Notes for Week 1/30/20 - 2/7/20

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Suppose we observe samples (y_i, x_i) for $i = 1, \dots, n$. Here $x_1, \dots, x_n$ are random design points, sampled independently from a distribution P with density p supported on an open set $\mathcal{X} \subset \mathbb{R}^d$. The responses $Y = \{y_1, \dots, y_n\}$ are then assumed to follow the model

$$y_i = f(x_i) + \varepsilon_i, \quad \varepsilon_i \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1) \tag{1}$$

Our task is to distinguish

$$\mathbf{H}_0 : f = f_0 := 0 \quad \text{vs} \quad \mathbf{H}_{\mathrm{a}} : f \neq f_0$$

and we worst-case risk to assess performance: for a test ϕ and function class $\mathcal{H}$,

$$\mathcal{R}_\epsilon(\phi; \mathcal{H}) = \mathbb{E}_{f=f_0}(\phi) + \sup_{f \in \mathcal{H}, \|f - f_0\|_{\mathcal{L}^2} \geq \epsilon} \mathbb{E}_f(1 - \phi).$$

Test statistics. For a graph G with Laplacian $L_G = D_G - A_G$, we arrange the eigenvalues in ascending order and denote them $\lambda_1(G) \leq \ldots \leq \lambda_n(G)$. We let $v_k(G)$ be the eigenvector associated with the kth eigenvalue $\lambda_k(G)$.

We define the graph Laplacian eigenvector projection test to be the magnitude of the projection of the responses y onto the first κ graph Laplacian eigenvectors,

$$\phi_{\mathrm{spec}}(G_{n,r}) := \mathbf{1}\bigl\{T_{\mathrm{spec}}(G_{n,r}) \geq \tau\bigr\}, \quad T_{\mathrm{spec}}(G_{n,r}) := \frac{1}{n}\sum_{k=1}^{\kappa}\left(\sum_{i=1}^{n} v_{k,i}(G_{n,r}) y_i\right)^2.$$

The action of the graph Laplacian $L_{n,r}f(x)$ (we use the abbreviation $L_{n,r} := L_{G_{n,r}}$ to avoid the double subscripts) can be viewed as approximating the action of a continuum weighted Laplacian operator $\Delta_P f(x)$. The latter operator is a 2nd order differential operator; however the graph Laplacian does a poor job of tracking the 2nd derivative of f when x is sufficiently close to the boundary of the domain $\mathcal{X}$. The problem is only exacerbated when we consider the iterated graph Laplacian operator $L_{n,r}^s f(x)$, and view it as an estimate of the continuum weighted Laplacian operator $\Delta_P^s f(x)$. To rectify the situation, we will assume that $f \in C_c^s(\mathcal{X})$, so that tracking derivatives near the boundary is no longer an issue. We let $\mathcal{X}_r := \{x \in \mathcal{X} : \mathrm{dist}(x, \mathcal{X}) > r\}$ denote the remainder of $\mathcal{X}$ once a tube of radius r around the boundary has been removed.

Lemma 1. *Fix integers $s \geq 0$ and $q \geq 1$, and index vector $k \in (n)^q$. Suppose that $f \in C_c^s(\mathcal{X})$, that $p \in C^0(\mathcal{X}; p_{\max})$, and additionally that $p \in C^{s-1}(\mathcal{X}; p_{\max})$ if $s \geq 2$.*

- *If $x \in \mathcal{X}_{qr}$ and $2q < s$, then there exist functions $f_{\ell,q} \in C_c^{s-\ell}(\mathcal{X})$ which additionally satisfy*

$$\|f_{\ell,q}\|_{C^{s-\ell}(\mathcal{X})} \leq c\,\|f\|_{C^s(\mathcal{X})}, \tag{2}$$

for each $\ell = 2q, \ldots, s-1$*, such that*

$$\left| \mathbb{E}\Big[D_k f(x)\Big] - \sum_{\ell=2q}^{s-1} f_{\ell,q}(x) r^\ell \right| \leq c r^s \|f\|_{C^s(\mathcal{X})} \quad \textit{if } 2q < s, \tag{3}$$

,

- *Otherwise if* $2q \geq s$ *or* $x \in \mathcal{X} \setminus \mathcal{X}_{qr}$*,*

$$\left| \mathbb{E}\Big[D_k f(x)\Big] \right| \leq c r^s \|f\|_{C^s(\mathcal{X})}. \tag{4}$$

Lemma 2 supplies an equivalent result when $f \in H_0^s(\mathcal{X})$. In this Lemma, we write $\mathbb{E}[D_k f] : \mathcal{X} \to \mathbb{R}$ for the mapping $x \mapsto \mathbb{E}[D_k f(x)]$. We will also denote $U_r = \mathcal{X} \setminus \mathcal{X}_r$.

Lemma 2. *Fix integers* $s \geq 0$ *and* $q \geq 1$*, and an index vector* $k \in (n)^q$*. Suppose that* $p \in C^0(\mathcal{X}; p_{\max})$*, and additionally that* $p \in C^{s-1}(\mathcal{X}; p_{\max})$ *if* $s \geq 2$*. Then there exists an* $r' > 0$ *such that for all* $0 < r < r'$*, the following statements hold for all* $f \in H_0^s(\mathcal{X})$*:*

- *The expected difference operator* $\mathbb{E}[D_k f]$ *belongs to* $\mathcal{L}^2(U_{qr})$*, with norm*

$$\Big\| \mathbb{E}\big[D_k f\big] \Big\|_{\mathcal{L}^2(U_{qr})} \leq c r^s \|f\|_{H^s(\mathcal{X})} \tag{5}$$

- *If* $2q \geq s$*, then additionally* $\mathbb{E}[D_k f]$ *belongs to* $\mathcal{L}^2(\mathcal{X}_{qr})$*, with norm*

$$\Big\| \mathbb{E}\big[D_k f\big] \Big\|_{\mathcal{L}^2(\mathcal{X}_{qr})} \leq c r^s \|f\|_{H^s(\mathcal{X})}. \tag{6}$$

  Otherwise there exist functions $f_\ell \in H_0^{s-\ell}(\mathcal{X})$*, which additionally satisfy*

$$\|f_\ell\|_{H^{s-\ell}(\mathcal{X})} \leq c \|f\|_{H^s(\mathcal{X})} \tag{7}$$

  for each $\ell = 2q, \ldots, s-1$*, such that*

$$\Big\| \mathbb{E}\big[D_k f\big] - \sum_{\ell=2q}^{s-1} r^\ell f_\ell \Big\|_{\mathcal{L}^2(\mathcal{X}_{qr})} \leq c r^s \|f\|_{H^s(\mathcal{X})} \tag{8}$$

1 Proofs

1.1 Proof of Lemma 2

One can interpret the conclusions of Lemma 2 as demonstrating that expected difference operators behave similarly to derivatives over $H_0^s(\mathcal{X})$. The proof of Lemma 2 is therefore naturally centered on taking Taylor expansions, but in order to do this, we must relate f to a function g which has classical derivatives.

