A Proofs

In this supplement, we present proofs for "Local Clustering of Density Upper Level Sets". Sections A.1 - A.3 detail the proof for Theorem 1. A.4 develops a bound of the form of (11), which we recall links the conductance function to mixing time; this will be necessary for both Theorems 2 and 3. A.5 and A.6 give the proof of Theorem 2, while A.7- A.10 give the proof of Theorem 3. A.11 gives some general concentration results used throughout, before we finish with the proof of Theorem 4 in A.12.

A.1 Volume estimates

We will fix $\mathcal{A} \subset \mathbb{R}^d$ to be an arbitrary set. To simplify expressions, for the σ-expansion $\mathcal{A}_\sigma$, we will write the set difference between $\mathcal{A}_\sigma$ and the $(\sigma + r)$-expansion $\mathcal{A}_{\sigma+r}$ as

$$\mathcal{A}_{\sigma,\sigma+r} := \{x : 0 < \mathrm{dist}(x, \mathcal{A}_\sigma) \leq r\},$$

where as a reminder $\mathrm{dist}(x, \mathcal{A}) = \min_{x' \in \mathcal{A}} \|x - x'\|$.

For notational ease, we write

$$\mathrm{cut}_{n,r} = \mathrm{cut}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}),\ \mu_K = \mathbb{E}(\mathrm{cut}_{n,r}),\ p_K = \frac{\mu_K}{\binom{n}{2}}$$
$$\mathrm{vol}_{n,r} = \mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}),\ \mu_V = \mathbb{E}(\mathrm{vol}_{n,r}),\ p_V = \frac{\mu_V}{\binom{n}{2}}$$

for the random variable, mean, and probability of cut size and volume, respectively.

We state Lemma 1 without proof, as it is trivial. We formally include it mainly to comment on its (potential) suboptimality; for sets $\mathcal{A}$ with diameter much larger than σ, the volume estimate of Lemma 1 will be quite poor.

Lemma 1. *For any $\delta > 0$ and $x \in \mathcal{A}_\sigma$,*

$$\sigma B \subset \mathcal{A}_\sigma, \quad \text{and } \nu(\mathcal{A}_\sigma + \delta B) \leq \nu\left(\left[1 + \frac{\delta}{\sigma}\right]\mathcal{A}_\sigma\right) = \left(1 + \frac{\delta}{\sigma}\right)^d \nu(\mathcal{A}_\sigma)$$

where $\mathcal{A}_\sigma = \mathcal{A} + \sigma B$ is the σ-expansion of $\mathcal{A}$.

We will need to carefully control the volume of the expansion set using the above estimate; Lemma 2 serves this purpose.

Lemma 2. *For any $0 \leq x \leq 1/2d$,*

$$(1 + x)^d \leq 1 + 2dx.$$

Proof. We take the binomial expansion of $(1+x)^d$:

$$\begin{aligned}(1+x)^d &= \sum_{k=0}^{d}\binom{d}{k}x^k \\ &= 1+dx+dx\left(\sum_{k=2}^{d}\frac{\binom{d}{k}x^{k-1}}{d}\right) \\ &\leq 1+dx+dx\left(\sum_{k=2}^{d}\frac{d^k}{(2d)^{k-1}d}\right) \\ &\leq 1+2dx.\end{aligned}$$

□

We will repeatedly employ Lemma 1 and Lemma 2 in tandem. As a first example, in Lemma 3, we use it to bound the ratio of $\nu(\mathcal{A}_\sigma)$ to $\nu(\mathcal{A}_{\sigma-r})$. This will be useful when we bound $\mathrm{vol}(\mathcal{C}_\sigma)$.

Lemma 3. *For σ, $\mathcal{A}_\sigma$ as in Lemma 1, let $r>0$ satisfy $r \leq \sigma/4d$. Then,*

$$\frac{\nu(\mathcal{A}_\sigma)}{\nu(\mathcal{A}_{\sigma-r})} \leq 2.$$

Proof. Fix $q=\sigma-r$. Then,

$$\begin{aligned}\nu(\mathcal{A}_\sigma) &= \nu(\mathcal{A}_{q+\sigma-q}) = \nu(\mathcal{A}_q + (\sigma-q)B) \\ &\leq \nu(\mathcal{A}_q + \frac{(\sigma-q)}{q}\mathcal{A}_q) = \left(1+\frac{\sigma-q}{q}\right)^d \nu(\mathcal{A}_q)\end{aligned}$$

where the inequality follows from Lemma 1. Of course, $\sigma-q=r$, and $\frac{r}{q} \leq \frac{1}{2d}$ for $r \leq \frac{1}{4d}$. The claim then follows from Lemma 2. □

A.2 Density-weighted cut and volume estimates

Lemma 4. *Under the conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) \leq 2\nu(\mathcal{C}_\sigma)\frac{rd}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. Recalling that f is the density function for $\mathbb{P}$, we have

$$\mathbb{P}(\mathcal{C}_{\sigma,\sigma+r}) = \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx \tag{A.1}$$

We partition $\mathcal{C}_{\sigma,\sigma+r}$ into slices, based on distance from $\mathcal{C}_\sigma$, as follows: for $k \in \mathbb{N}$,

$$\mathcal{T}_{i,k} = \left\{ x \in \mathbb{R}^d : t_{i,k} < \frac{\mathrm{dist}(x, \mathcal{C}_\sigma)}{r} \le t_{i+1,k} \right\}, \quad \mathcal{C}_{\sigma,\sigma+r} = \bigcup_{i=0}^{k-1} \mathcal{T}_{i,k}$$

where $t_i = i/k$ for $i = 0, \ldots, k-1$. As a result,

$$\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)dx = \sum_{i=0}^{k-1} \int_{\mathcal{T}_{i,k}} f(x)dx \le \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x).$$

We substitute

$$\nu(\mathcal{T}_{i,k}) = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B) - \nu(\mathcal{C}_\sigma + rt_{i,k}B) := \nu_{i+1,k} - \nu_{i,k}.$$

where for simplicity we've written $\nu_{i,k} = \nu(\mathcal{C}_\sigma + rt_{i+1,k}B)$. This, in concert with the upper bound

$$\max_{x \in \mathcal{T}_{i,k}} f(x) \le \lambda_\sigma - (rt_{i,k})^\gamma,$$

which follows from (A1) and (A2), yields

$$\begin{aligned} \sum_{i=0}^{k-1} \nu(\mathcal{T}_{i,k}) \max_{x \in \mathcal{T}_{i,k}} f(x) &\le \sum_{i=0}^{k-1} \Big\{ \nu_{i+1,k} - \nu_{i,k} \Big\} \Big(\lambda_\sigma - (rt_{i,k})^\gamma \Big) \\ &= \underbrace{\sum_{i=1}^{k} \nu_{i,k} \Big([\lambda_\sigma - (rt_{i,k})^\gamma] - [\lambda_\sigma - (rt_{i-1,k})^\gamma] \Big)}_{:=\Sigma_k} + \underbrace{\Big(\nu_{k,k} [\lambda_\sigma - r^\gamma] - \nu_{1,k}\lambda_\sigma \Big)}_{:=\xi_k} \end{aligned} \tag{A.2}$$

We first consider the term Σ_k. Here we use Lemma 1 to upper bound

$$\nu_{i,k} \le \mathrm{vol}(\mathcal{C}_\sigma) \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d$$

and so we can in turn upper bound Σ_k:

$$\Sigma_k \le \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \sum_{i=1}^{k} \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d \Big((t_{i-1,k})^\gamma - (t_{i,k})^\gamma \Big). \tag{A.3}$$

This, of course, is a Riemann sum, and as the inequality holds for all values of k it holds in the limit as well, which we compute to be

$$\begin{aligned} \lim_{k\to\infty} \sum_{i=1}^{k} \left(1 + \frac{rt_{i,k}}{\sigma} \right)^d \Big((t_{i-1,k})^\gamma - (t_{i,k})^\gamma \Big) &= \gamma \int_0^1 \left(1 + \frac{rt}{\sigma} \right)^d t^{\gamma-1} dt \\ &\overset{(i)}{\le} \gamma \int_0^1 \left(1 + \frac{2drt}{\sigma} \right) t^{\gamma-1} dt = \left(1 + \frac{\gamma 2dr}{\gamma+1} \right). \end{aligned}$$

where (i) follows from Lemma 2. We plug this estimate in to (A.3) and obtain

$$\lim_{k\to\infty} \Sigma_k \leq \mathrm{vol}(\mathcal{C}_\sigma) r^\gamma \left(1 + \frac{\gamma 2dr}{\gamma+1}\right).$$

We now provide an upper bound on ξ_k. It will follow the same basic steps as the bound on Σ_k, but will not involve integration:

$$\begin{aligned}
\xi_k &\overset{(ii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1+\frac{r}{\sigma}\right)^d (\lambda - r^\gamma) - \lambda\right\} \\
&\overset{(iii)}{\leq} \nu(\mathcal{C}_\sigma)\left\{\left(1+\frac{2dr}{\sigma}\right)(\lambda - r^\gamma) - \lambda\right\} = \nu(\mathcal{C}_\sigma)\left\{\frac{2dr}{\sigma}(\lambda - r^\gamma) - r^\gamma\right\}.
\end{aligned}$$

where (ii) follows from Lemma 1 and (iii) from Lemma 2. The final result comes from adding together the upper bounds on Σ_k and ξ_k and taking the limit as $k \to \infty$. □

Lemma 5. *Under the setup and conditions of Theorem 1, and for any $r < \sigma/2d$,*

$$p_K \leq \frac{4\lambda \nu_d r^{d+1} \nu(\mathcal{C}_\sigma) d}{\sigma}\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)$$

Proof. We can write $\mathrm{cut}_{n,r}$ as the sum of indicator functions,

$$\mathrm{cut}_{n,r} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{1}(x_i \in \mathcal{C}_{\sigma,\sigma+r}) \mathbf{1}(x_j \in B(x_i, r) \cap \mathcal{C}_\sigma) \tag{A.4}$$

and by linearity of expectation, we can obtain

$$p_K = \frac{\mu_K}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \in \mathcal{C}_{\sigma,\sigma+r}, x_j \in B(x_i, r) \cap \mathcal{C}_\sigma)$$

Writing this with respect to the density function f, we have

$$\begin{aligned}
p_K &= 2\int_{\mathcal{C}_{\sigma,\sigma+r}} f(x)\left\{\int_{B(x,r)\cap\mathcal{C}_\sigma} f(x')dx'\right\} dx \\
&\leq 2\nu_d r^d \lambda \int_{\mathcal{C}_{\sigma,\sigma+r}} f(x) dx
\end{aligned}$$

where the inequality follows from Assumption (A3), which implies that the density function $f(x') \leq \lambda$ for all $x' \in \mathcal{C}_\sigma \setminus \mathcal{C}$ (otherwise, x' would be in some $\mathcal{C}' \in \mathbb{C}_f(\lambda)$, which (A3) forbids). Then, upper bounding the integral using Lemma 5 gives the final result. □

