A Proofs

In this supplement, we present proofs for "Local Clustering of Density Upper Level Sets". Sections A.1 - A.3 detail the proof for Theorem 1. In Sections A.4 and A.5, we establish a link between the conductance and local spread, and mixing time; this will be necessary for Theorem 2. The proof of Theorem 2 is then carried out in Sections A.6 - A.10. A.11 gives some general concentration results we use throughout, before we finish with the proofs of Theorems 3 and 4, and Corollary 1, in Sections A.12 - A.14.

A.1 Volume estimates

Let $\mathcal{A} \subseteq \mathbb{R}^d$, and for $\sigma \geq 0$, write $\sigma B := B(0, \sigma) = \{x \in \mathbb{R}^d : \|x\| \leq \sigma\}$ for the closed ball of radius σ centered at the origin (and let $B^\circ(0, \sigma)$ denote the corresponding open ball). Let $\mathcal{A}_\sigma = \mathcal{A} + \sigma B$ be the direct sum of $\mathcal{A}$ and σB, $\mathcal{A}_\sigma = \{z = x + y : x \in \mathcal{A}, y \in \sigma B\}$.

Lemma 1. *If $\mathcal{A}$ is closed and bounded, then for any $\delta > 0$,*

$$\nu(\mathcal{A}_\sigma + \delta B) \leq \left(1 + \frac{\delta}{\sigma}\right)^d \nu(\mathcal{A}_\sigma).$$

Proof. We will show that for any $\epsilon > 0$,

$$\frac{\nu(\mathcal{A}_\sigma + \delta B)}{\nu(\mathcal{A}_\sigma)} \leq \frac{(\sigma + \delta + \epsilon)^d}{\sigma^d} \tag{A.1}$$

which is sufficient to prove the claim.

Fix $\epsilon > 0$. Our first goal is to find a finite collection $x_1, \ldots, x_N \in \mathbb{R}^d$ such that

$$\bigcup_{i=1}^N B(x_i, \sigma) \subseteq \mathcal{A}_\sigma \subset \bigcup_{i=1}^N B(x_i, \sigma + \epsilon). \qquad (N := N(\epsilon))$$

Observe that since $\mathcal{A}$ is closed and bounded, it is compact. As $B(x, \sigma)$ is compact, and the direct sum of two compact sets is itself compact, $\mathcal{A}_\sigma$ is compact. Moreover,

$$\mathcal{A}_\sigma \subset \bigcup_{x \in \mathcal{A}} B^\circ(x, \sigma + \epsilon)$$

so by compactness there exists $x_1, \ldots, x_N \in \mathcal{A}$ such that

$$\mathcal{A}_\sigma \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \epsilon).$$

By the triangle inequality, $\mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \epsilon + \delta)$. Of course, for each $x_i \in \mathcal{A}$, $B(x_i, \sigma) \in \mathcal{A}_\sigma$. Summarizing our findings, we have

$$\bigcup_{i=1}^N B(x_i, \sigma) \subseteq \mathcal{A}_\sigma, \ \mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N B^\circ(x_i, \sigma + \delta + \epsilon) \tag{A.2}$$

We next show a lower bound on $\nu(\mathcal{A}_\sigma)$. Partition $\mathcal{A}_\sigma$ using the balls $B(x_i, \sigma)$, meaning let $\mathcal{A}_\sigma^{(1)} := B(x_1, \sigma)$, $\mathcal{A}_\sigma^{(2)} := B(x_2, \sigma) \setminus B(x_1, \sigma)$, and continuing, so that

$$\mathcal{A}_\sigma^{(i)} := B(x_i, \sigma) \setminus \bigcup_{j=1}^{i-1} \mathcal{A}_\sigma^{(j)}. \qquad (i = 1, \ldots, N)$$

Observe that $\bigcup_{i=1}^N \mathcal{A}_\sigma^{(i)} = \bigcup_{i=1}^N B(x_i,\sigma)$, so by (A.2) $\mathcal{A}_\sigma \supseteq \bigcup_{i=1}^N \mathcal{A}_\sigma^{(i)}$. As $\mathcal{A}_\sigma^{(1)},\dots,\mathcal{A}_\sigma^{(N)}$ are non-overlapping,

$$\begin{aligned}\nu(\mathcal{A}_\sigma) &\geq \sum_{i=1}^N \nu(\mathcal{A}_\sigma^{(i)})\\ &= \sigma^d \nu_d \sum_{i=1}^N \frac{\nu(\mathcal{A}_\sigma^{(i)})}{\nu(B(x_i,\sigma))}\end{aligned}$$

We turn to proving an upper bound on $\nu(\mathcal{A}_\sigma + \delta B)$. Let $\mathcal{A}_{\sigma+\epsilon+\delta}^{(1)} := B(x_1, \sigma+\delta+\epsilon)$ and

$$\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)} := B(x_i,\sigma+\delta+\epsilon) \setminus \bigcup_{j=1}^{i-1} \mathcal{A}_{\sigma+\delta+\epsilon}^{(j)}. \qquad (i=2,\dots,N)$$

As $\bigcup_{i=1}^N \mathcal{A}_{\sigma+\delta+\epsilon}^{(i)} = \bigcup_{i=1}^N B(x_i,\sigma+\delta+\epsilon)$, by (A.2)

$$\mathcal{A}_\sigma + \delta B \subset \bigcup_{i=1}^N \mathcal{A}_{\sigma+\delta+\epsilon}^{(i)}$$

and therefore

$$\begin{aligned}\nu(\mathcal{A}_{\sigma+\delta}) &\leq \sum_{i=1}^N \nu\big(\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)}\big)\\ &= \sum_{i=1}^N \nu_d(\sigma+\delta+\epsilon)^d \frac{\nu(\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)})}{\nu(B(x_i,\sigma+\delta+\epsilon))}\\ &\leq \nu_d(\sigma+\delta+\epsilon)^d \sum_{i=1}^N \frac{\nu(\mathcal{A}_\sigma^{(i)})}{\nu(B(x_i,\sigma))}\end{aligned}$$

where the last inequality follows from Lemma 2. We have shown (A.1), and thus the claim. ☐

Lemma 2. *For $i=1,\dots,N$ and $A_\sigma^{(i)}, A_{\sigma+\delta+\epsilon}^{(i)}$ as in Theorem 1,*

$$\frac{\nu(\mathcal{A}_{\sigma+\delta+\epsilon}^{(i)})}{\nu(B(x_i,\sigma+\delta+\epsilon))} \leq \frac{\nu(\mathcal{A}_\sigma^{(i)})}{\nu(B(x_i,\sigma))}$$

Proof. Let $\delta' := \delta + \epsilon$. It will be sufficient to show that

$$\left(\mathcal{A}_{\sigma+\delta'}^{(i)} - \{x_i\}\right) \subseteq \left(1 + \frac{\delta'}{\sigma}\right) \cdot \left(\mathcal{A}_\sigma^{(i)} - \{x_i\}\right)$$

since then

$$\nu(\mathcal{A}_{\sigma+\delta'}^{(i)}) \leq \left(1+\frac{\delta'}{\sigma}\right)^d \nu(\mathcal{A}_\sigma^{(i)}) = \frac{\nu(B(x_i,\sigma+\delta'))}{\nu(B(x_i,\sigma))}\nu(\mathcal{A}_\sigma^{(i)}).$$

Assume without loss of generality that $x_i = 0$, and let $x \in \mathcal{A}_{\sigma+\delta'}^{(i)}$, meaning

$$\|x\| \leq \sigma+\delta', \;\; \|x - x_j\| > \sigma + \delta' \text{ for } j = 1,\dots,i-1. \qquad (A.3)$$

Letting $x' = \frac{\sigma}{\sigma+\delta'}x$, since $\|x\| \leq \sigma+\delta'$, $\|x'\| \leq \sigma$ and therefore $x' \in B(0,\sigma)$. Additionally observe that for any $j = 1,\dots,i-1$, by the triangle inequality

$$\|x' - x_j\| \geq \|x - x_j\| - \|x - x'\| > \sigma + \delta' - \frac{\delta'}{\sigma+\delta'}\|x\| \geq \sigma$$

and therefore $x' \notin B(x_j,\sigma)$ for any $j = 1,\dots,i-1$. So $x' \in \mathcal{A}_\sigma^{(i)}$. ☐

We will need to carefully control the volume of expansion sets using the estimate in Lemma 1; Lemma 3 serves this purpose.

Lemma 3. *For any* $0 \le x \le 1/2d$,

$$(1+x)^d \le 1 + 2\,dx$$
$$(1-x)^d \ge 1 - 2\,dx.$$

Proof. We take the binomial expansion of $(1+x)^d$:

$$\begin{aligned}
(1+x)^d &= \sum_{k=0}^{d} \binom{d}{k} x^k \\
&= 1 + dx + dx \left(\sum_{k=2}^{d} \frac{\binom{d}{k} x^{k-1}}{d} \right) \\
&\le 1 + dx + dx \left(\sum_{k=2}^{d} \frac{\binom{d}{k}}{(2d)^{k-1} d} \right) && \text{(since } x \le \tfrac{1}{2d}\text{)} \\
&\le 1 + dx + dx \left(\sum_{k=2}^{d} \frac{1}{2^{k-1}} \right) \le 1 + 2\,dx.
\end{aligned}$$

The proof for the corresponding lower bound on $(1-x)^d$ is symmetric. □

Let $\mathcal{C}_{\sigma,\sigma+r} := \{x : 0 < \mathrm{dist}(x, \mathcal{C}_\sigma) < r\}$, where $\mathcal{C}_\sigma$ is as in Theorem 1. Lemma 4 involves the bulk of the technical effort required to prove Theorem 1; it will be necessary to bound the expected cut size of $\mathcal{C}_\sigma[X]$ in $G_{n,r}$.

Lemma 4. *Under the conditions of Theorem 1, and for any* $0 < r \le \sigma/2d$,

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) \le \frac{2dr}{\sigma} \left(\lambda_\sigma - c_0 \frac{r^\gamma}{\gamma+1} \right) \nu(\mathcal{C}_\sigma)$$

Proof. We partition $\mathcal{C}_{\sigma,\sigma+r}$ into slices based on distance from $\mathcal{C}_\sigma$ as follows: for $k \in \mathbb{N}$,

$$\mathcal{T}_{i,k} = \left\{ x \in \mathcal{C}_{\sigma,\sigma+r} : t_{i,k} < \frac{\mathrm{dist}(x, \mathcal{C}_\sigma)}{r} \le t_{i+1,k} \right\}, \quad \mathcal{C}_{\sigma,\sigma+r} = \bigcup_{i=0}^{k-1} \mathcal{T}_{i,k}$$

where $t_i = i/k$ for $i = 0, \ldots, k-1$. As a result, for any $k \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) = \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)\,dx = \sum_{i=0}^{k-1} \int_{\mathcal{T}_{i,k}} f(x)\,dx \le \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x). \tag{A.4}$$

(A1) and (A2) imply the upper bound

$$\max_{x \in \mathcal{T}_{i,k}} f(x) \le \lambda_\sigma - c_0 (r t_{i,k})^\gamma,$$

and writing

$$\nu(\mathcal{T}_{i,k}) = \nu(\mathcal{C}_\sigma + r t_{i+1,k} B) - \nu(\mathcal{C}_\sigma + r t_{i,k} B) =: \nu_{i+1,k} - \nu_{i,k},$$

we have

$$\sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x\in\mathcal{T}_{i,k}} f(x) \leq \sum_{i=0}^{k-1} \Big\{\nu_{i+1,k} - \nu_{i,k}\Big\}\Big(\lambda_\sigma - c_0(rt_{i,k})^\gamma\Big)$$
$$= \sum_{i=1}^{k} \underbrace{\nu_{i,k}\Big([\lambda_\sigma - c_0(rt_{i-1,k})^\gamma] - [\lambda_\sigma - c_0(rt_{i,k})^\gamma]\Big)}_{:=\Sigma_k} + \underbrace{\Big(\nu_{k,k}\,[\lambda_\sigma - c_0 r^\gamma] - \nu_{1,k}\lambda_\sigma\Big)}_{:=\xi} \tag{A.5}$$

where the second equality comes from rearranging terms in the sum.

We first consider the term Σ_k. $\mathcal{C}$ has finite diameter by (A1), as otherwise $\int_{\mathcal{C}_\sigma} f(x)dx = \infty$. Letting $\overline{\mathcal{C}}$ be the closure of $\mathcal{C}$, we observe that $\overline{\mathcal{C}_\sigma} = \overline{\mathcal{C}} + \sigma B$, and moreover for any $\delta > 0$, $\nu(\overline{\mathcal{C}_\sigma} + \delta B) = \nu(\mathcal{C}_\sigma + \delta B)$ (as $\partial(\mathcal{C}_\sigma + \delta B)$ is Lipschitz and therefore has measure zero). As a result, for each $t_{i,k}, i = 1, \ldots, k$ we may apply Lemma 1 to $\overline{\mathcal{C}}$ and obtain

$$\nu_{i,k} = \nu(\mathcal{C}_\sigma + rt_{i,k}B) \leq \nu(\mathcal{C}_\sigma)\left(1 + \frac{rt_{i,k}}{\sigma}\right)^d \tag{A.6}$$

which in turn gives

$$\Sigma_k \leq c_0\nu(\mathcal{C}_\sigma)r^\gamma \sum_{i=1}^{k}\left(1 + \frac{rt_{i,k}}{\sigma}\right)^d\Big((t_{i,k})^\gamma - (t_{i-1,k})^\gamma\Big)$$
$$= c_0\nu(\mathcal{C}_\sigma)r^\gamma \sum_{i=1}^{k}\left(1 + \frac{ru_{i,k}^{1/\gamma}}{\sigma}\right)^d(u_{i,k} - u_{i,k-1}). \qquad (u_{i,k} = t_{i,k}^\gamma) \tag{A.7}$$

(A.7) is a Riemann sum, and taking the limit as $k \to \infty$ we obtain

$$\lim_{k\to\infty} c_0\nu(\mathcal{C}_\sigma)r^\gamma \sum_{i=1}^{k}\left(1 + \frac{ru_{i,k}^{1/\gamma}}{\sigma}\right)^d(u_{i,k} - u_{i,k-1}) = c_0\nu(\mathcal{C}_\sigma)r^\gamma \int_0^1 \left(1 + \frac{ru^{1/\gamma}}{\sigma}\right)^d du$$
$$\overset{(i)}{\leq} c_0\nu(\mathcal{C}_\sigma)r^\gamma \int_0^1 \left(1 + \frac{2dru^{1/\gamma}}{\sigma}\right) du$$
$$= c_0\nu(\mathcal{C}_\sigma)r^\gamma\left(1 + \gamma\frac{2dr}{(\gamma+1)\sigma}\right). \tag{A.8}$$

where (i) follows from Lemma 3 in light of the fact $r \leq \sigma/2d$.