Since $f \in H_0^s(\mathcal{X})$, there exists a sequence $(f_m) \subset C_c^s(\mathcal{X})$ such that $\|f_m - f\|_{H^s(\mathcal{X})} \to 0$ as $m \to \infty$ (Indeed f_m will be smooth for each m, but we will not need that fact.) Picking m large enough so that

$$\|f_m - f\|_{H^s(\mathcal{X})} \leq r^s \|f\|_{H^s(\mathcal{X})}$$

we have that for any $\widetilde{f} \in \mathcal{L}^2(\mathcal{X})$,

$$\begin{aligned}
\Big\| \mathbb{E}\Big[D_k f\Big] - \widetilde{f} \Big\|_{\mathcal{L}^2(\mathcal{X})} &\leq \Big\| \mathbb{E}\Big[D_k f_m\Big] - \widetilde{f} \Big\|_{\mathcal{L}^2(\mathcal{X})} + \Big\| \mathbb{E}\Big[D_k (f - f_m)\Big] \Big\|_{\mathcal{L}^2(\mathcal{X})} \\
&\leq \Big\| \mathbb{E}\Big[D_k f_m\Big] - \widetilde{f} \Big\|_{\mathcal{L}^2(\mathcal{X})} + c \|f - f_m\|_{\mathcal{L}^2(\mathcal{X})} \\
&\leq \Big\| \mathbb{E}\Big[D_k f_m\Big] - \widetilde{f} \Big\|_{\mathcal{L}^2(\mathcal{X})} + c r^s \|f\|_{H^s(\mathcal{X})}
\end{aligned}$$

Since $f_m \in C_c^s(\mathcal{X})$, it can be continuously extended to $g : \mathbb{R}^d \to \mathbb{R},\ g \in C_c^s(\mathcal{X})$ by taking $g(x) = 0$ for all $x \in \mathbb{R}^d \setminus \mathcal{X}$, such that $\|g\|_{H^s(\mathbb{R}^d)} = \|f_m\|_{H^s(\mathcal{X})}$ and $g = f$ everywhere on $\mathcal{X}$. Therefore $\mathbb{E}[D_k g] = \mathbb{E}[D_k f_m]$, and it suffices to prove the estimates (5)-(8) hold with respect to g.

Before we do so, let us establish some notation. When $s \geq 1$, since $g \in C_c^s(\mathbb{R}^d)$ it admits a Taylor expansion of the form

$$g(y) = \sum_{|\alpha|=0}^{s-1} g^{(\alpha)}(x)(z-x)^\alpha + \sum_{|\alpha|=s} (y-x)^\alpha G_\alpha(x,y),$$

for any $y, x \in \mathbb{R}^d$. When $s \geq 2$, since $p \in C_c^{s-1}(\mathcal{X})$ it also admits a Taylor expansion,

$$p(y) = \sum_{|\beta|=0}^{s-2} p^\beta(x)(y-x)^\beta + \sum_{|\beta|=s-1} (x-y)^s P_\beta(x,y).$$

for any $y, x \in \mathcal{X}$. In both cases we use the integral form of the remainders:

$$G_\alpha(x,y) = \int_0^1 g^{(\alpha)}\big(x + t(y-x)\big)(1-t)^{|\alpha|}\, dt$$
$$P_\beta(x,y) = \int_0^1 p^{(\beta)}\big(x + t(y-x)\big)(1-t)^{|\beta|}\, dt$$

where by Rademacher's Theorem $g^{(\alpha)}$ and $p^{(\beta)}$ exist almost everywhere, and the preceding integrals are therefore well defined.

It will also be helpful to introduce some notation. For $G : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, let

$$\Big(\mathbb{E}_\alpha[G]\Big)(x) := \int_{\mathcal{X}} (y-x)^\alpha G(x,y) K_r(y,x) p(y)\, dy, \qquad \mathbb{E}_\alpha(x) := \Big(\mathbb{E}_\alpha[1]\Big)(x)$$
$$\Big(\mathbb{I}_\alpha[G]\Big)(x) := \int_{\mathcal{X}} (y-x)^\alpha G(x,y) K_r(y,x)\, dy, \qquad \mathbb{I}_\alpha(x) := \Big(\mathbb{I}_\alpha[1]\Big)(x)$$

Additionally we define

$$I_\alpha := \int z^\alpha K\big(\|z\|\big)\, dz.$$

and note that when $B(x,r) \subset \mathcal{X}$, the following two facts are true: first, that $\mathbb{I}_{\alpha,x} = r^{|\alpha|} I_\alpha$, and second that $I_\alpha = 0$ when $|\alpha| = 1$.

We begin with $s = 0$. When $q = 1$, we have

$$\begin{aligned}
\big\|\mathbb{E}\big[D_k g\big]\big\|_{\mathcal{L}^2(\mathcal{X})} &= \int_{\mathcal{X}} \left[\int_{\mathcal{X}} \Big(g(y) - g(x)\Big) K_r(y,x) p(y)\, dy\right]^2 dx \\
&\leq p_{\max}^2 \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \Big(|g(y)| + |g(x)|\Big) K_r(y,x)\, dy\right]^2 dx \\
&\leq K_{\max}^2 p_{\max}^2 \int_{\mathbb{R}^d} \left[\int_{B(0,1)} |g(zr+x)| + |g(x)|\ dz\right]^2 dx
\end{aligned}$$

and the statement follows by Lemma 11. The same result holds (up to different constants) for $s = 0$ and general q by induction.

For $s \geq 1$, we first show the desired estimate over U_{qr}.