Lemma 6. *Under the setup and conditions of Theorem 1,*

$$p_V \geq \lambda_\sigma^2 \nu_d r^d \nu(\mathcal{C}_\sigma)$$

Proof. The proof will proceed similarly to Lemma 5. We begin by writing $\mathrm{vol}_{n,r}$ as the sum of indicator functions,

$$\mathrm{vol}_{n,r} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{1}(x_i \in \mathcal{C}_\sigma)\mathbf{1}(x_j \in B(x_i, r)) \tag{A.5}$$

and by linearity of expectation we obtain

$$p_V = \frac{\mu_V}{\binom{n}{2}} = 2 \cdot \mathbb{P}(x_i \in \mathcal{C}_\sigma, x_j \in B(x_i, r)).$$

Writing this with respect to the density function f, we have

$$\begin{aligned} p_V &= 2\int_{\mathcal{C}_\sigma} f(x)\left\{\int_{B(x,r)} f(x')dx'\right\}dx \\ &\geq 2\int_{\mathcal{C}_{\sigma-r}} f(x)\left\{\int_{B(x,r)} f(x')dx'\right\}dx \\ &\overset{(i)}{\geq} 2\lambda_\sigma^2 \nu_d r^d \int_{\mathcal{C}_{\sigma-r}} f(x)dx \end{aligned}$$

where (i) follows from the fact that $B(x, r) \subset \mathcal{C}_\sigma$ for all $x \in C_{\sigma-r}$, along with the lower bound in Assumption (A1). The claim then follows from Lemma 3. ☐

We now convert from bounds on p_K and p_V to probabilistic bounds on $\mathrm{cut}_{n,r}$ and $\mathrm{vol}_{n,r}$ in Lemmas 7 and 8. The key ingredient will be Lemma 23, Hoeffding's inequality for U-statistics; the proofs for both are nearly identical and we give only a proof for Lemma 7.

Lemma 7. *The following statement holds for any $\delta \in (0, 1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_K + \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.6}$$

with probability at least $1 - \delta$.

Lemma 8. *The following statement holds for any $\delta \in (0, 1]$: Under the setup and conditions of Theorem 1,*

$$\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} \geq p_V - \sqrt{\frac{\log(1/\delta)}{n}} \tag{A.7}$$

with probability at least $1 - \delta$.

Proof of Lemma 7. From (A.4), we see that $\mathrm{cut}_{n,r}$, properly scaled, can be expressed as an order-2 U-statistic,

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_K(x_i, x_j)$$

where

$$\phi_K(x_i,x_j) = \mathbf{1}(x_i \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_j \in B(x_i,r)\cap\mathcal{A}_\sigma)+\mathbf{1}(x_j \in \mathcal{A}_{\sigma,\sigma+r})\mathbf{1}(x_i \in B(x_j,r)\cap\mathcal{A}_\sigma).$$

From Lemma 23 we therefore have

$$\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} \leq p_k + \sqrt{\frac{\log(1/\delta)}{n}}$$

with probability at least $1-\delta$. □

A.3 Proof of Theorem 1

The proof of Theorem 1 is more or less given by Lemmas 5, 6, 7, and 8. All that remains is some algebra, which we take care of below.

Fix $\delta \in (0,1]$ and let $\delta' = \delta/2$. For sufficiently large n, note that by (6), $\mathrm{vol}_{n,r} \leq 1 - \mathrm{vol}_{n,r}$ with probability at least $1-\delta$, and therefore $\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{\mathrm{cut}_{n,r}}{\mathrm{vol}_{n,r}}$. Some trivial algebra gives us the expression

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) = \frac{p_K + \left(\frac{\mathrm{cut}_{n,r}}{\binom{n}{2}} - p_K\right)}{p_V + \left(\frac{\mathrm{vol}_{n,r}}{\binom{n}{2}} - p_V\right)} \tag{A.8}$$

We assume (A.6) and (A.7) hold with respect to δ', keeping in mind that this will happen with probability at least $1-\delta$. Along with (A.8) this means

$$\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}]) \leq \frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n}$$

for $\mathrm{Err}_n = \sqrt{\frac{\log(1/\delta')}{n}}$. Now, some straightforward algebraic manipulations yield

$$\begin{aligned}
\frac{p_K + \mathrm{Err}_n}{p_V - \mathrm{Err}_n} &= \frac{p_K}{p_V} + \left(\frac{p_K}{p_V - \mathrm{Err}_n} - \frac{p_K}{p_V}\right) + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \\
&= \frac{p_k}{p_V} + \frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}\left(\frac{p_K}{p_V} + 1\right) \\
&\leq \frac{p_K}{p_V} + 2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n}.
\end{aligned}$$

By Lemmas 5 and Lemma 6, we have

$$\frac{p_K}{p_V} \leq \frac{4rd}{\sigma}\frac{\lambda}{\lambda_\sigma}\frac{\left(\lambda_\sigma - \frac{r^\gamma}{\gamma+1}\right)}{\lambda_\sigma}$$

Then, the choice of

$$n \geq \frac{9\log(2/\delta)}{\epsilon^2}\left(\frac{1}{\lambda_\sigma^2 \nu(\mathcal{C}_\sigma)\nu_d r^d}\right)^2$$

implies $2\frac{\mathrm{Err}_n}{p_V - \mathrm{Err}_n} \leq \epsilon$.

A.4 Mixing time on graphs

For $N \in \mathbb{N}$ and a set V of N vertices, take $G = (V, E)$ to be an undirected and unweighted graph, with associated adjacency matrix $\mathbf{A}$, random walk matrix $\mathbf{W}$, and stationary distribution $\boldsymbol{\pi} = (\pi_u)_{u \in V}$ where $\pi_v = \frac{\mathbf{D}_{vv}}{\mathrm{vol}(V;G)}$. For $v \in V$,

$$q_{vu}^{(m)} = e_v \mathbf{W}^m e_u, \quad \mathbf{q}_v^{(m)} = \left(q_{vu}^{(m)}\right)_{u \in V}, \quad \mathbf{q}_v = (\mathbf{q}_{v\cdot}^{(1)}, \mathbf{q}_{v\cdot}^{(2)}, \ldots), \tag{A.9}$$

denote respectively the m-step transition probability, distribution, and sequence distributions of the random walk over G originating at v. Letting $\mathbf{q} = (\mathbf{q}_v)_{v \in V}$, the relative pointwise mixing time is thus

$$\tau_\infty(\mathbf{q}; G) := \tau_\infty(\mathbf{q}; G) = \min\left\{m : \forall u, v \in V, \frac{\left|q_{vu}^{(m)} - \boldsymbol{\pi}_u\right|}{\boldsymbol{\pi}_u} \leq 1/4\right\}$$

where we include the extra dependency on $\mathbf{q}$ as we will need to consider mixing of various random walks.

Two key quantities relate the mixing time to the expansion of subsets S of V. The *local spread* is defined to be

$$s(G) := \frac{9D_{\min}}{10}\pi_{\min}$$

for $D_{\min} := \min_{v \in V} \mathbf{D}_{vv}$ and $\pi_{\min} := D_{\min}/\mathrm{vol}(V;G)$.

where $\beta(S) := \inf_{v \in S} \mathbf{q}_v^{(1)}(S^c)$, and by convention we let $\mathbf{p}(S) = \sum_{u \in S} p_u$ for any distribution vector $\mathbf{p} = (p_u)_{u \in V}$ over V. We collect some necessary facts about the local spread in Lemma 9.

Lemma 9.
- *If $\boldsymbol{\pi}(S) \leq s(G)$, then for every $u \in S$, $\mathbf{q}_u^{(1)}(S^c) \geq 1/10$.*
- *For any $v, u \in V$, and $m \in N$ greater than 0, $q_{vu}^{(m)}/\pi_{\min} \leq 1/s(G)$.*

Proof. If $t = \boldsymbol{\pi}(S) \leq \frac{9D_{\min}}{10}\pi_{\min}$, divide both sides by $\pi_{\min}$ to obtain

$$|S| \leq \frac{9D_{\min}}{10}$$

which implies $\mathbf{q}_v^{(1)}(S^c) \geq 1/10$ for all $v \in S$. This implies the first statement.

The second statement follows from the fact $q_{vu}^{(m)} \leq 1/D_{\min}$ for any m. □

The local spread facilitates conversion between $\tau_\infty(\mathbf{q}_v; G)$ and the more easily manageable *total variation* mixing time, given by

$$\tau_1(\boldsymbol{\rho}; G) = \min\left\{ m : \forall v \in V, \|\boldsymbol{\rho}_v - \boldsymbol{\pi}\|_{TV} \leq 1/4 \right\}$$

where

$$\boldsymbol{\rho}_v^{(m)} = \frac{1}{m} \sum_{k=1}^{m+1} \mathbf{q}_v^m, \quad \boldsymbol{\rho}_v = \left(\boldsymbol{\rho}_v^{(1)}, \boldsymbol{\rho}_v^{(2)}, \boldsymbol{\rho}_v^{(3)} \ldots \right), \quad \boldsymbol{\rho} = (\boldsymbol{\rho}_v)_{v \in V} \tag{A.10}$$

and $\|\mathbf{p} - \boldsymbol{\pi}\|_{TV} = \sum_{v \in V} |p_v - \pi_v|$ is the total variation norm between distributions $\mathbf{p}$ and $\boldsymbol{\pi}$.

Lemma 10. *For* $\mathbf{q}$ *as in* (A.9) *and* $\boldsymbol{\rho}$ *as in* (A.10),

$$\tau_\infty(\mathbf{q}; G) \leq 2752 \tau_1(\boldsymbol{\rho}; G) \log\left(4 \max\left\{ 1, \frac{1}{s(G)} \right\} \right)$$

Proof. Masking dependence on the starting vertex v for the moment, let

$$\Delta_u^{(m)} = q_{vu}^{(m)} - \pi_u, \quad \delta_u^{(m)} = \frac{\Delta_u^{(m)}}{\pi_u}$$

and $\boldsymbol{\Delta}^{(m)} = (\Delta_u^{(m)})_{u \in V}$, $\boldsymbol{\delta}^{(m)} = (\delta_u^{(m)})_{u \in V}$. For a vector $\boldsymbol{\Delta} = (\Delta_u)_{u \in V}$, the $L^p(\boldsymbol{\pi})$ norm is given by

$$\|\boldsymbol{\Delta}\|_{L^p(\boldsymbol{\pi})} = \left(\sum_{u \in V} (\Delta_u)^p \, \pi_u \right)^{1/p}$$

To go between the $L^\infty(\boldsymbol{\pi})$ and $L^1(\boldsymbol{\pi})$ norms, we have

$$\begin{aligned}
\left\| \boldsymbol{\delta}^{(2m)} \right\|_{L^\infty(\pi)} &\overset{(i)}{\leq} \left\| \boldsymbol{\delta}^{(m)} \right\|_{L^2(\pi)}^2 \\
&= \left\| (\boldsymbol{\delta}^{(m)})^2 \right\|_{L^1(\pi)} \\
&\overset{(ii)}{\leq} \left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^1(\pi)} \left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^\infty(\pi)}
\end{aligned}$$

where (i) is a result of [2] and (ii) follows from Holder's inequality. Now, we upper bound the second factor on the right hand side by observing

$$\begin{aligned}
\left\| (\boldsymbol{\delta}^{(m)}) \right\|_{L^\infty(\pi)} &\leq \max\left\{ 1, \max_{u \in V} \frac{q_{vu}^{(m)}}{\pi_u} \right\} \\
&\overset{(iii)}{\leq} \max\left\{ 1, \frac{1}{s(G)} \right\}
\end{aligned}$$

where (iii) follows from Lemma 9.

