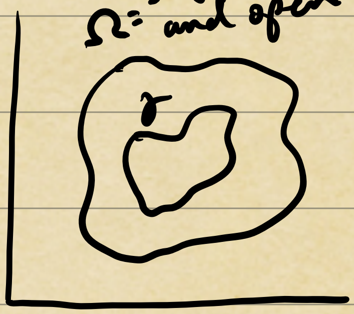
Isolated Singularities & Residue Calculus

Recap: Holomorphic functions are like Tolstoy's "happy families." If $f: \Omega \to \mathbb{C}$ is holomorphic:

Ω = simply connected and open



$\Rightarrow$ f is infinitely differentiable on Ω

$\Rightarrow$ f has uniformly convergent Taylor series in finite disks around each point $z \in \Omega$.

$\Rightarrow$ $f \neq 0$ has finitely many zeros in any closed & bd'd subset of Ω (like polynomials).

$\Rightarrow$ f is uniquely determined by its values on any subset of Ω w/ limit pt. in Ω. For example, on any curve or nonempty open region in Ω.

$\Rightarrow$ f is uniquely determined by the derivatives $f^{(n)}(z_0)$, $n = 0, 1, 2, \dots$ at any point $z_0 \in \Omega$.

Our most important tool to study holomorphic f:

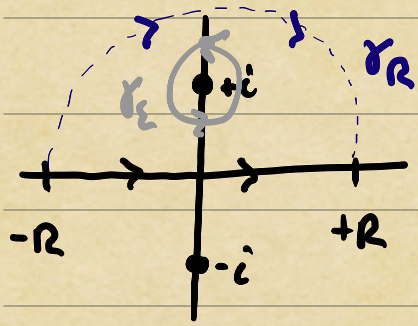
"Cauchy Integral Formulas"

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \qquad n = 0, 1, 2, \dots$$

Singularities of Complex Functions

Points at which f is <u>not</u> complex differentiable are called singularities of f. The location and "nature" of these singularities have a profound influence on the behavior of f. They are also useful for calculations...

Example $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{\infty} \underbrace{\frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]}_{f(x)} dx$



$$= \int_{\gamma_R} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz$$

As $R \to \infty$, $\displaystyle\left|\int_{\gamma_R} \frac{1}{z^2+1}\,dz\right| \leq \pi R \to 0.$ ← "contribution" to integral from point at infinity

On the other hand,

$$\int_{\gamma_\varepsilon} f(z)\,dz = \frac{1}{2i}\int_{\gamma_\varepsilon} \frac{1}{z-i}\,dz - \underbrace{\frac{1}{2i}\int_{\gamma_\varepsilon} \frac{1}{z+i}\,dz}_{=0 \text{ (Cauchy's Thm)}}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}} = \frac{2\pi i}{2i} = \pi$$ ← "contribution" to integral from singularity at $z = +i$

Laurent Series

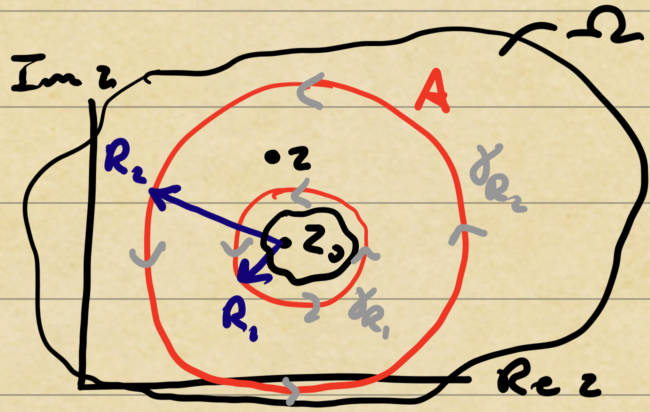
To make this approach systematic, we need to analyze the behavior of f near a singular point. Just as Taylor (power) series' give the local picture of a function holomorphic at a point, Laurent series' provide a local picture of f near an isolated singular point.

Thm Let $f(z)$ be holomorphic in Ω (Ω open, containing the annulus) the annulus $A = \{z : R_1 \leq |z - z_0| \leq R_2\}$. Then for any $z \in \operatorname{int}(A)$,

$$(*) \quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k, \text{ where } a_k = \frac{1}{2\pi i} \int_{\gamma_{R_2}} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

Here, $\gamma_{R_2} = \{z : |z| = R_2\}$ and $(*)$ defines a bijection from $\{a_k\}_{k=-\infty}^{+\infty}$ to $f : \Omega \to \mathbb{C}$ (rep. is unique).

