Notes for the week of 5/11/19 - 5/17/19

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For an undirected graph $G = (V, E)$, the lazy random walk over G is the Markov chain with transition probabilities given by $\mathbf{W} := \frac{\mathbf{I}+\mathbf{D}^{-1}\mathbf{A}}{2}$, stationary distribution π. Denote the m-step probability distribution of this random walk originating from a particular $v \in V$ as $q^{(m)} : V \times V \to [0, 1]$, $q^{(m)}(v, u) = e_v \mathbf{W}^m e_u$. We wish to upper bound the total variation distance between the distributions $q_v^{(m)} := q^{(m)}(v, \cdot)$ and π,

$$\left\| q_v^{(t+1)} - \pi \right\|_{TV} = \sum_{u \in V} \left| q_v^{(m)}(u) - \pi(u) \right|$$

using geometric properties of the graph $\mathbf{W}$. We introduce the *degree, cut* and *volume* functionals over a graph. For $u \in V$, $A \subseteq G$,

$$\mathrm{cut}(A; G) = \sum_{u \in A} \sum_{v \in A^c} \mathbf{1}((u, v) \in E), \quad \deg(u; G) = \sum_{v \in V} \mathbf{1}((u, v) \in E), \quad \mathrm{vol}(A; G) = \sum_{u \in A} \deg(u; G)$$

The *local spread* is defined as

$$s(G) := \frac{9}{10} \cdot \min_{u \in V} \{\deg(u; G)\} \cdot \min_{u \in V} \{\pi(v)\}$$

Letting *normalized cut* of $A \subseteq G$, $\Phi(A; G)$ be defined as

$$\Phi(A; G) = \frac{\mathrm{cut}(A, A^c; G)}{\min \{\mathrm{vol}(A; G), \mathrm{vol}(A^c; G)\}},$$

and the *conductance* is

$$\Phi(G) = \min_{A \subseteq V} \Phi(A; G).$$

Theorem 1. *For any $v \in V$,*

$$\left\| q_v^{(t+3)} - \pi \right\|_{TV} \leq \left\{ as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2 \min_{u \in V} \deg(u; G)} \right\} + \left(\frac{1}{1 - 2s(G)/9} \right) \left(1 - \frac{\Phi^2(G)}{2} \right)^t$$

As we will see, Theorem 1 is an essential step to providing an upper bound on the uniform mixing time, which is what we want. We justify this statement next, before moving on to proving Theorem 1.

Uniform mixing time. Consider the *uniform* distance [1] between $q_v^{(t)}$ and π, given by

$$d_{\mathrm{unif}}(q_v^{(t)}, \pi) = \max_{u \in V} \left\{ \frac{\pi(u) - q_v^{(t)}(u)}{\pi(u)} \right\}.$$

[1] Note d_{unif} is not a formally a distance as it is not symmetric.

Theorem 2. *Let* $\left\|q_v^{(t)} - \pi\right\|_{TV} \le \frac{1}{14}\max\left\{1, \frac{1}{s(G)}\right\}$. *Then,*

$$d_{unif}(q_v^{(t+3)}, \pi) \le \frac{1}{4}$$

Proof. Fix $u \in V$ and let $m \ge t+1$ be arbitrary. The stationarity of π gives

$$\begin{aligned}
\frac{\pi(u) - q_v^m(u)}{\pi(u)} &= \sum_{y \in V} \left(\pi(y) - q^{(m-1)}(v,y)\right)\left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) \\
&\overset{(i)}{=} \sum_{y \ne u} \left(\pi(y) - q^{(m-1)}(v,y)\right)\left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) + \frac{\pi(u) - q^{(m-1)}(v,u)}{\pi(u)}\left(\frac{1}{2} - \pi(u)\right) \\
&\le \sum_{y \ne u} \left(\pi(y) - q^{(m-1)}(v,y)\right)\left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) + \frac{\pi(u) - q^{(m-1)}(v,u)}{2\pi(u)} \qquad (1)
\end{aligned}$$

where (i) follows from $q^{(1)}(u,u) = \frac{1}{2}$.

Then

$$\begin{aligned}
\sum_{y \ne u} \left(\pi(y) - q^{(m-1)}(v,y)\right)\left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) &\le \left\|q_v^{(m-1)} - \pi\right\|_{TV} \max_{y \ne u}\left|\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right| \\
&\le \left\|q_v^{(m-1)} - \pi\right\|_{TV} \max\left\{1, \max_{y \ne u}\left\{\frac{q^{(1)}(y,u)}{\pi(u)}\right\}\right\} \\
&\le \left\|q_v^{(m-1)} - \pi\right\|_{TV} \max\left\{1, \frac{1}{s(G)}\right\}
\end{aligned}$$

since for $y \ne u$, $q^{(1)}(y,u) \le 1/\left(2\min_{u \in V}\deg(u;G)\right)$. As $m-1 \ge t$, it is well known **?** that the laziness of the random walk guarantees $\left\|q_v^{(m-1)} - \pi\right\|_{TV} \le \left\|q_v^{(t)} - \pi\right\|_{TV}$, and therefore

$$\sum_{y \ne u} \left(\pi(y) - q^{(m-1)}(v,y)\right)\left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) \le \frac{1}{14}.$$

