A Proofs

In this supplement, we present proofs for "Local Clustering of Density Upper Level Sets". Sections A.1 - A.3 detail the proof for Theorem 1. A.4 develops a bound of the form of (15), which we recall links the conductance function to mixing time; this will be necessary for both Theorems 2 and 3. A.5 and A.6 give the proof of Theorem 2, while A.7- A.10 give the proof of Theorem 3.

A.1 Volume estimates

We will fix $\mathcal{A} \subset \mathbb{R}^d$ to be an arbitrary set. To simplify expressions, for the σ-expansion $\mathcal{A}_\sigma$, we will write the set difference between $\mathcal{A}_\sigma$ and the $(\sigma + r)$-expansion $\mathcal{A}_{\sigma+r}$ as

$$\mathcal{A}_{\sigma,\sigma+r} := \{x : 0 < \mathrm{dist}(x, \mathcal{A}_\sigma) \leq r\},$$

where as a reminder $\mathrm{dist}(x, \mathcal{A}) = \min_{x' \in \mathcal{A}} \|x - x'\|$.

For notational ease, we write

$$\mathrm{cut}_{n,r} = \mathrm{cut}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}),\ \mu_K = \mathbb{E}(\mathrm{cut}_{n,r}),\ p_K = \frac{\mu_K}{\binom{n}{2}}$$
$$\mathrm{vol}_{n,r} = \mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}),\ \mu_V = \mathbb{E}(\mathrm{vol}_{n,r}),\ p_V = \frac{\mu_V}{\binom{n}{2}}$$

for the random variable, mean, and probability of cut size and volume, respectively.

We state Lemma 1 without proof, as it is trivial. We formally include it mainly to comment on its (potential) suboptimality; for sets $\mathcal{A}$ with diameter much larger than σ, the volume estimate of Lemma 1 will be quite poor.

Lemma 1. *For any $\delta > 0$ and $x \in \mathcal{A}_\sigma$,*

$$\sigma B \subset \mathcal{A}_\sigma, \quad \text{and } \nu(\mathcal{A}_\sigma + \delta B) \leq \nu\left(\left[1 + \frac{\delta}{\sigma}\right]\mathcal{A}_\sigma\right) = \left(1 + \frac{\delta}{\sigma}\right)^d \nu(\mathcal{A}_\sigma)$$

where $\mathcal{A}_\sigma = \mathcal{A} + \sigma B$ is the σ-expansion of $\mathcal{A}$.

We will need to carefully control the volume of the expansion set using the above estimate; Lemma 2 serves this purpose.

Lemma 2. *For any $0 \leq x \leq 1/2d$,*

$$(1 + x)^d \leq 1 + 2dx.$$

Proof. We take the binomial expansion of $(1+x)^d$:

$$\begin{aligned}(1+x)^d &= \sum_{k=0}^{d} \binom{d}{k} x^k \\ &= 1 + dx + dx \left(\sum_{k=2}^{d} \frac{\binom{d}{k} x^{k-1}}{d} \right) \\ &\leq 1 + dx + dx \left(\sum_{k=2}^{d} \frac{d^k}{(2d)^{k-1} d} \right) \\ &\leq 1 + 2dx.\end{aligned}$$

□

We will repeatedly employ Lemma 1 and Lemma 2 in tandem. As a first example, in Lemma 3, we use it to bound the ratio of $\nu(\mathcal{A}_\sigma)$ to $\nu(\mathcal{A}_{\sigma-r})$. This will be useful when we bound $\mathrm{vol}(\mathcal{C}_\sigma)$.

Lemma 3. *For σ, $\mathcal{A}_\sigma$ as in Lemma 1, let $r > 0$ satisfy $r \leq \sigma/4d$. Then,*

$$\frac{\nu(\mathcal{A}_\sigma)}{\nu(\mathcal{A}_{\sigma-r})} \leq 2.$$

Proof. Fix $q = \sigma - r$. Then,

$$\begin{aligned}\nu(\mathcal{A}_\sigma) &= \nu(\mathcal{A}_{q+\sigma-q}) = \nu(\mathcal{A}_q + (\sigma - q)B) \\ &\leq \nu(\mathcal{A}_q + \frac{(\sigma-q)}{q}\mathcal{A}_q) = \left(1 + \frac{\sigma - q}{q}\right)^d \nu(\mathcal{A}_q)\end{aligned}$$

where the inequality follows from Lemma 1. Of course, $\sigma - q = r$, and $\frac{r}{q} \leq \frac{1}{2d}$ for $r \leq \frac{1}{4d}$. The claim then follows from Lemma 2. □

A.2 Density-weighted cut and volume estimates

Lemma 4. *Under the conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) \leq 2\nu(\mathcal{C}_\sigma)\frac{rd}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. Recalling that f is the density function for $\mathbb{P}$, we have

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) = \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx \tag{A.1}$$

We partition $\mathcal{C}_{\sigma,\sigma+r}$ into slices, based on distance from $\mathcal{C}_\sigma$, as follows: for $k \in \mathbb{N}$,

$$\mathcal{T}_{i,k} = \left\{ x \in \mathbb{R}^d : t_{i,k} < \frac{\mathrm{dist}(x, \mathcal{C}_\sigma)}{r} \leq t_{i+1,k} \right\}, \quad \mathcal{C}_{\sigma,\sigma+r} = \bigcup_{i=0}^{k-1} \mathcal{T}_{i,k}$$

where $t_i = i/k$ for $i = 0, \ldots, k-1$. As a result,

$$\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx = \sum_{i=0}^{k-1} \int_{\mathcal{T}_{i,k}} f(x)dx \leq \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x).$$

We substitute

$$\nu(\mathcal{T}_{i,k}) = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B) - \nu(\mathcal{C}_\sigma + rt_{i,k}B) := \nu_{i+1,k} - \nu_{i,k}.$$

where for simplicity we've written $\nu_{i,k} = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B)$. This, in concert with the upper bound

$$\max_{x \in \mathcal{T}_{i,k}} f(x) \leq \lambda_\sigma - (rt_{i,k})^\gamma,$$

which follows from (A3) and (A4), yields

$$\begin{aligned} \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x) &\leq \sum_{i=0}^{k-1} \left\{ \nu_{i+1,k} - \nu_{i,k} \right\} \left(\lambda_\sigma - (rt_{i,k})^\gamma \right) \\ &= \sum_{i=1}^{k} \underbrace{\nu_{i,k} \left([\lambda_\sigma - (rt_{i,k})^\gamma] - [\lambda_\sigma - (rt_{i-1,k})^\gamma] \right)}_{:=\Sigma_k} + \underbrace{\left(\nu_{k,k}\,[\lambda_\sigma - r^\gamma] - \nu_{1,k}\lambda_\sigma \right)}_{:=\xi_k} \end{aligned} \tag{A.2}$$

We first consider the term Σ_k. Here we use Lemma 1 to upper bound

$$\nu_{i,k} \leq \mathrm{vol}(\mathcal{C}_\sigma) \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d$$

and so we can in turn upper bound Σ_k:

$$\Sigma_k \leq \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \sum_{i=1}^{k} \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d \left((t_{i-1,k})^\gamma - (t_{i,k})^\gamma \right). \tag{A.3}$$

This, of course, is a Riemann sum, and as the inequality holds for all values of k it holds in the limit as well, which we compute to be

$$\begin{aligned} \lim_{k \to \infty} \sum_{i=1}^{k} \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d \left((t_{i-1,k})^\gamma - (t_{i,k})^\gamma \right) &= \gamma \int_0^1 \left(1 + \frac{rt}{\sigma} \right)^d t^{\gamma-1} dt \\ &\overset{(i)}{\leq} \gamma \int_0^1 \left(1 + \frac{2drt}{\sigma} \right) t^{\gamma-1} dt = \left(1 + \frac{\gamma 2dr}{\gamma+1} \right). \end{aligned}$$

where (i) follows from Lemma 2. We plug this estimate in to (A.3) and obtain

$$\lim_{k\to\infty} \Sigma_k \leq \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \left(1 + \frac{\gamma 2dr}{\gamma+1}\right).$$

