The Fourier Transform (Pt 2)

Recap

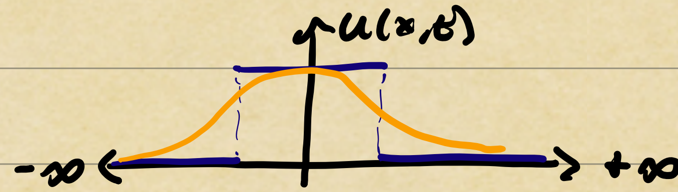
Given $f:\mathbb{R}\to\mathbb{R}$ with appropriate regularity & decay:

(1) $$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\xi x}\, dx$$ "Fourier Transform"

is the Fourier Transform of f, which satisfies

(2) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi.$$ "Inverse Fourier Transform"

Example: Heated Wire



$$\partial_t u = \partial_x^2 u, \quad u(x,0) = g(x), \qquad \lim_{x\to\pm\infty} u(x,t) = 0.$$

$$\Rightarrow \partial_t \hat{u}(\xi,t) = -\xi^2 \hat{u}(\xi,t) \quad \Rightarrow \hat{u}(\xi,t) = \underbrace{\hat{u}(\xi,0)}_{\hat{g}(\xi)}\, e^{-\xi^2 t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}(\xi)\, e^{-\xi^2 t} e^{i\xi x}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} g(y)\, e^{-i\xi y}\, dy\right] e^{-\xi^2 t} e^{i\xi x}\, d\xi$$

$$= \int_{-\infty}^{+\infty} \Big[\underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\xi^2 t + i\xi(x-y)} d\xi}_{K(x-y,t)} \Big] g(y)\, dy$$

$$\Rightarrow \quad u(x,t) = \int_{-\infty}^{+\infty} K(x-y,t)\, g(y)\, dy$$

$$\Rightarrow K(x-y,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\xi^2 t} e^{i\xi(x-y)} d\xi$$

= Fourier Transform of $e^{-\xi^2 t}$ evaluated at $x-y$ $\Rightarrow$ Gaussian in $x-y$.

{ See Lec. 3 notes

$$= \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/4t}$$

The solution at time $t>0$ is computed by convolving the initial condition g with the

"heat kernel" $\quad K(x-y,t) = \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/4t}$

which acts as a "solution operator" for heat eqn.

The Inversion Formula

"Inversion Formula"

We can establish (1)-(2) using complex analysis for a particularly useful class of functions. Let $a>0$, then $f \in \mathfrak{F}_a$ if

(i) f is holomorphic in the strip

$$S_a = \{ z \in \mathbb{C} : |\mathrm{Im}(z)| < a \}.$$

(ii) There exist constant $A>0$ s.t.

$$|f(x+iy)| \leq \frac{A}{1+x^2} \qquad x \in \mathbb{R},\ |y|<a.$$

Roughly, (i) and (ii) ensure that the improper integrals in (1)-(2) are well-defined. We have

by (ii) $$\lim_{R\to\infty} \left| \int_{-R}^{+R} f(x) e^{-i\xi x}\,dx \right| \leq A \int_{-R}^{R} (1+x^2)^{-1}\,dx < \infty$$