1.1.1 Boundary region

Take $q = 1$, and $k \in [n]$. We begin by relating the $\mathcal{L}^2$ norm of $\mathbb{E}[D_k g]$ over U_r to the $\mathcal{L}^2$ norm of g over U_{2r}, as follows:

$$
\begin{aligned}
\Big\|\mathbb{E}\big[D_k g\big]\Big\|_{\mathcal{L}^2(U_r)}^2 &= \int_{U_r}\bigg[\int_{\mathcal{X}}\big(g(y)-g(x)\big)K_r(x,y)p(y)\,dy\bigg]^2\,dx \\
&\leq p_{\max}^2\int_{U_r}\bigg[\int_{\mathcal{X}}\big(|g(y)|+|g(x)|\big)K_r(x,y)\,dy\bigg]^2\,dx \\
&\overset{(i)}{\leq} p_{\max}^2K_{\max}^2\int_{U_r}\bigg[\int_{B(0,1)\cap(\mathcal{X}-x)/r}\big(|g(zr+x)|+|g(x)|\big)\,dz\bigg]^2\,dx \\
&\overset{(ii)}{\leq} 2p_{\max}^2K_{\max}^2\int_{U_r}\nu_d^2\big(g(x)\big)^2\,dx + 2p_{\max}^2K_{\max}^2\nu_d\int_{U_r}\bigg[\int_{B(0,1)\cap(\mathcal{X}-x)/r}\big(g(zr+x)\big)^2\,dz\bigg]\,dx \\
&\leq 2p_{\max}^2K_{\max}^2\int_{U_r}\nu_d^2\big(g(x)\big)^2\,dx + 2p_{\max}^2K_{\max}^2\nu_d\int_{B(0,1)}\int_{U_r}\big(g(zr+x)\big)^2\,dz\,dx \\
&\leq 4p_{\max}^2K_{\max}^2\nu_d^2\,\|g\|_{\mathcal{L}^2(U_{2r})}^2
\end{aligned}
$$

where (i) follows from change of variables and (ii) from Young's and Jensen's inequality. Reasoning by induction, we see that it suffices to show that

$$
\|g\|_{\mathcal{L}^2(U_{(q+1)r})}^2 \leq cr^{2s}\,\|g\|_{H^s(\mathcal{X})}^2
$$

to establish (5). The previous inequality is established in Lemma 4 for all $r > 0$ sufficiently small, and we have therefore proved the desired estimate over the boundary.

1.1.2 Interior region

To show the desired bounds on $\mathcal{X}_{qr}$ when $s \geq 1$, we reason by induction on q.

Base case. In the base case $q = 1$, meaning $D_k g$ is only a single-difference operator. Since $s \geq 1$, replacing g by its Taylor expansion inside the first order expected difference operator $\mathbb{E}[D_k g(x)]$ yields

$$
\mathbb{E}\Big[D_k g(x)\Big] = \sum_{1\leq|\alpha|<s}\mathbb{E}_\alpha(x)\cdot g^{(\alpha)}(x) + \sum_{|\alpha|=s}\Big(\mathbb{E}_\alpha\big[G_\alpha\big]\Big)(x) \tag{9}
$$

When $s = 1$ only the second term in the previous expression is non-zero, and we therefore begin by analyzing this term, obtaining that for each $|\alpha| = s$,

$$
\begin{aligned}
\Big\|\Big(\mathbb{E}_\alpha\big[G_\alpha\big]\Big)\Big\|_{\mathcal{L}^2(\mathcal{X}_r)}^2 &= \Big\|\int(y-\cdot)^\alpha G_\alpha(\cdot,y)K_r(y,\cdot)p(y)\,dy\Big\|_{\mathcal{L}^2(\mathcal{X}_r)}^2 \\
&\leq r^{2s}p_{\max}^2K_{\max}^2\Big\|\int_{B(0,1)}G_\alpha(\cdot,zr+\cdot)\,dy\Big\|_{\mathcal{L}^2(\mathcal{X}_r)}^2 \\
&\leq r^{2s}p_{\max}^2K_{\max}^2\nu_d\int_{B(0,1)}\Big\|G_\alpha(\cdot,zr+\cdot)\Big\|_{\mathcal{L}^2(\mathcal{X}_r)}^2\,dz \\
&\leq r^{2s}p_{\max}^2K_{\max}^2\nu_d^2\,\Big\|g^{(\alpha)}\Big\|_{\mathcal{L}^2(\mathcal{X})}^2 \\
&\leq r^{2s}p_{\max}^2K_{\max}^2\nu_d^2\,\|g\|_{H^s(\mathcal{X})}^2\,;
\end{aligned}
$$

hence (6) follows when $q = 1, s = 1$.

When $s \geq 2$ we must analyze $\mathbb{E}_\alpha(x) \cdot g^{(\alpha)}(x)$, which we do by using the Taylor expansion of p. Since $B(x,r) \subset \mathcal{X}$, we recall that $\mathbb{I}_{\alpha+\beta,x} = r^{|\alpha|+|\beta|} I_{\alpha,\beta}$; thus

$$\begin{aligned}
\mathbb{E}_\alpha(x) &= \int (y-x)^\alpha K_r(y,x) p(y)\, dy \\
&= \sum_{|\beta|=0}^{s-2} p^{(\beta)}(x) \mathbb{I}_{\alpha+\beta,x} + \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[P_\beta\big]\Big)(x) \\
&= \sum_{|\beta|=0}^{s-2} p^{(\beta)}(x) r^{|\alpha|+|\beta|} I_{\alpha+\beta} + \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[P_\beta\big]\Big)(x).
\end{aligned}$$

Replacing $\mathbb{E}_\alpha(x)$ by this expansion in (9) gives

$$\mathbb{E}\Big[D_k g(x)\Big] = \sum_{|\alpha|=1}^{s-1} \sum_{|\beta|=0}^{s-2} r^{|\alpha|+|\beta|} I_{\alpha+\beta} g^{(\alpha)}(x) p^{(\beta)}(x) + \sum_{|\alpha|=1}^{s} \sum_{|\beta|=s-1} g^{(\alpha)}(x) \Big(\mathbb{I}_{\alpha+\beta}\big[P_\beta\big]\Big)(x) + \sum_{|\alpha|=s} \Big(\mathbb{E}_\alpha\big[G_\alpha\big]\Big)(x)$$

We now divide the sum in the first term based on the size of $|\alpha| + |\beta|$. The critical fact is that $I_{\alpha+\beta} = 0$ when $|\alpha| + |\beta| = 1$. When $s = 2$ this leaves

$$\mathbb{E}\Big[D_k g(x)\Big] = \sum_{|\alpha|=1}^{s} \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[P_\beta\big]\Big)(x) g^{(\alpha)}(x) + \sum_{|\alpha|=s} \Big(\mathbb{E}_\alpha\big[G_\alpha\big]\Big)(x).$$

