A Proofs

In this supplement, we present proofs for "Local Clustering of Density Upper Level Sets". We begin by providing technical lemmas, before moving on to proving the main results of the paper.

Throughout, we will fix $\mathcal{A} \subset \mathbb{R}^d$ to be an arbitrary set. To simplify expressions, for the σ-expansion $\mathcal{A}_\sigma$, we will write the set difference between $\mathcal{A}_\sigma$ and the $(\sigma + r)$-expansion $\mathcal{A}_{\sigma+r}$ as

$$\mathcal{A}_{\sigma,\sigma+r} := \{x : 0 < \rho(x, \mathcal{A}_\sigma) \leq r\},$$

where $\rho(x, \mathcal{A}) = \min_{x' \in \mathcal{A}} \|x - x'\|$.

For notational ease, we write

$$\mathrm{cut}_{n,r} = \mathrm{cut}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}), \ \mu_K = \mathbb{E}(\mathrm{cut}_{n,r}), \ p_K = \frac{\mu_K}{\binom{n}{2}}$$
$$\mathrm{vol}_{n,r} = \mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}), \ \mu_V = \mathbb{E}(\mathrm{vol}_{n,r}), \ p_V = \frac{\mu_V}{\binom{n}{2}}$$

for the random variable, mean, and probability of cut size and volume, respectively.

A.1 Technical Lemmas

We state Lemma 1 without proof, as it is trivial. We formally include it mainly to comment on its (potential) suboptimality; for sets $\mathcal{A}$ with diameter much larger than σ, the volume estimate of Lemma 1 will be quite poor.

Lemma 1. *For any $\sigma > 0$ and the σ-expansion $\mathcal{A}_\sigma = \mathcal{A} + \sigma B$,*

$$\sigma B \subset \mathcal{A}_\sigma, \quad \text{and } \nu(\mathcal{A} + \sigma B) \leq \nu((1+\sigma)\mathcal{A}) = (1+\sigma)^d \nu(\mathcal{A}).$$

We will need to carefully control the volume of the expansion set using the above estimate; Lemma 2 serves this purpose.

Lemma 2. *For any $0 \leq x \leq 1/2d$,*

$$(1+x)^d \leq 1 + 2dx.$$

The proof of Lemma 2 is based on approximation via Taylor series, and we omit it.

We will repeatedly employ Lemma 1 and Lemma 2 in tandem. As a first example, in Lemma 3, we use it to bound the ratio of $\nu(\mathcal{A}_\sigma)$ to $\nu(\mathcal{A}_{\sigma-r})$. This will be useful when we bound $\mathrm{vol}(\mathcal{C}_\sigma)$.

Lemma 3. *For σ, $\mathcal{A}_\sigma$ as in Lemma 1, let $r > 0$ satisfy $r \leq \sigma/4d$. Then,*

$$\frac{\nu(\mathcal{A}_\sigma)}{\nu(\mathcal{A}_{\sigma-r})} \leq 2.$$

Proof. Fix $q = \sigma - r$. Then,

$$\begin{aligned}\nu(\mathcal{A}_\sigma) &= \nu(\mathcal{A}_{q+\sigma-q}) = \nu(\mathcal{A}_q + (\sigma - q)B) \\ &\leq \nu(\mathcal{A}_q + \frac{(\sigma-q)}{q}\mathcal{A}_q) = \left(1 + \frac{\sigma - q}{q}\right)^d \nu(\mathcal{A}_q)\end{aligned}$$

where the inequality follows from Lemma 1. Of course, $\sigma - q = r$, and $\frac{r}{q} \leq \frac{1}{2d}$ for $r \leq \frac{1}{4d}$. The claim then follows from Lemma 2. □

The proof of Theorem 3 also depends on a parameter – which we term *discrete local spread* – to handle the mixing over very small steps. Formally, the discrete local spread $\pi_1(G)$ is given by

$$\pi_1(G) := \frac{d_{\min}(G)^2}{10\mathrm{vol}(V;G)} \tag{A.1}$$

where $d_{\min}(G) = \min_{v \in V} d(v)$ is the minimum degree in G. Intuitively, the discrete local spread gauges how much the walk given by $\mathbf{W}$ has mixed after one step, starting from any node v. We will denote $\pi_1(G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]])$ by $\widetilde{\pi}_{1,n}$.

Prove Lemma 16.

A.2 Cut and volume estimates

Lemma 4. *Under the conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) \leq 2\nu(\mathcal{C}_\sigma)\frac{rd}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. Recalling that f is the density function for $\mathbb{P}$, we have

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) = \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx \tag{A.2}$$

We partition $\mathcal{C}_{\sigma,\sigma+r}$ into slices, based on distance from $\mathcal{C}_\sigma$, as follows: for $k \in \mathbb{N}$,

$$\mathcal{T}_{i,k} = \left\{x \in \mathbb{R}^d : t_{i,k} < \frac{\rho(x, \mathcal{C}_\sigma)}{r} \leq t_{i+1,k}\right\}, \quad \mathcal{C}_{\sigma,\sigma+r} = \bigcup_{i=0}^{k-1} \mathcal{T}_{i,k}$$

where $t_i = i/k$ for $i = 0, \ldots, k-1$. As a result,

$$\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx = \sum_{i=0}^{k-1} \int_{\mathcal{T}_{i,k}} f(x)dx \leq \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x).$$

We substitute

$$\nu(\mathcal{T}_{i,k}) = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B) - \nu(\mathcal{C}_\sigma + rt_{i,k}B) := \nu_{i+1,k} - \nu_{i,k}.$$

where for simplicity we've written $\nu_{i,k} = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B)$. This, in concert with the upper bound

$$\max_{x\in\mathcal{T}_{i,k}} f(x) \leq \lambda_\sigma - (rt_{i,k})^\gamma,$$

which follows from (A1) and (A2), yields

$$\begin{aligned}\sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x\in\mathcal{T}_{i,k}} f(x) &\leq \sum_{i=0}^{k-1}\Big\{\nu_{i+1,k} - \nu_{i,k}\Big\}\Big(\lambda_\sigma - (rt_{i,k})^\gamma\Big) \\ &= \sum_{i=1}^{k} \underbrace{\nu_{i,k}\Big([\lambda_\sigma - (rt_{i,k})^\gamma] - [\lambda_\sigma - (rt_{i-1,k})^\gamma]\Big)}_{:=\Sigma_k} + \underbrace{\Big(\nu_{k,k}\,[\lambda_\sigma - r^\gamma] - \nu_{1,k}\lambda_\sigma\Big)}_{:=\xi_k}\end{aligned} \tag{A.3}$$

We first consider the term Σ_k. Here we use Lemma 1 to upper bound

$$\nu_{i,k} \leq \mathrm{vol}(\mathcal{C}_\sigma)\left(1 + \frac{rt_{i,k}}{\sigma}\right)^d$$

and so we can in turn upper bound Σ_k:

$$\Sigma_k \leq \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \sum_{i=1}^{k}\left(1 + \frac{rt_{i,k}}{\sigma}\right)^d \Big((t_{i-1,k})^\gamma - (t_{i,k})^\gamma\Big). \tag{A.4}$$