We now provide an upper bound on ξ_k. It will follow the same basic steps as the bound on Σ_k, but will not involve integration:

$$\begin{aligned}\xi_k &\overset{(ii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1+\frac{r}{\sigma}\right)^d(\lambda - r^\gamma) - \lambda\right\} \\ &\overset{(iii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1+\frac{2dr}{\sigma}\right)(\lambda - r^\gamma) - \lambda\right\} = \nu(\mathcal{C}_\sigma)\left\{\frac{2dr}{\sigma}(\lambda - r^\gamma) - r^\gamma\right\}.\end{aligned}$$

where (ii) follows from Lemma 1 and (iii) from Lemma 2. The final result comes from adding together the upper bounds on Σ_k and ξ_k and taking the limit as $k \to \infty$. $\square$

Lemma 5. *Under the setup and conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$p_K \leq \frac{4\lambda\nu_d r^{d+1}\nu(\mathcal{C}_\sigma)d}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. We can write $\mathrm{cut}_{n,r}$ as the sum of indicator functions,

$$\mathrm{cut}_{n,r} = \sum_{i=1}^n \sum_{j=1}^n \mathbf{1}(x_i \in \mathcal{C}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i, r) \cap \mathcal{C}_\sigma) \tag{A.4}$$

and by linearity of expectation, we can obtain

$$p_K = \frac{\mu_K}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \in \mathcal{C}_{\sigma,\sigma+r}, x_j \in B(x_i, r) \cap \mathcal{C}_\sigma)$$

Writing this with respect to the density function f, we have

$$\begin{aligned}p_K &= 2\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)\left\{\int_{B(x,r)\cap\mathcal{C}_\sigma} f(x')dx'\right\}dx \\ &\leq 2\nu_d r^d \lambda \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx\end{aligned}$$

where the inequality follows from Assumption (A1), which implies that the density function $f(x') \leq \lambda$ for all $x' \in \mathcal{C}_\sigma \setminus \mathcal{C}$ (otherwise, x' would be in some $\mathcal{C}' \in \mathbb{C}_f(\lambda)$, which (A1) forbids). Then, upper bounding the integral using Lemma 5 gives the final result. $\square$

Lemma 6. *Under the setup and conditions of Theorem 1,*

$$p_V \geq \lambda_\sigma^2 \nu_d r^d \nu(\mathcal{C}_\sigma)$$

Proof. The proof will proceed similarly to Lemma 5. We begin by writing $\mathrm{vol}_{n,r}$ as the sum of indicator functions,

$$\mathrm{vol}_{n,r} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{1}(x_i \in \mathcal{C}_\sigma)\mathbf{1}(x_j \in B(x_i, r)) \tag{A.5}$$

and by linearity of expectation we obtain

$$p_V = \frac{\mu_V}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \in \mathcal{C}_\sigma, x_j \in B(x_i, r)).$$

Writing this with respect to the density function f, we have

$$\begin{aligned}
p_V &= 2 \int_{\mathcal{C}_\sigma} f(x) \left\{ \int_{B(x,r)} f(x')dx' \right\} dx \\
&\geq 2 \int_{\mathcal{C}_{\sigma-r}} f(x) \left\{ \int_{B(x,r)} f(x')dx' \right\} dx \\
&\overset{(i)}{\geq} 2\lambda_\sigma^2 \nu_d r^d \int_{\mathcal{C}_{\sigma-r}} f(x)dx
\end{aligned}$$

where (i) follows from the fact that $B(x, r) \subset \mathcal{C}_\sigma$ for all $x \in C_{\sigma-r}$, along with the lower bound in Assumption (A3). The claim then follows from Lemma 3. $\square$

We now convert from bounds on p_K and p_V to probabilistic bounds on $\mathrm{cut}_{n,r}$ and $\mathrm{vol}_{n,r}$ in Lemmas 7 and 8. The key ingredient will be Lemma 23, Hoeffding's inequality for U-statistics; the proofs for both are nearly identical and we give only a proof for Lemma 7.

Lemma 7. *The following statement holds for any $\delta \in (0, 1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_K + \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.6}$$

with probability at least $1 - \delta$.

Lemma 8. *The following statement holds for any $\delta \in (0, 1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} \geq p_V - \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.7}$$

with probability at least $1 - \delta$.

Proof of Lemma 7. From (A.4), we see that $\mathrm{cut}_{n,r}$, properly scaled, can be expressed as an order-2 U-statistic,

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_K(x_i, x_j)$$

where

$$\phi_K(x_i, x_j) = \mathbf{1}(x_i \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i, r) \cap \mathcal{A}_\sigma) + \mathbf{1}(x_j \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_i \in B(x_j, r) \cap \mathcal{A}_\sigma).$$

From Lemma 23 we therefore have

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_k + \sqrt{\frac{\log(1/\delta)}{n}}$$

with probability at least $1 - \delta$. □

A.3 Proof of Theorem 1

The proof of Theorem 1 is more or less given by Lemmas 5, 6, 7, and 8. All that remains is some algebra, which we take care of below.

Fix $\delta \in (0, 1]$ and let $\delta' = \delta/2$. Noting that $\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{\mathrm{cut}_{n,r}}{\mathrm{vol}_{n,r}}$, some trivial algebra gives us the expression

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{p_K + \left(\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} - p_K\right)}{p_V + \left(\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} - p_V\right)} \tag{A.8}$$

We assume (A.6) and (A.7) hold with respect to δ', keeping in mind that this will happen with probability at least $1 - \delta$. Along with (A.8) this means

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) \leq \frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n}$$

for $\mathrm{Err}_n = \sqrt{\frac{\log(1/\delta')}{n}}$. Now, some straightforward algebraic manipulations yield

$$\begin{aligned}
\frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n} &= \frac{p_K}{p_V} + \left(\frac{p_K}{p_V - \mathrm{Err}_n} - \frac{p_K}{p_V}\right) + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \\
&= \frac{p_k}{p_V} + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}\left(\frac{p_K}{p_V} + 1\right) \\
&\leq \frac{p_K}{p_V} + 2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}.
\end{aligned}$$

By Lemmas 5 and Lemma 6, we have

$$\frac{p_K}{p_V} \leq \frac{4rd}{\sigma}\frac{\lambda}{\lambda_\sigma}\frac{\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)}{\lambda_\sigma}$$

Then, the choice of

$$n \geq \frac{9\log(2/\delta)}{\epsilon^2}\left(\frac{1}{\lambda_\sigma^2 \nu(\mathcal{C}_\sigma)\nu_d r^d}\right)^2$$

implies $2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \leq \epsilon$.

A.4 Mixing time on graphs

For $N \in \mathbb{N}$ and a set V of N vertices, take $G = (V, E)$ to be an undirected and unweighted graph, with associated adjacency matrix $\mathbf{A}$, random walk matrix $\mathbf{W}$, and stationary distribution $\boldsymbol{\pi} = (\pi_u)_{u\in V}$ where $\pi_v = \frac{\mathbf{D}_{vv}}{\mathrm{vol}(V;G)}$. For $v \in V$,

$$q_{vu}^{(m)} = e_v \mathbf{W}^m e_u, \quad \mathbf{q}_v^{(m)} = \left(q_{vu}^{(m)}\right)_{u\in V}, \quad \mathbf{q}_v = (\mathbf{q}_{v\cdot}^{(1)}, \mathbf{q}_{v\cdot}^{(2)}, \ldots), \tag{A.9}$$

denote respectively the m-step transition probability, distribution, and sequence distributions of the random walk over G originating at v. Letting $\mathbf{q} = (\mathbf{q}_v)_{v\in V}$, the relative pointwise mixing time is thus

$$\tau_\infty(\mathbf{q}; G) = \min\left\{m : \forall u, v \in V, \frac{\left|q_{vu}^{(m)} - \boldsymbol{\pi}_u\right|}{\boldsymbol{\pi}_u} \leq 1/4\right\}$$

Two key quantities relate the mixing time to the expansion of subsets S of V. The *local spread* is defined to be

$$s(G) := \frac{9D_{\min}}{10}\pi_{\min}$$

for $D_{\min} := \min_{v\in V} \mathbf{D}_{vv}$ and $\pi_{\min} := D_{\min}/\mathrm{vol}(V;G)$.

where $\beta(S) := \inf_{v\in S} \mathbf{q}_v^{(1)}(S^c)$, and by convention we let $\mathbf{p}(S) = \sum_{u\in S} p_u$ for any distribution vector $\mathbf{p} = (p_u)_{u\in V}$ over V. We collect some necessary facts about the local spread in Lemma 9.

