Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Taylor series. Finite difference formulas are often used to approximate derivatives from function samples on a grid. For example, the second-order centered difference approximation to the first derivative of a function f on an equally spaced grid with grid spacing $h > 0$ is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

(a) If $u(x)$ has three continuous derivatives, show that the centered difference formula approximates $u'(x)$ with accuracy $\mathcal{O}(h^2)$ as $h \to 0$. Derive an explicit upper bound for the approximation error using an appropriate Taylor polynomial.

(b) How does the bound in part (a) change if $u(x)$ is only twice continuously differentiable?

(c) Derive a fourth-order accurate centered difference formula to approximate $u'(x)$ from samples $u(x-2h), u(x-h), u(x), u(x+h), u(x+2h)$ with grid spacing $h > 0$. Here, "fourth-order" accurate means that your approximation should have accuracy $\mathcal{O}(h^4)$ as $h \to 0$ when $u(x)$ has five continuous derivatives.

(d) Use the second- and fourth-order finite-difference formulas to approximate the derivatives of the functions $\sin(2\pi x)$, $\cos(\pi(x-0.5))$, and $\sqrt{(1+\cos(2\pi x))^3}$ on an equispaced grid of $n = 500$ points on the periodic interval $[0, 1]$. Plot the approximation error for the derivative at each grid point and then plot the maximum absolute error on grids with $n = 100, 200, 300, \ldots, 10^4$ (use a logarithmic scale for both axes). Can you explain the behavior of the error for each function (e.g., why proportional to h^2, h^4, etc.)?

1) Fourier series. In the Fourier basis, a 2-periodic function $f(x)$ on $[-1, 1]$ is written as

$$f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\pi k x}, \qquad \text{where} \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi k x} f(x)\, dx.$$

(a) Show that if f is n-times continuously differentiable with $|f^{(n)}(x)| \leq M$ on the periodic interval $[-1, 1]$, then $|\hat{f}_k| \leq \sqrt{2}M/(\pi k)^n$. (**Hint:** integrate by parts.)

(b) If $f(x)$ is approximated by the truncated series $f_N(x) = (1/\sqrt{2}) \sum_{k=-N}^{N} \hat{f}_k e^{i\pi k x}$, how do you expect the approximation error $E_N = \max_{-1 \leq x \leq 1} |f(x) - f_N(x)|$ to scale as N is increased? Derive a rigorous bound for the approximation error.

(c) Compute the Fourier coordinates of $f(x) = \sin^3(\pi x)$, $g(x) = |x|$, and $h(x) = |\sin(\pi x)|^3$. Plot the magnitude of the Fourier coefficients $-250 \leq k \leq 250$ on a logarithmic scale. Based on the coefficient plots, roughly what accuracy do you expect if you approximate g and h by truncating their Fourier series, discarding terms with $|k| > 250$? Compare your observations with your work in part (a) and (b).