)} - \frac{\sqrt{N_\sigma(\ell)} v\big(\sigma(i)\big)}{\sqrt{\deg_G\big(\sigma(\ell)\big)\deg_G\big(\sigma(i)\big)}} \Big\} \frac{W_G\Big[\sigma(\ell), \sigma(i)\Big]}{N_\sigma(\ell) N_\sigma(i)} \\
&= \frac{1}{\sqrt{N_\sigma(\ell)}} \sum_{i=1}^{n+m} \Big\{ \frac{v\big(\sigma(\ell)\big)}{\deg_G\big(\sigma(\ell)\big)} - \frac{v\big(\sigma(i)\big)}{\sqrt{\deg_G\big(\sigma(\ell)\big)\deg_G\big(\sigma(i)\big)}} \Big\} \frac{W_G\Big[\sigma(\ell), \sigma(i)\Big]}{N_\sigma(i)} \\
&\overset{(i)}{=} \frac{1}{\sqrt{N_\sigma(\ell)}} \sum_{j=1}^{n} \Big\{ \frac{v\big(\sigma(\ell)\big)}{\deg_G\big(\sigma(\ell)\big)} - \frac{v\big(j\big)}{\sqrt{\deg_G\big(\sigma(\ell)\big)\deg_G\big(j\big)}} \Big\} W_G\Big[\sigma(\ell), j\Big] \\
&= \frac{1}{\sqrt{N_\sigma(\ell)}} \lambda v\big(\sigma(\ell)\big) = \lambda u(\ell),
\end{aligned}$$

where (i) follows from the substitution $j = \sigma(i)$.

Therefore every eigenvalue of G is also an eigenvalue of H. It follows immediately that the kth smallest eigenvalue of H must be no greater than the kth smallest eigenvalue of G. □

1.2 Weighted graphs

Lemma 3. *Let $G = ([n], E)$ be an unweighted and connected graph, and let A be an $n \times n$ symmetric matrix with entries $0 < A_{ij} < 1$. Let $H = ([n], W)$ be a weighted graph with weights*

$$W_{ij} = A_{ij} \times \mathbf{1}\{(i, j) \in G\}.$$

Then,

$$\lambda_k(N_H) \geq \min\{A_{ij}\} \cdot \lambda_k(N_G)$$

for all $k = 1, \ldots, n$.

Proof. Note that since A has strictly positive entries and G is connected, the degree of every vertex $i \in [n]$ is positive in both G and H; thus D_H and D_G are full rank, and so is $D_G^{1/2} D_H^{-1/2}$. By the Courant-Fischer

Theorem,

$$\begin{aligned}
\lambda_k(N_H) &= \min_V \Big\{ \max_v \Big\{ \frac{v^T N_H v}{v^T v} : v \in V \text{ and } v \neq 0 \Big\} : \dim(V) = k \Big\} \\
&= \min_V \Big\{ \max_u \Big\{ \frac{u^T L_H u}{u^T D_H u} : u \in D_H^{-1/2} V \text{ and } u \neq 0 \Big\} : \dim(V) = k \Big\} \qquad (\text{substituting } u = D_H^{-1/2} v) \\
&\geq \min\{A_{ij}\} \cdot \min_V \Big\{ \max_u \Big\{ \frac{u^T L_G u}{u^T D_G u} : u \in D_H^{-1/2} V \text{ and } u \neq 0 \Big\} : \dim(V) = k \Big\} \\
&= \min\{A_{ij}\} \cdot \min_V \Big\{ \max_w \Big\{ \frac{w^T N_G w}{w^T w} : w \in D_G^{1/2} D_H^{-1/2} V \text{ and } u \neq 0 \Big\} : \dim(V) = k \Big\} \\
&\qquad\qquad (\text{substituting } w = D_G^{1/2} u) \\
&= \min\{A_{ij}\} \cdot \lambda_k(N_G),
\end{aligned}$$

where the last inequality follows from Lemma 5. □

1.3 Tensor product graphs

For an unweighted graph $G = ([n], E)$, the random walk matrix P_G has entries

$$(P_G)_{ij} = \frac{1}{\deg_G(i)} \mathbf{1}\{(i,j) \in E\}.$$

Recall that the spectrum $\Lambda(P_G) = 1 - \Lambda(N_G)$ (see, e.g. (von Luxburg)).

