Notes for the week of 3/20/19 - 3/27/19

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For a given $\sigma > 0$ and some $\mathcal{C} \subset \mathbb{R}^d$, let $\mathcal{C}_\sigma = \mathcal{C} + B(0,\sigma)$ be the σ-expansion of $\mathcal{C}$. Fix $r > 0$. Let ν be the Lebesgue measure over Euclidean space $\mathbb{R}^d$, and $B(x,r)$ be a ball of radius r centered at x. Consider the *speedy r-ball walk*[1] over $\mathcal{C}_\sigma \subset \mathbb{R}^d$, defined by the following transition probability density function

$$\widetilde{P}_{\nu,r}(x;\mathcal{S}) := \frac{\nu(\mathcal{S} \cap B(x,r))}{\nu(\mathcal{C}_\sigma \cap B(x,r))} \qquad (x \in \mathcal{C}_\sigma, \mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

where $\mathfrak{B}(\mathcal{C}_\sigma)$ is the Borel σ-algebra of $\mathcal{C}_\sigma$.

Denote the stationary distribution for this Markov chain by $\pi_{\nu,r}$, which satisfies the relation [2]

$$\int_{\mathcal{C}_\sigma} \widetilde{P}_{\nu,r}(x;\mathcal{S}) d\pi_{\nu,r}(x) = \pi_{\nu,r}(\mathcal{S}). \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

Letting the *local conductance* be given by

$$\ell_{\nu,r}(x) := \frac{\nu(\mathcal{C}_\sigma \cap B(x,r))}{\nu(B(x,r))} \qquad (x \in \mathcal{C}_\sigma)$$

a bit of algebra verifies that

$$\pi_{\nu,r}(\mathcal{S}) = \frac{\int_{\mathcal{S}} \ell_{\nu,r}(x)}{\int_{\mathcal{C}_\sigma} \ell_{\nu,r}(x)}. \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

We next introduce the *ergodic flow*, $\widetilde{Q}_{\nu,r}$. Let $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{C}_\sigma$ be a partition of $\mathcal{C}_\sigma$. Then the ergodic flow between $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by

$$\widetilde{Q}_{\nu,r}(\mathcal{S}_1,\mathcal{S}_2) := \int_{\mathcal{S}_1} \widetilde{P}_{\nu,r}(x;\mathcal{S}_2) d\pi_{\nu,r}(x) \qquad (\mathcal{S}_1,\mathcal{S}_2 \in \mathfrak{B}(\mathcal{C}_\sigma))$$

and the *(continuous) conductance profile* is

$$\widetilde{\Phi}_{\nu,r}(t) := \min_{\substack{\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma) \\ 0 < \pi_{\nu,r}(\mathcal{S}) \leq t}} \frac{\widetilde{Q}_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\nu,r}(\mathcal{S})} \qquad (0 < t \leq 1/2)$$

1 Conductance over $\mathcal{C}_\sigma$

An essential step in upper bounding the mixing time over $G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})]$ is lower bounding the conductance profile $\widetilde{\Phi}_{\nu,r}(t)$.

[1] We call it 'speedy' because it only considers moves within $\mathcal{C}_\sigma$.

[2] See Section 4 for verification. In order to ensure a unique stationary distribution, we could consider only the *lazy* version of the ball walk. For the moment we ignore this technicality.

As in [2], we will assume $\mathcal{C}_\sigma$ is the image of some convex set K under a measure preserving, lipschitz function g.

Assumption 1 (Embedding)**.** *Assume there exists $K \subset \mathbb{R}^d$ convex space, mapping $g : K \to \mathcal{C}_\sigma$, and constant $L < \infty$ such that*

$$\forall x, y \in K, |g(x) - g(y)| \leq L |x - y|, \text{ and } \det(D_x g) = 1.$$

In other words, g is measure-preserving and L-Lipschitz.

