Series Solutions of ODEs

Recap

ODEs $\quad -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda w(x) u \quad$ "Sturm-Liouville"

"Regular": $\quad p, p', q, w$ continuous and $p > 0$, $w > 0$.

B.C.'s $\quad \underbrace{\alpha_1 u(0) + \alpha_2 u'(0) = 0}_{\text{at least one nonzero}}, \quad \underbrace{\beta_1 u(1) + \beta_2 u'(1) = 0}_{\text{at least one nonzero}}$

Regular S-L problems have compact resolvent, so ONB of eigvecs $u_1, u_2, u_3, \ldots$, and real, increasing eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$ (unbounded).

in fact, each λ_k has 1 lin. indep. vec

"Singular": S-L problems not regular $\Rightarrow$ called singular

Singular problems exhibit a much richer range of behavior. Moreover, many families of special functions arise as solutions to singular Sturm-Liouville eigenproblems.

Power Series Solutions

A useful technique for both regular & singular problems is to look for smooth power series solns:

$$u(x) = \sum_{n=1}^{\infty} a_n x^n$$

We'll try to find coefficients as follows

Example: $u' = 2xu$

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$
$$u'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$
$$2x\,u(x) = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + \cdots + 2a_n x^{n+1} + \cdots$$

For $u' - 2xu = 0$, we need

	1	x	x^2	$\cdots$	x^{n+1}
u'	a_1	$2a_2$	$3a_3$	$\cdots$	$(n+2)a_{n+2}$
$2xu$		$2a_0$	$2a_1$		$2a_n$

$\Rightarrow a_1 = 0 \qquad \Rightarrow a_2 = a_0 \qquad \Rightarrow a_3 = \frac{2}{3} a_1 = 0$

$$\Rightarrow \boxed{a_{n+2} = \frac{2}{n+2} a_n} \qquad n = 0, 1, 2, \ldots$$

recurrence for soln coeffs

Note that all odd terms vanish since $a_1 = 0$.

Set $n = 2m$, $\quad a_{2m} = \frac{2}{m} a_{2m-2} = \frac{1}{m} a_{2m-2}$

$$= \frac{1}{m}\frac{1}{m-1} a_{2m-4} = \cdots = \frac{1}{m!} a_0$$

$$\Rightarrow \boxed{u(x) = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!}}$$

Power series solution

We can check radius of convergence w/ ratio test:

$$L = \lim_{m\to\infty} \frac{\frac{x^{2m+2}}{(m+1)!}}{\frac{x^{2m}}{m!}} = \lim_{m\to\infty} \frac{x^2}{m+1} = 0 \quad \text{for any } |x| < \infty.$$

Therefore, power series converges absolutely and uniformly on any interval of the form $[-R, R]$.

Note that the series is simply the exponential series with $x \to x^2$, so we have

$$u(x) = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} = a_0 e^{x^2}$$

We can check that $u(x) = a_0 e^{x^2}$ indeed satisfies the ODE, since

$$u'(x) = (a_0 e^{x^2})' = 2x a_0 e^{x^2} = 2x u(x) \checkmark$$

This is the same solution we get from separation of variables

$$\frac{du}{u} = 2x\,dx \Rightarrow \ln u = x^2 + \ln \text{Const}$$

$$\Rightarrow u(x) = (\text{Const.}) e^{x^2}$$

Note the power series for $u(x)$ converges (abs.) for every complex z with $|z| < \infty$, a special situation associated w/ the fact that $u'(z)$ is complex differentiable for every $z \in \mathbb{C} \setminus \{\infty\}$.

Legendre's Eqn.

$$(1-x^2)u'' - 2xu + \ell(\ell+1)u = 0$$

$\hookleftarrow$ singular points at $x = \pm 1$

Assume power series ansatz:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \ell(\ell+1)\,u(x) = \sum_{n=0}^{\infty} \ell(\ell+1) a_n x^n$$

$$u'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad -2xu'(x) = \sum_{n=1}^{\infty} -2n\, a_n x^n$$

$$u''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad -x^2 u''(x) = \sum_{n=2}^{\infty} -n(n-1) a_n x^n$$

	const.	x	x^2	x^3	...	x^n ...
u''	$2a_2$	$6a_3$	$12a_4$	$20a_5$		$(n+2)(n+1)a_{n+2}$
$-x^2u''$			$-2a_2$	$-6a_3$		$-n(n-1)a_n$
$-2xu'$		$-2a_1$	$-4a_2$	$-6a_3$		$-2n a_n$
$\ell(\ell+1)u$	$\ell(\ell+1)a_0$	$\ell(\ell+1)a_1$	$\ell(\ell+1)a_2$	$\ell(\ell+1)a_3$		$\ell(\ell+1)a_n$

$$\Rightarrow 2a_2 + \underbrace{\ell(\ell+1)}_{\ell^2+\ell} a_0 = 0 \rightarrow a_2 = -\frac{\ell(\ell+1)}{2} a_0$$

$$\Rightarrow \quad 6a_3 + (l^2 + l - 2)a_1 = 0 \rightarrow a_3 = -\frac{(l-1)(l+2)}{\underset{3!\ \nearrow}{6}} a_1$$

$$\Rightarrow \quad 12a_4 + (l^2 + l - 6)a_2 = 0 \rightarrow a_4 = -\frac{l(l+1)(l-2)(l+3)}{\underset{4!\ \nearrow}{12}} a_0$$

