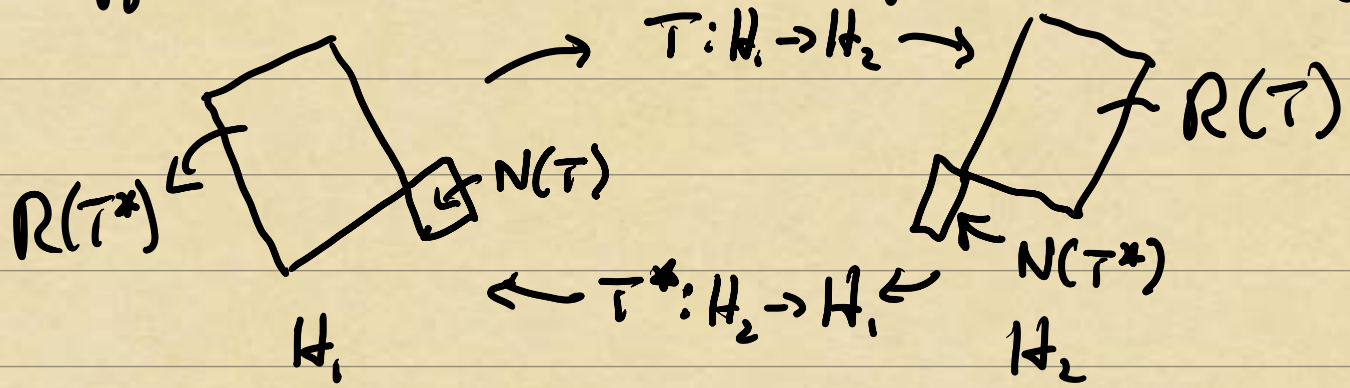
Bounded & Unbounded Operators

Suppose we have a linear operator $T: H_1 \to H_2$



Adjoint: $T^*: H_2 \to H_1$ s.t. $\langle Tx, y \rangle_{H_2} = \langle x, T^* y \rangle_{H_1}$

$N(T) = \{ x \in H_1 \mid Tx = 0 \}$, $N(T^*) = \{ y \in H_2 \mid Ty = 0 \}$

$$R(T) = \{ y \in H_2 \mid y = Tx,\ x \in H_1 \}$$

$$R(T^*) = \{ x \in H_1 \mid x = T^* y,\ y \in H_2 \}$$

Example: $T: L^2([-1,1]) \to L^2([-1,1])$

$[Tf](x) = \int_{-1}^{x} f(y)\,dy$ $\qquad$ $[T^* g](y) = \int_{y}^{1} g(x)\,dx$

$R(T) \neq L^2([-1,1])$ $\qquad$ $N(T^*) = \{0\}$

<u>Idea:</u> We know that if $L = \frac{d}{dx}$

$$\underbrace{L\,T}_{I}f = f, \qquad TLf = f \text{ if } Lf \in L^2([-1,1])$$

$$\Rightarrow \quad Tu = v \overset{\text{if } Lv \in L^2([-1,1])}{\Rightarrow} LTu = Lv \Rightarrow u = Lv$$

Differentiation in $L^2(\overset{\Omega}{[-1,1]})$

Question: Which functions can we differentiate in $L^2([-1,1])$, and still get $\frac{du}{dx} \in L^2([-1,1])$?

If $u \in C^1[-1,1]$, then $\frac{du}{dx} \in C[-1,1] \subset L^2([-1,1])$.

Can we expand the domain of $\frac{d}{dx}$ so that its range fills all of $L^2([-1,1])$?

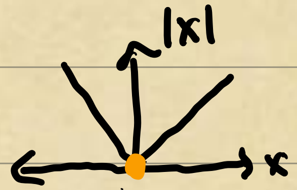
Def. We say that $f \in L^2([-1,1])$ has a weak derivative $g \in L^2([-1,1])$ if

$$(*) \qquad -\int_{-1}^{+1} f(x)\varphi'(x)\,dx = \int_{-1}^{+1} g(x)\varphi(x)\,dx$$

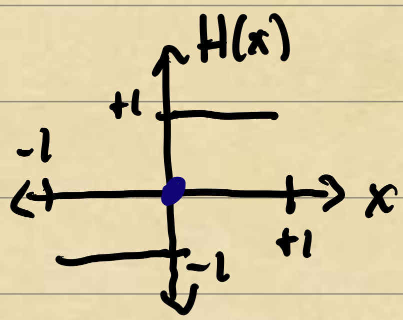
holds for any $\varphi \in C_0^1[-1,1] := \{\varphi \in C^1([-1,1]) \mid \varphi(\pm 1) = 0\}$.