We have already shown that the second term belongs to $\mathcal{L}^2(\mathcal{X})$, and provided an appropriate upper bound on its norm. The first term is similarly upper bounded, since for each term inside the sum

$$\left\|\mathbb{I}_{\alpha+\beta}\big[P_\beta\big] g^{(\alpha)}\right\|_{\mathcal{L}^2(\mathcal{X})}^2 \leq r^{2(|\alpha|+|\beta|)} p_{\max}^2 \left\|g^{(\alpha)}\right\|_{\mathcal{L}^2(\mathcal{X})}^2 \leq r^{2(|\alpha|+|\beta|)} p_{\max}^2 \|g\|_{H^s(\mathcal{X})}^2$$

thus establishing (4) when $s = 2$. Otherwise when $s > 2$, we rearrange

$$\sum_{|\alpha|=1}^{s} \sum_{|\beta|=0}^{s-2} r^{|\alpha|+|\beta|} I_{\alpha+\beta} g^{(\alpha)}(x) p^{(\beta)}(x) = \sum_{\ell=2}^{s-1} r^\ell \underbrace{\left\{\sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} g^{(\alpha)}(x) p^{(\beta)}(x)\right\}}_{:= g_{\ell,1}(x)} + \sum_{\ell=s+1}^{2s-2} r^\ell \sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} g^{(\alpha)}(x) p^{(\beta)}(x).$$

and therefore

$$\begin{aligned}
&\mathbb{E}\Big[D_k g(x)\Big] - \sum_{\ell=2}^{s-1} r^\ell g_{\ell,1}(x) = \\
&\quad \sum_{\ell=s+1}^{2s-2} r^\ell \sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} g^{(\alpha)}(x) p^{(\beta)}(x) + \sum_{|\alpha|=1}^{s} \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[P_\beta\big]\Big)(x) g^{(\alpha)}(x) + \sum_{|\alpha|=s} \Big(\mathbb{E}_\alpha\big[G_\alpha\big]\Big)(x)
\end{aligned}$$

On the left hand side, taking $\ell = |\alpha| + |\beta|$, note that for each $\ell < s$ the function $g_{\ell,1} \in C_c^{s-\ell}(\mathcal{X}) \subset H_0^{s-\ell}(\mathcal{X})$ and further

$$\|g_{\ell,1}\|_{H^{s-\ell}(\mathcal{X})} \leq c p_{\max} \|g\|_{H^s(\mathcal{X})}. \tag{10}$$

The right hand side consists of three terms, and we have already obtained sufficient estimates on the second and third term, so it remains to deal with the first term. We have that $g^{(\alpha)} \cdot p^{(\beta)} \in C_c^0(\mathcal{X}) \subset \mathcal{L}_0^2(\mathcal{X})$ and

$$\left\|g^{(\alpha)} p^{(\beta)}\right\|_{\mathcal{L}^2(\mathcal{X})} \leq p_{\max} \|g\|_{H^s(\mathcal{X})}, \tag{11}$$

establishing (3) when $s > 2$.

Induction step. We now assume that (6)-(8) hold with respect to g for all $k \in (n)^q$, and show the desired estimates on $\mathbb{E}[D_j D_k g]$ for all $(kj) \in (n)^{q+1}$.

We first consider the case where $s \le 2q$. Then,

$$\left\|\mathbb{E}\big[D_j D_k g\big]\right\|^2_{\mathcal{L}^2(X_{(q+1)r})} \le 2p_{\max}^2 K_{\max}^2 \nu_d^2 \left\|D_k g\right\|^2_{\mathcal{L}^2(X_{qr})} \le cr^{2s} \left\|g\right\|_{H^s(\mathcal{X})}$$

where the final inequality follows by hypothesis, and gives the desired estimate.

Otherwise $s \ge 2q+1$. We make use of the inductive hypothesis through the following three facts:

1. There exist functions $g_{2q,q}, \dots, g_{s-1,q}$ satisfying (8) such that

$$\left\|\mathbb{E}\big[D_k g\big] - \sum_{\ell=2q}^{s-1} r^\ell g_{\ell,q}\right\|_{\mathcal{L}^2(X_{qr})} \le cr^s \left\|g\right\|_{H^s(\mathcal{X})}$$

2. The functions $f_{\ell,q}$ belong to $H_0^{s-\ell}(\mathcal{X})$. Thus by hypothesis, when $\ell = s-1$ or $\ell = s-2$,

$$\left\|\mathbb{E}\big[D_j g_{\ell,q}\big]\right\|_{\mathcal{L}^2(\mathcal{X}_r)} \le cr^s \left\|f\right\|_{H^s(\mathcal{X})}.$$

3. Otherwise if $s - \ell > 2$, there exist further functions $g_{\ell,m,q}$ for $m = 2, \dots, s-\ell-1$ such that

$$\left\|\mathbb{E}\big[D_j f\big] - \sum_{m=2}^{s-\ell-1} r^m g_{\ell,m,q}\right\|_{\mathcal{L}^2(\mathcal{X}_r)} \le cr^s \left\|g\right\|_{H^s(\mathcal{X})}$$

   The functions $g_{\ell,m,q} \in H_0^{s-(\ell+m)}(\mathcal{X})$ additionally satisfy

$$\left\|g_{\ell,m,q}\right\|_{H^{s-(\ell+m)}(\mathcal{X})} \le c\left\|g_\ell\right\|_{H^{s-\ell}(\mathcal{X})} \le c\left\|g\right\|_{H^s(\mathcal{X})}.$$

Making use of the law of iterated expectation and the Fact 1, we have

$$\begin{aligned}
\mathbb{E}\Big[D_j D_k g(x)\Big] &= \mathbb{E}\Big[\Big(\mathbb{E}\big[D_k g(x_j)|x_j\big] - \mathbb{E}\big[D_k g(x)\big]\Big) K_r(x_j, x)\Big] \\
&= \sum_{\ell=2q}^{s-1} r^\ell \mathbb{E}\big[D_j g_\ell(x)\big] + \mathbb{E}\Big[\Big(\mathbb{E}\big[D_k g\big](x_j) - \sum_{\ell=2q}^{s-1} r^\ell g_\ell(x_j)\Big) K_r(x_j, x)\Big] + \mathbb{E}\big[D_k g\big](x) - \sum_{\ell=2q}^{s-1} r^\ell g_\ell(x).
\end{aligned} \tag{12}$$

The above expansion consists of three terms. By Fact 1, the third term has bounded norm

$$\left\|\mathbb{E}\big[D_k g\big] - \sum_{\ell=2q}^{s-1} r^\ell g_\ell\right\|_{\mathcal{L}^2(\mathcal{X}_{(q+1)r})} \le \left\|\mathbb{E}\big[D_k g\big] - \sum_{\ell=2q}^{s-1} r^\ell g_\ell\right\|_{\mathcal{L}^2(\mathcal{X}_{(q)r})} \le cr^s \left\|g\right\|_{H^s(\mathcal{X})}$$

By Fact 1 and Lemma 5, the same estimate holds with respect to the second term (up to constants).