Theorem 1. *Assume $\mathcal{C}_\sigma \subset \mathbb{R}^d$ satisfies Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and Lipschitz function g with Lipschitz constant $L < \infty$. Then, for any $0 < r < 2\sigma/\sqrt{d}$, the continuous conductance function of the speedy r-ball walk satisfies*

$$\widetilde{\Phi}_{\nu,r}(t) \geq \frac{r}{2^{12} D_K L \sqrt{d}}.$$

The proof of Theorem 1 naturally employs similar techniques to those in the convex setting (e.g. Theorem 5.2 in [4]), except it employs an isoperimetric inequality which holds for non-convex sets, from [1]. Let $\operatorname{dist}(\mathcal{S}, \mathcal{S}') = \inf_{x \in \mathcal{S}, y \in \mathcal{S}'} \|x - y\|$.

Lemma 1 (Isoperimetry of Lipschitz embeddings of convex sets.)**.** *Let $\Omega \subset \mathbb{R}^d$ satisfy Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and Lipschitz function g with Lipschitz constant $L < \infty$. Then, for any partition $(\Omega_1, \Omega_2, \Omega_3)$ of Ω,*

$$\nu(\Omega_3) \geq 2 \frac{\operatorname{dist}(\Omega_1, \Omega_2)}{L D_K} \min(\nu(\Omega_1), \nu(\Omega_2))$$

The ball walk may behave poorly near points of low local conductance. However, because $\mathcal{C}_\sigma$ is a σ-expanded set, for sufficiently small r all points will have high local conductance, so we do not need to worry about this problem.

Lemma 2. *Let $u \in \mathcal{C}_\sigma$. Then, for any $r < \frac{\sigma}{2\sqrt{d}}$,*

$$\ell_{\nu,r}(u) \geq \frac{6}{25}.$$

As is standard, we will also require that one-step distributions of nearby points be relatively similar.

Lemma 3 (One-step distributions)**.** *Let $u, v \in \mathcal{C}_\sigma$ be such that*

$$\|u - v\| \leq \frac{rt}{\sqrt{d}}$$

for some $0 < t < 1/8$, and further assume there exists $\ell > 0$ such that $\ell(u), \ell(v) \geq \ell$. Then,

$$\left\| \widetilde{P}_{\nu,r}(u; \cdot) - \widetilde{P}_{\nu,r}(v; \cdot) \right\|_{TV} \leq 1 + \frac{3\sqrt{3}t}{4\sqrt{2\pi}} - \ell$$

where for P, Q probabilities over measurable space $(\Omega, \mathfrak{M})$, the total variation distance between P and Q is

$$\|P - Q\|_{TV} = \sup_{A \in \mathfrak{M}} |P(A) - Q(A)|.$$

We delay proof of Lemmas 1 - 3 to subsequent sections. Armed with these lemmas, we are ready to prove our main result.

Proof of Theorem 1. Let $S_1 \cup S_2 = \mathcal{C}_\sigma$, and let $\ell = \inf_{x \in \mathcal{C}_\sigma} \ell_{\nu,r(x)}$. We will show that

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) \geq \frac{\sqrt{2\pi} r \ell^4}{24 D_K L \sqrt{d}} \min \{\pi_{\nu,r}(S_1), \pi_{\nu,r}(S_2)\}$$

Once we have shown this, Lemma 2 gives the bound $\ell \geq \frac{1}{76}$. Then, dividing both sides by $\pi_{\nu,r}(S_1)$ yields the desired result.

Now, consider the sets

$$S_1' = \left\{ x \in S_1 : \widetilde{P}_{\nu,r}(x; S_2) < \frac{\ell}{4} \right\}$$
$$S_2' = \left\{ x \in S_1 : \widetilde{P}_{\nu,r}(x; S_2) < \frac{\ell}{4} \right\}$$

and $S_3' = \mathcal{C}_\sigma \setminus S_1' \setminus S_2'$.

