Separable PDEs

Recap Solutions of singular ODEs may have singularities (solution or derivatives blow up) at the singular points. We can identify the singularity and construct solutions w/

Generalized Power Series
$$u(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Bessel's Equation
$$x^2 u'' + x u' + (x^2 - p^2) u = 0$$

Indicial Equation
$$r^2 - p^2 = 0 \quad \Rightarrow \quad r = \pm p$$

Coefficient Recurrence
$$a_{2m} = -\frac{a_{2m-1}}{4m(m+p)} \qquad m = 0, 1, 2, \ldots$$

Bessel Functions
$$J_p(x) = u_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1+p)} \left(\frac{x}{2}\right)^{2m+p} \qquad (*)$$

Here, $\Gamma(x)$ is the gamma function and u_p is normalized by $a_0 = \frac{1}{2^p \Gamma(1+p)} = \frac{1}{2^p p!}$ if p = integer

Properties of Bessel Functions

⇒ For each $\pm p \neq$ integer, $(*)$ gives two lin. indep. solutions. One blows up at $x=0$ while the other vanishes at $x=0$.

⇒ For $\pm p =$ integer, $(*)$ gives only one lin. indep. solution. A second lin. indep. soln. can be constructed with a logarithmic singularity.

⇒ Customary to write second lin. indep. solution as

"Neumann" or "Weber"
$$N_p(x) = \frac{\cos(\pi p)\, J_p(x) - J_{-p}(x)}{\sin(\pi p)} \qquad p \neq 1,2,3,\ldots$$

⇒ $J_p(x)$ and $N_p(x)$ behave much like $\sin(n\pi x)$ and $\cos(n\pi x)$, but on $x \geq 0$ with decaying envelope as $x \to \infty$. Can form other "named" functions via combos: Hankel, Hyperbolic, ...

Orthogonality

Bessel functions satisfy an orthogonality relationship, best understood as a weighted analogue of ONB $\{\sin n\pi x\}_{n=1}^{\infty}$.

$\sin(x), \cos(x)$ | $J_p(x), N_p(x)$
$\downarrow$ | $\downarrow$

zeros: $x = n\pi \quad n = 1, 2, \ldots$ | $x = \alpha_n \quad n = 1, 2, \ldots$

at $x=1$: $\sin(n\pi x)\big|_{x=1} = 0$ | at $x=1$: $J_p(\alpha_n x)\big|_{x=1} = 0$

Diff Eq.: $u'' + (n\pi)^2 u$ | $x(xu')' + (\alpha_n^2 x^2 - p^2)u = 0$

$\Rightarrow u(x) = \sin(n\pi x)$ | $\Rightarrow u(x) = J_p(\alpha_n x)$

orthog:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$$

$$\int_0^1 x\, J_p(\alpha_n x) J_p(\alpha_m x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} J_p'(\alpha_n)^2 & n = m \end{cases}$$

So $J_p(\alpha_n x)$ are analogues of $\sin(n\pi x)$

Bessel "modes" | Fourier modes

Fuch's Theorem

The modified power series for ODEs is a powerful tool for constructing solutions near/at a singular point. For 2nd Order,

$$(**) \qquad u'' + f(x) u' + g(x) u = 0$$

Thm
If $x f(x)$ and $x^2 g(x)$ have convergent power series (are analytic) at $x=0$, then $(**)$ has either

(1) Two lin. indep. modified power series soln's

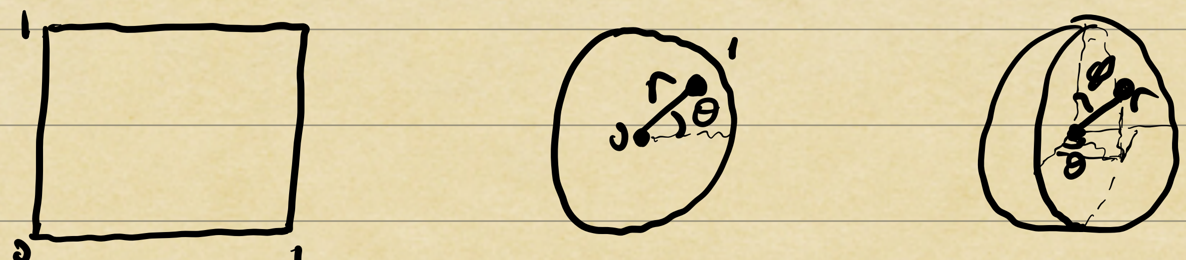
or

(2) One modified power series solution $S_1(x)$ and one lin. indep. solution $S_1(x)\ln(x) + S_2(x)$, where $S_2(x)$ is a second mod. power series.

$\Rightarrow$ The condition on $f(x)$ any $g(x)$ makes $x=0$ a "regular" singular point.