When $s = (2q+1)$ or $s = (2q+2)$, by Fact 2

$$\left\|\sum_{\ell=2q}^{s-1} r^\ell \mathbb{E}\big[D_j g_\ell\big]\right\|_{\mathcal{L}^2(\mathcal{X}_r)} \le cr^s \left\|g\right\|_{H^s(\mathcal{X})}$$

and the desired result (6) follows from the triangle inequality. Finally when $s > (2q+2)$, by using Facts 2 and 3 we obtain

$$\left\| \sum_{\ell=2q}^{s-1} r^{\ell} \mathbb{E}\big[D_j g_\ell\big] - \sum_{\ell=2q}^{s-3} \sum_{m=2}^{s-\ell-1} r^{\ell+m} g_{\ell,m,q} \right\|_{\mathcal{L}^2(X_r)} \leq c r^s \|g\|_{H^s(X)}$$

Rewriting the double sum as a single sum over $\ell + m = 2q, \ldots, s-1$ and plugging back in to (12) gives the desired result (8).

1.2 Proof of Lemma 1

The proof of Lemma 1 is centered on taking Taylor expansions of the function f and the density p. When $s \geq 1$, since $f \in C_c^s(\mathcal{X})$ it admits a Taylor expansion of the form

$$f(y) = \sum_{|\alpha|=0}^{s-1} f^{(\alpha)}(x)(z-x)^{\alpha} + \sum_{|\alpha|=s} (y-x)^{\alpha} F_{\alpha,x}(y),$$

and when $s \geq 2$, since $p \in C_c^{s-1}(\mathcal{X})$ it also admits a Taylor expansion,

$$p(y) = \sum_{|\beta|=0}^{s-2} p^{\beta}(x)(y-x)^{\beta} + \sum_{|\beta|=s} (x-y)^s G_{\beta}(x,y).$$

In both cases x and y are arbitrary points in $\mathcal{X}$, and we use the integral form of the remainders:

$$F_{\alpha}(x,y) = \int_0^1 f^{(\alpha)}(x + t(y-x))\,dt, \qquad \Big|F_{\alpha}(x,y)\Big| \leq \|f\|_{C^s(\mathcal{X})}$$
$$G_{\beta}(x,y) = \int_0^1 p^{(\beta)}(x + t(y-x))\,dt, \qquad \Big|G_{\beta}(x,y)\Big| \leq \|p\|_{C^{s-1}(\mathcal{X})}$$

where by Rademacher's Theorem $f^{(\alpha)}$ and $p^{(\beta)}$ exist almost everywhere, and the preceding integrals are therefore well defined.

It will also be helpful to introduce some notation. For $F : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, let

$$\Big(\mathbb{E}_{\alpha}[F]\Big)(x) := \int (y-x)^{\alpha} F(x,y) K_r(y,x) p(y)\,dy, \qquad \mathbb{E}_{\alpha,x} := \Big(\mathbb{E}_{\alpha}[1]\Big)(x)$$
$$\Big(\mathbb{I}_{\alpha}[F]\Big)(x) := \int (y-x)^{\alpha} F(x,y) K_r(y,x)\,dy, \qquad \mathbb{I}_{\alpha,x} := \Big(\mathbb{I}_{\alpha}[1]\Big)(x)$$

Additionally we define

$$I_{\alpha} := \int z^{\alpha} K\big(\|z\|\big)\,dz.$$

and note that when $B(x,r) \subset \mathcal{X}$, the following two facts are true: first, that $\mathbb{I}_{\alpha,x} = r^{|\alpha|} I_{\alpha}$, and second that $I_{\alpha} = 0$ when $|\alpha| = 1$.

The case when $s = 0$ follows from the boundedness of f, as a simple inductive argument yields

$$\Big|\mathbb{E}\Big[D_k f(x)\Big]\Big| \leq 2^q p_{\max}^q \|f\|_{C^0(\mathcal{X})}.$$

for any $x \in \mathcal{X}$.

For $s \geq 1$, we first show the desired estimate when $x \in \mathcal{X} \setminus \mathcal{X}_{qr}$.

1.2.1 Boundary point.

Since $f \in C_c^s(\mathcal{X})$, there exists an open set V compactly contained $V \subset \bar{V} \subset U$ such that $\mathrm{supp}(f) \subset V$. Furthermore, since $\mathrm{dist}(x, \partial\mathcal{X}) < r$ there exists a particular x_0 in $\mathcal{X} \setminus V$ such that $\|x_0 - x\| \leq r$. Since there exists some $\delta > 0$ such that $B(x_0, \delta) \subset \mathcal{X} \setminus V$, we know

$$f^{(\beta)}(x_0) = 0, \quad \text{for all } \beta$$

and as a result $f(x)$ must itself be quite small,

$$\begin{aligned}
\big|f(x)\big| &= \left|\sum_{|\alpha|=0}^{s-1} f^{(\alpha)}(x_0)(x - x_0)^\alpha + \sum_{|\alpha|=s} (x - x_0)^\alpha F_\alpha(x_0, x)\right| \\
&= \left|\sum_{|\alpha|=s} (x - x_0)^\alpha F_\alpha(x_0, x)\right| \\
&\leq cr^s \left|F_\alpha(x_0, x)\right| \leq cr^s \|f\|_{C^s(\mathcal{X})}.
\end{aligned}$$

Observe $\mathbb{E}\Big[D_k f(x)\Big] = 0$ unless $x_j \in B(x, qr)$ for each $j \in k$, which implies $x_j \in \mathcal{X} \setminus \mathcal{X}_{2qr}$. Thus,

$$\left|\mathbb{E}\Big[D_k f(x)\Big]\right| \leq 2^q p_{\max}^q \max_{\widetilde{x} \in x_k, x} \big\{|f(\widetilde{x})|\big\} \leq cr^s \|f\|_{C^s(\mathcal{X})},$$

establishing (4) for $x \in \mathcal{X} \setminus \mathcal{X}_{qr}$.

1.2.2 Interior point.

To show the desired result when $x \in \mathcal{X}_{qr}$ and $s \geq 1$, we use induction on q.

