Laplace Domain Analysis

Let $f:[0,\infty)\to\mathbb{C}$ be (piecewise) continuous and $|f(t)|\le Ce^{bt}$, for some $C,b\ge 0$, for all $t\ge 0$.

Laplace Transform
$$F(z)=\int_0^\infty f(t)e^{-zt}\,dt,\qquad z\in\mathbb{C}.$$

If f is (piecewise) continuously differentiable,

Inverse Transform
$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(z)e^{zt}\,dz,\qquad \gamma>b.$$

(If piecewise continuous, transform recovers the "average" $\lim_{\varepsilon\to 0}\frac{1}{2}(f(t+\varepsilon)-f(t-\varepsilon))$ at discontinuities.)

Laplace Domain Analysis provides a natural analogue of Fourier Analysis for initial-value prob's.

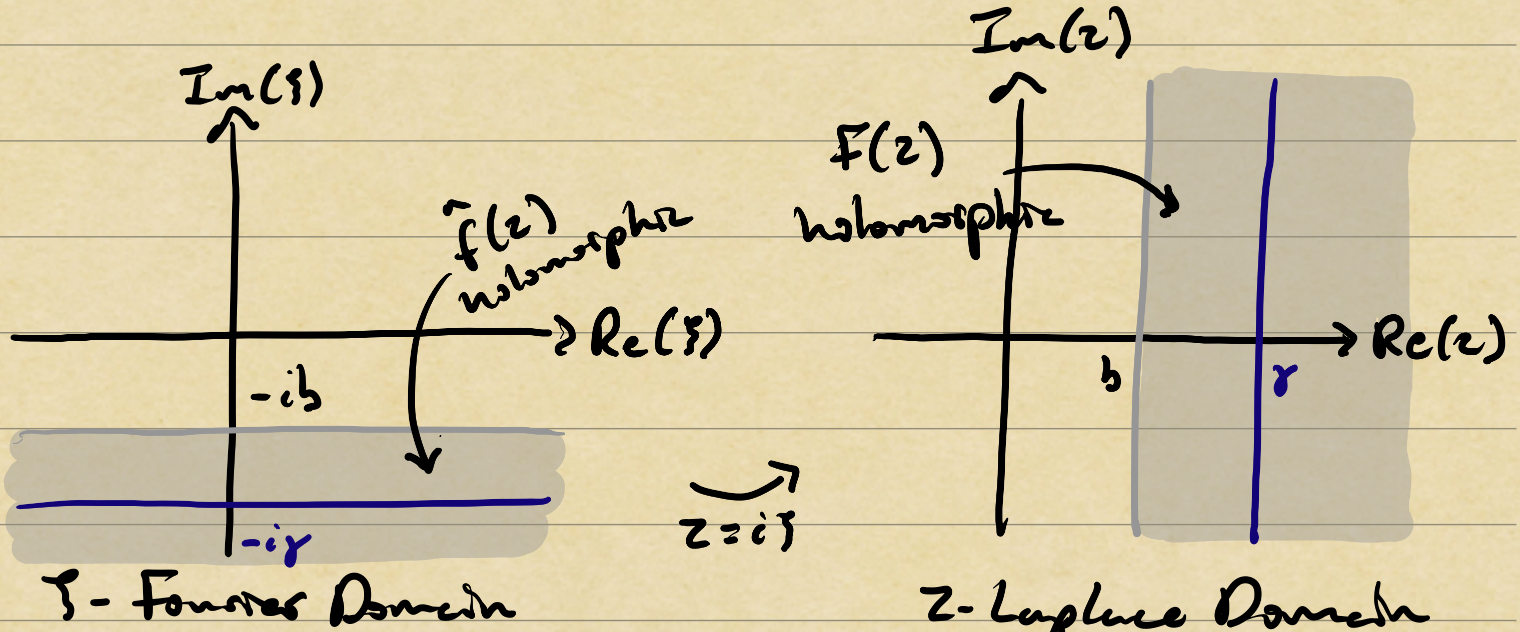
IVP $\quad u''(t)+k^2u(t)=f(t),\quad u(0)=a,\ u'(0)=b$

char. poly. $\qquad$ forcing $\qquad$ initial

LDA $\quad [z^2+k^2]U(z)=F(z)+az+b$

Bromwich Formula

The Complex Inversion ("Bromwich") formula can be derived from its Fourier counterpart:



$$\hat{f}(\xi) = \int_0^\infty f(t)\, e^{-i\xi t}\, dt \qquad\qquad F(z) = \int_0^\infty f(t)\, e^{-zt}\, dt$$

$$= \int_0^\infty \underbrace{\left[f(t)\, e^{\mathrm{Im}(\xi) t}\right]}_{|\cdot| \le C e^{(b - \mathrm{Im}\,\xi)t}} e^{-i\mathrm{Re}(\xi)t}\, dt$$

$k = \mathrm{Re}(\xi)$

So if $-\gamma = \mathrm{Im}(\xi) < -b$, the integral converges abs. and

$$\Rightarrow f(t)\, e^{-\gamma t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\underbrace{k - i\gamma}_{\xi = k - i\gamma})\, e^{ikt}\, dk$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k-i\gamma)e^{i(k-i\gamma)t}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty-i\gamma}^{+\infty-i\gamma} \hat{f}(\xi)e^{i\xi t}\,d\xi$$

If we substitute $z = i\xi$, we get the Laplace inversion formula

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)e^{zt}\,dz.$$

Note that $F(z)$ may also be analytic in regions with $\mathrm{Re}(z) < b$, but this will depend heavily on further structure in $f(t)$.

