The Singular Value Decomp. (Part 2)

Recap

Hilbert-Schmidt operator has form

$$[Kf](x) = \int_{\mathbb{R}^d} \underbrace{k(x,y)}_{\in L^2(\mathbb{R}^d \times \mathbb{R}^d)} f(y)\,dy, \qquad x \in \mathbb{R}^d.$$

with operator norm controlled by $k(x,y)$,

$$\|K\|_{L^2 \to L^2} \le \underbrace{\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |k(x,y)|^2 \,dx\,dy\right]^{1/2}}_{\substack{\text{called Hilbert-Schmidt} \\ \text{norm of } K,\ \|K\|_{HS} < \infty}} < \infty,$$

and adjoint $[K^* f](x) = \int_{\mathbb{R}^d} \overline{k(y,x)} f(y)\,dy$.

Every H-S op. has an SVD of the form

right/left singular vecs

$$[Kf](x) = \sum_{j=1}^{\infty} \sigma_j \langle f, v_j \rangle u_j(x), \qquad \begin{matrix} \langle u_i, u_j \rangle \\ \langle v_i, v_j \rangle \end{matrix} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

singular vals

which converges in $L^2(\mathbb{R}^d)$ for every $f \in L^2(\mathbb{R}^d)$.

Singular Values of HS ops

We can think of the SVD of K as an expansion of the kernel $K(x,y)$,

$$K(x,y) = \sum_{j\geq 1}^{\infty} \overset{\text{coeffs of expansion}}{\sigma_j} \underbrace{u_j(x)\overline{v_j(y)}}_{\text{orthonormal in } L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$\Rightarrow [Kf](x) = \int_{\mathbb{R}^d} \left(\sum_{j\geq 1}^{\infty} \sigma_j u_j(x) \overline{v_j(y)} \right) f(y)\, dy$$

$$= \sum_{j=1}^{\infty} \sigma_j \left[\int_{\mathbb{R}^d} \overline{v_j(y)} f(y)\, dy \right] u_j(x)$$

$$= \sum_{j\geq 1}^{\infty} \sigma_j \langle f, v_j \rangle u_j(x)$$

The L^2 norm of kernel $K(x,y)$ is

$$\infty > \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K(x,y)|^2\, dx\, dy = \iint_{\mathbb{R}^d \mathbb{R}^d} \left| \left(\sum_{j\geq 1}^{\infty} \sigma_j u_j(x) \overline{v_j(y)} \right) \right|^2 dx\, dy$$

$$= \sum_{j\geq 1}^{\infty} \sigma_j^2 \iint_{\mathbb{R}^d \mathbb{R}^d} |u_j(x) v_j(y)|^2\, dx\, dy = \sum_{j\geq 1} \sigma_j^2$$

Note that $\{\sigma_j\}_{j\geq 1}^{\infty}$ is square summable and $\sigma_j \xrightarrow{j\to\infty} 0$.

Hilbert-Schmidt Norm

Two (most?) common/useful matrix norms:

$$\|A\|_2 = \sup_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$ "2-norm" or "Operator norm"

largest sing. val.

$$\|A\|_F = \Big[\sum_{i,j} (A)_{ij}^2\Big]^{1/2} = \Big(\sum_{j=1}^{r} \sigma_j^2\Big)^{1/2}$$ "Frobenius norm"

For H-S operators the direct analogues are

$$\|K\| = \sup_{f\in L^2(\mathbb{R})} \frac{\|Kf\|}{\|f\|} = \sigma_1 \qquad \|K\|_{HS} = \Big[\sum_{i,j=1}^{\infty} \underbrace{|\langle Ke_j, e_i\rangle|}_{\text{matrix elements in ONB } \{e_i\}_{i=1}^{\infty}}^2\Big]^{1/2} = \Big[\sum_{j=1}^{\infty} \sigma_j^2\Big]^{1/2}$$

The singular value expansion of $K(x,y)$ gives us best rank-n approx to K in both norms:

$$K_n(x,y) = \sum_{j=1}^{n} \sigma_j u_j(x)\overline{v_j(y)}$$

minimizes $\|K - K_n\|$ and $\|K - K_n\|_{H\text{-}S}$ over all rank-n operators $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and

$$\|K-K_n\| = \sigma_{n+1}, \quad \|K-K_n\|_{HS} = \left[\sum_{i=n+1}^{\infty} \sigma_i^2\right]^{1/2}.$$

Since $\sum_{n=1} \sigma_n^2 < \infty$, $K_n \to K$ as $n \to \infty$ in both "operator" and "Hilbert-Schmidt" norms.

$\Rightarrow$ HS operators can be approximated in norm by finite-rank operators!

$\Rightarrow$ This property is called compactness. Compact operators have relatively simple spectral properties and often arise in connection w/smooth kernels as well as the inverses of "nice" diff ops.

<u>Existence & Uniqueness for HS ops</u>

Consider b'dd op $T = \lambda I + K$ where K is Hilbert-Schmidt (more generally, compact).

Analogous to matrices, we have Fredholm's alt.,

Either, a) for all $f \in L^2(\mathbb{R}^d)$, $Tu = f$ has a unique solution u with $\|u\| \leq M \|f\|$, i.e., T has a b'dd inverse
indep. of f

or b) The homogeneous eqn. $Tv = 0$ has nontrivial solutions $v \in L^2(\mathbb{R}^d)$. In this case $Tu = f$ has a unique soln. if and only if $\langle f, v \rangle = 0$ for all v in $N(T^*)$.

In particular, $\dim(N(T)) = \dim(N(T^*)) < \infty$ as long as $\lambda \neq 0$ for $T = \lambda I + K$.

Self-adjoint HS ops

If $K(x,y) = \overline{K(y,x)}$, K is self-adjoint.
In this case, the SVD has more structure:

Since $K = K^*$, "column" and "row" space are same.

The eigenvalue problem for K is

$$Ku = \lambda u \qquad u \in L^2(\mathbb{R}^d).$$

<u>Spectral Thm</u>

If $K: H \to H$ is selfadjoint compact, then there is an ONB $\{\varphi_k\}_{k=1}^{\infty} \subset H$ of eigenvectors of K. Moreover, eigenvalues are real and $\lambda_k \to 0$ as $k \to \infty$.

Note by Fred. Alt. that multiplicity $\dim(N(K-\lambda I))$ is <u>finite</u> for $\lambda \neq 0$.

We say that K has eigenvalue expansion

$$K(x,y) = \sum_{j=1}^{\infty} \lambda_k \varphi_j(x) \overline{\varphi_j(y)},$$

which coincides with SVD of K when $\lambda_k \geq 0$ (such K is called semidefinite).