Notes for the week of 4/8/19 - 4/12/19

Alden Green

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Let $\{x_1, x_2, \ldots\}$ be a sequence of points sampled independently from probability measure $\mathbb{P}$ with density function f. For each n, write $\mathbf{X}_n = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$. Given some $\lambda, \sigma > 0$, let

$$\mathcal{U} = \{x : f(x) \geq \lambda\}\,,\ \mathcal{C} = \text{one connected component of } \mathcal{U},\ \text{and } \mathcal{C}_\sigma = \mathcal{C} + B(0, \sigma)$$

Write $\widetilde{\mathbf{X}}_n = \mathcal{C}_\sigma[\mathbf{X}_n]$, $\widetilde{E}_n = \Big\{(i,j) : x_i, x_j \in \widetilde{\mathbf{X}}_n, \|x_i - x_j\|_2 \leq r\Big\}$ and let $\widetilde{G}_{n,r} = \big(\widetilde{\mathbf{X}}_n, \widetilde{E}_n\big)$. For a set $S \subseteq \widetilde{\mathbf{X}}_n$, the normalized cut of S within $\widetilde{G}_{n,r}$ can be defined as

$$\widetilde{\Phi}_{n,r}(S) := \frac{\widetilde{\mathrm{cut}}(S)}{\min\Big\{\widetilde{\mathrm{vol}}(S), \widetilde{\mathrm{vol}}(S^c)\Big\}},\ \text{where } \widetilde{\mathrm{cut}}(S) := \mathrm{cut}(S; \widetilde{G}_{n,r}),\ \widetilde{\mathrm{vol}}(S) := \mathrm{vol}(S; \widetilde{G}_{n,r})$$

and in this context $S^c = \widetilde{\mathbf{X}}_n \setminus S$ denotes the complement of S within $\widetilde{G}_{n,r}$. Then, the *graph conductance profile* over $\widetilde{G}_{n,r}$ is

$$\widetilde{\Phi}_{n,r}(t) := \min_{\substack{S \subseteq \widetilde{\mathbf{X}}_n: \\ 0 < \widetilde{\pi}_n(S) < t}} \widetilde{\Phi}_{n,r}(S)$$

where $\widetilde{\pi}_{n,r}(S) = \frac{\widetilde{\mathrm{vol}}(S)}{\widetilde{\mathrm{vol}}(\widetilde{\mathbf{X}}_n)}$. We will prove a lower bound on the graph conductance profile by a continuous analogue.

0.1 Continuous conductance

Let ν be the Lebesgue measure over Euclidean space $\mathbb{R}^d$, and $B(x,r)$ be a ball of radius r centered at x. For $\mathcal{S} \subset \mathbb{R}^d$ a Borel set,

$$\nu_{\mathbb{P}}(\mathcal{S}) := \int_{\mathcal{S}} f(x) dx$$

is the weighted volume.

The r-ball walk over $\mathcal{C}_\sigma$ is a Markov chain, with transition probability given by

$$\widetilde{P}_{\mathbb{P},r}(x; \mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S} \cap B(x,r))}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r))} \qquad (x \in \mathcal{C}_\sigma, \mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

where $\mathfrak{B}(\mathcal{C}_\sigma)$ is the Borel σ-algebra of $\mathcal{C}_\sigma$.

Denote the stationary distribution for this Markov chain by $\pi_{\mathbb{P},r}$, which is defined by the relation

$$\int_{\mathcal{C}_\sigma} \widetilde{P}_{\mathbb{P},r}(x; \mathcal{S}) d\pi_{\mathbb{P},r}(x) = \pi_{\mathbb{P},r}(\mathcal{S}). \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

Letting the *local conductance* be given by

$$\ell_{\mathbb{P},r}(x) := \nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r)) \qquad (x \in \mathcal{C}_\sigma)$$

a bit of algebra verifies that

$$\pi_{\mathbb{P},r}(\mathcal{S}) = \frac{\int_{\mathcal{S}} \ell_{\mathbb{P},r}(x) f(x) dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r}(x) f(x) dx}. \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

