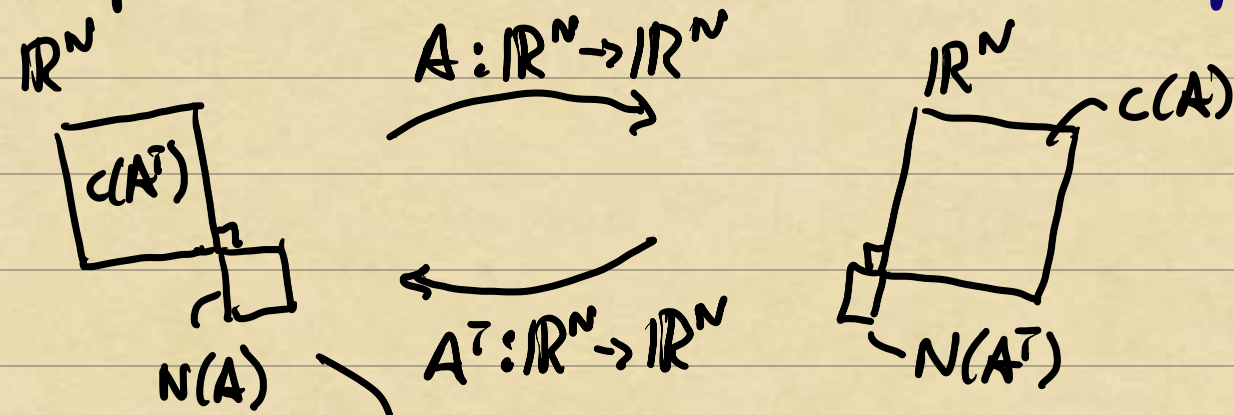
Linear Eqn's in Hilbert Spaces

Recap: The finite-dimensional theory of linear transformations can be "summarized" in a picture of the 4 fundamental subspaces:



$\frac{d}{dx}u - u = 0$

$Ax = 0 \to$ rows of A orthogonal to x

When is $Ax = b$ well-posed?

"data"

$b \in C(A)$ (i) Solutions exist

$N(A) = \{0\}$ (ii) Solution is unique

needed $A^{-1}: \mathbb{R}^N \to \mathbb{R}^N$ (iii) Solution to depend continuously on A, b

$\Rightarrow$ If A is invertible, (i)-(iii) are satisfied.

Q: When is A invertible?

$$\mathbb{R}^N = C(A) \iff N(A) = N(A^T) = \{0\}$$

2 Fundamental Subspaces

Suppose we have $T: H_1 \to H_2$, where H_1, H_2 are Hilbert spaces.

The analogue of "Column Space" is "Range"

$y_1, y_2 \in R(T)$

$\alpha T x_1 + T x_2$

$= T(\alpha x_1 + x_2)$ ✓

$$R(T) = \{ Tx \in H_2 \mid x \in H_1 \}.$$

The analogue of "Null space" is "Kernel"

$y_1, y_2 \in N(T)$

$T(\alpha_1 y_1 + y_2)$

$= \alpha_1 T y_1 + T y_2$

$= 0 + 0 = 0$ ✓

$$N(T) = \{ x \in H_1 \mid Tx = 0 \}$$

The 2 "Adjoint" Subspaces

The adjoint of $A: H_1 \to H_2$ is $A^*: H_2 \to H_1$

$x \in H_1$

$y \in H_2$

$$\langle Ax, y \rangle_{H_2} = \langle x, A^* y \rangle_{H_1}$$

Example: $A: \mathbb{R}^N \to \mathbb{R}^M$ $\qquad A^* = A^T$

$$\langle Ax, y\rangle = y^T A x = (A^T y)^T x = \langle x, A^T y\rangle$$

If $Ax = 0 \Rightarrow \langle Ax, y\rangle = 0 \iff \langle x, A^T y\rangle = 0$

Example: If $E: \mathbb{R}^N \to \overset{\text{(real)}}{H}$, then

$$\langle E c, \overset{\in H}{f}\rangle = \langle c, \overbrace{E^T f}^{\in \mathbb{R}^N}\rangle$$

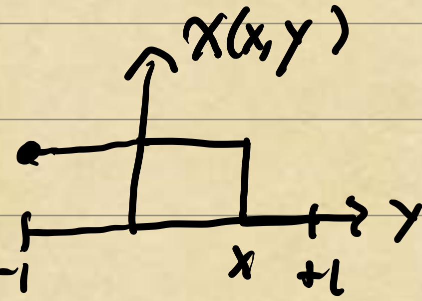
$$\begin{bmatrix} | & & | \\ e_1 & \cdots & e_N \\ | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \qquad\qquad \begin{bmatrix} \langle f, e_1\rangle \\ \vdots \\ \langle f, e_N\rangle \end{bmatrix}$$

Example: $[Tf](x) = \int_{-1}^{x} f(y)\,dy \qquad x \in [-1,1]$

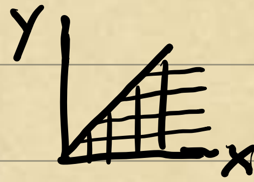
Consider $T: L^2([-1,1]) \to L^2([-1,1])$. For $g \in L^2([-1,1])$

$$\int_{-1}^{+1}\Big[\underbrace{\int_{-1}^{x} f(y)\,dy}_{[Tf](x)}\Big]g(x)\,dx \overset{?}{=} \int_{-1}^{+1} f(x)\,[T^* g](x)\,dx$$

$$= \int_{-1}^{+1}\Big[\int_{-1}^{+1} f(y)\,\chi(x,y)\,dy\Big]g(x)\,dx$$



$$= \int_{-1}^{+1}\left[\int_{-1}^{+1} g(x)\chi(x,y)\,dx\right] f(y)\,dy$$



$$= \int_{-1}^{+1}\Bigg[\underbrace{\int_{y}^{+1} g(x)\,dx}_{[T^*g](y)}\Bigg] f(y)\,dy$$

$$[Tf](x) = \int_{-1}^{x} f(y)\,dy, \quad [T^*g](y) = \int_{y}^{+1} g(x)\,dx$$

<u>Example</u> | $$[Tf](x) = \int_{-1}^{+1} K(x,y) f(y)\,dy$$

$$[T^*g](y) = \int_{-1}^{+1} K(x,y) g(x)\,dx$$

<u>A Cautionary Example:</u>

$C([-1,1]) \to C([-1,1])$

Consider $T : L^2([-1,1]) \to L^2([-1,1])$, defined

$$[Tf](x) = \int_{-1}^{x} f(y)\,dy$$

$$[T^*g](y) = \int_{y}^{+1} g(x)\,dx$$

Question 1. What is $N(T^*)$?

Find $u \in L^2([-1,1])$ s.t. $T^* u = 0$.

$\Rightarrow N(T^*) = \{0\}$

<u>Question 2.</u> What is $R(T)$?

$\Rightarrow R(T) \neq L^2([-1,1])$

(not every L^2 function has an L^2 derivative