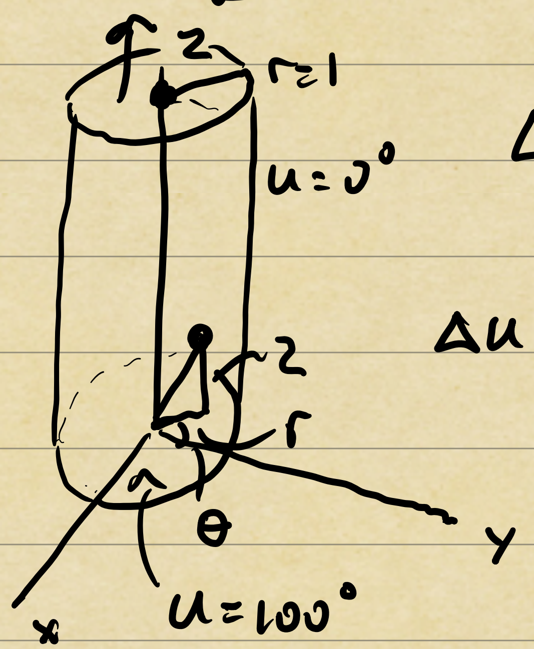
Separable PDEs

2^{nd} order ODEs often arise while solving classical PDEs of math. phys. on simple "separable" geometries: rectangle, disk, sphere, cone, etc.



$[0,1]\times[0,1]$ $\qquad$ $[0,1]\times[0,2\pi]$ $\qquad$ $[0,1]\times[0,2\pi]\times[0,\pi]$

Example 1: Temperature in a Cylinder



$$\Delta u = \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Delta u(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\left.\begin{aligned} &u(r,\theta,0) = 100 \\ &u(1,\theta,z) = 0 \\ &\lim_{z\to\infty} u(r,\theta,z) = 0 \end{aligned}\right\} \text{Boundary conditions}$$

Separable Solution: $u = R(r)\,\Theta(\theta)\,Z(z)$

$$\Rightarrow \quad \Delta u = \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

Each term depends only on one distinct variable.

$\Rightarrow$ They each must be equal to the same const.

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = K^2, \qquad Z(z) = \begin{cases} e^{Kz} \\ e^{-Kz} \end{cases}$$

Take $K > 0$ and $Z(z) = e^{-Kz}$ so $u(r,\theta,z) \to 0$, as $z \to \infty$

$\Rightarrow$ We are left with

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + K^2 = 0$$

$$\rightarrow \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + K^2 r^2 = 0$$

$\Rightarrow$ The angular term must be equal to a const.

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2 \qquad \Theta = \begin{cases} \sin(n\theta) \\ \cos(n\theta) \end{cases}$$

Need periodic in θ so choose $n =$ integer.

$\Rightarrow$ Finally, the radial term must be equal to

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 + K^2 r^2 = 0$$

$$\rightarrow \quad r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (K^2 r^2 - n^2)R = 0$$

This is just Bessel's equation with $r \to Kr$

$$R(r) = J_n(Kr)$$

Note we only take the smooth solution at the origin since we want finite temp.

$\Rightarrow$ To determine value(s) of K, we apply the zero B.C. at $r=1$

$$R_{n,m}(r) = J_n(\underbrace{\alpha_m^{(n)}}_{\text{zero of } J_n} r)$$

Full general Solution is combo of

$$u(r,\theta,z) = R_{n,m}(r)\,\Theta_n(\theta)\,Z_m(z)$$

$$\Rightarrow \quad u(r,\theta,z) = \begin{cases} J_n(\alpha_m^{(n)} r)\sin(n\theta)e^{-\alpha_m z} & n=1,2,3,\ldots \\ J_n(\alpha_m^{(n)} r)\cos(n\theta)e^{-\alpha_m z} & n=0,1,2,\ldots \end{cases}$$

We have enforced B.C.s on wall and at $z\to\infty$, but still need to enforce B.C.'s at $z=0$ (heated base of cylinder)

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{n,m} J_n(\alpha_m^{(n)} r)\cos(n\theta)e^{-\alpha_m z}$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{n,m} J_n(\alpha_m^{(n)} r)\sin(n\theta)e^{-\alpha_m z}$$

The idea now is to use orthogonality of the basis functions in r and θ directions to compute coeffs $a_{n,m}$ and $b_{n,m}$ when $z=0$.

$$100 = u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{n,m} J_n(\alpha_m^{(n)} r)\cos(n\theta)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{n,m} J_n(\alpha_m^{(n)} r)\sin(n\theta)$$

Clearly all the coeffs with $n=1,2,3,\ldots$ vanish since LHS is independent of θ.

We are left with $n=0$ $(\cos(0\theta)=1)$ and

$$100 = \sum_{m=1}^{\infty} a_m J_0(\alpha_m^{(0)} r)$$

$$\Rightarrow \quad a_m = \int_0^1 100 r J_0(\alpha_m^{(0)} r)\,dr \Big/ \int_0^1 r J_0(\alpha_m^{(0)} r)^2\,dr$$

Can simplify further using recurrence relations for Bessel functions, but for now, we have

$$u(r,\theta,z) = \sum_{m=1}^{\infty} a_m J_0(\alpha_m^{(0)} r) e^{-\alpha_m z}$$

where $\quad a_m = 100 \dfrac{\int_0^1 r J_0(\alpha_m^{(0)} r)\,dr}{\int_0^1 r J_0^2(\alpha_m^{(0)} r)\,dr}$

analogous to Fourier series solutions of BVP.

Using some identities for Bessel functions:

$$a_m = \frac{200}{\alpha_m^{(0)} J_1(\alpha_m^{(0)})}$$

zero of J_0