Please submit your solutions to the following problems on Gradescope by **6pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) A Sturm–Liouville problem. Consider the Sturm–Liouville eigenvalue problem

$$u''(x) - q(x)u(x) = \lambda u(x), \qquad u(0) = u(1) = 0.$$

Here, $q(x)$ is a continuous, real-valued, non-negative function and we are looking for eigenfunctions $u(x)$ and associated eigenvalues λ that solve this second-order differential equation.

(a) Show that any eigenvalue of the Sturm–Liouville problem must be strictly negative.

Hint: Integrate both sides against a carefully selected test function and integrate by parts to show that $-\lambda$ is the integral of a strictly positive function.

(b) Show that any two eigenfunctions with distinct eigenvalues must be orthogonal.

Hint: Use the fact that the differential operator $Lu = u'' - qu$ is self-adjoint.

(c) Let v_- satisfy $v_-'' - qv_- = 0$ subject to $v_-(0) = 0$ and v_+ satisfy $v_+'' - qv_+ = 0$ subject to $v_+(1) = 0$. The "Wronskian" $w = v_+'(x)v_-(x) - v_-'(x)v_+(x)$ of these two solutions is a nonzero constant. Define the integral operator $[Tf](x) = \int_0^1 k(x,y)f(y)\,dy$ with

$$k(x,y) = \begin{cases} w^{-1}v_-(x)v_+(y), & 0 \le x \le y \le 1, \\ w^{-1}v_+(x)v_-(y), & 0 \le y \le x \le 1. \end{cases}$$

Verify that T is a self-adjoint Hilbert–Schmidt operator.

(d) Let T be the integral operator in (c). Show that if f is continuous on $[0,1]$, then

$$[Tf]''(x) - q(x)[Tf](x) = f(x),$$

i.e., that T is a bounded inverse of the differential operator $[Lu](x) = u''(x) - q(x)u(x)$.

(e) Use your work in (a)-(d) to argue that each eigenfunction of T is an eigenvector of L and explain why this implies that the orthogonal eigenfunctions of L span $L^2([0,1])$.

2) Operators without eigenvectors.

(a) Show that the multiplication operator $T : L^2([0,1]) \to L^2([0,1])$ defined by $[Tf](x) = xf(x)$ is bounded and self-adjoint.

(b) However, show that T from (a) has no eigenvectors in $L^2([0,1])$. Why does the spectral theorem from Lecture 21 fail to apply here? What crucial property is T missing?

(c) Now suppose that $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and consider the operator $S\phi_k = k^{-1}\phi_{k+1}$. Show that S is compact, i.e., that there is a sequence of finite rank operators S_n such that $\|S - S_n\| \to 0$ as $n \to \infty$.

(d) Show that S has no eigenvectors in $\mathcal{H}$. Why does the spectral theorem from Lecture 10 fail to apply here? What crucial property is S missing?