Lemma 9.

- *If $\boldsymbol{\pi}(S) \leq s(G)$, then for every $u \in S$, $\mathbf{q}_u^{(1)}(S^c) \geq 1/10$.*
- *For any $v, u \in V$, and $m \in N$ greater than 0, $q_{vu}^{(m)}/\pi_{\min} \leq 1/s(G)$.*

Proof. If $t = \boldsymbol{\pi}(S) \leq \frac{9D_{\min}}{10}\pi_{\min}$, divide both sides by $\pi_{\min}$ to obtain

$$|S| \leq \frac{9D_{\min}}{10}$$

which implies $\mathbf{q}_v^{(1)}(S^c) \geq 1/10$ for all $v \in S$. This implies the first statement.

The second statement follows from the fact $q_{vu}^{(m)} \leq 1/D_{\min}$ for any m. □

The local spread facilitates conversion between $\tau_\infty(\mathbf{q}_v; G)$ and the more easily manageable *total variation* mixing time, given by

$$\tau_1(\boldsymbol{\rho}; G) = \min\left\{m : \forall v \in V, \|\boldsymbol{\rho}_v - \boldsymbol{\pi}\|_{TV} \leq 1/4\right\}$$

where

$$\boldsymbol{\rho}_v^{(m)} = \frac{1}{m}\sum_{k=1}^{m+1} \mathbf{q}_v^m, \quad \boldsymbol{\rho}_v = \left(\boldsymbol{\rho}_v^{(1)}, \boldsymbol{\rho}_v^{(2)}, \boldsymbol{\rho}_v^{(3)} \dots\right), \quad \boldsymbol{\rho} = (\boldsymbol{\rho}_v)_{v\in V} \tag{A.10}$$

and $\|\mathbf{p} - \boldsymbol{\pi}\|_{TV} = \sum_{v\in V} |p_v - \pi_v|$ is the total variation norm between distributions $\mathbf{p}$ and $\boldsymbol{\pi}$.

Lemma 10. *For* $\mathbf{q}$ *as in* (A.9) *and* $\boldsymbol{\rho}$ *as in* (A.10)*,*

$$\tau_\infty(\mathbf{q}; G) \leq 2752\tau_1(\boldsymbol{\rho}; G) \log\left(4 \max\left\{1, \frac{1}{s(G)}\right\}\right)$$

Proof. Masking dependence on the starting vertex v for the moment, let

$$\Delta_u^{(m)} = q_{vu}^{(m)} - \pi_u, \quad \delta_u^{(m)} = \frac{\Delta_u^{(m)}}{\pi_u}$$

and $\boldsymbol{\Delta}^{(m)} = (\Delta_u^{(m)})_{u\in V}$, $\boldsymbol{\delta}^{(m)} = (\delta_u^{(m)})_{u\in V}$. For a vector $\boldsymbol{\Delta} = (\Delta_u)_{u\in V}$, the $L^p(\boldsymbol{\pi})$ norm is given by

$$\|\boldsymbol{\Delta}\|_{L^p(\boldsymbol{\pi})} = \left(\sum_{u\in V} (\Delta_u)^p \pi_u\right)^{1/p}$$

To go between the $L^\infty(\boldsymbol{\pi})$ and $L^1(\boldsymbol{\pi})$ norms, we have

$$\begin{aligned}
\left\|\boldsymbol{\delta}^{(2m)}\right\|_{L^\infty(\pi)} &\overset{(i)}{\leq} \left\|\boldsymbol{\delta}^{(m)}\right\|^2_{L^2(\pi)} \\
&= \left\|(\boldsymbol{\delta}^{(m)})^2\right\|_{L^1(\pi)} \\
&\overset{(ii)}{\leq} \left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^1(\pi)} \left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)}
\end{aligned}$$

where (i) is a result of [2] and (ii) follows from Holder's inequality. Now, we upper bound the second factor on the right hand side by observing

$$\begin{aligned}
\left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)} &\leq \max\left\{1, \max_{u\in V} \frac{q_{vu}^{(m)}}{\pi_u}\right\} \\
&\overset{(iii)}{\leq} \max\left\{1, \frac{1}{s(G)}\right\}
\end{aligned}$$

where (iii) follows from Lemma 9.

Now, we leverage the following well-known fact [4]: for any $\epsilon > 0$, if $m \geq \tau_1(\mathbf{q}_v^{(m)}; G) \cdot \log(1/\epsilon)$ then

$$\left\| \mathbf{q}_v^{(m)} - \boldsymbol{\pi} \right\|_{TV} \leq \epsilon.$$

But $\left\| \mathbf{q}_v^{(m)} - \boldsymbol{\pi} \right\|_{TV}$ is exactly $\left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^1(\pi)}$. Therefore, picking

$$m_0 = \tau_1(\mathbf{q}_v^{(m)}; G) \cdot \log\left(4 \max\left\{1, \frac{1}{s(G)}\right\}\right)$$

implies $\left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^\infty(\pi)} \leq 1/4$ for all $m \geq 2m_0$. Then,

$$\left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^\infty(\pi)} = \sup_u \left\{ \frac{\left| q_{vu}^{(m)} - \boldsymbol{\pi}_u \right|}{\boldsymbol{\pi}_u} \right\}.$$

and since none of the above depended on a specific choice for v, the supremum can be taken over all starting vertices v as well. Thus $\tau_\infty(\mathbf{q}^{(m)}; G) \leq 2m_0$.

Finally, it is known [4] that

$$\tau_1(\mathbf{q}^{(m)}; G) \leq 1376 \tau_1(\boldsymbol{\rho}^{(m)}; G)$$

and so the desired result holds. □

The second key quantity is the *conductance function*

$$\Phi(t; G) := \min_{\substack{S \subseteq V, \\ \boldsymbol{\pi}(S) \leq t}} \Phi(S; G) \qquad (\pi_{\min} \leq t < 1) \tag{A.11}$$

where $\Phi(S; G)$ is the normalized cut of S in G given by (2).

Lemma 11 leverages the conductance function and local spread to produce an upper bound on the total variation distance between $\boldsymbol{\rho}_v^{(m)}$ and $\boldsymbol{\pi}$.

Lemma 11. *If $D_{\min} > 10$, for any $v \in V$:*

$$\left\| \boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi} \right\|_{TV} \leq \max\left\{ \frac{1}{4}, \frac{1}{10} + \frac{70}{m} \left(\frac{20}{9} + \int_{t=s'(G)}^{1/2} \frac{4}{t \Phi^2(t; G)} dt \right) \right\}$$

where $s'(G) = s(G)/9$.