Now, we leverage the following well-known fact [4]: for any $\epsilon > 0$, if $m \geq \tau_1(\mathbf{q}_v^{(m)}; G) \cdot \log(1/\epsilon)$ then

$$\left\|\mathbf{q}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV} \leq \epsilon.$$

But $\left\|\mathbf{q}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV}$ is exactly $\|(\boldsymbol{\delta}^{(m)})\|_{L^1(\pi)}$. Therefore, picking

$$m_0 = \tau_1(\mathbf{q}_v^{(m)}; G) \cdot \log\left(4 \max\left\{1, \frac{1}{s(G)}\right\}\right)$$

implies $\|(\boldsymbol{\delta}^{(m)})\|_{L^\infty(\pi)} \leq 1/4$ for all $m \geq 2m_0$. Then,

$$\left\|(\boldsymbol{\delta}^{(m)})\right\|_{L^\infty(\pi)} = \sup_u \left\{ \frac{\left|q_{vu}^{(m)} - \boldsymbol{\pi}_u\right|}{\boldsymbol{\pi}_u} \right\}.$$

and since none of the above depended on a specific choice for v, the supremum can be taken over all starting vertices v as well. Thus $\tau_\infty(\mathbf{q}^{(m)}; G) \leq 2m_0$.

Finally, it is known [4] that

$$\tau_1(\mathbf{q}^{(m)}; G) \leq 1376\tau_1(\boldsymbol{\rho}^{(m)}; G)$$

and so the desired result holds. $\square$

The second key quantity is the *conductance function*

$$\Phi(t; G) := \min_{\substack{S \subseteq V, \\ \boldsymbol{\pi}(S) \leq t}} \Phi(S; G) \qquad (\pi_{\min} \leq t < 1) \tag{A.11}$$

where $\Phi(S; G)$ is the normalized cut of S in G given by (1).

Lemma 11 leverages the conductance function and local spread to produce an upper bound on the total variation distance between $\boldsymbol{\rho}_v^{(m)}$ and $\boldsymbol{\pi}$.

Lemma 11. *If $D_{\min} > 10$, for any $v \in V$:*

$$\left\|\boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, \frac{1}{10} + \frac{70}{m}\left(\frac{20}{9} + \int_{t=s'(G)}^{1/2} \frac{4}{t\Phi^2(t; G)} dt\right)\right\}$$

where $s'(G) = s(G)/9$.

To prove Lemma 11 we first introduce a generalization of $\Phi(t; G) \cdot \Phi(t; G)$ known as a blocking conductance function. [1]

[1] For more details, see [4]

Definition 1 (Blocking Conductance Function of [4]). *For $t_0 \geq \pi_{\min}$, a function $\phi(t;G) : [t_0, 1/2] \to [0,1]$ is a* blocking conductance function *if for all $S \subset V$ with $\boldsymbol{\pi}(S) = t \in [t_0, 1/2]$, either of the following hold:*

1. Exterior inequality. *For all $y \in \left[\frac{1}{2}t, t\right] : \phi_{int}(S) \geq \phi(\max\{t_0, y\})$*
2. Interior inequality. *For all $y \in \left[t, \frac{3}{2}t\right] : \phi_{ext}(S) \geq \phi(\max\{y, 1-y\})$.*

where ϕ_{int} and ϕ_{ext} are defined respectively as

$$\phi_{int}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V;G)\left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

$$\phi_{ext}(S) = \sup_{\lambda \leq \boldsymbol{\pi}(S)} \min_{\substack{B \subset S^c \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V;G)\left[\boldsymbol{\pi}(S)\boldsymbol{\pi}(S^c)\right]^2}$$

Theorem 1 (Theorem 3.2 of [4]). *Consider $\phi(t;G) : [t_0, 1/2] \to [0,1]$ a blocking conductance function. Then, letting*

$$h^m(t_0) = \sup_{S : \boldsymbol{\pi}(S) < t_0} \left(\boldsymbol{\rho}_v^{(m)}(S) - \boldsymbol{\pi}(S)\right)$$

the following statement holds: if ϕ is a blocking conductance function,

$$\left\|\boldsymbol{\rho}_v^{(m)} - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, h^1(t_0) + \frac{70}{m}\left(\frac{1}{\phi(t_0;G)} + \int_{t=t_0}^{1/2} \frac{4}{t\phi(t;G)} dt\right)\right\}$$

Note that in [4] this theorem is stated with respect to h^0. However, in the subsequent proof it holds with respect to h^m, and it is observed that h^m is decreasing in m. For our purposes it is more useful to state it with respect to h^1, as we have done.

Proof of Lemma 11. Consider the function $\phi_0(t, G) : [s(G), 1/2] \to [0,1]$ defined by

$$\phi_0(t;G) = \begin{cases} \frac{1}{5}, & t = s'(G) \\ \frac{1}{4}\Phi^2(t;G), & t \in (s'(G), 1/2] \end{cases} \tag{A.12}$$

Lemma 12. *If $D_{\min} > 10$, ϕ_0 is a blocking conductance function.*

We take Lemma 12 as given, and defer the proof until after the proof of Lemma 11.

Lemma 12 and Theorem 1 together yield:

$$\left\|\boldsymbol{\rho}^t - \boldsymbol{\pi}\right\|_{TV} \leq \max\left\{\frac{1}{4}, h^1(s'(G)) + \frac{70}{m}\left(5 + \int_{t=s'(G)}^{1/2} \frac{4}{t\Phi^2(t;G)}\right)\right\}$$

Then, $h^1(s'(G)) \leq 1/10$ follows exactly from the proof of Lemma 9, except now $\boldsymbol{\pi}(S) \leq s'(G)$ results in the sharper bound of $\mathbf{q}_u^{(1)}(S^c) \geq 9/10$ for every $u \in S$. $\square$

Lemma 12. The condition $D_{\min} > 10$ ensures that $s(G) \geq \pi_{\min}$.

It is known that $\frac{1}{4}\Phi^2(x;G)$ satisfies the exterior inequality for all $t \in (\pi_{\min}, 1/2]$.

For $t = s'(G)$ we will instead use the interior inequality. For any S such that $\boldsymbol{\pi}(S) \leq s'(G)$, the following statement holds: for every $u \in S$, $\mathrm{cut}(u, S^c; G) \geq 9/10 \cdot \deg(u;G)$. Fixing $\lambda = \pi(S)/2$, we have

$$
\begin{aligned}
\phi_{int}(S) &\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{\lambda \mathrm{cut}(S \setminus B, S^c; G)}{\mathrm{vol}(V;G)\left[\lambda(1-\lambda)\right]^2} \\
&\geq \min_{\substack{B \subset S \\ \boldsymbol{\pi}(B) \leq \lambda}} \frac{9\lambda \sum_{u \in S \setminus B} \deg(u;G)}{10\mathrm{vol}(V;G)\left[\lambda(1-\lambda)\right]^2} \\
&\geq \frac{9\lambda^2}{20[\lambda^2(1-\lambda)^2]} \geq \frac{9}{20}.
\end{aligned}
$$

$\square$

A.5 Conductance function and local spread: non-convex case.

We begin with some notation. Write $\mathcal{C}_\sigma[\mathbf{X}] = \widetilde{\mathbf{X}}$, and $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$. For $S \subset \widetilde{\mathbf{X}}$, let $\widetilde{\mathrm{cut}}_{n,r}(S) = \mathrm{cut}(S; \widetilde{G}_{n,r})$ and similarly $\widetilde{\mathrm{vol}}_{n,r}(S) = \mathrm{vol}(S; \widetilde{G}_{n,r})$.

Consider $\mathbf{z} \subset \mathcal{C}_\sigma$ such that $\mathcal{N}_{\mathbf{z}} = \{B(z, r/3) : z \in \mathbf{z}\}$ is an internal covering of $\mathcal{C}_\sigma$, meaning $\mathcal{N}_{\mathbf{z}} \supseteq \mathcal{C}_\sigma$. Then, we write

$$
\begin{aligned}
\widetilde{B}_{\min} &= \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\min} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right| \\
\widetilde{B}_{\max} &= \min_{z \in \mathbf{z}} \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right|, \quad \widetilde{D}_{\max} = \min_{\widetilde{x} \in \widetilde{\mathbf{X}}} \left| \widetilde{\mathrm{cut}}_{n,r}(x) \right|
\end{aligned}
$$

Both the conductance function and local spread will depend heavily on these quantities. Lemma 13 collects the bounds we will need.

Lemma 13. *Let $\mathcal{C}_\sigma$ satisfy the conditions of Theorem 2. For sufficiently large*

n, and $r \le \sigma/4d$, each of the following bounds hold with probability $1-\delta$:

$$\widetilde{B}_{\max} \le \left(1 + \sqrt{3^d \frac{3(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma$$

$$\widetilde{B}_{\min} \ge \left(1 - \sqrt{3^d \frac{2(\log|\mathcal{N}_{\mathbf{z}}| + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$

$$\widetilde{D}_{\max} \le \left(1 + \sqrt{\frac{3(\log n + \log(1/\delta))}{n\nu_d r^d \Lambda_\sigma}}\right) n\nu_d r^d \Lambda_\sigma$$

$$\widetilde{D}_{\min} \ge \left(1 - \sqrt{\frac{2(\log n + \log(1/\delta))}{n\nu_d r^d \lambda_\sigma \beta_d}}\right) n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d$$

$$\left(1 - \sqrt{\frac{2\log(1/\delta)}{n\lambda_d \nu_d \sigma^d}}\right) n\lambda_\sigma \nu_d \sigma^d \le \widetilde{n} \le \left(1 + \sqrt{\frac{3\log(1/\delta)}{n\Lambda_\sigma \nu_d D^d}}\right) n\Lambda_d \nu_d D^d \tag{A.13}$$

where $\widetilde{n} = \left|\widetilde{\mathbf{X}}\right|$ and $\beta_d \ge 1/2^d$.

In particular, fix $\epsilon > 0$. Then, for n as specified in Theorem 2, the following event:

$$\widetilde{B}_{\max} \le (1+\epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \Lambda_\sigma, \quad \widetilde{D}_{\max} \le (1+\epsilon)\, n\nu_d r^d \Lambda_\sigma$$

$$\widetilde{B}_{\min} \ge (1-\epsilon)\, n\nu_d \left(\frac{r}{3}\right)^d \lambda_\sigma \beta_d, \quad \widetilde{D}_{\min} \ge (1-\epsilon)\, n\nu_d r^d \lambda_\sigma \beta_d$$

$$(1-\epsilon) n\lambda_\sigma \nu_d \sigma^d \le \widetilde{n} \le (1+\epsilon) n\Lambda_\sigma \nu_d D^d \tag{A.14}$$

occurs with probability at least $1-\delta$.

