Differential Operators

Recap

Unlike Hilbert-Schmidt integral operators, diff. operators are typically unbounded. We can study them - invertibility, diagonalization, etc. - by examining the resolvent operator

$$R(z) = (L - z)^{-1}$$

diff. op $L: D(L) \to H$

defined on $\rho(L) = \{ z \in \mathbb{C} \mid (L-z)^{-1} \text{ is a b'dd op} \}$.

The resolvent is often a compact operator or even Hilbert-Schmidt operator whose SVD, EVD encodes key information about L.

Example: $\quad L = -\frac{d^2}{dx^2} \qquad D(L) = C_0^2([0,1])$

Integral Reform $\quad (L - z)u = f \quad \Rightarrow (I - zK)u = Kf$

Resolvent Map $\quad R(z) = \sum_{j=1}^{\infty} \frac{\langle f, u_j \rangle}{\mu_j - z} u_j \qquad \begin{aligned} K u_j &= \lambda_j u_j \\ \mu_j &= 1/\lambda_j \end{aligned}$

The point of the integral reformulation is to use the spectral theorem for K to get an eigenbasis where we can calculate coord's of u from the data: "coords" of f and eigenvalues of K.

Now that we have constructed the resolvent map, let's point out a few key features:

1) The resolvent set is $\rho(L) = \mathbb{C} \setminus \{1/\lambda_j\}_{j=1}^{\infty}$. The spectrum is $\sigma(L) = \{1/\lambda_j\}_{j=1}^{\infty}$, which makes sense as $Ku_j = \lambda_j u_j \Rightarrow Lu_j = \frac{1}{\lambda_j} u_j$, i.e., $\sigma(L) =$ "eigenvalues of L" in this case.

2) Similarly, $R(z)u_j = \frac{1}{\mu_j - z} u_j$, so u_j is an eigenvector of the resolvent with eigenvalue $(\mu_j - z)^{-1}$.

3) Since K is Hilbert-Schmidt, $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$, and therefore $\sum_{j=1}^{\infty} \left|\frac{1}{\mu_j - z}\right|^2 = \sum_{j=1}^{\infty} \lambda_j^2 |1 - z\lambda_j|^{-2} < \infty$ for $z \in \rho(L)$.

4) This implies that the resolvent is a Hilbert Schmidt integral operator with kernel

also called "Green's function"

$$r(x,y;z) = \sum_{j=1}^{\infty} \frac{1}{\mu_j - z} u_j(x)\overline{u_j(y)}$$

This holds for every $z \in \rho(L)$.

5) When $z \in \mathbb{R} \setminus \lambda(L)$, resolvent is also selfadjoint since $r(x,y;z) = \overline{r(y,x;z)}$.

Operators w/compact resolvent

The situation in the example above is typical for a broad class of differential operators: Their resolvent is a compact operator that encodes spectral properties.

Q: What can we say about ops w/ compact resolvent?

Suppose that $R(z) = (L-z)^{-1}$ is <u>compact</u> & <u>self-adj</u> for some $z \in \mathbb{R}$. Then $z \in \rho(L)$ and

$$R(z)f = \sum_{j=1}^{\infty} \nu_j \langle u_j, f \rangle u_j$$

where $R(z)u_j = \nu_j u_j$ by spectral thm. and $\nu_j \to 0$ as $j \to \infty$, $\{u_j\}$ is an ONB.

$$\Rightarrow \quad (L-z)^{-1} u_j = \nu_j u_j$$
$$\Rightarrow \quad \nu_j (L-z) u_j = u_j$$
$$\Rightarrow \quad L u_j = \left(\frac{1}{\nu_j} + z\right) u_j$$

So every eigenvector of $R(z)$ is an eigenvector of L w/ eigenvalue $\mu_j = (z + 1/\nu_j)$.

Any self-adjoint operator $L : D(L) \to H$ with compact resolvent has

i) ONB of eigenvectors

ii) real eigenvalues $\to \infty$ as $j \to \infty$

iii) invertible if and only if $0 \notin \lambda(L)$

Let's look at some concrete examples

Regular Sturm-Liouville

$$[Lu](x) = -\frac{1}{w(x)}\left(\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x)\right)$$

where $\alpha_1 u(0) + \beta_1 u'(0) = 0 \qquad \alpha_2 u(1) + \beta_2 u'(1) = 0$ $\quad$ or $[-1,1]$ okay too

α_1, α_2 or β_1, β_2 not both zero

is a regular Sturm-Liouville problem if

p, p', q, w are continuous and $p, w > 0$ on $[0,1]$

$\Rightarrow L$ is self-adjoint w.r.t. $\langle f,g\rangle = \int_0^1 f(x)g(x)w(x)\,dx$

and has a compact resolvent.

$\Rightarrow$ Formed as in HW 5, kernel of resolvent built from null-functions satisfying one-sided B.C.'s.

$\Rightarrow$ RSL ops have real eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_n$$

with ONB of eigenfunctions.

SL problems appear often when solving PDE on simple geometries by separation of variables, e.g. square, disk, sphere, cone, etc.

Many special functions arise as soln's to <u>singular</u> SL problems. Need some complex analysis to study these.

<u>Example</u>: Bessel's equation (arises solving PDEs on disc)

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\Rightarrow (x y')' + \left(x - \frac{\nu^2}{x}\right) y = 0$$

<u>Example</u>: Legendre equation (arises solving PDEs in sphere)

$$(1-x^2) y'' - 2x y' + \nu(\nu+1) y = 0$$

$$\Rightarrow \left((1-x^2) y'\right)' + \nu(\nu+1) y = 0$$