Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Normed vector spaces. Denote the set of all polynomials on $[-1, 1]$ with real coefficients, of any degree, by $\mathbb{P}$ and define a map $\|\cdot\| : \mathbb{P} \to [0, \infty)$ by $\|p\| = \sup_{-1\leq x\leq 1} |p(x)|$.

(a) Verify that $\mathbb{P}$ is a vector space over $\mathbb{R}$ and that the map $\|\cdot\|$ is a norm on $\mathbb{P}$.

(b) Name one finite-dimensional subspace and one infinite-dimensional subspace of $\mathbb{P}$.

(c) Show that $\mathbb{P}$ is **not** complete with respect to the norm $\|\cdot\|$.

Hint: Construct a Cauchy sequence of polynomials whose limit is not a polynomial.

(d) Show that the limit of any Cauchy sequence in $\mathbb{P}$ is a continuous function on $[-1, 1]$.

Note: The space of continuous functions with the supremum norm is an example of a *Banach space*, a complete normed space. While MATH 6600 focuses on Hilbert spaces and the role of orthogonality, Banach spaces also occupy an important place in applied analysis.

2) Chebyshev Series. The Chebyshev polynomials provide a "nonperiodic analogue" of Fourier series for $[-1, 1]$. They are defined by $T_n(x) = \cos(n \arccos x)$, for $n = 0, 1, 2, 3, \ldots$.

Hint: The substitution $x = \cos\theta$ may be useful in completing some of the following exercises.

(a) Verify that $T_n(x)$ is a degree n polynomial by showing that it satisfies a three term recurrence: $T_0(x) = 1$, $T_1(x) = x$, and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.

(b) Show that $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)(1 - x^2)^{-1/2}dx$ defines an inner product on $\mathbb{P}$.

(c) Show that the Chebyshev polynomials form an orthogonal basis for $\mathbb{P}$.

Note: For the purpose of this question, you may interpret $\text{span}\{T_k\}_{k=0}^{\infty}$ as the set of all *finite* linear combinations of Chebyshev polynomials. This interpretation of span corresponds to a *Hamel* basis for the infinite-dimensional vector space $\mathbb{P}$.

(d) Given a continuous function $f : [-1, 1] \to \mathbb{R}$, give a formula for the *best approximation* to f in $\text{span}\{T_0, T_1, \ldots, T_N\}$ with respect to the norm induced by $\langle\cdot, \cdot\rangle$.

(e) Based on your work in Homework 1, if f has n continuous derivatives, then what rate of decrease do you expect for the best approximation error as $N \to \infty$ in part (d)?

Bonus (Optional). On homework 1, you may have noticed that the Fourier coefficients of some of the nonsmooth functions decayed at a faster rate than the upper bounds you derived. Can you derive a sharper upper bound for the decay rate of Fourier coefficients of a function whose nth derivative is piecewise continuous with finitely many points of discontinuity?