Proof. We observe that for any $s \le \sigma/4d$ and any $x \in \mathcal{C}_\sigma$,

$$\nu(B(x,s) \cap \mathcal{C}_\sigma) \ge \left(\frac{s}{2}\right)^d = s^d \beta_d.$$

(In fact, tighter bounds can be shown to hold, but we will not need them). Therefore, by (A1):

$$\lambda_\sigma \nu_d s^d \beta_d \le \mathbb{P}\left(B(z,s) \cap \mathcal{C}_\sigma\right) \le \Lambda_\sigma \nu_d s^d$$

In particular, this holds for $s = r$ and $s = r/3$, and for every $z \in \mathbf{z}$ as well as every $z \in \widetilde{\mathbf{X}}$. Now, by (A4) and (A1) we also have

$$\Lambda_\sigma \nu_d \sigma^d \le \mathbb{P}(\mathcal{C}_\sigma) \le \Lambda_\sigma \nu_d D^d$$

The proof of each statement in (A.13) then follows from application of Lemma 24.

To show (A.14), we note that $|\mathcal{N}_z|$ is less than the covering number of the D-ball in d dimensions. Therefore

$$|\mathcal{N}_z| \leq \left(\frac{6D}{r} + 1\right)^d.$$

It is then immediately apparent that n chosen as in Theorem 2 yields (A.14). $\square$

Now, we consider the conductance function and local spread computed over $\widetilde{G}_{n,r}$, which we refer to by

$$\widetilde{\Phi}_{n,r}(t) = \Phi(t; \widetilde{G}_{n,r}), \quad \widetilde{s}_{n,r} = s(\widetilde{G}_{n,r}).$$

where the restriction in the minimization problem of (A.11) is with respect to $\widetilde{\pi}_{n,r}$ the stationary distribution over $\widetilde{G}_{n,r}$.

We will bound $\widetilde{\Phi}_{n,r}(1/2)$ and $\widetilde{s}_{n,r}$ under the event that (A.14) holds, noting that this occurs with probability at least $1 - \delta$.

Lemma 14. *If the bounds given by* (A.14) *hold, then*

$$\widetilde{s}_{n,r} \geq \frac{9}{10} \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{D^d} \beta_d^2$$

Proof. The local spread can be written as

$$\widetilde{s}_{n,r} = \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \geq \frac{9}{10} \frac{\widetilde{D}_{\min}^2}{\widetilde{D}_{\max} \widetilde{n}}.$$

Then apply the relevant results of Lemma 13. $\square$

Lemma 15. *If the bounds given by* (A.14) *hold, then:*

$$\widetilde{\Phi}_{n,r}(1/2) \geq \frac{\lambda_\sigma (1-\epsilon) \beta_d}{4 \Lambda_\sigma (1+\epsilon) 3^d} \left(1 + \frac{(1-\epsilon) r^d \lambda_\sigma}{(1+\epsilon) D^d \Lambda_\sigma}\right)$$

Proof. Fix $S \subset \widetilde{\mathbf{X}}$ with $\widetilde{\pi}_{n,r}(S) \leq 1/2$. Partition $\mathcal{N}_{\mathbf{z}} = \mathcal{N}_{\mathbf{z}}^+ \cup \mathcal{N}_{\mathbf{z}}^-$, where

$$\begin{aligned}
\mathcal{N}_{\mathbf{z}}^- &= \left\{ B(z, r/3) : 2 \left| B(z, r/3) \cap S \right| \leq \left| B(z, r/3) \cap \widetilde{\mathbf{X}} \right| \right\} \\
\mathcal{N}_{\mathbf{z}}^+ &= \mathcal{N}_{\mathbf{z}} \setminus \mathcal{N}_{\mathbf{z}}^-
\end{aligned}$$

and correspondingly $S^- = \mathcal{N}_{\mathbf{z}}^- \cap S$, $S^+ = \mathcal{N}_{\mathbf{z}}^+ \cap S$, so

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} = \frac{\widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) + \widetilde{\mathrm{cut}}_{n,r}(S^+; \widetilde{G}_{n,r} \setminus S)}{\widetilde{\mathrm{vol}}_{n,r}(S^-) + \widetilde{\mathrm{vol}}_{n,r}(S^+)}.$$

It is immediately apparent that the following bounds hold for all $S \subset \widetilde{\mathbf{X}}$:

$$\begin{aligned}
\widetilde{\mathrm{cut}}_{n,r}(S^-; \widetilde{G}_{n,r} \setminus S) &\geq \frac{|S^-| \, \widetilde{B}_{\min}}{2} \\
\widetilde{\mathrm{vol}}_{n,r}(S^-) &\leq \left|S^-\right| \widetilde{D}_{\max} \\
\widetilde{\mathrm{vol}}_{n,r}(S^+) &\leq \widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \mathbf{1}(\left|N_{\mathbf{z}}^+\right| > 0)
\end{aligned}$$

If moreover $|N_{\mathbf{z}}^+| < |N_{\mathbf{z}}|$, then

$$\widetilde{\mathrm{cut}}_{n,r}(S^+; \widetilde{G}_{n,r} \setminus S) \geq \frac{\widetilde{B}_{\min}^2}{4} \mathbf{1}(\left|N_{\mathbf{z}}^+\right| > 0)$$

follows from the fact that the graph $H_{n,r} = (\mathbf{z}, E_H)$, with $(z_i, z_j) \in E_H$ if $\|z_i - z_j\| \leq r/3$, is connected. As a result, if $|N_{\mathbf{z}}^+| < |N_{\mathbf{z}}|$ we have

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)} \geq \frac{\widetilde{B}_{\min}}{4\widetilde{D}_{\max}} + \frac{\widetilde{B}_{\min}^2}{8\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})} \tag{A.15}$$

using the inequality $2(A+B)/(C+D) \geq A/C + B/D$ for A, B, C, D non-negative.

If, on the other hand , $|\mathcal{N}_{\mathbf{z}}^+| = \mathcal{N}_{\mathbf{z}}$, then (A.15) holds with respect to S^c. Then, because $\widetilde{\pi}_{n,r}(S) \leq 1/2$,

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^c)}{\widetilde{\mathrm{vol}}_{n,r}(S^c)} \leq \frac{\widetilde{\mathrm{cut}}_{n,r}(S)}{\widetilde{\mathrm{vol}}_{n,r}(S)}$$

and so we get the exact statement of (A.15). Noting, as in the proof of Lemma 14, that $\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) \leq \widetilde{n} \cdot \widetilde{D}_{\max}$, the relevant results of Lemma 13 yield the desired inequality. □

A.6 Proof of Theorem 2

Throughout this proof, we will condition on the events of Lemmas 14 and 15, namely

$$\begin{aligned}
\widetilde{s}_{n,r} &\geq \frac{9}{10} \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \frac{\lambda_\sigma^2}{\Lambda_\sigma^2} \frac{r^d}{D^d} \beta_d^2 \\
\widetilde{\Phi}_{n,r}(1/2) &\geq \frac{\lambda_\sigma(1-\epsilon)\beta_d}{4\Lambda_\sigma(1+\epsilon)3^d} \left(1 + \frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)
\end{aligned}$$

noting that for n as chosen in Theorem 3, this will occur with probability at least $1 - \delta$ (by Lemma 13).

As a reminder, we write $\mathcal{C}_\sigma[\mathbf{X}] = \widetilde{\mathbf{X}}$, and $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$. Fix arbitrary $v \in \widetilde{\mathbf{X}}$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}_{\widetilde{\mathbf{X}}}^t, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{\mathbf{q}}_n^{(2)}, \ldots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. (As we will see, this bound will hold over all such starting vertices $v \in \widetilde{\mathbf{X}}$.)

By Lemma 9,

$$\begin{aligned}\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\ &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, d\log\left(\frac{8D(1+\epsilon)^2\Lambda_\sigma^2}{r(1-\epsilon)^2\lambda_\sigma^2}\right)\right\}\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 11, we have that

$$\begin{aligned}\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right) \\ &\leq \frac{1400}{3}\left(5 + \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt\right)\end{aligned} \tag{A.16}$$

where $s_{\mathbb{P},r} = \frac{9}{10}\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\frac{\lambda_\sigma^2}{\Lambda_\sigma^2}\frac{r^d}{D^d}\beta_d^2$.(Since r remains constant, for sufficiently large n the lower bound on $\widetilde{D}_{\min}$ of Lemma 13 will be at least 10, and therefore Lemma 12 holds.)

Now, we can upper bound the average conductance integral:

$$\begin{aligned}\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)}dt &\leq \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(1/2)}dt \\ &\leq 64\frac{9^d\Lambda_\sigma^2(1+\epsilon)^2\beta_d^2}{\lambda_\sigma^2(1-\epsilon)^2}\left(1 + \frac{(1-\epsilon)r^d\lambda_\sigma}{(1+\epsilon)D^d\Lambda_\sigma}\right)^{-2} \log s_{\mathbb{P},r}.\end{aligned}$$

Plugging this in to (A.16) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (10).

A.7 Population-level conductance function: convex case.

When $\mathcal{C}$ is convex, we will make use of the theory developed in A.4 with respect to the conductance function $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$. First, however, we introduce a population-level analogue to $\Phi(t; G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})])$ over the set $\mathcal{C}_\sigma$, which we denote $\widetilde{\Phi}_{\mathbb{P},r}$. (In general, we will adopt the convention of using $\widetilde{f}$ to denote functionals computed with respect to $\mathcal{C}_\sigma$.)

For $\mathcal{S} \subset \mathbb{R}^d$

$$\nu_{\mathbb{P}}(\mathcal{S}) := \int_{\mathcal{S}} f(x)dx$$

is the weighted volume.

The r-ball walk over $\mathcal{C}_\sigma$ is a Markov chain. For $x \in \mathcal{C}_\sigma$ and $\mathcal{S}, \mathcal{S}' \subset \mathcal{C}_\sigma$, the transition probability is given by

$$\widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S} \cap B(x,r))}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r))}.$$

The stationary distribution $\pi_{\mathbb{P},r}$ thus satisfies

$$\int_{\mathcal{C}_\sigma} \widetilde{P}_{\mathbb{P},r}(x;\mathcal{S}) d\pi_{\mathbb{P},r}(x) = \pi_{\mathbb{P},r}(\mathcal{S})$$

for all $\mathcal{S} \in \mathcal{C}_\sigma$. A simple calculation yields

$$\ell_{\mathbb{P},r}(x) := \nu_{\mathbb{P}}(\mathcal{C}_\sigma \cap B(x,r)) \quad \pi_{\mathbb{P},r}(\mathcal{S}) := \frac{1}{\int_{\mathcal{C}_\sigma} f(x)\ell_{\mathbb{P},r}(x)dx} \int_{\mathcal{S}} f(x)\ell_{\mathbb{P},r}(x)dx,$$

and therefore the ergodic flow is

$$\begin{aligned}
\widetilde{Q}_{\mathbb{P},r}(\mathcal{S},\mathcal{S}') &:= \int_{\mathcal{S}} d\pi_{\mathbb{P},r}(x) P_{\mathbb{P},r}(x;\mathcal{S}')dx \\
&= \frac{1}{\int_{\mathcal{C}_\sigma} f(x)\ell_{\mathbb{P},r}(x)dx} \int_{\mathcal{S}} f(x) \left(\int_{\mathcal{S}' \cap B(x,r)} f(x')dx' \right) dx
\end{aligned}$$

The continuous conductance function is then

$$\begin{aligned}
\widetilde{\Phi}_{\mathbb{P},r}(t) &:= \min_{\substack{\mathcal{S} \subset \mathcal{C}_\sigma, \\ \pi_{\mathbb{P},r}(\mathcal{S}) \leq t}} \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} \\
&= \min_{\substack{\mathcal{S} \subset \mathcal{C}_\sigma, \\ \pi_{\mathbb{P},r}(\mathcal{S}) \leq t}} \frac{\int_{\mathcal{S}} f(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} f(x')dx' \right) dx}{\int_{\mathcal{S}} f(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x')dx' \right) dx}.
\end{aligned}$$

For $m > 0$ and $0 < t_0 < t_1 < \ldots < t_m < 1$, denote the *stepwise approximation to* g by $\bar{g}$, defined as

$$\bar{g}(t) = g(t_i), \quad \text{for } t \in [t_{i-1}, t_i] \tag{A.17}$$

The stepwise approximation will be important to showing the consistency results of Section A.9 can be translated to a uniform bound. Lemma 16 shows that the approximation will not overly degrade our estimates of the population-level conductance function.