To prove Lemma 11 we first introduce a generalization of $\Phi(t; G) \cdot \Phi(t; G)$ known as a blocking conductance function. [1]

[1] For more details, see [4]

Definition 1 (Blocking Conductance Function of [4]). *For $t_0 \geq \pi_{\min}$, a function $\phi(t; G) : [t_0, 1/2] \to [0, 1]$ is a* blocking conductance function *if for all $S \subset V$ with $\boldsymbol{\pi}(S) = t \in [t_0, 1/2]$, either of the following hold:*

1. Exterior inequality. *For all $y \in \left[\frac{1}{2}t, t\right] : \phi_{int}(S) \geq \phi(\max\{t_0, y\})$*
2. Interior inequality. *For all $y \in \left[t, \frac{3}{2}t\right] : \phi_{ext}(S) \geq \phi(\max\{y, 1-y\})$.*

where ϕ_{int} and ϕ_{ext} are defined respectively as

$$\phi_{int}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \text{cut}(S \setminus B, S^c; G)}{\text{vol}(V; G) \left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

$$\phi_{ext}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subset S^c \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \text{cut}(S \setminus B, S^c; G)}{\text{vol}(V; G) \left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

Theorem 1 (Theorem 3.2 of [4]). *Consider $\phi(t; G) : [t_0, 1/2] \to [0, 1]$ a blocking conductance function. Then, letting*

$$h^m(t_0) = \sup_{S : \boldsymbol{\pi}(S) < t_0} \left(\boldsymbol{\rho}_v^{(m)}(S) - \boldsymbol{\pi}(S)\right)$$

the following statement holds: if ϕ is a blocking conductance function,

$$\left\|\boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, h^1(t_0) + \frac{70}{m}\left(\frac{1}{\phi(t_0; G)} + \int_{t=t_0}^{1/2} \frac{4}{t\phi(t; G)} dt\right)\right\}$$

Note that in [4] this theorem is stated with respect to h^0. However, in the subsequent proof it holds with respect to h^m, and it is observed that h^m is decreasing in m. For our purposes it is more useful to state it with respect to h^1, as we have done.

Proof of Lemma 11. Consider the function $\phi_0(t, G) : [s(G), 1/2] \to [0, 1]$ defined by

$$\phi_0(t; G) = \begin{cases} \frac{1}{5}, & t = s'(G) \\ \frac{1}{4}\Phi^2(t; G), & t \in (s'(G), 1/2] \end{cases} \tag{A.12}$$

Lemma 12. *If $D_{\min} > 10$, ϕ_0 is a blocking conductance function.*

We take Lemma 12 as given, and defer the proof until after the proof of Lemma 11.

Lemma 12 and Theorem 1 together yield:

$$\left\|\boldsymbol{\rho}^t - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, h^1(s'(G)) + \frac{70}{m}\left(5 + \int_{t=s'(G)}^{1/2} \frac{4}{t\Phi^2(t; G)}\right)\right\}$$

Then, $h^1(s'(G)) \leq 1/10$ follows exactly from the proof of Lemma 9, except now $\boldsymbol{\pi}(S) \leq s'(G)$ results in the sharper bound of $\mathbf{q}_u^{(1)}(S^c) \geq 9/10$ for every $u \in S$. $\square$

Lemma 12. The condition $D_{\min} > 10$ ensures that $s(G) \geq \pi_{\min}$.

It is known that $\frac{1}{4}\Phi^2(x;G)$ satisfies the exterior inequality for all $t \in (\pi_{\min}, 1/2]$.

For $t = s'(G)$ we will instead use the interior inequality. For any S such that $\boldsymbol{\pi}(S) \leq s'(G)$, the following statement holds: for every $u \in S$, $\mathrm{cut}(u, S^c; G) \geq 9/10 \cdot \deg(u; G)$. Fixing $\lambda = \pi(S)/2$, we have

$$\begin{aligned}
\phi_{int}(S) &\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V;G)\,[\lambda(1-\lambda)]^2} \\
&\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{9\lambda \sum_{u \in S \setminus B} \deg(u;G)}{10\mathrm{vol}(V;G)\,[\lambda(1-\lambda)]^2} \\
&\geq \frac{9\lambda^2}{20[\lambda^2(1-\lambda)^2]} \geq \frac{9}{20}.
\end{aligned}$$

$\square$

A.5 Conductance function and local spread: non-convex case.

We begin with some notation. Write $\mathcal{C}_\sigma[\mathbf{X}] = \widetilde{\mathbf{X}}$, and $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$. For $S \subset \widetilde{\mathbf{X}}$, let $\widetilde{\mathrm{cut}}_{n,r}(S) = \mathrm{cut}(S; \widetilde{G}_{n,r})$ and similarly $\widetilde{\mathrm{vol}}_{n,r}(S) = \mathrm{vol}(S; \widetilde{G}_{n,r})$.

Consider $\mathbf{z} \subset \mathcal{C}_\sigma$ such that $\mathcal{N}_{\mathbf{z}} = \{B(z, r/3) : z \in \mathbf{z}\}$ is an internal covering of $\mathcal{C}_\sigma$, meaning $\mathcal{N}_{\mathbf{z}} \supseteq \mathcal{C}_\sigma$. Then, we write

$$\begin{aligned}
\widetilde{B}_{\min} &= \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\min} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right| \\
\widetilde{B}_{\max} &= \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\max} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right|
\end{aligned}$$

Both the conductance function and local spread will depend heavily on these quantities. Lemma 13 collects the bounds we will need.

Lemma 13. *Let $\mathcal{C}_\sigma$ satisfy the conditions of Theorem 2. For sufficiently large*

n, and $r \le \sigma/4d$, each of the following bounds hold with probability $1-\delta$:

$$\widetilde{B}_{\max} \le \left(1 + \sqrt{3^d \frac{3(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma$$

$$\widetilde{B}_{\min} \ge \left(1 - \sqrt{3^d \frac{2(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$

$$\widetilde{D}_{\max} \le \left(1 + \sqrt{\frac{3(\log n + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d r^d \Lambda_\sigma$$

$$\widetilde{D}_{\min} \ge \left(1 - \sqrt{\frac{2(\log n + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$

$$\left(1 - \sqrt{\frac{2\log(1/\delta)}{n\lambda_d \nu_d \sigma^d}}\right) n\lambda_\sigma \nu_d \sigma^d \le \widetilde{n} \le \left(1 + \sqrt{\frac{3\log(1/\delta)}{n\Lambda_\sigma \nu_d D^d}}\right) n\Lambda_d \nu_d D^d \tag{A.13}$$

where $\widetilde{n} = \left|\widetilde{\mathbf{X}}\right|$ and $\beta_d \ge 1/2^d$.

In particular, fix $\epsilon > 0$. Then, for n as specified in Theorem 2, the following event:

$$\widetilde{B}_{\max} \le (1+\epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma, \quad \widetilde{D}_{\max} \le (1+\epsilon)\, n\nu_d r^d \Lambda_\sigma$$

$$\widetilde{B}_{\min} \ge (1-\epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d, \quad \widetilde{D}_{\min} \ge (1-\epsilon)\, n\nu_d r^d \lambda_\sigma \beta_d$$

$$(1-\epsilon) n\lambda_\sigma \nu_d \sigma^d \le \widetilde{n} \le (1+\epsilon) n\Lambda_\sigma \nu_d D^d \tag{A.14}$$

occurs with probability at least $1-\delta$.

Proof. We observe that for any $s \le \sigma/4d$ and any $x \in \mathcal{C}_\sigma$,

$$\nu(B(x,s) \cap \mathcal{C}_\sigma) \ge \left(\frac{s}{2}\right)^d = s^d \beta_d.$$

(In fact, tighter bounds can be shown to hold, but we will not need them). Therefore, by (A3):

$$\lambda_\sigma \nu_d s^d \beta_d \le \mathbb{P}\left(B(z,s) \cap \mathcal{C}_\sigma\right) \le \Lambda_\sigma \nu_d s^d$$

In particular, this holds for $s = r$ and $s = r/3$, and for every $z \in \mathbf{z}$ as well as every $z \in \widetilde{\mathbf{X}}$. Now, by (A2) and (A3) we also have

$$\Lambda_\sigma \nu_d \sigma^d \le \mathbb{P}(\mathcal{C}_\sigma) \le \Lambda_\sigma \nu_d D^d$$

The proof of each statement in (A.13) then follows from application of Lemma 24.

