Notes on ‘An Elementary Introduction to Modern Convex Geometry’

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1 Setup

Let K be a convex body in $\mathbb{R}^d$. Denote the ratio of the d-dimensional unit ball by ν_d. Let $Q = [-1, 1]^d$ be the unit cube in $\mathbb{R}^d$.

Assuming symmetry of K will often greatly ease proofs.

Definition 1.1. We say K is *(centrally) symmetric* if $-x \in K$ whenever $x \in K$. This will also imply that K is the unit ball of some norm $\|\cdot\|_K$ on $\mathbb{R}^d$:

$$K = \{x : \|x\|_K \leq 1\}$$

Volume ratio. The volume ratio is used to prove reverse isoperimetric inequalities of the form we want. To define it, we first recall the concept of an *ellipsoid*: given $(e_j)_{j=1}^d$ an orthonormal basis of $\mathbb{R}^d$ and (a_j) positive numbers, the ellipsoid

$$\left\{x : \sum_{i=1}^{d} \frac{\langle x, e_j \rangle^2}{\alpha_j^2} \leq 1\right\}$$

has volume $\nu_d \prod \alpha_j$.

Definition 1.2. Let K be a convex body in $\mathbb{R}^d$. The *volume ratio* of K is

$$\mathrm{vr}(K) := \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathcal{E})}\right)^{1/d}$$

where $\mathcal{E}$ is the ellipsoid of maximal volume in K.

Let B_2^d be the d-dimensional unit Euclidean ball.

Definition 1.3. The *surface area* of a convex body $K \in \mathbb{R}^d$, ∂K, is defined by

$$\mathrm{vol}(\partial K) = \lim_{\epsilon \to 0} \frac{\mathrm{vol}(K + \epsilon B_2^d) - \mathrm{vol}(K)}{\epsilon}$$

2 Convolutions and Volume Ratios: The Reverse Isoperimetric Problem

Theorem 1. *Let K be a convex body and T a regular solid simplex in $\mathbb{R}^d$. Then there is an affine image of K whose volume is the same as that of T and whose surface area is no larger than that of T.*

We will prove a related, by far simpler, result.

Theorem 2. *Let K be a symmetric convex body, and T the d-dimensional unit cube. Then there is an affine image of K whose volume is the same as that of T and whose surface area is no larger than that of T.*

The result follows from a result showing that the unit cube $Q = [-1, 1]^d$ has the largest volume ratio among convex bodies.

Theorem 3. *Among symmetric convex bodies the cube has largest volume ratio.*

Proof of Theorem 2. We begin by recall that for Q

$$\mathrm{vol}(\partial Q) = 2d\mathrm{vol}(Q)^{(d-1)/d},$$

so we wish to show that any other convex body K has an affine image $\widetilde{K}$ such that

$$\mathrm{vol}(\partial \widetilde{K}) \leq 2d \ \mathrm{vol}(\widetilde{K})^{(d-1)/d}.$$

Choose $\widetilde{K}$ so that its maximal volume ellipsoid is B_2^n, the Euclidean ball of radius 1. Then, by Theorem 3, we have that

$$\mathrm{vr}(\widetilde{K}) \leq \mathrm{vr}(Q)$$

and since the maximal volume ellipsoid is the same for both $\widetilde{K}$ and Q, this implies $\mathrm{vol}(\widetilde{K}) \leq \mathrm{vol}(Q) = 2^d$.

Now, since $\widetilde{K} \supset B_2^d$, the surface area can be upper bounded

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\mathrm{vol}(\widetilde{K} + \epsilon \widetilde{K}) - \mathrm{vol}(\widetilde{K})}{\epsilon} &= \mathrm{vol}(\widetilde{K}) \lim_{\epsilon \to 0} \frac{(1+\epsilon)^d - 1}{\epsilon} \\
&= d \ \mathrm{vol}(\widetilde{K}) = d \ \mathrm{vol}(\widetilde{K})^{1/d} \mathrm{vol}(\widetilde{K})^{(d-1)/d} \\
&\leq 2d\mathrm{vol}(\widetilde{K})^{(d-1)/d}.
\end{aligned}$$

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