Differential Operators

So far our study of linear equations and linear operators has focused primarily on compact (in particular, Hilbert-Schmidt) operators.

- Bounded invertibility & Neumann series
- Fredholm alternative for compact ops
- Spectral theory of self-adjoint compact ops
- SVD for Hilbert-Schmidt ops

These tools give us powerful characterizations and explicit formulas for soln of linear eqns.

<u>But</u>

Many problems in applied math are formulated as differential eqn's and we have seen that differential operators are, in general, <u>not</u> compact or Hilbert-Schmidt.

How can we tackle unbounded ops?

The Resolvent

The key idea is a systematic generalization of "integral reformulation," that allows us to study diff ops using the theory of bounded / Compact / Hilbert-Schmidt operators.

Consider $L: D(A) \to H$ an unbounded operator with domain $D(A) \subset H$. For many diff ops, the resolvent

$$R(z) = (L - z)^{-1}$$

is not only a b'd inverse of $L - z$, it is also compact or even Hilbert-Schmidt.

Example: $L = -\frac{d^2}{dx^2}$ $\quad D(A) = C_0^2([0,1])$

self-adjoint

$$\begin{aligned} &(L - z)u = f \\ &u(0) = u(1) = 0 \end{aligned} \quad \underset{\text{HW4}}{\Longrightarrow} \quad u(x) - z\int_0^1 K(x,y)u(y)\,dy = \int_0^1 K(x,y) f(y)\,dy$$

$$\Downarrow$$

$$(I - zK)u = Kf$$

In HW4, we saw that

$$K(x,y) = \begin{cases} t(1-x), & t \leq x \\ x(1-t), & x \leq t \end{cases}$$

Hilbert-Schmidt
Self-adjoint

By the spectral theorem for Selfadjoint compact operators, we know that

$u_j = \sin(j\pi x)$
$\lambda_j = \frac{1}{(k\pi)^2}$

real

$$K u_j = \lambda_j u_j \qquad j = 1, 2, 3, \ldots$$

ONB

$$\langle u_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so any function in $H = L^2([0,1])$ has

$$u = \sum_{j=1}^{\infty} \langle u, u_j \rangle u_j \quad \text{and} \quad Ku = \sum_{j=1}^{\infty} \lambda_j \langle u, u_j \rangle u_j$$

Therefore, to find u, we need to find $\langle u, u_j \rangle$:

$$(I - zK) u = Kf$$

eigen basis

$$\sum_{j=1}^{\infty} (1 - z\lambda_j) \langle u, u_j \rangle u_j = \sum_{j=1}^{\infty} \lambda_j \langle f, u_j \rangle u_j$$

$$(1 - z\lambda_j) \langle u, u_j \rangle = \lambda_j \langle f, u_j \rangle$$

If $z \neq \lambda_j^{-1}$ for any $j = 1, 2, 3, \ldots$, then

$$\langle u, u_j \rangle = \frac{\lambda_j \langle f, u_j \rangle}{1 - z\lambda_j}$$

and
$$u = \sum_{j=1}^{\infty} \frac{\lambda_j \langle f, u_j \rangle}{1 - z\lambda_j} u_j = \sum_{j=1}^{\infty} \frac{\langle f, u_j \rangle}{\underbrace{1/\lambda_j}_{\text{eigenvals of } L} - z} u_j$$

In other words, we have found a linear map from data $f \to$ solution u.

$$u = (\underbrace{L - z}_{R(z) \text{ "resolvent"}})^{-1} f$$

$$R(z) f = \sum_{j=1}^{\infty} \frac{\langle f, u_j \rangle}{\mu_j - z} u_j \leftarrow \begin{cases} \mu_j = 1/\lambda_j \\ L u_j = \mu_j u_j \end{cases}$$

Resolvent has kernel w/ E.V. expansion

$$r(x, y; z) = \sum_{j=1}^{\infty} \underbrace{\frac{1}{\mu_j - z}}_{\to 0 \text{ as } j \to \infty} u_j(x) \overline{u_j(y)}$$

Note that resolvent is a compact, in fact, Hilbert-Schmidt operator because K is, so $\mu_j = 1/\lambda_j \to \infty$ and $\sum_{j=1}^{\infty} \left| \frac{1}{\mu_j - z} \right|^2 < \infty$