To show (A.14), we note that $|\mathcal{N}_z|$ is less than the covering number of the D-ball in d dimensions. Therefore

$$|\mathcal{N}_z| \leq \left(\frac{6D}{r} + 1\right)^d.$$

It is then immediately apparent that n chosen as in Theorem 2 yields (A.14). $\square$

Now, we consider the conductance function and local spread computed over $\widetilde{G}_{n,r}$, which we refer to by

$$\widetilde{\Phi}_{n,r}(t) = \Phi(t; \widetilde{G}_{n,r}), \quad \widetilde{s}_{n,r} = s(\widetilde{G}_{n,r}).$$

where the restriction in the minimization problem of (A.11) is with respect to $\widetilde{\pi}_{n,r}$ the stationary distribution over $\widetilde{G}_{n,r}$.

We will bound $\widetilde{\Phi}_{n,r}(1/2)$ and $\widetilde{s}_{n,r}$ under the event that (A.14) holds, noting that this occurs with probability at least $1-\delta$.

Lemma 14. *If the bounds given by* (A.14) *hold, then*

$$\widetilde{s}_{n,r} \geq \frac{9}{10} \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{D^d} \beta_d^2$$

Proof. The local spread can be written as

$$\widetilde{s}_{n,r} = \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \geq \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{D}_{\max}\widetilde{n}}.$$

Then apply the relevant results of Lemma 13. $\square$

Lemma 15. *If the bounds given by* (A.14) *hold, then:*

$$\widetilde{\Phi}_{n,r}(1/2) \geq \frac{\lambda_\sigma(1-\epsilon)\beta_d}{4\Lambda_\sigma(1+\epsilon)3^d}\left(1 + \frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)$$

Proof. Fix $S \subset \widetilde{\mathbf{X}}$ with $\widetilde{\pi}_{n,r}(S) \leq 1/2$. Partition $\mathcal{N}_{\mathbf{z}} = \mathcal{N}_{\mathbf{z}}^+ \cup \mathcal{N}_{\mathbf{z}}^-$, where

$$\begin{aligned}
\mathcal{N}_{\mathbf{z}}^- &= \left\{B(z, r/3) : 2\left|B(z,r/3) \cap S\right| \leq \left|B(z,r/3) \cap \widetilde{\mathbf{X}}\right|\right\} \\
\mathcal{N}_{\mathbf{z}}^+ &= \mathcal{N}_{\mathbf{z}} \setminus \mathcal{N}_{\mathbf{z}}^-
\end{aligned}$$

and correspondingly $S^- = \mathcal{N}_{\mathbf{z}}^- \cap S$, $S^+ = \mathcal{N}_{\mathbf{z}}^+ \cap S$, so

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} = \frac{\widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) + \widetilde{\mathrm{cut}}_{n,r}(S^+; \widetilde{G}_{n,r} \setminus S)}{\widetilde{\mathrm{vol}}_{n,r}(S^-) + \widetilde{\mathrm{vol}}_{n,r}(S^+)}.$$

It is immediately apparent that the following bounds hold for all $S \subset \widetilde{\mathbf{X}}$:

$$\begin{aligned}
\widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) &\geq \frac{|S^-| \, \widetilde{B}_{\min}}{2} \\
\widetilde{\mathrm{vol}}_{n,r}(S^-) &\leq \left|S^-\right| \widetilde{D}_{\max} \\
\widetilde{\mathrm{vol}}_{n,r}(S^+) &\leq \widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \mathbf{1}(\left|N_{\mathbf{z}}^+\right| > 0)
\end{aligned}$$

If moreover $|N_{\mathbf{z}}^+| < |N_{\mathbf{z}}|$, then

$$\widetilde{\mathrm{cut}}_{n,r}(S^+; \widetilde{G}_{n,r} \setminus S) \geq \frac{\widetilde{B}_{\min}^2}{4} \mathbf{1}(\left|N_{\mathbf{z}}^+\right| > 0)$$

follows from the fact that the graph $H_{n,r} = (\mathbf{z}, E_H)$, with $(z_i, z_j) \in E_H$ if $\|z_i - z_j\| \leq r/3$, is connected. As a result, if $|N_{\mathbf{z}}^+| < |N_{\mathbf{z}}|$ we have

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} \geq \frac{\widetilde{B}_{\min}}{4\widetilde{D}_{\max}} + \frac{\widetilde{B}_{\min}^2}{8\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \tag{A.15}$$

using the inequality $2(A+B)/(C+D) \geq A/C + B/D$ for A, B, C, D non-negative.

If, on the other hand , $|\mathcal{N}_{\mathbf{z}}^+| = \mathcal{N}_{\mathbf{z}}$, then (A.15) holds with respect to S^c. Then, because $\widetilde{\pi}_{n,r}(S) \leq 1/2$,

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^c)}{\widetilde{\mathrm{vol}}_{n,r}(S^c)} \leq \frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)}$$

and so we get the exact statement of (A.15). Noting, as in the proof of Lemma 14, that $\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \leq \widetilde{n} \cdot \widetilde{D}_{\max}$, the relevant results of Lemma 13 yield the desired inequality. □

A.6 Proof of Theorem 2

Throughout this proof, we will condition on the events of Lemmas 14 and 15, namely

$$\begin{aligned}
\widetilde{s}_{n,r} &\geq \frac{9}{10} \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{D^d} \beta_d^2 \\
\widetilde{\Phi}_{n,r}(1/2) &\geq \frac{\lambda_\sigma (1-\epsilon) \beta_d}{4\Lambda_\sigma (1+\epsilon) 3^d} \left(1 + \frac{(1-\epsilon) r^d \lambda_\sigma}{(1+\epsilon) D^d \Lambda_\sigma}\right)
\end{aligned}$$

noting that for n as chosen in Theorem 3, this will occur with probability at least $1 - \delta$ (by Lemma 13).

As a reminder, we write $\mathcal{C}_\sigma[\mathbf{X}] = \widetilde{\mathbf{X}}$, and $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$. Fix arbitrary $v \in \widetilde{\mathbf{X}}$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}_{\widetilde{\mathbf{X}}}^t, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{\mathbf{q}}_n^{(2)}, \ldots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. (As we will see, this bound will hold over all such starting vertices $v \in \widetilde{\mathbf{X}}$.)

By Lemma 9,

$$\begin{aligned}\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\le 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\ &\le 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, d\log\left(\frac{8D(1+\epsilon)^2\Lambda_\sigma^2}{r(1-\epsilon)^2\lambda_\sigma^2}\right)\right\}\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 11, we have that

$$\begin{aligned}\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\le \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right) \\ &\le \frac{1400}{3}\left(5 + \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right)\end{aligned} \tag{A.16}$$

where $s_{\mathbb{P},r} = \frac{9}{10}\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\frac{\lambda_\sigma^2}{\Lambda_\sigma^2}\frac{r^d}{D^d}\beta_d^2$.(Since r remains constant, for sufficiently large n the lower bound on $\widetilde{D}_{\min}$ of Lemma 13 will be at least 10, and therefore Lemma 12 holds.)

Now, we can upper bound the average conductance integral:

$$\begin{aligned}\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt &\le \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(1/2)}dt \\ &\le 64\frac{9^d\Lambda_\sigma^2(1+\epsilon)^2\beta_d^2}{\lambda_\sigma^2(1-\epsilon)^2}\left(1 + \frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)^{-2}\log s_{\mathbb{P},r}.\end{aligned}$$

Plugging this in to (A.16) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (14).

A.7 Population-level conductance function: convex case.

When $\mathcal{C}$ is convex, we will make use of the theory developed in A.4 with respect to the conductance function $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$. First, however, we introduce a population-level analogue to $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$ over the set $\mathcal{C}_\sigma$, which we denote $\widetilde{\Phi}_{\mathbb{P},r}$. (In general, we will adopt the convention of using $\widetilde{f}$ to denote functionals computed with respect to $\mathcal{C}_\sigma$.)

For $\mathcal{S} \subset \mathbb{R}^d$

$$\nu_{\mathbb{P}}(\mathcal{S}) := \int_{\mathcal{S}} f(x)dx$$

is the weighted volume.

