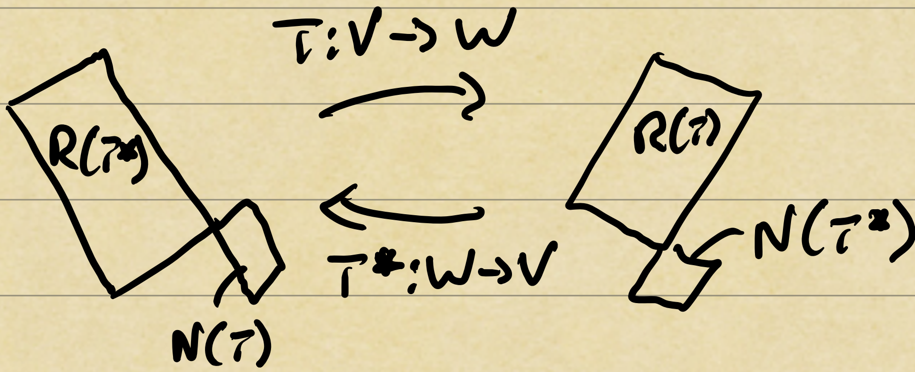
Hilbert Spaces & Their Transformations

Recap



In infinite dimensions, the four fundamental subspaces may not cover all of V and W.

A Hilbert space $H = (V, \langle \cdot, \cdot \rangle)$ is a complete inner product space, meaning that every Cauchy sequence has a limit in H.

$\Rightarrow$ $\{f_n\}_{n \geq 1}^{\infty}$ is Cauchy if for any $\varepsilon > 0$, there is an N s.t. $\|f_n - f_m\| < \varepsilon$ for all $n, m > N$.

$\Rightarrow$ $\{f_n\}_{n \geq 1}^{\infty}$ has a limit $f \in H$ if for any $\varepsilon > 0$, there is an N s.t. $\|f_n - f\| < \varepsilon$ for all $n > N$.

Any inner product space can be completed, by including limits of Cauchy seq., to make Hilbert.

Example: $L^2(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \int_\Omega |f|^2 dx < \infty \}$ ($\Omega \in \mathbb{R}^n$; or $\mathbb{C}$)

Note: Functions in $L^2(\Omega)$ are really equivalence classes that agree everywhere except measure zero sets, so that $\|f-g\| = \int_\Omega |f-g|^2 dx = 0$ implies $f = g \in L^2(\Omega)$, as required for a norm.

$\Rightarrow$ $L^2(\Omega)$ is the space we get when we complete $C^\infty(\Omega)$ with the norm

$$\|\cdot\| = \int_\Omega |f|^2 dx.$$

Similarly for $C^k(\Omega)$, $k \geq 1$.

$\Rightarrow$ Consequently, smooth functions are dense in $L^2(\Omega)$ meaning that for any $f \in L^2(\Omega)$, there is a Cauchy sequence of smooth functions $\{f_n\}_{n=1}^\infty \subset C^k(\Omega)$ s.t. $\lim_{n\to\infty} \underbrace{\|f_n - f\|}_{= \int_\Omega |f_n - f|^2 dx} = 0.$

Hilbert Basis

A Hilbert space is called separable if there is a countable dense subset, $\{\varphi_n\}_{n\geq 1}^{\infty}$, meaning that for any $f \in H$, we can find a subsequence $\{\varphi_{n_k}\}_{k\geq 1}^{\infty} \subset \{\varphi_n\}_{n\geq 1}^{\infty}$ s.t. $\lim_{k\to\infty} \|\varphi_{n_k} - f\| = 0$.

Almost every Hilbert space encountered in practice is separable!

Every Separable Hilbert space admits a countable orthonormal basis.

$$H \supset \{e_n\}_{n\geq 1}^{\infty} \quad \text{s.t.} \quad \langle e_j, e_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Any $f \in H$ has unique coordinates

coordinates of f

$$f = \langle f, e_1 \rangle e_1 + \langle f, e_2 \rangle e_2 + \cdots$$

where $\lim_{n\to\infty} \| f - \sum_{k\geq 1}^{n} \langle f, e_k \rangle e_k \| = 0.$

The countable ONB makes working in a separable Hilbert Space nice: we can develop very good analogues of tools from finite-dim linear algebra.

