Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Representations of a linear transformation. Let V and W be real vector spaces with bases $X = \{x_1, \ldots, x_n\} \subset V$ and $Y = \{y_1, \ldots, y_m\} \in W$, respectively. Suppose that a linear transformation $\mathcal{T} : V \to W$ has the matrix representation $T_{X\to Y} : \mathbb{R}^n \to \mathbb{R}^m$, which maps coordinates of $v \in V$ in the basis X to the coordinates of $Tv \in W$ in the basis Y.

Notation: Given quasimatrices $A : \mathbb{R}^n \to V$ and $B : \mathbb{R}^m \to W$ with columns $a_1, \ldots, a_n \in V$ and $b_1, \ldots, b_m \in W$, respectively, let $\mathcal{T}A$ denote the quasimatrix with columns $\mathcal{T}a_1, \ldots, \mathcal{T}a_n \in W$ and A^TB denote the $n \times m$ matrix with entries $(A^TB)_{ij} = \langle b_j, a_i \rangle$.

- **(a)** Show that $T_{X\to Y} = (Q_Y^T Q_Y)^{-1} Q_Y^T \mathcal{T} Q_X$, where Q_X and Q_Y are quasimatrices whose columns are the basis elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, respectively.
- **(b)** If $X' = \{x'_1, \ldots, x'_n\} \subset V$ and $Y' = \{y'_1, \ldots, y'_m\} \subset W$ are bases for V and W, derive a 'change-of-basis' formula that changes $T_{X\to Y}$ to $T_{X'\to Y'}$, the matrix representation of $\mathcal{T}$ that maps coordinates of v in the basis X' to those of Tv in the basis Y'.
- **(c)** How do the formulas in (a) and (b) simplify when X and Y are orthonormal bases?

2) Polynomial solutions to differential equations. Show that for each $n = 1, 2, 3, \ldots$, the following differential equations have a polynomial solution of degree n.

- (a) The Hermite differential equation: $y'' - 2xy' + 2ny = 0$.
- (b) The Laguerre differential equation: $xy'' + (1 - x)y' + ny = 0$.