Inversion, Regularity, & Decay

Recap

Given $f:\mathbb{R}\to\mathbb{R}$ with appropriate regularity & decay:

(1) $$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}\,dx$$ "Fourier Transform"

is the Fourier Transform of f, which satisfies

(2) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\,e^{i\xi x}\,d\xi.$$ "Inverse Fourier Transform"

Roughly speaking, smoothness in f leads to decay in $\hat{f}$ and vice versa. We define $f \in \mathfrak{F}_a$

$\Rightarrow$ f holomorphic in strip $S_a = \{z\in\mathbb{C} : |\operatorname{Im} z| < a\}$.

$\Rightarrow$ $|f(z)| \le A(1+(\operatorname{Re} z)^2)^{-1}$ for all $z\in S_a$.

A: some const. >0

Thm 1 If $f\in\mathfrak{F}_a$ for some $a>0$, then

$$|\hat{f}(\xi)| \le M e^{-b|\xi|}, \quad \xi\in\mathbb{R}.$$

By imposing regularity on f, we can control the decay of $\hat{f}$ to ensure that (2) is well-defined. This allows us to establish (1)-(2) for $f \in \mathfrak{F}_a$.

Theorem 2 If $f \in \mathfrak{F}_a$ for some $a > 0$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi, \quad \text{for all } x \in \mathbb{R}.$$

Pf $$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{0} \hat{f}(\xi) e^{i\xi x}\, d\xi}_{\mathrm{I}} + \underbrace{\frac{1}{2\pi}\int_{0}^{+\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi}_{\mathrm{II}}$$

Since $f \in \mathfrak{F}_a$, choose $0 < b < a$ and argue as in proof of Theorem 1 to express $(\xi > 0)$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(y - ib)\, e^{-i\xi(y - ib)}\, dy.$$

deform contour

Substitute this into II and calculate

$$\frac{1}{2\pi}\int_0^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi = \int_0^{\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y - ib)\, e^{-i\xi(y - ib - x)}\, dy\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y - ib)\left[\int_0^{\infty} e^{-i\xi(y - ib - x)}\, d\xi\right] dy$$

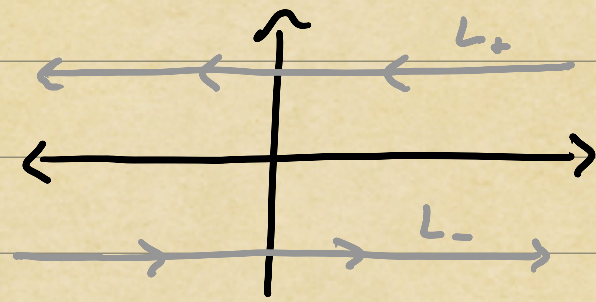
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y-ib) \lim_{L\to\infty}\left[\frac{e^{-i\xi(y-ib-x)}}{-b-i(y-x)}\right]_{\xi=0}^{\xi=L} dy$$

$$\lim_{L\to\infty}\left[\frac{e^{-i\xi(y-ib-x)}}{-b-i(y-x)}\right]_{\xi=0}^{\xi=L} = \lim_{L\to\infty}\left[\frac{1-e^{-bL}e^{-iL(y-x)}}{b+i(y-x)}\right]$$

$$= \frac{1}{b+i(y-x)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y-ib)}{b+i(y-x)}\, dy = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(y-ib)}{y-ib-x}\, dy$$

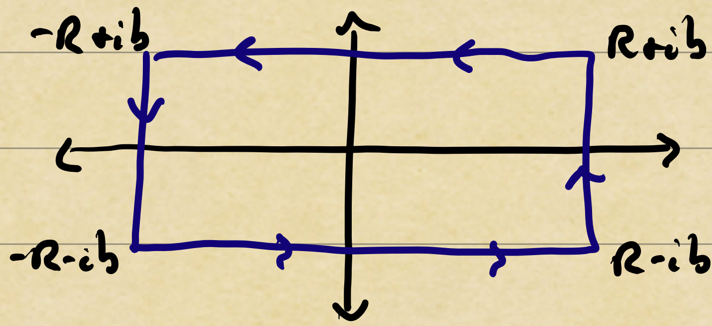
$$= \frac{1}{2\pi i}\int_{L_-} \frac{f(\zeta)}{\zeta - x}\, d\zeta \quad \text{where } L_- = \{\operatorname{Im} z = -b\}.$$



Similarly, for $\xi \geq 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{0} \hat{f}(\xi) e^{i\xi x}\, d\xi = \frac{1}{2\pi i}\int_{L_+} \frac{f(\zeta)}{\zeta - x}\, d\zeta$$

Now, consider the contour Γ_R and use



the Cauchy's integral formula to write

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(\xi)}{\xi - x} d\xi.$$

As $R \to \infty$, the integral over vertical sides $\to 0$:

$$\left| \int_{\pm R - ib}^{\pm R + ib} \frac{f(\xi)}{\xi - x} d\xi \right| \leq 2b \frac{A}{1+R^2} \left[\frac{1}{|R-x|} \right] \to 0 \text{ as } R \to \infty.$$

Therefore, in the limit $R \to \infty$, we have

$$f(x) = \frac{1}{2\pi i} \int_{L_-} \frac{f(\xi)}{\xi - x} d\xi + \frac{1}{2\pi i} \int_{L_+} \frac{f(\xi)}{\xi - x} d\xi$$

by calc. above

$$= \frac{1}{2\pi} \int_0^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi + \frac{1}{2\pi} \int_{-\infty}^{0} \hat{f}(\xi) e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x} d\xi.$$

In practice, Fourier inversion holds under much milder conditions. For example,

$\underline{\text{Thm 3}}$ Let $f: \mathbb{R} \to \mathbb{R}$ be piecewise continuous, absolutely integrable, with piecewise continuous f':

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) + f(x-\varepsilon)}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) e^{i\xi(x-y)} dy\, d\xi$$

Thm 3 shows that (1)-(2) hold at points of continuity of f and that Fourier inversion recovers the average at isolated discontinuities.

Bandlimited Functions

In signal processing and inverse problems, functions composed of Fourier modes w/ b'dd range of frequencies are called **bandlimited**:

$$f(x) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi)\, e^{i\xi x}\, d\xi$$

In other words, $\hat{f}(\xi) = 0$ when $|\xi| \geq B$.

This is an extreme case of "decay" in the Fourier Transform of f, so we might ask - "what regularity makes f bandlimited"?