An upper bound on ξ follows from largely the same logic, although it does not involve integration:

$$\xi \overset{(ii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1 + \frac{r}{\sigma}\right)^d(\lambda_\sigma - c_0r^\gamma) - \lambda_\sigma\right\}$$
$$\overset{(iii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1 + \frac{2dr}{\sigma}\right)(\lambda_\sigma - c_0r^\gamma) - \lambda_\sigma\right\} = \nu(\mathcal{C}_\sigma)\left\{\frac{2dr}{\sigma}(\lambda_\sigma - c_0r^\gamma) - c_0r^\gamma\right\}. \tag{A.9}$$

where (ii) follows from (A.6), and (iii) from Lemma 3. As the bounds in (A.4) and (A.5) hold for all k, these along with (A.8) and (A.9) imply the desired result. ☐

Lemma 5 will be necessary to lower bound the expected volume of $\mathcal{C}_\sigma[X]$ in $G_{n,r}$. Define the *uniform local conductance* $\ell_{\nu,r}(u)$ to be

$$\ell_{\nu,r}(u) = \nu\big(\mathcal{C}_\sigma \cap B(u,r)\big)$$

Lemma 5. *Let $u \in \mathcal{C}_\sigma$. Then, for any $0 < r \leq \frac{\sigma}{2\sqrt{d}}$,*

$$\ell_{\nu,r}(u) \geq \frac{6}{25}\nu_d r^d.$$

Proof. Since $u \in \mathcal{C}_\sigma$ there exists $x \in \mathcal{C}$ such that $u \in B(x,\sigma)$, and as $B(x,\sigma) \subseteq \mathcal{C}_\sigma$,

$$\nu\big(B(u,r) \cap B(x,\sigma)\big) \leq \nu\big(B(u,r) \cap \mathcal{C}_\sigma\big)$$

Without loss of generality, let $\|u - x\| = \sigma$; it is not hard to see that if $\|u - x\| < \sigma$, the volume of the overlap will only grow. Then, since $\|u - x\| = \sigma$, $B(u,r) \cap B(x,\sigma)$ contains a spherical cap of radius r and height

$$h = r - (r)^2/2\sigma = r\left(1 - \frac{r}{2\sigma}\right)$$

which by Lemma 6 has volume

$$\nu_{cap} = \frac{1}{2}\nu_d r^d I_{1-\alpha}\left(\frac{d+1}{2}, \frac{1}{2}\right)$$

with $\alpha = 1 - \frac{2rh - h^2}{r^2} = \frac{r^2}{4\sigma^2} \leq \frac{1}{8d}$.

Then by Lemmas 7 (applied with $t = 1$) and 8,

$$\begin{aligned} I_{1-\alpha}\left(\frac{d+1}{2}, \frac{1}{2}\right) &\geq 1 - \frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}\frac{3}{2\sqrt{d}} \\ &\geq 1 - \frac{3}{4}\sqrt{\frac{d+2}{\pi d}} \geq 1 - \frac{3}{4}\sqrt{\frac{3}{2\pi}}. \end{aligned}$$

□

The following formula for the volume of the spherical cap, stated in terms of the incomplete beta function, is well known. We include it without proof.

Lemma 6. *Let $\mathrm{Cap}_r(h)$ denote a spherical cap of radius r and height h. Then,*

$$\nu\big(\mathrm{Cap}_r(h)\big) = \frac{1}{2}\nu_d r^d I_{1-\alpha}(\frac{d+1}{2}; \frac{1}{2})$$

where

$$\alpha := 1 - \frac{2rh - h^2}{r^2}$$

and

$$I_{1-\alpha}(z,w) = \frac{\Gamma(z+w)}{\Gamma(z)\Gamma(w)}\int_0^{1-\alpha} u^{z-1}(1-u)^{w-1}du.$$

is the cumulative distribution function of a $\mathrm{Beta}(z,w)$ *distribution, evaluated at* $1-\alpha$.

Lemma 7. *For any $0 \leq t \leq 1$ and $\alpha \leq \frac{t^2}{8d}$,*

$$\int_0^{1-\alpha} u^{(d-1)/2}(1-u)^{-1/2}du \geq \frac{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{d}{2}+1\big)} - \frac{3t}{2\sqrt{d}}$$

Proof. We can write

$$\int_0^{1-\alpha} u^{(d-1)/2}(1-u)^{-1/2}du = \int_0^1 u^{(d-1)/2}(1-u)^{-1/2}du - \int_{1-\alpha}^1 u^{(d-1)/2}(1-u)^{-1/2}du$$

The first integral is simply the beta function, with

$$B(\frac{d+1}{2}, \frac{1}{2}) := \frac{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{d}{2}+1\big)}.$$

To upper bound the second integral, we apply the Taylor theorem with remainder to $(1-u)^{-1/2}$, obtaining

$$(1-u)^{-1/2} \leq \alpha^{-1/2} + \max_{u \in (1-\alpha,1)} \frac{\alpha}{2}(1-u)^{-3/2} = \frac{3}{2}\alpha^{-1/2}.$$

As a result,

$$\begin{aligned}
\int_{1-\alpha}^{1} u^{(d-1)/2}(1-u)^{-1/2} du &\leq \frac{3}{2}\alpha^{-1/2} \int_{1-\alpha}^{1} u^{(d-1)/2} du \\
&= \frac{3}{d+1}\alpha^{-1/2}\left(1-(1-\alpha)^{(d+1)/2}\right) \\
&\overset{(iv)}{\leq} \frac{3}{(d+1)}\alpha^{-1/2}(\alpha(d+1)) \\
&= 3\alpha^{1/2}.
\end{aligned}$$

where (iv) follows from Lemma 3, and the condition $\alpha \leq \frac{t^2}{8d}$. The result follows from the condition $\alpha \leq \frac{t^2}{8d}$. $\square$

Lemma 8 follows from $\Gamma(1/2) = \sqrt{\pi}$ and the upper bound $\Gamma(x+1)/\Gamma(x+s) \leq (x+1)^{1-s}$ for $s \in [0,1]$.

Lemma 8.

$$\frac{\Gamma\big(\frac{d}{2}+1\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \leq \sqrt{\frac{d+2}{2\pi}}$$

A.2 Density-weighted cut and volume estimates

For notational ease, we write

$$\begin{aligned}
\mathrm{cut}_{n,r} &= \mathrm{cut}(\mathcal{C}_\sigma[X]; G_{n,r}),\ \mu_K = \mathbb{E}(\mathrm{cut}_{n,r}),\ p_K = \frac{\mu_K}{\binom{n}{2}} \\
\mathrm{vol}_{n,r} &= \mathrm{vol}(\mathcal{C}_\sigma[X]; G_{n,r}),\ \mu_V = \mathbb{E}(\mathrm{vol}_{n,r}),\ p_V = \frac{\mu_V}{\binom{n}{2}} \\
\mathrm{vol}_{n,r}^c &= \mathrm{vol}(X \setminus \mathcal{C}_\sigma[X]; G_{n,r}),\ \mu_V^c = \mathbb{E}(\mathrm{vol}_{n,r}^c),\ p_V^c = \frac{\mu_V^c}{\binom{n}{2}}
\end{aligned}$$

for the random variable, mean, and probability of cut size and volume, respectively.

Lemma 9. *Under the setup and conditions of Theorem 1, and for any* $0 < r \leq \sigma/2d$,

$$p_K \leq \frac{4d\nu_d r^{d+1}\lambda}{\sigma}\left(\lambda_\sigma - c_0\frac{r^\gamma}{\gamma+1}\right)\nu(\mathcal{C}_\sigma)$$

Proof. We can write $\mathrm{cut}_{n,r}$ as a double sum,

$$\mathrm{cut}_{n,r} = \sum_{i=1}^{n}\sum_{j \neq i} \mathbf{1}(x_i \notin \mathcal{C}_\sigma)\mathbf{1}(x_j \in \mathcal{C}_\sigma)\mathbf{1}(\|x_i - x_j\| \leq r) \tag{A.10}$$

and by linearity of expectation, we obtain

$$p_K = \frac{\mu_K}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \notin \mathcal{C}_\sigma, x_j \in \mathcal{C}_\sigma, \|x_i - x_j\| \leq r). \qquad \text{(for each } i, j,\ i \neq j\text{)}$$

Writing this with respect to the density function f, we have

$$\begin{aligned} p_K &= 2 \int_{\mathbb{R}^d \setminus \mathcal{C}_\sigma} f(x) \mathbb{P}\big(B(x,r) \cap \mathcal{C}_\sigma\big)\, dx \\ &= 2 \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x) \mathbb{P}\big(B(x,r) \cap \mathcal{C}_\sigma\big)\, dx \\ &\leq 2\nu_d r^d \lambda \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)\, dx = 2\nu_d r^d \lambda \mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}). \end{aligned}$$

where the inequality follows from (A3), which implies $f(x) \leq \lambda$ for $x \in \mathcal{C}_\sigma \setminus \mathcal{C}$. Then, upper bounding the integral using Lemma 9 gives the final result. □

Lemma 10. *Under the setup and conditions of Theorem 1, and for any $0 < r \leq \sigma/2d$,*

$$p_V \geq \frac{12}{25} \lambda_\sigma^2 \nu_d r^d \nu(\mathcal{C}_\sigma)$$

Proof. The proof will proceed similarly to Lemma 9. We begin by writing $\mathrm{vol}_{n,r}$ as the sum of indicator functions,

$$\mathrm{vol}_{n,r} = \sum_{i=1}^{n} \sum_{j \neq i} \mathbf{1}(x_i \in \mathcal{C}_\sigma) \mathbf{1}(x_j \in B(x_i, r)) \tag{A.11}$$

and by linearity of expectation we obtain

$$p_V = \frac{\mu_V}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \in \mathcal{C}_\sigma, x_j \in B(x_i, r)). \qquad \text{(for any } i, j,\ i \neq j.\text{)}$$

Writing this with respect to the density function f, we have

$$\begin{aligned} p_V &= 2 \int_{\mathcal{C}_\sigma} f(x) \mathbb{P}(B(x,r))\, dx \\ &\geq 2 \int_{\mathcal{C}_\sigma} f(x) \mathbb{P}(B(x,r) \cap \mathcal{C}_\sigma)\, dx \end{aligned}$$

whence the claim then follows by Lemma 5. □

To employ Lemmas 9 and 10 in the proof of Theorem 1, we must relate the random variable

$$\Phi_{n,r}(\mathcal{C}_\sigma[X]) = \frac{\mathrm{cut}_{n,r}}{\min\left\{\mathrm{vol}_{n,r}, \mathrm{vol}_{n,r}^c\right\}}$$

to p_K and p_V.

In Lemma 11, we give probabilistic bounds on the $\mathrm{cut}_{n,r}$, $\mathrm{vol}_{n,r}$ and $\mathrm{vol}_{n,r}^c$ in terms of p_K and p_V. These bounds are a straightforward consequence of Lemma 30, Hoeffding's inequality for U-statistics.

Lemma 11. *For any $\delta \in (0, 1]$,*

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_K + \sqrt{\frac{\log(1/\delta)}{n}}, \ \textit{and}\ \frac{\mathrm{vol}_{n,r}}{\binom{n}{2}}, \frac{\mathrm{vol}_{n,r}^c}{\binom{n}{2}} \geq p_V - \sqrt{\frac{\log(1/\delta)}{n}}. \tag{A.12}$$

each with probability at least $1 - \delta$.

Proof of Lemma 11. From (A.10) and (A.11), we see that $\mathrm{cut}_{n,r}$ and $\mathrm{vol}_{n,r}$, properly scaled, can be expressed as order-2 U-statistics,

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_K(x_i, x_j), \quad \frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_V(x_i, x_j)$$

with kernels

$$\phi_K(x_i, x_j) = \mathbf{1}(x_i \in \mathcal{C}_\sigma, x_j \notin \mathcal{C}_\sigma, \|x_i - x_j\| \leq r) + \mathbf{1}(x_j \in \mathcal{C}_\sigma, x_i \notin \mathcal{C}_\sigma, \|x_i - x_j\| \leq r)$$
$$\phi_V(x_i, x_j) = \mathbf{1}(x_i \in \mathcal{C}_\sigma, \|x_i - x_j\| \leq r) + \mathbf{1}(x_j \in \mathcal{C}_\sigma, \|x_i - x_j\| \leq r).$$

Similarly,

$$\frac{\mathrm{vol}^c_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_{V^c}(x_i, x_j)$$

with kernel,

$$\phi_{V^c}(x_i, x_j) = \mathbf{1}(x_i \notin \mathcal{C}_\sigma, \|x_i - x_j\| \leq r) + \mathbf{1}(x_j \notin \mathcal{C}_\sigma, \|x_i - x_j\| \leq r).$$

From Lemma 30 we therefore have

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_K + \sqrt{\frac{\log(1/\delta)}{n}}, \quad \frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} \geq p_V - \sqrt{\frac{\log(1/\delta)}{n}}, \quad \frac{\mathrm{vol}^c_{n,r}}{\binom{n}{2}} \geq p_V^c - \sqrt{\frac{\log(1/\delta)}{n}}$$

each with probability at least $1 - \delta$. The claim follows in light of (7), which implies $p_V^c \geq p_V$. ☐

A.3 Proof of Theorem 1

The proof of Theorem 1 is more or less given by Lemmas 9, 10, and 11. All that remains is some algebra, which we take care of below.

Fix $\delta \in (0, 1]$ and let $\delta' = \delta/3$. We rewrite

$$\Phi_{n,r}(\mathcal{C}_\sigma[X]) = \frac{p_K + \left(\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} - p_K\right)}{p_V + \left(\frac{\min\{\mathrm{vol}_{n,r}, \mathrm{vol}^c_{n,r}\}}{\binom{n}{2}} - p_V\right)}. \tag{A.13}$$

Assume (A.12) holds with respect to δ', keeping in mind that this will happen with probability at least $1 - \delta$. Along with (A.13) this means

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) \leq \frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n}$$

for $\mathrm{Err}_n = \sqrt{\frac{\log(1/\delta')}{n}}$. Now, some straightforward algebraic manipulations yield

$$\begin{aligned}
\frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n} &= \frac{p_K}{p_V}\left(\frac{p_V}{p_V - \mathrm{Err}_n}\right) + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \\
&= \frac{p_K}{p_V} + \left(\frac{p_K}{p_V} + 1\right)\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \\
&\leq \frac{p_K}{p_V} + 2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}.
\end{aligned}$$

By Lemmas 9 and 10, we have

$$\frac{p_K}{p_V} \leq \frac{100rd}{12\sigma}\frac{\lambda}{\lambda_\sigma}\frac{\left(\lambda_\sigma - c_0\frac{r^\gamma}{\gamma+1}\right)}{\lambda_\sigma}.$$

Then, by the choice of sample size in (8),

$$n \geq \frac{(2+\epsilon)^2 \log\left(\frac{3}{\delta}\right)}{\epsilon^2 p_V^2}$$

which implies $2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \leq \epsilon$.