For x^n coeff: $(n+2)(n+1)a_{n+2} + (\underbrace{l^2 + l - n^2 - n}_{\substack{(l+n)(l-n)+(l-n) \\ (l-n)(l+n+1)}})a_n = 0$

$$\boxed{a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)} a_n} \qquad n = 0, 1, 2, \ldots$$

Our series expansion separates (formally) into two series w/ even and odd coeffs, respectively.

$$u(x) = a_0\left[1 - \frac{\overset{1}{l}\,(\overset{2}{l+1})}{2!}x^2 + \frac{\overset{1}{l}\,(\overset{2}{l+1})(\overset{-1}{l-2})(\overset{4}{l+3})}{4!}x^4 + \cdots\right]$$

$$+ a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 + \cdots\right]$$

By the ratio test, we can check the radius of convergence of our soln series:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+2}x^{n+2}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(l-n)(l+n+1)}{(n+2)(n+1)}}_{\sim \frac{n^2}{n^2}x^2 = x^2 \text{ to leading order}} x^2 \right| = x^2$$

So series converges absolutely and uniformly when

$$L = x^2 < 1 \implies x \in (-1, 1).$$

In general, they may not converge $\overset{\text{abs.}}{\wedge}$ when $x = \pm 1$. singular points

To illustrate the general structure of the series solution, consider the case $l = 0$.

$\Rightarrow$ The "a_0" series terminates after the constant term, since all other coeffs have a factor of l $(= 0)$.

$\Rightarrow$ The "a_1" series diverges at $x = -1$,

$$u(-1) = a_0 + a_1 \overbrace{\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \cdots\right)}^{\text{divergent}}$$

Therefore, the only smooth solution on $[-1,1]$ is

$$u_0(x) = a_0$$

Similarly, consider the case $\ell = 1$.

$\Rightarrow$ The "a_1" series terminates after $n=1$, since all following terms have a factor of $\ell - 1 \, (=0)$.

$\Rightarrow$ The "a_0" series diverges at $x = \pm 1$,

$$u(1) = a_1 \quad + a_0 \Big[1 + \underbrace{\sum_{n \geq 0}^{\infty} \frac{n-1}{n+1}}_{\text{diverges}} \Big]$$

Therefore, the only smooth soln. on $[-1,1]$ is

$$u_1(x) = a_1 x$$

These two cases illustrate the general trend.

When l is even, the "a_0" series terminates after $l=n$ terms and the "a_1" series diverges at $x=\pm 1$.

When l is odd, the "a_1" series terminates after $l=n$ terms and the "a_0" series diverges at $x=\pm 1$.

Therefore, for each integer $l \geq 0$, the Legendre equation has a single smooth linearly indep. solution in the form of a degree l polynomial.

If we normalize so that $u_l(1)=1$, these are precisely the Legendre polynomials, multiples of those we computed via Gram-Schmidt.

Generalized Power Series Method:

Recall that Euler's equation had the form:

$$x^2 u''(x) + a x u'(x) + b u(x) = 0$$

This is a simple example of a singular 2nd order ODE and recall that its solutions were not always smooth/b'dd at the singular point $x=0$.

$$u(x) = x^r \quad \Rightarrow \quad (r^2 + ar + b)\, u(x) = 0$$

$$\Rightarrow \quad r = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

The solutions may not always be non-negative integer powers of x. For example, $a = \frac{1}{2}$

$$\Rightarrow \quad u(x) = c_1 + c_2 x^{-1/2}$$

The term $x^{-1/2}$ is not differentiable at $x=0$ and has no convergent power series there.

To account for these potential singularities, we can use a generalized power series ansatz:

$$u(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Similar to Euler's equation, we will allow the ODE to "tell" us what r should be, revealing the form of the singularity (if any).

Example $\quad x^2u''(x) - 2xu'(x) + (x^2+2)u(x) = 0$

singular point at $x=0$

$$u(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$u'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

By matching coefficients of equal powers, we get

	x^r	x^{r+1}	x^{r+2}	$\cdots$	x^{r+n}
x^2u''	$r(r-1)a_0$	$(r+1)ra_1$	$(r+2)(r+1)a_2$		$(n+r)(n+r-1)a_n$
$-2xu'$	$-2ra_0$	$-2(r+1)a_1$	$-2(n+2)a_2$		$-2(n+r)a_n$
x^2u	0	0	a_0	$\cdots$	a_{n-2}
$2u$	$2a_0$	$2a_1$	$2a_2$		$2a_n$

The key to determining r is the balance of coeffs of x^r (just as for Euler's eqn):

$$(r(r-1) - 2r + 2)a_0 = 0$$

"indicial equation" $\Rightarrow$ $r^2 - 3r + 2 = 0$

$$r = -1 \quad \text{and} \quad r = -2$$

From here, we proceed to compute a series solution for each case separately

$$u_1(x) = \sum_{n=0}^{\infty} a_n r^{n-1} \qquad u_2(x) = \sum_{n=0}^{\infty} b_n r^{n-2}$$

The mechanics are identical to the earlier power series case, now that r has a concrete value. One derives a recurrence for the coeffs an (and, separately, b_n), solves the recurrence (if possible), and then simplifies the resulting power series (as far as possible).