Constant Coeff. Diff Ops

$$[Lu](x) = \left[a_n \frac{d^n}{dx^n} + \cdots + a_1 \frac{d}{dx} + a_0\right] u(x)$$

$$\widehat{[Lu]}(\xi) = \Big[\underbrace{a_n (i\xi)^n + \cdots + a_1 (i\xi) + a_0}_{\text{Characteristic polynomial}}\Big] \hat{u}(\xi)$$

Homogen. Solutions (null space)

$$[Lu](x) = 0 \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = 0$$

$$\Rightarrow u(x) = c_1 e^{i\xi_1 x} + \cdots + c_n e^{i\xi_n x}$$

where $\xi_1, \ldots, \xi_n$ satisfy $P(\xi_j) = 0 \quad (j = 1, \ldots, n)$

Inhomogeneous Problem

Compact support $[-M, M]$

$$[Lu](x) = f(x)$$

Fourier Transform

$$\Rightarrow \quad P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$$

Solution

$$\Rightarrow \quad u(x) = \frac{1}{2\pi} \int_L \frac{e^{izx}}{P(z)} \hat{f}(z)\, dz$$

avoid roots of $P(z)$

$|\mathrm{Im}(z)|$ b'dd on L

Solution operator $\Rightarrow$ $$u(x) = \frac{1}{2\pi}\int_L \frac{e^{izx}}{P(z)}\left[\int_{-M}^{M} f(y)\,e^{-izy}\,dy\right]dz$$

$$= \int_{-M}^{M} f(y)\left[\underbrace{\frac{1}{2\pi}\int_L \frac{e^{iz(x-y)}}{P(z)}\,dz}_{G(x-y)}\right]dy$$

$\Rightarrow$ $G(x-y)$ is the kernel of solution operator

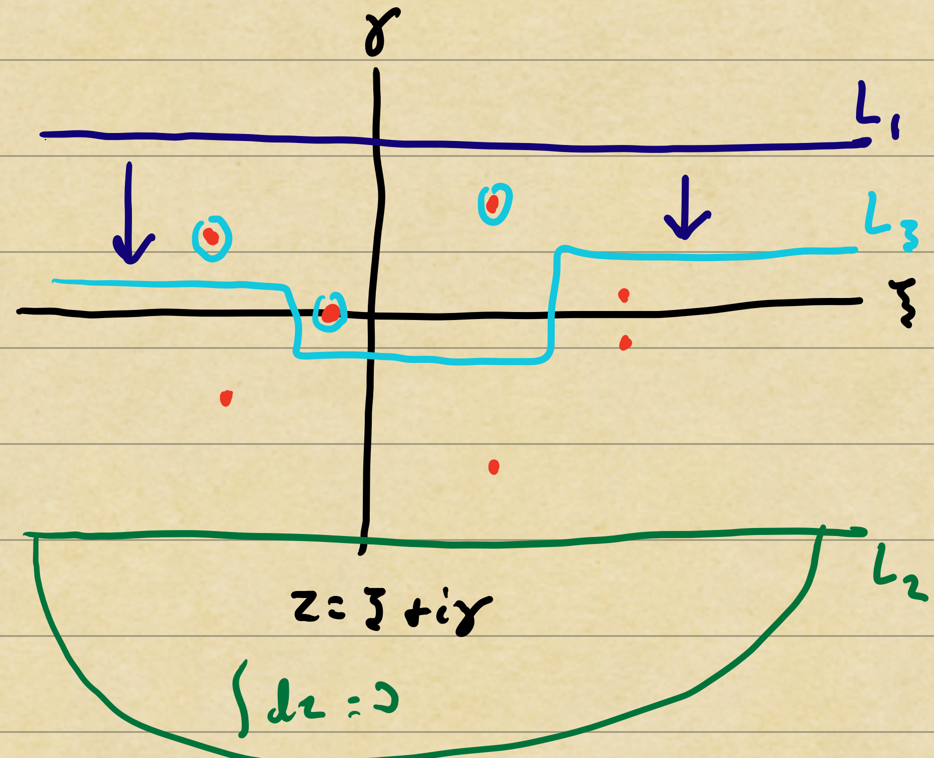
$w = x-y$

$$G(w) = \frac{1}{2\pi}\int_L [P(z)]^{-1} e^{iz\overbrace{w}^{<0}}\,dz$$

$\Rightarrow$ Loosely, $$\hat{G}(z) = \frac{1}{P(z)} = \frac{1}{a_n(z-\xi_1)\cdots(z-\xi_n)}$$