A.4 Mixing time on graphs

We recall some notation from the main text. For an undirected graph $G = (V, E)$, the lazy random walk over G is the Markov chain with transition probabilities given by $\mathbf{W} := \frac{\mathbf{I}+\mathbf{D}^{-1}\mathbf{A}}{2}$ and stationary distribution π. Denote the m-step probability distribution of this random walk originating from a particular $v \in V$ as $q^{(m)} : V \times V \to [0, 1]$, $q^{(m)}(v, u) = e_v \mathbf{W}^m e_u$. An important intermediate quantity used to bound the relative pointwise mixing time $\tau_\infty(G)$ is the total variation distance between the distributions $q_v^{(m)} := q^{(m)}(v, \cdot)$ and π,

$$\left\|q_v^{(t+1)} - \pi\right\|_{TV} = \sum_{u \in V} |q_v^{(m)}(u) - \pi(u)|$$

We recall the *degree, cut* and *volume* functionals over a graph. For $u \in V$, $S \subseteq V$,

$$\mathrm{cut}(A; G) = \sum_{u \in A}\sum_{v \in A^c} \mathbf{1}((u,v) \in E), \quad \deg(u; G) = \sum_{v \in V} \mathbf{1}((u,v) \in E), \quad \mathrm{vol}(A; G) = \sum_{u \in A} \deg(u; G)$$

The *local spread* is defined as

$$s(G) := \frac{9}{10} \cdot \min_{u \in V}\{\deg(u; G)\} \cdot \min_{u \in V}\{\pi(v)\}$$

Letting the *normalized cut* $\Phi(S; G)$ be defined as

$$\Phi(S; G) = \frac{\mathrm{cut}(S, S^c; G)}{\min\{\mathrm{vol}(S; G), \mathrm{vol}(S^c; G)\}},$$

the *conductance* is

$$\Phi(G) = \min_{S \subseteq V} \Phi(S; G).$$

Proposition 1 relates these geometric quantities to the TV distance between $q_v^{(t)}$ and stationary distribution π.

Proposition 1. *For any $v \in V$, and any $0 < a \leq 1$,*

$$\left\|q_v^{(t+3)} - \pi\right\|_{TV} \leq \left\{as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2\min_{u \in V}\deg(u; G)}\right\} + \left(\frac{1}{1 - 2as(G)} + \frac{1}{2as(G)}\right)\left(1 - \frac{\Phi^2(G)}{2}\right)^t$$

As we will see, Proposition 1 is an essential step to providing an upper bound on the uniform mixing time. We justify this statement next, before moving on to proving Proposition 1 in the following subsection.

Uniform mixing time. Consider the uniform distance [1] between $q_v^{(t)}$ and π, given by

$$d_{\text{unif}}(q_v^{(t)}, \pi) = \max_{u \in V} \left\{ \frac{\pi(u) - q_v^{(t)}(u)}{\pi(u)} \right\}.$$

Lemma 12. *Let* $\left\| q_v^{(t)} - \pi \right\|_{TV} \leq \frac{1}{14} \max\left\{1, \frac{1}{s(G)}\right\}$. *Then,*

$$d_{\text{unif}}(q_v^{(t+3)}, \pi) \leq \frac{1}{4}$$

Proof. Fix $u \in V$ and let $m \geq t+1$ be arbitrary. The stationarity of π gives

$$\begin{aligned}
\frac{\pi(u) - q_v^m(u)}{\pi(u)} &= \sum_{y \in V} \left(\pi(y) - q^{(m-1)}(v,y)\right) \left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) \\
&\overset{(i)}{=} \sum_{y \neq u} \left(\pi(y) - q^{(m-1)}(v,y)\right) \left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) + \frac{\pi(u) - q^{(m-1)}(v,u)}{\pi(u)} \left(\frac{1}{2} - \pi(u)\right) \\
&\leq \sum_{y \neq u} \left(\pi(y) - q^{(m-1)}(v,y)\right) \left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) + \frac{\pi(u) - q^{(m-1)}(v,u)}{2\pi(u)} \qquad \text{(A.14)}
\end{aligned}$$

where (i) follows from $q^{(1)}(u,u) = \frac{1}{2}$.

Then

$$\begin{aligned}
\sum_{y \neq u} \left(\pi(y) - q^{(m-1)}(v,y)\right) \left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) &\leq \left\| q_v^{(m-1)} - \pi \right\|_{TV} \max_{y \neq u} \left| \frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)} \right| \\
&\leq \left\| q_v^{(m-1)} - \pi \right\|_{TV} \max\left\{1, \max_{y \neq u}\left\{\frac{q^{(1)}(y,u)}{\pi(u)}\right\}\right\} \\
&\leq \left\| q_v^{(m-1)} - \pi \right\|_{TV} \max\left\{1, \frac{1}{s(G)}\right\}
\end{aligned}$$

since for $y \neq u$, $q^{(1)}(y,u) \leq 1/\left(2 \min_{u \in V} \deg(u; G)\right)$. As $m - 1 \geq t$, it is well known [3] that the laziness of the random walk guarantees $\left\| q_v^{(m-1)} - \pi \right\|_{TV} \leq \left\| q_v^{(t)} - \pi \right\|_{TV}$, and therefore

$$\sum_{y \neq u} \left(\pi(y) - q^{(m-1)}(v,y)\right) \left(\frac{q^{(1)}(y,u) - \pi(u)}{\pi(u)}\right) \leq \frac{1}{14}.$$

Plugging this in to (A.14) and taking maximum on both sides, we obtain

$$d_{\text{unif}}(q_v^{(m)}, \pi) \leq \frac{2}{7} + \frac{d_{\text{unif}}(q_v^{(m-1)}, \pi)}{2} \qquad \text{(A.15)}$$

The recurrence relation of (A.15) along with the initial condition $d_{\text{unif}}(q_v^{(t)}, \pi) \leq 1$ yields

$$d_{\text{unif}}(q_v^{(t+1)}, \pi) \leq \frac{8}{14} \Rightarrow d_{\text{unif}}(q_v^{(t+2)}, \pi) \leq \frac{10}{28} \Rightarrow d_{\text{unif}}(q_v^{(t+3)}, \pi) \leq \frac{1}{4}$$

and the claim is shown. ∎

[1] Note d_{unif} is not a formally a distance as it is not symmetric.

Together, Proposition 1 and Lemma 12 imply the main result of this section.

Proposition 2. *Assume* $\min_{u\in v}\deg(u;G)\geq 10$*. Then,*

$$\tau_\infty(G)\leq\frac{2}{\Phi^2(G)}\log\left(\frac{1440}{s(G)}\right)\log\left(\frac{14}{s(G)}\right)+3\log\left(\frac{14}{s(G)}\right)+3$$

Proof. Fix $a=\frac{1}{18}$, and let $\tau_0=\frac{2}{\Phi^2(G)}\log\left(\frac{80}{as(G)}\right)$. Then,

$$\left(1-\frac{\Phi^2(G)}{2}\right)^{\tau_0}\leq\exp(-\tau_0\Phi^2(G)/2)\leq\frac{as(G)}{80}$$

and so by Proposition 1,

$$\left\|q_v^{(\tau_0+3)}-\pi\right\|_{TV}\leq\max\left\{\frac{1}{20},\frac{1}{8}+\frac{1}{20}+\frac{1}{20}\right\}+\left(\frac{2}{as(G)}\right)\frac{as(G)}{80}\leq\frac{1}{4}.$$

It is a well known fact [4] that if $\left\|q_v^{(t)}-\pi\right\|_{TV}\leq\frac{1}{4}$, then for any $)<\epsilon<1$, $\left\|q_v^{(t\log_2(1/\epsilon))}-\pi\right\|_{TV}\leq\epsilon$. Therefore, letting $\tau_1=(\tau_0+3)\log(\frac{14}{s(G)})$,

$$\left\|q_v^{(\tau_1)}-\pi\right\|_{TV}\leq\frac{1}{14s(G)}$$

and so $d_{\text{unif}}(q_v^{(\tau_1+3)},\pi)\leq\frac{1}{4}$. □

A.5 Proof of Proposition 1.

For arbitrary starting distribution q (meaning $\text{supp}(q)\subseteq V$ and $\sum_{u\in V}q(u)=1$), and for $t\geq 0$ an integer, let $q^{(t)}$ be the t-step probability distribution of the lazy random walk with starting distribution q. Consider the distance function $h_q^{(t)},t\geq 0$,

$$h_q^{(t)}(x)=\max\left\{\sum_{u\in V}\left(q^{(m)}(u)-\pi(u)\right)w(u)\right\}$$

where the maximum is over all $w:V\to[0,1]$ such that $0\leq w(u)\leq 1$ for all u, and $\sum_{u\in V}w(u)\pi(u)=x$. Writing $h_v^{(t)}(x):=h_{e_v}^{(t)}(x)$ in a small abuse of notation, in Lemmas 13 and 15, we give an upper bound on $h_{e_v}^{(t)}(x)$ for all $0\leq x\leq 1$.

Remark 1. $h_q^{(t)}$ permits an equivalent relation. Order the elements of $V=\{u_1,\ldots,u_N\}$ $(N=|V|)$, such that

$$\frac{q^{(m)}(u_1)}{\pi(u_1)}\geq\frac{q^{(m)}(u_2)}{\pi(u_2)}\geq\ldots\geq\frac{q^{(m)}(u_N)}{\pi(u_N)}$$

and let $U_k=\{u_1,\ldots,u_k\}$. Then for any x, letting k satisfy $\pi(U_{k-1})<x<\pi(U_k)$, it can be shown that,

$$h_q^{(t)}(x)=\sum_{j=1}^{k-1}(q^{(m)}(u_j)-\pi(u_j))+\frac{x-\pi(U_{k-1})}{\pi(u_k)}\left(q^{(m)}(u_k)-\pi(u_k)\right).\tag{A.16}$$

The formulation on the right hand side of (A.16) has come to be known as the *Lovasz-Simonovits curve*.

Mixing over large sets. For $0 \leq \mu \leq 1$ and $\mu \leq x \leq 1-\mu$ let

$$\ell_\mu(x) = \frac{1-\mu-x}{1-2\mu} h_q^{(0)}(\mu) + \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu)$$

be the linear interpolator between $h_q^{(0)}(\mu)$ and $h_q^{(0)}(1-\mu)$.

Lemma 13. *For any $0 \leq \mu \leq 1/2$ and $\mu \leq x \leq 1-\mu$ and $t \geq 0$,*

$$h_q^{(t)}(x) \leq \ell_\mu(x) + \max\left\{\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu}, \frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1\right\}\left(1 - \frac{\Phi^2(G)}{2}\right)^t$$

Theorem 13 is a direct consequence of Theorem 1.2 of [3]. To state the latter, we introduce

$$C_\mu = \max\left\{\frac{h_q^{(0)}(x) - \ell_\mu(x)}{\sqrt{x-\mu}}, \frac{h_q^{(0)}(x) - \ell_\mu(x)}{\sqrt{1-x-\mu}} : \mu < x < 1-\mu\right\}$$

Theorem 1 (Theorem 1.2 of [3])**.** *For any $0 \leq \mu \leq \frac{1}{2}$, $\mu \leq x \leq 1-\mu$ and an integer $t \geq 0$,*

$$h_q^{(t)}(x) \leq \ell_\mu(x) + C_\mu \min\left\{\sqrt{x-\mu}, \sqrt{1-x-\mu}\right\}\left(1 - \frac{\Phi^2(G)}{2}\right)^t$$

Proof of Lemma 13. Fix $0 \leq \mu \leq \frac{1}{2}$. We will show that for all $\mu \leq x \leq 1-\mu$

$$h_q^{(0)}(x) - \ell_\mu(x) \leq \max\left\{\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu}, \frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1\right\}\min\left\{\sqrt{x-\mu}, \sqrt{1-x-\mu}\right\} \tag{A.17}$$

whence the claim follows by Theorem 1.

Note that $\ell_\mu(\mu) = h_q^{(0)}(\mu)$, and for $x \geq \mu$,

$$h_q^{(0)}(x) \leq h_q^{(0)}(\mu) + (x-\mu)\frac{h_q^{(0)}(\mu)}{\mu}$$

by the concavity of h_0 along with Lemma 14. Some basic algebra then yields

$$\begin{aligned}
h_q^{(0)}(x) - \ell_\mu(x) &\leq h_q^{(0)}(\mu) - \left(\frac{1-\mu-x}{1-2\mu} h_q^{(0)}(\mu) + \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu)\right) + \frac{h_q^{(0)}(\mu)}{\mu}(x-\mu) \\
&= \frac{x-\mu}{1-2\mu} h_q^{(0)}(\mu) + \frac{h_q^{(0)}(\mu)}{\mu}(x-\mu) - \frac{x-\mu}{1-2\mu} h_q^{(0)}(1-\mu) \\
&\leq \sqrt{x-\mu}\left(\frac{h_q^{(0)}(\mu)}{1-2\mu} + \frac{h_q^{(0)}(\mu)}{\mu}\right)
\end{aligned}$$

On the other hand, $\ell_\mu(1-\mu) = h_q^{(0)}(1-\mu)$, and by the concavity of $h_q^{(0)}$ and Lemma 14, for $x \leq 1-\mu$

$$h_q^{(0)}(x) \leq h_q^{(0)}(1-\mu) + (1-x-\mu).$$

Similar manipulations to above give the upper bound

$$h_q^{(0)}(x) - \ell_\mu(x) \leq \sqrt{1-\mu-x}\left(\frac{h_q^{(0)}(1-\mu)}{1-2\mu} + 1\right)$$

and (A.17) follows. ∎

Lemma 14. *The subdifferential $v(x)$ of $h_q^{(0)}(x)$ satisfies*

$$-1 \leq v(x) \leq \frac{h_q^{(0)}(\mu)}{\mu}$$

Mixing over small sets.