This, of course, is a Riemann sum, and as the inequality holds for all values of k it holds in the limit as well, which we compute to be

$$\begin{aligned}\lim_{k\to\infty}\sum_{i=1}^{k}\left(1 + \frac{rt_{i,k}}{\sigma}\right)^d \Big((t_{i-1,k})^\gamma - (t_{i,k})^\gamma\Big) &= \gamma\int_0^1 \left(1 + \frac{rt}{\sigma}\right)^d t^{\gamma-1}dt \\ &\overset{(i)}{\leq} \gamma\int_0^1 \left(1 + \frac{2drt}{\sigma}\right) t^{\gamma-1}dt = \left(1 + \frac{\gamma 2dr}{\gamma+1}\right).\end{aligned}$$

where (i) follows from Lemma 2. We plug this estimate in to (A.4) and obtain

$$\lim_{k\to\infty}\Sigma_k \leq \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \left(1 + \frac{\gamma 2dr}{\gamma+1}\right).$$

We now provide an upper bound on ξ_k. It will follow the same basic steps as the bound on Σ_k, but will not involve integration:

$$\begin{aligned}\xi_k &\overset{(ii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1 + \frac{r}{\sigma}\right)^d(\lambda - r^\gamma) - \lambda\right\} \\ &\overset{(iii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1 + \frac{2dr}{\sigma}\right)(\lambda - r^\gamma) - \lambda\right\} = \nu(\mathcal{C}_\sigma)\left\{\frac{2dr}{\sigma}(\lambda - r^\gamma) - r^\gamma\right\}.\end{aligned}$$

where (ii) follows from Lemma 1 and (iii) from Lemma 2. The final result comes from adding together the upper bounds on Σ_k and ξ_k and taking the limit as $k \to \infty$. $\square$

Lemma 5. *Under the setup and conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$p_K \leq \frac{4\lambda\nu_d r^{d+1}\nu(\mathcal{C}_\sigma)d}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. We can write $\mathrm{cut}_{n,r}$ as the sum of indicator functions,

$$\mathrm{cut}_{n,r} = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{1}(x_i \in \mathcal{C}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i,r)\cap\mathcal{C}_\sigma) \tag{A.5}$$

and by linearity of expectation, we can obtain

$$p_K = \frac{\mu_K}{\binom{n}{2}} = 2\cdot\mathbb{P}(x_i \in \mathcal{C}_{\sigma,\sigma+r}, x_j \in B(x_i,r)\cap\mathcal{C}_\sigma)$$

Writing this with respect to the density function f, we have

$$\begin{aligned}
p_K &= 2\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)\left\{\int_{B(x,r)\cap\mathcal{C}_\sigma} f(x')dx'\right\}dx \\
&\leq 2\nu_d r^d\lambda\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx
\end{aligned}$$

where the inequality follows from Assumption (A3), which implies that the density function $f(x') \leq \lambda$ for all $x' \in \mathcal{C}_\sigma \setminus \mathcal{C}$ (otherwise, x' would be in some $\mathcal{C}' \in \mathbb{C}_f(\lambda)$, which (A3) forbids). Then, upper bounding the integral using Lemma 5 gives the final result. $\square$

Lemma 6. *Under the setup and conditions of Theorem 1,*

$$p_V \geq \lambda_\sigma^2\nu_d r^d\nu(\mathcal{C}_\sigma)$$

Proof. The proof will proceed similarly to Lemma 5. We begin by writing $\mathrm{vol}_{n,r}$ as the sum of indicator functions,

$$\mathrm{vol}_{n,r} = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{1}(x_i \in \mathcal{C}_\sigma)\mathbf{1}(x_j \in B(x_i,r)) \tag{A.6}$$

and by linearity of expectation we obtain

$$p_V = \frac{\mu_V}{\binom{n}{2}} = 2\cdot\mathbb{P}(x_i \in \mathcal{C}_\sigma, x_j \in B(x_i,r)).$$

Writing this with respect to the density function f, we have

$$\begin{aligned} p_V &= 2\int_{\mathcal{C}_\sigma} f(x)\left\{\int_{B(x,r)} f(x')dx'\right\}dx \\ &\geq 2\int_{\mathcal{C}_{\sigma-r}} f(x)\left\{\int_{B(x,r)} f(x')dx'\right\}dx \\ &\overset{(i)}{\geq} 2\lambda_\sigma^2 \nu_d r^d \int_{\mathcal{C}_{\sigma-r}} f(x)dx \end{aligned}$$

where (i) follows from the fact that $B(x,r) \subset \mathcal{C}_\sigma$ for all $x \in C_{\sigma-r}$, along with the lower bound in Assumption (A1). The claim then follows from Lemma 3. $\square$

We now convert from bounds on p_K and p_V to probabilistic bounds on $\mathrm{cut}_{n,r}$ and $\mathrm{vol}_{n,r}$ in Lemmas 7 and 8. The key ingredient will be Lemma 9, Hoeffding's inequality for U-statistics; the proofs for both are nearly identical and we give only a proof for Lemma 7.

Lemma 7. *The following statement holds for any $\delta \in (0,1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_K + \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.7}$$

with probability at least $1-\delta$.

Lemma 8. *The following statement holds for any $\delta \in (0,1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} \geq p_V - \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.8}$$

with probability at least $1-\delta$.

Proof of Lemma 7. From (A.5), we see that $\mathrm{cut}_{n,r}$, properly scaled, can be expressed as an order-2 U-statistic,

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq n} \phi_K(x_i, x_j)$$

where

$$\phi_K(x_i,x_j) = \mathbf{1}(x_i \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i,r)\cap\mathcal{A}_\sigma) + \mathbf{1}(x_j \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_i \in B(x_j,r)\cap\mathcal{A}_\sigma).$$

From Lemma 9 we therefore have

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_k + \sqrt{\frac{\log(1/\delta)}{n}}$$

with probability at least $1-\delta$. $\square$

A.3 Proof of Theorem 1

The proof of Theorem 1 is more or less given by Lemmas 5, 6, 7, and 8. All that remains is some algebra, which we take care of below.

Fix $\delta \in (0, 1]$ and let $\delta' = \delta/2$. Noting that $\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{\mathrm{cut}_{n,r}}{\mathrm{vol}_{n,r}}$, some trivial algebra gives us the expression

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{p_K + \left(\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} - p_K\right)}{p_V + \left(\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} - p_V\right)} \tag{A.9}$$

We assume (A.7) and (A.8) hold with respect to δ', keeping in mind that this will happen with probability at least $1 - \delta$. Along with (A.9) this means

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) \leq \frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n}$$

for $\mathrm{Err}_n = \sqrt{\frac{\log(1/\delta')}{n}}$. Now, some straightforward algebraic manipulations yield

$$\begin{aligned}
\frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n} &= \frac{p_K}{p_V} + \left(\frac{p_K}{p_V - \mathrm{Err}_n} - \frac{p_K}{p_V}\right) + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \\
&= \frac{p_k}{p_V} + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}\left(\frac{p_K}{p_V} + 1\right) \\
&\leq \frac{p_K}{p_V} + 2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}.
\end{aligned}$$

By Lemmas 5 and Lemma 6, we have

$$\frac{p_K}{p_V} \leq \frac{4rd}{\sigma}\frac{\lambda}{\lambda_\sigma}\frac{\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)}{\lambda_\sigma}$$

Then, the choice of

$$n \geq \frac{9\log(2/\delta)}{\epsilon^2}\left(\frac{1}{\lambda_\sigma^2 \nu(\mathcal{C}_\sigma)\nu_d r^d}\right)^2$$

implies $2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \leq \epsilon$.

