Integral Operators: Hilbert-Schmidt

If we have a $T: D(T) \to H$, consider

$$(*) \qquad Tx = b$$

$\Rightarrow$ Equation $(*)$ is well-posed, we typically want T to have a b'dd inverse $T^{-1}: H \to H$.

Q: How do we find b'dd inverse for, e.g.,

$$(**) \quad \underbrace{u'(x) + v(x)u(x)}_{[Tu](x)} = f(x), \quad \text{s.t. } u(-1) = 0.$$

Integral Reformulation

$$[Kf](x) = \int_{-1}^{x} f(x)\,dx \quad \overset{g'(x)=f(x) \quad g(-1)=0}{\iff} \quad [Kf](x) = g(x)$$

$$\underbrace{u(x) + \int_{-1}^{x} v(y)u(y)\,dy}_{[I + K_v]u} = \int_{-1}^{x} f(y)\,dy$$

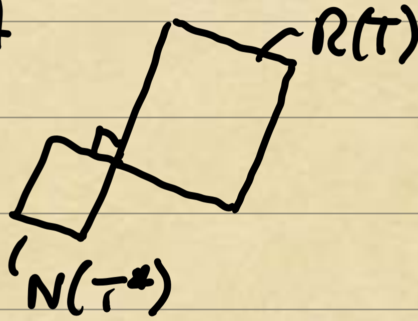
$$[I + K_v]u = Kf$$

Criteria for B'dd Invertibility

Suppose we have a b'dd op $T: H \to H$.
Then $T^{-1}: H \to H$ exists and is b'dd if

there is a $\delta > 0$, $\|T^* u\| \geq \delta \|u\|$ for all $u \in H$.

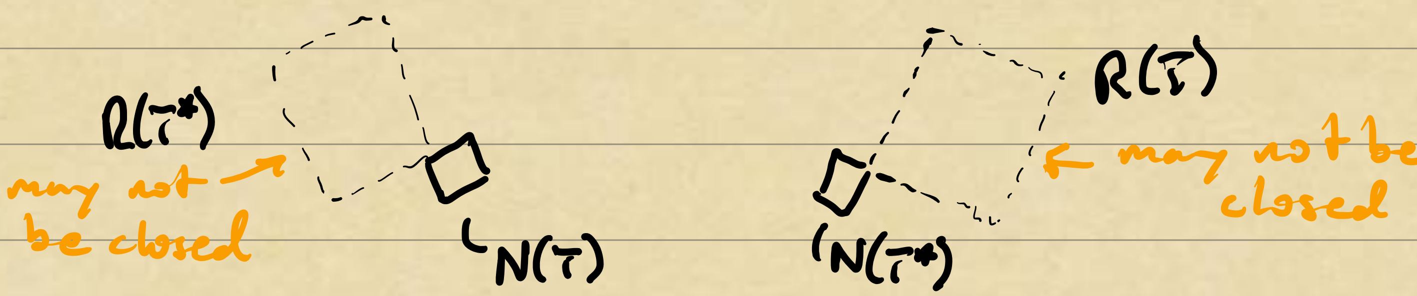
$T: H \to H$



<u>Pf</u> This is a consequence of "Closed range Thm."

For a general b'dd operator, we have

$$N(T^*) = R(T)^{\perp}, \quad N(T) = R(T^*)^{\perp}$$



If T has limited shrinking power
$\Rightarrow H = N(T^*) \cup R(T)$

Compare existence & uniqueness:

$\underline{A : \mathbb{R}^n \to \mathbb{R}^n}$ $\qquad$ $\underline{T : H \to H}$ b'dd

Existence: $A^T x = 0 \iff x = 0$ $\qquad$ $\|T^* u\| \geq \delta \|u\| \;\; \forall u \in H.$

Uniqueness: $A x = 0 \iff x = 0$ $\qquad$ $Tu = 0 \iff u = 0$

continuous on $[-1, 1]$

Example: $\underbrace{u(x) + \int_{-1}^{x} v(y)\, u(y)\, dy}_{[\underset{T}{I + K_v}]u} = \int_{-1}^{x} f(y)\, dy$

$= Kf$

Check b'dd | "amplification"

$\|Tu\| \leq \|u + K_v u\| \leq \|u\| + \|K_v u\|$

$\leq (1 + \|K_v\|)\|u\|$

$$\|K_v u\|^2 = \int_{-1}^{+1} \left| \int_{-1}^{x} v(y)\, u(y)\, dy \right|^2 dx$$

$$\leq \int_{-1}^{+1} \int_{-1}^{x} |v(y)|^2 |u(y)|^2 dy\, dx$$

$$\leq \int_{-1}^{+1} \int_{-1}^{+1} |v(y)|^2 |u(y)|^2 dy\, dx$$

$$\leq \max_{-1 \leq y \leq 1} |v(y)|^2 \int_{-1}^{+1} \int_{-1}^{+1} |u(y)|^2 dy\, dx$$

$$\leq 2\|V\|_\infty^2 \|u\|^2$$

$$\Rightarrow \|K_v\| = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|K_v u\|}{\|u\|} \leq \sqrt{2}\|V\|_\infty$$

$$\Rightarrow \quad \|T\| = \|I + K_v\| \leq 1 + \sqrt{2}\|V\|_\infty$$

<u>Check Coercivity</u> $\|T^* u\| \geq \delta \|u\|$ for any $u \in H$
"Shrinking"
δ doesn't depend on $u \in H$.

"reverse" triangle ineq.

$$\|T^* u\| \geq \|u + K_v^* u\| \geq \big|\|u\| - \|K_v^* u\|\big|$$

If $\|K_v^*\| = \sigma < 1 \Rightarrow \big|\|u\| - \|K_v^* u\|\big| \geq (1-\sigma)\|u\|$

$$[K_v^* u](x) = \int_x^1 v(y) u(y)\, dx$$

$$\Rightarrow \|K_v^*\| = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|K_v^* u\|}{\|u\|} \leq \sqrt{2}\|V\|_\infty \ < 1$$

$\Rightarrow$ If $\|v\|_\infty < 1/\sqrt{2}$, then $\|K_v\| < 1$.

$$\Rightarrow \quad \|T^* u\| \geq (1 - \sqrt{2}\|v\|_\infty)\|u\|$$

Hilbert-Schmidt Operators

A Hilbert-Schmidt integral operator has form

$$[Kf](x) = \int_\Omega k(x,y) f(y)\,dy$$

where the kernel $k : \Omega \times \Omega \to \mathbb{R}$ satisfies

$$\int_\Omega \int_\Omega |k(x,y)|^2 \,dx\,dy < \infty.$$

Hilbert-Schmidt integral ops provide a continuous analogue of matrices, with continuously indexed "rows" and "columns."