Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Weak derivatives. Recall that $f \in L^2([0,1])$ has a weak derivative $g \in L^2([0,1])$ if

$$\int_0^1 g(x)\phi(x)\,dx = -\int_0^1 f(x)\phi'(x)\,dx, \qquad \text{for every} \qquad \phi \in C_0^1([0,1]).$$

Here, $C_0^1([0,1])$ is the space of continuously differentiable functions with $u(0) = u(1) = 0$.

(a) Verify that if $f \in L^2([0,1])$ is continuously differentiable on $[0,1]$, its classical derivative is also a weak derivative.

(b) Show that the weak derivative of $f \in L^2([0,1])$ is unique in $L^2([0,1])$. That is, if $g \in L^2([0,1])$ and $h \in L^2([0,1])$ are both weak derivatives of f, then $\int_0^1 |g-h|^2 dx = 0$.

Hint: You may use the fact that $\int_0^1 |u(x)|^2 dx = 0$ if and only if $\int_0^1 u(x)\phi(x)\,dx = 0$ for every $\phi \in C_0^1([0,1])$. (If you want a challenge, try to prove this.)

(c) Verify that weak differentiation is a linear operation on $H^1([0,1])$ (the set of functions in $L^2([0,1])$ with square integrable weak derivatives).

(d) Establish a form of the *product rule* for weak derivatives. Show that if $g \in C_0^1([0,1])$ and $f \in H^1([0,1])$, then the weak derivative of their product satisfies $(fg)' = f'g + fg'$.

2) Boundary Conditions. Consider the first-order differential operator defined by

$$[Lu](x) = u'(x) + x\,u(x), \qquad u \in D(L) = \{g \in H^1([0,1]) \,|\, g(0) = 0.\}.$$

Note that the set $D(L)$, the domain of L, encodes a *boundary condition* for the solution.

(a) Calculate the adjoint of L, i.e., find an operator $L^\dagger$ and a domain $D(L^\dagger)$ such that $\int_0^1 v(x)[Lu](x)\,dx = \int_0^1 u(x)[L^\dagger v](x)\,dx$ holds for any $u \in D(L)$ and $v \in D(L^\dagger)$.

(b) Are there any nontrivial solutions to $Lu = 0$ in $D(L)$, or to $L^\dagger u = 0$ in $D(L^\dagger)$?

(c) Calculate the inverse of $L : D(L) \to L^2([0,1])$, that is, find an integral operator $K : L^2([0,1]) \to L^2([0,1])$ such that $Lu = f$ if and only if $u = Kf$. Is K bounded?

Hint: Use the method of integrating factors.

3) Existence and Uniqueness. Consider the second-order differential operator defined by

$$[Lu](\theta) = u''(\theta), \qquad \text{where} \qquad u \in D(L) = \{g \in H^2([0,1]) \,|\, g(0) = g(1)\,,\, g'(0) = g'(1)\}.$$

Here, $H^2([0,1])$ is the set of functions in $L^2([0,1])$ with two weak derivatives in $L^2([0,1])$.

(a) Calculate the adjoint of L, i.e., find an operator $L^\dagger$ and a domain $D(L^\dagger)$ such that $\int_0^1 v(\theta)[Lu](\theta)\,d\theta = \int_0^1 u(\theta)[L^\dagger v](\theta)\,d\theta$ holds for any $u \in D(L)$ and $v \in D(L^\dagger)$.

(b) Write down bases for the null space of L and its adjoint, respectively.

(c) Based on part (b), what condition must f satisfy for $Lu = f$ to have a solution?

(d) If a solution to $Lu = f$ exists, is it unique? Write down the general solution(s).