(Power) Series Solutions of ODEs

Diff. & Integral Eqns as examples of linear equations on spaces of functions:

$$(*) \qquad \left[\frac{d}{dx} - 1\right] u(x) = 0, \qquad u(0) = 1$$

Choosing a finite-dimensional subspace and basis, we can convert $(*)$ to a matrix equation. For example, we choose $\{1, x, x^2, \ldots, x^N\} \subset \mathbb{P}_N$:

$$(**) \qquad \begin{bmatrix} -1 & 1 & 0 & \cdots & & 0 \\ & -1 & 2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ & & & & -1 & N \\ \hline 1 & 0 & \cdots & & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \\ \\ a_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \\ \hline 1 \end{bmatrix}$$

$$\frac{d}{dx} 1 = 0$$

$$\frac{d}{dx} x = 1$$

$$\vdots$$

$$\frac{d}{dx} x^N = N x^N$$

$$u(x) = e^x \approx \sum_{k=0}^{N} \frac{1}{k!} x^k$$

Partial Sums converge absolutely & uniformly.

Question: When do Partial Sums converge?

Power Series Solutions

The "discretization" (**) is a useful way to generate approximate solutions to (*). In the limit $N \to \infty$, we get a power series

$$\tilde{u}(x) = \sum_{k=0}^{\infty} a_k x^k.$$

To check if the power series "solves" the ODE, we need to

i) Check if the power series converges, so that $\tilde{u}(x)$ is well-defined

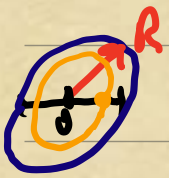
ii) Check that $\tilde{u}(x)$ satisfies the ODE

iii) " " the auxillary conditions.

In practice, for power series, we really only need to check i) and we ii) - iii) "for free"

A few facts about Power series:

i) Convergence. A power series absolutely and uniformly in an interval $[-r,r] \subset (-R,R)$.



(more generally, a disk of radius R in complex plane) i.e., for $|x|<R$. It diverges for $|x|>R$. May or May Not converge on the boundary.

$\Rightarrow$ R is called the radius of convergence.

$\Rightarrow$ Limit cases are $R=0$ and $R=\infty$.

Pf| If series converges absolutely at some x_* in $[-R,R]$. Then for any x with $|x| \le |x_*|$,

$$\left|\sum_{k=N+1}^{\infty} a_k x^k\right| \le \sum_{k=N+1}^{\infty} |a_k||x|^k \le \sum_{k=N+1}^{\infty} |a_k||x_*|^k \to 0 \text{ as } N\to\infty.$$

Tools $\Rightarrow$ Test convergence at x_* using comparison, ratio test, root test, or bound Cauchy sequence of partial sums directly.

Example Apply the ratio test to $\tilde{u}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Fixed x $\Rightarrow \lim_{k\to\infty} \left| \frac{x^{k+1}/(k+1)!}{x^k/k!} \right| = \lim_{k\to\infty} |x|/k+1 = 0$

$\Rightarrow$ For any $x \in \mathbb{R}$, series converges absolutely.

$\Rightarrow$ Radius of convergence $R = \infty$.

ii) Derivatives of Power Series. A power series can be differentiated term-wise anywhere in its disk of convergence:

$$\tilde{u}'(x) = \lim_{N\to\infty} \sum_{k=0}^{N} k a_k x^{k-1} = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

The convergence is absolute in $(-R,R)$ and uniform on $[-r,r] \subset (-R,R)$.

Example $\tilde{u}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\tilde{u}'(x) = \sum_{k=1}^{\infty} \frac{\cancel{k} x^{k-1}}{\underset{(k-1)!}{\cancel{k!}}} = \sum_{j=0} \frac{x^j}{j!}$

$$\tilde{u}'(x) - u(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \left(\frac{1}{k!} - \frac{1}{k!}\right) x^k = 0$$

After we check (i) convergence, (ii) - (iii) are satisfied by construction (matching coeffs of partial sums) and check by termwise diff

Modified Power Series

ODE solutions are not always as "smooth" as a power series, and we need to modify our approach to "see" these singularities.

"singular" numbers

Euler's Equation

$$x^2 u''(x) + a\, x\, u'(x) + b\, u(x) = 0$$

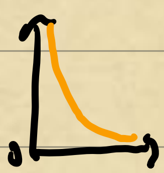
The origin is a singular point of this ODE.

x^r

$$\tilde{u}(x) = x^r \Rightarrow (r^2 + ar + b)\,\tilde{u}(x) = 0$$

To be true for all x $\Rightarrow$ $r_\pm = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$

Solutions are not always smooth. For $a = 1/2$



$$u(x) = c_1 + c_2 x^{-1/2}$$