We next introduce the *ergodic flow*, $\widetilde{Q}_{\mathbb{P},r}$. Let $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{C}_\sigma$ be a partition of $\mathcal{C}_\sigma$. Then the ergodic flow between $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by

$$\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}_1, \mathcal{S}_2) := \int_{\mathcal{S}_1} \widetilde{P}_{\mathbb{P},r}(x; \mathcal{S}_2) d\pi_{\mathbb{P},r}(x), \qquad (\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{B}(\mathcal{C}_\sigma))$$

the *(continuous) normalized cut* by

$$\widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S}) := \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}_1, \mathcal{S}_2)}{\min\{\pi_{\mathbb{P},r}(\mathcal{S}), \pi_{\mathbb{P},r}(\mathcal{S}^c)\}}, \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

and the *(continuous) conductance profile* by

$$\widetilde{\Phi}_{\nu,r}(t) := \min_{\substack{\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma) \\ 0 < \pi_{\mathbb{P},r}(\mathcal{S}) \leq t}} \widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S}) \qquad (0 < t \leq 1/2)$$

where $\mathcal{S}^c = \mathcal{C}_\sigma \setminus \mathcal{S}$.

To relate the graph and continuous conductance profiles, we introduce mappings between the data $\widetilde{\mathbf{X}}_n$ and the space $\mathcal{C}_\sigma$.

0.2 Transportation maps and TL^1 distance.

Let

$$\widetilde{\mathbb{P}}(\mathcal{S}) = \frac{\mathbb{P}(\mathcal{S})}{\mathbb{P}(\mathcal{C}_\sigma)}, \ \widetilde{\mathbb{P}}_n(\mathcal{S}) := \frac{1}{\widetilde{n}} \sum_{x_i \in \widetilde{\mathbf{X}}_n} \mathbf{1}(x_i \in \mathcal{S}) \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

be the (empirical) probability measures, conditional on $x \sim \mathbb{P}$ lying within $\mathcal{C}_\sigma$ (Here $\mathfrak{B}(\mathcal{C}_\sigma)$ is the Borel σ-algebra of $\mathcal{C}_\sigma$). A Borel map $T : \mathcal{C}_\sigma \to \widetilde{\mathbf{X}}_n$ is said to be a *transportation map* between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ if for arbitrary $\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma)$,

$$\widetilde{\mathbb{P}}(\mathcal{S}) = \widetilde{\mathbb{P}}_n(T(\mathcal{S})).$$

If a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$ satisfies $\|\mathrm{Id} - T_n\|_{L^1(\widetilde{\mathbb{P}})} = o_P(1)$, we refer to it as a sequence of *stagnating transportation maps*. Lemma 1 establishes that with probability one, such a sequence of stagnating transportation maps will exist. In fact, under suitable conditions, the convergence happens at rate $\left(\frac{\log n}{n}\right)^{1/d}$.

Lemma 1 (Adaptation of Proposition 5 of Garcia Trillos 2016)**.** *With probability one, there exists a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$, $T_n : \mathcal{C}_\sigma \to \widetilde{\mathbf{X}}_n$ such that the following statement holds:*

$$\limsup_{n \to \infty} \frac{\widetilde{n}^{1/d} \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}}{(\log \widetilde{n})^{p_d}} \leq C$$

where $\mathrm{Id}(x) = x$ is the identity mapping over $\mathcal{C}_\sigma$, C is a universal constant and $p_d = 3/4$ for $d = 2$ and $1/d$ for $d \geq 3$.

Definition 0.1. For a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1(\widetilde{\mathbb{P}}_n)$ and $u \in L^1(\widetilde{\mathbb{P}})$, we say that $\{u_n\}_{n\in\mathbb{N}}$ converges TL^1 to u if there exists a sequence of stagnating transportation maps $\{T_n\}_{n\in\mathbb{N}}$ such that

$$d^{TL^1}(u,u_n) := \int_{\mathcal{C}_\sigma} |u(x) - u_n \circ T_n(x)|\, d\widetilde{\mathbb{P}}(x) \xrightarrow{n} 0 \tag{1}$$

and denote it $u_n \overset{TL^1}{\to} u$.

Remark 1. Note that as written this is not a metric, as u and u_n lie in different spaces. Technically, we can resolve this by writing

$$d^{TL^1}((\widetilde{\mathbb{P}},u),(\widetilde{\mathbb{P}}_n,u_n)) = \inf_{\pi\in\Gamma(\widetilde{\mathbb{P}},\widetilde{\mathbb{P}}_n)} \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} |x-y| + |f(x)-g(y)|\, d\pi(x,y) \tag{2}$$

where γ is the space of couplings over the measures $\widetilde{\mathbb{P}}, \widetilde{\mathbb{P}}_n$. However, it can be shown that (2) converges to zero if and only if (1) is satisfied and

$$\widetilde{\mathbb{P}}_n \overset{w}{\to} \widetilde{\mathbb{P}}.$$

Since this additional condition will be satisfied with probability one, we simplify things by hereafter referring only to the condition in (1). See Garcia Trillos 15 for more details.