Poles at roots of $P(z)$

$G(x-y)=0$ when $x<y$



$G(t-\tau)=0$ $t<\tau$

$\Rightarrow$ Solution operators from different contours differ by residues of $\hat{G}(z)e^{iz(x-y)}$, which correspond directly to homog. solns:

$$\operatorname*{Res}_{z=\xi_k} \frac{e^{iz(x-y)}}{P(z)} = \lim_{z\to\xi_k} \frac{d^{m-1}}{dz^{m-1}}\left[(z-\xi_k)^m \frac{e^{iz(x-y)}}{P(z)}\right]$$

$$= Q(x-y)\, e^{i\xi_k(x-y)}$$

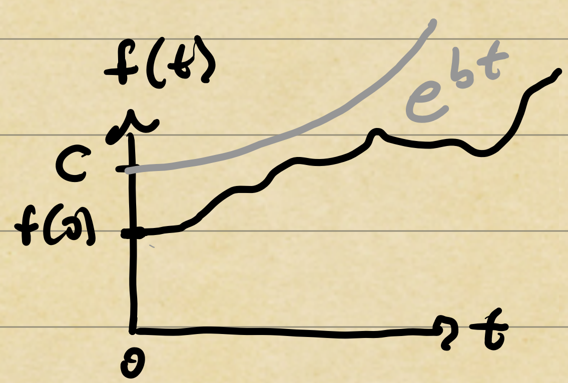
degree $\leq m-1$ poly whose coeffs depend on ξ_k.

$\Rightarrow$ If root ξ_k of $P(z)$ is simple ($m=1$), then residue is just a multiple of Fourier mode

$$e^{i\xi_k(x-y)} = \underbrace{e^{i\mathrm{Re}(\xi_k)(x-y)}}_{\text{frequency}}\ \underbrace{e^{-\mathrm{Im}(\xi_k)(x-y)}}_{\text{growth}}$$

$\Rightarrow$ If $m>1$, no longer pure Fourier modes

<u>Laplace Transform</u>



Let $f: \mathbb{R}_+ \to \mathbb{C}$ be (Piecewise) Continuous and $|f(t)| \leq Ce^{bt}$ on $t\in[0,\infty)$.

($\mathbb{R}_+$: $[0,\infty)$)

Then, the Laplace transform of $f(t)$ is

$$F(z) = \int_0^\infty f(t)\, e^{-zt}\, dt.$$

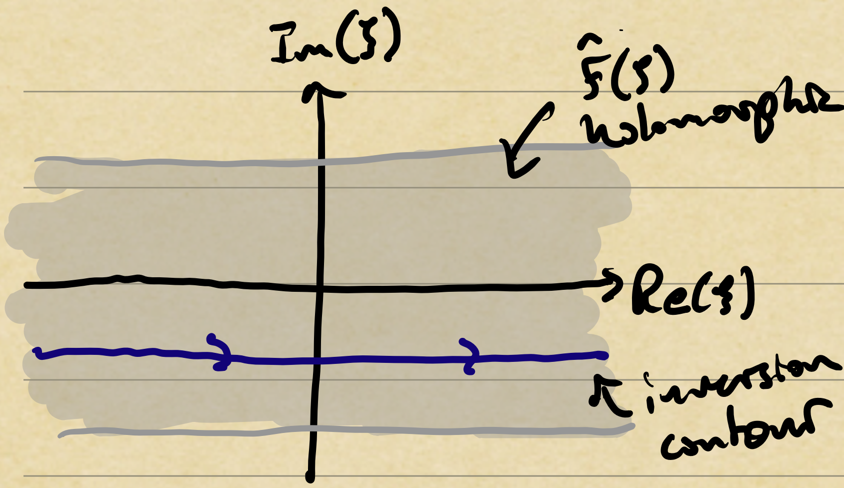
complex #

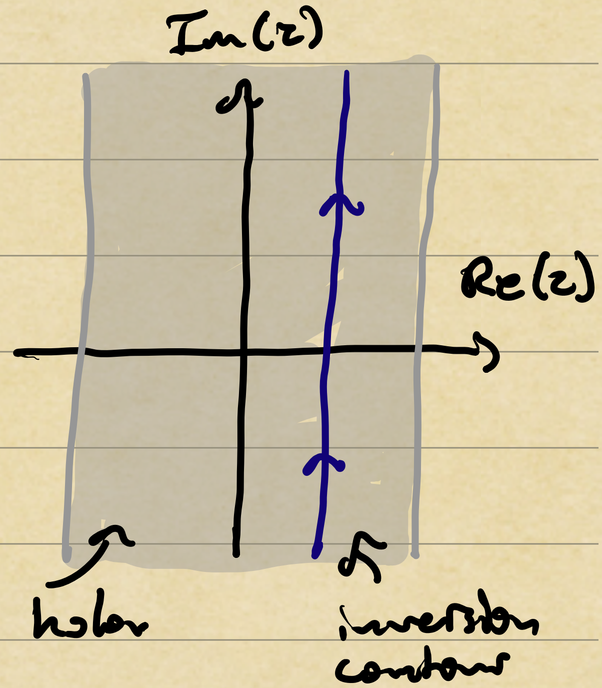
limits

decay exp.

