Notes for the week of 3/13/19 - 3/17/19

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For a given $\sigma > 0$ and some $\mathcal{C} \subset \mathbb{R}^d$, let $\mathcal{C}_\sigma = \mathcal{C} + B(0,\sigma)$ be the σ-expansion of $\mathcal{C}$. Fix $r > 0$. Let ν be the Lebesgue measure over Euclidean space $\mathbb{R}^d$, and $B(x,r)$ be a ball of radius r centered at x. Consider the *speedy r-ball walk*[1] over $\mathcal{C}_\sigma \subset \mathbb{R}^d$, defined by the following transition probability density function

$$\widetilde{P}_{\nu,r}(x;\mathcal{S}) := \frac{\nu(\mathcal{S} \cap B(x,r))}{\nu(\mathcal{C}_\sigma \cap B(x,r))} \qquad (x \in \mathcal{C}_\sigma, \mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

where $\mathfrak{B}(\mathcal{C}_\sigma)$ is the Borel σ-algebra of $\mathcal{C}_\sigma$.

Denote the stationary distribution for this Markov chain by $\pi_{\nu,r}$, which satisfies the relation [2]

$$\int_\Omega \widetilde{P}_{\nu,r}(x;\mathcal{S}) d\pi_{\nu,r}(x) = \pi_{\nu,r}(\mathcal{S}). \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

Letting the *local conductance* be given by

$$\ell_{\nu,r}(x) := \frac{\nu(\mathcal{C}_\sigma \cap B(x,r))}{\nu(B(x,r))} \qquad (x \in \mathcal{C}_\sigma)$$

a bit of algebra verifies that

$$\pi_{\nu,r}(\mathcal{S}) = \frac{\int_{\mathcal{S}} \ell_{\nu,r}(x)}{\int_{\mathcal{C}_\sigma} \ell_{\nu,r}(x)}. \qquad (\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma))$$

We next introduce the *ergodic flow*, $\widetilde{Q}_{\nu,r}$, defined by

$$\widetilde{Q}_{\nu,r}(\mathcal{S},\mathcal{S}') := \int_{\mathcal{S}} \widetilde{P}_{\nu,r}(x;\mathcal{S}) d\pi_{\nu,r}(x) \qquad (\mathcal{S},\mathcal{S}' \in \mathfrak{B}(\mathcal{C}_\sigma))$$

and the *(continuous) conductance function*

$$\widetilde{\Phi}_{\nu,r}(t) := \min_{\substack{\mathcal{S} \in \mathfrak{B}(\mathcal{C}_\sigma) \\ 0 < \pi_{\nu,r}(\mathcal{S}) \leq t}} \frac{\widetilde{Q}_{\nu,r}(\mathcal{S}, \mathcal{C}_\sigma \setminus \mathcal{S})}{\pi_{\nu,r}(\mathcal{S})} \qquad (0 < t \leq 1/2)$$

1 Conductance over $\mathcal{C}_\sigma$

An essential step in upper bounding the mixing time over $G_{n,r}[\mathcal{C}_\sigma(\mathbf{X})]$ is lower bounding the conductance function $\widetilde{\Phi}_{\nu,r}(t)$.

To do so, our main assumption will relate $\mathcal{C}_\sigma$ to a convex set via a Lipschitz transformation g.

[1] We call it 'speedy' because it only considers moves within $\mathcal{C}_\sigma$.

[2] As we will see, in this case the existence of a stationary distribution for the ball walk will be easily verifiable. In order to ensure uniqueness, we could consider only the *lazy* version of the ball walk. For the moment we ignore this technicality.

Assumption 1 (Embedding)**.** *Assume there exists $K \subset \mathbb{R}^d$ convex space, and biLipschitz measure preserving mapping $g : \mathbb{R}^d \to \mathbb{R}^d$:*

$$\exists L_\Omega, L_K > 0 : \forall x, y \in K, \frac{1}{L_K} |x - y| \leq |g(x) - g(y)| \leq L_{\mathcal{C}_\sigma} |x - y|, \det(D_x g) = 1$$

such that

$$\mathcal{C}_\sigma = g(K).$$

Theorem 1 (Uniform continuous conductance function)**.** *Assume $\mathcal{C}_\sigma \subset \mathbb{R}^d$ satisfies Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and biLipschitz function g with Lipschitz constants $L_{\mathcal{C}_\sigma}, L_K < \infty$. Then, for any $0 < r < 2\sigma/\sqrt{d}$, the continuous conductance function of the speedy r-ball walk satisfies*

$$\widetilde{\Phi}_{\nu,r}(t) \geq \frac{r}{2^{10} D_K L \sqrt{d}}.$$

2 Supporting theory.

Begin by recalling the isoperimetric inequality of Dyer and Frieze 1991.