1 Lower bound graph conductance profile

In order to lower bound the graph conductance profile $\widetilde{\Phi}_{n,r}(\cdot)$, we will need to split the analysis into two cases based on the size of $S \subseteq \widetilde{\mathbf{X}}_n$. Define the *graph local spread* to be

$$s(G) := \frac{9}{10} \min_{u\in V} \{\deg(u;G)\} \cdot \min_{u\in V} \{\pi(u;G)\}\,. \qquad (G=(V,E))$$

Theorem 1 states that for subsets of $\widetilde{\mathbf{X}}_n$ with volume at least $s(G)$, the graph normalized cut can be uniformly lower bounded by the continuous normalized cut. Sections 2 and 3 contains the proof of Theorem 1 along with other relevant results.

Theorem 1. *Let $\mathcal{C}_\sigma$ satisfy Assumption 1 with respect to Lipschitz constant L and convex set K with diameter D_K. Then,*

$$\liminf_{n\to\infty} \Big\{ \min_{S\in\mathcal{L}(\widetilde{G}_{n,r})} \widetilde{\Phi}_{n,r}(S) \Big\} \geq \frac{\lambda_\sigma^4 r}{\Lambda_\sigma^4 2^{12} D_K L \sqrt{d}}$$

where $\mathcal{L}(\widetilde{G}_{n,r}) = \Big\{ S \subseteq \widetilde{\mathbf{X}}_n : \widetilde{\pi}_{n,r}(S) \geq s(\widetilde{G}_{n,r}) \Big\}$

Lemma 2 shows that for the remaining small sets, the graph normalized cut is of constant order.

Lemma 2. *Let $G=(V,E)$ be an undirected graph, and $\mathcal{L}(G) = \{S \subseteq V : \pi(S;G) \geq s(G)\}$. Then,*

$$\min_{S\notin\mathcal{L}(G)} \Phi(S;G) \geq \frac{1}{10}.$$

Proof. Clearly, for any $u \in S$

$$\mathrm{cut}(\{u\}, S^c; G) \geq \deg(u;G) - |S| \tag{3}$$

Then, since $\pi(S;G) \leq s(G)$,

$$|S| \leq \pi(S;G)\cdot \frac{\mathrm{vol}(V;G)}{\min_{u\in V}\{\deg(u;G)\}} = \frac{\pi(S;G)}{\min_{u\in V}\{\pi(u;G)\}} \leq \frac{9}{10}\min_{u\in V}\{\deg(u;G)\}\,,$$

and therefore by (3), for any $u \in S$

$$\mathrm{cut}(\{u\}, S^c; G) \geq \deg(u; G) - \frac{9}{10}\min_{u \in V}\{\deg(u; G)\} \geq \frac{1}{10}\deg(u; G)$$

and the statement follows by summing over all $u \in S$. $\square$

Theorem 1 and Lemma 2 together yield Corollary 1, the main result of these notes.

Corollary 1 (Lower bound on graph conductance profile)**.** *Let $\mathcal{C}_\sigma$ satisfy Assumption 1 with respect to Lipschitz constant L and convex set K with diameter D_K. Then, with probability one the following asymptotic lower bound holds on the graph conductance function*

$$\liminf_{n \to \infty} \widetilde{\Phi}_{n,r}(t) \geq \min\left\{\frac{\lambda_\sigma^4 r}{\Lambda_\sigma^4 2^{12} D_K L\sqrt{d}}, \frac{1}{10}\right\} \qquad (0 \leq t \leq \tfrac{1}{2})$$

2 Proofs and Supporting Theory

2.1 Proof of Theorem 1.

By Lemma 1, with probability one there exists a sequence of stagnating transportation maps from $\widetilde{\mathbb{P}}$ to $\widetilde{\mathbb{P}}_n$, which we will denote $\{T_n\}_{n \in \mathbb{N}}$.

For $S \subseteq \widetilde{\mathbf{X}}_n$, let $T_n^{-1}(S) = \{x \in \mathcal{C}_\sigma : T_n(x) \in S\}$ be the preimage of T_n, and note that $T_n^{-1}(S^c) = \mathcal{C}_\sigma \setminus T_n^{-1}(S)$.