Suppose $\pi_{\nu,r}(S_1') < \pi_{\nu,r}(S_1)/2$. Then,

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) \geq \frac{\ell \pi_{\nu,r}(S_1)}{8}$$

Similarly, if $\pi_{\nu,r}(S_1') < \pi_{\nu,r}(S_1)/2$, then since

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) = \int_{S_2} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x)$$

a symmetric result holds.

So we can assume $\pi_{\nu,r}(S_1') \geq \pi_{\nu,r}(S_1)/2$, and likewise for S_2. Now, for every $u \in S_1', v \in S_2'$, we have that

$$\left\| \widetilde{P}_{\nu,r}(u; \cdot) - \widetilde{P}_{\nu,r}(v; \cdot) \right\|_{TV} \geq 1 - \widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_2) > 1 - \frac{\ell}{2}.$$

By Lemma 3, we therefore have

$$|u - v| \geq \frac{2\sqrt{2\pi} r \ell}{3\sqrt{3d}}.$$

and since $u \in S_1', v \in S_2'$ were arbitrary, the same inequality holds for $\operatorname{dist}(S_1', S_2')$. Therefore by Lemma 1

$$\operatorname{vol}(S_3') \geq \frac{2\sqrt{2\pi} r \ell}{3 D_K L \sqrt{3d}} \min \{\operatorname{vol}(S_1'), \operatorname{vol}(S_2')\}$$

We now prove the desired result:

$$\begin{aligned}
\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) &= \frac{1}{2} \big(\int_{S_2} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) \big) \\
&\geq \frac{\ell}{8} \pi_{\nu,r}(S_3') \\
&\geq \frac{\ell^2}{8 \nu(\mathcal{C}_\sigma)} \nu(S_3') \\
&\geq \frac{\sqrt{2} r \ell^3}{12 D_K L \sqrt{d} \nu(\mathcal{C}_\sigma)} \min \{\nu(S_1'), \nu(S_2')\} \\
&\geq \frac{\sqrt{2} r \ell^4}{12 D_K L \sqrt{d}} \min \{\pi_{\nu,r}(S_1'), \pi_{\nu,r}(S_2')\} \\
&\geq \frac{\sqrt{2} r \ell^4}{24 D_K L \sqrt{d}} \min \{\pi_{\nu,r}(S_1), \pi_{\nu,r}(S_2)\}.
\end{aligned}$$

$\square$

2 Supporting theory.

2.1 Proof of Lemma 1

The proof of Lemma 1 will hinge on the corresponding result in the convex setting, given in [3].

Theorem 2 (Isoperimetry of convex sets)**.** *Let (R_1, R_2, R_3) be a partition of a convex set $K \subset \mathbb{R}^d$. Then,*

$$\mathrm{vol}(R_3) \geq 2\frac{d(R_1, R_2)}{D_K} \min(\mathrm{vol}(R_1), \mathrm{vol}(R_2))$$

Proof of Lemma 1. For $\Omega_i, i = 1, 2, 3$, denote the preimage

$$R_i = \{x \in K : g(x) \in \Omega_i\}$$

For any $x \in R_1, y \in R_2$,

$$|x - y| \geq \frac{1}{L}\,|g(x) - g(y)| \geq \frac{1}{L}\mathrm{dist}(\Omega_1, \Omega_2).$$

Since $x \in \Omega_1$ and $y \in \Omega_2$ were arbitrary, we have

$$\mathrm{dist}(R_1, R_2) \geq \frac{1}{L}\mathrm{dist}(\Omega_1, \Omega_2).$$

By Theorem 2, therefore

$$\begin{aligned}\mathrm{vol}(R_3) &\geq 2\frac{\mathrm{dist}(R_1, R_2)}{D_K} \min\{\mathrm{vol}(R_1), \mathrm{vol}(R_2)\} \\ &\geq \frac{2}{D_K L}\mathrm{dist}(\Omega_1, \Omega_2) \min\{\mathrm{vol}(R_1), \mathrm{vol}(R_2)\}\end{aligned}$$

and by the measure-preserving property of g, this implies

$$\mathrm{vol}(\Omega_3) \geq \frac{2}{D_K L}\mathrm{dist}(\Omega_1, \Omega_2) \min\{\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2)\}.$$