<u>Example 1:</u> Harmonic Oscillator

Laplace Domain

$$\underbrace{[z^2+k^2]}_{\text{char. poly.}}U(z) = \underbrace{F(z)}_{\text{forcing}} + \underbrace{az+b}_{\text{initial}}$$

Solution in Lap. Dom.

$$U(z) = \frac{F(z)}{z^2+k^2} + \frac{az+b}{z^2+k^2}$$

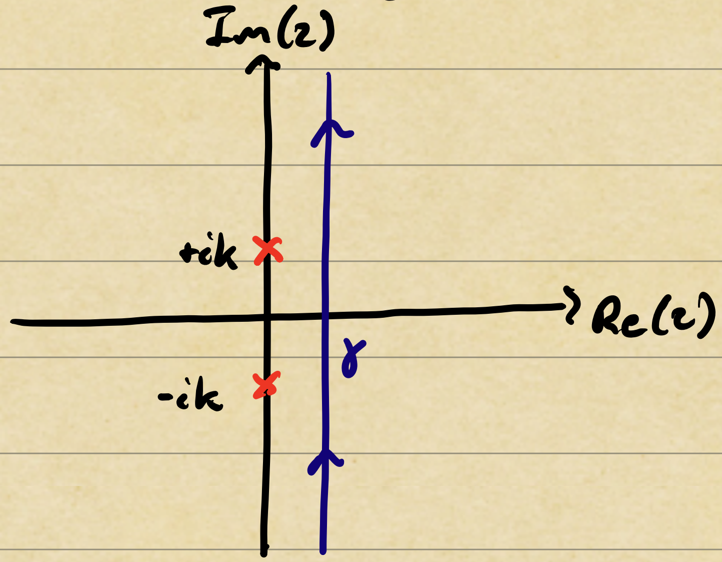
We can construct solutions and solution operators by applying the inverse transform.

$$u(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)}{z^2+k^2}\, e^{zt}\, dz \;+\; \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{az+b}{z^2+k^2}\, e^{zt}\, dz$$

Particular Soln. w/ $u(0) = u'(0) = 0$

Homogeneous Soln. w/ $u(0) = a$, $u'(0) = b$

The first term captures the oscillator's response to the forcing while the second term adjusts the particular solution to fit the initial conditions by adding in just the right combination of homog. soln's.



If $|f(t)|$ is bounded we can take any $\gamma > 0$ for the inverse transform.

Note that the second integrand is analytic except at $\pm ik$, poles of characteristic poly.

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{az+b}{z^2+k^2}\, e^{zt}\, dz = c_1(a,b)\, e^{ikt} + c_2(a,b)\, e^{-ikt}$$

residue theorem

Example 2: Linear Time-Invariant System

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underline{u} \qquad\quad A \qquad\quad \underline{u} \qquad\qquad\qquad \underline{u}(0) = \underline{x}$$

$$\underline{u} : [0,\infty) \to \mathbb{C}^n$$

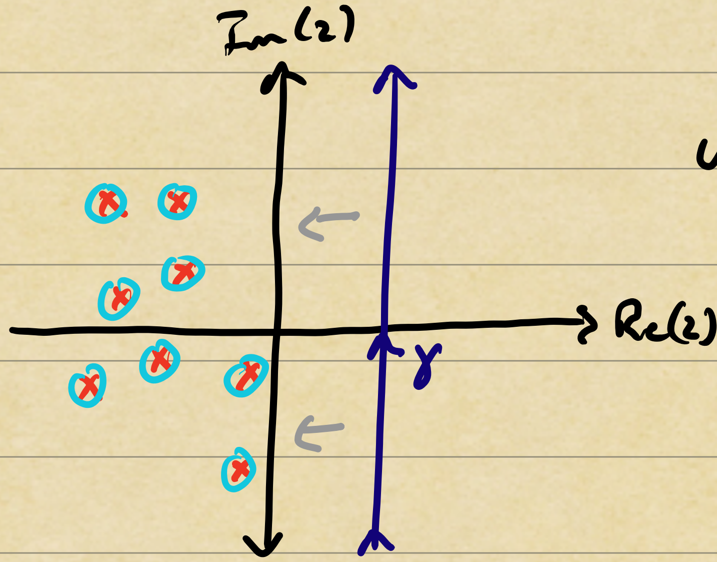
If we take the Laplace transform of each equation separately, we get

$$z\,\underline{U}(z) = A\underline{U}(z) - \underline{x} \qquad \underline{U} : \underset{\underset{\mathbb{C}}{\cup}}{\Omega} \to \mathbb{C}^n$$

$$\Rightarrow \qquad (A - zI)\underline{U}(z) = \underline{x}$$

$$\Rightarrow \qquad \underline{U}(z) = (A - zI)^{-1}\underline{x}, \qquad z \in \rho(A)$$

