Linear Transformations in Hilbert Space

Suppose we want to approximate $f \in C(\Omega)$, $\Omega \subset \mathbb{R}^d$,
but we only have samples at data points:

$$f_1 = f(x_1), \ldots, f_M = f(x_M).$$

We can choose a dictionary $E = [e_1 \cdots e_N]$,

$$\underbrace{\begin{bmatrix} e_1(x_1) & \cdots & e_N(x_1) \\ \vdots & & \vdots \\ e_1(x_M) & \cdots & e_N(x_M) \end{bmatrix}}_{E_{N,M}} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_M \end{bmatrix}$$

and approximate $\hat{f}(x) = c_1 e_1(x) + \cdots + c_N e_N(x)$.

If $N = M = \text{rank}(E_{N,N})$, $\hat{f}$ interpolates data.

If $N < M$, find best fit to data (regression).

Both dictionary and data distribution play an important role in accuracy/convergence.

Polynomial Interpolation $(N=M)$

Given distinct points $x_0, \dots, x_N \in [-1,1]$, every continuous function $f:[-1,1]\to\mathbb{R}$ has a unique polynomial interpolant of deg $\le N$:

$$p = P_x f.$$

The map $P_x : C[-1,1] \to \mathbb{P}_N$ is a linear projection of $C[-1,1]$ onto $\mathbb{P}_N$ with

$$\|P_x\| = \sup_{f\in C[-1,1]} \frac{\|P_x f\|}{\|f\|} \geq 1.$$

This amplification factor, the operator norm of P_x determines how far the interpolants can be from "best."

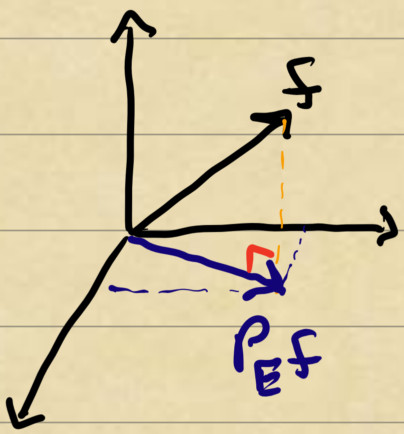
"Best"

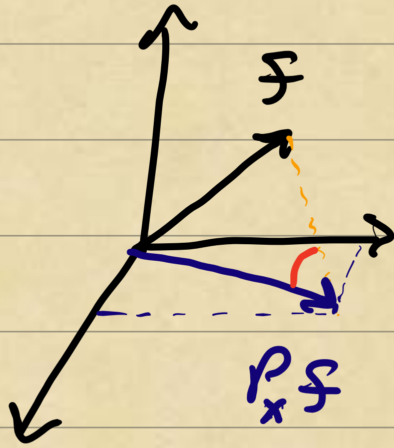
$$E = \begin{bmatrix} | & | & & | \\ 1 & x & \cdots & x^N \\ | & | & & | \end{bmatrix}$$

"Interpolant"

$E\underline{c} = P_E f$ ← orthogonal proj.

$E\underline{c} = P_x f$ ← oblique proj.





Claim: Given $X = \{x_0, \ldots, x_N\} \subset [-1,1]$ and $E = \{1, x, \ldots, x^N\}$, then

$$\|f - P_x f\| \le (1 + \|P_x\|)\|f - P_E f\|$$

interp. error $\qquad$ best fit error

pf| $\|f - P_x f\| \le \|f - P_E f\| + \|P_E f - P_x f\|$

Now, $P_x P_E f = P_E f$, so

$$\|P_E f - P_x f\| = \|P_x(P_E f - f)\| \le \|P_x\| \, \|f - P_E f\|$$

$$\Rightarrow \quad \|f - P_x f\| \le (1 + \|P_x\|)\|f - P_E f\|.$$

For "good" sets of interpolation nodes $\|P_x\| \sim \log(\#\text{ nodes})$ (slow growth)

Linear Transformations

Given vector spaces V & W, a map $T: V \to W$ is linear (a linear transformation) if

$$T(\alpha f + \beta g) = \alpha T f + \beta T g$$

__Example:__ $T: x \mapsto Ax$ where $A \in \mathbb{R}^{m \times n}$.

Q: What are the dimensions of V, W?

__Example:__ Differentiation on $C^1[-1,1]$.

$$\frac{d}{dx}(\alpha f(x) + \beta g(x)) = \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$$

Q: What is W (codomain) here?

__Example:__ Integral operators on $C[-1,1]$.

$$f(x) \mapsto \int_{-1}^{+1} k(x,y) f(y)\, dy$$

How can we use tools/ideas from lin. algebra to solve & analyze ODEs/PDEs?

Matrix Representations

If V, W are vector spaces of dimension $\dim(V)=n$ and $\dim(W)=m$, with bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$,

$$
\begin{aligned}
T(x) &= T(\alpha_1 x_1 + \cdots + \alpha_n x_n) \\
&= \alpha_1 T(x_1) + \cdots + \alpha_n T(x_n) \\
&= \alpha_1 (\beta_{11} y_1 + \cdots + \beta_{m1} y_m) \\
&\quad + \alpha_2 (\beta_{12} y_1 + \cdots + \beta_{m2} y_m) \\
&\quad \vdots \\
&\quad + \alpha_n (\beta_{1n} y_1 + \cdots + \beta_{mn} y_m)
\end{aligned}
$$

$$
= \begin{bmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix}}_{m \times n} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}
$$

The $(\beta)_{ij}$ matrix maps coords of x in basis

$\{x_1, \ldots, x_n\}$ to coords of $T(x)$ in basis $\{y_1, \ldots, y_m\}$.

<u>Example:</u> Differential Op on monomials of deg $\leq n$

$$\frac{d}{dx} 1 = 0 \qquad \frac{d}{dx} x = 1 \ \cdots \ \frac{d}{dx} x^n = n x^{n-1}$$

$$p(x) = a_0 1 + \cdots + a_n x^n \Rightarrow \frac{dp}{dx} = b_0 1 + \cdots + b_n x^n$$

$$\begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & n-1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$