Separable PDE (pt. 2)

Recap

Bessel functions share many similarities with Fourier sine and cosine modes.

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \end{cases}$$

roots of $\sin(x)$ at $n\pi$, $n = 1, 2, 3, \ldots$

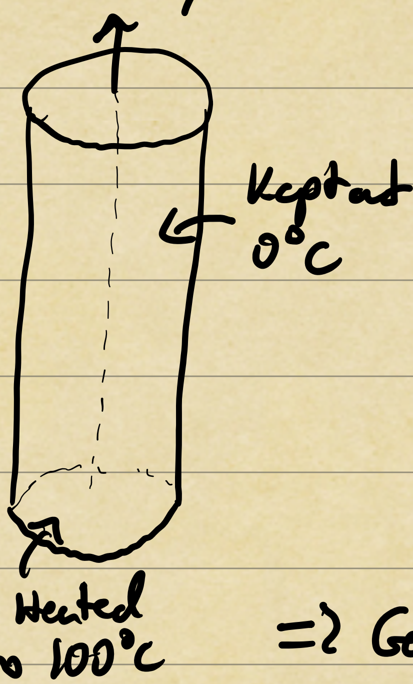
$$\int_0^1 x\, J_p(\alpha_n^{(p)} x) J_p(\alpha_m^{(p)} x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} J_p'(\alpha_n)^2 & n = m \end{cases}$$

roots of $J_p(x)$ at $\alpha_n^{(p)}$, $n = 1, 2, 3, \ldots$

The appear frequently when working in the disk, annuli, or cylindrical geometries.

Steady-State Heat in Cylinder

$$\Delta u = 0, \quad u\big|_{z=0} = 100, \quad u\big|_{r=1} = 0$$



Look for separable solutions:

$$u_{n,m}(r,\theta,z) = \begin{cases} J_n(\alpha_m^{(n)} r)\sin(n\theta)e^{-\alpha_m z} \\ \underbrace{J_n(\alpha_m^{(n)} r)}_{\text{radial}}\underbrace{\cos(n\theta)}_{\text{angular}}\underbrace{e^{-\alpha_m z}}_{\text{height}} \end{cases}$$

$\Rightarrow$ General Solution is linear combo of these separable solutions

Boundary Conditions

We have already "built in" the B.C.s

$$u|_{r=1} = 0 \quad \text{and} \quad \lim_{z\to\infty} u = 0$$

when solving the separated equations for the radial and cylindrical height variables.

To enforce the heated base, $u|_{z=0} = 100$, we will use orthogonality and matching.

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{n,m} J_n(\alpha_m^{(n)} r)\cos(n\theta) e^{-\alpha_m z}$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{n,m} J_n(\alpha_m^{(n)} r)\sin(n\theta) e^{-\alpha_m z}$$

Evaluating both sides at $z=0$, we get

$$100 = u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{n,m} J_n(\alpha_m^{(n)} r)\cos(n\theta)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{n,m} J_n(\alpha_m^{(n)} r)\sin(n\theta)$$

Clearly all the coeffs with $n = 1, 2, 3, \ldots$ vanish since LHS is independent of θ. Equivalently, multiply both sides by $\sin(n\theta)$ or $\cos(n\theta)$ and integrate. By orthogonality of Fourier modes, only the $n=0$ ($\cos(0\theta)=1$) mode can have nonzero coeffs.

We are left with $n=0$ ($\cos(0\theta)=1$) and

$$100 = \sum_{m=1}^{\infty} a_m J_0(\alpha_m^{(0)} r)$$

Taking inner products on both sides w/ $r J_0(\alpha_n^{(0)} r)$

$$a_m = \int_0^1 100 r J_0(\alpha_m^{(0)} r)\, dr \Big/ \int_0^1 r J_0(\alpha_m^{(0)} r)^2\, dr,$$

by orthogonality. Therefore, solution is

$$u(r, \theta, z) = \sum_{m=1}^{\infty} a_m J_0(\alpha_m^{(0)} r) e^{-\alpha_m z}$$

where $a_m = 100 \dfrac{\int_0^1 r J_0(\alpha_m^{(0)} r)\, dr}{\int_0^1 r J_0^2(\alpha_m^{(0)} r)\, dr} = \dfrac{200}{\alpha_m^{(0)} J_1(\alpha_m^{(0)})}$

some identities

Eigenpairs of the Disk Laplacian

The steady-state heat equation is a null-space problem: $\Delta u = 0$ s.t. B.C.s

Notice the very rich structure of the Laplacian's nullspace in a cylinder!

$$u_{n,m}(r,\theta,z) = \begin{cases} J_n(\alpha_m^{(n)} r)\sin(n\theta)e^{-\alpha_m z} \\ \underbrace{J_n(\alpha_m^{(n)} r)}_{\text{radial}}\underbrace{\cos(n\theta)}_{\text{angular}}\underbrace{e^{-\alpha_m z}}_{\text{height}} \end{cases}$$

Also note that the null functions are separable.

More generally, the eigenpairs of the Laplacian play an important role in many physical processes:

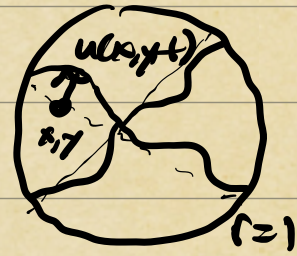
$$\Delta u = \lambda u \quad \text{s.t. B.C.s}$$

To motivate, let's look at a time-dependant PDE describing a vibrating 2D membrane.

Vibrating Circular Membrane

Wave Eqn. $\partial_t^2 u = c^2 \Delta u$

$\uparrow$ displacement $\uparrow$
$u(x,y,t)$



$u|_{r=1} = 0$
"Dirichlet"

sep. space & time

$$u(x,y,t) = F(x,y)\,T(t)$$

$$\Rightarrow \quad \underset{\text{spatial}}{\Delta F + k^2 F = 0}, \qquad \underset{\text{temporal}}{\ddot{T} + k^2 c^2 T = 0}$$

Because of the disk geometry in space,

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0$$

$$F(r,\theta) = R(r)\,\Theta(\theta)$$

separate radial & angular

$$\Rightarrow \quad \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} + k^2 = 0$$

$$\Rightarrow \quad \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} + k^2 r^2 = 0$$

Collecting soln.'s T, R and Θ gives

$$u_{n,m}(r,\theta,t) = \underbrace{J_n(\alpha_m^{(n)} r) \begin{Bmatrix} \sin(n\theta) \\ \cos(n\theta) \end{Bmatrix}}_{\text{Dirichlet Eigenfunctions of Lap on disk}} \underbrace{\begin{Bmatrix} \sin(\alpha_m^{(n)} c t) \\ \cos(\alpha_m^{(n)} c t) \end{Bmatrix}}_{\text{"harmonic" time-dependence}}$$

The frequencies of the drum head are given by the roots of the Bessel functions.

⇒ Fundamental Frequencies are sometimes called "normal modes."

⇒ Can experimentally observe nodes (zero level sets) of Dirichlet modes.