Theorem 2 (Isoperimetry of convex sets)**.** *Let (R_1, R_2, R_3) be a partition of a convex set $\Omega \subset \mathbb{R}^d$. Then,*

$$\mathrm{vol}(R_3) \geq 2\frac{d(R_1, R_2)}{D_K} \min(\mathrm{vol}(R_1), \mathrm{vol}(R_2))$$

The following result is from AbbasiYadkori18. It is an adaptation of Theorem 2 to hold in the case where $\Omega \subset \mathbb{R}^d$ is not convex, but is a Lipschitz embedding of a convex set in the sense of Assumption 1.

Lemma 1 (Isoperimetry of Lipschitz embeddings of convex sets.)**.** *Let $\Omega \subset \mathbb{R}^d$ satisfy Assumption 1 with respect to some convex set $K \subset \mathbb{R}^d$ and Lipschitz function g with Lipschitz constant $L < \infty$. Then, for any partition $(\Omega_1, \Omega_2, \Omega_3)$ of Ω,*

$$\mathrm{vol}(\Omega_3) \geq 2\frac{\mathrm{dist}(\Omega_1, \Omega_2)}{L D_K} \min(\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2))$$

Proof. For $\Omega_i, i = 1, 2, 3$, denote the preimage

$$R_i = \{x \in K : g(x) \in \Omega_i\}$$

For any $x \in R_1, y \in R_2$,

$$|x - y| \geq \frac{1}{L} |g(x) - g(y)| \geq \frac{1}{L} \mathrm{dist}(\Omega_1, \Omega_2).$$

Since $x \in \Omega_1$ and $y \in \Omega_2$ were arbitrary, we have

$$\mathrm{dist}(R_1, R_2) \geq \frac{1}{L} d(\Omega_1, \Omega_2).$$

By Theorem 2, therefore

$$\begin{aligned}\mathrm{vol}(R_3) &\geq 2\frac{\mathrm{dist}(R_1, R_2)}{D_K} \min(\mathrm{vol}(R_1), \mathrm{vol}(R_2)) \\ &\geq \frac{2}{D_K L} d(\Omega_1, \Omega_2) \min(\mathrm{vol}(R_1), \mathrm{vol}(R_2))\end{aligned}$$

and by the measure-preserving property of g, this implies

$$\mathrm{vol}(\Omega_3) \geq \frac{2}{D_K L} d(\Omega_1, \Omega_2) \min(\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2)).$$

□

Lemma 2 (One-step distributions)**.** *Let $u, v \in \mathcal{C}_\sigma$ be such that*

$$|u - v| \leq \frac{\sqrt{t}r}{L_K L_{\mathcal{C}_\sigma} \sqrt{d}}$$

for some $0 < t < 4/9$. Then,

$$\left\| \widetilde{P}_{\nu,r}(u;\cdot) - \widetilde{P}_{\nu,r}(v;\cdot) \right\|_{TV} \leq 1 - \frac{1}{300(9+8t)L_{C_\sigma}^d L_K^d}.$$

Proof of Lemma 2. Let $S_1 \cup S_2 = \mathcal{C}_\sigma$ be an arbitrary partition of $\mathcal{C}_\sigma$. We will show that

$$\widetilde{P}_{\nu,r}(u;S_1) - \widetilde{P}_{\nu,r}(v;S_1) \leq 1 - \frac{1}{300(9+8t)L_{C_\sigma}^d L_K^d}$$

The following manipulations reduce the problem to that of lower bounding the volume of the intersection of the balls $B(u,r)$ and $B(v,r)$ within $\mathcal{C}_\sigma$. They come directly from the proof of Lemma 3.6 in Kannan 97.

$$\widetilde{P}_{\nu,r}(u;S_1) - \widetilde{P}_{\nu,r}(v;S_1) = 1 - \widetilde{P}_{\nu,r}(u;S_2) - \widetilde{P}_{\nu,r}(v;S_1)$$

