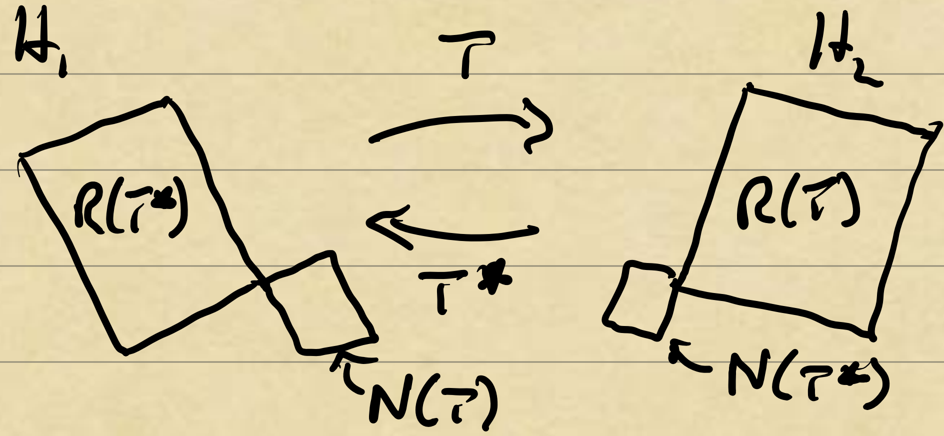
Operators w/ Bounded Inverse

Question: When is $Tx = b$ "well-posed"?

↑ Linear Transformation $T: H_1 \to H_2$



Recap

$\Rightarrow$ A function $f \in L^2(-1,1)$ is weakly differentiable (in $L^2(-1,1)$) if there is $g \in L^2(-1,1)$ such that

$$(*) \qquad -\int_{-1}^{+1} f(x)\varphi'(x)\,dx = \int_{-1}^{+1} g(x)\varphi(x)\,dx$$

for any $\varphi \in C_0^1(-1,1) = \{\varphi \in C^1 \mid \varphi(\pm 1) = 0\}$.

$\Rightarrow$ The map $Df = g$ is a linear transformation with range $R(D) = L^2(-1,1)$ and domain $\mathcal{D}(D) = \{f \in L^2(-1,1) \mid f, g \in L^2(-1,1) \text{ satisfy } (*)\} \subset L^2(-1,1)$.

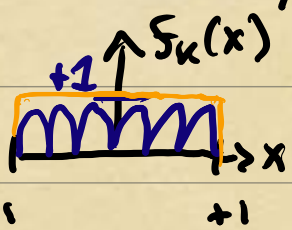
⇒ A linear operator $T : H_1 \to H_2$ is b'dd if

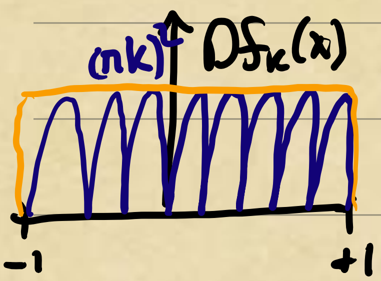
$$\|T\| = \sup_{f \in H_1} \frac{\|Tf\|_{H_2}}{\|f\|_{H_1}} < \infty$$

Example: Is the D b'dd (from $L^2(-1,1) \to L^2(-1,1)$)?

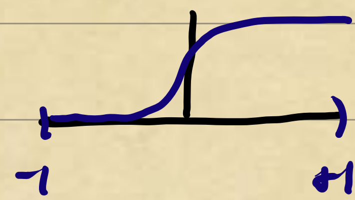
For $k = 1, 2, 3, 4, \ldots$ consider $f_k(x) = \sin(k\pi x)$.

$Df_k(x) = (\pi k)\sin(k\pi x)$

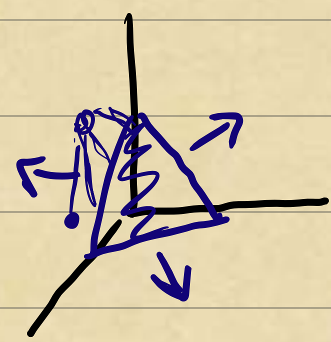




$$\frac{\|Df_k\|}{\|f_k\|} \to \infty$$



Weak Derivative operator is an unbounded op.



