Sturm-Liouville Problems

Recap Given an unbounded op $L:D(L) \to H$, the resolvent is $R(z) = (L-z)^{-1}$

resolvent set $\rho(L) = \{ z \in \mathbb{C} \mid (L-z) \text{ has b'dd inverse} \}$.

spectrum $\lambda(L) = \{ z \in \mathbb{C} \mid (L-z) \text{ has } \underline{\text{no}} \text{ b'dd inverse} \}$

By Neumann series, $\rho(L)$ is open and $\overset{\text{(closed)}}{\lambda(L)} = \mathbb{C} \setminus \rho(L)$.

If L is self-adjoint and $R(z)$ is compact for some $z \in \mathbb{R}$, then $\lambda(L) = \{\lambda_1, \lambda_2, \dots\}$ where

$$\underbrace{L u_j = \lambda_j u_j}_{\text{ONB}}$$

λ_j real and $\to \infty$ as $j \to \infty$

I.e. Spectrum is discrete, real, unb'dd and eigenvectors form an ONB for H.

Many differential operators have compact resolvent and can be studied through $R(z)$.

Sturm-Liouville

$$\underbrace{\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x)}_{Lu} = \lambda w(x)u(x)$$

$\alpha_1 u(0) + \alpha_2 u'(0) = 0$, $\beta_1 u(1) + \beta_2 u'(1) = 0$

α_1 or α_2 nonzero β_1 or β_2 nonzero

If p, p', q, w continuous and $p \geq \delta > 0$, $w > 0$ the S-L problem is regular. In this case

$\frac{1}{w(x)} L$ is unbounded and selfadjoint w.r.t.

$$\langle u,v\rangle = \int_0^1 u(x)v(x)w(x)\,dx$$

and has compact resolvent.

$\Rightarrow$ $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ real, <u>distinct</u> eigvals

$\Rightarrow$ $u_1, u_2, u_3, \ldots$ ONB for $L^2([0,1], w)$

Resolvent Kernel can be constructed explicitly using technique from HW5 Problem 1.

Sturm-Liouville problems often appear solving PDEs in simple geometries like square/cube, disk/sphere, etc.

Example: Fourier Series

$$-u''(x) = \lambda u(x) \qquad u(0) = u(1) = 0$$

$\Rightarrow$ const. coeff, so use ansatz $u(x) = e^{\alpha x}$

$$-u''(x) = \lambda u(x) \to (-\alpha^2 - \lambda) u(x) = 0$$

general solution

$$\alpha = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda} \to u(x) = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}$$

boundary conditions

$$0 = u(0) = c_1 e^{i\sqrt{\lambda}(0)} + c_2 e^{-i\sqrt{\lambda}(0)} = c_1 + c_2$$

$$\Rightarrow u(x) = c_1 (e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x})$$

$$0 = u(1) = c_1 (e^{i\sqrt{\lambda}(1)} - e^{-i\sqrt{\lambda}(1)}) \quad \Rightarrow \sqrt{\lambda} \text{ real so } \lambda > 0$$

$$= c_1 \sin(\sqrt{\lambda}) \quad \Rightarrow \sqrt{\lambda} = k\pi$$

$$\Rightarrow \quad u_k(x) = c\sin(k\pi x) \qquad k = 1, 2, 3, \ldots \left(\overset{-1,-2,-3}{\text{same soln}}\right)$$

$$\lambda_k = k^2\pi^2$$

Note that $k=0$ is trivial and $k, -k$ give $\overbrace{\sin kx, -\sin kx}^{\text{not lin. indep.}}$

converges in L^2

For any $f \in L^2([0,1])$, $\quad f(x) = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \hat{f}_k \sin(k\pi x)$

$$\hat{f}_k = \frac{1}{\sqrt{2}} \int_0^1 f(x) \sin(k\pi x)\, dx$$

Singular Sturm-Liouville problems

other Special functions often arise as solutions to singular Sturm-Liouville problems.

Example: Legendre polynomials (PDEs on sphere)

$$(1-x^2)u'' - 2xu' + \underbrace{l(l+1)}_{\lambda = \text{eigenvalue}} u = 0$$

$$\Rightarrow ((1-x^2)u')' + \quad (x^2 - l^2)u = 0$$

Example: Bessel functions (PDEs on disc)

$$x^2 u'' + xu' + (x^2 - \quad {}^2)u = 0$$

$$\Rightarrow (xu')' + \left(x - \frac{l^2}{x}\right)u = 0$$

These ODEs have a vanishing coeff in front of the highest derivative, which can (a) complicate analysis and (b) lead to interesting behavior of solutions near the "singular point."

To get started, let's examine a simple class of singular SL equations associated with Euler. These are "easy" to solve and illustrate some basic ideas about SSL problems.

$$x^2 u'' + 4xu' + 2u = 0 \qquad x > 0$$

ansatz $\Rightarrow u(x) = x^r$ (similar to exp ansatz)

$$\Rightarrow x^2(r(r-1)x^{r-2}) + 4(r x^{r-1}) + 2x^r = 0$$

$$\Rightarrow (r(r-1) + 4r + 2)x^r = 0$$

$$\Rightarrow r^2 + 3r + 2 = 0$$

$$\Rightarrow r_\pm = \frac{1}{2}\left[-3 \pm \sqrt{9-8}\right] = \begin{matrix} -1 \\ -2 \end{matrix}$$

$$\Rightarrow u(x) = c_1 x^{-1} + c_2 x^{-2}$$

Note that in this case solutions blow up at $\overset{\text{"singular point"}}{x=0}$

=> If roots are positive, polynomial solutions
=> Complex roots lead to oscillatory solutions
=> Repeated roots lead to logarithmic blow up

Power Series Solutions

More generally, can look for power series solutions, with ansatz

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

and solve for coeffs.

=> Easy to manipulate powers
and differentiate
=> Solve coeff by coeff
=> Effective if solutions are
very smooth (e.g. analytic)

=> This is our path of attack for SSL problems.