A.4 Concentration inequalities

Given a symmetric kernel function $k : \mathcal{X}^m \to \mathbb{R}$, and data $\{x_1, \ldots, x_n\}$, we define the *order-m U statistic* to be

$$U := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} k(x_{i_1}, \ldots, x_{i_m})$$

For Lemmas 9, let $X_1, \ldots, X_n \in \mathcal{X}$ be independent and identically distributed. We will additionally assume the order-m kernel function k satisfies the boundedness property $\sup_{x_1,\ldots,x_m} |k(x_1, \ldots, x_m)| \leq 1$.

Lemma 9 (Hoeffding's inequality for U-statistics.)**.** *For any $t > 0$,*

$$\mathbb{P}(|U - \mathbb{E}U| \geq t) \leq 2 \exp\left\{-\frac{2nt^2}{m}\right\}$$

Further, for any $\delta > 0$, we have

$$U \leq \mathbb{E}U + \sqrt{\frac{m \log(1/\delta)}{2n}},$$
$$U \geq \mathbb{E}U - \sqrt{\frac{m \log(1/\delta)}{2n}}$$

each with probability at least $1 - \delta$.

We will employ a sharper concentration inequality for $\sum_{i=1}^n X_i$.

Lemma 10. *Let $X_i \in \{0, 1\}$ for $i = 1, \ldots, n$ and let $\mu = \mathbb{E}(\sum_{i=1}^n X_i)$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i > (1+\epsilon)\mu\right) \leq \exp\left(\frac{-\delta^2 \mu}{3}\right)$$
$$\mathbb{P}\left(\sum_{i=1}^n X_i < (1-\epsilon)\mu\right) \leq \exp\left(\frac{-\delta^2 \mu}{2}\right)$$

A.5 Mixing time on graphs

For $N \in \mathbb{N}$ and a set V of N vertices, take $G = (V, E)$ to be an undirected and unweighted graph, with associated adjacency matrix $\mathbf{A}$, random walk matrix $\mathbf{W}$, and stationary distribution $\boldsymbol{\pi} = (\pi_u)_{u \in V}$ where $\pi_v = \frac{\mathbf{D}_{vv}}{\mathrm{vol}(V;G)}$. For $v \in V$,

$$q_{vu}^{(m)} = e_v \mathbf{W}^m e_u, \quad \mathbf{q}_v^{(m)} = \left(q_{vu}^{(m)}\right)_{u \in V}, \quad \mathbf{q}_v = (\mathbf{q}_{v\cdot}^{(1)}, \mathbf{q}_{v\cdot}^{(2)}, \ldots), \tag{A.10}$$

denote respectively the m-step transition probability, distribution, and sequence distributions of the random walk over G originating at v. Letting $\mathbf{q} = (\mathbf{q}_v)_{v \in V}$, the relative pointwise mixing time is thus

$$\tau_\infty(\mathbf{q}; G) = \min\left\{m : \forall u, v \in V, \frac{\left|q_{vu}^{(m)} - \boldsymbol{\pi}_u\right|}{\boldsymbol{\pi}_u} \leq 1/4\right\}$$

Two key quantities relate the mixing time to the expansion of subsets S of V. The *local spread* is defined to be

$$s(G) := \frac{9D_{\min}}{10}\pi_{\min}$$

for $D_{\min} := \min_{v\in V} \mathbf{D}_{vv}$ and $\pi_{\min} := D_{\min}/\mathrm{vol}(V;G)$.

where $\beta(S) := \inf_{v\in S} \mathbf{q}_v^{(1)}(S^c)$, and by convention we let $\mathbf{p}(S) = \sum_{u\in S} p_u$ for any distribution vector $\mathbf{p} = (p_u)_{u\in V}$ over V. We collect some necessary facts about the local spread in Lemma 11.

Lemma 11.
- *If $\boldsymbol{\pi}(S) \leq s(G)$, then for every $u \in S$, $\mathbf{q}_u^{(1)}(S^c) \geq 1/10$.*
- *For any $v, u \in V$, and $m \in N$ greater than 0, $q_{vu}^{(m)}/\pi_{\min} \leq 1/s(G)$.*

Proof. If $t = \boldsymbol{\pi}(S) \leq \frac{9D_{\min}}{10}\pi_{\min}$, divide both sides by $\pi_{\min}$ to obtain

$$|S| \leq \frac{9D_{\min}}{10}$$

which implies $\mathbf{q}_v^{(1)}(S^c) \geq 1/10$ for all $v \in S$. This implies the first statement.

The second statement follows from the fact $q_{vu}^{(m)} \leq 1/D_{\min}$ for any m. □

The local spread facilitates conversion between $\tau_\infty(\mathbf{q}_v; G)$ and the more easily manageable *total variation* mixing time, given by

$$\tau_1(\boldsymbol{\rho}; G) = \min\left\{ m : \forall v \in V, \|\boldsymbol{\rho}_v - \boldsymbol{\pi}\|_{TV} \leq 1/4 \right\}$$

where

$$\boldsymbol{\rho}_v^{(m)} = \frac{1}{m}\sum_{k=1}^{m+1} \mathbf{q}_v^m, \quad \boldsymbol{\rho}_v = \left(\boldsymbol{\rho}_v^{(1)}, \boldsymbol{\rho}_v^{(2)}, \boldsymbol{\rho}_v^{(3)} \ldots\right), \quad \boldsymbol{\rho} = (\boldsymbol{\rho}_v)_{v\in V} \tag{A.11}$$

and $\|\mathbf{p} - \boldsymbol{\pi}\|_{TV} = \sum_{v\in V} |p_v - \pi_v|$ is the total variation norm between distributions $\mathbf{p}$ and $\boldsymbol{\pi}$.

Lemma 12. *For $\mathbf{q}$ as in (A.10) and $\boldsymbol{\rho}$ as in (A.11),*

$$\tau_\infty(\mathbf{q}; G) \leq 2752\tau_1(\boldsymbol{\rho}; G) \log\left(4\max\left\{1, \frac{1}{s(G)}\right\}\right)$$

Proof. Masking dependence on the starting vertex v for the moment, let

$$\Delta_u^{(m)} = q_{vu}^{(m)} - \pi_u, \quad \delta_u^{(m)} = \frac{\Delta_u^{(m)}}{\pi_u}$$

and $\boldsymbol{\Delta}^{(m)} = (\Delta_u^{(m)})_{u\in V}$, $\boldsymbol{\delta}^{(m)} = (\delta_u^{(m)})_{u\in V}$. For a vector $\boldsymbol{\Delta} = (\Delta_u)_{u\in V}$, the $L^p(\boldsymbol{\pi})$ norm is given by

$$\|\boldsymbol{\Delta}\|_{L^p(\boldsymbol{\pi})} = \left(\sum_{u\in V} (\Delta_u)^p \pi_u\right)^{1/p}$$

To go between the $L^\infty(\boldsymbol{\pi})$ and $L^1(\boldsymbol{\pi})$ norms, we have

$$\begin{aligned}\left\|\boldsymbol{\delta}^{(2m)}\right\|_{L^\infty(\pi)} &\overset{(i)}{\leq} \left\|\boldsymbol{\delta}^{(m)}\right\|^2_{L^2(\pi)} \\ &= \left\|(\boldsymbol{\delta}^{(m)})^2\right\|_{L^1(\pi)} \\ &\overset{(ii)}{\leq} \left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^1(\pi)} \left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)}\end{aligned}$$

where (i) is a result of Benjamini and Morris and (ii) follows from Holder's inequality. Now, we upper bound the second factor on the right hand side by observing

$$\begin{aligned}\left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)} &\leq \max\left\{1, \max_{u\in V}\frac{q_{vu}^{(m)}}{\pi_u}\right\} \\ &\overset{(iii)}{\leq} \max\left\{1, \frac{1}{s(G)}\right\}\end{aligned}$$

where (iii) follows from Lemma 11.