Letting

$$\xi_n := \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P}, r_n^-}(x) f(x) dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P}, r_n^+}(x) f(x) dx}, \ \gamma_n(S) := \frac{\min\left\{\pi_{\mathbb{P}, r_n^-}(T_n^{-1}(S^c)), \pi_{\mathbb{P}, r_n^-}(T_n^{-1}(S^c))\right\}}{\min\left\{\pi_{\mathbb{P}, r_n^+}(T_n^{-1}(S^c)), \pi_{\mathbb{P}, r_n^+}(T_n^{-1}(S^c))\right\}}$$

where $r_n^{\pm} := r \pm \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$, by Lemma 5 and Corollary 2 we have that for all $S \subseteq \widetilde{\mathbf{X}}_n$,

$$\begin{aligned}\widetilde{\Phi}_{n,r}(S) &\geq \xi_n \gamma_n(S) \widetilde{\Phi}_{\mathbb{P}, r_n^-}(T_n^{-1}(S)) \\ &\geq \xi_n \gamma_n(S) \frac{\lambda_\sigma^4 r_n^-}{2^{12} \Lambda_\sigma^4 D_K L \sqrt{d}}.\end{aligned} \tag{4}$$

By Lemma 6, with probability one

$$\liminf_{n \to \infty} \xi_n = 1.$$

By Lemma 8, letting c be any constant satisfying $c > \frac{9\lambda_\sigma^4 \nu_d r^d}{50\Lambda_\sigma^2}$, there exists some $n \in \mathbb{N}$ such that for all $S \in \mathcal{L}(\widetilde{G}_{n,r})$,

$$\pi_{\mathbb{P},r}(T_n^{-1}(S)) \geq c > 0$$

and therefore by Lemma 7

$$\liminf_{n \to \infty} \left\{\inf_{S \in \mathcal{L}(\widetilde{G}_{n,r})} \gamma_n(S)\right\} = 1.$$

As Lemma 1 implies $r_n^- \to r$ with probability one, an application of Slutsky's Theorem to (4) completes the proof.

2.2 Graph functionals to continuous functionals.

Lemmas 3 and 4 provide the necessary bounds for the cut and vol functionals in terms of continuous analogues.

Lemma 3. *Let $S \subseteq \widetilde{\mathbf{X}}_n$, and let T_n be a transportation map between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$. Then, letting $\mathcal{S} = \{x \in \mathcal{C}_\sigma : T_n(x) \in S\}$*

$$\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{vol}}(S) \leq \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x)f(x)dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\pi_{\mathbb{P},r_n^+}(\mathcal{S})$$

where $r_n^+ = r + \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$.

Proof. Let $u : \widetilde{\mathbf{X}}_n \to \{0,1\}$ be the characteristic function for S, meaning

$$u(x) = \begin{cases} 1, & x \in S \\ 0, & \text{otherwise} \end{cases}$$

Now, we proceed

$$\begin{aligned}
\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{vol}}(S_n) &= \frac{1}{\widetilde{n}^2}\sum_{x_i,x_j\in\widetilde{\mathbf{X}}_n} \mathbf{1}(\|x_i - x_j\| \leq r)\,|u(x_i)| \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|x-y\| \leq r)\,|u(x)|\,d\widetilde{\mathbb{P}}_n(x)d\widetilde{\mathbb{P}}_n(y) \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|T_n(x) - T_n(y)\| \leq r)\,|u\circ T_n(x)|\,d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \\
&\leq \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|x-y\| \leq r_n^+)\,|u\circ T_n(x)|\,d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \qquad (5) \\
&= \int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)} 1d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

By definition we have $\frac{d\widetilde{\mathbb{P}}(x)}{d\mathbb{P}(x)} = \mathbb{P}(\mathcal{C}_\sigma)$. Therefore,

$$\begin{aligned}
\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)} 1d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x) &= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)} 1d\mathbb{P}(y)d\mathbb{P}(x) \\
&= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}} \ell_{\mathbb{P},r_n^+}(x)f(x)dx \\
&= \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r}(x)f(x)dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\pi_{\mathbb{P},r_n^+}(\mathcal{S})
\end{aligned}$$

which is the desired upper bound. The lower bound follows a similar proof, with the only change being (5), where r_n^+ is replaced by r_n^- and the inequality is reversed. □