Lemma 15. *Let $0 \le a \le 1$, and $t \ge 1$ an integer. Then for any $x \le as(G)$ or $x \ge 1 - as(G)$,*

$$h_v^{(t)}(x) \le \max\left\{as(G), \frac{1}{2^t} + \frac{9a}{20} + \frac{1}{2\min_{u\in V}\deg(u;G)}\right\}$$

Proof. First, we deal with the case $x \le as(G)$. Letting U_k be as in (A.16), we have

$$h_v^{(t)}(x) \le q_v^{(m)}(U_{k-1}) + q_v^{(m)}(u_k) \tag{A.18}$$

We will rely on the key fact that for any $u \ne v, t \ge 1$,

$$q_v^{(t)}(u) \le \frac{1}{2\min_{u\in V}\deg(u;G)} \tag{A.19}$$

On the other hand if $u = v$,

$$q_v^{(m)}(u) \le \frac{1}{2^t} + \frac{1}{2\min_{u\in V}\deg(u;G)}. \tag{A.20}$$

Therefore by (A.18), (A.19), and (A.20)

$$h_v^{(t)}(x) \le \frac{1}{2^t} + \frac{|U_k|}{2\min_{u\in V}\deg(u;G)} + \frac{1}{2\min_{u\in V}\deg(u;G)}.$$

Since $x \le as(G)$,

$$|U_k| \le \frac{x}{10\min_{u\in V}(\pi(v))} \le \frac{9a\min_{u\in V}\deg(u;G)}{10}.$$

For any $0 \le b \le 1$, $x \ge 1 - b$ implies $h_v^{(t)}(x) \le b$. Taking $b = as(G)$, the claim is shown. $\square$

Proof of Proposition 1. For any $A \subseteq V$ and any integer $t \ge 0$,

$$\max\left\{h_v^{(t)}(\pi(A)), h_v^{(t)}(1-\pi(A))\right\} \ge |q_v^{(t)}(A) - \pi(A)|$$

and taking max over both sides, we have

$$\max_{0\le x\le 1} h_v^{(t)}(x) \ge \left\|q_v^{(t)} - \pi\right\|_{TV}.$$

Letting $q = e_v W^3$, observe that $h_v^{(t+3)}(x) = h_v^{(t)}(x)$. Fix $\mu = as(G)$. Then, for $\mu \le x \le 1-\mu$, by Proposition 13,

$$\begin{aligned}
h_v^{(t+3)}(x) &= h_q^{(t)}(x)\\
&\le \max\left\{h_q^{(0)}(\mu), h_q^{(0)}(1-\mu)\right\} + \left(\frac{1}{1-2\mu} + \frac{1}{2\mu}\right)\left(1 - \frac{\Phi^2(G)}{2}\right)^t\\
&\le \max\left\{as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2\min_{u\in V}\deg(u;G)}\right\} + \left(\frac{1}{1-2as(G)} + \frac{1}{2as(G)}\right)\left(1 - \frac{\Phi^2(G)}{2}\right)^t
\end{aligned}$$

where the last inequality comes from application of Lemma 15 to $h_q^{(t)} = h_v^{(t+3)}$.

For $x \le \mu$ or $x \ge 1 - \mu$,

$$h_v^{(t+3)}(x) \le \max\left\{as(G), \frac{1}{8} + \frac{1}{20} + \frac{1}{2\min_{u\in V}\deg(u;G)}\right\}$$

and the proof is complete.

A.6 Population-level conductance profile.

We introduce the *r-ball walk*, a Markov chain over $\mathcal{C}_\sigma$ with transition probability given by

$$\widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) := \frac{\mathbb{P}(\mathcal{S}\cap B(x,r))}{\mathbb{P}(\mathcal{C}_\sigma \cap B(x,r))} \qquad (x\in\mathcal{C}_\sigma, \mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma))$$

where $\mathfrak{B}(\mathcal{C}_\sigma)$ is the Borel σ-algebra of $\mathcal{C}_\sigma$.

Denote the stationary distribution for this Markov chain by $\widetilde{\pi}_{\mathbb{P},r}$, which is defined by the relation

$$\int_{\mathcal{C}_\sigma} \widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) d\widetilde{\pi}_{\mathbb{P},r}(x) = \widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}). \qquad (\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma))$$

Letting the $\mathbb{P}$-*local conductance* be given by

$$\ell_{\mathbb{P},r}(x) := \mathbb{P}\big(\mathcal{C}_\sigma\cap B(x,r)\big) \qquad (x\in\mathcal{C}_\sigma)$$

a bit of algebra verifies that

$$\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}) = \frac{\int_{\mathcal{S}} \ell_{\mathbb{P},r}(x) f(x)\,dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r}(x) f(x)\,dx}. \qquad (\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma))$$

We next introduce the *ergodic flow*, $\widetilde{Q}_{\mathbb{P},r}$. Let $\mathcal{S}_1\cap\mathcal{S}_2 = \mathcal{C}_\sigma$ be a partition of $\mathcal{C}_\sigma$. Then the ergodic flow between $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by

$$\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}_1,\mathcal{S}_2) := \int_{\mathcal{S}_1} \widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}_2) d\widetilde{\pi}_{\mathbb{P},r}(x), \qquad (\mathcal{S}_1,\mathcal{S}_2\in\mathfrak{B}(\mathcal{C}_\sigma))$$

the $\mathbb{P}$-*(continuous) normalized cut* by

$$\widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S}) := \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S},\mathcal{S}^c)}{\min\{\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}),\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}^c)\}}, \qquad (\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma))$$

and the $\mathbb{P}$-*(continuous) conductance* by

$$\widetilde{\Phi}_{\mathbb{P},r} := \min_{\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma)} \widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S})$$

where $\mathcal{S}^c = \mathcal{C}_\sigma\setminus\mathcal{S}$, and we note this definition agrees with (12).

Proposition 3. *Let $\mathcal{C}$ satisfy Assumption (A1) for some $\lambda_\sigma\le\Lambda_\sigma$ and Assumption (A4) for some convex set $\mathcal{K}$ with diameter D, and measure-preserving mapping $g:\mathcal{K}\to\mathcal{C}_\sigma$ with biLipschitz constant L. Then, for any $0<r<\frac{\sigma}{2\sqrt{d}}$, the $\mathbb{P}$-continuous conductance of the r-ball walk satisfies*

$$\widetilde{\Phi}_{\mathbb{P},r}(t) > \frac{\lambda_\sigma^2 r}{2^{12}\Lambda_\sigma^2 DL\sqrt{d}}.$$

Similar results are already known (see e.g. [2]) when the density f is uniform (or log-concave) and $\mathcal{C}_\sigma$ is itself a convex set – indeed, in this case stronger versions of it exist, though we will not require them. We first prove an analogous result for the special case of $f\equiv 1$ everywhere on $\mathcal{C}_\sigma$. For $x\in\mathcal{C}_\sigma,\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma)$, let

$$\widetilde{P}_{\nu,r}(x;\mathcal{S}) := \frac{\nu(\mathcal{S}\cap B(x,r))}{\nu(\mathcal{C}_\sigma\cap B(x,r))}, \quad \pi_{\nu,r}(\mathcal{S}) = \frac{\int_{\mathcal{S}}\ell_{\nu,r}(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\nu,r}(x)\,dx}, \quad \widetilde{Q}_{\nu,r}(\mathcal{S}_1,\mathcal{S}_2) := \int_{\mathcal{S}_1}\widetilde{P}_{\nu,r}(x;\mathcal{S}_2)d\pi_{\nu,r}(x).$$

The uniform continuous normalized cut and conductance profile are then defined analogously to the weighted case,

$$\widetilde{\Phi}_{\nu,r}(\mathcal{S}) := \frac{\widetilde{Q}_{\nu,r}(\mathcal{S},\mathcal{S}^c)}{\min\{\pi_{\nu,r}(\mathcal{S}),\pi_{\nu,r}(\mathcal{S}^c)\}}, \quad \widetilde{\Phi}_{\nu,r}(t) := \min_{\mathcal{S}\in\mathfrak{B}(\mathcal{C}_\sigma)}\widetilde{\Phi}_{\nu,r}(\mathcal{S}).$$

Lemma 16. *Let $\mathcal{C}$ satisfy Assumption (A4) for some convex set $\mathcal{K}$ with diameter D, and measure-preserving mapping $g : \mathcal{K} \to \mathcal{C}_\sigma$ with biLipschitz constant L. Then, for any $0 < r < \frac{\sigma}{2\sqrt{d}}$, the uniform conductance of the r-ball walk satisfies*

$$\widetilde{\Phi}_{\nu,r} > \frac{r}{2^{12} D L \sqrt{d}}.$$

Most of the technical work needed to show Proposition 3 involves proving Lemma 16. We defer this work to the next subsection, and first show that Proposition 3 is a simple consequence of Lemma 16 along with (A1).

Proof of Proposition 3. Assume Lemma 16 holds. Since $t \leq 1/2$, for any Borel $\mathcal{S} \subseteq \mathcal{C}_\sigma$ with $\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}) \leq t$,

$$\widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S}) = \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{S}^c)}{\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S})}$$

By (A1) we obtain

$$\begin{aligned}
\frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{S}^c)}{\widetilde{\pi}_{\mathbb{P},r}(\mathcal{S})} &= \frac{\int_{\mathcal{S}} \mathbb{P}(S \cap B(x,r)) f(x)\, dx}{\int_{\mathcal{S}} \mathbb{P}(\mathcal{C}_\sigma \cap B(x,r)) f(x)\, dx} \\
&\geq \frac{\lambda_\sigma^2 \int_{\mathcal{S}} \nu(S \cap B(x,r))\, dx}{\Lambda_\sigma^2 \int_{\mathcal{S}} \nu(\mathcal{C}_\sigma \cap B(x,r))\, dx} \\
&\geq \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \widetilde{\Phi}_{\nu,r}(\mathcal{S})
\end{aligned}$$

and the statement follows by Lemma 16.

A.7 Proof of Lemma 16.

As is standard, the proof of a lower bound on the conductance relies on an isoperimetric inequality.

Lemma 17 (Isoperimetry of Lipschitz embeddings of convex sets.)**.** *Let $\mathcal{C}$ satisfy Assumption (A4) for some convex set $\mathcal{K}$ with diameter D, and measure-preserving mapping $g : \mathcal{K} \to \mathcal{C}_\sigma$ with biLipschitz constant L. Then, for any partition $(\Omega_1, \Omega_2, \Omega_3)$ of $\mathcal{C}_\sigma$,*

$$\nu(\Omega_3) \geq 2 \frac{\mathrm{dist}(\Omega_1, \Omega_2)}{L D_K} \min(\nu(\Omega_1), \nu(\Omega_2))$$

The proof of Lemma 17 from first principles is non-trivial, even in the convex setting, and is a primary technical contribution of the seminal work [3], extended by [1] among others. Once the result is shown in the case where Ω is convex, however, generalizing to the setting given by Assumption (A4) is not difficult.

Proof of Lemma 17. For $\Omega_i, i = 1, 2, 3$, denote the preimage

$$R_i = \{x \in K : g(x) \in \Omega_i\}$$

For any $x \in R_1, y \in R_2$,

$$|x - y| \geq \frac{1}{L} |g(x) - g(y)| \geq \frac{1}{L} \mathrm{dist}(\Omega_1, \Omega_2).$$

Since $x \in \Omega_1$ and $y \in \Omega_2$ were arbitrary, we have

$$\mathrm{dist}(R_1, R_2) \geq \frac{1}{L} \mathrm{dist}(\Omega_1, \Omega_2).$$

By Theorem 2.2 of [3],

$$\begin{aligned}\nu(R_3) &\geq 2\frac{\operatorname{dist}(R_1, R_2)}{D}\min\{\nu(R_1), \nu(R_2)\} \\ &\geq \frac{2}{DL}\operatorname{dist}(\Omega_1, \Omega_2)\min\{\nu(R_1), \operatorname{vol}(R_2)\}\end{aligned}$$

and by the measure-preserving property of g, this implies

$$\nu(\Omega_3) \geq \frac{2}{DL}\operatorname{dist}(\Omega_1, \Omega_2)\min\{\nu(\Omega_1), \nu(\Omega_2)\}.$$

$\square$

We will also need an upper bound on the distance between transition probability densities $\widetilde{P}_{\nu,r}(u,\cdot)$ and $\widetilde{P}_{\nu,r}(v,\cdot)$ for $u, v \in \mathcal{C}_\sigma$ close together.

Lemma 18 (One-step distributions)**.** *Let $u, v \in \mathcal{C}_\sigma$ be such that*

$$\|u - v\| \leq \frac{rt}{\sqrt{d}}$$

for some $0 < t < 1/8$, and further assume there exists $\ell > 0$ such that $\ell_{\nu,r}(u), \ell_{\nu,r}(v) \geq \ell\nu_d r^d$. Then,

$$\left\|\widetilde{P}_{\nu,r}(u;\cdot) - \widetilde{P}_{\nu,r}(v;\cdot)\right\|_{TV} \leq 1 + \frac{3\sqrt{3}t}{4\sqrt{2\pi}} - \ell$$

where $\|P - Q\|_{TV}$ is the total variation distance between probabilities P and Q.

The key result needed to show Lemma 18 is regarding the volume of the overlap $B(u, r) \cap B(v, r)$.

Lemma 19. *Let $u, v \in \mathbb{R}^d$ be points such that $\|u - v\| \leq t\frac{r}{\sqrt{d}}$ for some $0 < t < 1/8$. Then,*

$$\nu(B(u, r) \cap B(v, r)) \geq \nu_d r^d\left(1 - \frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)$$

Proof. We will treat only the case where $\|u - v\| = t\frac{r}{\sqrt{d}}$; if they are closer together the overlap of the volume will only increase. Then, it is not hard to see that $I = B(u, r) \cap B(v, r)$ is comprised of the union of two disjoint spherical caps, and thus

$$\nu(I) = 2\nu(\operatorname{Cap}_r(r(1 - \frac{t}{2\sqrt{d}}))).$$

From Lemma 6 we therefore obtain

$$\nu(I) = \nu_d r^d I_{1-\alpha}(\frac{d+1}{2}; \frac{1}{2})$$

where

$$\alpha = 1 - \frac{2r^2(1 - \frac{t}{2\sqrt{d}}) - r^2(1 - \frac{t}{2\sqrt{d}})^2}{r^2} = \frac{t^2}{4d}.$$

Expanding the incomplete beta function in integral form, we therefore have

$$\begin{aligned}\nu(I) &= \nu_d r^d \frac{\Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{1-\alpha} u^{(d-1)/2}(1 - u)^{-1/2}du \\ &\overset{(i)}{\geq} \nu_d r^d\left(1 - \frac{\Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{3t}{4\sqrt{d}}\right) \\ &\overset{(ii)}{\geq} \nu_d r^d\left(1 - \frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)\end{aligned}$$

where (i) follows from Lemma 7 (which we can validly apply since $\alpha \leq \frac{t^2}{2d}$), and (ii) from Lemma 8. $\square$

Proof of Lemma 18. Let $S_1 \cup S_2 = \mathcal{C}_\sigma$ be an arbitrary partition of $\mathcal{C}_\sigma$. We will show that

$$\widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_1) \leq 1 + \frac{3\sqrt{3}t}{4\sqrt{2\pi}} - \ell.$$

Since this will hold for arbitrary $S_1 \in \mathfrak{B}(\mathcal{C}_\sigma)$, it will hold for the infimum over all such S_1 as well, and therefore the same lower bound will hold for $\left\|\widetilde{P}_{\nu,r}(u;\cdot) - \widetilde{P}_{\nu,r}(v;\cdot)\right\|_{TV}$.

