Operator Exponentials

Recap: Suppose that $A : D(A) \to H$ is a self-adjoint operator w/compact resolvent.

$$Au_j = \lambda_j u_j \qquad j = 1, 2, 3, \ldots$$

completeness $\quad x = \sum_{j=1}^{\infty} \langle u_j, x \rangle u_j$

diagonalization $\quad Ax = \sum_{j=1}^{\infty} \lambda_j \langle u_j, x \rangle u_j$

Resolvent $\quad (A - z)^{-1} x = \sum_{j=1}^{\infty} (\lambda_j - z)^{-1} \langle u_j, x \rangle u_j$

Exponential $\quad e^{At} = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle u_j, x \rangle u_j \quad (?)$

Function $\quad f(A) = \sum_{j=1}^{\infty} f(\lambda_j) \langle u_j, x \rangle u_j \quad (?)$

Q: Why is $f(A)$ useful?

Q: When is $f(A)$ well-defined?

Q: What about if $A^* \neq A$?

Operator exponentials (Normal operators)

$$\frac{du}{dt} = Au \qquad u : \mathbb{R}_+ \to H \qquad \text{soln. map}$$
$$u|_{t=0} = g \in H$$

"Normal operator"

First, suppose that A has ONB of eig.-vec's.

$$u(t) = \sum_{j=1}^{\infty} c_j(t) u_j, \qquad c_j(t) = \langle u_j, u(t) \rangle$$

Solve for $c_j(t)$ by taking inner-products:

$$\langle u_j, \dot{u}(t) \rangle = \langle u_j, Au \rangle$$

$$\Rightarrow \quad \dot{c}_j(t) = \lambda_j \langle u_j, u(t) \rangle = \lambda_j c_j(t)$$

$$\Rightarrow \quad c_j(t) = e^{\lambda_j t} c_j(0)$$

$$\Rightarrow \quad u(t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle u_j, g \rangle u_j$$

Notice that since $\{u_j\}_{j\geq 1}^{\infty}$ are ONB, each coordinate of $u(t)$ evolves independently and the $\{u_j\}$'s cannot "add up" or "cancel out" b/c they are orthogonal

Q: What role do the eigenvalues play?

<u>Well-posedness</u> $\quad u(t) = e^{At} g$

First, we need $e^{\lambda_j t}$ b'dd as $j \to \infty$ (for each/any $t > 0$) so that

$$\|u(t)\|^2 = \sum_{j\geq 1}^{\infty} e^{2\lambda_j t} |\langle u_j, g \rangle|^2$$

$$\leq M(t) \sum_{j \geq 1}^{\infty} |\langle u_j, g \rangle|^2 \leq M \|g\|^2$$

$\Rightarrow$ This makes $e^{At} : g \to u(t)$ a b'dd operator

$\Rightarrow$ Also note it is linear: $e^{At}(\alpha f + g) = \alpha e^{At} f + e^{At} g$

So if $e^{\lambda_j t} \leq M(t)$ for $j = 1, 2, \ldots$, e^{At} is b'dd linear op.

This is true if $e^{\lambda_j} \leq M$, $Re(\lambda_j) \leq \alpha$, $j = 1, 2, 3, \ldots$

$\Rightarrow$ In this case $g \to e^{At} g$ is also continuous

$$\lim_{t \to 0} \| g - e^{At} g \| = 0$$

And since $e^{A(t+\Delta t)} = e^{At} e^{A\Delta t}$, we have

$$\lim_{t \to 0} \| u(t+\Delta t) - u(t) \| = 0$$

b/c $e^{A(t+\Delta t)} g - e^{At} g = e^{At}(e^{\Delta t} g - g) \to 0$ as $\Delta t \to 0$

These three properties make e^{At}

$\Rightarrow$ a solution operator for (*) and

$\Rightarrow$ make (*) well-posed

We say that $(e^{At})_{t \geq 0}$ is a

strongly continuous semigroup on H.

Qualitative Properties

$$u(t) = e^{At} g = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle u_j, g \rangle u_j$$

We can analyze further w/ Euler's identity

$$\lambda_j = \mu_j + i \nu_j$$

$$e^{\lambda_j t} = e^{\mu_j t}(\cos(\nu_j t) + i \sin(\nu_j t))$$

Real part of λ_j governs decay/growth:

$$\langle u_j, u(t) \rangle = e^{\mu_j t} \langle u_j, g \rangle (\cos(\nu_j t) + i \sin(\nu_j t))$$

$$\begin{aligned} \mathrm{Re}\,\lambda_j > 0 &\Rightarrow \text{growth} \\ \mathrm{Re}\,\lambda_j < 0 &\Rightarrow \text{decay} \\ \mathrm{Re}(\lambda_j) = 0 &\Rightarrow \text{modulus conserved} \end{aligned}$$

Imaginary part of λ_j governs frequency.

Self-adjoint A $\to$ Real Eigenvalues
$\Rightarrow$ Pure growth/decay

Heat on $x \in [-1,1]$ $\quad u_t = \Delta u \quad$ all modes decay

$\uparrow$

$\{\lambda_j < 0\}_{j=1}^{\infty} \quad u(t) \to 0$

Skew-adjoint A – Imag Eigenvalues

Schrodinger on $x \in [-1,1]$ $\quad u_t = i\Delta u \quad$ all modes conserve modulus $\|u(t)\|$ conserved.

$\uparrow$

$\{\operatorname{Re} \lambda_j = 0\}_{j=1}^{\infty}$