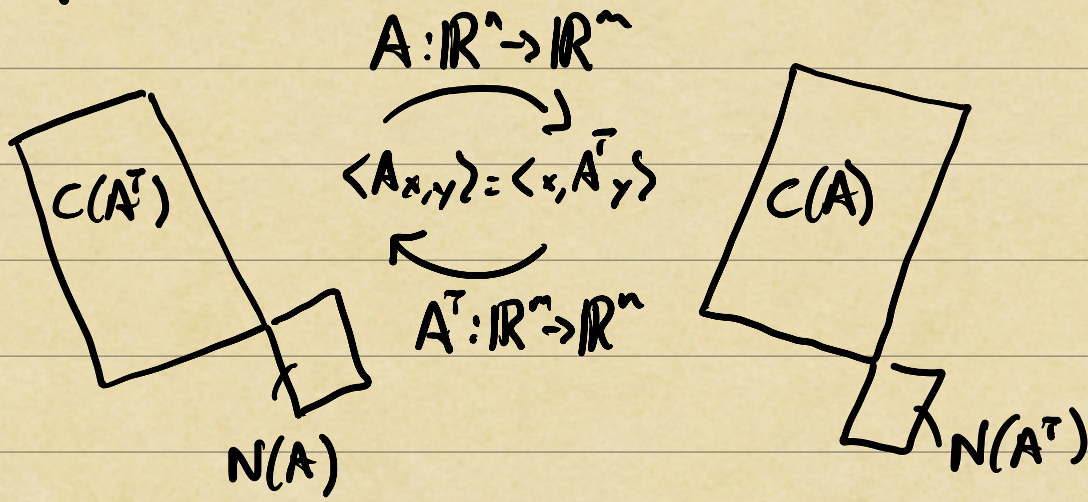
Linear Eqn's in a Hilbert Space

The theory of finite-dimensional linear transformations can be "summarized" in a picture of the four fundamental subspaces.



The output space can be partitioned into two subspaces of vectors

$$\mathbb{R}^m = C(A) \cup N(A^T)$$

These subspaces are orthogonal:

$$y_1 \in C(A) \text{ and } y_2 \in N(A^T) \iff \langle y_1, y_2\rangle = 0$$

$$\left(y_1 \in C(A) \iff y_1 = Ax_1 \text{ for some } x_1 \iff \langle Ax_1, y_2\rangle = \langle x_1, \overset{0}{\overset{\shortparallel}{A^* y_2}}\rangle = 0 \right)$$

Similarly, the input space can be partitioned into two orthogonal subspaces:

$$\mathbb{R}^n = C(A^T) \cup N(A).$$

We can rephrase the requirements for existance of solns to $Ax = b$.

$b \in C(A) \iff \langle b, y \rangle = 0$ for every $y \in N(A^T)$.

When $N(A^T)$ is low dimensional and a basis can be computed, the latter condition gives a concrete calculation to check if $b \in C(A)$.

Q: How does this work in infinite dimensional inner-product spaces?

$\Rightarrow$ It can be extremely useful but we need to be careful of a few inf-dim phenomena.

Example 1: $T = \frac{d}{dx} : C^\infty([0,1]_{per}) \to C^\infty([0,1]_{per})$

In Lecture 4, we calculated that

$$\underbrace{\int_0^1 \left[\frac{df}{dx}(x)\right] g(x)\,dx}_{\langle Tf, g\rangle} = \underbrace{\int_0^1 f(x)\left[-\frac{dg}{dx}(x)\right] dx}_{\langle f, T^* g\rangle}$$

So $T^* = -T$ and $N(T) = N(T^*) = \underbrace{\text{span}\{1\}}_{\text{"constants"}}$.

We might be tempted to conclude that

$f \in R(T)$ as long as $\langle 1, f\rangle = \int_0^1 f(x)\,dx = 0$.

In this case you would be right! But it doesn't always work out so nicely...

Example 2: Consider $T : f(x) \to \int_0^x f(y)\,dy$ as a map from $C([0,1]) \to C([0,1])$.

The adjoint is $T^* : f(x) \to -\int_x^1 f(y)\,dy$.

$$\Rightarrow \underbrace{\int_0^1 \left[\int_0^x f(y)\,dy\right] g(y)\,dy}_{\langle Tf, g\rangle} = \left[\left[\int_0^x f(y)\,dy\right]\left[\int_x^1 g(y)\,dy\right]\right]\Big|_0^1$$

$$- \int_0^1 f(x)\left[\int_x^1 g(y)\,dy\right]dx$$

$$= \underbrace{\int_0^1 f(x)\left[-\int_x^1 g(y)\,dy\right]dx}_{\langle f, T^* g\rangle}$$

There is no nonzero continuous function $f \in C([0,1])$ s.t.