Now, note that

$$\widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_1) = 1 - \widetilde{P}_{\nu,r}(u; S_2) - \widetilde{P}_{\nu,r}(v; S_1)$$

Let $I := B(u,r) \cap B(v,r)$. Then we have

$$\widetilde{P}_{\nu,r}(u; S_2) \geq \frac{1}{\nu(B(u,r))}\nu(S_2 \cap (B(u,r)) \geq \frac{1}{\nu(B(u,r))}\nu(S_2 \cap I)$$

with a symmetric inequality holding for $\widetilde{P}_{\nu,r}(v; S_1)$. As a result,

$$1 - \widetilde{P}_{\nu,r}(u; S_2) - \widetilde{P}_{\nu,r}(v; S_1) \leq 1 - \frac{1}{\nu_d r^d}\nu(\mathcal{C}_\sigma \cap I) \tag{A.21}$$

As (A.21) demonstrates, the overlap of the one-step distributions is related to the volume of the intersection between $B(u,r)$ and $B(v,r)$ within $\mathcal{C}_\sigma$.

From here, some simple manipulations yield

$$\begin{aligned}
\nu(\mathcal{C}_\sigma \cap I) &= \nu(I) - \nu(I \setminus \mathcal{C}_\sigma) \\
&\geq \nu(I) - \max\left\{\nu\big(B(u,r) \setminus \mathcal{C}_\sigma\big), \nu\big(B(v,r) \setminus \mathcal{C}_\sigma\big)\right\} \\
&\geq \nu_d r^d \left(1 - \frac{3\sqrt{3}t}{4\sqrt{2\pi}} - (1-\ell)\right) = \nu_d r^d \left(\ell - \frac{3\sqrt{3}t}{4\sqrt{2\pi}}\right)
\end{aligned} \tag{A.22}$$

where the last inequality follows from Lemma 19. (A.22) along with (A.21) then give the desired result. $\square$

Proof of Lemma 16. Let $S_1 \cup S_2 = \mathcal{C}_\sigma$, and let $\ell \geq 0$ satisfy $\ell \nu_d r^d \leq \ell_{\nu,r}(x)$. We will show that

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) \geq \frac{\sqrt{2\pi} r \ell^4}{24DL\sqrt{d}} \min\left\{\pi_{\nu,r}(S_1), \pi_{\nu,r}(S_2)\right\}$$

Once we have shown this, Lemma 5 gives the bound $\ell \geq \frac{6}{25}$. Then, dividing both sides by $\pi_{\nu,r}(S_1)$ yields the desired result.

Now, consider the sets

$$\begin{aligned}
S_1' &= \left\{x \in S_1 : \widetilde{P}_{\nu,r}(x; S_2) < \frac{\ell}{4}\right\} \\
S_2' &= \left\{x \in S_1 : \widetilde{P}_{\nu,r}(x; S_2) < \frac{\ell}{4}\right\}
\end{aligned}$$

and $S_3' = \mathcal{C}_\sigma \setminus S_1' \setminus S_2'$.

Suppose $\pi_{\nu,r}(S_1') < \pi_{\nu,r}(S_1)/2$. Then,

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) \geq \frac{\ell \pi_{\nu,r}(S_1)}{8}$$

Similarly, if $\pi_{\nu,r}(S_1') < \pi_{\nu,r}(S_1)/2$, then since

$$\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x) = \int_{S_2} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x)$$

a symmetric result holds.

So we can assume $\pi_{\nu,r}(S_1') \geq \pi_{\nu,r}(S_1)/2$, and likewise for S_2. Now, for every $u \in S_1', v \in S_2'$, we have that

$$\left\|\widetilde{P}_{\nu,r}(u; \cdot) - \widetilde{P}_{\nu,r}(v; \cdot)\right\|_{TV} \geq 1 - \widetilde{P}_{\nu,r}(u; S_1) - \widetilde{P}_{\nu,r}(v; S_2) > 1 - \frac{\ell}{2}.$$

By Lemma 18, we therefore have

$$\|u - v\| \geq \frac{2\sqrt{2\pi} r \ell}{3\sqrt{3d}}.$$

and since $u \in S_1', v \in S_2'$ were arbitrary, the same inequality holds for $\mathrm{dist}(S_1', S_2')$. Therefore by Lemma 17

$$\mathrm{vol}(S_3') \geq \frac{2\sqrt{2\pi} r \ell}{3DL\sqrt{3d}} \min\left\{\mathrm{vol}(S_1'), \mathrm{vol}(S_2')\right\}$$

We now prove the desired result:

$$\begin{aligned}
\int_{S_1} \widetilde{P}_{\nu,r}(x; S_2) &= \frac{1}{2}\big(\int_{S_2} \widetilde{P}_{\nu,r}(x; S_2) d\pi_{\nu,r}(x)\big) \\
&\geq \frac{\ell}{8} \pi_{\nu,r}(S_3') \\
&\geq \frac{\ell^2}{8\nu(\mathcal{C}_\sigma)} \nu(S_3') \\
&\geq \frac{\sqrt{2} r \ell^3}{12DL\sqrt{d}\nu(\mathcal{C}_\sigma)} \min\left\{\nu(S_1'), \nu(S_2')\right\} \\
&\geq \frac{\sqrt{2} r \ell^4}{12DL\sqrt{d}} \min\left\{\pi_{\nu,r}(S_1'), \pi_{\nu,r}(S_2')\right\} \\
&\geq \frac{\sqrt{2} r \ell^4}{24DL\sqrt{d}} \min\left\{\pi_{\nu,r}(S_1), \pi_{\nu,r}(S_2)\right\}.
\end{aligned}$$

A.8 Graph conductance and local spread.

Recall that $\widetilde{G}_{n,r} = G_{n,r}\big[\mathcal{C}_\sigma[X]\big]$ denotes the subgraph induced by $\mathcal{C}_\sigma[X]$. $\widetilde{\Phi}_{n,r}$, the graph conductance over $\widetilde{G}_{n,r}$ is defined in (11). Write $\widetilde{\pi}_{\min} := \min_{u \in \mathcal{C}_\sigma[X]} \widetilde{\pi}_{n,r}(u)$.

Proposition 4 (Lower bound on graph conductance profile)**.** *Let $\mathcal{C}$ satisfy Assumption (A1) for some $\lambda_\sigma \leq \Lambda_\sigma$ and Assumption (A4) for some convex set $\mathcal{K}$ with diameter D, and measure-preserving mapping $g : \mathcal{K} \to \mathcal{C}_\sigma$ with biLipschitz constant L. Then, with probability one the following lower bound holds on the graph conductance:*

$$\liminf_{n \to \infty} \widetilde{\Phi}_{n,r} > \frac{\lambda_\sigma^2 r}{\Lambda_\sigma^2 2^{12} DL\sqrt{d}}$$

In order to prove Proposition 4, we will need to split the analysis into two cases based on the size of $S \subseteq \mathcal{C}_\sigma[X]$. For any graph $G = (V, E)$, let $\mathcal{L}(G) = \{S \subseteq V : \pi(S) \geq s(G)\}$ (where as usual π denotes the stationary distribution of a random walk over G.)

In Lemma 20, we establish a uniform lower bound on the normalized cut of all sets $S \in \mathcal{L}(\widetilde{G}_{n,r})$.

Lemma 20. *Let $\mathcal{C}$ satisfy Assumption (A1) for some $\lambda_\sigma \leq \Lambda_\sigma$ and Assumption (A4) for some convex set $\mathcal{K}$ with diameter D, and measure-preserving mapping $g : \mathcal{K} \to \mathcal{C}_\sigma$ with biLipschitz constant L Then with probability one,*

$$\liminf_{n\to\infty} \left\{ \min_{S \in \mathcal{L}(\widetilde{G}_{n,r})} \widetilde{\Phi}_{n,r}(S) \right\} \geq \frac{\lambda_\sigma^2 r}{\Lambda_\sigma^2 2^{12} D L \sqrt{d}}$$

where $\mathcal{L}(\widetilde{G}_{n,r}) = \left\{ S \subseteq \mathcal{C}_\sigma[X] : \widetilde{\pi}_{n,r}(S) \geq s(\widetilde{G}_{n,r}) \right\}$.

The proof of Lemma 20 is nontrivial, and we devote the following subsection to it.

Lemma 21 shows that for the remaining small sets, the graph normalized cut is of constant order.

Lemma 21. *Let $G = (V, E)$ be an arbitrary undirected graph. Then, for non-empty subsets $S \subseteq V$,*

$$\min_{S \notin \mathcal{L}(G)} \Phi(S; G) \geq \frac{1}{10}.$$

Proof. The claim follows by simple manipulations:

$$\begin{aligned}
\Phi(S; G) &\geq \sum_{u \in S} \frac{\mathrm{cut}(u, S^c; G)}{\mathrm{vol}(S; G)} \\
&\geq \sum_{u \in S} \frac{\deg(u; G) - |S|}{\mathrm{vol}(S; G)} \\
&\geq \sum_{u \in S} \frac{\deg(u; G) - \frac{9}{10} \min_{u \in V} \deg(u; G)}{\mathrm{vol}(S; G)} && \text{(since } \pi(S) \leq s(G)) \\
&\geq \frac{1}{10} \sum_{u \in S} \frac{\deg(u; G)}{\mathrm{vol}(S; G)} = \frac{1}{10}.
\end{aligned}$$

□

Proposition 4 follows in light of the fact $\frac{\lambda_\sigma^2 r}{\Lambda_\sigma^2 2^{12} D L \sqrt{d}} < \frac{1}{10}$.

Lemma 22. *With probability one, the following bound holds:*

$$\liminf_{n\to\infty} s(\widetilde{G}_{n,r}) \geq \frac{\lambda_\sigma^2 \nu_d r^d}{20 \Lambda_\sigma}$$

Proof. We rewrite $s(\widetilde{G}_{n,r}) = \frac{9\widetilde{\deg}_{\min}^2}{10\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[X])}$ Therefore by Lemma 32, with probability one

$$\liminf_{n\to\infty} s(\widetilde{G}_{n,r}) \geq \frac{36 \lambda_\sigma r^d \nu_d}{625 \mathbb{P}(\mathcal{C}_\sigma) \Lambda_\sigma} \geq \frac{\lambda_\sigma^2 \nu_d r^d}{20 \Lambda_\sigma}.$$

□

A.9 Proof of Lemma 20

Let $\mathfrak{B}(\mathcal{C}_\sigma)$ be the Borel σ-algebra of $\mathcal{C}_\sigma$, and define the conditional probability measures

$$\widetilde{\mathbb{P}}(\mathcal{S}) = \frac{\mathbb{P}(\mathcal{S})}{\mathbb{P}(\mathcal{C}_\sigma)},\ \widetilde{\mathbb{P}}_n(\mathcal{S}) := \frac{1}{|\mathcal{C}_\sigma[X]|}\sum_{x_i \in \mathcal{C}_\sigma[X]} \mathbf{1}(x_i \in \mathcal{S}) \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

A Borel map $T : \mathcal{C}_\sigma \to \mathcal{C}_\sigma[X]$ is said to be a *transportation map* between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ if for any $S \subseteq \mathcal{C}_\sigma[X]$

$$\widetilde{\mathbb{P}}(T^{-1}(S)) = \widetilde{\mathbb{P}}_n(S).$$

where $T^{-1}(S) = \{x \in \mathcal{C}_\sigma : T(x) \in S\}$ is the preimage of T.

If a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} \|\mathrm{Id} - T_n\|_{L^1(\widetilde{\mathbb{P}})} = 0$ (where Id is the identity mapping), we refer to it as a sequence of *stagnating transportation maps*. Proposition 5 of [6] establishes that, for open connected domains with Lipschitz boundaries, such stagnating transportation maps exist. In particular, as $\mathcal{C}_\sigma[X]$ consists of $\widetilde{n} = |\mathcal{C}_\sigma[X]|$ points sampled independently from $\widetilde{\mathbb{P}}$, and $\mathcal{C}_\sigma$ is a connected domain with Lipschitz boundary, the following implication is immediate.

Lemma 23. *With probability one, there exists a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$, $T_n : \mathcal{C}_\sigma \to \mathcal{C}_\sigma[X]$ such that the following statement holds:*

$$\limsup_{n\to\infty} \frac{\widetilde{n}^{1/d}\,\|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}}{(\log \widetilde{n})^{p_d}} \le C \tag{A.23}$$

where $\mathrm{Id}(x) = x$ is the identity mapping over $\mathcal{C}_\sigma$, C is a universal constant and $p_d = 3/4$ for $d = 2$ and $1/d$ for $d \ge 3$.

Note that although $\mathcal{C}_\sigma$ is closed not open, as $\nu(\partial\mathcal{C}_\sigma) = 0$ we may apply Proposition 5 of [6] to $\mathcal{C}_\sigma^o$, and (A.23) will hold for arbitrary extension of T_n to $\mathcal{C}_\sigma$.

Graph cuts to continuous cuts. We use stagnating transportation maps to relate the discrete graph normalized cuts to the continuous normalized cuts discussed in Section A.6. We fix some notation, letting $r_n^\pm = r \pm \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$ for a transportation map T_n.

Lemmas 24 and 25 provide the necessary bounds for the cut and vol functionals in terms of continuous analogues.

Lemma 24. *Let $S \subseteq \mathcal{C}_\sigma[X]$ and T_n be a transportation map between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$. Then, letting $\mathcal{S} := T_n^{-1}(S)$,*

$$\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{vol}}_{n,r}(S) \le \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x) f(x)\,dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}).$$

Proof. Let $\varphi : \mathcal{C}_\sigma[X] \to \{0,1\}$ be the characteristic function for S, meaning

$$\varphi(x) = \begin{cases} 1, & x \in S \\ 0, & \text{otherwise} \end{cases}$$

Now, we proceed

$$\begin{aligned}
\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{vol}}_{n,r}(S) &\leq \frac{1}{\widetilde{n}^2}\sum_{x_i,x_j\in\mathcal{C}_\sigma[X]}\mathbf{1}(\|x_i-x_j\|\leq r)|\varphi(x_i)| \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|x-y\|\leq r)|\varphi(x)|d\widetilde{\mathbb{P}}_n(x)d\widetilde{\mathbb{P}}_n(y) \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|T_n(x)-T_n(y)\|\leq r)|\varphi\circ T_n(x)|d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \\
&\leq \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|x-y\|\leq r_n^+)|\varphi\circ T_n(x)|d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \\
&= \int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)}1d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

By definition we have $\frac{d\widetilde{\mathbb{P}}(x)}{d\mathbb{P}(x)} = \mathbb{P}(\mathcal{C}_\sigma)$. Therefore,

$$\begin{aligned}
\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)}1d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x) &= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\int_{\mathcal{C}_\sigma\cap B(x,r_n^+)}1d\mathbb{P}(y)d\mathbb{P}(x) \\
&= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx \\
&= \frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r}(x)f(x)\,dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S})
\end{aligned}$$

which is the desired upper bound. ☐

Lemma 25. *Let $S\subseteq\mathcal{C}_\sigma[X]$, and let T_n be a transportation map between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$. Then, letting $\mathcal{S}=T_n^{-1}(S)$,*

$$\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{cut}}_{n,r}(S)\geq\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}f(x)\,dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c).$$

Proof. As in the proof of Lemma 24, let $\varphi:\mathcal{C}_\sigma[X]\to\{0,1\}$ be the characteristic function for S.

