Quantum-Realizability of Empirical Models

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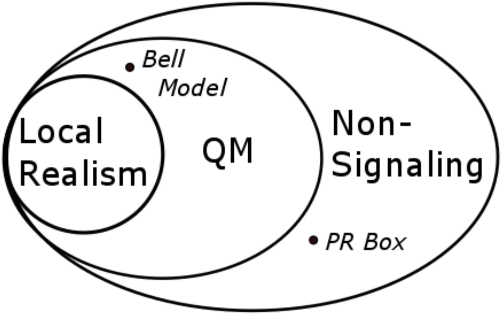
Relevant Questions

- What does it mean for a system to have classical correlation?
- What does it mean for a system to have quantum correlation and what limitations are there?
- Are there entangled states that act classically?



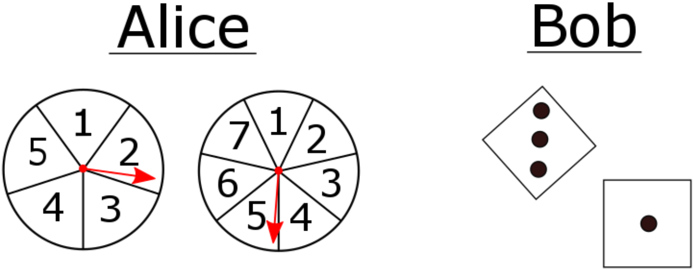
(2,2,2) Model

$$
\begin{tikzcd}
 & \rho \arrow[dl, "\rho_A"'] \arrow[dr, "\rho_B"] & \\
\mathsf{A} \arrow[d, "Q{,}R"'] & & \mathsf{B} \arrow[d, "S{,}T"] \\
\lambda = \pm 1 & & \lambda = \pm 1
\end{tikzcd}
$$

- State is distributed between two parties
- Each have a choice between two measurements which they make on the system simultaneously
- Each prospective measurement results in one of two possible outcomes

Classical Example

- Alice can choose between two spinners labeled 1-5 and 1-7
- Bob chooses between one fair die and one weighted to land on an even number 2/3 of the time

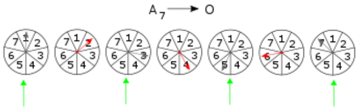


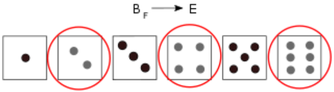
Classical Example cont.

Chances of each simultaneous roll of the die by Bob and spin of the spinner by Alice (or simultaneous 'measurement') being even and even, even and odd, etc., is described by the table:

	E,E	E,O	O,E	O,O
A_5, B_F	$1/5$	$1/5$	$3/10$	$3/10$
A_5, B_U	$4/15$	$2/15$	$6/15$	$1/5$
A_7, B_F	$3/14$	$3/14$	$4/14$	$4/14$
A_7, B_U	$6/21$	$3/21$	$8/21$	$4/21$

This table of probabilities is an example of an empirical model. Where the entries can be read as such:





$$P_{A_7,B_F \to O,E} = \quad 4/7 \quad \cdot \quad 1/2 \quad = 4/14$$

No-signalling

Any model that is classical or quantum-realizable must obey the no signaling condition, which states that the probability of a measurement having a certain outcome must be the same regardless of its context.

	E,E	E,O	O,E	O,O
A_5, B_F	1/5	1/5	3/10	3/10
A_5, B_U	4/15	2/15	6/15	1/5

For example, from the previous table,

$$P_{A_5 \to E} = P_{A_5, B_F \to E,E} + P_{A_5, B_F \to E,O} = P_{A_5, B_U \to E,E} + P_{A_5, B_U \to E,O} = 2/5$$

And for S,

	E,E	E,O	O,E	O,O
A_5, B_F	1/5	1/5	3/10	3/10
A_7, B_F	3/14	3/14	4/14	4/14

$$P_{B_F \to E} = P_{A_5, B_F \to E,E} + P_{A_5, B_F \to O,E} = P_{A_7, B_F \to E,E} + P_{A_7, B_F \to O,E} = 1/2$$

Global sections

A global section is the existence of probabilities for all measurements (i.e. Alice spinning both spinners and Bob rolling both dice) being done simultaneously.

For our classical example:

$A_5 A_7 B_F B_U$	P		$A_5 A_7 B_F B_U$	P
E,E,E,E	12/210		O,E,E,E	18/210
E,E,E,O	6/210		O,E,E,O	9/210
E,E,O,E	12/210		O,E,O,E	18/210
E,E,O,O	6/210		O,E,O,O	9/210
E,O,E,E	16/210		O,O,E,E	24/210
E,O,E,O	8/210		O,O,E,O	12/210
E,O,O,E	16/210		O,O,O,E	24/210
E,O,O,O	8/210		O,O,O,O	12/210

$$P_{A_7,B_F\to O,E} = P_{A_5,A_7,B_F,B_U\to E,O,E,E} + P_{A_5,A_7,B_F,B_U\to E,O,E,O} + P_{A_5,A_7,B_F,B_U\to O,O,E,E} + P_{A_5,A_7,B_F,B_U\to O,O,E,O} = 4/14$$

Classifying Models

- Any non-signaling model will have a global section over the reals
- If a model has a global section over the non-negative reals, it can be defined as classically-correlated

With this framework we can also look at models based on quantum states.

