Notes on "Geometric Random Walks: A Survey"

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Let $K \subset \mathbb{R}^d$ be a (convex) set and $\{P_u : u \in K\}$ be the transition probability density functions for a Markov chain with stationary distribution Q.

Definition 0.1 (Conductance)**.** Let $\phi(A)$, given by

$$\Phi(A) = \int_A P_u(K \setminus A) dQ(u), \ \phi(A) = \frac{\Phi(A)}{\min\{Q(A), Q(K \setminus A)\}}$$

be the normalized cut of A, and ϕ_s and ϕ, given by

$$\phi_s = \min_{A: s < Q(A) \leq \frac{1}{2}} \frac{\Phi(A)}{Q(A) - s}, \ \phi = \min_{A: 0 < Q(A) \leq 1/2} \frac{\Phi(A)}{Q(A)}$$

be the conductance profile and conductance, respectively.

Let $\ell(u) = 1 - P_u(\{u\})$ be the local conductance.

Lemma 1 (One-step distributions of nearby points)**.** *Let u, v be such that $|u - v| \leq \frac{t\delta}{\sqrt{d}}$ and $\ell(u), \ell(v) \geq \ell$. Then,*

$$\|P_u - P_v\|_{TV} \leq 1 + t - \ell$$

Theorem 1 (Conductance of Ball Walk)**.** *Let $K \subset \mathbb{R}^d$ be a convex body of diameter D such that, for every point $u \in K$, the local conductance of the ball walk with δ-steps is at least ℓ. Then,*

$$\phi \geq \frac{\ell^2 \delta}{16\sqrt{d} D}$$

Proof. We will show that for $S_1 \cup S_2 = K$ a partition into measurable sets,

$$\int_{S_1} P_x(S_2) dx \geq \frac{\ell^2 \delta}{16\sqrt{d} D} \min\{\text{vol}(S_1), \text{vol}(S_2)\}$$

Note that

$$\int_{S_1} P_x(S_2)dx = \int_{S_1} \left(\frac{\int_{S_2} \mathbf{1}(|x-x'| \le \delta)}{\int_K \mathbf{1}(|x-x'| \le \delta)} dx' dx \right)$$
$$= \frac{1}{\int_K \mathbf{1}(|x-x'| \le \delta)} \int_{S_2} \int_{S_1} \mathbf{1}(|x-x'| \le \delta) dx dx'$$
$$= \int_{S_2} P_{x'}(S_1) dx' \tag{1}$$

Now, write

$$S_1' = \left\{ x \in S_1 : P_x(S_2) \le \frac{\ell}{4} \right\}, \; S_2' = \left\{ x \in S_2 : P_x(S_1) \le \frac{\ell}{4} \right\}.$$

and note that

$$\int_{S_1} P_x(S_2) \ge \mathrm{vol}(S_1') \frac{\ell}{4}, \; \int_{S_2} P_x(S_1) \ge \mathrm{vol}(S_2') \frac{\ell}{4}$$

Therefore, if $\mathrm{vol}(S_1') \ge \frac{\mathrm{vol}(S_1)}{2}$, we have $\int_{S_1} P_x(S_2) \ge \mathrm{vol}(S_1)\frac{\ell}{8}$; moreover, by (1), if $\mathrm{vol}(S_2') \ge \frac{\mathrm{vol}(S_2)}{2}$ then $\int_{S_1} P_x(S_2) \ge \mathrm{vol}(S_2)\frac{\ell}{8}$, and under either case the desired result holds.

We proceed under the conditions $\mathrm{vol}(S_1') \le \frac{\mathrm{vol}(S_1)}{2}, \mathrm{vol}(S_2') \le \frac{\mathrm{vol}(S_2)}{2}$. Letting $S_3' = K \setminus (S_1' \cup S_2')$, we recall that for any such tripartition $R_1 \cup R_2 \cup R_3 = K$, we have (see Dyer and Frieze)

$$\mathrm{vol}(R_3) \ge \frac{2d(R_1, R_2)}{D} \min \{\mathrm{vol}(R_1), \mathrm{vol}(R_2)\}.$$

and therefore, given our choice of S_1', S_2', S_3',

$$\mathrm{vol}(S_3') \ge \frac{d(S_1, S_2)}{D} \min \{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}.$$

We upper bound $d(S_1, S_2)$ using Lemma 1. Pick arbitrary $u \in S_1', v \in S_2'$. Noting that,

$$\|P_u - P_v\|_{TV} \ge 1 - P_u(S_2) - P_v(S_1) \ge 1 - \frac{\ell}{2}$$

by Lemma 1

$$|u - v| \ge \frac{\ell\delta}{2\sqrt{d}}.$$

Since $u \in S_1', v \in S_2'$ were arbitrary, we have $d(S_1', S_2') \ge \ell\delta/2\sqrt{d}$, and therefore

$$\mathrm{vol}(S_3') \ge \frac{\ell\delta}{2\sqrt{d}D} \min \{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\} \tag{2}$$

Finally, if $x \in S_1 \cup S_3'$ then $P_x(S_2) \geq \ell/4$, and conversely if $x \in S_2 \cup S_3'$ then $P_x(S_1) \geq \ell/4$. As a result, we have

$$\begin{aligned} 2\int_{S_1} P_x(S_2)dx &= \int_{S_2} P_x(S_1)dx + \int_{S_1} P_x(S_2)dx \\ &\geq \frac{\ell \mathrm{vol}(S_3')}{4} \end{aligned}$$

and combining this with (2) gives the desired result. □