Diagonalizing Separable PDEs

Recap

Steady-State Heat Eqn.

$\Delta_{r,\theta} u + \partial_z^2 u = 0$

$u(r,\theta,0) = g(r,\theta)$

$u(1,\theta,z \sim = \lim_{z\to\infty} u(r,\theta,z)$

z-z cond., and "wall" cond

Solutions are linear combo of eigenfunctions of

$$u_{n,m}(r,\theta,z) = \begin{cases} J_n(\alpha_m^{(n)} r)\sin(n\theta)e^{-\alpha_m^{(n)} z} \\ J_n(\alpha_m^{(n)} r)\cos(n\theta)e^{-\alpha_m^{(n)} z} \end{cases}$$

Vibrating Circular Membrane

$\Delta_{r,\theta} u - c^2\partial_t^2 u = 0$

$u(r,\theta,0) = g(r,\theta)$

$u'(r,\theta,0) = h(r,\theta)$

$u(1,\theta,t) = 0$

2-t condition, and "edge"

Solutions are linear combo of eigenfunctions

$$u_{n,m}(r,\theta,t) = J_n(\alpha_m^{(n)} r)\begin{Bmatrix} \sin(n\theta) \\ \cos(n\theta) \end{Bmatrix}\begin{Bmatrix} \sin(\alpha_m^{(n)} ct) \\ \cos(\alpha_m^{(n)} ct) \end{Bmatrix}$$

In general, "separation of variables" constructs eigenfunctions of separable partial differential operators on sep. domains.

Eigenfunctions of Sep Ops

Eigenpairs of $\underset{\uparrow \text{ separable op.}}{L}u = \lambda u$ $\quad x \in \underset{\uparrow \text{ separable dom}}{\Omega}$

Separable domain: $\Omega = \Omega_1 \times \Omega_2$

Separable op: $L = L_1 + L_2$ s.t. if $u(x,y) = X(\overset{\in\Omega_1}{x})\,Y(\overset{\in\Omega_2}{y})$

$$L_1 X(x)Y(y) = Y(y) L_1 X(x), \quad L_2 X(x) Y(y) = X(x) L_2 Y(y)$$

If $L_1 X_j(x) = \mu_j X_j(x)$ & $L_2 Y_k(y) = \nu_k Y_k(y)$

Then $Lu_{j,k} = \lambda_{j,k} u_{j,k}$ where

$$u_{j,k}(x,y) = X_j(x) Y_k(y) \quad \text{and} \quad \lambda_{j,k} = \mu_j + \nu_k$$

So eigenpairs of L can be found by solving the eigenproblems for L_1, L_2, which is often simpler since these are lower dimensional problems (e.g. $\dim(\Omega_1) = \dim(\Omega_2) = 1$)

Orthogonality

If $\langle f,g\rangle_{D_1} = \int_{D_1} f(x)\overline{g(x)}\overset{w_1(x)}{\downarrow}dx$ and $\langle u,v\rangle_{D_2} = \int_{D_2} u(y)v(y)\overset{w_2(y)}{\downarrow}dy$

Then $\langle \tilde{f}, \tilde{g}\rangle_{\Omega} = \int_{D_2}\int_{D_1} \tilde{f}(x,y)\overline{\tilde{g}(x,y)}\, w_1(x)w_2(y)\,dx\,dy$

is an inner product on Ω and $\langle X_jY_k, X_nY_m\rangle_{\Omega} = \delta_{jn}\delta_{km}$

whenever $\langle X_j, X_n\rangle_{D_1} = \delta_{jn}$, $\langle Y_k, Y_m\rangle_{D_2} = \delta_{km}$

Nullspace problems

b/c $\Rightarrow \int_{D_2}\int_{D_1} X_j(x)\overline{X_n(x)}w_1(x)\,dx\; Y_k(y)\overline{Y_m(y)}w_2(y)\,dy$

Suppose $[Lu](x,y)=0$ $(x,y)\in\Omega$ with

$L = L_1 + L_2$ and $\Omega = D_1 \times D_2$ separable.

