Notes on "Fast Mixing Random Walks and Regularity of Incompressible Vector Fields"

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Begin by recalling the isoperimetric inequality of Dyer and Frieze 1991.

Theorem 1 (Isoperimetry of convex sets)**.** *Let $(\Omega_1, \Omega_2, \Omega_3)$ be a partition of a convex set Ω with unit volume. Then,*

$$\mathrm{vol}(\Omega_3) \geq 2\frac{d(\Omega_1, \Omega_2)}{D_\Omega} \min(\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2))$$

Here vol denotes d-dimensional volume, D_Ω denotes the diameter of Ω given by $D_\Omega = \max_{x,y \in \Omega} |x - y|$ where $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{R}^d$, and $d(\Omega_1, \Omega_2) = \min_{x \in \Omega_1, y \in \Omega_2} |x - y|$.

Assumption 1 (Embedding)**.** *Let Ω be a convex space with boundary $\partial\Omega$, and let Ω' be a bounded connected subset of $\mathbb{R}^d$. Assume Ω' is the image of Ω under a Lipschitz measure preserving mapping $g : \mathbb{R}^d \to \mathbb{R}^d$:*

$$\exists L_{\Omega'} > 0 : \forall x, y \in \Omega, |g(x) - g(y)| \leq L_{\Omega'} |x - y|, \det(D_x g) = 1$$

where $D_x g = (D_{x_i} g_j)_{i,j=1}^d$ is the Jacobian matrix of g evaluated at x.

Lemma 1 (Isoperimetry of non-convex sets.)**.** *Let Ω be a convex set with unit volume, and assume Ω' satisfies Assumption 1 with respect to $L_{\Omega'} > 0$. Then, for any partition $(\Omega'_1, \Omega'_2, \Omega'_3)$,*

$$\mathrm{vol}(\Omega'_3) \geq \frac{2}{D_\Omega L_{\Omega'}} d(\Omega'_1, \Omega'_2) \min\{\mathrm{vol}(\Omega'_1), \mathrm{vol}(\Omega'_2)\}$$

Proof. For $\Omega'_i, i = 1, 2, 3$, define the pre-image

$$\Omega_i = \{x \in \Omega : g(x) \in \Omega_i\}$$

where $g : \Omega \to \Omega'$ is a $L_{\Omega'}$-Lipschitz measure preserving mapping. For any $x_1 \in \Omega_1, x_2 \in \Omega_2$,

$$|x - y| \geq \frac{1}{L_{\Omega'}} |g(x) - g(y)| \geq \frac{1}{L_{\Omega'}} d(\Omega'_1, \Omega'_2).$$

Since $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ were arbitrary, we have

$$d(\Omega_1, \Omega_2) \geq \frac{1}{L_{\Omega'}} d(\Omega'_1, \Omega'_2).$$

By Theorem 1,

$$\begin{aligned}
\mathrm{vol}(\Omega_3) &\geq 2\frac{d(\Omega_1, \Omega_2)}{D_\Omega} \min(\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2)) \\
&\geq \frac{2}{D_\Omega L_{\Omega'}} d(\Omega'_1, \Omega'_2) \min(\mathrm{vol}(\Omega_1), \mathrm{vol}(\Omega_2))
\end{aligned}$$

and by the measure-preserving property of g, this implies

$$\mathrm{vol}(\Omega'_3) \geq \frac{2}{D_\Omega L_{\Omega'}} d(\Omega'_1, \Omega'_2) \min(\mathrm{vol}(\Omega'_1), \mathrm{vol}(\Omega'_2))$$

□