For graphs $G_1 = ([N], E_1)$ and $G_2 = ([M], E_2)$, the tensor $H = G_1 \otimes G_2$ is defined over the vertex set $[N] \times [M]$, with an edge between (i,k) and (j,l) if and only if i is connected to j in G_1 and k is connected to l in G_2.

Lemma 4. *Let $G_1 = ([N], E_1)$ and $G_2 = ([M], E_2)$ be unweighted graphs, and let $H = G_1 \otimes G_2$. Then, the spectrum*

$$\Lambda(N_H) = \Big\{ \lambda_k(N_{G_1}) + \lambda_k(N_{G_2}) - \lambda_k(N_{G_1}) \cdot \lambda_\ell(N_{G_1}) : (k, \ell) \in [N] \times [M] \Big\}$$

Proof. We will abbreviate $P_1 := P_{G_1}$ and $P_2 := P_{G_2}$. Let v_k and u_ℓ satisfy

$$P_1 v_k = \lambda_k v_k, \quad P_2 u_\ell = \lambda_\ell u_\ell.$$

Let $w : \mathbb{R}^{N \times M}$ be defined by $w_{ij} = v_{k,i} u_{\ell,j}$. We will show that w is an eigenvector of P_H satisfying

$$P_H w = \lambda_k \lambda_\ell \cdot w.$$

To see this, note that

$$\begin{aligned}
\deg_H\big((i,j)\big) &= \sum_{i'=1}^{N} \sum_{j'=1}^{M} \mathbf{1}\Big\{(i,j) \sim (i',j') \text{ in } H\Big\} \\
&= \sum_{i'=1}^{N} \sum_{j'=1}^{M} \mathbf{1}\Big\{i \sim i' \text{ in } G_1\Big\} \mathbf{1}\Big\{j \sim j' \text{ in } G_2\Big\} \\
&= \deg_{G_1}(i) \deg_{G_2}(j),
\end{aligned}$$

and therefore

$$\begin{aligned}
\big(P_H w\big)_{ij} &= \sum_{i'=1}^{N}\sum_{j'=1}^{M} \frac{w_{i'j'}}{\deg_H(i',j')}\mathbf{1}\Big\{(i,j)\sim(i',j') \text{ in } H\Big\}\\
&= \sum_{i'=1}^{N}\sum_{j'=1}^{M} \frac{v_{ki'}u_{\ell j'}}{\deg_{G_1}(i')\deg_{G_2}(j')}\mathbf{1}\Big\{i\sim i' \text{ in } G_1\Big\}\mathbf{1}\Big\{j\sim j' \text{ in } G_2\Big\}\\
&= \Bigg(\sum_{i'=1}^{N}\sum_{j'=1}^{M} \frac{v_{ki'}}{\deg_{G_1}(i')}\mathbf{1}\Big\{i\sim i' \text{ in } G_1\Big\}\Bigg)\Bigg(\sum_{i'=1}^{N}\sum_{j'=1}^{M} \frac{u_{kj'}}{\deg_{G_2}(j')}\mathbf{1}\Big\{j\sim j' \text{ in } G_1\Big\}\Bigg)\\
&= \lambda_k\lambda_\ell v_{ki}u_{\ell j}.
\end{aligned}$$

This characterizes the spectrum $\Lambda(P_H)$. The claim of Lemma 4 follows upon recalling that the spectrum $\Lambda(N_G) = 1 - \Lambda(N_G)$ for $G = H$, $G = G_1$ and $G = G_2$, so that

$$\begin{aligned}
\lambda_{k,\ell}(N_H) &= 1 - \lambda_{k,\ell}(P_H)\\
&= 1 - \lambda_k(P_{G_1})\lambda_\ell(P_{G_2})\\
&= 1 - \big(1-\lambda_k(N_{G_1})\big)\big(1-\lambda_\ell(N_{G_2})\big)\\
&= \lambda_k(N_{G_1}) + \lambda_\ell(N_{G_2}) - \lambda_k(N_{G_1})\lambda_\ell(N_{G_2}).
\end{aligned}$$

□

1.4 Variational lemmas

We will use the following fact repeatedly. We state and prove it formally as a sanity check. For a symmetric $n\times n$ matrix A, and a non-zero vector $v\in\mathbb{R}^n$, the Rayleigh quotient is

$$R_A(v) = \frac{v^TAv}{v^Tv}$$

We let $\lambda_1(A)\le\ldots\le\lambda_n(A)$ be the eigenvalues of A, sorted in ascending order. For a subspace $V\subseteq\mathbb{R}^n$ and operator $D:\mathbb{R}^n\to\mathbb{R}^n$, let $DV := \{Dv : v\in V\}$.