□

2.2 Proof of Lemma 3

Let $S_1 \cup S_2 = \mathcal{C}_\sigma$ be an arbitrary partition of $\mathcal{C}_\sigma$. We will show that

$$\widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_1) \leq 1 + \frac{3\sqrt{3}t}{4\sqrt{2\pi}} - \ell.$$

Since this will hold for arbitrary $S_1 \in \mathfrak{B}(\mathcal{C}_\sigma)$, it will hold for the infimum over all such S_1 as well, and therefore the same lower bound will hold for $\left\|\widetilde{P}_{\nu,r}(u; \cdot) - \widetilde{P}_{\nu,r}(v; \cdot)\right\|_{TV}$.

Now, note that

$$\widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_1) = 1 - \widetilde{P}_{\nu,r}(u; S_2) - \widetilde{P}_{\nu,r}(v; S_1)$$

Let $I := B(u, r) \cap B(u, r)$. Then we have

$$\widetilde{P}_{\nu,r}(u; S_2) \geq \frac{1}{\nu(B(u, r))}\nu(S_2 \cap (B(u, r)) \geq \frac{1}{\nu(B(u, r))}\nu(S_2 \cap I)$$

with a symmetric inequality holding for $\widetilde{P}_{\nu,r}(v;S_1)$. As a result,

$$1-\widetilde{P}_{\nu,r}(u;S_2)-\widetilde{P}_{\nu,r}(v;S_1)\le 1-\frac{1}{\nu_d r^d}\nu(\mathcal{C}_\sigma\cap I) \tag{1}$$

As (1) demonstrates, the overlap of the one-step distributions is related to the volume of the intersection between $B(u,r)$ and $B(v,r)$ within $\mathcal{C}_\sigma$.

From here, some simple manipulations yield

$$\begin{aligned}\nu(\mathcal{C}_\sigma\cap I)&=\nu(I)-\nu(I\setminus\mathcal{C}_\sigma)\\&\ge\nu(I)-\max\left\{\nu\big(B(u,r)\setminus\mathcal{C}_\sigma\big),\nu\big(B(v,r)\setminus\mathcal{C}_\sigma\big)\right\}\\&\ge\nu_d r^d\left(1-\frac{3\sqrt{3}t}{4\sqrt{2\pi}}-(1-\ell)\right)=\nu_d r^d\left(\ell-\frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)\end{aligned} \tag{2}$$

where we delay proof of the last inequality until the following section. (2) along with (1) then give the desired result.

2.3 Proof of (2)

The following formula for the volume of the spherical cap, stated in terms of the incomplete beta function, is well known. We include it without proof.

Lemma 4. *Let* $\mathrm{Cap}_r(h)$ *denote a spherical cap of radius* r *and height* h*. Then,*

$$\nu\big(\mathrm{Cap}_r(h)\big)=\frac{1}{2}\nu_d r^d I_{1-\alpha}\Big(\frac{d+1}{2};\frac{1}{2}\Big)$$

where

$$\alpha:=1-\frac{2rh-h^2}{r^2}$$

and

$$I_{1-\alpha}(z,w)=\frac{\Gamma(z+w)}{\Gamma(z)\Gamma(w)}\int_0^{1-\alpha}u^{z-1}(1-u)^{w-1}du.$$

is the cumulative distribution function of a $\mathrm{Beta}(z,w)$ *distribution, evaluated at* $1-\alpha$*.*

Lemma 5. *Let* $u,v\in\mathbb{R}^d$ *be points such that* $|u-v|\le t\frac{r}{\sqrt{d}}$ *for some* $0<t<1/8$*. Then,*

$$\nu(B(u,r)\cap B(v,r))\ge\nu_d r^d\left(1-\frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)$$

(2) follows immediately from Lemma 5 along with the definition of ℓ in Lemma 3. To prove Lemma 5, we will rely on the following result, which will also be useful to lower bound the local conductance.

