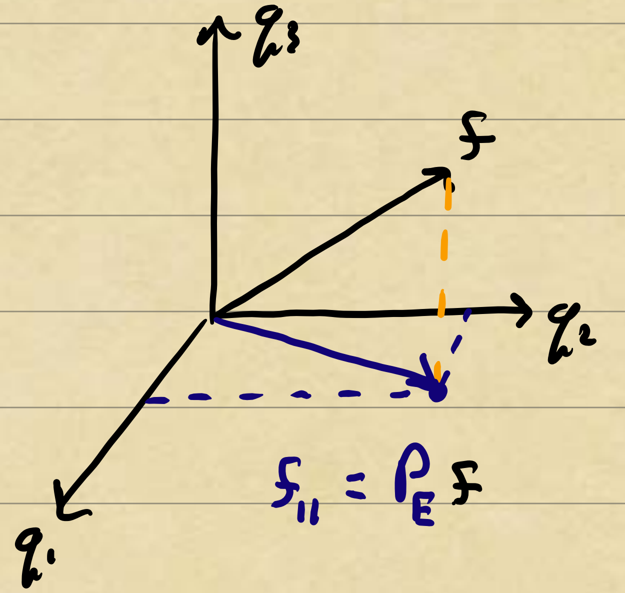
Best Approx. in Hilbert Space

Goal: Minimize $\|f - Ec\|$ in Hilbert norm.

$\uparrow$ $E: \mathbb{R}^N \to H$ (quasimatrix)



Two step procedure:

$\Rightarrow$ Project $f_{\parallel} = P_E f$

$\Rightarrow$ Solve $Ec = f_{\parallel}$

Here, $P_E : H \to \mathrm{col}(E)$ is the orthogonal projection of H onto $\mathrm{col}(E)$, characterized by

i) $f \to P_E f$ is linear

ii) $P_E f = f$ when $f \in \mathrm{col}(E)$

iii) $P_E f = 0$ when $f \in \mathrm{col}(E)^{\perp}$

iv) $\|P_E f\| \leq \|f\|$ for all $f \in H$.

Objective $\|f - Ec\|$ is minimized when $Ec = P_E f$.

Today: Orthonormal Basis in Hilbert Space

Orthonormal Basis (Finite-Dimension)

An orthonormal basis for a subspace $V_N \subset H$ with dimension $N<\infty$ is a basis $\{\underset{\sim}{q}_1, \dots, \underset{\sim}{q}_N\} \subset V_N$ such that $\langle \underset{\sim}{q}_i, \underset{\sim}{q}_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.

Claim: The orthogonal projection onto V_N can be computed explicitly via

$$P_N f = \sum_{k=1}^{N} \langle f, \underset{\sim}{q}_k \rangle \underset{\sim}{q}_k .$$

pf| For any $f \in H$, we have that

$$f = f_N + f_N^{\perp}, \text{ where } f_N \in V_N,\ f_N^{\perp} \in V_N^{\perp},$$

and $P_N f = f_N$ by definition. Since $\{\underset{\sim}{q}_1, \dots, \underset{\sim}{q}_N\}$ is a basis for V_N, there are unique scalars $\alpha_1, \dots, \alpha_N$ s.t.

$$f_N = \alpha_1 \underset{\sim}{q}_1 + \dots + \alpha_N \underset{\sim}{q}_N .$$

Using the orthonormality of $\{q_1, \dots, q_N\}$, and the linearity of $\langle \cdot, \cdot \rangle$, we have

$$\langle f_N, q_j \rangle = \alpha_1 \underbrace{\langle q_1, q_j \rangle}_{=0} + \dots + \alpha_j \underbrace{\langle q_j, q_j \rangle}_{=1} + \dots + \alpha_N \underbrace{\langle q_N, q_j \rangle}_{=0}$$

$$= \alpha_j$$

Therefore, $f_N = \sum_{k=1}^{N} \langle f, q_k \rangle q_k$ as claimed.

If $q_1, \dots, q_N$ are orthogonal, but not normalized ($\|q_k\| \neq 1$), then normalize $\tilde{q}_k = \frac{q_k}{\|q_k\|}$.

$$P_N f = \sum_{k=1}^{N} \frac{\langle f, q_k \rangle}{\langle q_k, q_k \rangle} q_k.$$

<u>Example:</u> Fourier Series : Best Approx. in $L^2([-1,1])$.