Lemma 4. *Let $S \subseteq \widetilde{\mathbf{X}}_n$, and let T_n be a transportation map between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$. Then, letting $\mathcal{S} = \{x \in \mathcal{C}_\sigma : T_n(x) \in S\}$,*

$$\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{cut}}(S) \geq \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}f(x)dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c)$$

where $r_n^- = r - \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$

Proof. Let $u : \widetilde{\mathbf{X}}_n \to \{0,1\}$ be the characteristic function for S, meaning

$$u(x) = \begin{cases} 1, & x \in S \\ 0, & \text{otherwise} \end{cases}$$

We proceed according to a very similar set of steps as Lemma 3:

$$\begin{aligned}
\frac{1}{\widetilde{n}^2}\widetilde{\text{cut}}(S) &= \frac{1}{\widetilde{n}^2}\sum_{x_i,x_j\in\widetilde{\mathbf{X}}_n} \mathbf{1}(\|x_i - x_j\| \le r)\,|u(x_i) - u(x_j)| \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|x - y\| \le r)\,|u(x) - u(y)|\, d\widetilde{\mathbb{P}}_n(x) d\widetilde{\mathbb{P}}_n(y) \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|T_n(x) - T_n(y)\| \le r)\,|u\circ T_n(x) - u\circ T_n(y)|\, d\widetilde{\mathbb{P}}(x) d\widetilde{\mathbb{P}}(y) \\
&\ge \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma} \mathbf{1}(\|x - y\| \le r_n^-)\,|u\circ T_n(x) - u\circ T_n(y)|\, d\widetilde{\mathbb{P}}(x) d\widetilde{\mathbb{P}}(y) \\
&= \int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)} d\widetilde{\mathbb{P}}(y) d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

We conclude similarly to the proof of Lemma 3,

$$\begin{aligned}
\int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)} d\widetilde{\mathbb{P}}(y) d\widetilde{\mathbb{P}}(x) &= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)} d\mathbb{P}(y) d\mathbb{P}(x) \\
&= \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-} f(x)dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c).
\end{aligned}$$

□

Lemma 5. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of transportation maps from $\widetilde{\mathbb{P}}$ to $\widetilde{\mathbb{P}}_n$, and let*

$$r_n^- = r - \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})},\ r_n^+ = r + \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}.$$

Fix $S \subseteq \widetilde{\mathbf{X}}_n$. Then, letting $\mathcal{S} = \{x \in \mathcal{C}_\sigma : T_n(x) \in S\}$,

$$\widetilde{\Phi}_{n,r}(S) \ge \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}\frac{\min\left\{\pi_{\mathbb{P},r_n^-}(\mathcal{S}),\pi_{\mathbb{P},r_n^-}(\mathcal{S}^c)\right\}}{\min\left\{\pi_{\mathbb{P},r_n^+}(\mathcal{S}),\pi_{\mathbb{P},r_n^+}(\mathcal{S}^c)\right\}}\widetilde{\Phi}_{\mathbb{P},r_n^-}(\mathcal{S}) \tag{6}$$

Proof. By Lemmas 3 and 4,

$$\frac{\widetilde{\text{cut}}(S)}{\widetilde{\text{vol}}(S)} \ge \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c)}{\pi_{\mathbb{P},r_n^+}(\mathcal{S})}$$

But, noting that $\mathcal{S}^c = \{x \in \mathcal{C}_\sigma : T_n(x) \in S^c\}$, Lemmas 3 and 4 also imply

$$\frac{\widetilde{\text{cut}}(S^c)}{\widetilde{\text{vol}}(S^c)} \ge \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S}^c,\mathcal{S})}{\pi_{\mathbb{P},r_n^+}(\mathcal{S}^c)}$$

and as $\widetilde{Q}_{\mathbb{P},r_n^-}(\cdot,\cdot)$ is symmetric in its arguments we obtain

$$\frac{\widetilde{\text{cut}}(S)}{\min\left\{\widetilde{\text{vol}}(S),\widetilde{\text{vol}}(S^c)\right\}} \ge \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c)}{\min\left\{\pi_{\mathbb{P},r_n^+}(\mathcal{S}),\pi_{\mathbb{P},r_n^+}(\mathcal{S}^c)\right\}},$$

and the proof is complete. □

2.3 Perturbation asymptotics.

In light of Lemma 1, the error incurred in (6) by the use of r_n^+ and r_n^- as opposed to r is asymptotically negligible.