Now, we leverage the following well-known fact (PhD thesis of Montenegro): for any $\epsilon > 0$, if $m \geq \tau_1(\mathbf{q}_v^{(m)}; G)\cdot\log(1/\epsilon)$ then

$$\left\|\mathbf{q}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV} \leq \epsilon.$$

But $\left\|\mathbf{q}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV}$ is exactly $\|(\boldsymbol{\delta}^{(m)})\|_{L^1(\pi)}$. Therefore, picking

$$m_0 = \tau_1(\mathbf{q}_v^{(m)}; G)\cdot\log\left(4\max\left\{1, \frac{1}{s(G)}\right\}\right)$$

implies $\|(\boldsymbol{\delta}^{(m)})\|_{L^\infty(\pi)} \leq 1/4$ for all $m \geq 2m_0$. Then,

$$\left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)} = \sup_u\left\{\frac{\left|q_{vu}^{(m)} - \boldsymbol{\pi}_u\right|}{\boldsymbol{\pi}_u}\right\}.$$

and since none of the above depended on a specific choice for v, the supremum can be taken over all starting vertices v as well. Thus $\tau_\infty(\mathbf{q}^{(m)}; G) \leq 2m_0$.

Finally, it is known (PhD thesis of Montenegro) that

$$\tau_1(\mathbf{q}^{(m)}; G) \leq 1376\tau_1(\boldsymbol{\rho}^{(m)}; G)$$

and so the desired result holds. $\square$

The second key quantity is the *conductance function*

$$\Phi(t; G) := \min_{\substack{S \subseteq V, \\ \boldsymbol{\pi}(S) \leq t}} \Phi(S; G) \qquad (\pi_{\min} \leq t < 1) \tag{A.12}$$

where $\Phi(S; G)$ is the normalized cut of S in G given by (3).

Lemma 13 leverages the conductance function and local spread to produce an upper bound on the total variation distance between $\boldsymbol{\rho}_v^{(m)}$ and $\boldsymbol{\pi}$.

Lemma 13. *If $D_{\min} > 10$, for any $v \in V$:*

$$\left\| \boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi} \right\|_{TV} \leq \max \left\{ \frac{1}{4}, \frac{1}{10} + \frac{70}{m} \left(\frac{20}{9} + \int_{t=s'(G)}^{1/2} \frac{4}{t\Phi^2(t; G)} dt \right) \right\}$$

where $s'(G) = s(G)/9$.

To prove Lemma 13 we first introduce a generalization of $\Phi(t; G) \cdot \Phi(t; G)$ known as a blocking conductance function. [1]

Definition 1 (Blocking Conductance Function of PhD thesis of Montenegro)**.** *For $t_0 \geq \pi_{\min}$, a function $\phi(t; G) : [t_0, 1/2] \to [0, 1]$ is a* blocking conductance function *if for all $S \subset V$ with $\boldsymbol{\pi}(S) = t \in [t_0, 1/2]$, either of the following hold:*

1. Exterior inequality. *For all $y \in \left[\frac{1}{2}t, t\right] : \phi_{int}(S) \geq \phi(\max\{t_0, y\})$*
2. Interior inequality. *For all $y \in \left[t, \frac{3}{2}t\right] : \phi_{ext}(S) \geq \phi(\max\{y, 1-y\})$.*

where ϕ_{int} and ϕ_{ext} are defined respectively as

$$\phi_{int}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subseteq S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V; G) \left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

$$\phi_{ext}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subseteq S^c \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V; G) \left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

Theorem 1 (PhD thesis of Montenegro Theorem 3.2)**.** *Consider $\phi(t; G) : [t_0, 1/2] \to [0, 1]$ a blocking conductance function. Then, letting*

$$h^m(t_0) = \sup_{S : \boldsymbol{\pi}(S) < t_0} \left(\boldsymbol{\rho}_v^{(m)}(S) - \boldsymbol{\pi}(S) \right)$$

the following statement holds: if ϕ is a blocking conductance function,

$$\left\| \boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi} \right\|_{TV} \leq \max \left\{ \frac{1}{4}, h^1(t_0) + \frac{70}{m} \left(\frac{1}{\phi(t_0; G)} + \int_{t=t_0}^{1/2} \frac{4}{t\phi(t; G)} dt \right) \right\}$$

[1] For more details, see PhD thesis of Montenegro

Note that in PhD thesis of Montenegro this theorem is stated with respect to h^0. However, in the subsequent proof it holds with respect to h^m, and it is observed that h^m is decreasing in m. For our purposes it is more useful to state it with respect to h^1, as we have done.

Proof of Lemma 13. Consider the function $\phi_0(t, G) : [s(G), 1/2] \to [0, 1]$ defined by

$$\phi_0(t; G) = \begin{cases} \frac{1}{5}, & t = s'(G) \\ \frac{1}{4}\Phi^2(t; G), & t \in (s'(G), 1/2] \end{cases} \tag{A.13}$$

Lemma 14. *If $D_{\min} > 10$, ϕ_0 is a blocking conductance function.*

We take Lemma 14 as given, and defer the proof until after the proof of Lemma 13.

Lemma 14 and Theorem 1 together yield:

$$\left\|\boldsymbol{\rho}^t - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, h^1(s'(G)) + \frac{70}{m}\left(5 + \int_{t=s'(G)}^{1/2} \frac{4}{t\Phi^2(t; G)}\right)\right\}$$

Then, $h^1(s'(G)) \leq 1/10$ follows exactly from the proof of Lemma 11, except now $\boldsymbol{\pi}(S) \leq s'(G)$ results in the sharper bound of $\mathbf{q}_u^{(1)}(S^c) \geq 9/10$ for every $u \in S$. $\square$

Lemma 14. The condition $D_{\min} > 10$ ensures that $s(G) \geq \pi_{\min}$.

It is known that $\frac{1}{4}\Phi^2(x; G)$ satisfies the exterior inequality for all $t \in (\pi_{\min}, 1/2]$.

For $t = s'(G)$ we will instead use the interior inequality. For any S such that $\boldsymbol{\pi}(S) \leq s'(G)$, the following statement holds: for every $u \in S$, $\mathrm{cut}(u, S^c; G) \geq 9/10 \cdot \deg(u; G)$. Fixing $\lambda = \pi(S)/2$, we have

$$\begin{aligned}
\phi_{int}(S) &\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V; G)\left[\lambda(1-\lambda)\right]^2} \\
&\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{9\lambda \sum_{u \in S \setminus B} \deg(u; G)}{10\mathrm{vol}(V; G)\left[\lambda(1-\lambda)\right]^2} \\
&\geq \frac{9\lambda^2}{20[\lambda^2(1-\lambda)^2]} \geq \frac{9}{20}.
\end{aligned}$$

$\square$

A.6 Conductance function and local spread: non-convex case.

