Solution Operators for ODEs

The Fourier Transform diagonalizes constant coefficient linear differential operators and allows us to construct solution operators for a large class of ODE/PDE.

$$[Lu](x) = \left[a_n \frac{d^n}{dx^n} + \cdots + a_1 \frac{d}{dx} + a_0\right] u(x)$$

Fourier Transform

$$\widehat{[Lu]}(\xi) = \underbrace{\left[a_n (i\xi)^n + \cdots + a_1 (i\xi) + a_0\right]}_{\text{degree } n \text{ poly in } \xi} \hat{u}(\xi)$$

The polynomial $P(\xi) = a_n(i\xi)^n + \cdots + a_1(i\xi) + a_0$ is called the characteristic polynomial and its behavior in the complex plane is intimately linked to ODE/PDE involving L:

$$[Lu](x) = 0 \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = 0$$

$$[Lu](x) = \lambda u(x) \quad \Rightarrow \quad [P(\xi) - \lambda]\hat{u}(\xi) = 0$$

$$[Lu](x) = f(x) \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$$

$$[\partial_t u](x) = [Lu](x) \quad \Rightarrow \quad \partial_t \hat{u}(\xi) = P(\xi)\hat{u}(\xi)$$

The solution of these problems boils down to solving for $\hat{u}(\xi)$ and inverting the transform. E.g., formally

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{f}(\xi)}{P(\xi)} e^{i\xi x}\, d\xi$$

inserting $\hat{f}(\xi)$ and swapping integration limits

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y) \underbrace{\left[\int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)}}{P(\xi)}\, d\xi\right]}_{G(x,y)} dy$$

The Kernel $G(x,y)$ acts as an operator, mapping the "data" f to the solution u. Depending on the context, it is known as a Green's function, a fundamental solution, or a propagator (for time-dependent problems).

Let's take a look at an example to

a) Rigorously construct u and G w/tools from complex analysis.

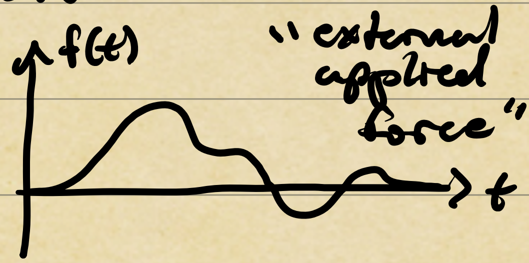
b) Examine how properties of G & $\hat{G}$ in complex plane influence soln's u.

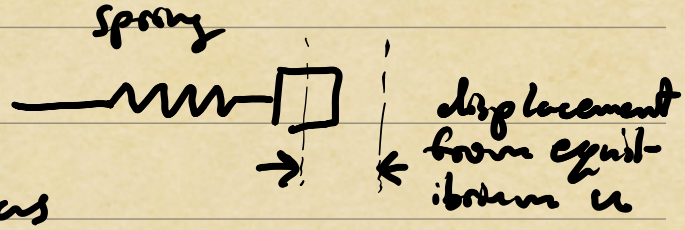
Example: Forced Oscillator

$$u''(t) + k^2 u(t) = f(t)$$

spring const. (k)

"forcing" ($f(t)$)





We'll assume that $f(t)$ has compact support in $[0,T]$ and f, f', f'' are cont.

Fourier Domain
$$(-\xi^2 + k^2)\hat{u}(\xi) = \hat{f}(\xi)$$
well-defined since f has compact supp.

Formally,
$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{f}(\xi)}{k^2 - \xi^2} e^{i\xi t}\, d\xi$$

But integrand has, in general, poles at $\pm k$. These singularities are <u>not</u> integrable so $u(t)$ is <u>not</u> well-defined for any t.

<u>Idea:</u> Move integrand into complex plane to avoid singularities of $(k^2 - \xi^2)^{-1}$.

Define
$$u(t) := \frac{1}{2\pi}\int_{L} \frac{\hat{f}(z)}{k^2 - z^2} e^{izt}\, dz, \quad t > 0.$$

where $L = \{z = x+iy \,;\, y = a > 0,\ x \in \mathbb{R}\}$.