Lemma 6. *For any* $0\le t\le 1$ *and* $\alpha\le\frac{t^2}{4d}$*,*

$$\int_0^{1-\alpha}u^{(d-1)/2}(1-u)^{-1/2}du\ge\frac{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{d}{2}+1\big)}-\frac{3t}{4\sqrt{d}}$$

Proof. We can write

$$\int_0^{1-\alpha}u^{(d-1)/2}(1-u)^{-1/2}du=\int_0^{1}u^{(d-1)/2}(1-u)^{-1/2}du-\int_{1-\alpha}^{1}u^{(d-1)/2}(1-u)^{-1/2}du$$

The first integral is simply the beta function, with

$$B(\frac{d+1}{2},\frac{1}{2}) := \frac{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{d}{2}+1\big)}.$$

To upper bound the second integral, we apply the Taylor theorem with remainder to $(1-u)^{-1/2}$, obtaining

$$(1-u)^{-1/2} \le \alpha^{-1/2} + \max_{u\in(1-\alpha,1)} \frac{\alpha}{2}(1-u)^{-3/2} = \frac{3}{2}\alpha^{-1/2}.$$

As a result,

$$\begin{aligned}\int_{1-\alpha}^{1} u^{(d-1)/2}(1-u)^{-1/2}du &\le \frac{3}{2(d+1)}\alpha^{-1/2}\int_{1-\alpha}^{1} u^{(d-1)/2}du \\ &= \frac{3}{2(d+1)}\alpha^{-1/2}\left(1-(1-\alpha)^{(d+1)/2}\right) \\ &\le \frac{3}{2(d+1)}\alpha^{-1/2}(\alpha(d+1)) = \frac{3}{2}\alpha^{1/2}.\end{aligned}$$

and the result follows from the condition $\alpha \le \frac{t^2}{2d}$. □

Proof of Lemma 5. We will treat only the case where $|u-v| = t\frac{r}{\sqrt{d}}$; if they are closer together the overlap of the volume will only increase. Then, it is not hard to see that $I = B(u,r)\cap B(v,r)$ is comprised of the union of two disjoint spherical caps, and thus

$$\nu(I) = 2\nu(\mathrm{Cap}_r(r(1-\frac{t}{2\sqrt{d}}))).$$

From Lemma 4 we therefore obtain

$$\nu(I) = \nu_d r^d I_{1-\alpha}(\frac{d+1}{2};\frac{1}{2})$$

where

$$\alpha = 1 - \frac{2r^2(1-\frac{t}{2\sqrt{d}}) - r^2(1-\frac{t}{2\sqrt{d}})^2}{r^2} = \frac{t^2}{4d}.$$

Expanding the incomplete beta function in integral form, we therefore have

$$\begin{aligned}\nu(I) &= \nu_d r^d \frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}\int_0^{1-\alpha} u^{(d-1)/2}(1-u)^{-1/2}du \\ &\overset{(i)}{\ge} \nu_d r^d\left(1 - \frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}\frac{3t}{4\sqrt{d}}\right) \\ &\overset{(ii)}{\ge} \nu_d r^d\left(1-\frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)\end{aligned}$$

where (i) follows from Lemma 6 (which we can validly apply since $\alpha \le \frac{t^2}{2d}$), and (ii) from Lemma 7.

