Fourier Series & Quadrature: Applications

Recap

Consider a 2π-periodic signal $f:[0,2\pi]\to\mathbb{C}$.

Fourier series

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}$$

Fourier coeffs

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$$

If f is holomorphic in 2π-period strip

$$S = \{\theta + iy : \theta\in[0,2\pi]_{per},\ y\in[-a,a]\}$$

and satisfies $\sup_{z\in S}|f(z)|\le M$, then

coeffs

$$|\hat{f}_k| \le M e^{-a|k|}$$

Phase error

$$\sup_{\theta\in[0,2\pi]}\Big|f(\theta) - \sum_{|k|\le N}\hat{f}_k e^{ik\theta}\Big| \le \frac{2Me^{-a(N+1)}}{1-e^{-a}}$$

Trapezoid error

$$\Big|\int_0^{2\pi} f(\theta)\,d\theta - \frac{1}{N}\sum f\Big(2\pi\frac{k}{N}\Big)\Big| \le \frac{4\pi M}{1-e^{-aN}} e^{-aN}$$

These exp. accurate approx.'s lead to fast algo's.

<u>**Example:**</u> Solve $-u''(x) = f(x)$, $x \in [0, 2\pi]$

where $\int_0^{2\pi} u(x)\,dx = 0$ and f is periodic, smooth, mean-zero

Note that if f is smooth, so must be u!

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx} \quad \text{and} \quad f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx}$$

$$\Rightarrow \quad -(e^{ikx})'' = k^2 e^{ikx}$$

$$\Rightarrow \quad k^2 \hat{u}_k = \hat{f}_k$$

$k=0$ be careful!

$$\Rightarrow \quad u(x) = \sum_{k=-\infty}^{+\infty} \frac{\hat{f}_k}{k^2} e^{ikx}$$

$\int_0^{2\pi} u(x)\,dx = 0$

$(\hat{f}_k = 0)$

In practice, we can develop fast and accurate numerical approximations to $u(x)$ by computing Fourier coeffs of f and forming the truncated Fourier series for u.

"Pseudo"

<u>Fourier Spectral Method</u>

1) Compute $\hat{f}_k \approx \frac{1}{N}\sum_{j=1}^{N} f(2\pi\frac{j}{N})\, e^{-ik\frac{2\pi j}{N}} = \tilde{f}_k$

for $k=-N,\dots,N$

2) Compute $u(2\pi j/N) \approx \sum_{j=-N}^{N} \frac{\tilde{f}_k}{k^2}\, e^{ik\,2\pi j/N}$

for $j=1,\dots,N$

$\Rightarrow$ Output is approximate samples of $u(x)$ on grid $2\pi\frac{1}{N},\dots,2\pi$

$\Rightarrow$ Exponentially accurate for smooth periodic right-hand sides.

$\Rightarrow$ Both steps require only $O(N\log N)$ arithmetic operations using FFT.

This is fine for constant coefficient problems, but what about variable coefficients?

$$a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x)$$

variable coeffs

Example: $-u''(x) + \cos(x)u(x) = f(x)$

$$-u''(x) = \sum_{k=-\infty}^{+\infty} k^2 \hat{u}_k e^{ikx} \qquad f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx}$$

What is the fourier series of $\cos(x)u(x)$?

$$\cos(x)u(x) = \cos(x)\sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx}$$

$$= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx}$$

$$= \sum_{k=-\infty}^{+\infty} \frac{\hat{u}_k}{2}\left(e^{i(k+1)x} + e^{i(k-1)x}\right)$$

$$= \sum_{k=-\infty}^{+\infty} \frac{1}{2}\left(\hat{u}_{k+1} + \hat{u}_{k-1}\right)e^{ikx}$$

$$\Rightarrow \quad k^2\hat{u}_k + \frac{1}{2}\left(\hat{u}_{k+1} + \hat{u}_{k-1}\right) = \hat{f}_k, \quad k = 0, \pm 1, \pm 2, \ldots$$

For a numerical scheme, we need to truncate and solve $2N+1 \times 2N+1$ system of eqn's.