$$[T^* f](x) = -\int_x^1 f(y)\,dy = 0$$

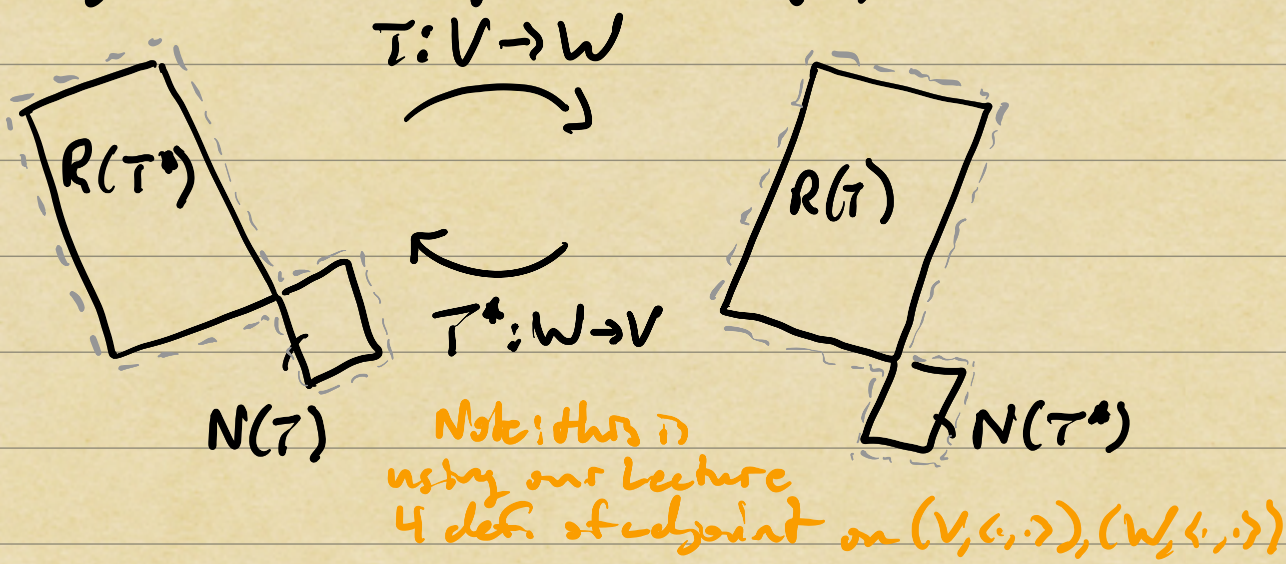
So the null space is empty: $N(T^*) = \{0\}$.

However, the range of T is <u>not</u> all of $C([0,1])$! This is b/c, by F.T.C.

$$[Tf](x) = \int_0^x f(y)\,dy \in C^1([0,1])!$$

So there is no continuous solution to $Tu = f$ if f is continuous but not differentiable.

The problem here is that the range of T is not <u>all</u> of $C([0,1])$, even though its null space is empty.



In infinite dimensions, there may be "gaps" between the fundamental subspaces and the whole space. In general

$$W \neq R(T) \cup N(T).$$

To proceed, we will add a new requirement for $(V, \langle \cdot, \cdot \rangle)$: completeness. The point is to identify a useful set of spaces and operators where we can check existance & uniqueness for $Tu = f$ by looking at $N(T)$ and $N(T^*)$.

Hilbert Spaces

Let $H = (V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say that H is complete if every Cauchy sequence has a limit in H.

Cauchy sequence

Given $\{f_n\}_{n=1}^{\infty}$ and any $\varepsilon > 0$, there is N s.t.

$$\|f_n - f_m\| < \varepsilon \text{ for all } n, m > N.$$

Limit of a sequence in H

There is an $f \in H$ s.t., for any $\varepsilon > 0$ there is an N s.t. $\|f_n - f\| < \varepsilon \ \forall n > N$.
We write $\lim_{n \to \infty} f_n = f$.

"If f_n, f_m get closer and closer together, they must get closer and closer to an element of the Hilbert space."

Note that $\|f\| = \sqrt{\langle f, f \rangle}$ is the norm induced by $\langle \cdot, \cdot \rangle$ on V here.

Example: $L^2([0,1]) = \{ f : \underbrace{\int_0^1 |f(x)|^2 dx}_{\text{Lebesgue integral}} < \infty \}$

$\Rightarrow$ The Hilbert spaces $L^2(\Omega) = \{ f : \Omega \to \mathbb{C} : \int_\Omega |f|^2 dx < \infty \}$ ($\Omega \subset \mathbb{R}^n$, or $\mathbb{R}$) are perhaps the most used function spaces in applied math!

$\Rightarrow$ Functions in $L^2(\Omega)$ are actually "equivalence classes" of functions that agree everywhere on Ω except sets of Lebesgue measure zero. The idea is that $\int_\Omega |f-g| dx = 0$ means $f = g \in L^2(\Omega)$, but the Lebesgue integral allows f and g to differ at measure zero sets (like at a single point), which don't contribute to the "area under the curve."

$\Rightarrow$ $L^2([0,1])$ is the completion of the following spaces with $\|\cdot\| = \sqrt{\int_0^1 (\cdot)^2 dx}$.
$\mathbb{P} = \{\text{poly on } [0,1]\}$, $C(\Omega)$, $C^1(\Omega)$, ..., $C^\infty(\Omega)$.

Any inner-product space $(V, \langle\cdot,\cdot\rangle)$ can be turned into a Hilbert space by taking its completion w.r.t. induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Essentially this means adding in any "missing" limits of Cauchy sequences.

Operators on Hilbert Spaces

With the notion of completeness, we have gone a long way toward patching up "gaps" in our inf-dim vector spaces.

The next step is to identify some suitable classes of operators on a Hilbert space like $L^2(\Omega)$, which we can use to study and solve partial differential, integral, and other linear eqn's in applied math.