Simulating Multi-valued Grover Search in Cirq

Project Statement

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November 5, 2021

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- Relatively new generalizations of Grover's search algorithm apply to multi-valued functions [2][3]
- Google's Cirq SDK allows simulation of qudit circuits
- New and emerging field of many-valued qudit algorithms offers lots of space to work on novel projects

Binary Grover Search (Simple Overview)

$$|+\rangle^{\otimes 3} = \begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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Note that here we are omitting normalization constants and repeated iterations of the oracle and Grover diffusion operator

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- A natural generalization then is to encode several different 'types of solutions' in our 'database' and search for one
- These different 'types' can be naturally encoded by the dth roots of unity as opposed to just 1 and -1

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$$|+_d\rangle^{\otimes n} \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{d^n}} \sum_{k=0}^{d^n-1} \omega_d^{f(k)} |k\rangle$$

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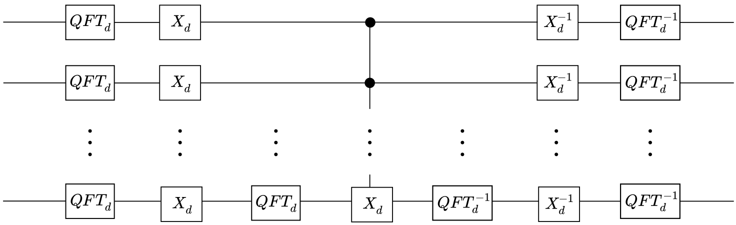
Maximilian Hunt and Samuel Hunt have recently published *Grovers Algorithm and Many-Valued Quantum Logic* (December 2020, [2]). They generalize the Grover diffusion operator to qudits and multi-valued functions using the circuit below:



For gates see `https://alexheilman.com/qis/qudits`

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$$\mathbf{Z_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}$$

Generalized Grover Search (Simple Overview)

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Again we omit normalization and gritty details

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The repository can be found at
`https://github.com/tuc56407/QuditScripts`

```
---------BEGIN-----------
Circuit:
0 (d=3): ───[QFT]───F───[QFT]───[X]────────────o─────[X-]─────[QFT-]───M('q0')─────────────
                    │                          │
1 (d=3): ───[QFT]───F───[QFT]───[X]───[QFT]───[X]───[QFT-]───[X-]─────[QFT-]─────M('q1')───
-------------------------
measurements: q0=2 q1=2
output vector: (-0.143-0.99j)|22⟩
-------------------------
Results:
q0=0010220222220221111212212020020211021010121222112002022000221012112212102020220101122111211101222002
q1=0111220002221221212201202100021222020101022201010110122201222102212200212120220100022001202000022110
-------------------------
state at step 0: [0.333+0.j 0.333+0.j 0.333+0.j 0.333+0.j 0.333+0.j 0.333+0.j 0.333+0.j
 0.333+0.j 0.333+0.j]
state at step 1: [ 0.333+0.j     0.333+0.j     0.333+0.j    -0.167-0.289j -0.167-0.289j
 -0.167-0.289j -0.167-0.289j -0.167-0.289j -0.167+0.289j]
state at step 2: [ 0.   -0.385j  0.167-0.096j -0.167-0.096j  0.667+0.192j -0.167-0.096j
  0.   +0.192j  0.333+0.192j -0.   +0.192j  0.167-0.096j]
state at step 3: [ 0.167-0.096j  0.333+0.192j -0.   +0.192j -0.167-0.096j  0.   -0.385j
  0.167-0.096j  0.   +0.192j  0.667+0.192j -0.167-0.096j]
state at step 4: [ 0.289+0.167j  0.   +0.j     0.   -0.333j  0.   -0.333j  0.   -0.j
 -0.289+0.167j  0.289+0.167j -0.289+0.5j    0.   -0.333j]
state at step 5: [ 0.289+0.167j  0.   +0.j     0.   -0.333j -0.289+0.167j  0.   -0.333j
  0.   -0.j     0.289+0.167j -0.289+0.5j    0.   -0.333j]
state at step 6: [-0.167-0.096j -0.333+0.192j -0.   +0.192j  0.   +0.192j  0.667+0.192j
 -0.167-0.096j  0.167-0.096j  0.333+0.192j  0.   +0.192j]
state at step 7: [ 0.385+0.333j -0.096+0.167j  0.   +0.j    -0.481-0.167j -0.096+0.167j
  0.   +0.j    -0.481+0.167j  0.192-0.j    -0.289-0.167j]
state at step 8: [0.   +0.612j 0.354+0.612j 0.177+0.306j 0.   +0.j    0.   +0.j
 0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j   ]
state at step 9: [0. +0.j    0.5+0.866j 0. +0.j    0. +0.j    0. +0.j    0. +0.j
 0. +0.j    0. +0.j    0. +0.j   ]
```

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- Multi-valued Grover search is recent development in field
- Progress in simulating/testing new work in Cirq

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What Hasn't Been Done?/Where to Go?

- The recent papers only generalize strict Grover search algorithm (i.e. oracles with only one relevant solution)
- Natural to then generalize multi-valued amplitude amplification (Grover search for multiple solutions)
- With this it would be required to generalize quantum counting to the multi-valued paradigm
- These together would effectively give an algorithm for evaluating and counting roots of multivariate polynomial over finite fields (as they can be encoded in these types of quantum states[1])

References I

[1] Paul Appel, Alexander J Heilman, Ezekiel W Wertz, et al. "Finite-Function-Encoding Quantum States". In: *arXiv preprint arXiv:2012.00490* (2020).

[2] Samuel Hunt and Maximilien Gadouleau. "Grover's Algorithm and Many-Valued Quantum Logic". In: *arXiv preprint arXiv:2001.06316* (2020).

[3] Yale Fan. "Applications of multi-valued quantum algorithms". In: *arXiv preprint arXiv:0809.0932* (2008).

Links for work in progress
Code: `https://github.com/tuc56407/QuditScripts`
Overview: `https://alexheilman.com/qis/qudits`