We begin with some notation. Write $\mathcal{C}_\sigma[\mathbf{X}] = \widetilde{\mathbf{X}}$, and $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$. For $S \subset \widetilde{\mathbf{X}}$, let $\widetilde{\mathrm{cut}}_{n,r}(S) = \mathrm{cut}(S; \widetilde{G}_{n,r})$ and similarly $\widetilde{\mathrm{vol}}_{n,r}(S) = \mathrm{vol}(S; \widetilde{G}_{n,r})$.

Consider $\mathbf{z} \subset \mathcal{C}_\sigma$ such that $\mathcal{N}_{\mathbf{z}} = \{B(z, r/3) : z \in \mathbf{z}\}$ is a covering of $\mathcal{C}_\sigma$, meaning $\mathcal{N}_{\mathbf{z}} \supseteq \mathcal{C}_\sigma$. Then, we write

$$\widetilde{B}_{\min} = \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\min} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right|$$
$$\widetilde{B}_{\max} = \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\max} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right|$$

Both the conductance function and local spread will depend heavily on these quantities. Lemma 15 collects the bounds we will need.

Lemma 15. *For sufficiently large n, each of the following bounds hold with probability $1 - \delta$:*

$$\widetilde{B}_{\max} \leq \left(1 + \sqrt{3^d \frac{3(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma$$
$$\widetilde{B}_{\min} \geq \left(1 - \sqrt{3^d \frac{2(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$
$$\widetilde{D}_{\max} \leq \left(1 + \sqrt{\frac{3(\log n + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d r^d \Lambda_\sigma$$
$$\widetilde{D}_{\min} \geq \left(1 - \sqrt{\frac{2(\log n + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$
$$\left(1 - \sqrt{\frac{2\log(1/\delta)}{n\lambda_d \nu_d \sigma^d}}\right) n\lambda_d \nu_d \sigma^d \leq \widetilde{n} \leq \left(1 + \sqrt{\frac{3\log(1/\delta)}{n\Lambda_d \nu_d D^d}}\right) n\Lambda_d \nu_d D^d$$

where $\widetilde{n} = \left|\widetilde{\mathbf{X}}\right|$ and $\beta_d = (1 - c_d \frac{r^2}{\sigma^2})$.

In particular, fix $\epsilon > 0$. Then, for n as specified in Theorem 3:

$$\widetilde{B}_{\max} \leq (1 + \epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma, \quad \widetilde{D}_{\max} \leq (1 + \epsilon)\, n\nu_d r^d \Lambda_\sigma$$
$$\widetilde{B}_{\min} \geq (1 - \epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d, \quad \widetilde{D}_{\min} \geq (1 - \epsilon)\, n\nu_d r^d \lambda_\sigma \beta_d$$
$$(1 - \epsilon) n\lambda_\sigma \nu_d \sigma^d \leq \widetilde{n} \leq (1 + \epsilon) n\Lambda_\sigma \nu_d D^d$$

each with probability at least $1 - \delta$.

The proof comes directly from Lemma 10, and we omit it.

Now, we consider the conductance function and local spread computed over $\widetilde{G}_{n,r}$, which we refer to by

$$\widetilde{\Phi}_{n,r}(t) = \Phi(t; \widetilde{G}_{n,r}), \quad \widetilde{s}_{n,r} = s(\widetilde{G}_{n,r}).$$

Lemma 16. *For $\mathcal{C}_\sigma$ and n satisfying the conditions of Theorem 3:*

$$\widetilde{s}_{n,r} \geq \frac{9}{10} \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{D^d} \beta_d^2$$

with probability at least $1-\delta$.

Proof. The local spread can be written as

$$\widetilde{s}_{n,r} = \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \geq \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{D}_{\max} \widetilde{n}}.$$

Then apply the relevant results of Lemma 15. □

Lemma 17. *For $\mathcal{C}_\sigma$ and n satisfying the conditions given by Theorem 3, the following statement holds:*

$$\widetilde{\Phi}_{n,r}(1/2) \geq \frac{\lambda_\sigma (1-\epsilon) \beta_d}{4 \Lambda_\sigma (1+\epsilon) 3^d} \left(1 + \frac{(1-\epsilon) r^d \lambda_\sigma}{(1+\epsilon) D^d \Lambda_\sigma} \right)$$

with probability at least $1-\delta$.

Proof. Fix $S \subset \widetilde{\mathbf{X}}$ with $\widetilde{\pi}_{n,r}(S) \leq 1/2$. Partition $\mathcal{N}_{\mathbf{z}} = \mathcal{N}_{\mathbf{z}}^+ \cup \mathcal{N}_{\mathbf{z}}^-$, where

$$\begin{aligned} \mathcal{N}_{\mathbf{z}}^- &= \left\{ B(z, r/3) : 2 \left| B(z, r/3) \cap S \right| \leq \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right| \right\} \\ \mathcal{N}_{\mathbf{z}}^+ &= \mathcal{N}_{\mathbf{z}} \setminus \mathcal{N}_{\mathbf{z}}^- \end{aligned}$$

and correspondingly $S^- = \mathcal{N}_{\mathbf{z}}^- \cap S$, $S^+ = \mathcal{N}_{\mathbf{z}}^+ \cap S$, so

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} = \frac{\widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) + \widetilde{\mathrm{cut}}_{n,r}(S^+; \widetilde{G}_{n,r} \setminus S)}{\widetilde{\mathrm{vol}}_{n,r}(S^-) + \widetilde{\mathrm{vol}}_{n,r}(S^+)}.$$

It is immediately apparent that the following bounds hold for all $S \subset \widetilde{\mathbf{X}}$:

$$\begin{aligned} \widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) &\geq \frac{|S^-| \, \widetilde{B}_{\min}}{2} \\ \widetilde{\mathrm{vol}}_{n,r}(S^-) &\leq \left| S^- \right| \widetilde{D}_{\max} \\ \widetilde{\mathrm{vol}}_{n,r}(S^+) &\leq \widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \mathbf{1}(\left| N_{\mathbf{z}}^+ \right| > 0) \end{aligned}$$

If moreover $|N_{\mathbf{z}}^{+}| < |N_{\mathbf{z}}|$, then

$$\widetilde{\mathrm{cut}}_{n,r}(S^{+}; \widetilde{G}_{n,r} \setminus S) \geq \frac{\widetilde{B}_{\min}^{2}}{4} \mathbf{1}(\left|N_{\mathbf{z}}^{+}\right| > 0)$$

follows from the fact that the graph $H_{n,r} = (\mathbf{z}, E_H)$, with $(z_i, z_j) \in E_H$ if $\|z_i - z_j\| \leq r/3$, is connected. As a result, if $|N_{\mathbf{z}}^{+}| < |N_{\mathbf{z}}|$ we have

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} \geq \frac{\widetilde{B}_{\min}}{4\widetilde{D}_{\max}} + \frac{\widetilde{B}_{\min}^{2}}{8\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \tag{A.14}$$

using the inequality $2(A+B)/(C+D) \geq A/C + B/D$ for A, B, C, D non-negative.

