Operators on Hilbert Spaces

Recap The Hilbert space of square-integrable funct's

$$L^2(\Omega) = \{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}) \mid \int_\Omega |f(x)|^2 dx < \infty \}$$

is an infinite-dimensional analogue of $\mathbb{R}^n$.

Differentiation (weak) and integration define linear transformations of $L^2(\Omega)$.

A linear transformation $T : H_1 \to H_2$ is bounded if

$$\|T\| = \sup_{f \in H_1} \frac{\|Tf\|_{H_2}}{\|f\|_{H_1}}.$$

If $H_1 = H_2$ we call T an operator on H_1.

$\Rightarrow$ Integration is a b'dd operator on $L^2(\Omega)$.

$\Rightarrow$ Differentiation is unbounded on $L^2(\Omega)$

Example: Differentiation vs. Integration on $L^2([0,2\pi])$.

$$\frac{d}{dx} : \underbrace{D}_{\text{"domain of } \frac{d}{dx}\text{"}} \to L^2([0,2\pi])$$

"domain of $\frac{d}{dx}$" $= \{u \in H^1([0,2\pi]) \mid u(0)=0\}$

For $k = 1, 2, 3, \ldots$

$$\frac{d}{dx}(\sin(kx)) = -k\cos(kx)$$

$$\frac{\|\frac{d}{dx}(\sin kx)\|_{L^2}}{\|\sin kx\|_{L^2}} = \frac{\sqrt{\pi k^2}}{\sqrt{\pi}} = k \quad (= 1, 2, 3, \ldots)$$

So $\frac{d}{dx}$ is "unbounded" on $L^2([0,2\pi])$.

Meanwhile, its inverse is b'dd on $L^2([0,2\pi])$.

$$[Tf](x) = \int_0^x f(y)\,dy$$

$$\|Tf\|_{L^2}^2 = \int_0^{2\pi} \left|\int_0^x f(y)\,dy\right|^2 dx \underset{\text{Jensen's inequality}}{\leq} 2\pi \int_0^{2\pi}\int_0^x |f(x)|^2\,dy\,dx$$

$$\leq 2\pi \left[\int_0^{2\pi} |f|^2 dx\right]\left[\int_0^{2\pi} dy\right]$$

$$= (2\pi)^2 \|f\|_{L^2}^2$$

$$\Rightarrow \|T\|_{L^2 \to L^2} \leq 2\pi \quad (T \text{ b'dd})$$

Bounded Inverse

The inverse of differential operators can often be expressed as bounded integral operators. This is good news b/c bounded operators are continuous.

A linear transformation $T: H_1 \to H_2$ between Hilbert spaces H_1, H_2 is continuous if

$$\lim_{n\to\infty} x_n = x \in H_1 \implies \lim_{n\to\infty} Tx_n = Tx \in H_2$$

Equivalently, by linearity only need to check sequences $\to 0 \in H_1$.

Thm A linear transformation $T: H_1 \to H_2$ is continuous if and only if it is b'dd.

$$\Rightarrow \quad \|T(x-x_n)\|_{H_2} \le \|T\|_{H_1\to H_2} \|x-x_n\|_{H_1}$$

$\Leftarrow$ Find proof in attached reading

Suppose we have $T: D \to H$ unbounded with b'dd inverse $T^{-1}: H \to H$.

Then, the solution to $Tu = f \in H$ is well-posed in the sense that

$$T\tilde{u} = f + \underbrace{e}_{\text{small perturbation}} \in H \quad \Rightarrow \quad \|u - \tilde{u}\| \to 0 \text{ as } \|e\| \to 0$$

$\Rightarrow$ We have that $u - \tilde{u} = T^{-1}e$ and since $\|T^{-1}\| < \infty$, it is continuous

$$\|u - \tilde{u}\|_H \leq \|T^{-1}\|_{H \to H} \|e\|_H \to 0 \qquad \text{as } \|e\|_H \to 0$$

Similarly, if $(T + \underbrace{E}_{\text{small perturbation}})\tilde{u} = f$, we have that

$$\|\tilde{u} - u\| \to 0 \quad \text{as } \|E\|_{H \to H} \to 0.$$

This is called "stability of b'dd invertibility" and follows from Neumann series for $(T+E)^{-1}$

$$(T+E)^{-1} = (I + T^{-1}E)^{-1}T^{-1} = T^{-1}\left(\sum_{k=0}^{\infty}(T^{-1}E)^k\right)$$

converges if $\|T^{-1}E\| < 1$ (e.g., if $\|E\| < \frac{1}{\|T^{-1}\|}$).

Integral Reformulation

Even when the inverse operator is not known (or known to exist) we can often reformulate differential equations (involving unb'dd ops) into integral equations (involving b'dd ops).

Example:

$$\frac{du}{dx} + \overset{\text{continuous}}{v(x)}\, u(x) = f(x) \qquad u(-1) = 0$$

$$\Rightarrow \quad u(x) + \int_{-1}^{x} v(y)u(y)\,dy = f(x] \qquad (*)$$

$$[Tu](x) = u(x) + \int_{-1}^{x} v(y)u(y)\,dy$$ b'dd integral operator

If we solve $(*)$, we solve ODE.

B'dd Ops

B'dd ops on $\mathcal{H}$ are much like matrices. We have that

$$N(T) = R(T^*)^\perp \text{ and } N(T^*) = R(T)^\perp$$

In general, $R(T) \neq N(T^*)^\perp$ b/c $R(T) \subset \mathcal{H}$ may not be a <u>closed</u> subspace. There may be "small gaps" so that $x \in N(T^*)^\perp$ is <u>not</u> in $R(T)$. However, $R(T) = \mathcal{H}$ for a b'dd operator if $\exists\, \delta > 0$ s.t.

$$\|T^* u\| \geq \delta \|u\| \text{ for all } u \in \mathcal{H}.$$

(A consequence of the "closed range" theorem, a foundational result in functional analysis)

This generalizes the condition for square matrices $A: \mathbb{R}^n \to \mathbb{R}^n$:

$$R(A) = \mathbb{R}^n \iff A^T x = 0 \text{ iff } x = 0$$

We can compare conditions for invertibility:

$A: \mathbb{R}^n \to \mathbb{R}^n$	(invertible?) $Ax = b \in H$, $Tu = f \in H$	$T: H \to H$ b'dd
Existance:	$A^T x = 0 \iff x = 0$	$\|Tu\| \geq \delta \|u\| \ \forall u \in H$ ($\delta > 0$ indep. of u)
Uniqueness:	$Ax = 0 \iff x = 0$	$Tu = 0 \iff u = 0$

Of course, for $\mathbb{R}^n$, fundamental thm of linear algebra assures us that $Ax = 0$ has nontrivial solutions IFF $A^T x = 0$ does.

We'll soon see why ($A^T x = 0 \iff x = 0$) is naturally replaced by ($\|Tu\| \geq \delta \|u\| \ \forall u \in H$) when we study the singular value decomp.