# Best Approx. in Hilbert Space

**Question:** How to choose $c_1, c_2, \dots, c_n$ so that $f - f_n$ is as "small" as possible?

$$(*) \qquad \underset{\underset{\sim}{f}}{\begin{bmatrix} | \\ f(x) \\ | \end{bmatrix}} \approx \underset{\underset{\sim}{E}}{\begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}} \underset{\underset{\sim}{c}}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}$$

**Idea:** Develop "linear algebra for functions" to solve $(*)$ in a "least-squares" sense.

A $\underset{\mathbb{R}}{\text{(real}}$ or $\underset{\mathbb{C}}{\text{complex)}}$ **Hilbert space** $H$ is

i) a **vector space** (over $\mathbb{R}$ or $\mathbb{C}$)

ii) equipped with an **inner product** $\langle \cdot, \cdot \rangle$

iii) **complete** (Cauchy sequences converge)

We can solve $(*)$ if we work in a Hilbert Space.

# Inner Product Spaces

A vector space $V$ with inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$) is an inner product space.

The inner product induces a norm $\| \cdot \| : V \to \mathbb{R}_+$

$$\|x\| = \sqrt{\langle x, x \rangle} \quad \text{for each } x \in V.$$

Two vectors $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$ and we write $x \perp y$.

Note that if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

pf] $\langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$
$\qquad = \|x\|^2 + 0 + 0 + \|y\|^2$

The Cauchy-Schwarz inequality allows us to control the size of inner products w/norms:

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

**Example:** Consider the inner product

$$\langle f,g\rangle \in \int_{-1}^{+1} f(x)\overline{g(x)}\,dx.$$

$\Rightarrow$ The induced norm is $\|f\| = \left(\int_{-1}^{+1} |f(x)|^2 dx\right)^{1/2}$.

$\Rightarrow$ The Cauchy-Schwarz inequality reads

$$\left|\int_{-1}^{+1} f(x)\overline{g(x)}\,dx\right| \leq \left(\int_{-1}^{+1}|f(x)|^2 dx\right)^{1/2}\left(\int_{-1}^{+1}|g(x)|^2 dx\right)^{1/2}$$

**Question:** What are some examples of orthogonal pairs of functions?

$x \qquad x^2 \qquad \int_{-1}^{+1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{+1} = 0$

$\Rightarrow$ If $f$ is odd and $g$ is even, then

$$\int_{-1}^{+1} f(x)\overline{g(x)}\,dx = \int_{-1}^{+1} \overset{\text{odd } h(x) = -h(-x)}{h(x)}\,dx = \int_{-1}^{0} h(x)\,dx + \int_{0}^{1} h(x)\,dx$$

$x \to -x,\ dx \to -dx$

$$= \int_{1}^{0} h(-x)(-dx) + \int_0^1 h(x)\,dx = \int_0^1 h(-x) + h(x)\,dx = 0$$

$\Rightarrow$ If $f(x) = \cos n\theta$ and $g(x) = \cos 2n\theta$

$$\int_{-1}^{+1} \cos n\theta \cos 2n\theta \, d\theta = \int_{-1}^{+1} \tfrac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) \tfrac{1}{2}\left(e^{i2n\theta} + e^{-i2n\theta}\right) d\theta$$

$$= \tfrac{1}{4}\int_{-1}^{+1} e^{i3n\theta}\, d\theta + \tfrac{1}{4}\int_{-1}^{+1} e^{-in\theta}\, d\theta + \tfrac{1}{4}\int_{-1}^{+1} e^{in\theta}\, d\theta + \tfrac{1}{4}\int_{-1}^{+1} e^{-i3n\theta}\, d\theta$$

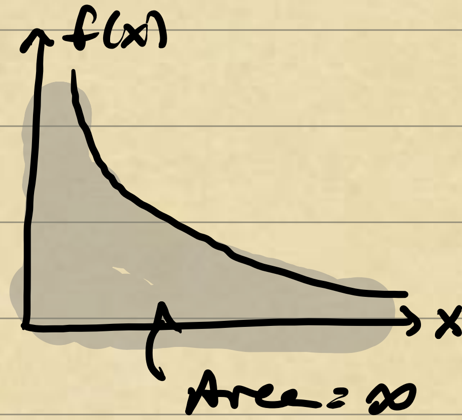
$$= \frac{1}{4}\left[ \frac{e^{i3n\theta}}{3ni}\bigg|_{-1}^{+1} + \frac{e^{-in\theta}}{-ni}\bigg|_{-1}^{+1} + \frac{e^{in\theta}}{ni}\bigg|_{-1}^{+1} + \frac{e^{-i3n\theta}}{-3ni}\bigg|_{-1}^{+1} \right]$$

$$= 0$$

## Completeness

In normed & inner product spaces of functions, we must wrestle with the fact that some functions have "infinite" length.

**Example:** $f(x) = 1/\sqrt{x}$




$$\|f\|^2 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x} = \lim_{\varepsilon \to 0} \frac{-1}{x}\bigg|_{\varepsilon}^{1}$$

$$= \infty$$

Therefore, we typically restrict to $\{f : \|f\| < \infty\}$.

We now face the question of which infinite linear combinations we will allow in our function space. The basic idea of completeness is that we should allow all infinite linear combinations which converge to something with finite norm.

Consider a vector space $V$ w/ inner product $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ and induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

A sequence $v_1, v_2, v_3, \ldots \in V$ is Cauchy if for every $\varepsilon > 0$, there is an $N$ s.t. for all $m,n > N$

$$\|v_m - v_n\| < \varepsilon.$$

A complete space $V$ is one in which every Cauchy sequence converges to an element of $V$.

For example, suppose that the linear combo's

$$\underbrace{v_n}_{\in V} = \sum_{k=1}^{n} \underbrace{\alpha_k}_{\in\mathbb{C}} \, f_k$$

form a Cauchy sequence. Then, completeness requires that there is an element $f \in V$ s.t.

$$\lim_{n\to\infty} \| v_n - f \| = 0.$$

Therefore, we can identify $f$ with the infinite linear combination (series)

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n + \cdots = \sum_{k=1}^{\infty} a_k f_k.$$

In particular, any series that converges absolutely in the Hilbert space norm, i.e.,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k \| f_k \| = 0$$

is Cauchy, and, therefore, converges to some limit in $V$: there is an $f \in V$ s.t.

$$f = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k f_k.$$