□

2.4 Proof of Lemma 2

Proof. Since $u \in \mathcal{C}_\sigma$ there exists $x \in \mathcal{C}$ such that $u \in B(x, \sigma)$, and as a result

$$\nu\big(B(u,r) \cap \mathcal{C}_\sigma\big) \geq \nu\big(B(u,r) \cap B(x,\sigma)\big)$$

Without loss of generality, let $|u - x| = \sigma$; it is not hard to see that if $|u - x| < \sigma$, the volume of the overlap will only grow. Then, since $|u - x| = \sigma$, $B(u,r) \cap B(x,\sigma)$ contains a spherical cap of radius r and height

$$h = r - (r)^2/2\sigma = r\left(1 - \frac{r}{2\sigma}\right)$$

which by Lemma 4 has volume

$$\nu_{cap} = \frac{1}{2}\nu_d r^d I_{1-\alpha}\left(\frac{d+1}{2}, \frac{1}{2}\right)$$

with $\alpha = 1 - \frac{2rh - h^2}{r^2} = \frac{r^2}{4\sigma^2} \leq \frac{1}{4d}$.

Then by Lemmas 6 (applied with $t = 1$) and 7,

$$\begin{aligned} I_{1-\alpha}\left(\frac{d+1}{2}, \frac{1}{2}\right) &\geq 1 - \frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \frac{3}{4\sqrt{d}} \\ &\geq 1 - \frac{3}{4}\sqrt{\frac{d+2}{2\pi d}} \geq 1 - \frac{3}{4}\sqrt{\frac{3}{2\pi}}. \end{aligned}$$

□

3 Additional Lemmas

Lemma 7 follows from $\Gamma(1/2) = \sqrt{\pi}$ and the upper bound $\Gamma(x+1)/\Gamma(x+s) \leq (x+1)^{1-s}$ for $s \in [0,1]$ (Gautschi's inequality).

Lemma 7.

$$\frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \leq \sqrt{\frac{d+2}{2\pi}}$$

4 Stationary distribution.

For completeness, we verify that $\pi_{\nu,r}$ is in fact a stationary distribution of the chain given by $\widetilde{P}_{\nu,r}$. Note that

$$\frac{d\pi_{\nu,r}(x)}{dx} \propto \ell_{\nu,r}(x).$$

Let $\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma)$. Then,

$$\begin{aligned}
\int_{\mathcal{C}_\sigma} \widetilde{P}_{\nu,r}(x;\mathcal{S}) d\pi_{\nu,r}(x) &\propto \int_{\mathcal{C}_\sigma} \widetilde{P}_{\nu,r}(x;\mathcal{S}) \ell_{\nu,r}(x) dx \\
&= \int_{\mathcal{C}_\sigma} \frac{\nu(\mathcal{S} \cap B(x,r))}{\nu(B(x,r))} dx \\
&= \int_{\mathcal{C}_\sigma} \int_{\mathcal{S}} \frac{\mathbf{1}(\|x-x'\| \le r)}{\nu(B(x,r))} dx dx' \\
&= \int_{\mathcal{S}} \int_{\mathcal{C}_\sigma} \frac{\mathbf{1}(\|x-x'\| \le r)}{\nu(B(x,r))} dx dx' \\
&= \int_{\mathcal{S}} \ell_{\nu,r}(x) dx
\end{aligned}$$

Since π is a probability, we know the normalized constant must be $1/\int_{\mathcal{C}_\sigma} \ell_{\nu,r}(x)dx$.

5 Notation

- For a set $K \subset \mathbb{R}^d$, $D_K = \max_{x,y\in K} |x-y|$, where $|x-y|$ is the Euclidean norm between of $x-y \in \mathbb{R}^d$.
- ν_d is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^d$.
- $D_x g = (D_{x_i} g_j)_{i,j=1}^d$ is the Jacobian matrix of g evaluated at x.
- $g(K) = \{y \in \mathbb{R}^d : g(x) = y \text{ for some } x \in K\}$ is the image of K under g.
- For measures P, Q over $(\Sigma, \mathcal{F})$, $\|P-Q\|_{TV} = \sup_{A\in\mathcal{F}} |P(A) - Q(A)|$.

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