Lemma 16.

- *For any function f monotone decreasing in t on the interval $[t_0, t_m]$, $\bar{f}(t) \leq f(t)$ for all $t \in [t_0, t_m]$.*
- *Fix*
$$g(t) = \log\left(\frac{1}{t}\right) \text{ for } x \in [t_0, 1/2]$$
If for all i in $1, \ldots, m$, $(t_i - t_{i-1}) \leq t_0/2$, then $\bar{g}(t) \geq g(t)/2$.

Proof. The first statement is immediately obvious, and we turn to proving the second.

The upper bound $g(t) \geq \bar{g}(t)$ follows immediately from the fact that $g(t)$ is a decreasing function along with the first statement.

By the concavity of the log function,

$$\bar{g}(t) = \log\left(\frac{1}{t_i}\right) \geq \log\left(\frac{1}{t}\right) - \frac{(t_i - t)}{t}.$$

As a result,

$$\bar{g}(t) - \frac{g(t)}{2} \geq \frac{\log\left(\frac{1}{t}\right)}{2} - \frac{(t_i - t)}{t} \geq 1/2 - 1/2 = 0.$$

□

The following theorem is found in [3]. It gives a bound population-level conductance function over convex bodies, when the density is uniform.

Theorem 2 (Restatement of [3] Theorem 4.6)**.** *Let $K \subset \mathbb{R}^d$ be a convex body of diameter D. Then for any $\mathcal{S} \subset K$ with $\pi_{\nu,r}(\mathcal{S}) \leq 1/2$,*

$$\frac{Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\nu,r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{1}{\pi_{\nu,r}(\mathcal{S})}\right)\right\}. \tag{A.18}$$

Lemma 17. *Under the conditions on $\mathcal{C}_\sigma$ given by Theorem 3, the following bounds hold:*

- *for $0 < t < 1/2$,*

$$\widetilde{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D}\log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

- *Let*

$$M = \frac{2^{d+1}D^d\Lambda_\sigma^2}{r^d\lambda_\sigma^2}$$

  and $t_i = (i+1)/M$ for $i = 0, \ldots, m-1$. Then, for $1/M < t < 1/2$

$$\overline{\Phi}_{\mathbb{P},r}(t) > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{162\sqrt{d}D}\mathrm{Log}\left(\frac{\Lambda_\sigma^2}{\lambda_\sigma^2 t}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

  where $\overline{\Phi}_{\mathbb{P},r}(t)$ is defined as in (A.17) with respect to $t_0, \ldots t_{M-1}$, and $\mathrm{Log}(A/t) = \max\{\log(1 + 2A), \log(A/t)\}$.

Before we prove Lemma 17, note that the choice of M is made to ensure t_0 is greater than the local spread of $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$, as we will see in Section A.8.

Proof of Lemma 17. We note that

$$\pi_{\mathbb{P},r}(S) \leq \pi_{\nu,r}(S) \cdot \frac{\Lambda_\sigma^2}{\lambda_\sigma^2}, \quad Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \geq Q_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S}) \cdot \frac{\lambda_\sigma^2}{\Lambda_\sigma^2}$$

Plugging these estimates in to (A.18) gives

$$\frac{Q_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} > \min\left\{\frac{1}{288\sqrt{d}}, \frac{r}{81\sqrt{d}D} \log\left(1 + \frac{\lambda_\sigma^2}{\Lambda_\sigma^2 \pi_{\mathbb{P},r}(\mathcal{S})}\right)\right\} \cdot \frac{\lambda_\sigma^4}{\Lambda_\sigma^4}$$

and since the right hand side is decreasing in $\pi_{\mathbb{P},r}(\mathcal{S})$, the desired lower bound holds on $\widetilde{\Phi}_{\mathbb{P},r}(t)$. The bound on $\overline{\Phi}_{\mathbb{P},r}(t)$ then follows from $\mathrm{Log}(A/t) \leq \log(1+1/t)$ for all $0 < t < 1/2$ and application of Lemma 16. $\square$

A.8 Consistency of local spread and conductance function: convex case.

The introduction of the stepwise approximation allows us to make use of Lemma 18, which gives us (pointwise) consistency of the discrete graph functionals $\widetilde{\Phi}_{n,r}(t)$ to the continuous functionals $\widetilde{\Phi}_{\mathbb{P},r}(t)$.

We use $\omega_r(1)$ to denote a term which goes to infinity as $r \to 0$, and likewise $o_r(1)$ to denote a term which goes to 0 as $r \to 0$.

Lemma 18. *Fix $0 < t < 1/2$. Under the conditions on $\mathcal{C}_\sigma$ given by Theorem 3, the following statement holds: with probability one, as $n \to \infty$,*

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \geq \min\left\{\widetilde{\Phi}_{\mathbb{P},r}(t), c_d r \omega_r(1)\right\} \tag{A.19}$$

where c_d is a constant which may depend on the dimension d (as well as the distribution $\mathbb{P}$), but not r.

As a consequence, for M and $(t_i)_{i=0}^{M-1}$ defined as in Lemma 17, we have that

$$\liminf_{n\to\infty} \overline{\Phi}_{n,r}(t) \geq \min\left\{\overline{\Phi}_{\mathbb{P},r}(t), c_d r \omega_r(1)\right\} \tag{A.20}$$

We defer the proof of pointwise consistency to Section A.9. For now, we show that (A.20) is immediately implied by (A.19).

Proof of (A.20). We take as given that for any $0 < t < 1/2$,

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) \geq \widetilde{\Phi}_{\mathbb{P},r}(t).$$

In particular, for sufficiently large n this will occur for each of $t_0, t_1, \ldots, t_m$ and therefore

$$\liminf_{n\to\infty} \overline{\Phi}_{n,r} \geq \overline{\Phi}_{\mathbb{P},r}$$

uniformly over $[1/m, 1/2]$. $\square$

A.9 Pointwise consistency of conductance function: convex case.

We will rely heavily on results of [5], which prove the same result but consider only a pointwise result on $\widetilde{\Phi}_{n,r}(1/2)$ rather than over the entire conductance function.

Let $\widetilde{\mathbf{X}} = \mathcal{C}_\sigma[\mathbf{X}] = \{\widetilde{x}_1, \ldots, \widetilde{x}_{\widetilde{n}}\}$, and $\widetilde{n} = \left|\widetilde{\mathbf{X}}\right|$. Then

$$\widetilde{\mathbb{P}}_n := \frac{1}{\widetilde{n}} \sum_{\widetilde{x}_i \in \widetilde{\mathbf{X}}} \delta_{\widetilde{x}_i}$$

is the empirical distribution of $\widetilde{\mathbf{X}}$. Likewise, for $\mathcal{S} \subset \mathcal{C}_\sigma$ let $\widetilde{\mathbb{P}}$ be the conditional distribution $\mathbb{P}(x \in S | x \in \mathcal{C}_\sigma)$, given by

$$\widetilde{\mathbb{P}}(\mathcal{S}) := \frac{\nu_{\mathbb{P}}(\mathcal{S})}{\nu_{\mathbb{P}}(\mathcal{C}_\sigma)}.$$

A Borel map $T : \mathcal{C}_\sigma \to \widetilde{\mathbf{X}}$ is a *transportation map* between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ if

$$\widetilde{\mathbb{P}}(\mathcal{S}) = \widetilde{\mathbb{P}}_n(T(\mathcal{S}))$$

for all $\mathcal{S} \in \mathcal{C}_\sigma$.

Lemma 19 (Proposition 5 of [5])**.** *There exists a sequence of transportation maps $(T_{\widetilde{n}})$ from $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}_n$ such that*

$$\limsup_{\widetilde{n} \to \infty} \frac{\widetilde{n}^{1/d} \left\|\mathrm{Id} - T_{\widetilde{n}}\right\|_{L^\infty(\widetilde{\mathbb{P}})}}{(\log \widetilde{n})^{p_d}} \leq C$$

where $p_d = 1/d$ for $d \geq 3$ and $3/4$ if $d = 2$.

These are referred to stagnating transportation maps. We refer the curious reader to [5] for more details.

For $S \subset \widetilde{\mathbf{X}}$, we will denote $\mathrm{vol}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{vol}}_{n,r}(S)$, and likewise $\mathrm{cut}(S; \widetilde{G}_{n,r})$ by $\widetilde{\mathrm{cut}}_{n,r}(S)$.

Consider a sequence of sets $(S_{\widetilde{n}})_{\widetilde{n} \in \mathbb{N}}$, with $u_{\widetilde{n}} = \mathbf{1}_{S_{\widetilde{n}}}$ the characteristic function of $S_{\widetilde{n}}$. Similarly, for $\mathcal{S} \subset \mathcal{C}_\sigma$ let $u = \mathbf{1}_{\mathcal{S}}$.

Definition 2. *For a sequence $(u_{\widetilde{n}}) \in L^1(\widetilde{\mathbb{P}}_{\widetilde{n}})$ and $u \in L^1(\widetilde{\mathbb{P}})$, we say $(u_{\widetilde{n}})$ converges TL^1 to u if there exists a sequence of stagnating transportation maps $(T_{\widetilde{n}})$ such that*

$$\int_{\mathcal{C}_\sigma} |u(x) - (u_{\widetilde{n}}) \circ T_{\widetilde{n}}(x)| \, d\widetilde{\mathbb{P}}(x) \to 0$$

and denote it $u_{\widetilde{n}} \overset{TL^1}{\to} u$.

