Series Solutions of ODEs (pt. 2)

Recap

Power series solutions of ODEs:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

find coeffs

Good way to find smooth solutions near a singular point of a singular ODE.

Legendre's Equation

$$(1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0$$

singular points at $x = \pm 1$

recurrence for coeffs

$$a_{n+2} = -\frac{(\ell-n)(\ell+n+1)}{(n+2)(n+1)} a_n \qquad n = 0, 1, 2, \ldots$$

Legendre Polynomials $u_\ell(1) = 1$

$$u_\ell(x) = \sum_{n=0}^{\ell} \underbrace{2^\ell \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell}}_{= a_n} x^n$$

$\Rightarrow$ Power series terminates at degree $n > \ell$ and the smooth solutions of Legendre's eqn. are the Legendre Polynomials.

Generalized Power Series Method:

Recall that Euler's equation had the form:

$$x^2u''(x) + axu'(x) + bu(x) = 0$$

This is a simple example of a singular 2^{nd} order ODE and recall that its solutions were not always smooth/b'dd at the singular point $x=0$.

$$u(x) = x^r \quad \Rightarrow \quad (r^2 + ar + b)u(x) = 0$$

$$\Rightarrow \quad r = \frac{1}{2}\left(-a \pm \sqrt{a^2-4b}\right)$$

The solutions may not always be non-negative integer powers of x. For example, $a = 1/2$

$$\Rightarrow \quad u(x) = c_1 + c_2 x^{-1/2}$$

The term $x^{-1/2}$ is not differentiable at $x=0$

and has no convergent power series there.

To account for these potential singularities, we can use a generalized power series ansatz:

$$u(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

r can be any real # and if $r \neq 0, 1, 2, \ldots$, then soln. has a singularity at $x=0$.

Similar to Euler's equation, we will allow the ODE to "tell" us what r should be, revealing the form of the singularity (if any).

<u>Example</u> $\quad x^2 u''(x) + 4xu'(x) + (x^2+2)u(x) = 0$

singular point at $x=0$

$$u(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \rightarrow x^2 u(x) = \sum_{n=0}^{\infty} a_n x^{n+r+2}$$

$$u'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \rightarrow xu'(x) = \sum_{n=0} (n+r) a_n x^{n+r}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \rightarrow x^2 u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

By matching coefficients of equal powers, we get

	x^r	x^{r+1}	x^{r+2}	$\cdots$	x^{r+n}
x^2u''	$r(r-1)a_0$	$(r+1)ra_1$	$(r+2)(r+1)a_2$		$(n+r)(n+r-1)a_n$
$4xu'$	$4ra_0$	$4(r+1)a_1$	$4(r+2)a_2$	$\cdots$	$4(n+r)a_n$
x^2u	0	0	a_0		a_{n-2}
$2u$	$2a_0$	$2a_1$	$2a_2$		$2a_n$

The key to determining r is the balance of coeffs of x^r (just as for Euler's eqn):

$$(r(r-1)+4r+2)a_0 = 0$$

"indicial equation" $\Rightarrow$ $r^2+3r+2=0 \Rightarrow (r+2)(r+1)=0$

$$r=-1 \text{ and } r=-2$$

From here, we proceed to compute a series solution for each case separately

$$u_1(x) = \sum_{n=0}^{\infty} a_n r^{n-1} \qquad u_2(x) = \sum_{n=0}^{\infty} b_n r^{n-2}$$

To illustrate, we will consider $r=-1$

$r=-1$: $u_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$

$\Rightarrow$ Choice of r makes coeff of x^r vanish.

$\Rightarrow$ Coeff of $x^{r+1} = x^0$ vanishes only if $a_1 = 0$.

$\Rightarrow$ From coeff of x^{n-1} gives (after algebra)

$$a_n = -\frac{1}{n(n+1)} a_{n-2} \qquad n \geq 2$$

Since $a_1 = 0$, all odd index coeffs vanish

For even n, we have $a_n = \frac{(-1)^{n/2}}{(n+1)!} a_0$ and

the generalized power series soln is

$$u_1(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m-1}$$

$$= a_0 x^{-2} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

$$= a_0 \frac{\sin x}{x^2}$$ $\Leftarrow$ singular at $x=0$

Bessel's Equation

Bessel function

$$x^2 u'' + x u' + (x^2 - p^2) u = 0$$

order of Bessel function

Easier to solve in S-L form:

$$x(xu')' + (x^2 - p^2) u = 0$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$u'(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} \rightarrow \quad x u'(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$$

$$(xu')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

$$x(xu')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}$$

	x^s	x^{s+1}	x^{s+2}	---	x^{s+n}
$x(xu')'$	$s^2 a_0$	$(1+s)^2 a_1$	$(2+s)^2 a_2$		$(n+s)^2 a_n$
$x^2 u$	0	0	a_0	---	a_{n-2}
$-p^2 u$	$-p^2 a_0$	$-p^2 a_1$	$-p^2 a_2$		$-p^2 a_n$

Indicial Eqn: $s^2 - p^2 = 0 \implies s = \pm p$

Case $s = +p$

$\implies$ Matching coeffs of x^{s+1} gives $a_1 = 0$.

$\implies$ Matching coeffs of x^{s+n}, $n \geq 2$, gives

$$\left[(n+s)^2 - p^2\right] a_n + a_{n-2} = 0$$

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2} = -\frac{a_{n-2}}{n(n+2p)}$$

$$\overset{\shortparallel}{p}$$

$\implies$ Since $a_1 = 0$, all odd terms vanish. We can write

$n = 0, 2, 4, \ldots$

$$n = 2m \qquad a_{2m} = -\frac{a_{2m-2}}{2m(2m+2p)} = -\frac{a_{2(m-1)}}{4m(m+p)}$$

$m = 0, 1, 2, \ldots$

$\implies$ We can solve the recurrence and simplify the coeffs using the Gamma function, $\Gamma(p+1) = p\Gamma(p)$

$$\text{If} \quad a_0 = \frac{1}{2^p \Gamma(1+p)} = \frac{1}{2^p p!}$$

(*) $$u_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1+p)} \left(\frac{x}{2}\right)^{2m+p}$$

In fact, the above procedure works out identically for $s=-p$ b/c only $s^2 = (\pm p)^2 = p^2$ appears in the matching conditions. So we get

$$u_{(-p)}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1-p)} \left(\frac{x}{2}\right)^{2m-p}$$

Equivalently, we can just think of p as ranging over both pos. and neg. real ~~#~~s in (*).