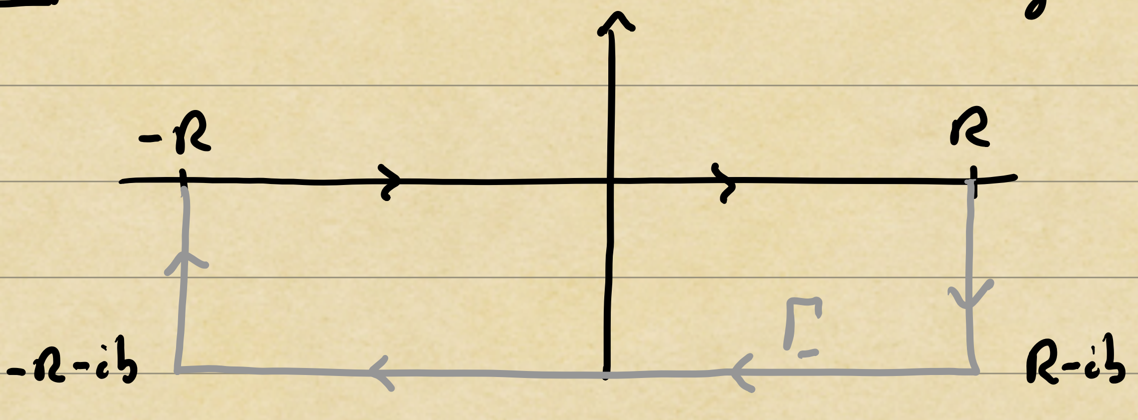
On the other hand (i)-(ii) imply that $\hat{f}(\xi)$ is also b'dd & decays rapidly as $|\xi| \to \infty$.

Theorem 1 If $f \in \mathfrak{F}_a$ for some $a>0$, then there is a constant $M \geq 0$ s.t., for any $0<b<a$,

$$|\hat{f}(\xi)| \leq M e^{-b|\xi|}.$$

Note: this is an analogue of the result on exp. decay of Fourier coeffs for smooth periodic f.

Pf Consider the contour Γ given by



and suppose that $\xi \geq 0$. By Cauchy's Thm

$$0 = \int_\Gamma f(z) e^{-i\xi z}\, dz = \int_{-R}^{R} f(x) e^{-i\xi x}\, dx + \int_{\Gamma_-} f(z) e^{-i\xi z}\, dz.$$

By (ii), we can show that the integrals over the vertical sides $\to 0$ as $R \to \infty$. We have

$|\Gamma|$

$$\lim_{R\to\infty}\left|\int_{\pm R-ib}^{\pm R} f(z)e^{-i\xi z}\,dz\right| \le \lim_{R\to\infty}\frac{bA}{1+R^2} = 0$$

by (ii) and $|e^{-i\xi z}|<1$ for $\xi>0$, $\operatorname{Im} z<0$.

On the lower horizontal contour,

$$\lim_{R\to\infty}\left|\int_{-R}^{+R} f(x-ib)e^{-i\xi(x-ib)}\,dx\right| \le \lim_{R\to\infty}\int_{-R}^{+R}\frac{A}{1+x^2}e^{-\xi b}\,dx$$

$$= A\pi e^{-\xi b}$$

Equating the two horizontal contours, we have

$$|\hat{f}(\xi)| = \lim_{R\to\infty}\left|\int_{-\infty}^{+\infty} f(x)e^{-i\xi x}\,dx\right| \le A\pi e^{-\xi b}, \quad \xi>0.$$

An analogous argument with a contour in the upper half-plane for the case $\xi<0$ (the case $\xi=0$ follows directly from (ii)) shows

$$|\hat{f}(\xi)| \le A\pi e^{\xi b}, \quad \xi<0.$$

Therefore, $\hat{f}(\xi)$ decays rapidly and the improper integral in (2) is well-defined:

$$\lim_{R\to\infty}\left|\frac{1}{2\pi}\int_{-R}^{+R}\hat{f}(\xi)e^{i\xi x}\,d\xi\right| \le \frac{A}{2}\int_{-R}^{+R}e^{-b|\xi|}\,d\xi < \infty.$$

We can now prove the inversion formula.

<u>Theorem 2</u>| If $f \in \mathfrak{F}_a$ for some $a>0$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi, \quad \text{for all } x\in\mathbb{R}.$$

<u>Pf</u>| $$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{0}\hat{f}(\xi)e^{i\xi x}\,d\xi}_{\mathrm{I}} + \underbrace{\frac{1}{2\pi}\int_{0}^{+\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi}_{\mathrm{II}}$$

Since $f\in\mathfrak{F}_a$, choose $0<b<a$ and argue as in proof of Theorem 1 to express ($\xi>0$)

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(y-ib)e^{-i\xi(y-ib)}\,dy.$$