If, on the other hand , $|\mathcal{N}_{\mathbf{z}}^{+}| = \mathcal{N}_{\mathbf{z}}$, then (A.14) holds with respect to S^c. Then, because $\widetilde{\pi}_{n,r}(S) \leq 1/2$,

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^c)}{\widetilde{\mathrm{vol}}_{n,r}(S^c)} \leq \frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)}$$

and so we get (A.14). Noting, as in the proof of Lemma 16, that $\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \leq \widetilde{n} \cdot \widetilde{D}_{\max}$, the relevant results of Lemma 15 yield the desired inequality. □

A.7 Population-level conductance function: convex case.

We will make use of the above theory with respect to the conductance function $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$. First, however, we introduce a population-level analogue to $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$ over the set $\mathcal{C}_\sigma$, which we denote $\widetilde{\Phi}_{\mathbb{P},r}$. (In general, we will adopt the convention of using $\widetilde{f}$ to denote functionals computed with respect to $\mathcal{C}_\sigma$.)

For $\mathcal{S} \subset \mathbb{R}^d$

$$\nu_{\mathbb{P}}(\mathcal{S}) := \int_{\mathcal{S}} f(x) dx$$

is the weighted volume.

The r-ball walk over $\mathcal{C}_\sigma$ is a Markov chain. For $x \in \mathcal{C}_\sigma$ and $\mathcal{S}, \mathcal{S}' \subset \mathcal{C}_\sigma$ the transition probability is given by

$$\widetilde{P}_{\mathbb{P},r}(x; \mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S} \cap B(x,r))}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r))}, \quad \widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{S}') := \int_{x \in \mathcal{S}} f(x) P_{\mathbb{P},r}(x; \mathcal{S}') dx,$$

stationary distribution defined by

$$\ell_{\mathbb{P},r}(x) := \frac{\nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r))}{\nu_{\mathbb{P}}(B(x,r))}, \quad \pi_{\mathbb{P},r}(\mathcal{S}) := \frac{1}{\int_{\mathcal{C}_\sigma} f(x) \ell_{\mathbb{P},r}(x) dx} \int_{\mathcal{S}} f(x) \ell_{\mathbb{P},r}(x) dx$$

and corresponding conductance function

$$\widetilde{\Phi}_{\mathbb{P},r}(t) := \min_{\substack{\mathcal{S} \subset \mathcal{C}_\sigma, \\ \pi_{\mathbb{P},r}(\mathcal{S}) \leq t}} \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})}.$$

For $m > 0$ and $0 < t_0 < t_1 < \ldots < t_m < 1$, denote the *stepwise approximation to g* by $\bar{g}$, defined as

$$\bar{g}(t) = g(t_i), \quad \text{for } t \in [t_{i-1}, t_i] \tag{A.15}$$

The stepwise approximation will be important to showing the consistency results of Section (A.7) hold across the entire conductance function. Lemma 18 shows that the approximation will not overly degrade our estimates of the population-level conductance function.

Lemma 18.
- *For any function f monotone decreasing in t on the interval $[t_0, t_m]$, $\bar{f}(t) \leq f(t)$ for all $t \in [t_0, t_m]$.*
- *Fix*

$$g(t) = \log\left(\frac{1}{t}\right) \text{ for } x \in [t_0, 1/2]$$

If for all i in $1, \ldots, m$, $(t_i - t_{i-1}) \leq t_0/2$, then $\bar{g}(t) \geq g(t)/2$.

Proof. The first statement is immediately obvious, and we turn to proving the second.

The upper bound $g(t) \geq \bar{g}(t)$ follows immediately from the fact that $g(t)$ is a decreasing function along with the first statement.

By the concavity of the log function,

$$\bar{g}(t) = \log\left(\frac{1}{t_i}\right) \geq \log\left(\frac{1}{t}\right) - \frac{(t_i - t)}{t}.$$

As a result,

$$\bar{g}(t) - \frac{g(t)}{2} \geq \frac{\log\left(\frac{1}{t}\right)}{2} - \frac{(t_i - t)}{t} \geq 1/2 - 1/2 = 0.$$

□

Theorem 2 (Restatement of Kannan 2004 Theorem 4.6)**.** *Let $K \subset \mathbb{R}^d$ be a convex body of diameter D. Then for any $\mathcal{S} \subset K$ with $\pi_{\nu,r}(\mathcal{S}) \leq 1/2$,*

$$\frac{Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\nu,r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{1}{\pi_{\nu,r}(\mathcal{S})}\right)\right\}. \tag{A.16}$$

Lemma 19. *Under the conditions on $\mathcal{C}_\sigma$ given by Theorem 3, the following bounds hold:*

- *for $0 < t < 1/2$,*

$$\widetilde{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

- *Let*
$$M = \frac{2^{d+1} D^d \Lambda_\sigma^2}{r^d \lambda_\sigma^2}$$
and $t_i = (i+1)/M$ *for* $i = 0, \ldots, m-1$. *Then, for* $1/M < t < 1/2$
$$\overline{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{162\sqrt{d}D} \operatorname{Log}\left(\frac{\Lambda_\sigma^2}{\lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$
where $\overline{\Phi}_{\mathbb{P},r}(t)$ *is defined as in* (A.15) *with respect to* $t_0, \ldots t_{M-1}$, *and* $\operatorname{Log}(A/t) = \max\{\log(1+2A), \log(A/t)\}$.

Before we prove Lemma 19, note that the choice of M is made to ensure t_0 is greater than the local spread of $G_{n,r}\left[\mathcal{C}_\sigma[\mathbf{X}]\right]$, as we will see in Section A.8.

Proof of Lemma 19. We note that
$$\pi_{\mathbb{P},r}(S) \leq \pi_{\nu,r}(S) \cdot \frac{\Lambda_\sigma^2}{\lambda_\sigma^2}, \quad Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \geq Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \cdot \frac{\lambda_\sigma^2}{\Lambda_\sigma^2}$$

Plugging these estimates in to (A.16) gives
$$\frac{Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D} \log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 \pi_{\mathbb{P},r}(\mathcal{S})}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$
and since the right hand side is decreasing in $\pi_{\mathbb{P},r}(\mathcal{S})$, the desired lower bound holds on $\widetilde{\Phi}_{\mathbb{P},r}(t)$. The bound on $\overline{\Phi}_{\mathbb{P},r}(t)$ then follows from $\operatorname{Log}(A/t) \leq \log(1+1/t)$ for all $0 < t < 1/2$ and application of Lemma 18. ∎

A.8 Consistency of local spread and conductance function: convex case.

The introduction of the stepwise approximation allows us to make use of Lemma 20, which gives us (pointwise) consistency of the discrete graph functionals $\widetilde{\Phi}_{n,r}(t)$ to the continuous functionals $\widetilde{\Phi}_{\mathbb{P},r}(t)$.

