Notes for the week of 4/27/19 - 5/3/19

Alden Green

April 29, 2019

Let $\mathcal{A} \subseteq \mathbb{R}^d$, and for $\sigma \geq 0$, write $\sigma B := B(0,\sigma) = \left\{x \in \mathbb{R}^d : \|x\| \leq \sigma\right\}$ for the closed ball of radius σ centered at the origin (and let $B^\circ(0,\sigma)$ denote the corresponding open ball). Let $\mathcal{A}_\sigma = \mathcal{A} + \sigma B$ be the direct sum of $\mathcal{A}$ and σB.

Theorem 1. *If $\mathcal{A}$ is closed and bounded, then for any $\delta > 0$,*

$$\nu(\mathcal{A}_\sigma + \delta B) \leq \left(1 + \frac{\delta}{\sigma}\right)^d \nu(\mathcal{A}_\sigma).$$

Proof. We will show that for any $\epsilon > 0$,

$$\frac{\nu(\mathcal{A}_\sigma + \delta B)}{\nu(\mathcal{A}_\sigma)} \leq \frac{(\sigma + \delta + \epsilon)^d}{\sigma^d} \tag{1}$$

which is sufficient to prove the claim.

Fix $\epsilon > 0$. Our first goal is to find a finite collection $x_1, \ldots, x_N \in \mathbb{R}^d$ such that

$$\bigcup_{i=1}^N B(x_i, \sigma) \subseteq \mathcal{A}_\sigma \subset \bigcup_{i=1}^N B(x_i, \sigma + \epsilon). \tag{$N := N(\epsilon)$}$$

Observe that since $\mathcal{A}$ is closed and bounded, it is compact. As $B(x,\sigma)$ is compact, and the direct sum of two compact sets is itself compact, $\mathcal{A}_\sigma$ is compact. Moreover,

$$\mathcal{A}_\sigma \subset \bigcup_{x \in \mathcal{A}} B^\circ(x, \sigma + \epsilon)$$

so by compactness there exists $x_1, \ldots, x_N \in \mathcal{A}$ such that

$$\mathcal{A}_\sigma \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \epsilon).$$

By the triangle inequality, $\mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \epsilon + \delta)$. Of course, for each $x_i \in \mathcal{A}$, $B(x_i, \sigma) \in \mathcal{A}_\sigma$. Summarizing our findings, we have

$$\bigcup_{i=1}^N B(x_i, \sigma) \subseteq \mathcal{A}_\sigma,\ \mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \delta + \epsilon) \tag{2}$$

We proceed by giving a lower bound on $\nu(\mathcal{A}_\sigma)$. Partition $\mathcal{A}_\sigma$ using the balls $B(x_i,\sigma)$, meaning let $\mathcal{A}_\sigma^{(1)} := B(x_1,\sigma)$, $\mathcal{A}_\sigma^{(2)} := B(x_2,\sigma) \setminus B(x_1,\sigma)$, and continuing, so that

$$\mathcal{A}_\sigma^{(i)} := B(x_i,\sigma) \setminus \bigcup_{j=1}^{i-1} \mathcal{A}_\sigma^{(j)}. \qquad (i = 1,\dots,N)$$

Of course, by (2) $\mathcal{A}_\sigma \supseteq \bigcup_{i=1}^N \mathcal{A}_\sigma^{(i)}$. Therefore,

$$\begin{aligned}
\nu(\mathcal{A}_\sigma) &\geq \sum_{i=1}^N \nu(\mathcal{A}_\sigma^{(i)}) \\
&= \sigma^d \nu_d \sum_{i=1}^N \frac{\nu(\mathcal{A}_\sigma^{(i)})}{\nu(B(x_i,\sigma))} \\
&=: \sigma^d \nu_d \sum_{i=1}^N c_i.
\end{aligned}$$

Now we turn to proving an upper bound on $\nu(\mathcal{A}_\sigma + \delta B)$. Let $\mathcal{A}_{\sigma+\epsilon+\delta}^{(1)} := B(x_1, \sigma+\delta+\epsilon)$ and

$$\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)} := B(x_i, \sigma+\delta+\epsilon) \setminus \bigcup_{j=1}^{i-1} \mathcal{A}_{\sigma+\delta+\epsilon}^{(j)}. \qquad (i = 1,\dots,N)$$

By (2),

$$\mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N \mathcal{A}_{\sigma+\delta+\epsilon}^{(i)}$$

and as a result

$$\begin{aligned}
\nu(\mathcal{A}_{\sigma+\delta}) &\leq \sum_{i=1}^N \mathcal{A}_{\sigma+\delta+\epsilon}^{(i)} \\
&= \sum_{i=1}^N \nu_d(\sigma+\delta+\epsilon)^d \frac{\nu(\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)})}{\nu(B(x_i,\sigma+\delta+\epsilon))} \\
&\leq \nu_d(\sigma+\delta+\epsilon)^d \sum_{i=1}^N c_i
\end{aligned}$$

where the last inequality follows from Lemma 1. We have shown (1), and thus the claim. □

1 Additional Theory

Lemma 1. *For $i = 1,\dots,N$ and $A_\sigma^{(i)}, A_{\sigma+\delta+\epsilon}^{(i)}$ as in Theorem 1,*

$$\frac{\nu(\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)})}{\nu(B(x_i,\sigma+\delta+\epsilon))} \leq \frac{\nu(\mathcal{A}_\sigma^{(i)})}{\nu(B(x_i,\sigma))}$$

Proof. Let $\delta' := \delta + \epsilon$. It will be sufficient to show that

$$\left(\mathcal{A}^{(i)}_{\sigma+\delta'} - \{x_i\}\right) \subseteq \left(1 + \frac{\delta'}{\sigma}\right) \cdot \left(\mathcal{A}^{(i)}_{\sigma} - \{x_i\}\right)$$

since then

$$\nu(\mathcal{A}^{(i)}_{\sigma+\delta'}) \leq \left(1 + \frac{\delta'}{\sigma}\right)^d \nu(\mathcal{A}_\sigma) = \frac{\nu(B(x_i, \sigma + \delta'))}{\nu(B(x_i, \sigma))} \nu(\mathcal{A}_\sigma).$$

Assume without loss of generality that $x_i = 0$, and let $x \in \mathcal{A}^{(i)}_{\sigma+\delta'}$, meaning

$$\|x\| \leq \sigma + \delta', \;\; \|x - x_j\| > \sigma + \delta' \text{ for } j = 1, \ldots, i-1. \tag{3}$$

Letting $x' = \frac{\sigma}{\sigma+\delta'} x$, since $\|x\| \leq \sigma + \delta'$, $\|x'\| \leq \sigma$ and therefore $x' \in B(0, \sigma)$. Additionally observe that for any $j = 1, \ldots, i-1$, by the triangle inequality

$$\|x' - x_j\| \geq \|x - x_j\| - \|x - x'\| > \sigma + \delta' - \frac{\delta'}{\sigma + \delta'} \|x\| \geq \sigma$$

and therefore $x' \notin B(x_j, \sigma)$ for any $j = 1, \ldots, i-1$. So $x' \in \mathcal{A}^{(i)}_{\sigma}$. $\square$