$$
\begin{bmatrix}
N & \frac{1}{2} & & & & & & & \\
\frac{1}{2} & \ddots & \ddots & & & & & & \\
& \ddots & 4 & 1 & & & & & \\
& & \frac{1}{2} & 0 & \frac{1}{2} & & & & \\
& & & 1 & 4 & \ddots & & & \\
& & & & \ddots & \ddots & \frac{1}{2} & & \\
& & & & & \frac{1}{2} & N^2 & &
\end{bmatrix}
\begin{bmatrix} \hat{u}_{-N} \\ \vdots \\ \hat{u}_{-1} \\ \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix}
=
\begin{bmatrix} \hat{f}_{-N} \\ \vdots \\ \hat{f}_{-1} \\ \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_N \end{bmatrix}
$$

$$
D \qquad \underline{\hat{u}} = \underline{\hat{f}}
$$

$$
u(x) = \sum_{k=-N}^{N} \hat{u}_k e^{ikx} \qquad x \in [0, 2\pi]
$$

$\Rightarrow$ Compute $\{\hat{f}_k\}_{k=-N}^{N}$ in $\mathcal{O}(N \log_2 N)$ FLOPS with FFT-based trap rule.

$\Rightarrow$ Solve tridiagonal system in $\mathcal{O}(N)$ FLOPS with banded Cholesky (Gauss. elim.) fancy symm.

$\Rightarrow$ Evaluate $u(x)$ on equispaced grid in $\mathcal{O}(N \log_2 N)$ with FFT.

The key to this scheme being "fast" is that the linear system for the Fourier coeffs was banded (tridiagonal).

"fast" = requires FLOPS $\sim N \log_2 N$

What about more general coefficients, like $c(x) = \exp(-\cos^2(x))$?

<u>Idea:</u> If coeffs are smooth and periodic, linear system never gets "too expensive."

Multiplication in "Fourier Space"

Consider the multiplication of two functions f and g given by Fourier series:

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta} \quad \text{and} \quad g(\theta) = \sum_{k=-\infty}^{+\infty} \hat{g}_k e^{ik\theta}$$

What is the Fourier series for $[fg](\theta)$?

$$[fg](\theta) = \left(\sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}\right)\left(\sum_{k=-\infty}^{+\infty} \hat{g}_k e^{ik\theta}\right)$$

$$= \sum_{j=-\infty}^{+\infty} \hat{f}_j \left[\sum_{k=-\infty}^{+\infty} \hat{g}_k e^{ik\theta}\right] e^{ij\theta}$$

$$= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \hat{f}_j \hat{g}_k e^{i(k+j)\theta}$$

$$= \sum_{n=-\infty}^{+\infty} \left(\sum_{j+k=n} \hat{f}_j \hat{g}_k \right) e^{in\theta}$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{\left(\sum_{k=-\infty}^{+\infty} \hat{f}_{n-k} \hat{g}_k \right)}_{\text{"Discrete convolution"} \,=\, (\widehat{fg})_n} e^{in\theta}$$

To multiply two Fourier series, we perform a discrete convolution on their Fourier coeffs.

$$M_f g = \underbrace{\begin{bmatrix} \ddots & & \hat{f}_{-1} & \hat{f}_0 & \hat{f}_1 & \hat{f}_2 & \ddots & & \\ \hat{f}_{-N} & \cdots & & \hat{f}_{-1} & \hat{f}_0 & \hat{f}_1 & \cdots & \hat{f}_N & \\ & & & & \ddots & \ddots & \ddots & \hat{f}_1 & \\ & & & & & & \hat{f}_{-1} & \hat{f}_0 & \end{bmatrix}}_{\text{"Toeplitz" Operator/Matrix}} \begin{bmatrix} \hat{g}_{-N} \\ \vdots \\ \hat{g}_{-1} \\ \hat{g}_0 \\ \hat{g}_1 \\ \vdots \\ \hat{g}_N \end{bmatrix}$$

Note that if $|\hat{f}_k| \le M e^{-a|k|}$, the matrix entries decay exponentially away from the diagonal.