Lemma 20. *Fix* $0 < t < 1/2$. *Under the conditions on* $\mathcal{C}_\sigma$ *given by Theorem 3, the following statement holds: with probability one, as* $n \to \infty$,
$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \geq \widetilde{\Phi}_{\mathbb{P},r}(t) \tag{A.17}$$
As a consequence, for M *and* $(t_i)_{i=0}^{M-1}$ *defined as in Lemma 19, we have that*
$$\liminf_{n\to\infty} \overline{\Phi}_{n,r} \geq \overline{\Phi}_{\mathbb{P},r} \tag{A.18}$$

We defer the proof of pointwise consistency to Section A.9. For now, we show that (A.18) is immediately implied by (A.17).

Proof of (A.18). We take as given that for any $0 < t < 1/2$,

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \geq \widetilde{\Phi}_{\mathbb{P},r}(t).$$

In particular, this will occur for $t_0, t_1, \ldots, t_m$ and therefore

$$\liminf_{n\to\infty} \overline{\Phi}_{n,r} \geq \overline{\Phi}_{\mathbb{P},r}$$

uniformly over $[1/m, 1/2]$. □

A.9 Proof of pointwise consistency of conductance function: convex case.

Throughout, let $\widetilde{n} = |\mathcal{C}_\sigma[\mathbf{X}]|$, $\widetilde{\mathbf{X}} = \mathcal{C}_\sigma[\mathbf{X}] = \{\widetilde{x}_1, \ldots, \widetilde{x}_{\widetilde{n}}\}$, and

$$\widetilde{\mathbb{P}}_n := \frac{1}{\widetilde{n}} \sum_{\widetilde{x}_i \in \widetilde{\mathbf{X}}} \delta_{\widetilde{x}_i}$$

be the empirical distribution of $\widetilde{\mathbf{X}}$. For $S \subset \widetilde{\mathbf{X}}$, we will denote $\mathrm{vol}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{vol}}_{n,r}(S)$, and likewise $\mathrm{cut}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{cut}}_{n,r}(S)$.

Consider a sequence of sets $(S_{\widetilde{n}})_{\widetilde{n}\in\mathbb{N}}$, with $u_{\widetilde{n}} = \mathbf{1}_{S_{\widetilde{n}}}$ the characteristic function of $S_{\widetilde{n}}$. Similarly, for $\mathcal{S} \subset \mathcal{C}_\sigma$ let $u = \mathbf{1}_{\mathcal{S}}$.

Give Lemma re: stagnating transportation maps and define TL^1 convergence.

Lemma 21. *If* $(u_{\widetilde{n}}) \overset{TL^1}{\to} u$,

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \widetilde{Q}_{\mathbb{P},r}(S, \mathcal{C}_\sigma \setminus S) \cdot \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x) \right)$$

Proof. We note immediately that $n \to \infty$ implies $\widetilde{n} \to \infty$ with probability one.

Now, we can write

$$\begin{aligned}
\frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} &= \frac{1}{\widetilde{n}^2} \sum_{i=1}^{\widetilde{n}} \sum_{j=1}^{\widetilde{n}} u_{\widetilde{n}}(\widetilde{x}_i)\big(1 - u_{\widetilde{n}}(\widetilde{x}_j)\mathbf{1}(\|\widetilde{x}_i - \widetilde{x}_j\| \leq r)\big) \\
&= \int_{\mathcal{C}_\sigma} \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} u_{\widetilde{n}}(x)\big(1 - u_{\widetilde{n}}(x')\big) d\mathbb{P}_n(x') \right) d\mathbb{P}_n(x) \\
&= \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} 1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x') d\mathbb{P}(x') \right) d\mathbb{P}(x).
\end{aligned}$$

Note that

$$\lim_{n\to\infty} \nu\big(B(T_{\widetilde{n}}(x),r) \setminus B(x,r)\big) = 0$$
$$\lim_{n\to\infty} \nu\big(B(x,r) \setminus B(T_{\widetilde{n}}(x),r)\big) = 0.$$

Therefore

$$\lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} 1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x') d\mathbb{P}(x') = \lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(x,r)} 1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x') d\mathbb{P}(x')$$

and by an application of the bounded convergence theorem

$$\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} 1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x') d\mathbb{P}(x') \right) d\mathbb{P}(x) =$$
$$\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\mathbb{P}(x') \right) d\mathbb{P}(x).$$

Letting

$$\mathcal{I}_n^1 = \int_{\mathcal{C}_\sigma} (u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (u(x') - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\mathbb{P}(x') \right) d\mathbb{P}(x)$$
$$\mathcal{I}_n^2 = \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x) - u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\mathbb{P}(x') \right) d\mathbb{P}(x)$$

we have

$$\int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\mathbb{P}(x') \right) d\mathbb{P}(x) =$$
$$\int_{\mathcal{C}_\sigma} u(x) \left\{ \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) d\mathbb{P}(x') \right\} d\mathbb{P}(x) + \mathcal{I}_n^1 + \mathcal{I}_n^2. \tag{A.19}$$

Recalling the definition of $\widetilde{Q}_{\mathbb{P},r}$, we have

$$\begin{aligned}
\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) &= \int_{\mathcal{S}} f(x) \widetilde{P}_{\mathbb{P},r}(x; \mathcal{C}_\sigma \setminus \mathcal{S}) \\
&= \frac{1}{\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x') dx'} \int_{\mathcal{S}} f(x) \int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} f(x') dx' dx \\
&= \frac{1}{\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x') dx'} \int_{\mathcal{C}_\sigma} f(x) u(x) \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) f(x') dx' dx
\end{aligned} \tag{A.20}$$

Since $(u_{\widetilde{n}}) \overset{TL^1}{\to} u$, another application of the bounded convergence theorem yields $\lim_{n\to\infty} \mathcal{I}_n^1 = \lim_{n\to\infty} \mathcal{I}_n^2 = 0$. Therefore by (A.19) and (A.20) the final result holds. $\square$

A.10 Proof of Theorem 3

Throughout this proof, we will condition on the events of Lemmas 16 and 17, namely

$$\begin{aligned}\widetilde{s}_{n,r} &\geq \frac{9}{10}\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\frac{\lambda_\sigma^2}{\Lambda_\sigma^2}\frac{r^d}{D^d}\beta_d^2 \\ \widetilde{\Phi}_{n,r}(1/2) &\geq \frac{\lambda_\sigma(1-\epsilon)\beta_d}{4\Lambda_\sigma(1+\epsilon)3^d}\left(1+\frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)\end{aligned}$$

noting that for n as chosen in Theorem 3 this will occur with probability at least $1-\delta$.

We refer to the subgraph $G_{n,r}\left[\mathcal{C}_\sigma[\mathbf{X}]\right]$ as $\widetilde{G}_{n,r}$. Fix arbitrary $v = x_i \in \mathcal{C}_\sigma[\mathbf{X}]$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}_{\mathcal{C}_\sigma[\mathbf{X}]}^t, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{q}_n^{(2)}, \ldots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$.

By Lemma 11,

$$\begin{aligned}\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\ &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, 4d\log\left(\frac{D\Lambda_\sigma^2}{\beta_d\lambda_\sigma^2}\right)\right\}\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 13, we have that

$$\begin{aligned}\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right) \\ &\leq \frac{1400}{3}\left(5 + \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right) \qquad \text{(A.21)}\end{aligned}$$

where $s_{\mathbb{P},r} = \frac{\lambda_\sigma^2}{\Lambda_\sigma^2}\frac{r^d}{(2D)^d}$.(Since r remains constant, for sufficiently large n with probability at least $1-\delta$, $\widetilde{D}_{\min} > 10$.)

