Hilbert-Schmidt Operators

Recap

B'dd $\overset{\text{(linear)}}{\text{operators}}$ on a Hilbert space

$$T: H \to H, \quad \|T\| = \sup_{f \in H} \frac{\|Tf\|}{\|f\|} < \infty$$

have limited "amplifying" power on H.

Operators with b'dd inverse on H lead to "well-posed" equations and

Neumann series

$$(T+E)^{-1} = T^{-1} \sum_{k=0}^{\infty} ET^{-1} \quad \text{if} \quad \|ET^{-1}\| < 1$$

Differential operators on $L^2([-1,1])$ are <u>not</u> (usually) b'dd, but they (usually) have b'dd inverse given by integral op.

Integral reformulation: trade ODE/PDE for integral equation.

$$u(-1) = 0, \quad u'(x) + v(x)u(x) = f(x) \iff u(x) + \int_{-1}^{x} v(y)u(y)\,dy = \int_{-1}^{x} f(y)\,dy$$

Criteria for b'dd invertibility

Matrix $A: \mathbb{R}^n \to \mathbb{R}^n$ | B'dd Op. $T: H \to H$

Soln Exists

$A^T x = 0 \iff x = 0$

$\Rightarrow C(A) = \mathbb{R}^n$

$\|T^* u\| \geq \delta \|u\|$ ($\delta > 0$)

$\Rightarrow R(T) = H$

Soln unique

$Ax = 0 \iff x = 0$

$\Rightarrow Ax = Ay = b$

$\Rightarrow A(x-y) = 0$

$\Rightarrow x - y = 0$

$Tu = f \iff u = 0$

$\Rightarrow Tu = Tv = f$

$\Rightarrow T(u-v) = 0$

$\Rightarrow u - v = 0$

For T^{-1} b'dd, we need $N(T^*) = \{0\}$ but this alone is not enough. We have to rule out T^* shrinking vectors by an arbitrary amount, like this:

$$\{v_n\} \subset H \text{ w/ } \|v_n\| = 1 \text{ s.t. } T^* v_n \to 0 \in H.$$

When this happens, $R(T) \subset H$ is <u>not</u> the whole of H and $b \perp N(T^*) = \{0\}$ doesn't imply that $b \in R(T)$. Another way to see what is happening: T^{-1} is an unb'dd op which cannot be defined on all of H.

<u>Example:</u> $Tf = \int_{-1}^{x} f(y)\,dx : L^2([-1,1]) \to L^2([-1,1])$

has inverse $T^{-1}u = \frac{du}{dx} : \{u \in H^1([-1,1]) \mid u(-1) = 0\}$
$\to L^2([-1,1])$

The image of T is <u>not</u> all of L^2, and the inverse is <u>not</u> a b'dd operator on L^2.

There is no $v \neq 0 \in \{L^2([-1,1]) \mid u(-1) = 0\}$ s.t. $T^* v = 0$, but the sequence $v_k = \sin(\pi k x)$ has $\|v_k\|_{L^2}^2 = 1$ for $k = 1, 2, 3, \ldots$

$$T^* v_k = \int_x^{-1} \sin(\pi k y)\,dy = \frac{1}{\pi k}\left[\cos(\pi k x) \pm 1\right]$$

$$\|T^* v_n\|_{L^2}^2 = 2\left(\frac{1}{\pi k}\right)^2 \to 0 \quad \text{as } k \to \infty.$$

$\Rightarrow T^*$ is <u>not</u> b'dd below.

⇒ T is not b'dd below b/c there are oscillatory functions that "cancel out" when they are integrated over many parts of the domain.

⇒ This feature is common in integral ops and, in particular, inverses of diff. ops. b/c diff ops are (usually) unbounded.

So, diff ops typically have b'dd inverses, but we have to be careful how we use these for, e.g., integral reformulation. We want the resulting equations to be well-posed.

real-valued
continuous
b'dd on [-1,1]

Example: $u'(x) + v(x)u(x) = f(x) \qquad u(-1) = 0$

Int. Ref. $\Rightarrow \quad \underbrace{u(x) + \int_{-1}^{x} v(y)u(y)\,dy}_{} = \int_{-1}^{x} f(y)\,dy$

$T = I + K \Rightarrow Ku = \int_{-1}^{x} vu\,dy$

The inverse of $\frac{d}{dx}$ is not b'dd below, but T^* is b'dd below w/right conditions on v.

$$\left|\underbrace{\int_x^1 v(y)u(y)\,dy}_{K^*u}\right|^2 \le 2\int_x^1 |v(y)|^2|u(y)|^2\,dy$$

Jensen's inequality (again)

$$\le 2\sup_{-1\le y\le 1}|v(y)|^2\,\|u\|_{L^2}^2$$

integrate over both sides $\Rightarrow$ $$\|K^*u\|_{L^2} \le \underbrace{\left[4\sup_{-1\le y\le 1}|v(y)|^2\right]}_{= \|K\|}\|u\|_{L^2}^2$$

Since $T^* = I + K^* : H \to H$, if

$$\|K^*\| = 2\sup_{-1\le y\le 1}|v(y)| < 1, \text{ then}$$

$$\|T^*u\| = \|u + K^*u\| \ge \big|\|u\| - \|K^*u\|\big| \ge \overbrace{(1-\|K^*\|)}^{\text{“}\delta\text{”}}\|u\| > 0$$

"reverse triangle ineq."

In other words, the "shrinking power" of K is offset by the identity term.

Since T is b'dd and coercive (b'dd below) it has a b'dd inverse! So the integral equation is well-posed in that there is a unique soln, cont. depend. on init. data.

Integral & Hilbert-Schmidt Operators

Many of the integral operators we encounter in practice are associated with a reasonably "nice" kernel function on domain $\Omega \subset \mathbb{R}^d$.

$$[Kf](x) = \int_\Omega k(x,y)\, f(y)\, dy$$

If $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, then K is Hilbert-Schmidt.

Example:

$$[Kf](x) = \int_{-1}^{x} v(y) f(y)\, dy = \int_{-1}^{1} \underbrace{\mathbb{1}(y<x)\, v(y)}_{k(x,y)} f(y)\, dy$$

Hilbert-Schmidt operators are as "close to matrices" as you can reasonably expect in inf. dim. spaces.

Many of the factorizations & tools for matrices carry over to Hilbert-Schmidt matrices directly.