Linear Operators & Equations

With the basic equipment of vector spaces, norms, and inner products, we turn to the analysis of linear eqns:

$$(*) \qquad \underset{\substack{\uparrow \\ \text{linear transformation}}}{A}x = b$$

where $A : V \to W$, $b \in W$ are given. We want to start with simple questions like

classical "well posed" criteria
- ⇒ When does $(*)$ have a solution?
- ⇒ How many solutions are there?
- ⇒ Do the solutions depend continuously on the data (i.e., on b, A, etc.)?

Of course, we also want to look at techniques for calculating/approximating solutions.

Later, we will look at more detailed "analytic" properties of solutions (beyond linear algebra).

Our goal is to use tools from "linear algebra" to study things like differential and integral equations. To calibrate our expectations, let's start in familiar territory.

<u>Example:</u> $A \in \mathbb{R}^{m \times n}$ $(A: \mathbb{R}^n \to \mathbb{R}^m)$

<u>Q1:</u> When does $Ax = b$ have a solution?

$\Rightarrow$ When $b \in C(A) = \{$span of cols. of $A\}$

<u>Q2:</u> If $b \in C(A)$, how many solutions does $Ax = b$ have?

$\Rightarrow$ If columns of A are lin. indep., then solution is unique.

$\Rightarrow$ If columns of A are lin. dep., then there are infinitely many solns.

(can always add solutions of $Ay = 0$)

Q3: Does x depend continuously on A, b?

$\Rightarrow$ For $m=n$, if A is invertible, <u>yes.</u>

<u>continuity in b</u>

$$Ax = b + \overset{\text{small perturbation}}{e} \qquad \|e\| \ll 1$$

$$x = A^{-1}b + A^{-1}e \qquad \|A^{-1}e\| \leq \|A^{-1}\|\|e\| \to 0 \text{ as } \|e\| \to 0$$

<u>continuity in A</u>

$$(A+E)x = b \quad \text{as } \|E\| \to 0 \text{ ?}$$

$$x = (A+E)^{-1}b = A^{-1}\overbrace{(I + A^{-1}E + O(\|E\|^2))}^{\text{"Neumann series"}}b$$

$$= A^{-1}b + O(\|E\|)$$

$$\to 0 \quad \text{as } \|E\| \to 0 \ \checkmark$$

$\Rightarrow$ Otherwise, in general, <u>not</u> continuously dependent on data A, b.

Q: How do these answers change if

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$$

vectors in inner product space $(V, \langle \cdot, \cdot \rangle)$

is a quasimatrix?

Analogous to the matrix case, the "column space" and "null space" of $A: V \to W$ play a key role in existence & uniqueness.

Let $A: V \to W$ be a linear transformation between two vector spaces. The null space of A is

$$N(A) = \{x \in V : Ax = 0\}.$$

The null space is a subspace of V, b/c

$$x, y \in V \text{ and } Ax = Ay = 0 \Rightarrow A(\overset{\in V}{x+y}) = Ax + Ay = 0$$

The range of A is a subspace of W:

$$R(A) = \{y \in W : y = Ax \text{ for some } x \in V\}.$$

For a unique solution to $Ax = b$, we want

$$\underset{\text{existence?}}{b \in R(A)} \qquad \text{and} \qquad \underset{\text{uniqueness?}}{N(A) = \{0\}}$$

(complicated in inf. dim!)

Example: $\frac{d^2}{dx} : \mathbb{P}_n \to \mathbb{P}_n$

Does $\frac{d^2}{dx} u(x) = p(x) \in \mathbb{P}_n$ have a soln?

$\Rightarrow$ Only if $p \in \mathbb{P}_{n-2} \subset \mathbb{P}_n$

If $p \in \mathbb{P}_{n-2}$, is the solution unique?