Base case. In the base case $q = 1$, meaning $D_k f$ is only a single-difference operator. Since $s \geq 1$, replacing f by its Taylor expansion inside the first order expected difference operator $\mathbb{E}[D_k f(x)]$ yields

$$\mathbb{E}\Big[D_k f(x)\Big] = \sum_{1 \leq |\alpha| < s} \mathbb{E}_{\alpha,x} \cdot f^{(\alpha)}(x) + \sum_{|\alpha|=s} \Big(\mathbb{E}_\alpha\big[F_\alpha\big]\Big)(x) \tag{13}$$

When $s = 1$ only the second term in the previous expression is non-zero, and we therefore begin by analyzing this term, obtaining that for any $x \in \mathcal{X}$ and each $|\alpha| = s$,

$$\begin{aligned}
\left|\Big(\mathbb{E}_\alpha\big[F_\alpha\big]\Big)(x)\right| &= \left|\int (y - x)^\alpha F_\alpha(x, y) K_r(y, x) p(y)\, dy\right| \\
&\leq r^s p_{\max} \left|F_\alpha(x, y)\right| \\
&\leq r^s p_{\max} \|f\|_{C^s(\mathcal{X})}.
\end{aligned} \tag{14}$$

Therefore $\mathbb{E}_\alpha\big[F_\alpha\big] \in C^0(\mathcal{X})$ additionally satisfies (4), and the claim of Lemma 1 is established for $q = 1$ and $s = 1$.

When $s \geq 2$ we must analyze $\mathbb{E}_{\alpha,x} \cdot f^{(\alpha)}(x)$, which we do by using the Taylor expansion of p. Since $B(x, r) \subset \mathcal{X}$, we recall that $\mathbb{I}_{\alpha+\beta,x} = r^{|\alpha|+|\beta|} I_{\alpha,\beta}$; thus

$$\begin{aligned}
\mathbb{E}_{\alpha,x} &= \int (y - x)^\alpha K_r(y, x) p(y)\, dy \\
&= \sum_{|\beta|=0}^{s-2} p^{(\beta)}(x) \mathbb{I}_{\alpha+\beta,x} + \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[G_\beta\big]\Big)(x) \\
&= \sum_{|\beta|=0}^{s-2} p^{(\beta)}(x) r^{|\alpha|+|\beta|} I_{\alpha+\beta} + \sum_{|\beta|=s-1} \Big(\mathbb{I}_{\alpha+\beta}\big[G_\beta\big]\Big)(x).
\end{aligned}$$

Replacing $\mathbb{E}_{\alpha,x}$ by this expansion in (13) gives

$$\mathbb{E}\Big[D_k f(x)\Big] = \sum_{|\alpha|=1}^{s-1}\sum_{|\beta|=0}^{s-2} r^{|\alpha|+|\beta|} I_{\alpha+\beta} f^{(\alpha)}(x) p^{(\beta)}(x) + \sum_{|\alpha|=1}^{s}\sum_{|\beta|=s-1}\Big(\mathbb{I}_{\alpha+\beta}\big[G_\beta\big]\Big)(x) f^{(\alpha)}(x) + \sum_{|\alpha|=s}\Big(\mathbb{E}_\alpha\big[F_\alpha\big]\Big)(x)$$

We now divide the sum in the first term based on the size of $|\alpha|+|\beta|$. The critical fact is that $I_{\alpha+\beta}=0$ when $|\alpha|+|\beta|=1$. When $s=2$ this leaves

$$\mathbb{E}\Big[D_k f(x)\Big] = \sum_{|\alpha|=1}^{s}\sum_{|\beta|=s-1}\Big(\mathbb{I}_{\alpha+\beta}\big[G_\beta\big]\Big)(x) f^{(\alpha)}(x) + \sum_{|\alpha|=s}\Big(\mathbb{E}_\alpha\big[F_\alpha\big]\Big)(x).$$

In (14) we have already shown that the second term belongs to $C^0(\mathcal{X})$, and provided an appropriate upper bound on its norm. Similar analysis shows that when $1 \le |\alpha| \le s$ and $|\alpha|+|\beta| \ge s$,

$$\begin{aligned}\Big\|\mathbb{I}_{\alpha+\beta}\big[G_\beta\big] f^{(\alpha)}\Big\|_{C^0(\mathcal{X})} &\le r^{|\alpha|+|\beta|} p_{\max}\Big\|f^{(\alpha)}\Big\|_{C^0(\mathcal{X})}\\ &\le r^{|\alpha|+|\beta|} p_{\max}\|f\|_{C^s(\mathcal{X})}\\ &\le r^s p_{\max}\|f\|_{C^s(\mathcal{X})},\end{aligned} \tag{15}$$

taking care of the first term, and establishing (4) when $s=2$. Otherwise when $s>2$, we rearrange

$$\sum_{|\alpha|=1}^{s}\sum_{|\beta|=0}^{s-2} r^{|\alpha|+|\beta|} I_{\alpha+\beta} f^{(\alpha)}(x) p^{(\beta)}(x) = \sum_{\ell=2}^{s-1} r^\ell \underbrace{\Bigg\{\sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} f^{(\alpha)}(x) p^{(\beta)}(x)\Bigg\}}_{:=f_{\ell,1}(x)} + \sum_{\ell=s+1}^{2s-2} r^\ell \sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} f^{(\alpha)}(x) p^{(\beta)}(x).$$

and therefore

$$\begin{aligned}&\left|\mathbb{E}\Big[D_k f(x)\Big] - \sum_{\ell=2}^{s-1} r^\ell f_{\ell,1}(x)\right| = \\ &\sum_{\ell=s+1}^{2s-2} r^\ell \sum_{|\alpha|+|\beta|=\ell} I_{\alpha+\beta} f^{(\alpha)}(x) p^{(\beta)}(x) + \sum_{|\alpha|=1}^{s}\sum_{|\beta|=s-1}\Big(\mathbb{I}_{\alpha+\beta}\big[G_\beta\big]\Big)(x) f^{(\alpha)}(x) + \sum_{|\alpha|=s}\Big(\mathbb{E}_\alpha\big[F_\alpha\big]\Big)(x)\end{aligned}$$

On the left hand side, taking $\ell = |\alpha|+|\beta|$, note that for each $\ell < s$ the function $f_{\ell,1} \in C_c^{s-\ell}(\mathcal{X})$ and further

$$\|f_{\ell,1}\|_{C^{s-\ell}(\mathcal{X})} \le c p_{\max}\|f\|_{C^s(\mathcal{X})}. \tag{16}$$

The right hand side consists of three terms, and we have already obtained sufficient estimates on the second and third term in (14) and (15), respectively. It remains to deal with the first term. We have that $f^{(\alpha)}\cdot p^{(\beta)} \in C_c^0(\mathcal{X})$ and

$$\Big\|f^{(\alpha)} p^{(\beta)}\Big\|_{C^0(\mathcal{X})} \le p_{\max}\|f\|_{C^s(\mathcal{X})}, \tag{17}$$

establishing (3) when $s>3$.