Denote the intersection $I := B(u,r) \cap B(u,r)$. Then we have

$$\widetilde{P}_{\nu,r}(u;S_2) \geq \frac{1}{\nu(B(u,r))}\nu(S_2 \cap (B(u,r)) \geq \frac{1}{\nu(B(u,r))}\nu(S_2 \cap I)$$

with a symmetric inequality holding for $\widetilde{P}_{\nu,r}(v;S_1)$. As a result,

$$1 - \widetilde{P}_{\nu,r}(u;S_2) - \widetilde{P}_{\nu,r}(v;S_1) \geq \frac{1}{\nu_d r^d}\nu(\mathcal{C}_\sigma \cap I) \tag{1}$$

For shorthand, let $\widetilde{\nu}(\mathcal{S}) = \nu(\mathcal{S} \cap \mathcal{C}_\sigma)$. We proceed, making repeated use of Assumption 1 along with Lemma 3,

$$\begin{aligned}
\widetilde{\nu}\Big(B(u,r) \cap B(v,r)\Big) &\geq \nu\left(g\Big(B(x, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap B(y, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big)\right) \\
&= \nu\Big(B(x, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap B(y, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big) \\
&\geq \min\left\{\nu\Big(B(x, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big), \nu\Big(B(y, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big)\right\} \cdot \frac{3}{9+8t} \qquad (2) \\
&\geq \min\left\{\widetilde{\nu}\Big(B(u, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big), \widetilde{\nu}\Big(B(v, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big)\right\} \cdot \frac{3}{9+8t} \\
&\geq \min\left\{\nu\Big(B(u, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big), \nu\Big(B(v, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big)\right\} \cdot \frac{1}{10(9+8t)} \qquad (3)
\end{aligned}$$

where (2) follows from Lemma 8 and (3) from Lemma 9. Plugging (3) back into (1) – and noting that Lemma 9 implies $\ell \geq \frac{1}{30}$ – we have

$$\widetilde{P}_{\nu,r}(u,A) - \widetilde{P}_{\nu,r}(v,A) \leq 1 - \frac{1}{300(9+8t)L_{C_\sigma}^d L_K^d}.$$

Since this holds for any $A \subset \mathcal{C}_\sigma$, it holds over the supremum over all such A, therefore the desired statement is shown. $\square$

Lemma 3 (Lipschitz balls and local conductance.). *Under the assumption(s) of Theorem 2, for $x, y \in K$ and $u = g(x), v = g(y)$ such that $|u - v| \leq \frac{rt}{\sqrt{d}}$*

- $B(u,r) \cap B(v,r) \cap \mathcal{C}_\sigma \supseteq g\Big(B(x, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap B(y, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big)$
- $|x - y| \leq \frac{tr}{L_K\sqrt{d}}$.
- $\nu\Big(B(x, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big) \geq \widetilde{\nu}\Big(B(u, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big)$ *and similarly* $\nu\Big(B(y, \frac{r}{L_{\mathcal{C}_\sigma}}) \cap K\Big) \geq \widetilde{\nu}\Big(B(v, \frac{r}{L_{\mathcal{C}_\sigma} L_K})\Big)$.

Proof. See page 18 of handwritten notes. □

Lemma 4 (One-step distributions over convex sets.). *Let $K \subset \mathbb{R}^d$ be a convex set, and $u, v \in K$, be such that $|u - v| \leq \frac{tr}{\sqrt{d}}$ and $\ell(u), \ell(v) \leq \ell$. Then,*

$$\|P_{\nu,r}(x;\cdot) - P_{\nu,r}(y;\cdot)\|_{TV} \leq 1 + t - \ell$$

where $P_{\nu,r}(x; A) = \frac{\nu(B(x,r)\cap A)}{\nu(B(x,r)\cap K)}$.