State $|00\rangle$

For the product state,

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

The value of each table entry may be described by

$$P_{x,y} = \frac{1}{4}(1 + (-1)^x u_3)(1 + (-1)^y v_3)$$

where u is either the vector associated with Q or R and v is for S or T. Thus the table will have the form

	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
Q,S	$(1+q_3)(1+s_3)$	$(1+q_3)(1-s_3)$	$(1-q_3)(1+s_3)$	$(1-q_3)(1-s_3)$
Q,T	$(1+q_3)(1+t_3)$	$(1+q_3)(1-t_3)$	$(1-q_3)(1+t_3)$	$(1-q_3)(1-t_3)$
R,S	$(1+r_3)(1+s_3)$	$(1+r_3)(1-s_3)$	$(1-r_3)(1+s_3)$	$(1-r_3)(1-s_3)$
R,T	$(1+r_3)(1+t_3)$	$(1+r_3)(1-t_3)$	$(1-r_3)(1+t_3)$	$(1-r_3)(1-t_3)$

With each entry being multiplied by a factor of $\frac{1}{4}$ so the rows sum to one

State $|00\rangle$ Conditions

The only conditions that must be met in order for the table to be realizable with the state $|00\rangle$ and basic measurements are:

- *No signalling condition*
- *Every row* $P_{00} \cdot P_{11} = P_{01} \cdot P_{10}$

State $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

For the maximally entangled two-qubit state,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

It can be shown that the table entries of an empirical model based off of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and basic measurements will have the form,

$$P_{x,y} = 1 + (-1)^{x+y}(u_1 v_1 - u_2 v_2 + u_3 v_3)$$

And thus will have a table of the form

	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
Q,S	$\frac{1}{4}(1+X_{qs})$	$\frac{1}{4}(1-X_{qs})$	$\frac{1}{4}(1-X_{qs})$	$\frac{1}{4}(1+X_{qs})$
Q,T	$\frac{1}{4}(1+X_{qt})$	$\frac{1}{4}(1-X_{qt})$	$\frac{1}{4}(1-X_{qt})$	$\frac{1}{4}(1+X_{qt})$
R,S	$\frac{1}{4}(1+X_{rs})$	$\frac{1}{4}(1-X_{rs})$	$\frac{1}{4}(1-X_{rs})$	$\frac{1}{4}(1+X_{rs})$
R,T	$\frac{1}{4}(1+X_{rt})$	$\frac{1}{4}(1-X_{rt})$	$\frac{1}{4}(1-X_{rt})$	$\frac{1}{4}(1+X_{rt})$

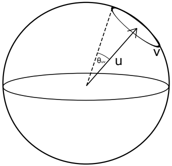
where $X_{uv} = u_1 v_1 - u_2 v_2 + u_3 v_3$

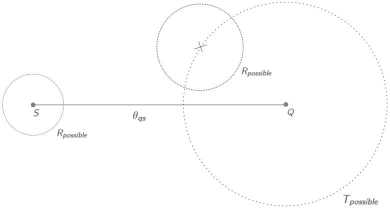
State $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ Conditions

X_{uv}'s can be taken as dot products of reflected vectors and thus used find an angle between the measurement vectors $\vec{u}$ and $\vec{v}$ from the table entries.

$$\theta_{uv} = cos^{-1}(X_{uv}) = cos^{-1}(4 * P_{00} - 1)$$

These angles must then meet a certain inequality, as the sum of the smaller angles cannot be less than the largest angle, otherwise the remaining measurement vector could not exist.





State $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ Conditions

The criteria that must be met in order for a table to be realizable with the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ are:

- *No signalling condition*
- *Every row* $P_{00} = P_{11}, P_{01} = P_{10}$
- $\theta_{QR} + \theta_{RS} + \theta_{QT} \geq \theta_{QS}$

(*Where* θ_{QS} *is the greatest angle*)

Werner State Investigation

Entangled states allow for non-classical correlations, a valid question then is whether there exist entangled states that act classically. We consider the state:

$$\rho_W = p \, |s\rangle \langle s| + (1 - p) \frac{\mathbb{I}}{4}$$

$$(|s\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle))$$

It can be shown via the PPT criterion that the Werner State, ρ_W, is definitively entangled for $p > \frac{1}{3}$. Further, by Tsirelson's bound it can be shown that ρ_W is definitively non-classical for $p \geq \frac{1}{\sqrt{2}}$.

Werner State Investigation

The probability of outcomes (x, y) associated with measurements U, V on the Werner State is:

$$P_{u,v \to x,y} = \frac{1}{4}(1 + p\,(xy\chi_{UV}))$$

Where $\chi_{UV} = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$, and x, y are either 1 or -1.

The Werner state has been shown to be classically correlated up to $p = \frac{1}{2}$, as there exists a global section over the non-negative reals for $p \leq \frac{1}{2}$:

$$P_{Q,R,S,T \to w,x,y,z}$$

$$= \frac{1}{16}(1 - p\,(\underbrace{wy\chi_{QS} + wz\chi_{QT} + xy\chi_{RS} + xz\chi_{RT} - wx\chi_{QR} - yz\chi_{ST}}_{(*)}))$$

It can be shown the quantity $(*)$ cannot exceed the value 2 and thus the entire value of $P_{Q,R,S,T}$ is non-negative for $p \leq \frac{1}{2}$. Thus, some entangled quantum states act classically regardless of the measurements chosen.

THANKS!

Thank You especially to Dr. Lyons, Dr. Walck, and you for your time!

Interested in applying or seeing other results?
go to
http://quantum.lvc.edu/mathphys