$\Rightarrow$ No, because $u(x) + \underbrace{\alpha + \beta x}_{\frac{d^2}{dx^2}(\alpha+\beta x)=0} = v(x)$
also satisfies $\frac{d^2}{dx} v = p(x)$

Example: $\frac{d}{dx} : C^\infty([0,1)_{\text{periodic}}) \to C^\infty([0,1)_{\text{periodic}})$

Does $\frac{d}{dx} u(x) = f(x)$ have a solution?

$$\int_0^{+1} \frac{d}{dx} u(x)\, dx = \int_0^{+1} f(x)\, dx$$

$$\Rightarrow \underbrace{\Big[u\Big]_0^{+1}}_{0 \text{ (periodicity)}} = \int_0^{+1} f(x)\, dx$$

$\Rightarrow$ Solution exists only if $\int_0^1 f(x)\,dx = 0$.

E.g. if f is orthogonal to 1 (or span$\{1\}$)

in inner product $\langle g, h\rangle = \int_0^1 g(x)h(x)\,dx$.

$\Rightarrow$ Again solution is not unique b/c $\frac{d}{dx}(\text{const}) = 0$.

For a deeper understanding of these conditions, we need to look at the adjoint.

The Adjoint of a linear Operator

Given a linear operator $T: V \to W$
on an inner product space $(V, \langle\cdot,\cdot\rangle_V), (W, \langle\cdot,\cdot\rangle_W)$
the adjoint of T is the operator T^* s.t.

$$\langle Tu, v\rangle = \langle u, T^* v\rangle \qquad u, v \in V.$$

* (We will need to refine this definition, take this as first attempt to generalize $A^T \in \mathbb{R}^{n\times m}$)

Example: $A \in \mathbb{R}^{m\times n}$ $\quad A^* = A^T \in \mathbb{R}^{n \times m}$

(or $A \in \mathbb{C}^{m\times n}$ $\quad A^* = \bar{A}^T = A^H$ (conjugate transpose))

$$\langle Ax, y\rangle = y^T(Ax) = (A^T y)^T x = \langle x, A^T y\rangle$$

Example: $T = \frac{d}{dx} : C^\infty([0,1)_{\text{periodic}}) \to C^\infty([0,1)_{\text{periodic}})$

$$\langle Tf, g\rangle = \int_0^1 \left[\frac{d}{dx} f(x)\right] g(x)\,dx$$

$$= \underbrace{g(1)f(1) - g(0)f(0)}_{=0 \text{ (periodicity)}} - \int_0^1 f(x)\left[\frac{d}{dx} g(x)\right] dx$$

$$= \langle f, -Tg\rangle$$

$\Rightarrow$ $T^* = -T$ example of a skew-Hermitian operator (generalizes skew-Hermitian matrix)

Example: $T = \frac{d}{dx} : C^1([0,1]) \to C([0,1])$

if $f, g \in C^1([0,1])$ $\langle Tf, g\rangle = \int_0^1 \frac{df}{dx} g(x)\,dx = \underbrace{f(x)g(x)\Big|_0^1}_{} - \int_0^1 f(x)\frac{dg}{dx}\,dx$

$= -\langle f, Tg\rangle$ $\quad$ ← must vanish

[e.g. if $f(0) = 0$ and $g(1) = 0$] ← for T^* exist.

$\Rightarrow$ We need to restrict the domain of T to, e.g., $\{f \in C^1([0,1]) : f(0) = 0\}$ and then T^* domain is $\{g \in C^1([0,1]) : g(1) = 0\}$.

$\Rightarrow$ Boundary conditions become part of the domain of T, T^* and play a key role in well-posedness.

$\Rightarrow$ Note that $-T = T^*$ are <u>not</u> skew-Hermitian, b/c they do <u>not</u> have the same domain in this example.

$\Rightarrow$ What would adjoint of $\frac{d}{dx}$ be if we had $f(0) = f(1) = 0$? How would this effect solutions to $\frac{d}{dx} u = f$?

The adjoint plays a key role in the theory of $Ax = b$ for linear maps. Next lecture, we will wrap up some technical loopholes in our study of well-posedness in inf-dims.