__Example:__ $\ell^2 = \{(\overset{\in\mathbb{R}}{a_1}, \overset{\in\mathbb{R}}{a_2}, \dots) \mid \sum_{k=1}^{\infty} |a_k|^2 < \infty\}$

real-valued sequence

E.g., if $\{e_1, e_2, \dots,\} \subset L^2(\Omega)$ is an ONB, then every $f \in L^2(\Omega)$ is uniquely identified by its coordinate $\hat{f}_1 = \langle f, e_1 \rangle$, $\hat{f}_2 = \langle f, e_2 \rangle, \dots$

__Q:__ Why must $\sum_{k=1}^{\infty} |\langle f, e_1 \rangle|^2 < \infty$?

__sketch__ $\Rightarrow \underset{<\infty}{\|f\|^2} = \langle f, f \rangle = \langle f, \sum_{k=1}^{\infty} \langle f_k, e_k \rangle e_k \rangle = \sum_{k=1}^{\infty} |\langle f_k, e_k \rangle|^2$

So choosing an ONB for a separable Hilbert space allows us to work in ℓ^2.

$\Rightarrow$ Every separable Hilbert space is isomorphic to ℓ^2: inner-product-preserving bijective maps from $H \to \ell^2$.

Linear Operators

Lets return to our study of differential and integral operators as linear transformations, now acting on a Hilbert space $\mathcal{H} = (V, \langle \cdot, \cdot \rangle)$.

Q! What does it mean to differentiate a function in $L^2(\Omega)$?

Idea: Use integration-by-parts against smooth functions to extend the notion of a derivative as a linear op. on $L^2(\Omega)$.

$f \in L^2([-1,1])$ has weak derivative g if for all $\varphi \in C_0^1([-1,1]) = \{\varphi \in C^1([-1,1]) \mid \varphi(\pm 1) = 0\}$

$$-\int_{-1}^{+1} f \varphi' \, dx = \int_{-1}^{+1} \varphi g \, dx$$

In particular, if $f \in C^1([-1,1])$, then $g = f'$ is the usual (classical) derivative.

$\Rightarrow$ Easy to check that weak derivative of f is unique and defines a linear transformation, if it exists.

Class of functions in $L^2([0,1])$ with L^2-weak derivatives is itself a Hilbert space:

$$H^1([0,1]) = \{ f \in L^2([0,1]) \mid f' \text{(weak)} \in L^2([0,1]) \}$$

$\Rightarrow$ Note that $H^1([0,1]) \neq L^2([0,1])$! There are functions in L^2 not weakly diff.

Bounded Operators

A bounded linear operator on Hilbert space is a linear transformation $T: H_1 \to H_2$ s.t.

$$M = \sup_{f \in H} \frac{\|Tf\|_{H_1}}{\|f\|_{H_2}} < \infty$$

M is called the norm of T, denoted $\|T\|_{H_1 \to H_2}$

Example: $T_1 = \frac{d}{dx}$ on $L^2([0,1])$ is not b'dd

For $k = 1, 2, 3, \ldots$

$$\frac{d}{dx}(\sin(kx)) = -k\cos(kx)$$

$$\int_0^1 (\sin kx)^2 dx = \frac{1}{2} - \frac{\sin(2k)}{4k} < 1$$

$$k\int_0^1 (\cos kx)^2 dx = \frac{1}{2}k + \sin(2k)$$

$$\frac{\|T_1(\sin kx)\|_{L^2}}{\|\sin kx\|_{L^2}} \to \infty \quad \text{as } k \to \infty$$

We say that $T : H^1([0,1]) \to L^2([0,1])$ is an unbounded operator with domain $H^1([0,1]) \subset L^2([0,1])$.

Example: If we think of T_2 as mapping $H^1([0,1])$ onto $L^2([0,1])$ and use the norm

$$\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}^2}$$

weak derivative

$$\|T\|_{H^1 \to L^2} = \sup_{f \in H^1} \frac{\|Tf\|_{L^2}}{\|f\|_{H_1}} = \sup_{f \in H^1} \frac{\|f'\|_{L^2}}{\sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}^2}} < 1$$

Now, b'dd op. from $H^1 \to L^2$.

<u>Example:</u> $Tf = \int_{-1}^{x} f(y)\,dy$

$$\int_{-1}^{+1} \left| \int_{-1}^{x} f(y)\,dy \right|^2 dx \leq \int_{-1}^{+1} \int_{-1}^{+1} |f(y)|^2\,dy\,dx$$

$$= \|f\|^2 \left[\int_{-1}^{+1} 1\,dx \right]$$

$$= \|f\|^2 \cdot 2$$

So $\|T\|_{L^2 \to L^2} \leq \sqrt{2}$

It is usually simplest to work w/ b'dd operator from $H \to H$ (same space) when possible.