Induction Step. We now assume that (3) and (4) hold for all $k \in (n)^q$ and $x \in \mathcal{X}_{qr}$, and prove the desired estimates hold with respect to $\mathbb{E}[D_j D_k f(x)]$ for all $(kj) \in (n)^{q+1}$ and $x \in \mathcal{X}_{(q+1)r}$.

We first consider the case where $s \le 2q$. Here,

$$\left|\mathbb{E}\Big[D_j D_k f(x)\Big]\right| \le 2p_{\max}\sup_{x\in\mathcal{X}}\Big\{\mathbb{E}[D_k f(x)]\Big\} \le c r^s\|f\|_{C^s(\mathcal{X})}$$

–with the last inequality following by the inductive hypothesis–and (4) is established.

Otherwise $s \geq 2q+1$. Our strategy will be to apply the inductive hypothesis twice, once to expand the qth order operator D_k and once to expand the first order operator D_j. To achieve the former, we first note that by the law of iterated expectation

$$\mathbb{E}\Big[D_j D_k f(x)\Big] = \mathbb{E}\Big[\Big(\mathbb{E}\big[D_k f(x_j)|x_j\big] - \mathbb{E}\big[D_k f(x)\big]\Big) K_r(x_j, x)\Big]. \tag{18}$$

We now apply the inductive hypothesis to both $\mathbb{E}\Big[D_k f(x)\Big]$ and $\mathbb{E}\Big[D_k f(x_j)|x_j\Big]$, which states that

$$\left|\mathbb{E}\Big[D_k f(x)\Big] - \sum_{\ell=2q}^{s-1} r^\ell f_{\ell,q}(x)\right|, \left|K_r(x_j,x)\left(\mathbb{E}\Big[D_k f(x_j)|x_j\Big] - \sum_{\ell=2q}^{s-1} r^\ell f_{\ell,q}(x_j)\right)\right| \leq c r^s \|f\|_{C^s(\mathcal{X})}.$$

We emphasize that we may imply the inductive hypothesis in the latter case since $x \in \mathcal{X}_{(q+1)r}$ implies either $K_r(x_j, x) = 0$, or $x_j \in \mathcal{X}_{qr}$. Plugging back in to (18) gives

$$\left|\mathbb{E}\Big[D_j D_k f(x)\Big] - \sum_{\ell=2q}^{s-1} r^\ell \mathbb{E}\Big[D_j f_{\ell,q}(x)\Big]\right| \leq c r^s \|f\|_{C^s(\mathcal{X})}. \tag{19}$$

Next, we note that $f_{\ell,q} \in C_c^{s-\ell}(\mathcal{X})$, for each $\ell = 2q, \ldots, s-1$. We now apply the inductive hypothesis again, this time to $\mathbb{E}[D_j f_{\ell,q}(x)]$. If $s - \ell \leq 2$,

$$\left|\mathbb{E}\Big[D_j f_{\ell,q}(x)\Big]\right| \leq c r^{s-\ell} \|f_{\ell,q}\|_{C^{s-\ell}(\mathcal{X})} \leq c r^{s-\ell} \|f\|_{C^s(\mathcal{X})}.$$

Therefore when $s = 2q+1$ or $s = 2q+2$, by (19)

$$\left|\mathbb{E}\Big[D_j D_k f(x)\Big]\right| \leq \sum_{\ell=2q}^{s-1} r^\ell \left|\mathbb{E}\Big[D_j f_{\ell,q}(x)\Big]\right| + c r^s \|f\|_{C^s(\mathcal{X})} \leq c r^s \|f\|_{C^s(\mathcal{X})}$$

completing the proof of (4).

On the other hand if $s - \ell > 2$, there exist functions $\widetilde{f}_{\ell+\widetilde{\ell},q} \in C_c^{s-\ell-\widetilde{\ell}}(\mathcal{X})$ satisfying $\left\|\widetilde{f}_{\ell+\widetilde{\ell},q}\right\|_{C^{s-\ell}(\mathcal{X})} \leq c \|f_{\ell,q}\|_{C^s(\mathcal{X})}$ such that

$$\left|\mathbb{E}\Big[D_j f_{\ell,q}(x)\Big] - \sum_{\ell=2}^{s-\ell-1} r^{\widetilde{\ell}} f_{\ell+\widetilde{\ell},q}(x)\right| \leq c r^{(s-\ell-\widetilde{\ell})} \|f_{\ell,q}\|_{C^{s-\ell}(\mathcal{X})} \leq c r^{(s-\ell-\widetilde{\ell})} \|f\|_{C^s(\mathcal{X})}.$$

Plugging in to (19), we have

$$\left|\mathbb{E}\Big[D_j D_k f(x) - \sum_{\ell=2q}^{s-3} \sum_{\widetilde{\ell}=2}^{s-\ell-1} \widetilde{f}_{\ell+\widetilde{\ell},q}(x) r^{\ell+\widetilde{\ell}}\Big]\right| \leq c r^s \|f\|_{C^s(\mathcal{X})}$$

and rewriting the sum inside the absolute value over $\ell + \widetilde{\ell} = 2q+2, \ldots, s-1$, we have completed the proof of (3).

2 Additional Results

To show equation (5) in Lemma 2, it suffices to prove an equivalent bound on the $\mathcal{L}^2$ norm $\|f\|_{\mathcal{L}^2(U_{2qr})}$. We show this in the case where the domain is the positive halfspace in $\mathbb{R}^d$, that is $\mathcal{X} = R^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$.

Lemma 3. *Suppose $g \in C^s_c(\mathbb{R}^d_+) \cap H^s(\mathbb{R}^d_+)$. Then*

$$\|g\|_{L^2(U_r)} \leq 2r^s \|g\|_{H^s(\mathbb{R}^d_+)} \tag{20}$$

Proof. We immediately take advantage of the axis-oriented structure of $\mathbb{R}^d_+$. Write $x \in U_{2qr}$ as $x = (x', te_d)$, where $x' \in \mathbb{R}^{d-1} \cap \{x_d = 0\}$ lies on the plane at $x_d = 0$, and $0 \leq |t| \leq r$. Since g is compactly supported on $\mathbb{R}^d_+$, by taking a Taylor expansion we obtain

$$\begin{aligned} g(x) &= \sum_{\ell=0}^{s-1} \frac{\partial^\ell}{\partial e_d^\ell} g(x') t^\ell + \int_0^t \frac{\partial^s}{\partial e_d^s} g(x' + he_d) t^{s-1}\, dh \\ &= \int_0^t \frac{\partial^s}{\partial e_d^s} g(x' + he_d) h^{s-1}\, dh \end{aligned}$$