We proceed according to a very similar set of steps as Lemma 24:

$$\begin{aligned}
\frac{1}{\widetilde{n}^2}\widetilde{\mathrm{cut}}_{n,r}(S) &= \frac{1}{\widetilde{n}^2}\sum_{x_i,x_j\in\mathcal{C}_\sigma[X]}\mathbf{1}(\|x_i-x_j\|\leq r)|\varphi(x_i)-\varphi(x_j)| \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|x-y\|\leq r)|\varphi(x)-\varphi(y)|d\widetilde{\mathbb{P}}_n(x)d\widetilde{\mathbb{P}}_n(y) \\
&= \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|T_n(x)-T_n(y)\|\leq r)|\varphi\circ T_n(x)-\varphi\circ T_n(y)|d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \\
&\geq \iint_{\mathcal{C}_\sigma\times\mathcal{C}_\sigma}\mathbf{1}(\|x-y\|\leq r_n^-)|\varphi\circ T_n(x)-\varphi\circ T_n(y)|d\widetilde{\mathbb{P}}(x)d\widetilde{\mathbb{P}}(y) \\
&= \int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)}d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

We conclude similarly to the proof of Lemma 24,

$$\begin{aligned}\int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)} d\widetilde{\mathbb{P}}(y)d\widetilde{\mathbb{P}}(x) &= \frac{1}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\int_{\mathcal{S}^c\cap B(x,r_n^-)} d\mathbb{P}(y)d\mathbb{P}(x)\\ &= \frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)d\mathbb{P}(x)}{\mathbb{P}(\mathcal{C}_\sigma)^2}\int_{\mathcal{S}}\frac{\mathbb{P}\big(\mathcal{S}^c\cap B(x,r_n^-)\big)}{\ell_{\mathbb{P},r_n^-}(x)}d\widetilde{\pi}_{\mathbb{P},r}(x)\\ &= \frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}f(x)\,dx}{\mathbb{P}(\mathcal{C}_\sigma)^2}\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c).\end{aligned}$$

□

Lemma 26. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of transportation maps from $\widetilde{\mathbb{P}}$ to $\widetilde{\mathbb{P}}_n$, and fix $S\subseteq\mathcal{C}_\sigma[X]$. Then, letting $\mathcal{S}=T_n^{-1}(S)$,*

$$\widetilde{\Phi}_{n,r}(S)\geq\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}\frac{\min\Big\{\widetilde{\pi}_{\mathbb{P},r_n^-}(\mathcal{S}),\widetilde{\pi}_{\mathbb{P},r_n^-}(\mathcal{S}^c)\Big\}}{\min\Big\{\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}),\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}^c)\Big\}}\widetilde{\Phi}_{\mathbb{P},r_n^-}(\mathcal{S}) \tag{A.24}$$

Proof. By Lemmas 24 and 25,

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)}\geq\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c)}{\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S})}.$$

Then, noting that $\mathcal{S}^c=T_n^{-1}(S^c)$, Lemmas 24 and 25 also imply

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^c)}{\widetilde{\mathrm{vol}}_{n,r}(S^c)}\geq\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S}^c,\mathcal{S})}{\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}^c)}$$

and as $\widetilde{Q}_{\mathbb{P},r_n^-}(\cdot,\cdot)$ is symmetric in its arguments we obtain

$$\frac{\widetilde{\mathrm{cut}}(S)}{\min\Big\{\widetilde{\mathrm{vol}}(S),\widetilde{\mathrm{vol}}(S^c)\Big\}}\geq\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}\frac{\widetilde{Q}_{\mathbb{P},r_n^-}(\mathcal{S},\mathcal{S}^c)}{\min\Big\{\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}),\widetilde{\pi}_{\mathbb{P},r_n^+}(\mathcal{S}^c)\Big\}},$$

and the proof is complete. □

Perturbation analysis. If $r_n^-=r=r_n^+$, (A.24) would simplify to $\widetilde{\Phi}_{n,r}(S)\geq\widetilde{\Phi}_{\mathbb{P},r}(\mathcal{S})$. We show that this conclusion is robust to asymptotically negligible perturbations of r.

Lemma 27 (Continuity of local conductance)**.** *For $\{T_n\}_{n\in\mathbb{N}}$ a sequence of stagnating transportation maps,*

$$\limsup_{n\to\infty}\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}=1$$

Proof. Letting $\mathcal{R}_n(x):=\{x'\in\mathcal{C}_\sigma : x'\in B(x,r_n^+), x'\notin B(x,r_n^-)\}$, we have

$$\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx=\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx+\int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n(x)}f(y)f(x)\,dy\,dx.$$

and therefore

$$\frac{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^+}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}=1+\frac{\int_{\mathcal{C}_\sigma}\int_{\mathcal{R}_n(x)}f(y)f(x)\,dy\,dx}{\int_{\mathcal{C}_\sigma}\ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}.$$

We upper bound the remainder term

$$\int_{\mathcal{C}_\sigma} \int_{\mathcal{R}_n(x)} f(y) f(x) \, dy \, dx \leq \mathbb{P}(\mathcal{C}_\sigma) \Lambda_\sigma \nu_d \big((r_n^+)^d - r^d \big) \to 0$$

where the convergence happens as $n \to \infty$ by the stagnating property of $\{T_n\}_{n \in \mathbb{N}}$.

We apply a similar analysis to the denominator. By Lemma 5

$$\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P}, r_n^-}(x) f(x) \, dx = \int_{\mathcal{C}_\sigma} \int_{B(x, r_n^-)} f(y) f(x) \, dy \, dx \geq \frac{6}{25} \mathbb{P}(\mathcal{C}_\sigma) \lambda_\sigma (r_n^-)^d \nu_d \to \mathbb{P}(\mathcal{C}_\sigma) \lambda_\sigma^2 r^d \nu_d > 0$$

where the convergence happens as $n \to \infty$ by the stagnating property of $\{T_n\}_{n \in \mathbb{N}}$.

Then the desired result follows from an application of Slutsky's Theorem. □

Lemma 28 (Continuity of stationary distribution)**.** *Let $c > 0$ be a fixed constant, and $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of stagnating transportation maps. Then,*

$$\liminf_{n \to \infty} \frac{\min \left\{ \widetilde{\pi}_{\mathbb{P}, r_n^-}(\mathcal{S}), \widetilde{\pi}_{\mathbb{P}, r_n^-}(\mathcal{S}^c) \right\}}{\min \left\{ \widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S}), \widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S}^c) \right\}} = 1$$

uniformly over all sets $\mathcal{S} \subseteq \mathfrak{B}(\mathcal{C}_\sigma)$ satisfying $\min \{ \widetilde{\pi}_{\mathbb{P}, r}(\mathcal{S}), \widetilde{\pi}_{\mathbb{P}, r}(\mathcal{S}^c) \} > c$.

Proof. It will be sufficient to show that

$$\liminf_{n \to \infty} \frac{\widetilde{\pi}_{\mathbb{P}, r_n^-}(\mathcal{S})}{\widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S})} \quad \text{and} \quad \liminf_{n \to \infty} \frac{\widetilde{\pi}_{\mathbb{P}, r_n^-}(\mathcal{S}^c)}{\widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S}^c)} = 1,$$

and we will show only the former, the proof of the latter being identical.

The proof proceeds similarly to Lemma 27. Letting

$$\mathcal{R}_n(x) := \left\{ x' \in \mathcal{S} : x' \in B(x, r_n^+), x' \notin B(x, r_n^- \right\}$$

we may rewrite

$$\frac{\widetilde{\pi}_{\mathbb{P}, r_n^-}(\mathcal{S})}{\widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S})} = 1 - \frac{\int_{\mathcal{S}} \int_{\mathcal{R}_n(x)} f(y) f(x) \, dy \, dx}{\widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S})}.$$

By the stagnating property of T_n,

$$\liminf_{n \to \infty} \int_{\mathcal{S}} \int_{\mathcal{R}_n(x)} f(y) f(x) \, dy \, dx \leq \mathbb{P}(\mathcal{S}) \Lambda_\sigma \big((r_n^+)^d - (r_n^-)^d \big) \to 0$$

where the convergence occurs as $n \to \infty$. On the other hand, by hypothesis

$$\limsup_{n \to \infty} \widetilde{\pi}_{\mathbb{P}, r_n^+}(\mathcal{S}) \geq c > 0.$$

and the result follows by Slutsky's Theorem. □

Lemma 29 (Stationary distribution lower bound)**.** *Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of stagnating transportation maps from $\mathbb{P}$ to $\mathbb{P}_n$. Then with probability one,*

$$\liminf_{n \to \infty} \min_{S \in \mathcal{L}(\widetilde{G}_{n,r})} \widetilde{\pi}_{\mathbb{P}, r}(T_n^{-1}(S)) \geq \frac{\lambda_\sigma^2 \nu_d r^d}{20 \Lambda_\sigma}$$

Proof. Fix $\epsilon > 0$, and let $S \in \mathcal{L}(\widetilde{G}_{n,r})$ be arbitrary, writing $\mathcal{S} := T_n^{-1}(S)$. Letting $\mathcal{R}_n(x)$ be as in the proof of Lemma 28, we have

$$\widetilde{\pi}_{n,r}(S) \leq \widetilde{\pi}_{\mathbb{P},r_n^-}(\mathcal{S}) + \frac{\int_{\mathcal{S}} \int_{\mathcal{R}_n(x)} f(y)f(x)\,dy\,dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}. \tag{A.25}$$

Clearly $\widetilde{\pi}_{\mathbb{P},r_n^-}(\mathcal{S}) \leq \widetilde{\pi}_{\mathbb{P},r}(\mathcal{S})$. Moreover

$$\frac{\int_{\mathcal{S}} \int_{\mathcal{R}_n(x)} f(y)f(x)\,dy\,dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x)f(x)\,dx} \leq \frac{\Lambda_\sigma}{\lambda_\sigma}\left(\frac{r_n^+ - r_n^-}{r_n^-}\right)^d$$

which along with (A.25) implies

$$s(\widetilde{G}_{n,r}) \leq \widetilde{\pi}_{n,r}(S) \leq \widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}) + \frac{\Lambda_\sigma}{\lambda_\sigma}\left(\frac{r_n^+ - r_n^-}{r_n^-}\right)^d \to \widetilde{\pi}_{\mathbb{P},r}(\mathcal{S}).$$

where the first lower bound comes from the fact $S \in \mathcal{L}(\widetilde{G}_{n,r})$, and the convergence is as $n \to \infty$ by the stagnating property of $\{T_n\}_{n\in\mathbb{N}}$.

By Lemma 22, with probability one as $n \to \infty$,

$$s(\widetilde{G}_{n,r}) \geq \frac{\lambda_\sigma^2 \nu_d r^d}{20\Lambda_\sigma}$$

and the claim is shown. ◻

Proof of Lemma 20 By Lemma 23, with probability one there exists a sequence of stagnating transportation maps from $\widetilde{\mathbb{P}}$ to $\widetilde{\mathbb{P}}_n$, which we will denote $\{T_n\}_{n\in\mathbb{N}}$.

Let $S \subseteq \mathcal{C}_\sigma[X]$ be arbitrary, and define

$$\xi_n := \frac{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^-}(x)f(x)\,dx}{\int_{\mathcal{C}_\sigma} \ell_{\mathbb{P},r_n^+}(x)f(x)\,dx},\ \gamma_n(S) := \frac{\min\left\{\widetilde{\pi}_{\mathbb{P},r_n^-}(T_n^{-1}(S^c)), \widetilde{\pi}_{\mathbb{P},r_n^-}(T_n^{-1}(S^c))\right\}}{\min\left\{\widetilde{\pi}_{\mathbb{P},r_n^+}(T_n^{-1}(S^c)), \widetilde{\pi}_{\mathbb{P},r_n^+}(T_n^{-1}(S^c))\right\}}$$

where $r_n^{\pm} := r \pm \|\mathrm{Id} - T_n\|_{L^\infty(\widetilde{\mathbb{P}})}$. By Proposition 3 and Lemma 26 we have that

$$\begin{aligned}\widetilde{\Phi}_{n,r}(S) &\geq \xi_n \gamma_n(S) \widetilde{\Phi}_{\mathbb{P},r_n^-}(T_n^{-1}(S)) \\ &\geq \xi_n \gamma_n(S) \frac{\lambda_\sigma^2 r_n^-}{2^{12}\Lambda_\sigma^2 DL\sqrt{d}}.\end{aligned} \tag{A.26}$$

By Lemma 27, with probability one

$$\liminf_{n\to\infty} \xi_n = 1.$$

By Lemma 29, letting c be any constant satisfying $0 < c < \frac{\lambda_\sigma^2 \nu_d r^d}{20\Lambda_\sigma}$, with probability one there exists some $m \in \mathbb{N}$ such that for all $n \geq m$,

$$\min_{S \in \mathcal{L}(\widetilde{G}_{n,r})} \widetilde{\pi}_{\mathbb{P},r}(T_n^{-1}(S)) \geq c > 0$$

and therefore by Lemma 28

$$\liminf_{n\to\infty} \left\{ \inf_{S\in\mathcal{L}(\widetilde{G}_{n,r})} \gamma_n(S) \right\} = 1.$$

As Lemma 23 implies $r_n^- \to r$ with probability one, an application of Slutsky's Theorem to (A.26) completes the proof.

A.10 Proof of Theorem 2

Since r is fixed, $\min_{u \in V} \deg(u; \widetilde{G}_{n,r}) \to \infty$ with n. Therefore for sufficiently large n we may apply Proposition 2 and obtain

$$\tau_\infty(\widetilde{G}_{n,r}) \leq \frac{2}{\widetilde{\Phi}_{n,r}^2} \log\left(\frac{1440}{s(\widetilde{G}_{n,r})}\right) \log\left(\frac{14}{s(\widetilde{G}_{n,r})}\right) + 3\log\left(\frac{14}{s(\widetilde{G}_{n,r})}\right) + 3$$

and the claim follows from Proposition 4 and Lemma 22.

A.11 Concentration inequalities

Given a symmetric kernel function $K : \mathbb{R}^d \to \mathbb{R}$ with $\sup_{x,y \in \mathbb{R}^d} |K(x,y)| \leq 1$, and independent and identically distributed data $\{x_1, \ldots, x_n\}$, we define the *order-2 U statistic* to be

$$U := \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} K(x_i, x_j).$$

Lemma 30 (Hoeffding's inequality for U-statistics.)**.** *For any $t > 0$,*

$$\mathbb{P}(|U - \mathbb{E}U| \geq t) \leq 2\exp\left\{-\frac{2nt^2}{m}\right\}$$

Further, for any $\delta > 0$, we have

$$U \leq \mathbb{E}U + \sqrt{\frac{\log(1/\delta)}{n}},$$
$$U \geq \mathbb{E}U - \sqrt{\frac{\log(1/\delta)}{n}}$$

each with probability at least $1 - \delta$.

We will employ a sharper inequality to bound the sum of independent Bernoulli random variables.