Plugging this in to (1) and taking maximum on both sides, we obtain

$$d_{\text{unif}}(q_v^{(m)}, \pi) \le \frac{2}{7} + \frac{d_{\text{unif}}(q_v^{(m-1)}, \pi)}{2} \qquad (2)$$

The recurrence relation of (2) along with the initial condition $d_{\text{unif}}(q_v^{(t)}, \pi) \le 1$ yields

$$d_{\text{unif}}(q_v^{(t+1)}, \pi) \le \frac{8}{14} \Rightarrow d_{\text{unif}}(q_v^{(t+2)}, \pi) \le \frac{10}{28} \Rightarrow d_{\text{unif}}(q_v^{(t+3)}, \pi) \le \frac{1}{4}$$

and the claim is shown. □

1 Proof of Theorem 1.

For arbitrary starting distribution q (meaning $\text{supp}(q) \subseteq V$ and $\sum_{u \in V} q(u) = 1$), and for $t \ge 0$ an integer, consider the distance function $h_q^{(t)}, t \ge 0$,

$$h_q^{(t)}(x) = \max\left\{\sum_{u \in V}\left(q^{(m)}(u) - \pi(u)\right)w(u)\right\}$$

where the maximum is over all $w : V \to [0,1]$ such that $0 \le w(u) \le 1$ for all u, and $\sum_{u\in V} w(u)\pi(u) = x$. Writing $h_v^{(t)}(x) := h_{e_v}^{(t)}(x)$ in a small abuse of notation, in Theorem 3 and Lemma 2, we give an upper bound on $h_{e_v}^{(t)}(x)$ for all $0 \le x \le 1$.

Remark 1. $h_q^{(t)}$ permits an equivalent relation. Order the elements of $V = \{u_1, \dots, u_N\}$ $(N = |V|)$, such that

$$\frac{q^{(m)}(u_1)}{\pi(u_1)} \ge \frac{q^{(m)}(u_2)}{\pi(u_2)} \ge \dots \ge \frac{q^{(m)}(u_N)}{\pi(u_N)}$$

and let $U_k = \{u_1, \dots, u_k\}$. Then for any x, letting k satisfy $\pi(U_{k-1}) < x < \pi(U_k)$, it can be shown that,

$$h_q^{(t)}(x) = \sum_{j=1}^{k-1}(q^{(m)}(u_j) - \pi(u_j)) + \frac{x - \pi(U_{k-1})}{\pi(u_k)}\left(q^{(m)}(u_k) - \pi(u_k)\right). \tag{3}$$

The formulation on the right hand side of (3) has come to be known as the *Lovasz-Simonovits curve*.

Mixing over large sets. For $0 \le \mu \le 1$ and $\mu \le x \le 1-\mu$ let

$$\ell_\mu(x) = \frac{1-\mu-x}{1-2\mu} h_q^{(0)}(\mu) + \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu)$$

be the linear interpolator between $h_q^{(0)}(\mu)$ and $h_q^{(0)}(1-\mu)$.

Theorem 3. *For any $0 \le \mu \le 1/2$ and $\mu \le x \le 1-\mu$ and $t \ge 0$,*

$$h_q^{(t)}(x) \le \ell_\mu(x) + \max\left\{\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu}, \frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1\right\}\left(1 - \frac{\Phi^2(G)}{2}\right)^t$$

Theorem 3 is a direct consequence of Theorem 4. To state the latter, we introduce

$$C_\mu = \max\left\{\frac{h_q^{(0)}(x) - \ell_\mu(x)}{\sqrt{x-\mu}}, \frac{h_q^{(0)}(x) - \ell_\mu(x)}{\sqrt{1-x-\mu}} : \mu < x < 1-\mu\right\}$$

Theorem 4 (Theorem 1.2 of Lovasz-Simonovits 1993)**.** *For any $0 \le \mu \le \frac{1}{2}$, $\mu \le x \le 1-\mu$ and an integer $t \ge 0$,*

$$h_q^{(t)}(x) \le \ell_\mu(x) + C_\mu \min\left\{\sqrt{x-\mu}, \sqrt{1-x-\mu}\right\}\left(1 - \frac{\Phi^2(G)}{2}\right)^t$$

Proof of Theorem 3. Fix $0 \le \mu \le \frac{1}{2}$. We will show that for all $\mu \le x \le 1-\mu$

$$h_q^{(0)}(x) - \ell_\mu(x) \le \max\left\{\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu}, \frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1\right\}\min\left\{\sqrt{x-\mu}, \sqrt{1-x-\mu}\right\} \tag{4}$$

whence the claim follows by Theorem 4.

