Series Solutions for ODEs

A linear transformation $T : \underbrace{V \to W}_{\text{vector spaces}}$ has

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg \qquad f, g \in V$$

If V and W have bases $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, respectively, then T has a matrix representation $T_{X \to Y} : \mathbb{C}^n \to \mathbb{C}^m$.

$$T(\overbrace{a_1 x_1 + \cdots + a_n x_n}^{V}) = \overbrace{b_1 y_1 + \cdots + b_m y_m}^{W}$$

$$\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{m1} & \cdots & T_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$T_{X \to Y} \quad \underline{a} \quad = \quad \underline{b}$$

Question: Can we use tools from linear algebra to solve & analyze diff. eq. & integral eq.?

Idea: Choose a basis and reduce solving linear diff. eq./integral eq. to matrix algebra.

Example: Differential operator $\frac{d}{dx}$ on $\mathbb{P}_N$.

Take $V = \mathbb{P}_N$ and $W = \mathbb{P}_{N-1}$ with monomial bases:

$$\frac{d}{dx} 1 = 0, \quad \frac{d}{dx} x = 1, \quad \dots, \quad \frac{d}{dx} x^N = N x^{N-1}$$

$$(*) \qquad \overset{(N-1)\times N}{\underset{T_{x\to x}}{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & N \end{bmatrix}}} \underset{\underline{a}}{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}} = \underset{\underline{b}}{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{bmatrix}}$$

Question: Given $p \in \mathbb{P}_{N-1}$, solve $u' = p$.

$\Rightarrow$ Solve the upper triangular linear system by back substitution.

$$a_N = \frac{b_{N-1}}{N}, \quad a_{N-1} = \frac{b_{N-2}}{N-1}, \quad \dots, \quad a_1 = b_0$$

The coefficient a_0 is undetermined so

$$u(x) = \underset{\substack{\uparrow \\ \text{free}}}{a_0} + \underset{\substack{\uparrow \\ =a_1}}{b_0} x + \dots + \underset{\substack{\uparrow \\ =a_{N-1}}}{\frac{b_{N-2}}{N-1}} x^{N-1} + \underset{\substack{\uparrow \\ =a_N}}{\frac{b_{N-1}}{N}} x^N$$

$\Rightarrow$ To obtain a unique solution for a first-order ODE, we typically need to specify an auxillary equation, e.g., B.C.'s:

$$u' = p, \text{ such that } u(0) = 1.$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 1 \\ \hline b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{bmatrix}$$

The linear system is now invertible and the ODE has a unique soln:

$$u(x) = 1 + b_0 x + \cdots + \frac{b_{N-1}}{N} x^N.$$

Series Solutions

In general, solutions to ODEs need not be polynomials. However, we can often construct a sequence of polynomial approximations that converge to the true soln.

Example: Find an approximate solution to $u' = u$ in $\mathbb{P}_N$.

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & 0 \quad N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \\ a_N \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \\ a_{N-1} \end{bmatrix}$$

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2}, \quad a_3 = \frac{a_2}{3} \quad \dots, \quad a_N = \frac{a_{N-1}}{N}$$

"recurrence relation" $\Rightarrow \quad k a_k = a_{k-1}, \quad k = 1, 2, 3, \dots, N$

$$a_1 = a_0, \quad a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_0}{3 \cdot 2}, \quad \dots, \quad a_N = \frac{a_0}{N!}$$

$$\Rightarrow \quad u(x) \approx \underset{\substack{\uparrow \\ \text{free}}}{a_0} \sum_{k=0}^{N} \frac{1}{k!} x^k = u_N(x)$$

In the limit $N \to \infty$, does $u_N \to u$, the exact solution of the differential eq.?

Define $\quad u_N(x) = a_0 \sum_{k=0}^{N} \frac{x^k}{k!}$

Claim: Given any interval $[a, b] \subset \mathbb{R}$, the partial sums $\{u_N\}_{N=1}^{\infty}$ converge uniformly to a continuous function on $[a,b]$.