We then bound the squared $\mathcal{L}^2$ norm of g using Hardy's inequality,

$$\begin{aligned} \int_{\mathcal{X}} \big(g(x)\big)^2\, dx &= \int_{\mathbb{R}^{d-1}} \int_0^r \left(\int_0^t \frac{\partial^s}{\partial e_d^s} g(x' + he_d) h^{s-1}\, dh \right)^2 dt\, dx' \\ &\leq r^{2s} \int_{\mathbb{R}^{d-1}} \int_0^r \left(\frac{1}{t} \int_0^t \frac{\partial^s}{\partial e_d^s} g(x' + he_d)\, dh \right)^2 dt\, dx' \\ &\leq 4r^{2s} \int_{\mathbb{R}^{d-1}} \int_0^\infty \left(\frac{\partial^s}{\partial e_d^s} g(x' + te_d) \right)^2 dt\, dx' \\ &\leq 4r^{2s} \|g\|^2_{H^s(\mathbb{R}^d_+)}. \end{aligned}$$

□

Now we show a similar result when $\mathcal{X} \subset \mathbb{R}^d$ is assumed to be a bounded open set with Lipschitz boundary. The proof is slightly more complicated, but the key ideas are the same.

Lemma 4. *Let $\mathcal{X} \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. For any $g \in C^\infty_c(\mathcal{X})$, we have that*

$$\|g\|^2_{\mathcal{L}^2(U_r)} \leq cr^s \|g\|_{H^s(\mathcal{X})}$$

for all $r > 0$ sufficiently small.

Proof. Fix $x_0 \in \partial\mathcal{X}$, and let $Q_d(x_0, r)$ be the d-dimensional cube centered at x_0 of side length r. We will show that for all sufficiently small $r > 0$,

$$\|g\|_{\mathcal{X} \cap \mathcal{L}^2(Q_d(x_0,r))} \leq cr^s \|g\|_{H^s(Q_d(x_0,r))} \tag{21}$$

The Lemma then follows by taking a finite covering of ∂X – possible since X is assumed to be bounded – in a similar manner to e.g. Theorem 18.1 of (Leoni) or Theorem 1 in 5.5 of (Evans).

We begin by straightening the boundary. Since $\mathcal{X}$ has a Lipschitz boundary, for any $x_0 \in \partial X$ there exists a rigid motion $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ with $T(x_0) = 0$, a Lipschitz continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}^d$, and a radius $r'_0 > 0$ such that, setting $y = \Phi(x)$ and writing $y = (y', y_d)$, we have that for all $r < r_0$,

$$\Phi(\mathcal{X} \cap Q(x_0, r)) = \{y \in Q(0, r) : y_d > \gamma(y')\}.$$

Fixing $0 < r < r_0/[\mathrm{Lip}(\gamma)\sqrt{d}]$, we have that $\gamma(y') > -r$ for all $y' \in Q_{d-1}(x_0, r)$; writing $\widetilde{g}(y) = g(T^{-1}(y))$, taking a Taylor expansion of $\widetilde{g}(y)$ around $\widetilde{g}((y', \gamma(y')))$ thus yields

$$\begin{aligned}
\widetilde{g}(y) &= \sum_{\ell=1}^{s-1} g^{(\ell e_d)}\big((y', \gamma(y'))\big)(y_d - \gamma(y'))^\ell + \int_{\gamma(y')}^{y_d} \widetilde{g}^{(se_d)}\big((y', he_d)\big) h^{s-1}\, dh \\
&= \int_{\gamma(y')}^{y_d} \widetilde{g}^{(se_d)}\big((y', he_d)\big) h^{s-1}\, dh
\end{aligned}$$

where the second equality follows from the assumption $g \in C_c^\infty(\mathcal{X})$. We now analyze the $\mathcal{L}^2$ norm over $Q_d(x_0, r)$,

$$\begin{aligned}
\|g\|^2_{\mathcal{X} \cap \mathcal{L}^2(Q_d(x_0,r))} &= \|\widetilde{g}\|^2_{\mathcal{L}^2(\Phi(\mathcal{X} \cap Q(0,r)))} \\
&= \int_{Q_{d-1}(0,r)} \int_{\gamma(y')}^{r} \left[\int_{\gamma(y')}^{y_d} \widetilde{g}^{(se_d)}\big((y', he_d)\big) h^{s-1}\, dh\right]^2 dy_d\, dy' \\
&\overset{(i)}{\leq} (2r)^{2(s-1)} \int_{Q_{d-1}(0,r)} \int_{\gamma(y')}^{r} \big(y_d - \gamma(y')\big)^2 \left[\frac{1}{y_d - \gamma(y')} \int_{\gamma(y')}^{y_d} \widetilde{g}^{(se_d)}\big((y', he_d)\big)\, dh\right]^2 dy_d\, dy' \\
&\overset{(ii)}{\leq} (2r)^{2(s-1)} \int_{Q_{d-1}(0,r)} \int_{\gamma(y')}^{r} \big(y_d - \gamma(y')\big) \int_{\gamma(y')}^{y_d} \left[\widetilde{g}^{(se_d)}\big((y', he_d)\big)\right]^2 dh\, dy_d\, dy' \\
&\overset{(iii)}{\leq} (2r)^{2s-1} \int_{Q_{d-1}(0,r)} \int_{\gamma(y')}^{r} \int_{\gamma(y')}^{r} \left[\widetilde{g}^{(se_d)}\big((y', he_d)\big)\right]^2 dh\, dy_d\, dy' \\
&\leq (2r)^{2s} \int_{Q_{d-1}(0,r)} \int_{\gamma(y')}^{r} \left[\widetilde{g}^{(se_d)}\big((y', he_d)\big)\right]^2 dh\, dy' \\
&\overset{(iv)}{\leq} (2r)^{2s} \int_{Q(x_0,r)} \left[g^{(se_d)}(x)\right]^2 dx \leq (2r)^{2s} \|g\|^2_{H^s(B(x_0,r))}
\end{aligned}$$

where (i) follows since $0 < y_d - \gamma(y') < r - \gamma(y') < 2r$, (ii) follows by Jensen's inequality, (iii) follows since $y_d < r$, and (iv) follows from a change of variables. This completes the proof of Lemma 4. $\square$

The following Lemma helps us deal with remainder terms in the Sobolev case.

Lemma 5. *Suppose $f \in \mathcal{L}^2(U_r)$ and $k \in (n)$. Then,*

$$\left\|\mathbb{E}\big[D_k f\big]\right\|_{\mathcal{L}^2(U)}, \left\|\mathbb{E}\big[f K_r(x_k, \cdot)\big]\right\|_{\mathcal{L}^2(U)} \leq c \left\|f\right\|_{\mathcal{L}^2(U_r)}$$