Now, we can upper bound the average conductance integral:

$$\begin{aligned}\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt &\leq \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(1/2)}dt \\ &\leq 64\frac{9^d\Lambda_\sigma^2(1+\epsilon)^2\beta_d^2}{\lambda_\sigma^2(1-\epsilon)^2}\left(1+\frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)^{-2}\log\left(\frac{D\Lambda_\sigma^2}{\beta_d\lambda_\sigma^2}\right).\end{aligned}$$

Plugging this in to (A.21) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (15).

A.11 Proof of Theorem 3

Throughout this proof, we will refer to the subgraph $G_{n,r}\left[\mathcal{C}_\sigma[\mathbf{X}]\right]$ as $\widetilde{G}_{n,r}$.

Fix arbitrary $v = x_i \in \mathcal{C}_\sigma[\mathbf{X}]$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}^t_{\mathcal{C}_\sigma[\mathbf{X}]}, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{q}_n^{(2)}, \dots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$.

By Lemmas 11 and 16,

$$\begin{aligned}
\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq 2752 \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\
&\leq 2752 \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, 4d \log\left(\frac{2D\Lambda_\sigma^2}{\lambda_\sigma^2}\right)\right\}
\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 13, we have that

$$\limsup_{n \to \infty} \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \leq \limsup_{n \to \infty} \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t \widetilde{\Phi}^2_{n,r}(t)} dt\right) \tag{A.22}$$

(Since r remains constant, for sufficiently large n, $\mathbf{D}_{xx} > C$ will be fulfilled for any $x \in \mathcal{C}_\sigma[\mathbf{X}]$, and any $C < \infty$.) We set aside the constant term for the moment and turn to the integral. By Lemma 16,

$$\limsup_{n \to \infty} \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t \widetilde{\Phi}^2_{n,r}(t)} dt \leq \limsup_{n \to \infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t \widetilde{\Phi}^2_{n,r}(t)} dt$$

where $s_{\mathbb{P},r} = \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{(2D)^d}$. We now replace the discrete conductance function $\widetilde{\Phi}_{n,r}$ by the stepwise approximation to the continuous conductance function, $\overline{\Phi}_{n,r}$:

$$\begin{aligned}
\limsup_{n \to \infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t \widetilde{\Phi}^2_{n,r}(t)} dt &\overset{(i)}{\leq} \limsup_{n \to \infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t \overline{\Phi}^2_{n,r}(t)} dt \\
&= \int_{s_{\mathbb{P},r}}^{1/2} \limsup_{n \to \infty} \frac{4}{t \overline{\Phi}^2_{n,r}(t)} dt \\
&\overset{(ii)}{\leq} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t \overline{\Phi}^2_{\mathbb{P},r}(t)} dt
\end{aligned}$$

where (i) follows from Lemma 18 and (ii) from Lemma 20 (along with the continuous mapping theorem). Now, we make use of Lemma 19:

$$\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{\mathbb{P},r}^2(t)} dt \leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt + \int_{s_{\mathbb{P},r}}^{1/2} \frac{81 dD^2}{r^2 t \mathrm{Log}(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt \right)$$

$$\leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \underbrace{\int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt}_{:=\mathcal{J}_1} + 81 \underbrace{\int_{s_{\mathbb{P},r}}^{\lambda_\sigma^2/(4\Lambda_\sigma^2)} \frac{dD^2}{r^2 t \log(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt}_{:=\mathcal{J}_2} + 81 \underbrace{\int_{\lambda_\sigma^2/(4\Lambda_\sigma^2)}^{1/2} \frac{dD^2}{r^2 t \log(1 + \frac{4\lambda_\sigma^2}{\Lambda_\sigma^2})} dt}_{:=\mathcal{J}_3} \right)$$

Computing a few simple integrals yields the following upper bounds on $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$:

$$\begin{aligned}
\mathcal{J}_1 &\leq d^2 \log\left(\frac{2D\Lambda_\sigma^2}{r\lambda_\sigma^2}\right) \\
\mathcal{J}_2 &\leq \frac{dD^2}{r^2}\left[\log(2d) + \log\left(\log\left(\frac{2D}{r}\right)\right)\right] \\
\mathcal{J}_3 &\overset{(iii)}{\leq} 2\frac{dD^2}{r^2}\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\log\left(4\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\right)
\end{aligned}$$

where (iii) uses the upper bound $\frac{1}{\log(1+x)} \leq \frac{1}{x}$.

Plugging these bounds in to (A.22) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (15).

B OTHER STUFF

Lemma 22 (Bernstein's inequality for U-statistics)**.** *Additionally, assume* $\sigma^2 = \mathrm{Var}\,(k(X_1, \ldots, X_m)) < \infty$*. Then for any* $\delta > 0$*,*

$$\mathbb{P}(U - \mathbb{E}U \geq t) \leq \exp\left\{-\frac{n}{2m}\frac{t^2}{\sigma^2 + t/3}\right\},$$

Moreover if $\sigma^2 \leq \mu/n$*,*

$$\begin{aligned}
U &\leq \mathbb{E}U \cdot \left(1 + \max\left\{\sqrt{\frac{2m\log(1/\Delta)}{\mu}}, \frac{2m\log(1/\Delta)}{3\mu}\right\}\right), \\
U &\geq \mathbb{E}U \cdot \left(1 - \max\left\{\sqrt{\frac{2m\log(1/\Delta)}{\mu}}, \frac{2m\log(1/\Delta)}{3\mu}\right\}\right)
\end{aligned}$$

each with probability at least $1 - \Delta$*.*

Multiplicative bound: As $\tilde{k}(x_1, x_2)$ is the sum of two Bernoulli random variables with negative covariance (since $\mathbf{1}(x_i \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i, r) \cap \mathcal{A}_\sigma) = 1$ implies $\mathbf{1}(x_j \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_i \in B(x_j, r) \cap \mathcal{A}_\sigma) = 0$ and vice versa), we can upper bound $\mathrm{Var}\left(\tilde{k}(x_1, x_2)\right) \leq \widetilde{p}$, where we recall

$$\widetilde{p} = 2 \cdot \mathbb{P}\left(\mathbf{1}(x_1 \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_2 \in B(x_1, r) \cap \mathcal{A}_\sigma)\right)$$

From Lemma 22, we therefore have

$$\frac{\widetilde{\mathcal{E}}}{\binom{n}{2}} \leq \widetilde{p} + \max\left\{\sqrt{\frac{4\log(1/\Delta)\widetilde{p}}{n}}, \frac{4\log(1/\Delta)}{3n}\right\}$$

with probability at least $1 - \Delta$.

Multiplicative bound: The two terms on the right hand side are both distributed Bernoulli$(p/2)$. Moreover, since $\mathbf{1}(x_i \in A_\sigma) = 1$ implies $\mathbf{1}(x_j \in A_\sigma) = 0$, they have negative covariance. We can therefore upper bound $\mathrm{Var}(k'(x_i, x_j)) \leq p$, and so from Lemma 22, we have

$$\frac{\mathcal{V}}{\binom{n}{2}} \geq p - \max\left\{\sqrt{\frac{4\log(1/\Delta)p}{n}}, \frac{4\log(1/\Delta)}{3n}\right\}$$

with probability at least $1 - \Delta$.