Hilbert-Schmidt Operators

Recap

Hilbert Space

The linear eqn. $Tu = f$, where $T: H \to H$ is a b'dd linear operator is well-posed if

$$\| T^{*} u \| \geq \delta \|u\| \text{ for all } u \in H.$$

We can often convert differential eqn.'s (unb'dd ops) into integral eqn's (b'dd ops)

$$u(x) + \int_\Omega K(x,y)\, u(y)\, dy = f(x) \qquad x \in \Omega \subset \mathbb{R}^d.$$

This is a Fredholm integral eqn. (2^{nd} kind), which is associated w/operator $T = I + K$.

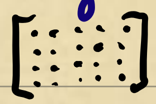
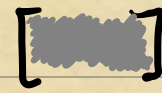
The integral op. $[Ku](x) = \int_\Omega K(x,y)\, u(y)\, dy$ is called Hilbert-Schmidt if $K \in L^2(\Omega \times \Omega)$:

$$\|K\|_{HS}^2 = \int_\Omega \int_\Omega |k(x,y)|^2\, dx\, dy < \infty.$$

The norm $\|K\|_{HS}$ is the Hilbert-Schmidt norm.

Properties of Hilbert-Schmidt Operators

Hilbert-Schmidt operators are continuous analogues of matrices:

$$A_{ij} \ (1 \le i, j \le n) \quad \Longleftrightarrow \quad k(x,y) \qquad x, y \in \Omega$$

$$(Av)_i = \sum_{j=1}^{n} A_{ij} v_j \quad \Longleftrightarrow \quad [Ku](x) = \int_\Omega k(x,y)\, u(y)\, dy$$

Fact 1: If $f : L^2(\Omega)$, then $f \to Kf$ is integrable on Ω and defined for a.e. $x \in \Omega$.

$$\Rightarrow \quad [Kf](x) = \int_\Omega k(x,y)\, f(y)\, dy \quad \text{is well-defined.}$$

Fact 2: The operator $K : L^2(\Omega) \to L^2(\Omega)$ is b'dd

$$\|K\| \le \left[\int_\Omega \int_\Omega |k(x,y)|^2\, dx\, dy \right]^{1/2} = \|K\|_{HS}$$

Fact 3: The adjoint of $K : L^2(\Omega) \to L^2(\Omega)$ is

$$[K^* f](x) = \int_\Omega k(y,x)\, f(y)\, dy \qquad x \in \Omega.$$

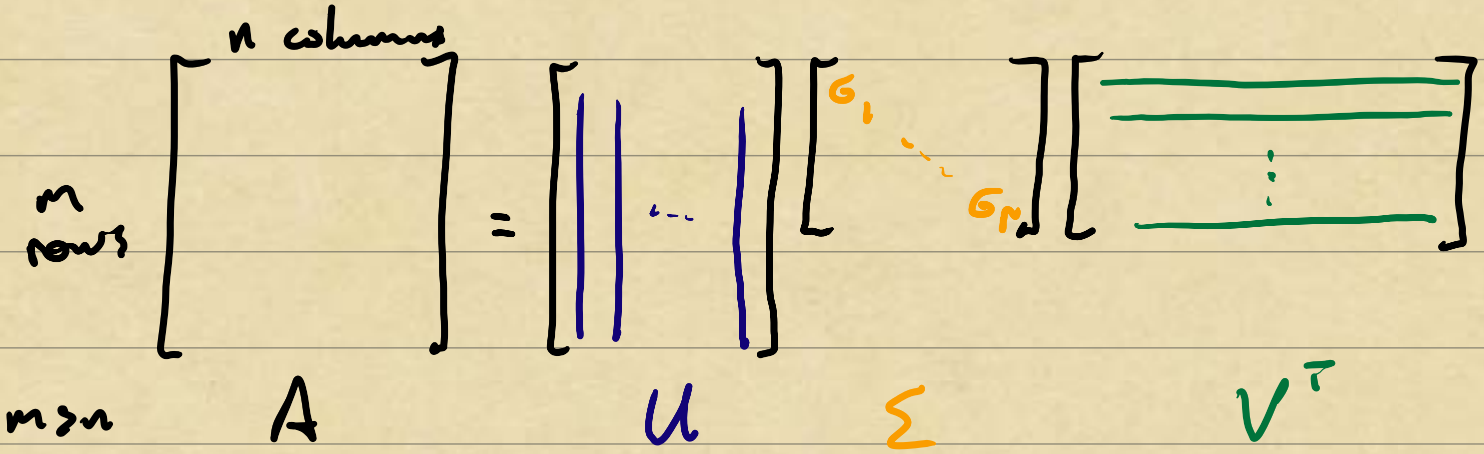
Pf Sketch

$$\langle Kf, g\rangle = \int_\Omega \left[\int_\Omega k(x,y) f(y)\,dy\right] g(x)\,dx$$

