Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Orthogonal Projection Again. Let $E : \mathbb{R}^N \to \mathcal{H}$ be a quasimatrix, whose linearly independent columns are elements of a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, i.e.,

$$E = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_N \\ | & & | \end{bmatrix}.$$

(a) Given $f \in \mathcal{H}$, show that $\mathbf{c} \in \mathbb{R}^N$ minimizes $\|E\mathbf{c} - f\|$ if and only if $E^T E\mathbf{c} = E^T f$, where $E^T E$ and $E^T f$ are the matrix and vector, respectively, whose components are

$$(E^T E)_{ij} = \langle e_j, e_i \rangle, \qquad \text{and} \qquad (E^T f)_j = \langle f, e_j \rangle.$$

(b) Is the Gram matrix, $E^T E$, from part (a) invertible? Explain your reasoning.

(c) Verify that $E(E^T E)^{-1} E^T f$ is the orthogonal projection of f onto $\text{span}(e_1, \ldots, e_N)$.

2) Best Dictionary Approximation. Use the Chebfun system in MATLAB to compare the numerical accuracy of two formulas for best approximation of a function $f : [-1, 1] \to \mathbb{R}$:

$$\mathbf{c}_1 = (E_N^T E_N)^{-1} E_N^T f, \qquad \text{and} \qquad \mathbf{c}_2 = R_N^{-1} Q_N^T f.$$

Here, $E_N = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}$ is the quasimatrix of monomials up to degree N and $E_N = Q_N R_N$ is the QR decomposition of E_N with respect to the $L^2([-1, 1])$ inner product.

Note: You may find the MATLAB code `demo01.m` on the course repository useful.

(a) Given $f(x) = 1/(1 + 20x^2)$, plot the relative error $\|E_N \mathbf{c}_1 - f\|/\|f\|$ for dictionary sizes $N = 10, 20, 30, \ldots, 800$. Use a base-10 logarithmic scale for the relative error axis.

(b) Repeat the experiment in part (a) for $\|E_N \mathbf{c}_2 - f\|$ and compare the error curves.

(c) Plot the condition numbers of the matrices R_N and $E_N^T E_N$ for dictionary sizes $N = 10, 20, 30, \ldots, 800$. Use a base-10 logarithmic scale for the condition number axis.

(d) Interpret the error curves in parts (a) and (b) in light of your experiments in part (c).

3) Interpolation. Use the Chebfun system in MATLAB to compare the approximation accuracy of polynomial interpolants in (i) equally spaced points and (ii) Legendre points:

`xi = linspace(-1,1,N).';` and `xii = legpts(N);`.

Use a Legendre basis to form the generalized Vandermonde matrices for interpolation.

(a) Given $f(x) = 1/(1 + 20x^2)$, plot the relative error $\|f - p_N\|/\|f\|$ in both interpolants for $N = 10, 20, 30, \ldots, 500$. Use a base-10 logarithmic scale for the relative error axis.

(b) Plot the condition number of the generalized Vandermonde matrices for both interpolants for $N = 10, 20, 30, \ldots, 500$. Use a base-10 logarithmic scale for the condition number axis.

(c) Interpret the error curves in part (a) in light of your experiments in part (b).