Note that $\ell_\mu(\mu) = h_q^{(0)}(\mu)$, and for $x \ge \mu$,

$$h_q^{(0)}(x) \le h_q^{(0)}(\mu) + (x-\mu)\frac{h_q^{(0)}(\mu)}{\mu}$$

by the concavity of h_0 along with Lemma 1. Some basic algebra then yields

$$\begin{aligned} h_q^{(0)}(x) - \ell_\mu(x) &\le h_q^{(0)}(\mu) - \left(\frac{1-\mu-x}{1-2\mu} h_q^{(0)}(\mu) + \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu) \right) + \frac{h_q^{(0)}(\mu)}{\mu}(x-\mu) \\ &= \frac{x-\mu}{1-2\mu} h_q^{(0)}(\mu) + \frac{h_q^{(0)}(\mu)}{\mu}(x-\mu) - \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu) \\ &\le \sqrt{x-\mu}\left(\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu} \right) \end{aligned}$$

On the other hand, $\ell_\mu(1-\mu) = h_q^{(0)}(1-\mu)$, and by the concavity of $h_q^{(0)}$ and Lemma 1, for $x \le 1-\mu$

$$h_q^{(0)}(x) \le h_q^{(0)}(1-\mu) + (1-x-\mu).$$

Similar manipulations to above give the upper bound

$$h_q^{(0)}(x) - \ell_\mu(x) \le \sqrt{1-\mu-x}\left(\frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1 \right)$$

and (4) follows. □

Lemma 1. *The subdifferential $v(x)$ of $h_q^{(0)}(x)$ satisfies*

$$-1 \le v(x) \le \frac{h_q^{(0)}(\mu)}{\mu}$$

Mixing over small sets.

Lemma 2. *Let $0 \le a \le 1$, and $t \ge 1$ an integer. Then for any $x \le as(G)$ or $x \ge 1 - as(G)$,*

$$h_v^{(t)}(x) \le \max\left\{ as(G), \frac{1}{2^t} + \frac{9a}{20} + \frac{1}{2\min_{u\in V} \deg(u;G)} \right\}$$

Proof. First, we deal with the case $x \le as(G)$. Letting U_k be as in (3), we have

$$h_v^{(t)}(x) \le q_v^{(m)}(U_{k-1}) + q_v^{(m)}(u_k) \tag{5}$$

We will rely on the key fact that for any $u \ne v, t \ge 1$,

$$q_v^{(t)}(u) \le \frac{1}{2\min_{u\in V} \deg(u;G)} \tag{6}$$

On the other hand if $u = v$,

$$q_v^{(m)}(u) \le \frac{1}{2^t} + \frac{1}{2\min_{u\in V} \deg(u;G)}. \tag{7}$$

Therefore by (5), (6), and (7)

$$h_v^{(t)}(x) \le \frac{1}{2^t} + \frac{|U_k|}{2\min_{u\in V} \deg(u;G)} + \frac{1}{2\min_{u\in V} \deg(u;G)}.$$

Since $x \le as(G)$,

$$|U_k| \le \frac{x}{10\min_{u\in V}(\pi(v))} \le \frac{9a\min_{u\in V} \deg(u;G)}{10}.$$

For any $0 \le b \le 1$, $x \ge 1-b$ implies $h_v^{(t)}(x) \le b$. Taking $b = as(G)$, the claim is shown. □

Proof of Theorem 1. For any $A \subseteq V$ and any integer $t \geq 0$,

$$\max\left\{h_v^{(t)}(\pi(A)), h_v^{(t)}(1-\pi(A))\right\} \geq \left|q_v^{(t)}(A) - \pi(A)\right|$$

and taking max over both sides, we have

$$\max_{0\leq x\leq 1} h_v^{(t)}(x) \geq \left\|q_v^{(t)} - \pi\right\|_{TV}.$$

Letting $q = e_v W^3$, observe that $h_v^{(t+3)}(x) = h_v^{(t)}(x)$. Fix $\mu = \frac{1}{9}s(G)$. Then, for $\mu \leq x \leq 1-\mu$, by Theorem 3,

$$\begin{aligned}
h_v^{(t+3)}(x) &= h_q^{(t)}(x)\\
&\leq \max\left\{h_q^{(0)}(\mu), h_q^{(0)}(1-\mu)\right\} + \left(\frac{1}{1-2\mu} + \frac{1}{2\mu}\right)\left(1 - \frac{\Phi^2(G)}{2}\right)^t\\
&\leq \max\left\{as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2\min_{u\in V}\deg(u;G)}\right\} + \left(\frac{1}{1-2s(G)/9}\right)\left(1 - \frac{\Phi^2(G)}{2}\right)^t
\end{aligned}$$

where the last inequality comes from application of Lemma 2 to $h_q^{(t)} = h_v^{(t+3)}$.

For $x \leq \mu$ or $x \geq 1-\mu$,

$$h_v^{(t+3)}(x) \leq \max\left\{as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2\min_{u\in V}\deg(u;G)}\right\}$$

and the proof is complete.