Notes on "Random Walks and an $O^\star(n^5)$ Volume Algorithm for Convex Bodies"

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Let x, y be two points with $|x - y| \leq \frac{\delta}{\sqrt{n}}$, and set

$$C = (x + B') \cap (y + B'). \qquad (B' = \delta B)$$

Consider the 'moons'

$$M_x = (x + B') \setminus (y + B'),\ M_y = (y + B') \setminus (x + B')$$

and set

$$R_x = M_x \cap (x - y + C),\ R_y = M_y \cap (y - x + C).$$

Finally, let C' be obtained by blowing up $\frac{1}{2}(x + y)$ from its center by a factor of $1 + \frac{4n}{n-1}$.

Lemma 1.

$$M_x \setminus R_x \subseteq C'$$

Lemma 2. *For every convex body K containing x and y,*

$$\mathrm{vol}(K \cap (M_x \setminus R_x)) \leq (e - 1)\mathrm{vol}(K \cap C)$$

Proof. Without loss of generality let $x - y = 0$, so that $C' = (1 + 4n/(n - 1))C$. By Lemma 1, we have

$$K \cap (M_x \setminus R_x) \subseteq K \cap C'$$

Then, since K is convex, $0 \in K$. So if $z \in K$, then αz is in K for any $0 \leq \alpha \leq 1$. As a result, we have that $(K \cap C') \subseteq (1 + 4n/(n - 1))(K \cap C)$. Thus,

$$\mathrm{vol}(K \cap (M_x \setminus R_x)) \leq \mathrm{vol}(K \cap C') \leq (1 + \frac{4n}{n-1})^n \mathrm{vol}(K \cap C) \leq (e - 1)\mathrm{vol}(K \cap C).$$

$\square$

Lemma 3. *For every convex body K,*

$$\mathrm{vol}(K \cap C)^2 \geq \mathrm{vol}(K \cap R_x)\mathrm{vol}(K \cap R_y)$$

Proof. By the Brunn-Minkowski Theorem,

$$g(u) = \mathrm{vol}((u + C) \cap K)$$

is log-concave. Therefore

$$g(0) \geq g(x - y)g(y - x)$$

and the statement follows. $\square$

Lemma 4. *For every convex body K containing x and y,*

$$\mathrm{vol}(K \cap C) \geq \frac{1}{e+1} \min \{\mathrm{vol}((x + B') \cap K), \mathrm{vol}((y + B') \cap K)\}$$

Proof. We have that

$$\mathrm{vol}(K \cap R_x) + \mathrm{vol}(K \cap M_x \setminus R_x) = \mathrm{vol}(K \cap M_x)$$

and so by Lemma 2,

$$\mathrm{vol}(K \cap R_x) \geq \mathrm{vol}(K \cap M_x) - (e-1)\mathrm{vol}(K \cap C) = \mathrm{vol}((x + B') \cap K) - e(\mathrm{vol}(K \cap C))$$

with a corresponding lower bound holding for $\mathrm{vol}(K \cap R_y)$. We then apply Lemma 3 to obtain

$$\mathrm{vol}(K \cap C) \geq \min \{\mathrm{vol}((x + B') \cap K), \mathrm{vol}((y + B') \cap K)\} - e(\mathrm{vol}(K \cap C))$$

from which the desired result is apparent. $\square$