Note that $u(t)$ is now well-defined since

$$\hat{f}(z) = \int_0^T f(t)\, e^{-izt}\, dt$$

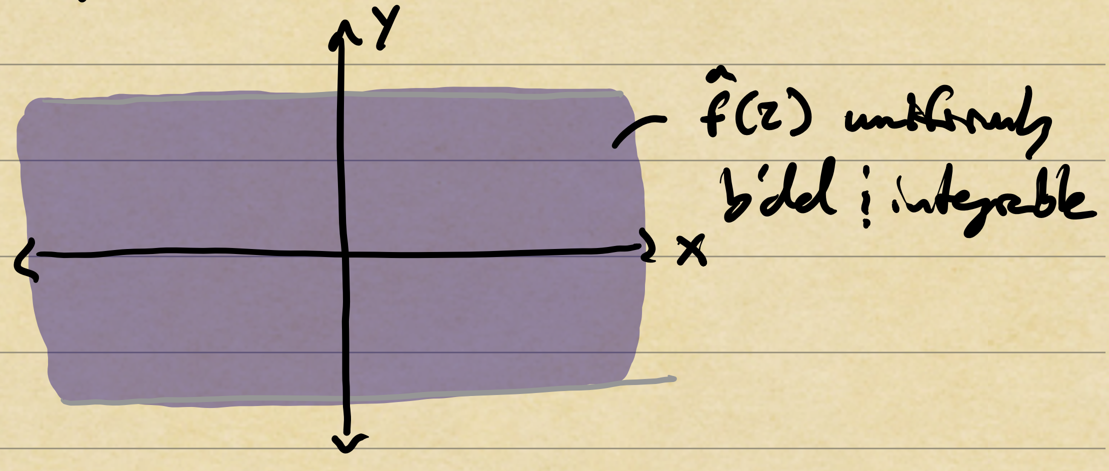
is an entire function by P-W Thm and

$$|\hat{f}(x+iy)| = \left| \int_0^T f(t)\, \frac{e^{-izt}}{-iz}\, dt \right|$$

$$= \left| \int_0^T f(t)\, \frac{e^{-izt}}{-z^2}\, dt \right|$$

$$\le \frac{e^{yT}}{x^2+y^2} \int_0^T |f(t)|\, dt \le \frac{C(y)}{1+x^2}$$

So $\hat{f}(z)$ is uniformly b'dd and integrable on any strip of finite half-width $a>0$



Consequently, the contour integral is well-def:

$$u(t) := \frac{1}{2\pi}\int_L \frac{\hat{f}(z)}{k^2-z^2}\, e^{izt}\,dz, \quad t > 0.$$

Now, we claim that $u(t)$ satisfies the ODE:

$$u''(t) + k^2 u(t) = f(t).$$

First, we calculate the left-hand side,

$$u''(t) + k^2 u(t) = \frac{1}{2\pi}\int_L \frac{-z^2\hat{f}(z)}{k^2-z^2}\, e^{izt}\,dz + \frac{k^2}{2\pi}\int_L \frac{\hat{f}(z)}{k^2-z^2}\, e^{izt}\,dz$$

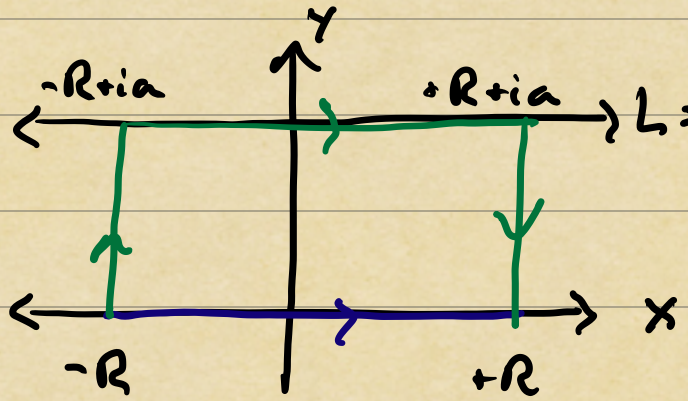
$$= \frac{1}{2\pi}\int_L \cancelto{=1}{\frac{k^2-z^2}{k^2-z^2}}\, \hat{f}(z)\, e^{izt}\,dz$$

$$= \frac{1}{2\pi}\int_L \hat{f}(z)\, e^{izt}\,dz$$

But since $\hat{f}(z)$ is entire and satisfies

$$|\hat{f}(x+iy)| \leq \frac{C(y)}{1+x^2}$$

We can deform the contour back to $\mathbb{R}$



By Cauchy's thm and bound on $\hat{f}(x+iy)$ the contribution from vertical contours vanishes as $R \to \infty$ and we get

$$\frac{1}{2\pi}\int_L \hat{f}(z)\, e^{izt}\, dz = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{i\xi t}\, d\xi = f(t)$$

Therefore, $u''(t) + k^2 u(t) = f(t)$ as claimed.

In effect, we have exploited the smoothness of the right-hand side to construct a solution in the complex Fourier domain.

Can we go further and construct the solution operator from our formula

$$u(t) = \frac{1}{2\pi}\int_L \frac{e^{ikz}}{k^2 - z^2}\, \hat{f}(z)\, dz,$$

by bringing the contour back to $\mathbb{R}$?