3 Proof of (2)

This section is closely related to the results of Kannan 97. We will build, through several lemmas, to (2). Some preliminary notation: Fix $x, y \in K$, and denote $r' = r/L_K$. Define

$$C = B(x, r') \cap B(y, r')$$

the 'moons'

$$M_x = B(x, r') \setminus B(y, r'),\ M_y = B(y, r') \setminus B(x, y')$$

and set

$$R_x = M_x \cap (x - y + C),\ R_y = M_y \cap (y - x \cap C).$$

Lemma 5. *Let x, y be two points in K such that $|x - y| < \sqrt{t}r'/\sqrt{d}$. Let C' be the blowup of C around its center $\frac{1}{2}(x + y)$ by $\alpha^{-1} := \frac{4d+3t}{4d-t}$. Then*

$$M_x \setminus R_x \subseteq C'$$

Proof. Assume without loss of generality that $x = -y$, and let $z \in M_x \setminus R_x$. Write $z = \mu x + w$ where $w \perp x$. Then,

- $|z - x| \leq r'$, since $z \in B(x, r')$.
- $|z - y| > r'$, since $z \in B(y, r')$.
- $|z - 3x| > r'$, since $z \notin R_x$ means that either $|z - 3x| > r'$ or $|z - x| > r'$, and we know that second inequality will never hold.

As a result, we have that $\mu \in (0, 2)$.

Now, if $|\alpha z - y| \leq \delta$, this would imply $\alpha z \in C$, and therefore $z \in C'$. We do some straightforward, if tedious, algebra to obtain the desired result:

$$\begin{aligned}
|\alpha z - y|^2 &= |\alpha\mu x + \alpha w + x|^2 \\
&= (\alpha\mu + 1)^2 |x|^2 + \alpha^2 |w|^2 \\
&\leq (\alpha\mu + 1)^2 |x|^2 + \alpha^2((r')^2 - (\mu - 1)^2 |x|^2) \\
&= (\alpha\mu + 1)^2 \frac{t(r'^2)}{d} + \alpha^2(\mu - 1)^2 \frac{t(r'^2)}{d} + \alpha^2 (r')^2 \\
&= (r')^2 \frac{t}{d} \left((4\frac{d}{t} + 3)\alpha^2 + 4\alpha + 1 \right) \\
&= (r')^2
\end{aligned}$$

where the last line follows from our choice of α. □

Lemma 6. *Under the notation and conditions of Lemma 5, we have*

$$\mathrm{vol}(K \cap (M_x \setminus R_x)) \leq (1 + \frac{8t}{3})\mathrm{vol}(K \cap C)$$

whenever $0 < t < 4/9$.

Proof. From Lemma 5, we have

$$\begin{aligned}
\mathrm{vol}(K \cap (M_x \setminus R_x)) &\leq \mathrm{vol}(K \cap C') \\
&\leq \mathrm{vol}((\alpha^{-1})(K \cap C)) \\
&= (1 + \frac{4t}{4d - t})^d \mathrm{vol}(K \cap C) \\
&\overset{(i)}{\leq} \left(1 + \frac{8td}{4d - t}\right) \mathrm{vol}(K \cap C) \\
&\leq (1 + \frac{8t}{3})\mathrm{vol}(K \cap C)
\end{aligned}$$

where (i) follows from a first order Taylor expansion of $(1 + x)^d$ about $x = 0$. □

The following lemma is taken directly from Kannan 97.

Lemma 7. *For every convex body K,*

$$\mathrm{vol}(K \cap C)^2 \geq \mathrm{vol}(K \cap R_x)\mathrm{vol}(K \cap R_y)$$

Lemma 8. *Under the notation and conditions of Lemma 5*

$$\mathrm{vol}(K \cap C) \geq \frac{3}{9 + 8t} \min\left\{\mathrm{vol}(B(x, r') \cap K), \mathrm{vol}(B(y, r'))\right\}$$

Proof. From Lemma 6, we have

$$\mathrm{vol}(K \cap R_x) \geq \mathrm{vol}(K \cap M_x) - (1 + \frac{8t}{3})\mathrm{vol}(K \cap C)$$

which, by the identity $B(x, r') = C \cup M_x$, further implies

$$\mathrm{vol}(K \cap R_x) \geq \mathrm{vol}(K \cap B(x, r')) - (2 + \frac{8t}{3})\mathrm{vol}(K \cap C)$$

with a symmetric inequality holding fro $K \cap R_y$. Applying Lemma 7 we obtain

$$\mathrm{vol}(K \cap C) \geq \min\left\{\mathrm{vol}(K \cap B(x, r')), \mathrm{vol}(K \cap B(y, r'))\right\} - (2 + \frac{8t}{3})\mathrm{vol}(K \cap C)$$