Lemma 31. *Let $X_i \in \{0,1\}$ for $i = 1, \ldots, n$ and let $\mu = \sum_{i=1}^n \mathbb{E}(X_i)$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i > (1+\epsilon)\mu\right) \leq \exp\left(\frac{-\epsilon^2 \mu}{3}\right)$$
$$\mathbb{P}\left(\sum_{i=1}^n X_i < (1-\epsilon)\mu\right) \leq \exp\left(\frac{-\epsilon^2 \mu}{2}\right)$$

We will require asymptotic bounds on several different degree and volume functionals, which follow from Lemmas 30 and 31. Let

$$\widetilde{\deg}_{\min} := \min_{u \in \mathcal{C}_\sigma[X]} \deg(u; G_{n,r}[\mathcal{C}_\sigma[X]]), \quad \widetilde{\deg}_{\max} := \max_{u \in \mathcal{C}_\sigma[X]} \deg(u; G_{n,r}[\mathcal{C}_\sigma[X]])$$
$$\deg'_{\min} := \min_{u \in \mathcal{C}'_\sigma[X]} \deg(u; G_{n,r}), \qquad \deg_{\max} := \max_{u \in \mathcal{C}_\sigma[X]} \deg(u; G_{n,r})$$
$$\deg_{\min} := \min_{u \in X} \deg(u; G_{n,r}).$$

Lemma 32. *Under the conditions of Theorem 4, for any $\epsilon > 0$, each of the following bounds hold with probability tending to one as $n \to \infty$:*

$$\frac{\widetilde{\deg}_{\min}}{n}, \frac{\deg'_{\min}}{n} \geq (1-\epsilon)\frac{6}{25}\lambda_\sigma r^d \nu_d \tag{A.27}$$

$$\frac{\deg_{\min}}{n} \geq \frac{6}{25}(1-\epsilon)\lambda_{\min} r^d \nu_d \tag{A.28}$$

$$\frac{\deg_{\max}}{n} \leq (1+\epsilon)\Lambda_\sigma r^d \nu_d \tag{A.29}$$

$$\frac{|\mathcal{C}_\sigma[X]|}{n} \leq (1+\epsilon)\mathbb{P}(\mathcal{C}_\sigma) \tag{A.30}$$

$$\frac{1}{2}\mathrm{vol}_0 \leq \mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]) \leq \frac{3}{2}\mathrm{vol}_0 \tag{A.31}$$

Additionally, with probability one,

$$\liminf_{n\to\infty} \frac{\widetilde{\deg}_{\min}}{n} \geq \frac{6}{25}\lambda_\sigma r^d \nu_d$$

$$\limsup_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[X])}{n(n-1)} \leq \mathbb{P}(\mathcal{C}_\sigma)\Lambda_\sigma \nu_d r^d$$

Proof. We note that for any $x \in \mathcal{C}_\sigma \cup \mathcal{C}'_\sigma$, by (A1) and (A6) along with Lemma 5,

$$\frac{6}{25}\lambda_\sigma r^d \nu_d \leq \mathbb{P}(B(x,r))$$

Then, by Lemma 31 along with a union bound

$$\mathbb{P}\left(\widetilde{\deg}_{\min} \leq n(1-\epsilon)\frac{6}{25}\lambda_\sigma r^d \nu_d\right) \leq n \exp\left\{-\frac{6n\epsilon^2\lambda_\sigma r^d \nu_d}{50}\right\} \overset{n\to\infty}{\longrightarrow} 0.$$

As the same bound holds for $\deg'_{\min}$, (A.27) is shown. (A.28) and (A.29) follow from similar reasoning. (A.30) follows directly from the law of large numbers.

Finally, note that

$$\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]) = \sum_{i=1}^{n}\sum_{j\neq i}^{n} \mathbf{1}(x_i \in \mathcal{C}_\sigma)\mathbf{1}(\|x_i - x_j\| \leq r)$$

can be rewritten as a U-statistic, and furthermore

$$\int_{\mathcal{C}_\sigma} \mathbb{P}(B(x,r))\,dx = \mathbb{E}\left(\mathbf{1}(x_i \in \mathcal{C}_\sigma)\mathbf{1}(\|x_i - x_j\| \leq r)\right).$$

Therefore by Lemma 30

$$|\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]) - n(n-1)\int_{\mathcal{C}_\sigma}\mathbb{P}(B(x,r))\,dx| \leq \frac{3}{20}n(n-1)\int_{\mathcal{C}_\sigma}\mathbb{P}(B(x,r))\,dx \tag{A.32}$$

with probability at least $1 - 2\exp(-\frac{3}{20}n\int_{\mathcal{C}_\sigma}\mathbb{P}(B(x,r)\,dx) \to 0$ as $n \to \infty$. Then, application of the triangle inequality to (A.32), along with the range $\mathrm{vol}_0 \in [3/4, 5/4]\cdot n(n-1)\int_{\mathcal{C}_\sigma}\mathbb{P}(B(x,r))\,dx$ given by (13), yields

$$|\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]) - \mathrm{vol}_0| \leq \left(\frac{3}{20} + \frac{1}{4}\right)n(n-1)\int_{\mathcal{C}_\sigma}\mathbb{P}(B(x,r))\,dx \leq \frac{1}{2}\mathrm{vol}_0.$$

By similar reasoning to (A.32), for any $\epsilon > 0$, letting $\mu = \int_{\mathcal{C}_\sigma} \mathbb{P}(B(x,r)\,dx) \geq \mathbb{E}\left(\frac{\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[X])}{n(n-1)}\right)$ we have

$$\mathbb{P}\left(\frac{\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[X])}{n(n-1)} \geq (1+\epsilon)\mu\right) \leq \exp(-n\epsilon^2\mu).$$

The final two statements therefore both follow from the Borel-Cantelli Lemma. $\square$

A.12 Proof of Theorem 3

In this section, we prove both Theorem 3 and the relevant portions of Corollary 1.

Lemma 33 is a restatement of Lemma 3.4 of [7], translated into notation consistent with the rest of this paper. It is stated with respect to an arbitrary $A \subseteq G = (V,E)$, and a good set $A^g \subseteq A$ with $\mathrm{vol}(A^g; G) \geq \mathrm{vol}(A;G)/2$. [2]

Lemma 33. *Let $\epsilon \geq 0$, $v \in A^g$, $\alpha \leq \frac{\Psi(A;G)}{9}$, and $\mathrm{vol}_0 \in [1-c, 1+c]\mathrm{vol}(A;G)$ for $0 < c < 1$. Consider taking sweep cuts of the approximate PPR vector $p^{(\epsilon)} := p^{(\epsilon)}(v,\alpha;G)$ of the form*

$$S_\beta := \left\{u \in V : p^{(\epsilon)} \geq \beta \frac{\deg(u;G)}{\mathrm{vol}_0}\right\}$$

For any $\beta < \frac{3}{5(1+c)}$, and $\epsilon \leq \frac{1}{10}\mathrm{vol}(A;G)$ the following bounds hold:

$$\mathrm{vol}(S_\beta \setminus A; G) \leq \frac{2\Phi(A;G)}{\alpha\beta(1-c)}\mathrm{vol}(A;G), \tag{A.33}$$

$$\mathrm{vol}(A \setminus S_\beta; G) \leq \left(\frac{2\Phi(A;G)}{\alpha(\frac{3}{5} - \beta(1+c))} + 8\Phi(A;G)\right)\mathrm{vol}(A;G), \tag{A.34}$$

Proof of Theorem 3. Observe that by Theorem 2 and the upper bounds $\alpha \leq \frac{\mathbf{\Psi}(\theta)}{9}, r < \sigma/2d$ given in (13), with probability one as $n \to \infty$, $\alpha \leq \frac{\Psi_{n,r}(\mathcal{C}_\sigma[X])}{9}$. Moreover,

(i) we restrict $v \in \mathcal{C}_\sigma[X]^g$

(ii) the output of Algorithm 1 $\widehat{C} = S_\beta$ for some $1/40 < \beta < 1/11$

(iii) by Lemma 32, $\mathrm{vol}_0 \in [1/2, 3/2]\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])$

(iv) the PPR vector $p(v,\alpha;G_{n,r})$ is simply an approximate PPR vector with $\epsilon = 0$.

All conditions of Lemma 33 are therefore satisfied, and consequently by (A.33), (A.34), Theorem 1, and the lower bound $\alpha \geq \mathbf{\Psi}(\theta)/10$, the following bounds hold with probability tending to 1 as $n \to \infty$:

$$\mathrm{vol}_{n,r}(\widehat{C} \setminus \mathcal{C}_\sigma[X]) \leq 1600\kappa(\mathcal{C})\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]), \quad \text{and} \quad \mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X] \setminus \widehat{C}) \leq 52\kappa(\mathcal{C})\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])$$

Noting that

$$\begin{aligned}
\mathrm{vol}_{n,r}(\widehat{C} \setminus \mathcal{C}_\sigma[X]) &\geq |\widehat{C} \setminus \mathcal{C}_\sigma[X]|\, \deg_{\min} \\
\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X] \setminus \widehat{C}) &\geq |\mathcal{C}_\sigma[X] \setminus \widehat{C}| \widetilde{\deg}_{\min}, \text{ and} \\
\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X]) &\leq |\mathcal{C}_\sigma[X]| \max_{x \in \mathcal{C}_\sigma[X]} \deg_{\max}
\end{aligned}$$

the claim follows in light of Lemma 32.

[2] The sweep cuts of Lemma 3.4 of [7] are stated directly with respect to $\mathrm{vol}(A;G)$, rather than vol_0; this discrepancy leads to the extra factors of $1-c$ and $1+c$ which appear in our bounds as opposed to theirs.

Proof of Corollary 1 misclassification rates. The choice $\epsilon = \frac{1}{20\mathrm{vol}_0}$, along with Lemma 32 and the upper bound on vol_0 given by (13), together imply that $\epsilon < \frac{1}{10}\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])$ with probability tending to one as $n \to \infty$. We may therefore apply Lemma 33 with respect to $p^{(\epsilon)}$. All other aspects of the proof of Theorem 3 may be applied exactly as stated.

A.13 Proof of Theorem 4

We begin with some notation. Let $\mathbf{D}$ and $\mathbf{W}$ be the degree and (lazy) random walk matrices for the neighborhood graph $G_{n,r}$. Similarly, let $\widetilde{\mathbf{D}}$ and $\widetilde{\mathbf{W}}$ be the degree and (lazy) random walk matrices for the induced subgraph $\widetilde{G}_{n,r}$. Consider $\overline{\pi}_{n,r} : \mathcal{C}_\sigma[X] \to [0,1]$ given by

$$\overline{\pi}_{n,r}(x) := \frac{\widetilde{\deg}_{n,r}(x)}{\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])}.$$

(Note that $\overline{\pi}_{n,r}$ is distinct from $\widetilde{\pi}_{n,r}$, as we normalize by $\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])$ rather than $\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[X])$.)

Lemma 34 provides uniform error bounds on $p(u) - \overline{\pi}_{n,r}(u)$ over all $x \in X$ (where in an abuse of notation we consider $\overline{\pi}_{n,r}(x) = 0$ for $x \in X \setminus \mathcal{C}_\sigma[X]$).

Lemma 34. *Let $0 < r < \sigma$ and $\alpha \leq \frac{\Psi_{n,r}(\mathcal{C}_\sigma[X])}{9}$. Then the following statement holds: there exists a good set $\mathcal{C}_\sigma[X]^g \subseteq \mathcal{C}_\sigma[X]$ with $\mathrm{vol}(\mathcal{C}_\sigma[X]^g; G_{n,r}) \geq \mathrm{vol}(\mathcal{C}_\sigma[X]; G_{n,r})/2$ such that the following bounds hold with respect to $p := p(v, \alpha; G_{n,r})$ for any $v \in \mathcal{C}_\sigma[X]^g$:*

- *For each $u \in \mathcal{C}[X]$,*
$$p(u) \geq \frac{4}{5}\overline{\pi}_{n,r}(u) - \frac{20\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\Psi_{n,r}(\mathcal{C}_\sigma[X])\widetilde{\deg}_{\min}} \tag{A.35}$$

- *Let $\mathcal{C}' \neq \mathcal{C} \in \mathbb{C}_f(\lambda)$ satisfy (A3) with respect to $\mathcal{C}$. Then for each $u' \in \mathcal{C}'[X]$,*
$$p(u') \leq \frac{20\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\Psi_{n,r}(\mathcal{C}_\sigma[X])\,\deg_{\min}'}. \tag{A.36}$$

Proof. The proof of Lemma 34 is lengthy, but not difficult. Given starting distribution q with $\mathrm{supp}(q) \subseteq \mathcal{C}_\sigma[X]$, we let
$$\widetilde{p}_q = \alpha q + (1-\alpha)\widetilde{p}_q\widetilde{\mathbf{W}} \tag{A.37}$$
be the PPR vector originating from q over $\widetilde{G}_{n,r}$. (When the starting distribution $q = e_v$ is a point mass at a seed node $v \in \mathcal{C}_\sigma[X]$, we write $\widetilde{p}_v := \widetilde{p}_{e_v}$ in a slight abuse of notation).

Our analysis will involve *leakage* and *soakage* vectors, defined by

$$\ell_t := e_v(\mathbf{W}\widetilde{\mathbf{I}})^t(\mathbf{I} - \mathbf{D}^{-1}\widetilde{\mathbf{D}}),\ \ \ell := \sum_{t=0}^{\infty}(1-\alpha)^t\ell_t,$$
$$s_t := e_v(\mathbf{W}\widetilde{\mathbf{I}})^t(\mathbf{W}\widetilde{\mathbf{I}}^c),\ \ s := \sum_{t=0}^{\infty}(1-\alpha)^t s_t. \tag{A.38}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\widetilde{\mathbf{I}}$ is an $n \times n$ diagonal matrix with $\widetilde{\mathbf{I}}_{xx} = 1$ if $x \in \mathcal{C}_\sigma[X]$ and 0 otherwise, and $\widetilde{\mathbf{I}}^c = \mathbf{I} - \widetilde{\mathbf{I}}$.

In words, for $u \in \mathcal{C}_\sigma[X]$, $\ell_t(u)$ is the probability that a random walk over $G_{n,r}$ originating from $v \in \mathcal{C}_\sigma[X]$ stays within $\widetilde{G}_{n,r}$ for t steps, arriving at u on the tth step, and then leaks out of $\mathcal{C}_\sigma[X]$ on the $t+1$th step. For $w \in X \setminus \mathcal{C}_\sigma[X]$, $\ell_t(w) = 0$. By contrast, for w again in $X \setminus \mathcal{C}_\sigma[X]$, $s_t(w)$ is the probability that a random walk originating from v stays within $\mathcal{C}_\sigma[X]$ for t steps, and then is soaked up into w on the $t+1$ step, while $s_t(u) = 0$ for all $u \in \mathcal{C}_\sigma[X]$. The vectors ℓ and s then give the total mass leaked and soaked, respectively, by the PPR vector.