Example: Is $[Tf](x) = \int_{-1}^{x} f(y)\,dy$ b'dd on L^2?

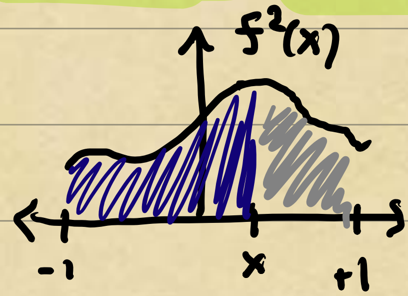
$$\|Tf\|^2 = \int_{-1}^{+1} \Big[\overbrace{\int_{-1}^{x} f(y)\,dy}^{[Tf](x)}\Big]^2 dx$$

Jensen's inequality

φ is convex, then

$$\varphi\left[\int_a^b g(y)\,dy\right] \le \int_a^b \varphi(g(y))\,dy$$

$$\le \int_{-1}^{+1} \left[\int_{-1}^{x} f(y)^2\,dy\right] dx$$



$$\le \int_{-1}^{+1} \Big[\underbrace{\int_{-1}^{+1} f(y)^2\,dy}_{\|f\|^2}\Big] dx$$

$$\le 2\|f\|^2$$

$$\Rightarrow \quad \|T\| = \sup_{f\in L^2} \frac{\|Tf\|}{\|f\|} \le \sqrt{2}$$

Operators w/ Bounded Inverse

weakly differentiable functions

Weak differentiation: $\quad D: \mathcal{D}(D) \to L^2(-1,1)$

Cumulative Integration: $\quad T: L^2(-1,1) \to L^2(-1,1)$

$\Rightarrow$ Essentially,* T is an inverse for D.

* See Homework 6 to work out details.

Inverses of differential operators are often/typically b'dd in practice.

Question: What implications does a b'dd inverse hold for well-posedness?

$$(**) \qquad Tx = b \qquad T : \mathcal{D}(T) \to L^2(-1,1)$$

Suppose that $T^{-1} : L^2(-1,1) \to L^2(-1,1)$ is b'dd.

existence
uniqueness $\Rightarrow \quad x = T^{-1}b$

Does x depend continuously on T, b?

continuity on B

$$T x_e = b + \overset{e \in L^2(-1,1)}{e} \Rightarrow x_e = \underbrace{T^{-1}b}_{x} + T^{-1}e$$

$$\Rightarrow \|x_e - x\| \leq \|T^{-1}e\| \qquad \frac{\|T^{-1}e\|}{\|e\|} \leq \|T^{-1}\|$$

$$\Rightarrow \|x_e - x\| \leq \|T^{-1}\| \|e\|$$

$$\Rightarrow \text{As } \overset{(\|e\| \to 0)}{e \to 0}, \ \|x_e - x\| \to 0.$$

continuity on T

Remarkably, $(T + \overset{\|E\| \ll 1}{E})^{-1}$ exists and is b'dd for all E with $\|T^{-1}E\| < 1$ (e.g., $\|E\| \leq 1/\|T^{-1}\|$)

$$(T+E)^{-1} = T^{-1}\left(\sum_{k=0}^{\infty} (T^{-1}E)^k\right)$$ Neumann Series

$$(T+E)x_E = b \Rightarrow x_E = (T+E)^{-1}b$$
$$= T^{-1}b + \underbrace{\mathcal{O}(\|E\|)}_{\to 0 \text{ as } \|E\| \to 0}$$

$\Rightarrow$ Finite-dimensions, invertibility implies well-posed.

$\Rightarrow$ Infinite-dimensions, b'dd invertibility implies well-posed

Integral Reformulation

(***) $u'(x) + v(x)u(x) = f(x)$ $\qquad u(-1) = 0$

$\Rightarrow \quad g'(x) = f(x), \quad g(-1) = 0$

$$[Tf](x) = g(x) = \int_{-1}^{x} f(x)\,dx$$

Apply T to (***) $\Rightarrow$
$$u(x) + \underbrace{\int_{-1}^{x} v(y)u(y)\,dy}_{[T_v u](x)} = \int_{-1}^{x} f(y)\,dy$$
$$u(x) + [T_v u](x) = h(x)$$

$$[I + T_v]u = h$$