Lemma 20. *If* $(u_{\widetilde{n}}) \overset{TL^1}{\to} u$*, with probability one*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S})\cap B(x,r)} \widetilde{f}(x')dx' \right) dx$$

where $\widetilde{f}$ *is the density function of* $\widetilde{\mathbb{P}}$ *over* $\mathcal{C}_\sigma$*.*

Proof. We note immediately that $n \to \infty$ implies $\widetilde{n} \to \infty$ with probability one. Now, we can write

$$\begin{aligned}
\frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} &= \frac{1}{\widetilde{n}^2} \sum_{i=1}^{\widetilde{n}} \sum_{j=1}^{\widetilde{n}} u_{\widetilde{n}}(\widetilde{x}_i)\big(1 - u_{\widetilde{n}}(\widetilde{x}_j)\mathbf{1}(\|\widetilde{x}_i - \widetilde{x}_j\| \le r)\big) \\
&= \int_{\mathcal{C}_\sigma} \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} u_{\widetilde{n}}(x)\big(1 - u_{\widetilde{n}}(x')\big) d\widetilde{\mathbb{P}}_n(x') \right) d\widetilde{\mathbb{P}}_n(x) \\
&= \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x).
\end{aligned}$$

Note that, for any $x \in \mathcal{C}_\sigma$,

$$\begin{aligned}
\lim_{n\to\infty} \nu\big(B(T_{\widetilde{n}}(x),r) \setminus B(x,r)\big) = 0 \\
\lim_{n\to\infty} \nu\big(B(x,r) \setminus B(T_{\widetilde{n}}(x),r)\big) = 0.
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') = \lim_{n\to\infty} \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x').$$

An application of the bounded convergence theorem yields

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(T_{\widetilde{n}}(x),r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) = \\
&\lim_{n\to\infty} \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x).
\end{aligned}$$

Letting

$$\begin{aligned}
\mathcal{I}_n^1 &= \int_{\mathcal{C}_\sigma} (u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (u(x') - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) \\
\mathcal{I}_n^2 &= \int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x) - u(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x)
\end{aligned}$$

we have

$$\int_{\mathcal{C}_\sigma} (u_{\widetilde{n}} \circ T_{\widetilde{n}}(x)) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u_{\widetilde{n}} \circ T_{\widetilde{n}}(x'))\, d\widetilde{\mathbb{P}}(x') \right) d\widetilde{\mathbb{P}}(x) =$$

$$\int_{\mathcal{C}_\sigma} u(x) \left\{ \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) d\widetilde{\mathbb{P}}(x') \right\} d\widetilde{\mathbb{P}}(x) + \mathcal{I}_n^1 + \mathcal{I}_n^2. \tag{A.21}$$

Recalling that $u = 1_{\mathcal{S}}$, we can see

$$\int_{\mathcal{C}_\sigma} u(x) \left\{ \int_{\mathcal{C}_\sigma \cap B(x,r)} (1 - u(x')) d\widetilde{\mathbb{P}}(x') \right\} d\widetilde{\mathbb{P}}(x) = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} \widetilde{f}(x') dx' \right) dx \tag{A.22}$$

Since $(u_{\widetilde{n}}) \overset{TL^1}{\to} u$, another application of the bounded convergence theorem yields $\lim_{n\to\infty} \mathcal{I}_n^1 = \lim_{n\to\infty} \mathcal{I}_n^2 = 0$. Therefore by (A.21) and (A.22) the final result holds. □

Lemma 21. *If $u_{\widetilde{n}} \overset{TL^1}{\to} u$, then*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} \widetilde{f}(x') dx' \right) dx$$

with probability one.

Proof. We note that

$$\begin{aligned} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} &= \frac{1}{\widetilde{n}^2} \sum_{i=1}^{\widetilde{n}} \sum_{j=1}^{\widetilde{n}} u_{\widetilde{n}}(\widetilde{x}_i) \mathbf{1}(\|\widetilde{x}_i - \widetilde{x}_j\| \leq r) - \frac{1}{\widetilde{n}} \\ &= \int_{\mathcal{C}_\sigma} \int_{\mathcal{C}_\sigma \cap B(x,r)} \left(u_{\widetilde{n}}(x) d\widetilde{\mathbb{P}}_n(x') \right) d\widetilde{\mathbb{P}}(x) - \frac{1}{\widetilde{n}} \end{aligned}$$

Of course, $\lim_{n\to\infty} \frac{1}{\widetilde{n}} = 0$, and so

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} = \lim_{n\to\infty} \int_{\mathcal{C}_\sigma} \left(\mathcal{C}_\sigma \cap B(x,r) u_{\widetilde{n}}(x) d\widetilde{\mathbb{P}}_n(x') \right) d_{\widetilde{\mathbb{P}}}(x)$$

The proof then proceeds analogously to Lemma 20. □

Lemma 22 can be found in [1] (Theorem 3.1) or [5] (Lemma 23).

Lemma 22. *If $u_{\widetilde{n}} \overset{TL^1}{\not\to} u$ for some $u \in L^1\nu$, with probability one:*

$$\liminf_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(\mathcal{S}_{\widetilde{n}})}{\widetilde{n}^2} \geq c_d r^{d+1} \omega_r(1)$$

where c_d is a constant which does not depend on r but may depend on $\mathcal{C}_\sigma$ and f, and $\omega_r(1) \to \infty$ as $r \to 0$.

Proof of (A.19). Let $(S^\star_{\widetilde{n}})$

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S^\star_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S^\star_{\widetilde{n}})} = \widetilde{\Phi}_{n,r}(t), \quad \widetilde{\pi}_{n,r}(S_{\widetilde{n}}) \leq t$$

be the sequence of minimizers of the normalized cut with stationary distribution at most t in the graph $\widetilde{G}_{n,r}$. Denote $u^\star_{\widetilde{n}} = 1_{S^\star_{\widetilde{n}}}$, and assume that $u^\star_{\widetilde{n}} \overset{TL^1}{\to} u$, for $u = 1_{\mathcal{S}}$, $\mathcal{S} \subset \mathcal{C}_\sigma$. Then, by Lemmas 20 and 21,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})} &= \frac{\int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} \widetilde{f}(x')dx'\right) dx}{\int_{\mathcal{S}} \widetilde{f}(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} \widetilde{f}(x')dx'\right) dx} \\
&\overset{(i)}{=} \frac{\int_{\mathcal{S}} f(x) \left(\int_{(\mathcal{C}_\sigma \setminus \mathcal{S}) \cap B(x,r)} f(x')dx'\right) dx}{\int_{\mathcal{S}} f(x) \left(\int_{\mathcal{C}_\sigma \cap B(x,r)} f(x')dx'\right) dx} \quad = \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})}.
\end{aligned}$$

where (i) holds because the normalization factors present in $\widetilde{f}$ cancel. From Lemma 21, we also have that $\lim_{n\to\infty} \widetilde{\pi}_{n,r}(S_{\widetilde{n}}) = \pi_{\mathbb{P},r}(\mathcal{S})$, and therefore $\pi_{\mathbb{P},r}(\mathcal{S}) \leq t$. As a result,

$$\liminf_{n\to\infty} \widetilde{\Phi}_{n,r}(t) = \frac{\widetilde{Q}_{\mathbb{P},r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\mathbb{P},r}(\mathcal{S})} \geq \widetilde{\Phi}_{\mathbb{P},r}(t).$$

On the other hand, if $u^\star_{\widetilde{n}}$ does not converge TL^1, then

$$\frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} \geq c_d r^{d+1} \omega_r(1)$$

Additionally,

$$\frac{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})}{\widetilde{n}^2} \leq \frac{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})}{\widetilde{n}^2}$$

and

$$\limsup_{n\to\infty} \frac{\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r})}{\widetilde{n}^2} \leq \nu_d r^d \Lambda_\sigma$$

As a result,

$$\liminf_{n\to\infty} \frac{\widetilde{\mathrm{cut}}_{n,r}(S_{\widetilde{n}})}{\widetilde{\mathrm{vol}}_{n,r}(S_{\widetilde{n}})} \geq c_d r \omega_r(1).$$

□

A.10 Proof of Theorem 3

Throughout this proof, we will refer to the subgraph $G_{n,r}[\mathcal{C}_\sigma[\mathbf{X}]]$ as $\widetilde{G}_{n,r}$.

Fix arbitrary $v = x_i \in \mathcal{C}_\sigma[\mathbf{X}]$, and let

$$\widetilde{\mathbf{q}}_n^{(m)} = e_v \mathbf{W}_{\mathcal{C}_\sigma[\mathbf{X}]}^t, \quad \widetilde{\mathbf{q}}_n = (\widetilde{q}_n^{(1)}, \widetilde{q}_n^{(2)}, \ldots)$$

Our goal is to upper bound $\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$.

By Lemmas 9 and 14,

$$\begin{aligned}\tau_\infty(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, \log\left(\frac{4}{\widetilde{s}_{n,r}}\right)\right\} \\ &\leq 2752\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \max\left\{2, 4d\log\left(\frac{2D\Lambda_\sigma^2}{\lambda_\sigma^2}\right)\right\}\end{aligned}$$

We now upper bound $\tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r})$. From Lemma 11, we have that

$$\limsup_{n\to\infty} \tau_1(\widetilde{\mathbf{q}}_n; \widetilde{G}_{n,r}) \leq \limsup_{n\to\infty} \frac{1400}{3}\left(5 + \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt\right) \tag{A.23}$$

(Since r remains constant, for sufficiently large n, $\mathbf{D}_{xx} > C$ will be fulfilled for any $x \in \mathcal{C}_\sigma[\mathbf{X}]$, and any $C < \infty$.) We set aside the constant term for the moment and turn to the integral. By Lemma 14,

$$\limsup_{n\to\infty} \int_{\widetilde{s}_{n,r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt \leq \limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt$$

where $s_{\mathbb{P},r}$ is as in the proof of Theorem 2. We now replace the discrete conductance function $\widetilde{\Phi}_{n,r}$ by the stepwise approximation to the continuous conductance function, $\overline{\Phi}_{n,r}$:

$$\begin{aligned}\limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\widetilde{\Phi}_{n,r}^2(t)} dt &\overset{(i)}{\leq} \limsup_{n\to\infty} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{n,r}^2(t)} dt \\ &= \int_{s_{\mathbb{P},r}}^{1/2} \limsup_{n\to\infty} \frac{4}{t\overline{\Phi}_{n,r}^2(t)} dt \\ &\overset{(ii)}{\leq} \int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{\mathbb{P},r}^2(t)} dt + c_d^2 \log(s_{\mathbb{P},r}) \frac{1}{r^2} o_r(1)\end{aligned}$$

where (i) follows from Lemma 16 and (ii) from Lemma 18 (along with the continuous mapping theorem). Now, we make use of Lemma 17:

$$\int_{s_{\mathbb{P},r}}^{1/2} \frac{4}{t\overline{\Phi}_{\mathbb{P},r}^2(t)} dt \leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt + \int_{s_{\mathbb{P},r}}^{1/2} \frac{81 d D^2}{r^2 t \mathrm{Log}(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt \right)$$

$$\leq \frac{\Lambda_\sigma^8}{\lambda_\sigma^8} \cdot \left(331776 \underbrace{\int_{s_{\mathbb{P},r}}^{1/2} \frac{d}{t} dt}_{:=\mathcal{J}_1} + 81 \underbrace{\int_{s_{\mathbb{P},r}}^{\lambda_\sigma^2/(4\Lambda_\sigma^2)} \frac{dD^2}{r^2 t \log(\frac{\Lambda_\sigma^2}{t\lambda_\sigma^2})} dt}_{:=\mathcal{J}_2} + 81 \underbrace{\int_{\lambda_\sigma^2/(4\Lambda_\sigma^2)}^{1/2} \frac{dD^2}{r^2 t \log(1 + \frac{4\lambda_\sigma^2}{\Lambda_\sigma^2})} dt}_{:=\mathcal{J}_3} \right)$$

Computing a few simple integrals yields the following upper bounds on $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$:

$$\begin{aligned} \mathcal{J}_1 &\leq d^2 \log\left(\frac{2D\Lambda_\sigma^2}{r\lambda_\sigma^2}\right) \\ \mathcal{J}_2 &\leq \frac{dD^2}{r^2}\left[\log(2d) + \log\left(\log\left(\frac{2D}{r}\right)\right)\right] \\ \mathcal{J}_3 &\overset{(iii)}{\leq} 2\frac{dD^2}{r^2}\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\log\left(4\frac{\Lambda_\sigma^2}{\lambda_\sigma^2}\right) \end{aligned}$$

where (iii) uses the upper bound $\frac{1}{\log(1+x)} \leq \frac{1}{x}$.

