Notes on 'BLOCKING CONDUCTANCE AND MIXING IN RANDOM WALKS'

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1 SETUP

Continuous state Markov chains. Let $(K, \mathcal{A})$ be a σ-algebra. For every $u \in K$, let P_u be a probability measure on $(K, \mathcal{A})$, and assume that for every $S \in \mathcal{A}$, the value $P_u(S)$ is a measurable function of u. We call P_u the **1-step transition probability density** out of u. The triple $(K, \mathcal{A}, \{P_u : u \in K\})$ is a **Markov chain**.

Let $\mathcal{M}$ be a continuous state Markov chain with no-atoms. Let the **random-stopping Markov chain** ρ^m be the distribution induced by choosing an integer $Y \in \{0, 1, \ldots, m-1\}$ uniformly at random, then stopping after Y steps.

Random walks on geometric sets. We specify $\mathcal{M}$ to be the **ball walk in** K with step size r. In this walk, at every step, we go from the current point $x \in K$, to a random point in $B(x, r) \cap K$ where $B(x, r)$ is the ball of radius r with center x.

Note that the stationary distribution of $\mathcal{M}$ is not exactly the uniform measure. Define **local conductance**, $\ell(x)$, by

$$\ell(x) = \frac{\mathrm{vol}(K \cap B(x, r))}{\mathrm{vol}(B(x, r))}$$

for $x \in K$. Then, letting normalizing constant E be

$$E = \int_L \ell(x) dx$$

the stationary distribution π of the ball walk has density function ℓ / E.

Long-run behavior. To reason about the long-run convergence of ρ, we will need to understand what distribution ρ is convergence towards.

Assumption 1. We will assume that our Markov chain is ergodic, time-reversible, atom-free, and has stationary distribution π.

Intuitively, the rate of convergence is affected by how quickly the random walk can escape bottlenecks. The **ergodic flow** is defined for $X, Y \in \mathcal{A}$ by

$$Q(X,Y) := \int_X P_x(Y) d\pi(x).$$

Then, we define the **conductance function** $\Psi(t)$ for $t \in (0,1)$ to be

$$\Psi(t) := \min_{\substack{S \subset V, \\ \pi(S)=t}} Q(S, K \setminus S).$$

Large conductance functions mean that there are less pronounced bottlenecks, and therefore ρ should mix faster.

We will also need to formalize what type of convergence we are referring to. Define the **total variation distance** d_{TV} between two measures μ and ν defined over $(K, \mathcal{A})$ to be

$$d_{TV}(\mu, \nu) = \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|.$$

Define the **total variation mixing time** τ_{TV} of a sequence of distributions $\rho^1, \rho^2, \ldots$ converging to stationary distribution π to be

$$\min \left\{ T : d_{TV}(\rho^t, \pi) \leq 1/4 \text{ for all } t \geq T \right\}.$$

Local spread. Markov chains on geometric sets will have the nice property that the chain quickly disperses probability out of small sets. To quantify this, we introduce $\xi(A)$ to be

$$\xi(A) := \inf \{P_u(K \setminus A) : u \in A\} \quad \xi(t) = \inf \{\xi(A) : \pi(A) = t\} \quad (0 \leq t \leq 1).$$

Intuitively, on geometric sets, $\xi(t)$ should be large when t is small. Define the **local spread**,π_1, to be

$$\pi_1 := \inf \{t : \xi(t) \leq 1/10\}$$

Part of the theory developed allows us to ignore sets S of stationary probability less than π_1.

2 RESULTS

Theorem 1. *Consider a continuous state Markov chain $\mathcal{M}$ which satisfies Assumption 1, and further assume $\xi(a) > 0$ for some $a > 0$. Suppose that for some $A, B > 0$, an inequality of the form*

$$\Psi(x) \geq \min \{Ax, Bx \ln(1/x)\} \tag{1}$$

holds for all $a \leq x \leq 1/2$. Then

$$d(\rho^m, \pi) \leq \frac{8 \ln m}{m\xi(a)} + \frac{1}{m}\left(\frac{32}{A^2}\ln\left(\frac{1}{a}\right) + \frac{100}{B^2}\right)$$

To use Theorem 1, we will need to both find a and assert (1) holds for $x \geq a$.

Theorem 2. *The local spread of the proper move walk satisfies $\pi_1 \geq \frac{1}{2}(r/D)^{2d}$.*

Theorem 3. *Suppose that K is a convex set in $\mathbb{R}^d$ with diameter at most D, containing the unit ball $B = B(0,1)$, and further that the step size for the ball walk in K satisfies $r < \frac{D}{100}$. For $0 < x < 1/2$, we have*

$$\Psi(x) > \min\left\{\frac{1}{288\sqrt{d}}x, \frac{r}{81\sqrt{d}D}x \ln(1 + 1/2)\right\} \tag{2}$$

Theorem 3 states that the ball walk satisfies (1), and Theorem 2 gives us values for $\xi(a)$ and a. Deploying Theorem 1 therefore gives us

$$\tau_{TV} = \mathcal{O}\left(d\left(\frac{D}{r}\right)^2 + d^2 \ln(D/r)\right)$$

3 SUPPLEMENTAL RESULTS

4 PROOFS