Methods of Applied Math

MATH-6600 FALL 2024

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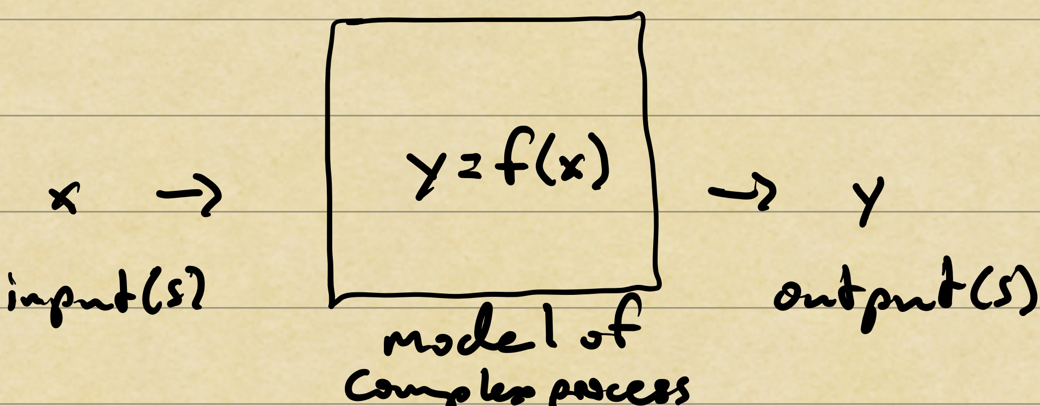
github.com/ajhPHROS/MATH-6600

Linked via LMS

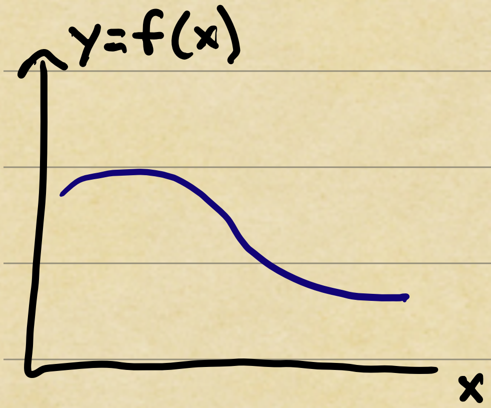
Piazza - questions, discussion, announcement

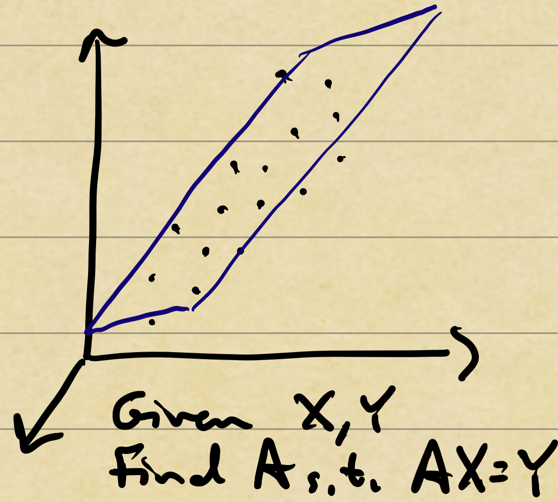
Gradescope - release and submit homework

"Study relationships among complex processes..."



Aim: Develop toolkit to analyze complex models





$\Longrightarrow$

Complexity?

The models of applied math have grown increasingly complex & sophisticated.

This course is about time-tested tools to break them down, analyze their structure, and gain insight into these rich descriptions of our world.

Course Roadmap

Linear (Part 1)

- Linear algebra w/functions
- Linear transformations (∞-dim)
- Stationary Eqs. & Equilibrium
- Linear Evolution Equations

Nonlinear (Part 2)

- Phase-plane (Poincaré) geometry
- Variational Calculus
- Fixed Point Thms
- Koopman Theory

Vector Spaces (Linear Spaces)

A vector space is a nonempty set V, closed under vector addition & scalar mult.

$$\underset{\mathbb{R} \text{ or } \mathbb{C}}{\text{scalar}}$$

Vector Addition

If $x, y \in V \Rightarrow x+y \in V$

$(x+y)+z = x+(y+z)$

$x+y = y+x$

$\exists\, 0 \in V$ s.t. $x+0 = x \ \forall x \in V$

$\forall x \in V\ \exists -x$ s.t. $x+(-x) = 0$

Scalar Multiplication

$x \in V \Rightarrow \alpha x \in V$

$1x = x$

$\alpha(\beta x) = (\alpha\beta)x$

$\alpha(x+y) = \alpha x + \alpha y$

$(\alpha+\beta)x = \alpha x + \beta x$

Essentially, $\underbrace{\alpha x + \beta y + \cdots + \gamma z}_{\text{linear combinations always stay in the space}} \in V$

Example: $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j=1,\ldots,n\}$

$$\alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}$$

Example: $C[-1,1] = \{f:[-1,1]\to\mathbb{R} : f \text{ continuous on } [-1,1]\}$

$\lim_{x\to y} f(x) = f(y),\ y\in[-1,1]$

$h(x) = \alpha f(x) + \beta g(x)$ is continuous when f, g are.

Q: What about $C^n[-1,1] = \{f\in C[-1,1] : f', \dots, f^{(n)} \in C[-1,1]\}$, is this a vector space?

Example: $\mathbb{P}_n = \{\underbrace{a_0 + a_1 x + \cdots + a_n x^n}_{\text{polynomials of degree } \le n} : a_i \in \mathbb{R} \text{ for } i=0,\dots,n\}$

$x\in[-1,1]$

The last two spaces, C^n and $\mathbb{P}_n$, are subspaces of C, the space of continuous functions.

