More on the SVD of H-S ops

A Hilbert-Schmidt operator has

$$[Kf](x) = \int_\Omega \underbrace{K(x,y)}_{L^2(\Omega\times\Omega) \text{ kernel}} f(y)\,dy, \quad \Omega \subset \mathbb{R}^d$$

$K : L^2(\Omega) \to L^2(\Omega)$ b'dd with SVD

$$K(x,y) = \sum_{j=1}^{r} \sigma_j u_j(x) v_j(y) \quad (r \in \mathbb{N} \cup \{\infty\})$$

$$\begin{array}{cc} & \leftarrow y \rightarrow \\ \begin{array}{c}\uparrow\\ x\\ \downarrow\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\end{array} = \sigma_1 \begin{bmatrix}\uparrow\\ u_1\\ \downarrow\end{bmatrix}\begin{bmatrix}\leftarrow v_1 \rightarrow\end{bmatrix} + \sigma_2 \begin{bmatrix}\uparrow\\ u_2\\ \downarrow\end{bmatrix}\begin{bmatrix}\leftarrow v_2 \rightarrow\end{bmatrix} + \cdots + \sigma_r \begin{bmatrix}\uparrow\\ u_r\\ \downarrow\end{bmatrix}\begin{bmatrix}\leftarrow v_r \rightarrow\end{bmatrix}$$

$\{u_j\}$ are an ONB for $R(K)$.
$\{v_j\}$ are an ONB for $R(K^*)$.
$\{\sigma_j\}$ couple $v_j \in R(K^*)$ to $u_j \in R(K)$

$$[Kf](x) = \sum_{j=1}^{\infty} \sigma_j \langle f, v_j \rangle u_j(x)$$

Low-Rank Approximation

The singular values decay as $j \to \infty$:

$$\infty \overset{\text{H-S prop.}}{>} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |k(x,y)|^2 \, dx\, dy$$

$$\overset{\text{SVD}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big|\sum_{j=1}^{\infty} \sigma_j \underbrace{u_j(x) v_j(y)}\Big|^2 \, dx\, dy$$

Note: $\{u_j\}$ ONB for $\underset{\subset L^2(\Omega)}{R(K)}$, $\{v_j\}$ ONB for $\underset{\subseteq L^2(\Omega)}{R(K^*)}$

$\Rightarrow \{u_j(x) v_j(y)\}$ is ONB for $\underset{L^2(\Omega)\times L^2(\Omega)}{R(K)\times R(K^*)}$

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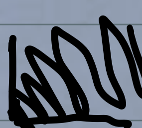

$$\Omega \Rightarrow \iint_{\Omega\Omega} u_j(x) v_j(y)\, u_i(x) v_i(y)\, dx\, dy$$

$$= \left(\int_\Omega u_j(x) u_i(x)\, dx\right)\left(\int_\Omega v_j(y) v_i(y)\, dy\right)$$

$$= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$= \sum_{j=1}^{\infty} \sigma_j^2 \int_\Omega \int_\Omega |u_j(x) v_j(x)|^2 \, dx\, dy$$

$$= \sum_{j=1}^{\infty} \sigma_j^2 \quad \Rightarrow \{\sigma_j\} \text{ square summable}$$

$\sigma_j^2 \sim 1/j \quad \rightarrow \quad \sigma_j^2$ decay faster than $1/j$

As $j \to \infty$, $\sigma_j^2 = o(1/j) \iff \lim_{j\to\infty} \dfrac{\sigma_j^2}{1/j} = \lim_{j\to\infty} j\sigma_j^2 = 0$

Analogues of Matrix Norms

Two common/useful matrix norms:

$$\|A\|_2 = \sup_{x\in\mathbb{R}^N} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$ "2-norm" "Operator Norm" "spectral norm"

$$\le$$

$$\|A\|_F = \sqrt{\left[\sum_{i,j} |A_{ij}|^2\right]} = \sqrt{\sum_{i=1}^{r} \sigma_k^2}$$ "Frobenius norm"

For Hilbert-Schmidt Operators:

$$\|K\| = \sup_{f\in L^2(\Omega)} \frac{\|Kf\|_{L^2}}{\|f\|_{L^2}} = \sigma_1$$ "Operator" Norm

$$\le$$

$$\|K\|_{HS} = \sqrt{\int_\Omega\int_\Omega |k(x,y)|^2\,dx\,dy} = \sqrt{\sum_{j=1}^{\infty} \sigma_j^2}$$ "Hilbert Schmidt norm"

What does SVD say about operator norm?

$$\|Kf\|^2 = \Big\|\sum_{j=1}^{\infty} \sigma_j \langle f, v_j\rangle u_j(x)\Big\|^2 \leq \underbrace{\|K\|_{HS}}_{\sqrt{\sum \sigma_j^2}} \|f\|$$

$$\leq \sum_{j=1}^{\infty} \sigma_j^2 |\langle f, v_j\rangle|^2 \underbrace{\|v_j\|^2}_{=1}$$

$$= \sigma_1^2 |\langle f, v_1\rangle|^2 + \sigma_2^2 |\langle f, v_2\rangle|^2$$

$$+ \cdots + \sigma_j^2 |\langle f, v\rangle|^2 + \cdots$$

$$\leq \sigma_1^2 \sum_{j=1}^{\infty} |\langle f, v_j\rangle|^2$$

$$= \sigma_1^2 \|f\|^2$$

$$\Rightarrow \|Kf\| \leq \sigma_1 \|f\|$$

The "amplifying power" of K is controlled by the leading singular value σ_1.

$$\|Kv_1\| = \sigma_1 \|v_1\| \Rightarrow \|K\| = \sigma_1$$

Eckart-Young for H-S Ops

$$K(x,y) = \sum_{j=1}^{\infty} \sigma_j u_j(x) v_j(y)$$

Goal: Find $K_n(x,y) = \sum_{j=1}^{n} \mu_j \phi_j(x) \theta_j(y)$

that minimize $\|K - K_n\|_*$ $\quad * =$ operator or HS

$$\Rightarrow K(x) - K_n(x) = \sum_{j=n+1}^{\infty} \sigma_{n+1} u_j(x) v_j(y)$$

$$\|K - K_n\| = \sigma_{n+1}, \quad \|K - K_n\|_{HS} = \sqrt{\sum_{j=n+1}^{\infty} \sigma_j^2}$$

N^{th} Partial Sum of SVE for K is the best possible rank N approximation of K, measured w.r.t. $\|\cdot\|$ or $\|\cdot\|_{HS}$.