[deform contours]

Substitute this into II and calculate

$$\frac{1}{2\pi}\int_{0}^{\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi = \int_0^\infty \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y-ib)e^{-i\xi(y-ib-x)}\,dy\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y-ib)\left[\int_0^\infty e^{-i\xi(y-ib-x)}\,d\xi\right]dy$$

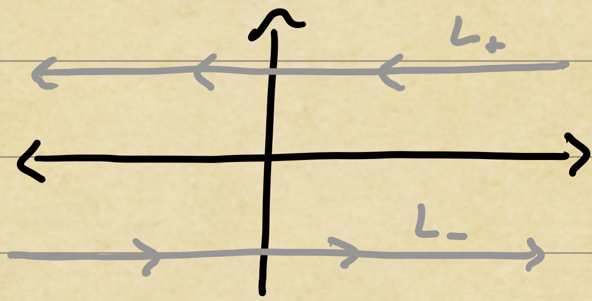
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y-ib)\lim_{L\to\infty}\left[\frac{e^{-i\xi(y-ib-x)}}{-b-i(y-x)}\right]_{\xi=0}^{\xi=L} dy$$

$$\lim_{L\to\infty}\left[\frac{e^{-i\xi(y-ib-x)}}{-b-i(y-x)}\right]_{\xi=0}^{\xi=L} = \lim_{L\to\infty}\left[\frac{1-e^{-bL}e^{-iL(y-x)}}{b+i(y-x)}\right]$$

$$= \frac{1}{b+i(y-x)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{f(y-ib)}{b+i(y-x)}\,dy = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(y-ib)}{y-ib-x}\,dy$$

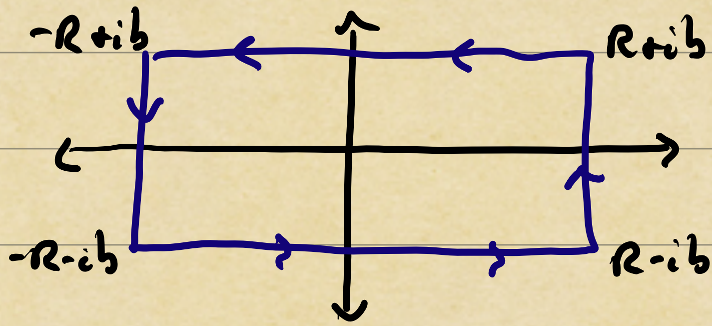
$$= \frac{1}{2\pi i}\int_{L_-}\frac{f(\zeta)}{\zeta-x}\,d\zeta \quad \text{where } L_- : \{\operatorname{Im} z = -b\}.$$



Similarly, for $\xi \geq 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{0}\hat{f}(\xi)e^{i\xi x}\,d\xi = \frac{1}{2\pi i}\int_{L_+}\frac{f(\zeta)}{\zeta-x}\,d\zeta$$

Now, consider the contour Γ_R and use



the Cauchy's integral formula to write

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(\zeta)}{\zeta - x}\, d\zeta.$$

As $R \to \infty$, the integral over vertical sides $\to 0$:

$$\left| \int_{\pm R - ib}^{\pm R + ib} \frac{f(\zeta)}{\zeta - x}\, d\zeta \right| \leq 2b \frac{A}{1+R^2} \left[\frac{1}{|R-x|} \right] \to 0 \text{ as } R \to \infty.$$

Therefore, in the limit $R \to \infty$, we have

$$f(x) = \frac{1}{2\pi i} \int_{L_-} \frac{f(\zeta)}{\zeta - x}\, d\zeta + \frac{1}{2\pi i} \int_{L_+} \frac{f(\zeta)}{\zeta - x}\, d\zeta$$

by calc. above

$$= \frac{1}{2\pi} \int_0^\infty \hat{f}(\xi) e^{i\xi x}\, d\xi + \frac{1}{2\pi} \int_{-\infty}^0 \hat{f}(\xi) e^{i\xi x}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi.$$