The r-ball walk over $\mathcal{C}_\sigma$ is a Markov chain. For $x \in \mathcal{C}_\sigma$ and $\mathcal{S}, \mathcal{S}' \subset \mathcal{C}_\sigma$, the transition probability is given by

$$\widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S} \cap B(x,r))}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r))}.$$

The stationary distribution $\pi_{\mathbb{P},r}$ thus satisfies

$$\int_{\mathcal{C}_\sigma} \widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) d\pi_{\mathbb{P},r}(x) = \pi_{\mathbb{P},r}(\mathcal{S})$$

for all $\mathcal{S} \in \mathcal{C}_\sigma$. A simple calculation yields

$$\ell_{\mathbb{P},r}(x) := \nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r)) \quad \pi_{\mathbb{P},r}(\mathcal{S}) := \frac{1}{\int_{\mathcal{C}_\sigma} f(x)\ell_{\mathbb{P},r}(x)dx} \int_{\mathcal{S}} f(x)\ell_{\mathbb{P},r}(x)dx,$$

and therefore the ergodic flow is

$$\begin{aligned}
\widetilde{Q}_{\mathbb{P},r}(\mathcal{S},\mathcal{S}') &:= \int_{\mathcal{S}} d\pi_{\mathbb{P},r}(x) P_{\mathbb{P},r}(x;\mathcal{S}')dx \\
&= \frac{1}{\int_{\mathcal{C}_\sigma} f(x)\ell_{\mathbb{P},r}(x)dx} \int_{\mathcal{S}} f(x) \left(\int_{\mathcal{S}' \cap B(x,r)} f(x')dx' \right) dx
\end{aligned}$$

The continuous conductance function is then

$$\begin{aligned}
\widetilde{\Phi}_{\mathbb{P},r}(t) &:= \min_{\substack{\mathcal{S} \subset \mathcal{C}_\sigma, \\ \pi_{\mathbb{P},r}(\mathcal{S}) \le t}} \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} \\
&= \min_{\substack{\mathcal{S} \subset \mathcal{C}_\sigma, \\ \pi_{\mathbb{P},r}(\mathcal{S}) \le t}} \frac{\int_{\mathcal{S}} f(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} f(x')dx' \right) dx}{\int_{\mathcal{S}} f(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x')dx' \right) dx}.
\end{aligned}$$

For $m > 0$ and $0 < t_0 < t_1 < \ldots < t_m < 1$, denote the *stepwise approximation to* g by $\bar{g}$, defined as

$$\bar{g}(t) = g(t_i), \quad \text{for } t \in [t_{i-1}, t_i] \tag{A.17}$$

The stepwise approximation will be important to showing the consistency results of Section A.9 can be translated to a uniform bound. Lemma 16 shows that the approximation will not overly degrade our estimates of the population-level conductance function.

Lemma 16.
- *For any function f monotone decreasing in t on the interval $[t_0, t_m]$, $\bar{f}(t) \le f(t)$ for all $t \in [t_0, t_m]$.*
- *Fix*
$$g(t) = \log\left(\frac{1}{t}\right) \ \textit{for } x \in [t_0, 1/2]$$
If for all i in $1, \ldots, m$, $(t_i - t_{i-1}) \le t_0/2$, then $\bar{g}(t) \ge g(t)/2$.

Proof. The first statement is immediately obvious, and we turn to proving the second.

The upper bound $g(t) \geq \bar{g}(t)$ follows immediately from the fact that $g(t)$ is a decreasing function along with the first statement.

By the concavity of the log function,

$$\bar{g}(t) = \log\left(\frac{1}{t_i}\right) \geq \log\left(\frac{1}{t}\right) - \frac{(t_i - t)}{t}.$$

As a result,

$$\bar{g}(t) - \frac{g(t)}{2} \geq \frac{\log\left(\frac{1}{t}\right)}{2} - \frac{(t_i - t)}{t} \geq 1/2 - 1/2 = 0.$$

□

The following theorem is found in [3]. It gives a bound population-level conductance function over convex bodies, when the density is uniform.

Theorem 2 (Restatement of [3] Theorem 4.6)**.** *Let $K \subset \mathbb{R}^d$ be a convex body of diameter D. Then for any $\mathcal{S} \subset K$ with $\pi_{\nu,r}(\mathcal{S}) \leq 1/2$,*

$$\frac{Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\nu,r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{1}{\pi_{\nu,r}(\mathcal{S})}\right)\right\}. \tag{A.18}$$

Lemma 17. *Under the conditions on $\mathcal{C}_\sigma$ given by Theorem 3, the following bounds hold:*

- *for $0 < t < 1/2$,*

$$\widetilde{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

- *Let*

$$M = \frac{2^{d+1} D^d \Lambda_\sigma^2}{r^d \lambda_\sigma^2}$$

  and $t_i = (i+1)/M$ for $i = 0, \ldots, m-1$. Then, for $1/M < t < 1/2$

$$\overline{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{162\sqrt{d}D}\mathrm{Log}\left(\frac{\Lambda_\sigma^2}{\lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

  where $\overline{\Phi}_{\mathbb{P},r}(t)$ is defined as in (A.17) with respect to $t_0, \ldots t_{M-1}$, and $\mathrm{Log}(A/t) = \max\{\log(1 + 2A), \log(A/t)\}$.

Before we prove Lemma 17, note that the choice of M is made to ensure t_0 is greater than the local spread of $G_{n,r}\left[\mathcal{C}_\sigma[\mathbf{X}]\right]$, as we will see in Section A.8.

Proof of Lemma 17. We note that

$$\pi_{\mathbb{P},r}(S) \le \pi_{\nu,r}(S) \cdot \frac{\Lambda_\sigma^2}{\lambda_\sigma^2}, \quad Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \ge Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \cdot \frac{\lambda_\sigma^2}{\Lambda_\sigma^2}$$

Plugging these estimates in to (A.18) gives

$$\frac{Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 \pi_{\mathbb{P},r}(\mathcal{S})}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

and since the right hand side is decreasing in $\pi_{\mathbb{P},r}(\mathcal{S})$, the desired lower bound holds on $\widetilde{\Phi}_{\mathbb{P},r}(t)$. The bound on $\overline{\Phi}_{\mathbb{P},r}(t)$ then follows from $\mathrm{Log}(A/t) \le \log(1+1/t)$ for all $0 < t < 1/2$ and application of Lemma 16. ☐

A.8 Consistency of local spread and conductance function: convex case.

The introduction of the stepwise approximation allows us to make use of Lemma 18, which gives us (pointwise) consistency of the discrete graph functionals $\widetilde{\Phi}_{n,r}(t)$ to the continuous functionals $\widetilde{\Phi}_{\mathbb{P},r}(t)$.

We use $\omega_r(1)$ to denote a term which goes to infinity as $r \to 0$, and likewise $o_r(1)$ to denote a term which goes to 0 as $r \to 0$.

Lemma 18. *Fix $0 < t < 1/2$. Under the conditions on $\mathcal{C}_\sigma$ given by Theorem 3, the following statement holds: with probability one, as $n \to \infty$,*

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \ge \min\left\{\widetilde{\Phi}_{\mathbb{P},r}(t), c_d r \omega_r(1)\right\} \tag{A.19}$$

where c_d is a constant which may depend on the dimension d (as well as the distribution $\mathbb{P}$), but not r.

As a consequence, for M and $(t_i)_{i=0}^{M-1}$ defined as in Lemma 17, we have that

$$\liminf_{n\to\infty} \overline{\Phi}_{n,r} \ge \min\left\{\overline{\Phi}_{\mathbb{P},r}(t), c_d r \omega_r(1)\right\} \tag{A.20}$$

We defer the proof of pointwise consistency to Section A.9. For now, we show that (A.20) is immediately implied by (A.19).