Pf| Take $\gamma_{R_1} = \{z : |z| = R_1\}$,



we have by Cauchy's Formula

(1) $$f(z) = \frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(\xi)}{\xi - z}\,dz - \frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(\xi)}{\xi - z}\,d\xi$$

On γ_{R_2}, we expand (b/c $|z-z_0| < |\xi - z_0|$ for all $\xi \in \gamma_{R_2}$)

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)}\left[\frac{1}{1 - \left(\frac{z-z_0}{\xi-z_0}\right)}\right] = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(\xi-z_0)^{k+1}}$$

On γ_{R_1}, we expand (b/c $|z-z_0| > |\xi - z_0|$ for all $\xi \in \gamma_{R_1}$)

$$\frac{1}{\xi - z} = \frac{-1}{z-z_0}\left[\frac{1}{1 - \left(\frac{\xi-z_0}{z-z_0}\right)}\right] = \frac{-1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^k$$

$$= -\sum_{k=0}^{\infty} \frac{(z-z_0)^{-k-1}}{(\xi-z_0)^{-k}} = -\sum_{k=1}^{\infty} \frac{(z-z_0)^{-k}}{(\xi-z_0)^{-k+1}}$$

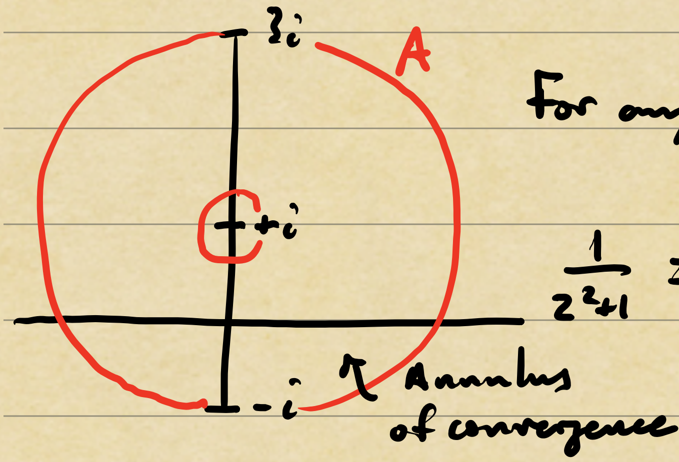
Note that for fixed $z \in \mathrm{int}(A)$, both series converge uniformly. We can integrate term-by-term in (1) to get

$$f(z) = \sum_{k=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi\right](z-z_0)^k + \sum_{k=1}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(\xi)}{(\xi-z_0)^{-k+1}}\,d\xi\right](z-z_0)^{-k}$$

$$= \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k \quad \text{with } a_k = \frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi$$

Note that γ_{R_1} can be deformed to γ_{R_2} when $k \leq -1$ b/c $f(\xi)(\xi - z_0)^{k-1}$ is holomorphic in $\Omega \supset A$.

Example Expand $f(z) = \dfrac{1}{z^2+1}$ around $z = +i$.



For any $0 < |z-i| < 2$, we have

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{(z-i)(2i+z-i)}$$

expand in powers of $(z-i)$

$$= \frac{1}{2i(z-i)\left(1+\frac{z-i}{2i}\right)} = \frac{1}{2i(z-i)}\sum_{k=0}^{\infty}\left(-\frac{z-i}{2i}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2i)^{k+1}}(z-i)^{k-1}$$

$$\Rightarrow \quad f(z) = \sum_{k=-1}^{\infty}\frac{(-1)^{k+1}}{(2i)^{k+2}}(z-i)^k, \qquad 0<|z-i|<2.$$

Example Expand $g(z) = e^{1/z}$ around $z=0$.

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!} \quad \Rightarrow \quad e^{1/z} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{0}\frac{z^k}{k!}$$

$f(z) = \frac{1}{z^2+1}$ has only one "negative" power of z in its expansion around $z=i$, while $g(z) = e^{1/z}$ has infinitely many negative powers of z at $z=0$.

This leads to very different behavior of $f(z)$ and $g(z)$ in the vicinity of their singular pts.