A subspace W of V is a nonempty set $W\subseteq V$ that is closed under linear combos:

$$x, y \in W \implies \alpha x + \beta y \in W$$

linear combos stay in subspace

space of all polys.
- odd, even
- inf dim.

Can you think of other subspaces of $C[-1,1]$?

Dependence, Span, Dimension

Its useful to have a sense of the intrinsic "size" or dimension of a vector space. In some sense, how much information do we need to store and manipulate its elements?

A set $S = \{x_1, \dots, x_n\} \subset V$ is called linearly independent if the only way to get

$$\underbrace{\alpha_1 x_1 + \cdots \alpha_n x_n = 0}_{\text{linear combo that "cancels"}}$$

is if all of the scalars $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Otherwise, we say that S is linearly dependent.

The span of $S = \{x_1, \dots, x_n\} \subset V$ is the subspace

$$W = \underbrace{\{x \in V : x = \alpha_1 x_1 + \cdots + \alpha_n x_n\}}_{\text{all possible linear combos of } x_1, \dots, x_n}$$

Q i (a) Is $\{1, x, x^2, \ldots, x^n\}$ lin. indep.?

(b) What is the span of $\{1, x, \ldots, x^n\}$?

$\{1, x, \ldots, x^n\}$ is an example of a basis for $\mathbb{P}_n$.

A set $S = \{x_1, \ldots, x_n\} \subset V$ is a basis for V if it is both:

i) linearly independent, and

ii) the span of S is V.

If S is a basis for V, then for each $x \in V$, there is a unique set of scalars s.t.

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

coordinates of x in the basis S

Any basis for V must have the same # of elements and this # is the dimension of V.

Q: What is the dimension of $\mathbb{P}_n$?

Q: What is the dimension of the space of <u>all</u> polynomials, $\mathbb{P} = \bigcup_{n \geq 0}^{\infty} \mathbb{P}_n$?

<u>Example:</u> Given a polynomial $p \in \mathbb{P}_n$, how do we find its coordinates in the basis $\{1, x, \ldots, x^n\}$? [Hint: use Taylor's Thm]

$$p(x) = p(0) + p'(0)x + \cdots + \frac{p^{(n)}(0)}{n!} x^n$$

Coefficients are the derivatives of p at $x=0$.

<u>Inner Products & Norms</u>

Given a vector space V over $\mathbb{C}$ (or $\mathbb{R}$), an inner product is a map

$$\langle \, , \, \rangle : V \times V \to \mathbb{C}$$

that satisfies the following criteria:

complex conjugate

i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

iii) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

iv) $\langle x, x \rangle \geq 0$ and $= 0$ IFF $x = 0$.

<u>Example:</u> $V = \mathbb{C}^n$, $\langle x, y \rangle = y^* x = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$

<u>Example:</u> $V = \mathbb{C}[-1,1]$, $\langle f, g \rangle = \int_{-1}^{+1} f(x) \overline{g(x)} \, dx$

A norm on V is a map $\|\cdot\| : V \to [0, \infty)$ s.t.

i) $\|x\| \geq 0$ and $= 0$ IFF $x = 0$

ii) $\|\alpha x\| = |\alpha| \|x\|$

iii) $\|x + y\| \leq \|x\| \|y\|$

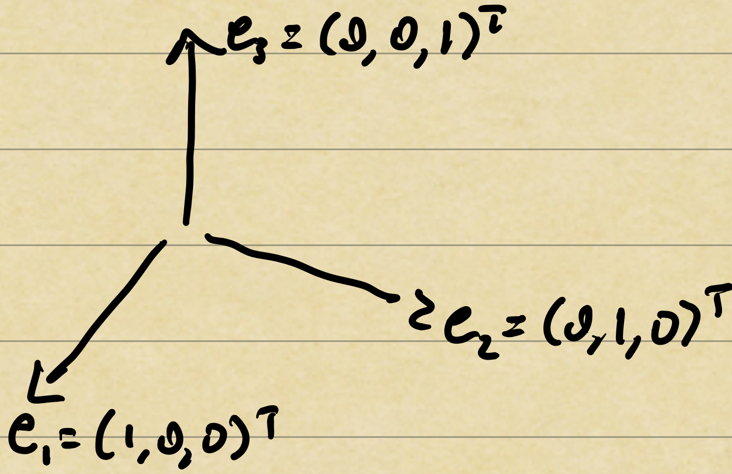
An inner product on V induces a norm by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

A vector with $\|x\| = 1$ is called a unit vector.
Two vectors with $\langle x, y \rangle = 0$ are orthogonal.

Orthogonal Bases

In an inner product space, orthogonal bases are the gold standard. They generalize the usual cartesian coordinate grid



An **orthogonal basis** $S = \{x_1, \dots, x_n\}$ is a basis with

$$\langle x_i, x_j \rangle = 0 \quad \text{when } i \neq j$$

All Basis vectors are orthogonal to each other

S is **orthonormal** if it is orthogonal and

$$\|x_i\| = 1 \quad \text{for } i = 1, \dots, n$$

Orthogonal unit vectors

Coordinates of $x \in V$ are easy to compute in an orthogonal basis $S = \{x_1, \dots, x_n\}$:

$$x = \frac{\langle x, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 + \frac{\langle x, x_2 \rangle}{\langle x_2, x_2 \rangle} x_2 + \cdots + \frac{\langle x, x_n \rangle}{\langle x_n, x_n \rangle} x_n$$

coordinates are orthogonal projections onto basis vectors (more on this later)

<u>Q</u>: Can you construct an orthonormal basis for $\mathbb{P}_n$ (say, $n = 4$)? How?