pf/ As shown in HW2 (Problem 1(c)) for the interval $[-1,1]$, the sequence of partial sums $\{u_N\}_{N=1}^{\infty}$ forms a Cauchy sequence in $\mathbb{P} = \bigcup_{N=1}^{\infty} \mathbb{P}_N$ with respect to norm

$$\|f\| = \sup_{a \le x \le b} |f(x)|.$$

As shown in HW2 (Problem 1(d)), this implies that $\{u_N\}_{N=1}^{\infty}$ converges uniformly to a continuous function $u:[a,b] \to \mathbb{R}$:

$$\|u_N - u\| = \sup_{a \le x \le b} |u_N(x) - u(x)| \to 0 \text{ as } N \to \infty.$$

Note: The limit function is, of course, the classical exponential e^x. The series also converges uniformly on compact subsets of the complex plane, where

$$e^z = e^x(\cos y + i \sin y), \quad z = \underset{\text{real}}{x} + i\underset{\text{imag.}}{y} \in \mathbb{C}.$$

To check that the limit function satisfies the diff. eq. $u'(x) = u(x)$, we differentiate through term-by-term:

$$u'(x) = a_0 \sum_{k=1}^{\infty} \frac{k x^{k-1}}{k!} = a_0 \sum_{j=0}^{\infty} \frac{x^j}{j!} = u(x).$$

Note that term-by-term differentiation is justified because the power series converges absolutely for any $x \in \mathbb{R}$, i.e., has radius-of-convergence $R = \infty$ and a power series can be differentiated term-by-term at any point inside its interval of convergence.

This method of constructing power series solutions to diff. eq., by constructing a sequence of polynomial solutions, is called the "method of Frobenius." Don't forget to check if/where power series converges!

Example: solve $u'(x) = 2x\,u(x)$

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$
$$u'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + N a_N x^{N-1}$$
$$2x\,u(x) = \qquad 2a_0 x + 2a_1 x^2 + \cdots + 2a_{N-2} x^{N-1}$$
$$+ 2a_{N-1} x^N + 2a_N x^{N+1}$$

We can derive a recurrence for the coefficients of our power series soln. by matching coefficients of each basis element $\{1, x, x^2, \ldots, x^N\}$.

	1	x	x^2	$\cdots$	x^{N-1}	x^N	x^{N+1}
u'	a_1	$2a_2$	$3a_3$	$\cdots$	Na_N		
$2xu$		$2a_0$	$2a_1$	$\cdots$	$2a_{N-2}$	$2a_{N-1}$	$2a_N$

can't match these in $\mathbb{P}_N$

$$a_k = \frac{2}{k} a_{k-2} \qquad k = 2, 3, 4, \ldots, N$$

Moreover, $a_1 = 0$ so all odd powers of x vanish. Set $k = 2j$ and we have

$$a_{2j} = \frac{a_{2j-2}}{j} \quad \Rightarrow \quad a_{2j} = \frac{a_0}{j!} \qquad j = 1, 2, 3, \ldots$$

$$u(x) \approx a_0 \sum_{j=0}^{N} \frac{x^{2j}}{j!} = u_N(x)$$

We can check, as in previous example, that $u_N \xrightarrow{N\to\infty} u$ absolutely and uniformly for any $x \in [a,b] \subset \mathbb{R}$ and that, by term-by-term differentiation, the limit function $u(x)$ satisfies $u'(x) = 2x\,u(x)$.

<u>Note 1:</u> The limit function is $u(x) = e^{x^2}$.

<u>Note 2:</u> The recurrence relation for coeffs of the approximation $u_N \in \mathbb{P}_N$ is equiv. to the linear system:

$$\underbrace{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \\ & & & 0 & N \end{bmatrix}}_{N\times(N+1)} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & & & \\ 2 & 0 & 0 & & \\ & 2 & 0 & 0 & \\ & & \ddots & \ddots & \ddots \\ & & 2 & 0 & 0 \end{bmatrix}}_{N\times(N+1)} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

$N\times(N+1)$ nullspace problem

$$\Rightarrow \begin{bmatrix} 0 & 1 & & & \\ 2 & 0 & 2 & & \\ & 2 & 0 & 3 & \\ & & \ddots & \ddots & \ddots \\ & & 2 & 0 & N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

move 1^{st} column to right-hand side and use a_0 as free parameter $\Rightarrow$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ 0 & 2 & & & \\ & 0 & 3 & & \\ & & \ddots & \ddots & \\ & & & 0 & N \end{bmatrix}}_{\substack{N \times N \\ \text{invertible}}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \underset{\substack{\uparrow \\ \text{free} \\ \text{param.}}}{a_0} \begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ 1-parameter family of soln's.