Lemma 5. *Let A be an $n\times n$ matrix, and let $D:\mathbb{R}^n\to\mathbb{R}^n$ be a full rank linear operator. Then,*

$$\lambda_k(A) = \min_V\Big\{\max_v\big\{R_A(v) : v\in DV \text{ and } v\neq 0\big\} : \dim(V) = k\Big\}$$

Proof. We know from the Courant-Fischer Theorem that

$$\lambda_k(A) = \min_V\Big\{\max_v\big\{R_A(v) : v\in V \text{ and } v\neq 0\big\} : \dim(V) = k\Big\}$$

Let (v_k, V_k) satisfy

$$R_A(v_k) = \lambda_k(A), \quad\text{and}\quad v_k = \operatorname*{argmax}_v\Big\{R_A(v) : v\in V_k \text{ and } v\neq 0\Big\}$$

Now, let $U_* = D^{-1}V_k$; since D is a full rank operator U_* is well-defined and $\dim(U_*) = \dim(V_k) = k$. Clearly $V_k = DU_*$, and therefore $v_k\in DU_*$. As a result

$$\begin{aligned}
\min_V\Big\{\max_v\big\{R_A(v) : v\in DV \text{ and } v\neq 0\big\} : \dim(V) = k\Big\} &\le \max_v\big\{R_A(v) : v\in DU_* \text{ and } v\neq 0\big\}\\
&= \max_v\big\{R_A(v) : v\in V_k \text{ and } v\neq 0\big\}\\
&= R_A(v_k)\\
&= \lambda_k(A).
\end{aligned}$$

On the other hand, let

$$U_k := \underset{U}{\operatorname{argmin}}\Big\{\max_v\big\{R_A(v) : v \in DU \text{ and } v \neq 0\big\} : \dim(U) = k\Big\}$$

and let $V_* = DU_k$. Since D is full rank $\dim(V_*) = \dim(U_*) = k$, and therefore

$$\begin{aligned}
\lambda_k(A) &= \min_V\Big\{\max\big\{R_A(v) : v \in V \text{ and } v \neq 0\big\} : \dim(V) = k\Big\} \\
&\leq \max\big\{R_A(v) : v \in V_* \text{ and } v \neq 0\big\} \\
&= \max\big\{R_A(v) : v \in DU_k \text{ and } v \neq 0\big\} \\
&= \min_U\Big\{\max_v\big\{R_A(v) : v \in DU \text{ and } v \neq 0\big\} : \dim(U) = k\Big\}
\end{aligned}$$

□

Among other things, Lemma 5 allows us to compare the spectrum of N_G and L_G, in terms of the minimum and maximum degree of G.

Lemma 6. *Let $G = ([N], E)$ be an unweighted connected graph. Then,*

$$\frac{\lambda_k(G)}{\deg_{\max}(G)} \leq \widetilde{\lambda}_k(G) \leq \frac{\lambda_k(G)}{\deg_{\min}(G)}$$

Proof. The following manipulations establish the lower bound,

$$\begin{aligned}
\lambda_k(N_G) &= \min_V\Big\{\max_v\Big\{\frac{v^T N_G v}{v^T v} : v \in V \text{ and } v \neq 0\Big\} : \dim(V) = k\Big\} \\
&= \min_V\Big\{\max_u\Big\{\frac{u^T L_G u}{u^T D_G u} : u \in D^{-1/2}V \text{ and } u \neq 0\Big\} : \dim(V) = k\Big\} \\
&\geq \frac{1}{\deg_{\max}(G)} \cdot \min_V\Big\{\max_u\Big\{\frac{u^T L_G u}{u^T u} : u \in D^{-1/2}V \text{ and } u \neq 0\Big\} : \dim(V) = k\Big\} \\
&= \frac{\lambda_k(G)}{\deg_{\max}(G)},
\end{aligned}$$

where the last inequality follows by Lemma 5. The upper bound follows by similar steps, upon replacing $\deg_{\max}(G)$ by $\deg_{\min}(G)$ in the previous expression and reversing the inequality. □