Note that this is just the Fourier Transform of $f(t)$ with complex "frequency" parameter

$$i\xi \to z$$





<u>**Example**</u> $u''(t) + k^2 u(t) = f(t)$

$$u(0) = a \qquad u'(0) = b$$

$$\int_0^\infty u''(t)\, e^{-zt}\, dt = -u'(0) + z \int_0^\infty u'(t)\, e^{-zt}\, dt$$

$$= -u'(0) - z u(0) + z^2 \underbrace{\int_0^\infty u(t)\, e^{-zt}\, dt}_{U(z)}$$

initial data

$$\underbrace{[z^2+k^2]}_{\text{characteristic poly}} \mathcal{U}(z) = \underbrace{F(z)}_{\text{forcing}} + \underbrace{az+b}_{\text{initial condition}}$$

Q:] How do invert the Laplace Transform?

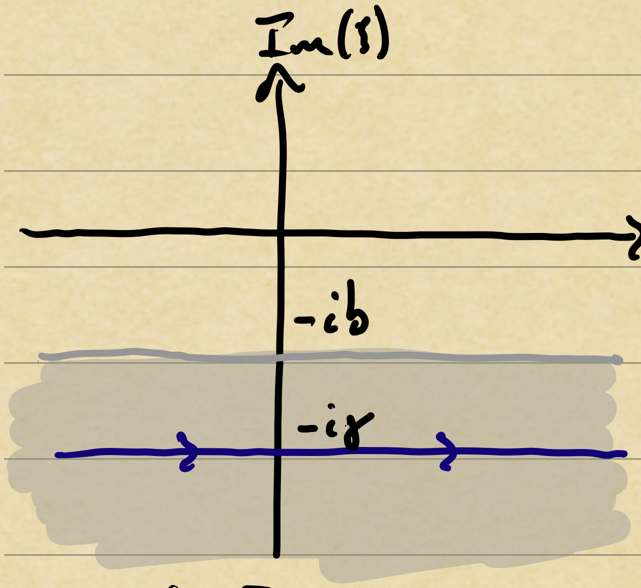
Laplace Inversion Formula (Bromwich Integral)

Let $f: \mathbb{R}_+ \to \mathbb{C}$ be (piecewise) continuous with (piecewise) continous derivative. If $|f(t)| \leq \underbrace{C}e^{bt}$, then for any $\gamma > b$

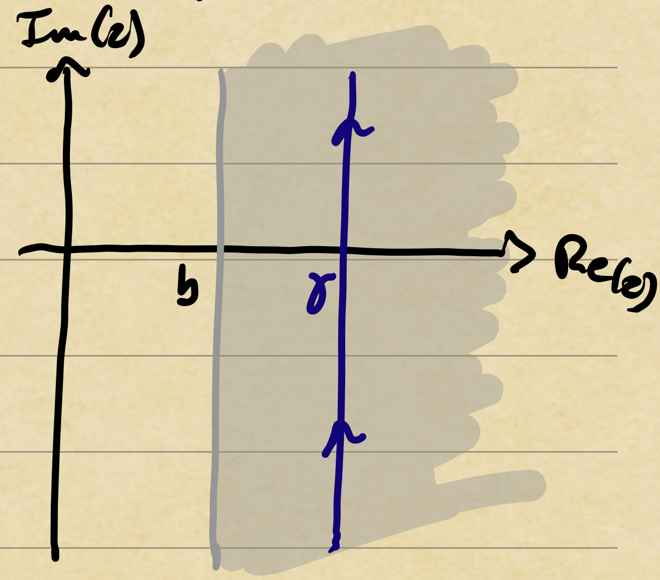
(i) $F(z)$ is holomorphic for $Re(z) > b$.

$$\text{(ii)} \quad \underbrace{\lim_{\varepsilon \to 0^+} \left[\frac{f(t+\varepsilon) + f(t-\varepsilon)}{2} \right]}_{\text{avg. "jump" at } f(t)} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(z) e^{zt} \, dz$$

Proof Sketch/ Use Fourier Transform and a "one-sided" variant of Paley-Wiener thm.



$i\xi = z$



ξ - Fourier Domain

z - Laplace Domain

$$\hat{f}(\xi) = \int_0^\infty f(t) e^{-i\xi t} dt$$

$$F(z) = \int_0^\infty f(t) e^{-zt} dt$$