Linear Transformations

Recap ⇒ Orthogonal bases generalize the usual Cartesian coordinate system in 2D, 3D, ..., ND.

$$\langle e_i, e_j \rangle = \begin{cases} \|e_i\|^2 & i = j \\ 0 & i \neq j \end{cases}$$

If $\|e_i\|^2 = 1$, the basis is orthonormal.

⇒ Gram-Schmidt constructs ONB for V, given basis $\{x_1, \dots, x_n\}$ and inner product $\langle \cdot, \cdot \rangle$.

⇒ To map between x and it's coordinates in a basis $\{x_1, \dots, x_n\} \subset V$, we write

$$x = \overset{X}{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}} \overset{\alpha}{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}$$

columns of quasimatrix are basis vectors

coordinates of x in basis

Note 1: Gram matrix $X^T X$ invertible b/c $\{x_1, \dots, x_n\}$ lin. indep.

$$\underline{\alpha} = (X^T X)^{-1} (X^T x)$$

$(X^T X)_{ij} = \langle x_j, x_i \rangle$ $\quad X^T x = \begin{pmatrix} \langle x, x_1 \rangle \\ \vdots \\ \langle x, x_n \rangle \end{pmatrix}$

Note 2: When $\{x_1, \dots, x_n\}$ is ONB $X^T X = I$

Now, suppose we have two bases, with

$$x = \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\underset{\sim}{\alpha}} \quad \text{and} \quad x = \underbrace{\begin{bmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}_{\underset{\sim}{\beta}}$$

Q: How are the coordinates $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ related?

$$X^T X \underset{\sim}{\alpha} = X^T Y \underset{\sim}{\beta}$$

$$\Rightarrow \quad \underset{\sim}{\alpha} = \overbrace{\underbrace{(X^T X)^{-1} X^T}_{\text{"pseudoinverse"}} Y}^{\text{"Change-of-basis"}} \underbrace{\underset{\sim}{\beta}}_{}$$

"pseudoinverse" $\xrightarrow{\text{apply to}}$ x

Note: On the computer, caution is required applying formulas w/ $(X^T X)^{-1}$ (see MATH-6800).

Example: What is the change-of-basis matrix that converts Legendre coeffs to monomial?

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \tfrac{1}{2}(3x^2 - 1), \ \ldots$$

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -1/2 & 0\\0 & 1 & 0\\0 & 0 & \frac{3}{2}\end{bmatrix}}_{\text{change-of-basis}}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

$$p(x) = \alpha_1 1 + \alpha_2 x + \alpha_3 x^2 \quad = \beta_1 p_0(x) + \beta_2 p_1(x) + \beta_3 p_2(x)$$

$$= \beta_1(1) + \beta_2(x) + \beta_3 \tfrac{1}{2}(3x^2 - 1)$$

$$= \underbrace{(\beta_1 - \tfrac{1}{2}\beta_3)}_{=\alpha_1} 1 + \underbrace{\beta_2}_{=\alpha_2} x + \underbrace{\tfrac{3}{2}\beta_3}_{=\alpha_3} x^2$$

Q: How would you convert back?

$$\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix} = \begin{bmatrix}1 & -1/2 & 0\\0 & 1 & 0\\0 & 0 & 3/2\end{bmatrix}^{-1}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

Linear Transformations

Given vector spaces V, W, a map $T: V \to W$ is linear (= "linear transformation") if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

Linear combo of inputs $\Rightarrow$ Linear combo of outputs

<u>Example:</u> $T: x \mapsto Ax$ where $A \in \mathbb{R}^{m \times n}$

Q: What are the dimensions of V, W here?

<u>Example:</u> Differentiation on $C^1([-1,1])$:

$$\frac{d}{dx}\left(\alpha f(x) + \beta g(x)\right) = \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$$

<u>Example:</u> "Divergence" operators on $C^1(\mathbb{R}^2)$:

$$\nabla \cdot \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \partial_x f_1(x,y) + \partial_y f_2(x,y)$$

<u>Example:</u> Integral operators on $C([-1,1])$:

$$f(x) \to \int_{-1}^{+1} k(x,y) f(y)\, dy$$

We'll ask, how can linear algebra tools help us study these operators and eqn's?

Matrix Representations

If V, W are vector spaces of dimension

$$\dim(V) = n \qquad \dim(W) = m$$

with bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, we can immediately write down a matrix representing a linear transformation

$$T : V \to W.$$

$$\begin{aligned}
T(x) &= T(\alpha_1 x_1 + \cdots + \alpha_n x_n) \\
&= \alpha_1 T(x_1) + \cdots + \alpha_n T(x_n) \\
&= \alpha_1 (\beta_{11} y_1 + \cdots + \beta_{n1} y_n) \\
&\quad + \alpha_2 (\beta_{12} y_1 + \cdots + \beta_{n2} y_n) \\
&\quad \vdots \\
&\quad + \alpha_n (\beta_{1n} y_1 + \cdots + \beta_{nn} y_n) \\
&= \begin{bmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{bmatrix} \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}
\end{aligned}$$

Map from coordinates of x to coordinates of $y = T(x)$ is the matrix $(B)_{ij} = B_{ij}$.

Each column of this matrix is

jth column of matrix $\quad B_j = (Y^T Y)^{-1} Y^T \underbrace{T(x_j)}_{y}$

or

$$B = (\underbrace{Y^T Y}_{\text{In this context}})^{-1} Y^T T X \qquad (TX)_j = T(x_j)$$

If Y is an ONB, then simplifies to

$$B = Y^T T X$$

<u>Example:</u> Diff op on monomials, deg $\leq n$

$$\frac{d}{dx} 1 = 0, \quad \frac{d}{dx} x = 1, \quad \frac{d}{dx} = 2x, \ldots, \frac{d}{dx} = nx^{n-1}$$

$$\begin{bmatrix} a_0' \\ a_1' \\ \vdots \\ a_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & n-1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$