Plugging these bounds in to (A.23) gives the desired upper bound on $\tau_\infty(\widetilde{q}_n, \widetilde{G}_{n,r})$, which translates to the lower bound of (10).

A.11 Concentration inequalities

Given a symmetric kernel function $k : \mathcal{X}^m \to \mathbb{R}$, and data $\{x_1, \ldots, x_n\}$, we define the *order-m U statistic* to be

$$U := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} k(x_{i_1}, \ldots, x_{i_m})$$

For Lemmas 23, let $X_1, \ldots, X_n \in \mathcal{X}$ be independent and identically distributed. We will additionally assume the order-m kernel function k satisfies the boundedness property $\sup_{x_1,\ldots,x_m} |k(x_1, \ldots, x_m)| \leq 1$.

Lemma 23 (Hoeffding's inequality for U-statistics.)**.** *For any $t > 0$,*

$$\mathbb{P}(|U - \mathbb{E}U| \geq t) \leq 2\exp\left\{-\frac{2nt^2}{m}\right\}$$

Further, for any $\delta > 0$, we have

$$U \le \mathbb{E}U + \sqrt{\frac{m \log(1/\delta)}{2n}},$$
$$U \ge \mathbb{E}U - \sqrt{\frac{m \log(1/\delta)}{2n}}$$

each with probability at least $1 - \delta$.

We will employ a sharper concentration inequality for $\sum_{i=1}^n X_i$.

Lemma 24. *Let $X_i \in \{0, 1\}$ for $i = 1, \ldots, n$ and let $\mu = \mathbb{E}(\sum_{i=1}^n X_i)$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i > (1+\epsilon)\mu\right) \le \exp\left(\frac{-\delta^2 \mu}{3}\right)$$
$$\mathbb{P}\left(\sum_{i=1}^n X_i < (1-\epsilon)\mu\right) \le \exp\left(\frac{-\delta^2 \mu}{2}\right)$$

Let $\mathbf{p} = (p_u)_{u \in \mathbf{X}}$ denote the PPR vector computed over $G_{n,r}$ (where for ease of reading we suppress dependence on the hyperparameter α and seed node v.) Recalling that $\widetilde{\pi}_{n,r}$ is the stationary distribution over $\widetilde{G}_{n,r}$, we write $\widetilde{\pi}_{n,r}(u)$ to denote the stationary distribution evaluated at $u \in \mathcal{C}_\sigma[X]$.

A.12 Proof of Theorem 4

Lemma 25 formalizes the intuitively sensible statement that PPR , can be uniformly lower bounded over clusters with small mixing time relative to their normalized cut. Importantly, it adds a corresponding upper bound for PPR outside of these clusters.

Lemma 25. *Consider running Algorithm 1 with any $r < \sigma$ and*

$$\frac{\Psi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{10} \le \alpha \le \frac{\Psi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{9}. \tag{A.24}$$

There exists a good set $\mathcal{C}_\sigma[\mathbf{X}]^g \subseteq \mathcal{C}_\sigma[\mathbf{X}]$ with $\mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}]^g) \ge \mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}])/2$ such that the following statements hold for all $v \in \mathcal{C}_\sigma[\mathbf{X}]^g$:

- *For all $u \in \mathcal{C}[\mathbf{X}]$,*

$$p_u \ge \frac{4}{5}\widetilde{\pi}_{n,r}(u) - \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{\Psi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])\widetilde{D}_{\min}} \tag{A.25}$$

- *For all $u' \in \mathcal{C}'_\sigma[\mathbf{X}]$,*

$$p_{u'} \le \frac{2\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{\Psi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])\widetilde{D}_{\min}} \tag{A.26}$$

Proof. We will write $\mathbf{W}_n = \mathbf{D}_n \mathbf{A}_n^{-1}$ for the transition probability matrix over $G_{n,r}$, and let $\widetilde{\mathbf{D}}_n$ and $\widetilde{\mathbf{W}}_n$ be the degree and random walk matrices for the subgraph $\widetilde{G}_{n,r}$.

We introduce *leakage* and *soakage* vectors, defined by

$$\begin{aligned}
\ell_t &:= e_v(\mathbf{W}_n \widetilde{\mathbf{I}}_n)^t(\mathbf{I}_n - \mathbf{D}_n^{-1}\widetilde{\mathbf{D}}_n) \\
\ell &:= \sum_{t=0}^{\infty}(1-\alpha)^t \ell_t \\
s_t &:= e_v(\mathbf{W}_n \widetilde{\mathbf{I}}_n)^t(\mathbf{W}_n \widetilde{\mathbf{I}}_n^c) \\
s &:= \sum_{t=0}^{\infty}(1-\alpha)^t s_t
\end{aligned}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\widetilde{\mathbf{I}}_n$ is a diagonal matrix with $(\widetilde{\mathbf{I}}_n)_{uu} = 1$ if $u \in \mathcal{C}_\sigma[\mathbf{X}]$ and $\widetilde{\mathbf{I}}_n^c = \mathbf{I}_n - \widetilde{\mathbf{I}}_n$.

Roughly, the proof will unfold in four steps. The first two will result in the lower bound of (A.25), while the latter two will imply the upper bound in (A.26).

1. For $u \in \mathcal{C}'[\mathbf{X}]$, use the results of [6] to produce the lower bound

   $$\mathbf{p}(u) \geq 4/5\widetilde{\pi}_{n,r}(u) - \widetilde{\mathbf{p}}_\ell(u)$$

   where

   $$\widetilde{\mathbf{p}}_\ell = \alpha\ell + (1-\alpha)\widetilde{\mathbf{p}}_\ell \widetilde{\mathbf{W}}_n$$

   is the PPR random walk over $\widetilde{G}_{n,r}$, and ℓ has bounded norm $||\ell||_1 \leq 2\frac{\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{\alpha}$.

2. Since $r < \sigma$, for any $u \in \mathcal{C}[\mathbf{X}]$ there are no edges between u and $G_{n,r}/\mathcal{C}_\sigma[\mathbf{X}]$. Therefore, the page-rank vector $\widetilde{\mathbf{p}}_\ell$ will not assign more than $||\ell||_1/d_{\min}(\mathcal{C}_\sigma[\mathbf{X}])$ probability mass to any vertex in $\mathcal{C}'[\mathbf{X}]$. This observation will conclude our proof of (A.25).

3. For vertices $u' \in G_{n,r}/\mathcal{C}_\sigma[\mathbf{X}]$, we can upper bound $p_v(u) \leq p_s(u')$. In particular, this hold for all $u' \in \mathcal{C}'[\mathbf{X}]$.

4. Since $r < \sigma$, there are no edges between u' and $G/\mathcal{C}'[\mathbf{X}]$. Therefore, the page-rank vector p_s will assign no more than $||s||_1/d_{\min}(\mathcal{C}_\sigma[\mathbf{X}])$ probability mass to any vertex in $\mathcal{C}'[\mathbf{X}]$. Additionally, s has bounded norm $||s||_1 \leq ||\ell||_1$. This will conclude our proof of (A.26), and hence Lemma 25.

Step 1 We will begin by restating some results of [6].

For seed node v, we write

$$\widetilde{\mathbf{p}}_v = \alpha e_v + (1-\alpha)\widetilde{\mathbf{p}}_v \widetilde{\mathbf{W}}_n \tag{A.27}$$

$$= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(e_v \widetilde{\mathbf{W}}_n^t \right) \tag{A.28}$$

From Lemma 3.1 of [6] we have that there exists a good set $\mathcal{C}_\sigma[\mathbf{X}]^g \subseteq \mathcal{C}_\sigma[\mathbf{X}]$ with $\mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}]^g) \geq \mathrm{vol}(\mathcal{C}_\sigma[\mathbf{X}])/2$ for all $v \in \mathcal{C}_\sigma[\mathbf{X}]^g$, $u \in \mathcal{C}_\sigma[\mathbf{X}]$

$$p_u \geq \widetilde{\mathbf{p}}_v(u) - \widetilde{\mathbf{p}}_\ell(u)$$

$$||\ell||_1 \leq \frac{2\widetilde{\Phi}_{n,r}}{\alpha} \tag{A.29}$$

where $\widetilde{\mathbf{p}}_v = (\widetilde{\mathbf{p}}_v(u))$ and likewise for $\widetilde{\mathbf{p}}_\ell = (\widetilde{\mathbf{p}}_\ell(u))$. (This will be the only time we need to restrict ourselves to this 'good set'.)

Moreover if, as we have specified, $\alpha \leq \widetilde{\Psi}_{n,r}/9$, Lemma 3.2 of [6] yields a lower bound on $\widetilde{p}$

$$\widetilde{\mathbf{p}}_v(u) \geq \frac{4}{5}\widetilde{\pi}_{n,r}(u). \tag{A.30}$$

Step 2 We turn to upper bounding $\widetilde{\mathbf{p}}_\ell(u)$. For any $u \in \mathcal{C}[\mathbf{X}]$, we have

$$\begin{aligned}
\widetilde{\mathbf{p}}_\ell(u) &= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\ell \widetilde{\mathbf{W}}_n^t \right)(u) \\
&= \|\ell\|_1 \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\frac{\ell}{\|\ell\|_1} \widetilde{\mathbf{W}}_n^t \right)(u) \\
&\overset{\text{(i)}}{=} \|\ell\|_1 \alpha \sum_{t=1}^{\infty} (1-\alpha)^t \left(\frac{\ell}{\|\ell\|_1} \widetilde{\mathbf{W}}_n^t \right)(u) \\
&\overset{\text{(ii)}}{\leq} \|\ell\|_1 \frac{1}{\widetilde{D}_{\min}}
\end{aligned} \tag{A.31}$$

where we use $\left(\ell \widetilde{\mathbf{W}}_n^t \right)(u)$ to denote $\ell \widetilde{\mathbf{W}}_n^t e_u$.