Then, solutions are $u_{j,k}(x,y) = X_j(x)Y_k(y)$

such that $\lambda_{j,k} = \mu_j + \nu_k = 0$.

If $\mu_j \neq \nu_k$ for any eigenvalues of L_1, L_2, then L is one-to-one and there are no nontrivial solutions to $Lu=0$.

Stationary Problems

$$[Lu](x,y) = f(x,y) \qquad (x,y) \in \Omega$$

$L = L_1 + L_2$ and $\Omega = D_1 + D_2$ separable

First find eigenpairs of L:

$$u_{j,k}(x,y) = X_j(x) Y_k(y) \quad \text{and} \quad \lambda_{j,k} = \mu_j + \nu_k$$

where $L_1 X_j = \mu_j X_j$ and $L_2 Y_k = \nu_k Y_k$.

Then, expand RHS in eigenbasis (if ONB)

$$f(x,y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle f, u_{j,k} \rangle_{\Omega} u_{j,k}(x,y)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k} X_j(x) Y_k(y)$$

$$c_{j,k} = \langle f, u_{j,k} \rangle_{\Omega}$$

$$= \int_{D_2}\int_{D_1} f(x,y) \overline{X_j(x)}\, \overline{Y_j(y)}\, w_1(x) w_2(y)\, dx\, dy$$

Time-Dependent Problems

$[\partial_t u](x,y,t) = [Lu](x,y,t) \quad (x,y) \in \Omega$

init. cond. $u(x,y,0) = g(x,y)$

$L = L_1 + L_2$ and $\Omega = D_1 + D_2$ separable

First find eigenpairs of L:

$u_{j,k}(x,y) = X_j(x) Y_k(y)$ and $\lambda_{j,k} = \mu_j + \nu_k$

where $L_1 X_j = \mu_j X_j$ and $L_2 Y_k = \nu_k Y_k$.

Then expand $u(x,y,t)$ in eigenbasis:

$$u(x,y,t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t)\, u_{j,k}(x,y)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(t)\, X_j(x) Y_k(y)$$

How do we find the coeffs $c_{j,k}(t)$?

$\Rightarrow$ Expand $\partial_t u(x,y,t)$ and match!

$$\partial_t u(x,y,t) = \sum_{j\geq 1}^{\infty} \sum_{k=1}^{\infty} \dot{c}_{j,k}(t) X_j(x) Y_k(y)$$

$\Rightarrow$ Matching coeffs (take inner product w/ $X_n Y_m$)

$$\underbrace{\sum_{j\geq 1}^{\infty} \sum_{k=1}^{\infty} \dot{c}_{j,k}(t) X_j(x) Y_k(y)}_{\partial_t u} = \underbrace{\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k} c_{j,k}(t) X_j(x) Y_k(y)}_{Lu}$$

$$\dot{c}_{j,k} = \lambda_{j,k} c_{j,k}$$

$$\Rightarrow \quad c_{j,k}(t) = e^{\lambda_{j,k} t} c_{j,k}(0) = e^{(\mu_j+\nu_k)t} c_{j,k}(0)$$

How to compute $c_{j,k}(0)$?

$\Rightarrow$ match coeffs of init cond.

$$u(x,y,0) = \sum_{j\geq 1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}(0) X_j(x) Y_j(y)$$

$$c_{j,k}(0) = \int_{D_2}\int_{D_1} g(x,y)\, \overline{X_j(x) Y_j(y)}\, w_1(x) w_2(y)\, dx\, dy$$

How does this procedure adapt to $\overset{\partial_t^2}{\text{waves}}$?

Operator Exponential

$\partial_t u = Lu \quad \Rightarrow \quad u(t) = e^{Lt} g$

$u|_{t=0} = g$

$\Rightarrow$ Beyond eigenpairs

$\Rightarrow$ Forcing (Duhamel's principle)