Lemma 6 (Continuity of local conductance)**.** *Letting* $\{T_n\}_{n\in\mathbb{N}}$ *be a sequence of stagnating transportation maps, and* $r_n^{\pm} = r \pm \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$*, with probability one the following holds:*

$$\limsup_{n\to\infty} \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx} = 1$$

Proof. Let $\mathcal{R}_n(x) := \{x' \in \mathcal{C}_\sigma : x' \in B(x, r_n^+), x' \notin B(x, r_n^-)\}$, we have

$$\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx = \int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx + \int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n} f(y)f(x)dydx.$$

and therefore

$$\frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx} = 1 + \frac{\int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n} f(y)f(x)dydx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx}.$$

We upper bound the remainder term

$$\int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n} f(y)f(x)dydx \le P(\mathcal{C}_\sigma)\Lambda_\sigma\big((r_n^+)^d - r^d\big)\nu^d$$

and taking limits as $n \to \infty$ we obtain with probability one

$$\limsup_{n\to\infty} \int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n} f(y)f(x)dydx \le \limsup_{n\to\infty} P(\mathcal{C}_\sigma)\Lambda_\sigma\big((r_n^+)^d - r^d\big)\nu^d = 0$$

by the stagnating property of $\{T_n\}_{n\in\mathbb{N}}$.

We apply a similar analysis to the denominator.

$$\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx = \int_{\mathcal{C}_\sigma}\int_{B(x,r_n^-)} f(y)f(x)dydx \ge \frac{6}{25}\mathbb{P}(\mathcal{C}_\sigma)\lambda_\sigma (r_n^-)^d \nu_d$$

and therefore by the stagnating property of $\{T_n\}_{n\in\mathbb{N}}$ and Lemma 9,

$$\liminf_{n\to\infty} \int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x) f(x)dx = \frac{6}{25}\mathbb{P}(\mathcal{C}_\sigma)\lambda_\sigma^2 r^d \nu_d > 0$$

again with probability one.

The desired result then follows from an application of Slutsky's Theorem. ∎

Lemma 7 (Continuity of stationary distribution)**.** *Let* $c > 0$ *be a fixed constant,* $\{T_n\}_{n\in\mathbb{N}}$ *be a sequence of stagnating transportation maps, and* $r_n^{\pm} = r \pm \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$*. With probability one the following statement holds:*

$$\liminf_{n\to\infty} \frac{\min\left\{\pi_{\mathbb{P},r_n^-}(\mathcal{S}), \pi_{\mathbb{P},r_n^-}(\mathcal{S}^c)\right\}}{\min\left\{\pi_{\mathbb{P},r_n^+}(\mathcal{S}), \pi_{\mathbb{P},r_n^+}(\mathcal{S}^c)\right\}} = 1$$

uniformly over all sets $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{C}_\sigma)$ *satisfying* $\min\{\pi_{\mathbb{P},r}(\mathcal{S}), \pi_{\mathbb{P},r}(\mathcal{S}^c)\} > c$.

Proof. It will be sufficient to show that

$$\liminf_{n\to\infty} \frac{\pi_{\mathbb{P},r_n^-}(\mathcal{S})}{\pi_{\mathbb{P},r_n^+}(\mathcal{S})} \text{ and} \liminf_{n\to\infty} \frac{\pi_{\mathbb{P},r_n^-}(\mathcal{S}^c)}{\pi_{\mathbb{P},r_n^+}(\mathcal{S}^c)} = 1.$$

and we will show only that $\liminf_{n\to\infty} \frac{\pi_{\mathbb{P},r_n^-}(\mathcal{S})}{\pi_{\mathbb{P},r_n^+}(\mathcal{S})} = 1$. The result for $\mathcal{S}^c$ is identical.

The proof proceeds similarly to Lemma 6. Letting

$$\mathcal{R}_n(x) := \left\{x' \in \mathcal{S} : x' \in B(x, r_n^+), x' \notin B(x, r_n^-\right\}$$