(i) follows from the fact that since $r < \sigma$, $\mathrm{cut}(\mathcal{C}[\mathbf{X}], G_{n,r}/\mathcal{C}_\sigma[\mathbf{X}]; G_{n,r}) = 0$. Therefore $(\mathbf{D}_n^{-1})_{uu}(\widetilde{\mathbf{D}}_n)_{uu} = 1$, and as a result

$$(\ell \widetilde{\mathbf{W}}_n^0)(u) = \ell(u) = 0.$$

To see (ii), let $q = \frac{\ell}{\|\ell\|_1} \widetilde{\mathbf{W}}_n^{t-1}$. Then

$$
\begin{aligned}
\left(\frac{\ell}{\|\ell\|_1}\widetilde{\mathbf{W}}_n^t\right)(u) &= \left(q\widetilde{\mathbf{W}}_n\right)(u) \\
&\leq \|q\|_1 \|\widetilde{\mathbf{W}}_{\cdot u}\|_\infty \\
&\overset{\text{(iii)}}{\leq} \frac{1}{\widetilde{D}_{\min}}.
\end{aligned}
$$

where $\widetilde{\mathbf{W}}_{\cdot u}$ is the uth column of $\widetilde{\mathbf{W}}_n$. (iii) then follows from the fact that any vertex in $\mathcal{C}[\mathbf{X}]$ is connected only to vertices in $\mathcal{C}_\sigma[\mathbf{X}]$, and therefore every entry of $\widetilde{\mathbf{W}}_{\cdot u}$ is either 0 or at most $1/\widetilde{D}_{\min}$.

Combined, (A.31), (A.30), and (A.29) imply

$$
p_v(u) \geq \frac{4}{5}\widetilde{\pi}_{n,r}(u) - 18\frac{\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{\widetilde{D}_{\min}\alpha}.
$$

for any $v \in \mathcal{C}_\sigma[\mathbf{X}]$.

Step 3 To get the corresponding upper bound on $p_v(u')$, we will use the soakage vectors s and s_t. We will first argue that s is a worse starting distribution – meaning it puts uniformly more mass outside the cluster – than simply starting at v.

Lemma 26. *For all $u' \notin \mathcal{C}_\sigma[\mathbf{X}]$,*

$$
\mathbf{p}_v(u') \leq \mathbf{p}_s(u'). \tag{A.32}
$$

Proof. We have

$$
\begin{aligned}
\mathbf{p}_v(u') &= \alpha \sum_{T=0}^{\infty} (1-\alpha)^T (e_v \mathbf{W}_n^T)(u) \\
&\overset{(i)}{=} \alpha \sum_{T=1}^{\infty} (1-\alpha)^T (e_v \mathbf{W}_n^T)(u')
\end{aligned}
$$

where (i) follows from $v \in \mathcal{C}_\sigma$, $u \notin \mathcal{C}_\sigma$ and therefore $e_v(u) = 0$.

Lemma 27 allows us to make the transition to sums of soakage vectors.

Lemma 27. *Let $G = (V, E)$ be a graph, with associated random walk matrix W.*

For any $T \geq 1$, q vector, $S \subset V$, and $s_t = s_t(S^c, q)$

$$
qW^T = \sum_{t=0}^{T-1} s_t W^{T-t-1} + q(WI_S)^T \tag{A.33}
$$

We prove Lemma 27 after completing the proof of Lemma 26.

Now, along with the fact $u \notin \mathcal{C}_\sigma$, we have

$$\left(e_v \mathbf{W}_n^T\right)(u') = \sum_{t=0}^{T-1} \left(s_t \mathbf{W}_n^{T-t-1}\right)(u')$$

and so

$$\begin{aligned}
\mathbf{p}_v(u) &= \alpha \sum_{T=1}^{\infty} (1-\alpha)^T \left(\sum_{t=0}^{T-1} s_t \mathbf{W}^{T-t-1}\right)(u') \\
&= \alpha \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} (1-\alpha)^T \left(s_t \mathbf{W}^{T-t-1}\right)(u') \\
&= \alpha \sum_{t=0}^{\infty} \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta+t+1} \left(s_t \mathbf{W}_n^{\Delta}\right)(u') \\
&\leq \alpha \sum_{t=0}^{\infty} \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta+t} \left(s_t \mathbf{W}_n^{\Delta}\right)(u') \\
&= \alpha \sum_{\Delta=0}^{\infty} (1-\alpha)^{\Delta} \left(s \mathbf{W}_n^{\Delta}\right)(u') \\
&= \mathbf{p}_s(u')
\end{aligned}$$

$\square$

Proof of Lemma 27. Proceed by induction. When $T = 1$,

$$\begin{aligned}
qW &= q(WI_S) + q(WI_{S^c}) \\
&= q(WI_S)^T + s_0
\end{aligned}$$

Assume true for T_0. For $T = T_0 + 1$,

$$
\begin{aligned}
qW^T &= qW^{T_0}W \\
&= \left\{ \sum_{t=0}^{T_0-1} s_t W^{T_0-1-t} + q(WI_S)^{T_0} \right\} W \\
&= \sum_{t=0}^{T_0-1} s_t W^{T-1-t} + q(WI_S)^{T_0}(WI_S + WI_{S^c}) \\
&= \sum_{t=0}^{T-1} s_t W^{T-1-t} + q(WI_S)^T
\end{aligned}
$$

□

Step 4 Just as we upper bounded the probability mass $\widetilde{\mathbf{p}}_\ell$ could assign to any one vertex, we can upper bound

$$
\begin{aligned}
\mathbf{p}_s(u') &= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(s\mathbf{W}_n^t \right)(u') \\
&= \|s\|_1 \alpha \sum_{t=0}^{\infty} (1-\alpha)^t \left(\frac{s}{\|s\|_1} \mathbf{W}_n^t \right)(u') \\
&= \|s\|_1 \alpha \sum_{t=1}^{\infty} (1-\alpha)^t \left(\frac{s}{\|s\|_1} \mathbf{W}_n^t \right)(u') \\
&\leq \|s\|_1 \frac{1}{\widetilde{D}_{\min}}.
\end{aligned} \tag{A.34}
$$

Finally, letting $q_t = e_v(\mathbf{W}_n \widetilde{\mathbf{I}}_n)^t$ for ease of notation, we have

$$
\begin{aligned}
\|s_t\|_1 &= \|q_t(\mathbf{W}_n \widetilde{\mathbf{I}}_n)\|_1 \\
&= \sum_{u' \in \mathbf{X}} \sum_{u \in \mathbf{X}} q_t(u)(\mathbf{W}_n \widetilde{\mathbf{I}}_n)(u, u') \\
&= \sum_{u' \in \mathbf{X}/\mathcal{C}_\sigma[\mathbf{X}]} \sum_{u \in \mathcal{C}_\sigma[\mathbf{X}]} \frac{q_t(u)}{(\mathbf{D}_n)_{uu}} \mathbf{1}(e_{u,u'} \in G_{n,r}) \\
&= \sum_{u \in \mathcal{C}_\sigma[\mathbf{X}]} \frac{q(u)\left((\mathbf{D}_n)_{uu} - (\widetilde{\mathbf{D}}_n)_{uu} \right)}{(\mathbf{D}_n)_{uu}} \\
&= \|q_t(I - \mathbf{D}_n^{-1} \widetilde{\mathbf{D}}_n)\|_1 = \|\ell_t\|_1.
\end{aligned}
$$

and as a result $\|s\|_1 = \|\ell\|_1$. Combining with $\|\ell\|_1 \leq 2\frac{\widetilde{\Phi}_{n,r}}{\alpha}$ and (A.34) yields the desired upper bound.

$\square$

Lemma 28. *Let $\mathcal{C}_\sigma$ satisfy the conditions of Theorem 4. For $r < \sigma$, the following statements hold with probability tending to one as $n \to \infty$:*

$$
\begin{aligned}
D_{\min}(\mathcal{C}_\sigma[X]; \widetilde{G}_{n,r}) &\geq \frac{1}{2}\nu_d r^d \lambda_\sigma \\
D_{\max}(\mathcal{C}_\sigma[X]; \widetilde{G}_{n,r}) &\leq 2\nu_d r^d \Lambda_\sigma \\
\widetilde{\mathrm{vol}}_{n,r}(\widetilde{G}_{n,r}) &\leq 2\nu(\mathcal{C}_\sigma)\Lambda_\sigma
\end{aligned}
$$

where $D_{\min}(\mathcal{C}_\sigma[X]; \widetilde{G}_{n,r})$ is the minimum degree of any vertex $v \in \mathcal{C}_\sigma[X]$ in the subgraph $\widetilde{G}_{n,r}$, and analogously for $D_{\max}(\mathcal{C}_\sigma[X]; \widetilde{G}_{n,r})$.

The statement follows immediately from Lemma 13.

Proof of Theorem 4 We note that by Theorems 1 and 2,

$$
\kappa_2(\mathcal{C}) \geq \frac{\Phi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}{\Psi_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])}.
$$

As a result Lemma 25 implies

$$
\begin{aligned}
p_u &\geq \frac{4}{5}\widetilde{\pi}_{n,r}(u) - \frac{18\kappa_2(\mathcal{C})}{\widetilde{D}_{\min}} \qquad (u \in \mathcal{C}[\mathbf{X}]) \\
p_{u'} &\leq \frac{18\kappa_2(\mathcal{C})}{\widetilde{D}_{\min}} \qquad (u' \in \mathcal{C}'[\mathbf{X}])
\end{aligned}
\tag{A.35}
$$

We turn to bounding $\widetilde{\pi}_{n,r}(u)$. Clearly,

$$
\begin{aligned}
\widetilde{\pi}_{n,r}(u) &\geq \frac{\widetilde{D}_{\min}}{\widetilde{\mathrm{vol}}_{n,r}(\mathcal{C}_\sigma[\mathbf{X}])} \\
&\geq \frac{\widetilde{D}_{\min}}{\widetilde{n}\widetilde{D}_{\max}}.
\end{aligned}
$$

Application of Lemma 28 then yields

$$
\widetilde{\pi}_{n,r}(u) \geq 8\frac{\lambda_\sigma}{\nu(\mathcal{C}_\sigma)\Lambda_\sigma^2} \tag{A.36}
$$

as well as

$$
\frac{1}{\widetilde{D}_{\min}} \leq \frac{2}{\nu_d r^d \lambda_\sigma} \tag{A.37}
$$

with probability tending to 1 as $n \to \infty$, for all $u \in \mathcal{C}[\mathbf{X}]$ (indeed, all $u \in \mathcal{C}_\sigma[\mathbf{X}]$.)

Combining (A.35), (A.36) and (A.37), along with the requirement on $\kappa_2(\mathcal{C})$ given by (14), we have

$$p_u \geq 3/5 \frac{\lambda_\sigma}{\nu(\mathcal{C}_\sigma)\Lambda_\sigma^2}$$
$$p_{u'} \leq 1/5 \frac{\lambda_\sigma}{\nu(\mathcal{C}_\sigma)\Lambda_\sigma^2}$$

for any $u \in \mathcal{C}$, $u' \in \mathcal{C}'$. As a result, if $\pi_0 \in (2/5, 3/5) \cdot \frac{\lambda_\sigma}{\nu(\mathcal{C}_\sigma)\Lambda_\sigma^2}$, as $n \to \infty$ with probability tending to one any sweep cut of the form of (4), including the output set $\widehat{C}$, will successfully recover $\mathcal{C}$ in the sense of (5).

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