Linear Approximation: Truncation Errors

Idea: Build up functions by linear combinations.

$$f(x) \approx \underbrace{c_1 e_1(x) + c_2 e_2(x) + \cdots + c_n e_n(x)}_{f_n(x)}$$

Question: How big is the truncation error $f(x) - f_n(x)$?

Example: Taylor Polynomial: Remainder Theorem.

$$f_n(x) \approx f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)$$

Thm: If f has $n+1$ continuous derivatives in a neighborhood $I_\delta = [x-\delta, x+\delta]$ of x_0, then $(n \geq 1)$

$$\text{(R)} \qquad f(x) - f_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt, \quad x \in I_\delta.$$

The remainder in (R) leads to the upper bound

$$\text{(E)} \qquad \sup_{x \in I_\delta} |f(x) - f_n(x)| \leq \frac{\delta^{n+1}}{(n+1)!} \sup_{x \in I_\delta} |f^{(n+1)}(x)|.$$

Pf/ Fundamental Thm of Calculus and integration-by-parts. See Lecture 2 notes.

Key Observations about Taylor Series:

1) The Taylor polynomial always interpolates f at x_0, e.g., $f(x_0) = f_n(x_0)$.

2) The order of interpolation is n, i.e., from (R)-(E), we have that

$$f(x) - f_n(x) \le \overset{\text{indep. of } x}{C} |x - x_0|^{n+1}, \quad \text{for } x \text{ near } x_0.$$

We write $|f(x) - f_n(x)| = O(|x-x_0|^n)$ as $x \to x_0$.

3) If all derivatives of $f(x)$ on I_δ exist and grow slower than $(n+1)!/\delta^{n+1}$, then f converges uniformly to f on I_δ, i.e.,

$$\sup_{x \in I_\delta} |f(x) - f_n(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Example: Convergence of Fourier Series.

$$f_N(x) = \frac{1}{\sqrt{2}} \sum_{k=-N}^{+N} \hat{f}_k e^{i\pi k x}, \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{+1} f(x) e^{-i\pi k x} dx.$$

Thm/ If $f, f', \ldots, f^{(n)}$ are continuous and periodic on $[-1,1]$ with $\sup_{x\in[-1,1]} |f^{(n)}(x)| \leq M$, then

$$(n \geq 1) \qquad |\hat{f}_k| \leq \frac{\sqrt{2}\, M}{(\pi k)^n} \qquad k = \pm 1, \pm 2, \ldots$$

The truncation error for f_N satisfies

$$(n \geq 2) \qquad |f(x) - f_n(x)| \leq \frac{2\sqrt{2}\, M}{(n-1)\pi^n N^{n-1}}$$

Pf/ Integrate definition of $\hat{f}_k$ by parts to bound coefficients. Bound series for $f(x) - f_N(x)$ with integral estimate. See HW 1 solutions.

Note: For certain functions with piecewise continuous derivatives, like $|x|$, the theorem can be sharpened to provide $\mathcal{O}(k^{-n-1})$ coeff. decay and $\mathcal{O}(N^{-n})$ trunc. err.

Key Observations about Fourier Series:

1) Unlike Taylor series, Fourier Series converge uniformly for all continuously differentiable, periodic functions.

2) The rate of convergence for Fourier Series (as $N \to \infty$) improves with each additional continuous derivative of f.

Theme: "Smooth" functions, with many continuous derivatives of modest growth, are often well-approximated by linear combinations of simple functions like polynomials. In contrast, "rough" functions, with singularities or fewer continuous derivatives, may suffer from slow convergence rates or non-convergence.

Q: How can we systematically reason about approximation/truncation errors?

Approximation in a Hilbert Space

To introduce a general framework for linear approximation, it's helpful to adapt some ideas from linear algebra to function spaces.

Lin. Alg. => Abstract vector spaces & subspaces

Lin. Alg. => Norm (measure "size") & inner product ("angle")

Lin. Alg. => Projection, basis, and coordinate transforms

New!
Analysis => Infinite series, convergence, completeness

We'll use these tools to place Fourier Series and Taylor series in a larger context and answer some practical questions about function approximation, recovery, interpolation, and the impact of noise and uncertainty.

Note: These foundational ideas are used

throughout modern applied math and we will build on them throughout the course, as we move onto linear transformations (linear ODE/PDE) and nonlinear maps (dynamical systems, nonlinear PDE, optimization).