Modified Power Series Solutions

ODE solutions are not always "smooth," and may not be resolved by a power series.

Euler's Eqn.

$$x^2 u''(x) + a x u'(x) + b u(x) = 0$$

$= 0$ at origin

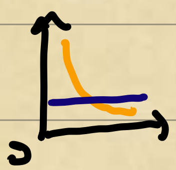
The origin is a singular point of this ODE.

$$r x^{r-1}, \quad r(r-1)x^{r-2}$$

$$\tilde{u}(x) = x^r \quad \Rightarrow \quad (\underbrace{r(r-1) + ar + b}_{r^2 + (a-1)r + b = 0})\,\tilde{u}(x) = 0$$

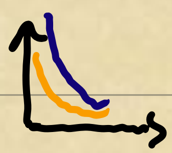
$$r_\pm = \frac{1}{2}\left(1 - a \pm \sqrt{(1-a)^2 - 4b}\right)$$

Solutions are not always smooth! E.g.



$$u(x) = c_1 1 + c_2 x^{-1/2} \qquad (a = 1/2,\ b = -1/2)$$

two lin. indep.



$$u(x) = c_1 x^{-1} + c_2 x^{-2} \qquad (a = 5,\ b = 2)$$

The singularities is determined by the indicial eqn.

$$r^2 + ar + b = 0$$

$$(r - r_+)(r - r_-) = 0 \Rightarrow r_+ = r_-$$

x^r $\Rightarrow$ If $\mathrm{Re}(r) \geq 0$, then the soln is bdd at $x = 0$.

$\Rightarrow$ If $\mathrm{Im}(r) \neq 0$, solutions oscillate

$\Rightarrow$ Repeated roots, 2^{nd} solution has a log term

$$u(x) = c_1 x^r + c_2 x^r \log x$$

Modified Power Series

Idea: Modify basis set to "factor" out the singular behavior:

$$\{1, x, \ldots, x^N\} \Rightarrow \{\overset{x^r 1}{x^r}, \overset{x^r x}{x^{r+1}}, \ldots, \overset{x^r x^N}{x^{r+N}}\}$$

Equivalently, use a power series ansatz with singular factor

$$\tilde{u}(x) = \underbrace{x^r}_{\text{singular part}} \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\text{smooth part}}$$

Need to find power series coeffs & exponent.

Example: $x^2u''(x) + 4xu'(x) + (x^2+2)u(x) = 0$

$\uparrow$ singular point at $x=0$

$$2u(x) = 2x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2a_n x^{n+r}$$

$$x^2u(x) = \sum_{n=0}^{\infty} a_n x^{n+r+2}$$

x^{r+2}

$$u'(x) = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$4xu'(x) = \sum_{n=0}^{\infty} 4(n+r)a_n x^{n+r}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r}$$

$\Rightarrow$ Match coeffs of x^{n+r} $(n \geq 0)$

coeffs of	x^r	x^{r+1}	x^{r+2}		x^{r+n}
x^2u''	$r(r-1)a_0$	$(r+1)ra_1$	$(r+2)(r+1)a_2$		$(n+r)(n+r-1)a_n$
$4xu'$	$4ra_0$	$4(r+1)a_1$	$4(r+2)a_2$	$\cdots$	$4(n+r)a_n$
x^2u	0	0	a_0		a_{n-2}
$2u$	$2a_0$	$2a_1$	$2a_2$		$2a_n$
(RHS)	0	0	0		0

depend on r!

$$\begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$L(r)\,\underline{a} = \underline{0} \qquad \text{(nonlinear eigenvalue problem)}$$

$$(L-\rho(r)I)\,\underline{a} = \underline{0}$$

$$(\underbrace{r(r-1)+4r+2}_{=0})\,a_0 = 0$$

$$r^2+3r+2=0 \Rightarrow (r+2)(r+1)=0$$

$$r=-1 \text{ and } r=-2$$

$$\tilde{u}_+(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n \qquad \tilde{u}_-(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n$$

$$r=-1 \Rightarrow a_n = -\frac{1}{n(n+1)}a_{n-2} \qquad a_0 = \text{free},\ a_1 = 0$$

$$a_n = \begin{cases} \dfrac{(-1)^{n/2}}{(n+1)!}a_0, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$m = \frac{n}{2}$

$$\tilde{u}_+(x) = a_0 x^{-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m}$$

$$= a_0 x^{-2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= a_0 \frac{\sin x}{x^2} \quad \Leftarrow \text{singular at } x=0$$

Fuch's Theorem

$$(*) \qquad u'' + f(x)u' + g(x)u = 0$$

Idea: $x^2u'' + axu' + bu = 0 \Rightarrow u'' + \frac{a}{x}u' + \frac{b}{x^2}u = 0$

Thm: If $xf(x)$ and $x^2g(x)$ have convergent power series at $x=0$, then $(*)$ has either

a) 2 Lin. Indep. solutions of form

$$u_\pm(x) = x^{r_\pm}\sum_{n=0}^{\infty} a_n^\pm x^n$$

b) 1 Modified Power series soln. $S_1(x)$ and another lin indep. soln of form $S_1(x)\log(x) + S_2(x)$.