Rewriting

$$\frac{\pi_{\mathbb{P},r_n^-}(\mathcal{S})}{\pi_{\mathbb{P},r_n^+}(\mathcal{S})} = 1 - \frac{\int_{\mathcal{S}} \int_{\mathcal{R}_n} f(y)f(x)dydx}{\pi_{\mathbb{P},r_n^+}(\mathcal{S})}$$

we have that

$$\liminf_{n\to\infty} \int_{\mathcal{S}} \int_{\mathcal{R}_n} f(y)f(x)dydx \le \mathbb{P}(\mathcal{S})\Lambda_\sigma \liminf_{n\to\infty} \big((r_n^+)^d - (r_n^-)^d\big)\nu^d = 0$$

where the equality occurs with probability one. On the other hand by hypothesis

$$\limsup_{n\to\infty} \pi_{\mathbb{P},r_n^+}(\mathcal{S}) \ge c > 0.$$

and the result follows by Slutsky's Theorem. ☐

Lemma 8 (Stationary distribution lower bound)**.** *With probability one, the following statement holds: let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of stagnating transportation maps from $\mathbb{P}$ to $\mathbb{P}_n$. Then, for any $\epsilon > 0$, there exists some $m \in \mathbb{N}$ such that for all $n \ge m$,*

$$\min_{S\in\mathcal{L}(\widetilde{G}_{n,r})} \pi_{\mathbb{P},r}(T_n^{-1}(S)) \ge \frac{9\lambda_\sigma^4 \nu_d r^d}{50\Lambda_\sigma^2} - \epsilon$$

Proof. Fix $\epsilon > 0$, and let $S \in \mathcal{L}(\widetilde{G}_{n,r})$ be arbitrary. Write $\mathcal{S} := T_n^{-1}(S)$.

We can upper bound $\widetilde{\pi}_{n,r}(S)$ by $\pi_{\mathbb{P},r}(\mathcal{S})$ plus a remainder term.

$$\begin{aligned}\widetilde{\pi}_{n,r}(S) &\le \frac{\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma} \mathbf{1}(\|x-x'\| \le r_n^+)f(x')f(x)dx'dx}{\int_{\mathcal{C}_\sigma}\int_{\mathcal{C}_\sigma} \mathbf{1}(\|x-x'\| \le r_n^+)f(x')f(x)dx'dx} \\ &= \pi_{\mathbb{P},r_n^-}(\mathcal{S}) + \frac{\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma} \mathbf{1}(r_n^- \le \|x-x'\| \le r_n^+)f(x')f(x)dx'dx}{\int_{\mathcal{C}_\sigma}\int_{\mathcal{C}_\sigma} \mathbf{1}(\|x-x'\| \le r_n^+)f(x')f(x)dx'dx} \end{aligned} \tag{7}$$

Clearly $\pi_{\mathbb{P},r_n^-}(\mathcal{S}) \le \pi_{\mathbb{P},r}(\mathcal{S})$. Moreover

$$\frac{\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma} \mathbf{1}(r_n^- \le \|x-x'\| \le r_n^+)f(x')f(x)dx'dx}{\int_{\mathcal{C}_\sigma}\int_{\mathcal{C}_\sigma} \mathbf{1}(\|x-x'\| \le r_n^+)f(x')f(x)dx'dx} \le \frac{\Lambda_\sigma}{\lambda_\sigma}\left(\frac{r_n^+ - r_n^-}{r_n^-}\right)^d$$

and by (7), we have

$$s(\widetilde{G}_{n,r}) \le \widetilde{\pi}_{n,r}(S) \le \pi_{\mathbb{P},r}(\mathcal{S}) + \frac{\Lambda_\sigma}{\lambda_\sigma}\left(\frac{r_n^+ - r_n^-}{r_n^-}\right)^d$$

The remainder term $\frac{\Lambda_\sigma}{\lambda_\sigma}\left(\frac{r_n^+ - r_n^-}{r_n^-}\right)^d$ is independent of S and asymptotically $o_p(1)$ by Lemma 1. An application of Lemma 11 completes the proof. ☐

2.4 Continuous conductance function.

In Theorem 2, we restate a necessary result from the 3/20 weekly notes.

Assumption 1 (Embedding). *Assume there exists $K \subset \mathbb{R}^d$ convex space, mapping $g : K \to \mathcal{C}_\sigma$, and constant $L < \infty$ such that*

$$\forall x, y \in K, |g(x) - g(y)| \leq L\,|x - y|\,, \text{ and } \det(D_x g) = 1.$$

In other words, g is measure-preserving and L-Lipschitz.

Theorem 2. *Assume $\mathcal{C}_\sigma \subset \mathbb{R}^d$ satisfies Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and Lipschitz function g with Lipschitz constant $L < \infty$. Then, for any $0 < r < \sigma/2\sqrt{d}$, the continuous conductance function of the speedy r-ball walk satisfies*

$$\widetilde{\Phi}_{\nu,r}(t) \geq \frac{r}{2^{12} D_K L \sqrt{d}}.$$

Corollary 2 follows almost immediately for Theorem 2.