<u>Idea:</u> Make integration-by-parts the defining characteristic of what it means to be diff.

<u>Example:</u> Is $f(x) = |x|$ weakly differentiable?



$$\int_{-1}^{+1} |x|\,\varphi'(x)\,dx = \int_{-1}^{0} (-x)\,\varphi'(x)\,dx + \int_{0}^{+1} x\,\varphi'(x)\,dx$$



$$= \overset{=0}{\cancel{-1\,\varphi(0)}} - \int_{-1}^{0} (-1)\,\varphi(x)\,dx + \cancel{1\,\varphi(0)} - \int_{0}^{1} (1)\,\varphi(x)\,dx$$

$$= -\left[\int_{-1}^{0} (-1)\,\varphi(x) + \int_{0}^{1} (+1)\,\varphi(x)\,dx\right]$$

$$= -\int_{-1}^{+1} H(x)\,\varphi(x)\,dx$$

$\Rightarrow$ The weak derivative is well-defined (unique $-$ $\int_{-1}^{+1} |g(x) - \tilde{g}(x)|^2 dx = 0$) and it is a linear transformation.

$\Rightarrow$ It generalizes the classical derivative so that if $f' = g \in C[-1,1]$, then the $L^2([-1,1])$ weak derivative of f is g.

$\Rightarrow$ Product rule, chain rule, etc. all have natural analogues for weak derivatives.

<u>Def.</u> Denote the linear map from $f \to g$, its L^2 weak derivative (when it exists) by

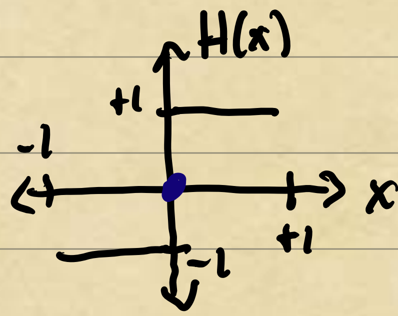
$$Df = g \quad \text{where} \quad f, g \text{ satisfy } (*).$$

$\Rightarrow$ The range $R(D) = L^2([-1,1])$, but what about the domain of D?

$$\mathcal{D}(D) = \{ f \in L^2[-1,1] \mid Df \in L^2[-1,1] \text{ exists} \}.$$

<u>Question:</u> Is $\mathcal{D}(D) = L^2([-1,1])$?

<u>Example:</u> $H(x) = \begin{cases} +1 & 0 < x < 1 \\ -1 & -1 < x < 1 \end{cases}$



$$\underbrace{-\int_{-1}^{+1} H(x)\varphi'(x)\,dx}_{\neq 0} = \underbrace{\int_{-1}^{+1} F(x)\varphi(x)\,dx}_{= 0}$$

$$\Rightarrow \mathcal{D}(D) \neq L^2([-1,1]), \text{ but}$$

$$\mathcal{D}(D) = H^1([-1,1]) = \{ f \in L^2([-1,1]) \mid Df \in L^2([-1,1]) \text{ exists} \}$$

classic diff. — weak diff — square integrable

$$C^1([-1,1]) \subset H^1([-1,1]) \subset L^2([-1,1])$$

Bounded Operators

A bounded linear transformation $T: H_1 \to H_2$ has limited "amplification" power:

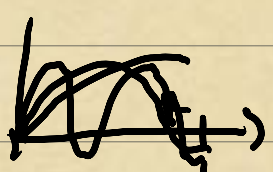
$$\|T\|_{H_1 \to H_2} = \sup_{f \in H_1} \frac{\|Tf\|_{H_2}}{\|f\|_{H_1}} < \infty$$

Example: Weak Differentiation $D: H^1(0,1) \to L^2(0,1)$

Consider $f_k(x) = \sin(kx)$ $\quad k = 1, 2, 3, \ldots$

$$\|f_k\|^2 = \int_0^1 |f_k(x)|^2 dx = \int_0^1 \sin^2(kx)\, dx = \frac{1}{2} - \frac{\sin(2k)}{4k}$$

$$\to 1/2$$



≤ 1

$$[Df_k](x) = k\cos(kx)$$

$$\|Df_k\|^2 = k^2\int_0^1 \cos^2(kx)\,dx = \frac{1}{2}k + \sin(2k) \to \infty$$

$$\Rightarrow \qquad \frac{\|Df_k\|}{\|f_k\|} \to \infty \quad \text{as} \quad k \to \infty.$$