A continuous, periodic function f on $[-1, 1]$ has

$$f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{i\pi k x}, \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{+1} f(x) e^{-i\pi k x} dx.$$

Consider the best approximation of f in the subspace $V_N = \{e^{-i\pi N x}, \dots, e^{+i\pi N x}\}$ of

$$L^2([-1,1]) = \left\{ f : [-1,1] \to \mathbb{R} \,/\, \int_{-1}^{+1} |f(x)|^2 dx < \infty \right\},$$

which is a Hilbert space w/inner product

$$\langle f, g \rangle = \int_{-1}^{+1} f(x)\overline{g(x)}\, dx.$$

Since $\{e^{ikx}\}_{k=-\infty}^{+\infty}$ are pairwise orthogonal,

$$f(x) = \underbrace{\frac{1}{\sqrt{2}} \sum_{k=-N}^{+N} \hat{f}_k e^{i\pi k x}}_{f_N} + \underbrace{\frac{1}{\sqrt{2}} \sum_{|k|>N} \hat{f}_k e^{i\pi k x}}_{f_N^{\perp}}$$

where $f_N \in V_N$ and $f_N^{\perp} \in V_N^{\perp}$. Therefore, $\langle f - f_N, g \rangle = 0$ for all $g \in V_N$ and f_N is the best approximation of f in V_N. Moreover, if P_N is the orthogonal projection onto V_N,

$$P_N f = \frac{1}{\sqrt{2}} \sum_{k=-N}^{N} \hat{f}_k e^{i\pi k x}, \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{+1} f(x) e^{-i\pi k x}.$$

Fourier modes $\{e^{-i\pi N x}, \dots, e^{+i\pi N x}\}$ are an ONB for V_N.

Orthonormal Bases (Infinite Dimension)

Given a Hilbert Space H, an orthonormal set $\{q_n\}_{n=1}^{\infty}$ satisfies $\langle q_i, q_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

A set $S \subset H$ is called dense in H if for any $f \in H$, for any $\varepsilon > 0$, there is a $\varphi \in S$ s.t. $\|f - \varphi\| < \varepsilon$. Equivalently, there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \|f - \varphi_n\| = 0.$$

An orthonormal set $\{q_n\}_{n=1}^{\infty} \subset H$ is an orthonormal basis for H if $\operatorname{span}\{q_n\}_{n=1}^{\infty}$ is dense in H. In other words, each $f \in H$ is a limit of finite linear combos of $\{q_n\}$,

$$\lim_{N \to \infty} \|f - \sum_{k=1}^{N} c_k q_k\| = 0.$$

We write $f = \sum_{k=1}^{\infty} c_k q_k$ and the coefficients are again given by $c_k = \langle f, q_k \rangle$.

For any orthonormal set $\{q_k\}_{k=1}^{\infty}$,

Bessel's Inequality $$\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, q_k \rangle|^2.$$

If $\{q_k\}_{k=1}^{\infty}$ is an ONB, then

Parseval's Identity $$\|f\|^2 = \sum_{k=1}^{\infty} |\langle f, q_k \rangle|^2.$$

<u>**Example:**</u> Every $f \in L^2([-1,1])$ has

$$f = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} c_k e^{i\pi k x}, \quad \text{where} \quad c_k = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} f(x) e^{-i\pi k x}\, dx.$$

<u>**Caution:**</u> The series converges in the L^2 norm:

$$\lim_{N \to \infty} \int \Big| f(x) - \underbrace{\frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} c_k e^{i\pi k x}}_{S_N} \Big|^2 dx = 0.$$

This is sometimes called convergence "in mean," but it does <u>not</u> imply that $S_N(x) \to f(x)$ pointwise (for each $x \in [-1,1]$). Convergence may fail at certain points unless f is smooth.

Example: What is the orthogonal projection of $f \in L^2([-1,1])$ onto the subspace of even functions $\{g \in H \,/\, g(x) = g(-x)\}$?

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)}_{P_V f} + \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Quasimatrix Notation

We can write orthogonal projectors in a compact matrix form using quasimatrices.

$$P_N f = \sum_{k=1}^{N} \langle f, q_k \rangle q_k = Q Q^* f$$

Here, Q and $Q^* f$ are defined as

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_N \\ | & & | \end{bmatrix}, \qquad Q^* f = \begin{bmatrix} \langle f, q_1 \rangle \\ \vdots \\ \langle f, q_N \rangle \end{bmatrix}$$

Quasimatrix Vector