Corollary 2. *Assume $\mathcal{C}_\sigma \subset \mathbb{R}^d$ satisfies Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and Lipschitz function g with Lipschitz constant $L < \infty$. Then, for any $0 < r < \sigma/2\sqrt{d}$, the continuous conductance function of the speedy r-ball walk satisfies*

$$\widetilde{\Phi}_{\mathbb{P},r}(t) \geq \frac{\lambda_\sigma^4 r}{2^{12} \Lambda_\sigma^4 D_K L \sqrt{d}}.$$

where we recall $\lambda_\sigma = \inf_{x \in \mathcal{C}_\sigma} f(x)$ and $\Lambda_\sigma = \sup_{x \in \mathcal{C}_\sigma} f(x)$.

3 Other results

We state Lemma 9 without proof. The proof in the uniform case can be found in the 3/20 notes.

Lemma 9. *Let $x \in \mathcal{C}_\sigma$. Then, for any $r < \frac{\sigma}{2\sqrt{d}}$,*

$$\ell_{\mathbb{P},r}(x) \geq \frac{6\lambda_\sigma^2}{25} r^d \nu_d$$

and for any $r > 0$,

$$\ell_{\mathbb{P},r}(x) \leq \Lambda_\sigma^2 r^d \nu_d$$

Lemma 10. *Let*

$$\mu' = \frac{6(n-1)\lambda_\sigma^2}{25} r^d \nu_d,\ \ \mu = (n-1)\Lambda_\sigma^2 r^d \nu_d.$$

Then for any $\delta \in [0, 1]$,

$$\Pr\left(\min_{x_i \in \widetilde{\mathbf{X}}_n} \deg(x_i; \widetilde{G}_{n,r}) \leq (1-\delta)\mu'\right) \leq (n-1)\exp\{-\delta^2 \mu'/2\}$$

$$\Pr\left(\max_{x_i \in \widetilde{\mathbf{X}}_n} \deg(x_i; \widetilde{G}_{n,r}) \geq (1+\delta)\mu\right) \leq (n-1)\exp\{-\delta^2 \mu/3\}$$

Proof. For each $x_i \in \widetilde{\mathbf{X}}_n$, letting $Y_{ij} = \mathbf{1}(x_j \in \mathcal{C}_\sigma, \|x_i - x_j\| \leq r)$ we can write $\deg(x_i; \widetilde{G}_{n,r}) = \sum_{i \neq j} Y_{ij}$. Note that $\mathbb{E}(Y_{ij}|x_i) = \ell_{\mathbb{P},r}(x_i)$, and $Y_i j$ and $Y_{ij'}$ are independent for all $j \neq j'$. Therefore the desired result follows from Lemmas 9 and 12 along with a union bound. □

Lemma 11. *With probability one, the following statement holds:*

$$\liminf_{n\to\infty} s(\widetilde{G}_{n,r}) \geq \frac{9\lambda_\sigma^4 \nu_d r^d}{50\Lambda_\sigma^2}$$

Proof. We rewrite

$$\begin{aligned} s(\widetilde{G}_{n,r}) &= \frac{9\left[\min_{x\in\widetilde{\mathbf{X}}_n}\left\{\widetilde{\deg}_{n,r}(x)\right\}\right]^2}{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{\mathbf{X}}_n)} \\ &\geq \frac{9\left[\min_{x\in\widetilde{\mathbf{X}}_n}\left\{\widetilde{\deg}_{n,r}(x)\right\}\right]^2}{n\max_{x\in\widetilde{\mathbf{X}}_n}\left\{\widetilde{\deg}_{n,r}(x)\right\}} \end{aligned}$$

The statement follows by Lemma 10 and the Borel-Cantelli Lemma. □

Lemma 12 (Multiplicative Chernoff Bound)**.** *Let $p', p \in [0,1]$, and let $Y_1, \dots, Y_n$ be independent $\{0,1\}$-valued random variables with $p' \leq E(Y_i) \leq p$ for all $i = 1, \dots, n$. Then, letting $\mu = pn$ and $\mu' = p'n$,*

$$\Pr\left(\sum_{i=1}^n Y_i \leq (1-\delta)\mu'\right) \leq \exp\{-\delta^2\mu'/2\}$$
$$\Pr\left(\sum_{i=1}^n Y_i \geq (1+\delta)\mu\right) \leq \exp\{-\delta^2\mu/3\}$$

for any $\delta \in [0,1]$.