We first prove (A.35), and begin by restating some results of [7], adapted to our notation. By Lemma 3.1 of [7], there exists a good set $\mathcal{C}_\sigma[X]^g \subseteq \mathcal{C}_\sigma[X]$ with $\mathrm{vol}(\mathcal{C}_\sigma[X]^g; G_{n,r}) \geq \mathrm{vol}(\mathcal{C}_\sigma[X]; G_{n,r})/2$ such that for every $v \in \mathcal{C}_\sigma[X]^g$

$$p(u) \geq \widetilde{p}_v(u) - \widetilde{p}_\ell(u), \quad \text{and} \quad \|\ell\|_1 \leq \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha}. \tag{A.39}$$

(The result $\|\ell\|_1 \leq \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha}$ is the only result in the proof of Theorem 4 which relies on the restriction $v \in \mathcal{C}_\sigma[X]^g$.)

If additionally $\alpha \leq \frac{\Psi_{n,r}(\mathcal{C}_\sigma[X])}{9}$, then by Corollary 3.3 of [7], for every $u \in \mathcal{C}_\sigma[X]$

$$\widetilde{p}(u) \geq \frac{4}{5}\overline{\pi}_{n,r}(u)$$

and along with (A.39), we obtain

$$p(u) \geq \frac{4}{5}\overline{\pi}_{n,r}(u) - \widetilde{p}_\ell(u).$$

We proceed to show the upper bound $\widetilde{p}_\ell(u) \leq \|\ell\|_1 / \widetilde{\deg}_{\min}$, whence (A.35) follows by (A.39). We note two facts regarding $\widetilde{p}_\ell(u)$, which hold for all $u \in \mathcal{C}[X]$.

1. Since $r < \sigma$, $(u, w) \notin G_{n,r}$ for any $w \notin \mathcal{C}_\sigma$. As a result, for all $t \geq 1$, $\ell_t(u) = 0$ and by extension, $\ell(u) = 0$ as well.
2. For any q such that $\sum_{w \in \mathcal{C}_\sigma[X]} q(w) \leq 1$, and any $t \geq 1$,

$$\begin{aligned} q\widetilde{\mathbf{W}}^t(u) &\leq \|q\|_1 \, ||\widetilde{\mathbf{W}}_{\cdot u}||_\infty \\ &\leq \frac{1}{\widetilde{\deg}_{\min}} \end{aligned} \tag{A.40}$$

   where $\widetilde{\mathbf{W}}_{\cdot u}$ is the uth column of $\widetilde{\mathbf{W}}$, and the last inequality follows from the fact $(u, w) \in \widetilde{G}_{n,r}$ implies $w \in \mathcal{C}_\sigma$, and therefore $\deg(w; \widetilde{G}_{n,r}) \geq \widetilde{\deg}_{\min}$.

These facts, along with some basic algebra, lead to the desired lower bound on $\widetilde{p}_\ell(u)$ for every $u \in \mathcal{C}[X]$:

$$\begin{aligned} \widetilde{p}_\ell(u) &= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\ell \widetilde{\mathbf{W}}^t\right)(u) \\ &= \|\ell\|_1 \, \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\frac{\ell}{\|\ell\|_1} \widetilde{\mathbf{W}}^t\right)(u) \\ &= \|\ell\|_1 \, \alpha \sum_{t=1}^{\infty} (1-\alpha)^t \left(\frac{\ell}{\|\ell\|_1} \widetilde{\mathbf{W}}^t\right)(u) \\ &\leq \frac{\|\ell\|_1}{\widetilde{\deg}_{\min}}. \end{aligned}$$

and (A.35) is proved.

We turn to showing (A.36). By Lemma 35, for all $u' \notin \mathcal{C}_\sigma[X]$,

$$p_v(u') \leq p_s(u').$$

Note that by (A3), for every $u \in \mathcal{C}_\sigma[X]$, $(u', u) \notin E$ and therefore $s(u') = 0$. Some manipulations, similar to those in the preceding part of the proof, yield a lower bound on $p_v(u')$ in terms of $\|s\|_1$:

$$
\begin{aligned}
p_s(u') &= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(s\mathbf{W}^t\right)(u') \\
&= \|s\|_1 \, \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\frac{s}{\|s\|_1}\mathbf{W}^t\right)(u') \\
&= \|s\|_1 \, \alpha \sum_{t=1}^{\infty} (1-\alpha)^t \left(\frac{s}{\|s\|_1}\mathbf{W}^t\right)(u') \\
&\leq \frac{\|s\|_1}{\deg'_{\min}}
\end{aligned}
$$

where the last inequality follows from precisely the same reasoning as (A.40). The claim follows in light of Lemma 37, along with (A.39). □

We turn now to the proof of Theorem 4. Let $\widetilde{\pi}_{\min} := \min_{u \in \mathcal{C}[X]} \widetilde{\pi}_{n,r}(u)$.

Proof of Theorem 4.

In the proof that follows, every inequality will hold with probability tending to one as $n \to \infty$.

Observe that by Theorem 2 and the upper bounds $\alpha \leq \frac{\Psi}{9}$ and $r < \sigma/2d$ given in (13), with probability one as $n \to \infty$, $\alpha \leq \frac{\Psi_{n,r}(\mathcal{C}_\sigma[X])}{9}$. As a result, with probability one as $n \to \infty$, the conditions of Theorem 4 subsume the requirements of Lemma 34, and by that lemma, the following inequalities hold for each $u \in \mathcal{C}[X]$, $u' \in \mathcal{C}'[X]$:

$$
\begin{aligned}
p(u) &\geq \frac{4}{5}\overline{\pi}_{n,r}(u) - \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha\widetilde{\deg}_{\min}} \\
p(u') &\leq \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha \deg'_{\min}}
\end{aligned}
\tag{A.41}
$$

We will show that under the conditions of Theorem 4, the following bounds hold with probability tending to one as $n \to \infty$:

$$
\begin{aligned}
\frac{2\widetilde{\Phi}_{n,r}(\mathcal{C}_\sigma[X])}{\alpha\widetilde{\deg}_{\min}} &\leq \frac{\widetilde{\deg}_{\min}}{125\mathrm{vol}_{n,r}(\mathcal{C}_\sigma X)} \\
\frac{2\widetilde{\Phi}_{n,r}(\mathcal{C}_\sigma[X])}{\alpha \deg'_{\min}} &\leq \frac{\deg'_{\min}}{125\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])}
\end{aligned}
\tag{A.42}
$$

The bounds in (A.42) along with (A.41) imply that for each $u \in \mathcal{C}[X]$,

$$
\begin{aligned}
\frac{p(u)}{\deg_{n,r}(u)} &\geq \frac{4}{5}\frac{\widetilde{\deg}_{n,r}(u)}{\deg_{n,r}(u)\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} - \frac{\widetilde{\deg}_{\min}}{125\deg_{n,r}(u)\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} \\
&\geq \frac{22}{125\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} && \text{(by (A.28) and (A.29), applied with } \epsilon = \tfrac{1}{47}) \\
&\geq \frac{44}{375\mathrm{vol}_0}. && \text{(by (A.31))}
\end{aligned}
$$

Similarly, for each $u' \in \mathcal{C}'[X]$,

$$\begin{aligned}
\frac{p(u')}{\deg_{n,r}(u')} &\leq \frac{\deg'_{\min}}{\deg_{n,r}(u')125\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} \\
&\leq \frac{1}{125\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} \\
&\leq \frac{8}{375\mathrm{vol}_0}. && \text{(by (A.31))}
\end{aligned}$$

and as every sweep cut S_β under consideration in Algorithm 1 satisfies $8/375 < \frac{\beta}{\mathrm{vol}_0} < 44/375$, (6) must hold for the output $\widehat{C} = S_{\beta^*}$.

It remains to prove (A.42). Apply Lemma 32 with $\epsilon = 1/11$ to obtain

$$\frac{\deg'^2_{\min}}{\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])}, \frac{\widetilde{\deg}^2_{\min}}{\mathrm{vol}_{n,r}(\mathcal{C}_\sigma[X])} \geq \left(\frac{6}{25}\right)^2 \frac{\lambda_\sigma^2 \nu_d r^d}{\Lambda_\sigma \mathbb{P}(\mathcal{C}_\sigma)} \tag{A.43}$$

where the bound holds with probability tending to one as $n \to \infty$.

By Theorem 1 and the lower bound $\alpha \geq \frac{1}{10\boldsymbol{\Psi}}$ given by (13), with probability tending to one as $n \to \infty$,

$$\frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha} \leq 20\kappa_1(\mathcal{C}). \tag{A.44}$$

(A.44), (A.43) and (17) together show (A.42).

Proof of Corollary 1: consistent cluster estimation. Using (18) with $\epsilon = \frac{1}{20}\mathrm{vol}_0$ along with (A.35) we obtain

$$p^{(\epsilon)}(u) \geq \frac{4}{5}\overline{\pi}_{n,r}(u) - \frac{\deg_{n,r}(u)}{20\mathrm{vol}_0} - \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[X])}{\alpha\widetilde{\deg}_{\min}}, \quad \text{for all } u \in \mathcal{C}[X]$$

and similar reasoning as used in the proof of Theorem 4 yields

$$\frac{p^{(\epsilon)}(u)}{\deg_{n,r}(u)} \geq \frac{15}{125\mathrm{vol}_0} \geq \frac{1}{11\mathrm{vol}_0}, \quad \text{for all } u \in \mathcal{C}[X].$$

On the other hand $p^{(\epsilon)}(u) \leq p(u)$. Therefore $\frac{p^{(\epsilon)}(u')}{\deg_{n,r}(u')} \leq \frac{8}{375\mathrm{vol}_0}$, and the claim is proved.

A.14 Linear Algebra Facts

We state here a number of basic facts which follow from matrix manipulations, which are used in the proof of Theorem 4.

Lemma 35. *For any $v \in \mathcal{C}_\sigma[X]$ and $u \notin \mathcal{C}_\sigma[X]$,*

$$p_v(u) \leq p_s(u)$$

where s is defined as in (A.38) and depends implicitly upon v.

Proof. The statement follows from Lemma 36 along with a series of algebraic manipulations,

$$\begin{aligned}
p_v(u) &= \alpha \sum_{T=0}^{\infty} (1-\alpha)^t \left(e_v \mathbf{W}^T\right)(u) \\
&= \alpha \sum_{T=1}^{\infty} (1-\alpha)^t \left(e_v \mathbf{W}^T\right)(u) \\
&\leq \alpha \sum_{T=1}^{\infty} (1-\alpha)^T \left(\sum_{t=0}^{T-1} s_t \mathbf{W}^{T-t-1}\right)(u) && \text{(Lemma 36)} \\
&= \alpha \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} (1-\alpha)^T \left(s_t \mathbf{W}^{T-t-1}\right)(u) \\
&= \alpha \sum_{t=0}^{\infty} \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta+t+1} \left(s_t \mathbf{W}^{\Delta}\right)(u) \\
&\leq \alpha \sum_{t=0}^{\infty} \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta+t} \left(s_t \mathbf{W}^{\Delta}\right)(u) \\
&= \alpha \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta} \left(s \mathbf{W}^{\Delta}\right)(u) \\
&= p_s(u)
\end{aligned}$$

□

Let $s_t := q(\mathbf{W}\mathbf{I}_S)^t(\mathbf{W}(\mathbf{I}_{S^c}))$ be the soakage vector out of $S \subseteq V$, where $\mathbf{I}_S$ is a $|V| \times |V|$ diagonal matrix with $(\mathbf{I}_S)_{uu} = 1$ if $u \in S$ and 0 otherwise, and $\mathbf{I}_{S^c} := \mathbf{I} - \mathbf{I}_S$.

Lemma 36. *Let $G = (V, E)$ be an unweighted, undirected graph with associated random walk matrix $\mathbf{W}$. For any $T \in \mathbb{N}, T \geq 1$, $q \in \mathbb{R}^{|V|}$, and $S \subseteq V$*

$$q\mathbf{W}^T = \sum_{t=0}^{T-1} s_t \mathbf{W}^{T-t-1} + q(\mathbf{W}\mathbf{I}_S)^T \tag{A.45}$$

In particular, if $u \in V \setminus S$, then

$$q\mathbf{W}^T(u) = \sum_{t=0}^{T-1} \left(s_t \mathbf{W}^{T-t-1}\right)(u) \tag{A.46}$$

Proof. We show (A.45), from which (A.46) is an immediate consequence.

To show (A.45), we proceed by induction on T. When $T = 1$,

$$q\mathbf{W} = q\mathbf{W}\mathbf{I}_S + q\mathbf{W}\mathbf{I}_{S^c} = q\mathbf{W}\mathbf{I}_S + s_0.$$

Then, for $T \in \mathbb{N},\ T \geq 2$,

$$\begin{aligned}
q\mathbf{W}^T &= q\mathbf{W}^{T-1}\mathbf{W} \\
&= \left\{\sum_{t=0}^{T-2} s_t \mathbf{W}^{T-2-t} + q(\mathbf{W}\mathbf{I}_S)^{T-1}\right\}\mathbf{W} && \text{(by the inductive hypothesis)} \\
&= \sum_{t=0}^{T-2} s_t \mathbf{W}^{T-1-t} + q(\mathbf{W}\mathbf{I}_S)^{T-1}(\mathbf{W}\mathbf{I}_S + \mathbf{W}\mathbf{I}_{S^c}) \\
&= \sum_{t=0}^{T-1} s_t \mathbf{W}^{T-1-t} + q(\mathbf{W}\mathbf{I}_S)^{T-1}(\mathbf{W}\mathbf{I}_S)
\end{aligned}$$

and the proof is complete. □

Lemma 37. *Letting s_t, ℓ_t s and ℓ be as in (A.38),*

$$\|s_t\|_1 = \|\ell_t\|_1\,,\ \textit{for each } t \geq 0$$

and therefore $\|s\|_1 = \|\ell\|_1$.

Proof. By the definition of s_t and ℓ_t, we have

$$\begin{aligned}
\|s_t\|_1 &= \left\|q_t(\mathbf{W}\widetilde{\mathbf{I}}^c)\right\|_1 \\
&= \sum_{u\in X}\sum_{u'\in X} q_t(u)(\mathbf{W}\widetilde{\mathbf{I}}^c)(u,u') \\
&= \sum_{u\in\mathcal{C}_\sigma[X]}\ \sum_{u'\in\mathcal{C}_\sigma[X]^c} \frac{q_t(u)}{(\mathbf{D})_{uu}}\mathbf{1}((u,u')\in G_{n,r}) \\
&= \sum_{u\in\mathcal{C}_\sigma[X]} \frac{q_t(u)\left((\mathbf{D})_{uu} - (\widetilde{\mathbf{D}})_{uu}\right)}{(\mathbf{D})_{uu}} \\
&= \left\|q_t(\mathbf{I} - \mathbf{D}^{-1}\widetilde{\mathbf{D}})\right\|_1 = \|\ell_t\|_1\,.
\end{aligned}$$

□

B Additional Experiments

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