and the desired result follows after some rearrangement. □

4 Proof of (3)

Lemma 9. *Let $u \in \mathcal{C}_\sigma = \mathcal{C} + \sigma B$ for some $\mathcal{C} \subseteq \mathbb{R}^d$. Then, for any $r' < \frac{\sigma}{4d}$,*

$$\nu(B(u, r') \cap \mathcal{C}_\sigma) \geq \nu(B(u, r'))\frac{1}{30}.$$

Proof. Since $u \in \mathcal{C}_\sigma$ there exists $v \in \mathcal{C}$ such that $u \in B(v, \sigma)$. Writing $r' := \frac{r}{L_{\mathcal{C}_\sigma} L_K}$, we have that

$$\widetilde{\nu}\big(B(u, r')\big) \geq \nu\big(B(u, r') \cap B(v, \sigma)\big)$$

The volume of such an intersection is clearly minimized when $|u - v| = \sigma$; in this case the intersection is formed by the union of two spherical caps. We will examine the larger of these two spherical caps, the cap of radius r and height

$$h = r' - (r')^2/2\sigma = r'\left(1 - \frac{r'}{2\sigma}\right)$$

Then, the volume of the cap ν_{cap} is known to be

$$\nu_{cap} = \frac{1}{2}\nu_d r^d I_{1-\alpha}(\frac{d+1}{2}; \frac{1}{2})$$

where

$$\alpha := 1 - \frac{2r'h - h^2}{(r')^2} \leq \frac{r'}{2\sigma}$$

and $I(\cdot;\cdot)$ represents the incomplete beta function:

$$I_\alpha(z, w) = \frac{\Gamma(z + w)}{\Gamma(z)\Gamma(w)} \int_0^\alpha u^{z-1}(1 - u)^{w-1} du.$$

Therefore,

$$\begin{aligned}
\nu_{cap} &= \frac{1}{2}\nu_d(r')^d \frac{\Gamma(d/2 + 1)}{\Gamma((d+1)/2)\Gamma(1/2)} \int_0^\alpha u^{(d-1)/2}(1 - u)^{-1/2} du \\
&\geq \nu_d(r')^d \frac{1}{2\sqrt{\pi}} \frac{\Gamma(d/2 + 1)}{\Gamma((d+1)/2)} \int_0^\alpha u^{(d-1)/2}(1 - u)^{-1/2} du && (\Gamma(1/2) = \sqrt{\pi}) \\
&\geq \nu_d(r')^d \frac{1}{2\sqrt{\pi}} \sqrt{\frac{d}{2}} \int_0^\alpha u^{(d-1)/2}(1 - u)^{-1/2} du. && \text{(Gautschi's inequality)}
\end{aligned}$$

Turning our attention to the relevant integral, letting $v = 1 - u$ and $\beta = 1/4d$ we obtain

$$\begin{aligned}
\int_0^\alpha u^{(d-1)/2}(1 - u)^{-1/2} du &= \int_\alpha^1 (1 - v)^{(d-1)/2} v^{-1/2} dv \\
&\geq \int_\beta^{2\beta} (1 - v)^{(d-1)/2} v^{-1/2} dv && (\alpha \leq \beta \leq 2\beta \leq 1) \\
&\geq 2(1 - 2\beta)^{(d-1)/2}\sqrt{\beta}(\sqrt{2} - 1) \\
&\geq \frac{1}{2}\sqrt{\frac{1}{d}}(\sqrt{2} - 1). && \text{(Lemma 2)}
\end{aligned}$$

Combining the pieces, we have

$$\nu_{cap} \geq \nu_d (r')^d \frac{(\sqrt{2}-1)}{2} \cdot \frac{1}{2\sqrt{2\pi}} \geq \frac{1}{30} \nu_d (r')^d.$$

$\square$

5 Notation

- For a set $K \subset \mathbb{R}^d$, $D_K = \max_{x,y \in K} |x-y|$, where $|x-y|$ is the Euclidean norm between of $x-y \in \mathbb{R}^d$.
- ν_d is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^d$.
- $D_x g = (D_{x_i} g_j)_{i,j=1}^d$ is the Jacobian matrix of g evaluated at x.
- $g(K) = \left\{ y \in \mathbb{R}^d : g(x) = y \text{ for some } x \in K \right\}$ is the image of K under g.
- For measures P, Q over $(\Sigma, \mathcal{F})$, $\|P-Q\|_{TV} = \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$.