Time-Dependent PDEs

Recap

The steady-state heat equation is a null-space problem: $\Delta u = 0$ s.t. B.C.s

Notice the very rich structure of the Laplacian's nullspace in a cylinder!

$$u_{n,m}(r,\theta,z) = \begin{cases} \underbrace{J_n(\alpha_m^{(n)} r)}_{\text{radial}} \underbrace{\sin(n\theta)}_{\text{angular}} \underbrace{e^{-\alpha_m z}}_{\text{height}} \\ J_n(\alpha_m^{(n)} r) \cos(n\theta) e^{-\alpha_m z} \end{cases}$$

Also note that the null functions are separable.

$$\Delta u = \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\substack{\text{Laplacian} \\ \text{on disk}}} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$e_{n,m}(r,\theta) = \begin{cases} J_n(\alpha_m^{(n)} r)\sin(n\theta) \\ J_n(\alpha_m^{(n)} r)\cos(n\theta) \end{cases}$$

eigenfunctions of Laplace on the disk.

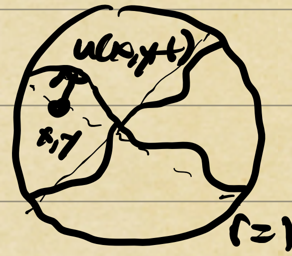
So the nullspace of Δ on the cylinder is a product of Laplace eigenfunctions on the disk coupled to exponentially decay along z-axis.

Vibrating Circular Membrane

Wave Eqn. $\quad \partial_t^2 u = c^2 \Delta u$

wave speed

displacement $u(x,y,t)$



sep. space & time

$$u(x,y,t) = F(x,y)\,T(t)$$

$u|_{r=1} = 0$
"Dirichlet"

$$\Rightarrow \quad \Delta F + k^2 F = 0, \qquad \ddot{T} + k^2 c^2 T = 0$$

spatial $\qquad\qquad$ temporal

$$u_{n,m}(r,\theta,t) = J_n(\alpha_m^{(n)} r) \begin{Bmatrix} \sin(n\theta) \\ \cos(n\theta) \end{Bmatrix} \begin{Bmatrix} \sin(\alpha_m^{(n)} c t) \\ \cos(\alpha_m^{(n)} c t) \end{Bmatrix}$$

Dirichlet Eigenfunctions of Lap on disk

"harmonic" time-dependence (sign diff compared to heat equation)

The general solutions are products of Laplace eigenfunctions on the disk and temporal "harmonics." Each eigenfunction corresponds to a time-dependent solution which simply oscillates at frequency $\frac{\alpha_m^{(n)} c}{2\pi}$.

The fundamental frequencies of the drum head are given by the roots of the Bessel functions.

⇒ Fundamental frequencies are sometimes called "normal modes."

⇒ Can experimentally observe nodes (zero level sets) of Dirichlet modes.

⇒ See demo in matlab (course repo.).

Diagonalization

It's useful to understand "separation-of-variables" as a special case of diagonalization.

$$L_1 u + L_2 u = 0 \qquad \Omega = \underbrace{D_1 \times D_2}_{\text{1D domains}}$$

If $u(x,y) = X(x)Y(y)$, then we should choose X, Y to be eigenfunctions of L_1, L_2 with equal and opposite eigenvalues.

$$L_1 X = \lambda X \qquad L_2 Y = -\lambda Y$$

$$\Rightarrow \quad L_1 XY + L_2 XY = \lambda XY - \lambda XY = 0 \checkmark$$

Then form general solution from combo.

$$u(x,y) = \sum_k c_k X_k(x) Y_k(y)$$

with coeffs c_k chosen to meet B.C.'s.

In fact, we can take this further and find <u>all</u> eigenfunctions/values of

$$L = L_1 + L_2$$

They are simply $\lambda_{j,k} = \lambda_j^{(1)} + \lambda_k^{(2)}$

and $u_{j,k}(x,y) = X_j(x) Y_k(y)$, formed

from eigenpairs of L_1 and L_2.

This provides a powerful framework for solving stationary & time-dependent PDEs.

<u>Stationary</u>

$$L u = f \qquad \text{on} \quad \Omega = D_1 \times D_2$$
$$L = L_1 + L_2$$

Expand $\quad u(x,y) = \sum_k c_k u_k(x,y)$

B.C.S two options

$$f(x,y) = \sum_k \hat{f}_k u_k(x,y)$$

$\Rightarrow$ Basis
$\Rightarrow$ coeffs

$$\Rightarrow \quad c_k \lambda_k = \hat{f}_k \quad \Rightarrow \quad c_k = \frac{\hat{f}_k}{\lambda_k}$$

$$\Rightarrow \quad u(x,y) = \sum_k \frac{\hat{f}_k}{\lambda_k} u_k(x,y)$$

where $\quad u_k(x,y) = X_{k_1}(x)\, Y_{k_2}(y)$
$\qquad \lambda_k = \lambda_{k_1} + \lambda_{k_2}$

Sep. var. provides eigenfunctions from 1D prob.

Time-dependent

$$\partial_t u = Lu \qquad \Omega = D_1 \times D_2$$
$$u|_{t=0} = g \qquad L = L_1 + L_2$$

$$u(x,y,t) = \sum_k c_k(t)\, u_k(x,y)$$

$$\dot{c}_k = \lambda_k \quad \Rightarrow \quad c_k = e^{\lambda_k t}$$

$$u(x,y,t) = \sum_k c_k(0)\, e^{\lambda_k t}\, u_k(x,y)$$
$$\Uparrow$$
$$g = \sum_k c_k(0)\, u_k(x,y)$$

$\Rightarrow$ Initial condition determines $c_k(0) = \langle g, u_k \rangle$

$\Rightarrow$ time-dependence determined by eigvals

$\Rightarrow$ Combo of fundamental modes is now time-dependent

$\Rightarrow$ Usually use basis satisfying B.C.s