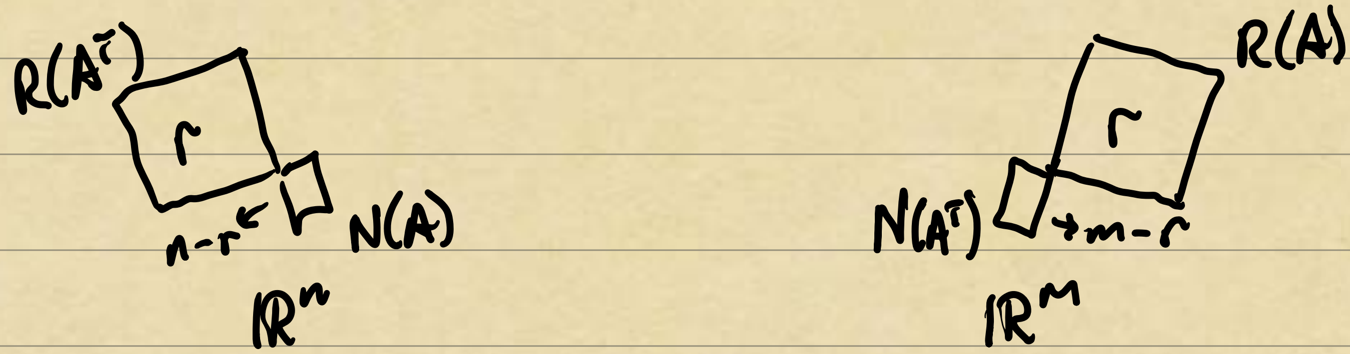
$$= \int_\Omega \Big[\underbrace{\int_\Omega k(x,y) g(x)\,dx}_{K^* g}\Big] f(y)\,dy = \langle f, K^* g\rangle$$

The Singular Value Decomposition (SVD)

The SVD of a matrix uses ONB bases for input/output domain that "make A diagonal"



Here, r = rank of A $(1 \le r \le \min(n,m))$.



U = ONB for column space of A
Σ = "Size" of coupling between v_j and u_j
V = ONB for row space of A

Given $x \in \mathbb{R}^n$, $x = Vc = \underbrace{V(V^T}_{\text{Orth. Proj. onto } R(A^T)}x)$

Given $x = \alpha v_j$, we get

$$Ax = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_r^T & - \end{bmatrix} \begin{bmatrix} | \\ \alpha v_j \\ | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ pos}$$

$$= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_j \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ pos} = \sigma_j \alpha u_j$$

In this way we can unpack $A = U \Sigma V^T$

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T$$

$$m[\,]\,[\;^{n}\;] = [m \times n]$$

SVD for Hilbert-Schmidt Ops

We can think of the SVD of K as a separable expansion of the kernel $K(x,y)$.

$$K(x,y) = \sum_{j=1}^{r} \sigma_j u_j(x) v_j(y)$$

$\{u_j\}_{j=1}^{r}$ are $L^2(\Omega)$-ONB for $R(K)$

$\{v_j\}_{j=1}^{r}$ are $L^2(\Omega)$-ONB for $R(K^*)$

$\{\sigma_j\}_{j=1}^{r}$ couple $v_j(x)$ to $u_j(x)$.

In general, r can be any positive integer or ∞.

If $r < \infty$ (finite), K is called "Finite Rank."

If $r=\infty$, The series converges in $L^2(\Omega \times \Omega)$

$$\lim_{\ell \to \infty} \int_\Omega \int_\Omega \left| K(x,y) - \sum_{j=1}^{\ell} \sigma_j u_j(x) v_j(x) \right|^2 dx\,dy = 0.$$

Hilbert-Schmidt ops are a very important example of compact operators.

The Spectral Theorem

For self-adjoint HS ops, we have

$$K(x,y) = K(y,x) \qquad x,y \in \Omega.$$

Thm If K is self-adjoint ($K = K^*$) and compact, then K has real eigenvalues and an ONB $\{u_j\}_{j\geq 1}^{\infty} \subset L^2(\Omega)$:

$$Kf = \sum_{j=1}^{\infty} \lambda_j \langle f, u_j \rangle u_j .$$

Here, $Ku_j = \lambda_j u_j$ and $\lambda_j \to 0$.