Operator Exponentials (Pt. 2)

Recap

Suppose that $A : D(A) \to H$ is a normal operator w/compact resolvent.

$$Au_j = \lambda_j u_j \qquad j = 1, 2, 3, \ldots$$

{ ONB }

operator exponential

$$e^{At} g = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle u_j, g \rangle u_j$$

If $\sup_j \mathrm{Re}(\lambda_j) = M < \infty$, e^{At} is well-defined

$\Rightarrow$ b'dd linear operator for each $t \geq 0$.

$$\Rightarrow \quad e^{A(t+\Delta t)} = e^{At} e^{A\Delta t} = e^{A\Delta t} e^{At}$$

and $e^{A(0)} = I$

$\Rightarrow$ strongly continuous w.r.t. t

$$\lim_{\Delta t \to 0} \| e^{A(t+\Delta t)} g - e^{At} g \| = 0$$

$(e^{At})_{t \geq 0}$ is a strongly continuous semigroup.

Time-evolution

The operator exponential arises naturally in the solution of time-dependendent probs.

$$\left.\begin{aligned} \dot{u} &= Au \\ u\big|_{t=0} &= g \end{aligned}\right. \quad \overset{\text{solution}}{\Longrightarrow} \quad u(t) = e^{At} g$$

When A generates a strongly continuous semigroup $(e^{At})_{t\geq 0}$, the problem is well-posed:

$\Rightarrow$ Soln. exists for all $t \geq 0$.

$\Rightarrow$ Soln. depends continuously on the data (A, g).

When A is normal w/compact resolvent, we can analyze behavior of $u(t)$ by looking at eigenvalues of A.

Eigenvalue Analysis

$$u(t) = e^{At} g = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle u_j, g \rangle u_j$$

$$\lambda_j = \underbrace{\mu_j}_{\text{real}} + i \underbrace{\nu_j}_{\text{imag}}$$

$$e^{\lambda_j t} = \underbrace{e^{\mu_j t}}_{\text{growth/decay}} \underbrace{(\cos(\nu_j t) + i \sin(\nu_j t))}_{\text{oscillations}}$$

Growth/Decay/Oscillation of each mode u_j is governed by corresponding eigenvalue.

$$\langle u_j, u(t) \rangle = \langle u_j, g \rangle \left[e^{\mu_j t} (\cos(\nu_j t) + i \sin(\nu_j t)) \right]$$

important role in, e.g., stability analysis

$\operatorname{Re} \lambda_j > 0 \Rightarrow$ growth

$\operatorname{Re} \lambda_j < 0 \Rightarrow$ decay

$\operatorname{Re} \lambda_j = 0 \Rightarrow$ modulus conserved

Imaginary part of λ_j governs frequency.

Example 1: Heat Eqn. on $\Omega = \text{disk}$

$u_t = \Delta u$ $\quad\quad$ Δ is self-adjoint neg. def.

$u|_{t=0} = g$ $\quad\quad$ $\Rightarrow$ real eigenvalues < 0

$u|_{\partial\Omega} = 0$ $\quad\quad$ $\Rightarrow$ all modes decay

$$\lim_{t\to\infty} u(t) = 0$$

Q: What does the solution look like $t \gg 1$?

$$0 > \lambda_1 > \lambda_2 > \cdots$$

$$u(t) = \sum_{j\geq 1}^{\infty} e^{\lambda_j t} \langle g, u_j \rangle u_j = e^{\lambda_1 t} \langle g, u_1 \rangle u_1 + \underbrace{O(e^{\lambda_2 t})}_{\text{exp. small as } t\to\infty}$$

So leading eigenvector dominates the profile of $u(t)$ at large times.

$\Rightarrow$ The heat semigroup is an example of a contraction semigroup $\|u(t)\| \leq \|g\|$

Example 2: Quantum Particle-in-a-box (Schrödinger)

$$\Omega = [-1,1]^2$$

$u_t = i\Delta u$

$u|_{t=0} = g$

$u|_{\partial\Omega} = 0$

$H^* = -H^*$

$\Rightarrow$ Skew-adjoint operator

$\Rightarrow$ pure imaginary eigenvalues

$\Rightarrow$ Solution norm conserved

$$\|u(t)\|^2 = \sum_{j=1}^{\infty} |\langle u_j, g\rangle|^2 \underbrace{|e^{i\lambda_j t}|^2}_{=1} = \|g\|^2$$

$\Rightarrow$ The L^2-norm of $u(t)$ is a probability density, so $\|g\| = \|u(t)\|$, $t \geq 0$, means that total probability is conserved.

$\Rightarrow$ In earlier wave examples, $\|u(t)\|$ is associated w/ elastic energy.

In general, the eigenvalues/eigenvectors of normal operators w/ compact resolvent give a complete and highly interpretable description of $(e^{At})_{t\geq 0}$ and associated problems

Duhamel's Formula

The operator exponential also appears in time-evolution problems w/ "forcing"

$$u_t - Au = f \qquad \Rightarrow \qquad u(t) = e^{At}g + \int_0^t e^{A\tau} f(\tau)\,d\tau$$
$$u|_{t=0} = g$$

Non-Normality

When eigenvectors of A do not form an ONB, eigenvalues may mislead

$$u(t) = \sum_{j=1}^{\infty} e^{\lambda_j t} c_j u_j$$

Coeffs c_j are no longer computed by orthogonal projection and dynamics along different modes can interact to cause transient behavior.

$\Rightarrow$ transient growth of $\|u\|$ even when $\mathrm{Re}(\lambda_j) < 0$ $j = 1, 2, \ldots$

$\Rightarrow$ Asymptotics still governed by λ_j's.

$\Rightarrow$ Semigroup theory and pseudospectra focus on the <u>resolvent</u>