Proof of (A.20). We take as given that for any $0 < t < 1/2$,

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \ge \widetilde{\Phi}_{\mathbb{P},r}(t).$$

In particular, for sufficiently large n this will occur for each of $t_0, t_1, \ldots, t_m$ and therefore

$$\liminf_{n\to\infty} \overline{\Phi}_{n,r} \ge \overline{\Phi}_{\mathbb{P},r}$$

uniformly over $[1/m, 1/2]$. ☐

A.9 Pointwise consistency of conductance function: convex case.

We will rely heavily on results of [5], which prove the same result but consider only a pointwise result on $\widetilde{\Phi}_{n,r}(1/2)$ rather than over the entire conductance function.

Let $\widetilde{\mathbf{X}} = \mathcal{C}_\sigma[\mathbf{X}] = \{\widetilde{x}_1, \ldots, \widetilde{x}_{\widetilde{n}}\}$, and $\widetilde{n} = \left|\widetilde{\mathbf{X}}\right|$. Then

$$\widetilde{\mathbb{P}}_n := \frac{1}{\widetilde{n}} \sum_{\widetilde{x}_i \in \widetilde{\mathbf{X}}} \delta_{\widetilde{x}_i}$$

is the empirical distribution of $\widetilde{\mathbf{X}}$. Likewise, for $\mathcal{S} \subset \mathcal{C}_\sigma$ let $\widetilde{\mathbb{P}}$ be the conditional distribution $\mathbb{P}(x \in S | x \in \mathcal{C}_\sigma)$, given by

$$\widetilde{\mathbb{P}}(\mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S})}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma)}.$$

A Borel map $T : \mathcal{C}_\sigma \to \widetilde{\mathbf{X}}$ is a *transportation map* between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ if

$$\widetilde{\mathbb{P}}(\mathcal{S}) = \widetilde{\mathbb{P}}_n(T(\mathcal{S}))$$

for all $\mathcal{S} \in \mathcal{C}_\sigma$.

Lemma 19 (Proposition 5 of [5])**.** *There exists a sequence of transportation maps $(T_{\widetilde{n}})$ from $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ such that*

$$\limsup_{\widetilde{n} \to \infty} \frac{\widetilde{n}^{1/d} \left\| \mathrm{Id} - T_{\widetilde{n}} \right\|_{L^\infty(\widetilde{\mathbb{P}})}}{(\log \widetilde{n})^{p_d}} \leq C$$

where $p_d = 1/d$ for $d \geq 3$ and $3/4$ if $d = 2$.

These are referred to stagnating transportation maps. We refer the curious reader to [5] for more details.

For $S \subset \widetilde{\mathbf{X}}$, we will denote $\mathrm{vol}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{vol}}_{n,r}(S)$, and likewise $\mathrm{cut}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{cut}}_{n,r}(S)$.

Consider a sequence of sets $(S_{\widetilde{n}})_{\widetilde{n} \in \mathbb{N}}$, with $u_{\widetilde{n}} = \mathbf{1}_{S_{\widetilde{n}}}$ the characteristic function of $S_{\widetilde{n}}$. Similarly, for $\mathcal{S} \subset \mathcal{C}_\sigma$ let $u = \mathbf{1}_{\mathcal{S}}$.

Definition 2. *For a sequence $(u_{\widetilde{n}}) \in L^1(\widetilde{\mathbb{P}}_{\widetilde{n}})$ and $u \in L^1(\widetilde{\mathbb{P}})$, we say $(u_{\widetilde{n}})$ converges TL^1 to u if there exists a sequence of stagnating transportation maps $(T_{\widetilde{n}})$ such that*

$$\int_{\mathcal{C}_\sigma} |u(x) - (u_{\widetilde{n}}) \circ T_{\widetilde{n}}(x)| \, d\widetilde{\mathbb{P}}(x) \to 0$$

and denote it $u_{\widetilde{n}} \overset{TL^1}{\to} u$.

Lemma 20. *If $(u_{\widetilde{n}}) \overset{TL^1}{\to} u$, with probability one*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} \widetilde{f}(x') dx' \right) dx$$

where $\widetilde{f}$ is the density function of $\widetilde{\mathbb{P}}$ over $\mathcal{C}_\sigma$.

Proof. We note immediately that $n \to \infty$ implies $\widetilde{n} \to \infty$ with probability one. Now, we can write

$$\begin{aligned}
\frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} &= \frac{1}{\widetilde{n}^2} \sum_{i=1}^{\widetilde{n}} \sum_{j=1}^{\widetilde{n}} u_{\widetilde{n}}(\widetilde{x}_i)\big(1 - u_{\widetilde{n}}(\widetilde{x}_j)\mathbf{1}(\|\widetilde{x}_i - \widetilde{x}_j\| \le r)\big) \\
&= \int_{\mathcal{C}_\sigma} \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} u_{\widetilde{n}}(x)\big(1 - u_{\widetilde{n}}(x')\big) d\widetilde{\mathbb{P}}_n(x') \right) d\widetilde{\mathbb{P}}_n(x) \\
&= \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x).
\end{aligned}$$

Note that, for any $x \in \mathcal{C}_\sigma$,

$$\begin{aligned}
\lim_{n\to\infty} \nu\big(B(T_{\widetilde{n}}(x), r) \setminus B(x,r)\big) = 0 \\
\lim_{n\to\infty} \nu\big(B(x,r) \setminus B(T_{\widetilde{n}}(x), r)\big) = 0.
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') = \lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x').$$

An application of the bounded convergence theorem yields

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) = \\
&\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x).
\end{aligned}$$

Letting

$$\begin{aligned}
\mathcal{I}_n^1 &= \int_{\mathcal{C}_\sigma} (u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (u(x') - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) \\
\mathcal{I}_n^2 &= \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x) - u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

we have

$$\int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) =$$
$$\int_{\mathcal{C}_\sigma} u(x) \left\{ \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) d\widetilde{\mathbb{P}}(x') \right\} d\widetilde{\mathbb{P}}(x) + \mathcal{I}_n^1 + \mathcal{I}_n^2. \tag{A.21}$$

Recalling that $u = 1_{\mathcal{S}}$, we can see

$$\int_{\mathcal{C}_\sigma} u(x) \left\{ \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) d\widetilde{\mathbb{P}}(x') \right\} d\widetilde{\mathbb{P}}(x) = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} \widetilde{f}(x') dx' \right) dx \tag{A.22}$$

Since $(u_{\widetilde{n}}) \stackrel{TL^1}{\to} u$, another application of the bounded convergence theorem yields $\lim_{n\to\infty} \mathcal{I}_n^1 = \lim_{n\to\infty} \mathcal{I}_n^2 = 0$. Therefore by (A.21) and (A.22) the final result holds. □

Lemma 21. *If $u_{\widetilde{n}} \stackrel{TL^1}{\to} u$, then*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} \widetilde{f}(x') dx' \right) dx$$

with probability one.

Proof. We note that

$$\begin{aligned} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} &= \frac{1}{\widetilde{n}^2} \sum_{i=1}^{\widetilde{n}} \sum_{j=1}^{\widetilde{n}} u_{\widetilde{n}}(\widetilde{x}_i) \mathbf{1}(\|\widetilde{x}_i - \widetilde{x}_j\| \le r) - \frac{1}{\widetilde{n}} \\ &= \int_{\mathcal{C}_\sigma} \int_{\mathcal{C}_\sigma \cap B(x,r)} \left(u_{\widetilde{n}}(x) d\widetilde{\mathbb{P}}_n(x') \right) d\widetilde{\mathbb{P}}(x) - \frac{1}{\widetilde{n}} \end{aligned}$$

Of course, $\lim_{n\to\infty} \frac{1}{\widetilde{n}} = 0$, and so

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \lim_{n\to\infty} \int_{\mathcal{C}_\sigma} \left(\mathcal{C}_\sigma \cap B(x,r) u_{\widetilde{n}}(x) d\widetilde{\mathbb{P}}_n(x') \right) d_{\widetilde{\mathbb{P}}}(x)$$

The proof then proceeds analogously to Lemma 20. □

Lemma 22 can be found in [1] (Theorem 3.1) or [5] (Lemma 23).

