Best Approx. in Hilbert Space

Idea: Build up functions by linear combo's.

$$f(x) \approx \underbrace{c_1 e_1(x) + c_2 e_2(x) + \cdots + c_n e_n(x)}_{= f_n(x)}$$

Question: How to choose $c_1, c_2, \ldots, c_n$ so that $f - f_n$ is as "small" as possible?

$$(*) \qquad \underset{\underline{f}}{\begin{bmatrix} | \\ f(x) \\ | \end{bmatrix}} \approx \underset{\underline{E}}{\begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}} \underset{\underline{c}}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}$$

Goal: Develop "linear algebra for functions" to solve $(*)$ in a "least-squares" sense.

(formally) $$\underline{c} = (E^* E)^{-1} E^T \underline{f} \qquad (F1)$$

The error in the best approximation is

(formally) $$\underline{f} - \underline{E}\underline{c} = (I - E(E^* E)^{-1} E^T) f \qquad (F2)$$

Intro to Hilbert Spaces

A (real or complex) Hilbert space H is

i) a vector space (over $\mathbb{R}$ or $\mathbb{C}$)

ii) equipped with an inner product $\langle \cdot , \cdot \rangle$

iii) complete (Cauchy sequences converge)

In this course, we will focus exclusively on separable Hilbert spaces, which are precisely those that have a countable orthonormal basis.

Let's unpack the new terms in blue, and see how they allow us to "solve" the best approximation problem & give meaning to (F1-F2).

Vector Spaces

A vector space is a nonempty set V, closed under addition and scalar multiplication.

<u>Vector Addition</u>

$x, y \in V \Rightarrow x + y \in V$

$(x+y)+z = x+(y+z)$

$x+y = y+x$

$\exists\, 0 \in V$ s.t. $x + 0 = x \;\forall x \in V$

$\forall x \in V \;\exists -x$ s.t. $x + (-x) = 0$

<u>Scalar Multiplication</u>

$x \in V \Rightarrow \underset{\in \mathbb{R} \text{ or } \in \mathbb{C}}{\alpha} x \in V$

$\alpha(\beta x) = (\alpha\beta)x$

$\alpha(x+y) = \alpha x + \alpha y$

$(\alpha+\beta)x = \alpha x + \beta x$

$1x = x$

Essentially, $\underbrace{\alpha x + \beta y + \cdots + \gamma z}_{\text{finite linear combo always in the space } V} \in V$

<u>Example:</u> $C[-1,1] = \{ f : [-1,1] \to \mathbb{R} \mid f \text{ continuous on } [-1,1] \}$

$h(x) = \alpha f(x) + \beta g(x)$ also continuous

<u>Question:</u> Can you think of other vector spaces whose elements are functions?

Example: $C^n[-1,1] = \{ f \in C[-1,1] \mid f', f'', \ldots, f^{(n)} \in C[-1,1] \}$

$$(k=1,2,\ldots,n) \quad \frac{d^k}{dx^k}\left(\alpha f(x) + \beta g(x)\right) = \alpha \frac{d^k f}{dx^k}(x) + \beta \frac{d^k g}{dx^k}(x)$$

$\Rightarrow$ If f, g have n continuous derivatives, then all linear combinations do too.

Example: $\mathbb{P}_n = \{ p \in C[-1,1] \mid p(x) = a_0 + a_1 x + \cdots + a_n x^n \}$

scalar coeffs

$$\alpha p(x) + \beta q(x) = (\alpha a_0 + \beta b_0) + \cdots + (\alpha a_n + \beta b_n) x^n$$

$= a_0 + a_1 x + \cdots + a_n x^n$

$b_0 + b_1 x + \cdots + b_n x^n$

$\Rightarrow$ Combinations of degree (at most) n poly's are also degree (at most) n poly's.

The last two examples are subspaces of $C[-1,1]$ = space of continuous functions.

A subspace W of V is a nonempty set $W \subseteq V$ that is closed under linear combo's.

Question: What are other subspaces of $C[-1,1]$?

zero at boundaries $\Rightarrow$ $C_0[-1,1] = \{ f \in C[-1,1] \mid f(-1) = f(1) = 0 \}$

odd functions $\Rightarrow$ $C_0[-1,1] = \{ f \in C[-1,1] \mid f(-x) = -f(x) \ \forall x \in [-1,1] \}$

Dimension, Span, Dependence

It's useful to have a sense of the intrinsic "size" of a vector space or subspace. In some sense, how much information do we need to store and manipulate its elements?

A set $S = \{x_1, \dots, x_n\} \subset V$ is called linearly independent if the only way to get

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

is if all of the scalars $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Otherwise, S is called linearly dependent.

The span of $S = \{x_1, x_2, \dots, x_n\} \subseteq V$ is the subspace formed via linear combo's

$$W = \{x \in V \mid x = \alpha_1 x_1 + \cdots + \alpha_n x_n\}$$

all possible linear combo's

Question: Is $\{1, x, \dots, x^n\}$ linearly indep.?

Yes! $0 = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \cdots + \alpha_n x^n \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Question: What is the span of $\{1, x, \dots, x^n\}$?

$$\begin{aligned} \operatorname{span}\{1, x, \dots, x^n\} &= \{f \in C[-1,1] \mid a_0 + a_1 x + \cdots + a_n x^n\} \\ &= \mathbb{P}_n \end{aligned}$$

A set $S \subset V$ is a basis for V if

i) linearly independent, and

ii) the span of S is V.

If S is a basis for V, then for each $x \in V$, there is a unique set of scalars s.t.

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

unique coordinates of x in basis S

Any basis S for V has the same # of elements and this # is the dimension of V.

If V has no finite basis set, it is infinite-dimensional. In this course, all infinite-dimensional spaces of interest will have a countable basis set $\{e_j\}_{j=1}^{\infty}$, i.e., a basis with elements indexed by integers.

<u>Example:</u> The vector space of polynomials $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$ is infinite-dimensional with basis

$$S = \{x^k\}_{k=0}^{\infty}.$$

Norms & Inner Products

Given a vector space V, a norm on V is a map $\|\cdot\|: V \to [0,\infty)$ that satisfies

i) $\|x\| \geq 0$, and $\|x\| = 0$ IFF $x = 0$.

ii) $\|\alpha x\| = |\alpha| \|x\|$

iii) $\|x+y\| \leq \|x\| + \|y\|$

An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$) that satisfies the following criteria:

i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

iii) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

iv) $\langle x, x \rangle \geq 0$ and $[= 0$ IFF $x = 0]$

Question: What is $\langle x, \alpha y \rangle$ in terms of $\langle x, y \rangle$?

$\Rightarrow \langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \bar{\alpha} \overline{\langle y, x \rangle} = \bar{\alpha} \langle x, y \rangle$

$\Rightarrow$ "conjugate linear" in second argument

Example: $V = \mathbb{C}^n$ $\quad \langle x, y \rangle = y^* x = \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

$$[\bar{y}_1 \ \bar{y}_2 \cdots \bar{y}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example: $V = C[-1, 1]$ $\quad \langle f, g \rangle = \int_{-1}^{+1} f(x) \overline{g(x)} \, dx$

The inner product on V always induces a norm defined by $\|x\| = \sqrt{\langle x, x \rangle}$.