Lemma 22. *If $u_{\widetilde{n}} \stackrel{TL^1}{\not\to} u$ for some $u \in L^1\nu$, with probability one:*

$$\liminf_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(\mathcal{S}_{\widetilde{n}})}{\widetilde{n}^2} \ge c_d r^{d+1} \omega_r(1)$$

where c_d is a constant which does not depend on r but may depend on $\mathcal{C}_\sigma$ and f, and $\omega_r(1) \to \infty$ as $r \to 0$.

Proof of (A.19). Let $(S^\star_{\widetilde{n}})$

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^\star_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S^\star_{\widetilde{n}})} = \widetilde{\Phi}_{n,r}(t), \quad \widetilde{\pi}_{n,r}(S_{\widetilde{n}}) \leq t$$

be the sequence of minimizers of the normalized cut with stationary distribution at most t in the graph $\widetilde{G}_{n,r}$. Denote $u^\star_{\widetilde{n}} = 1_{S^\star_{\widetilde{n}}}$, and assume that $u^\star_{\widetilde{n}} \overset{TL^1}{\to} u$, for $u = 1_{\mathcal{S}}$, $\mathcal{S} \subset \mathcal{C}_\sigma$. Then, by Lemmas 20 and 21,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})} &= \frac{\int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S})\cap B(x,r)} \widetilde{f}(x')dx'\right) dx}{\int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} \widetilde{f}(x')dx'\right) dx} \\
&\overset{(i)}{=} \frac{\int_{\mathcal{S}} f(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S})\cap B(x,r)} f(x')dx'\right) dx}{\int_{\mathcal{S}} f(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x')dx'\right) dx} \quad = \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})}.
\end{aligned}$$

where (i) holds because the normalization factors present in $\widetilde{f}$ cancel. From Lemma 21, we also have that $\lim_{n\to\infty} \widetilde{\pi}_{n,r}(S_{\widetilde{n}}) = \pi_{\mathbb{P},r}(\mathcal{S})$, and therefore $\pi_{\mathbb{P},r}(\mathcal{S}) \leq t$. As a result,

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) = \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} \geq \widetilde{\Phi}_{\mathbb{P},r}(t).$$

On the other hand, if $u^\star_{\widetilde{n}}$ does not converge TL^1, then

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} \geq c_d r^{d+1} \omega_r(1)$$

Additionally,

$$\frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} \leq \frac{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})}{\widetilde{n}^2}$$

and

$$\limsup_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})}{\widetilde{n}^2} \leq \nu_d r^d \Lambda_\sigma$$

As a result,

$$\liminf_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})} \geq c_d r \omega_r(1).$$

□

A.10 Proof of Theorem 3

Throughout this proof, we will refer to the subgraph $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$.

Fix arbitrary $v = x_i \in \mathcal{C}_\sigma[\mathbf{X}]$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}_{\mathcal{C}_\sigma[\mathbf{X}]}^t, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{q}_n^{(2)}, \dots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$.

By Lemmas 9 and 14,

$$\begin{aligned}\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq 2752 \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\ &\leq 2752 \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, 4d \log\left(\frac{2D\Lambda_\sigma^2}{\lambda_\sigma^2}\right)\right\}\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 11, we have that

$$\limsup_{n\to\infty} \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \leq \limsup_{n\to\infty} \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt\right) \tag{A.23}$$

(Since r remains constant, for sufficiently large n, $\mathbf{D}_{xx} > C$ will be fulfilled for any $x \in \mathcal{C}_\sigma[\mathbf{X}]$, and any $C < \infty$.) We set aside the constant term for the moment and turn to the integral. By Lemma 14,

$$\limsup_{n\to\infty} \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt \leq \limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt$$

where $s_{\mathbb{P},r}$ is as in the proof of Theorem 2. We now replace the discrete conductance function $\widetilde{\Phi}_{n,r}$ by the stepwise approximation to the continuous conductance function, $\overline{\Phi}_{n,r}$:

$$\begin{aligned}\limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt &\overset{(i)}{\leq} \limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{n,r}^2(t)} dt \\ &= \int_{s_{\mathbb{P},r}}^{1/2} \limsup_{n\to\infty} \frac{4}{t\overline{\Phi}_{n,r}^2(t)} dt \\ &\overset{(ii)}{\leq} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{\mathbb{P},r}^2(t)} dt + c_d^2 \log(s_{\mathbb{P},r}) \frac{1}{r^2} o_r(1)\end{aligned}$$

where (i) follows from Lemma 16 and (ii) from Lemma 18 (along with the continuous mapping theorem). Now, we make use of Lemma 17:

$$
\begin{aligned}
\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{\mathbb{P},r}^2(t)} dt &\leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt + \int_{s_{\mathbb{P},r}}^{1/2} \frac{81 dD^2}{r^2 t \mathrm{Log}(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt \right) \\
&\leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \underbrace{\int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt}_{:=\mathcal{J}_1} + 81 \underbrace{\int_{s_{\mathbb{P},r}}^{\lambda_\sigma^2/(4\Lambda_\sigma^2)} \frac{dD^2}{r^2 t \log(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt}_{:=\mathcal{J}_2} + 81 \underbrace{\int_{\lambda_\sigma^2/(4\Lambda_\sigma^2)}^{1/2} \frac{dD^2}{r^2 t \log(1 + \frac{4\lambda_\sigma^2}{\Lambda_\sigma^2})} dt}_{:=\mathcal{J}_3} \right)
\end{aligned}
$$

Computing a few simple integrals yields the following upper bounds on $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$:

$$
\begin{aligned}
\mathcal{J}_1 &\leq d^2 \log\left(\frac{2D\Lambda_\sigma^2}{r\lambda_\sigma^2}\right) \\
\mathcal{J}_2 &\leq \frac{dD^2}{r^2}\left[\log(2d) + \log\left(\log\left(\frac{2D}{r}\right)\right)\right] \\
\mathcal{J}_3 &\overset{(iii)}{\leq} 2\frac{dD^2}{r^2}\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\log\left(4\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\right)
\end{aligned}
$$

where (iii) uses the upper bound $\frac{1}{\log(1+x)} \leq \frac{1}{x}$.

Plugging these bounds in to (A.23) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (14).

A.11 Concentration inequalities

Given a symmetric kernel function $k : \mathcal{X}^m \to \mathbb{R}$, and data $\{x_1, \ldots, x_n\}$, we define the *order-m U statistic* to be

$$
U := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} k(x_{i_1}, \ldots, x_{i_m})
$$

For Lemmas 23, let $X_1, \ldots, X_n \in \mathcal{X}$ be independent and identically distributed. We will additionally assume the order-m kernel function k satisfies the boundedness property $\sup_{x_1,\ldots,x_m} |k(x_1, \ldots, x_m)| \leq 1$.

Lemma 23 (Hoeffding's inequality for U-statistics.)**.** *For any $t > 0$,*

$$
\mathbb{P}(|U - \mathbb{E}U| \geq t) \leq 2\exp\left\{-\frac{2nt^2}{m}\right\}
$$

Further, for any $\delta > 0$, we have

$$U \leq \mathbb{E}U + \sqrt{\frac{m \log(1/\delta)}{2n}},$$
$$U \geq \mathbb{E}U - \sqrt{\frac{m \log(1/\delta)}{2n}}$$

each with probability at least $1 - \delta$.

We will employ a sharper concentration inequality for $\sum_{i=1}^n X_i$.

Lemma 24. *Let $X_i \in \{0, 1\}$ for $i = 1, \ldots, n$ and let $\mu = \mathbb{E}(\sum_{i=1}^n X_i)$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i > (1+\epsilon)\mu\right) \leq \exp\left(\frac{-\delta^2 \mu}{3}\right)$$
$$\mathbb{P}\left(\sum_{i=1}^n X_i < (1-\epsilon)\mu\right) \leq \exp\left(\frac{-\delta^2 \mu}{2}\right)$$

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