The solution operator in the Laplace domain is the resolvent of A! The resolvent is well-defined & analytic in z except at eigenvalues of A.



$$\underline{u}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left[(A-zI)^{-1}\underline{x}\right] e^{zt}\,dz$$

$$= \sum_{k=1}^{n} \operatorname*{Res}_{z=\lambda_k} \left[(A-zI)^{-1}\underline{x}\right]e^{zt} + \underbrace{\lim_{\gamma'\to-\infty} \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \left[(A-zI)^{-1}\underline{x}\right]e^{zt}\,dz}_{=0}$$

If $A = \overset{\text{eigenvector matrix}}{V}\underset{\text{eigenvalue matrix}}{\Lambda}V^{-1}$ is diagonalizable, then

$$\operatorname*{Res}_{z=\lambda_k}\left[(A-zI)^{-1}\underline{x}\right]e^{zt} = e^{\lambda_k t}(\underline{w}_k^*\underline{x})\underline{v}_k$$

where $\underline{v}_k = k^{th}$ column of V, $\underline{w}_k^* = k^{th}$ column of V^{-1}.

$$\underline{u}(t) = \sum_{k=1}^{n} e^{\lambda_k t}(\underline{w}_k^*\underline{x})\underline{v}_k$$

This is the spectral representation of $u(t)$ and

$$S(t) = \sum_{k=1}^{n} e^{\lambda_k t}\,\underline{v}_k\underline{w}_k^*$$

is the spectral representation of the solution operator $S(t)$, which maps $\underline{u}(0) = \underline{x} \to \underline{u}(t)$.

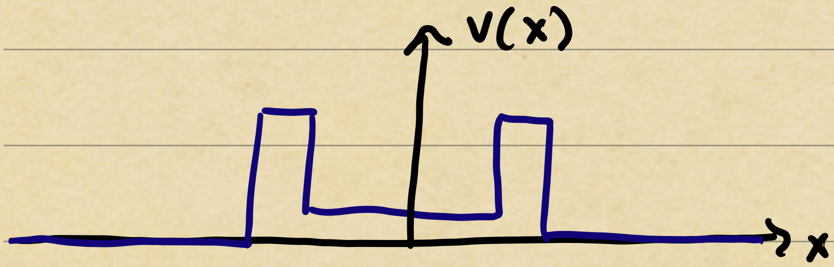
This Laplace Domain analysis extends easily to infinite-dimensional analogues of $u'(t) = Au(t)$, where A is an operator on an infinite-dimensional space. E.g.,

$$[Au] = \frac{d^2u}{dx^2} \qquad u : [0,\infty) \to C^2([-1,1])$$

twice cont. differentiable functions on $[-1,1]$

Example: Quantum Scattering

$$i\,\partial_t u(x,t) = \underbrace{\partial_x^2 u(x,t) + V(x)\,u(x,t)}_{[Lu](x,t)}, \qquad \underbrace{u(x,0) = h(x)}_{\text{initial condition}},$$

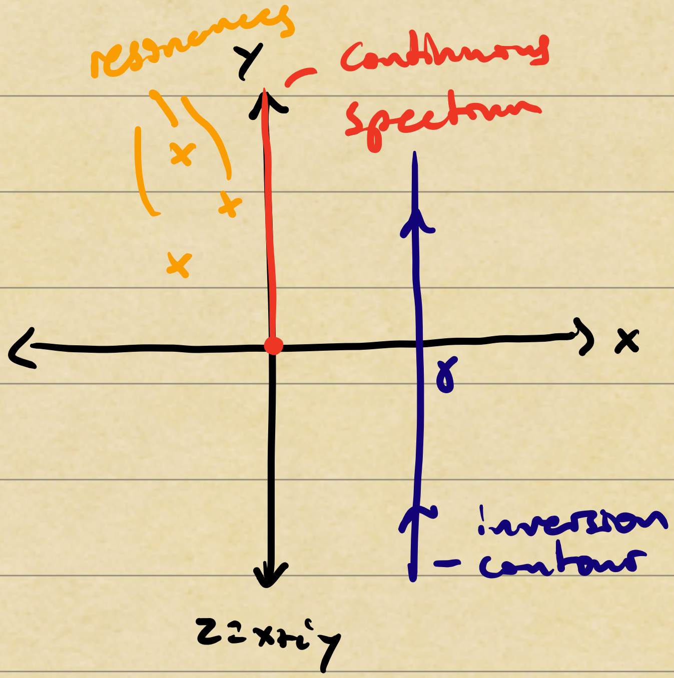


Quantum particles "scatter" from potential barrier.

Laplace Domain

$$z\,U(x,z) = L\,U(x,z) - h(x)$$

$$U(x,z) = (L - zI)^{-1} h(x)$$



For certain h, $(L - zI)^{-1} h$ is analytic over cont. spectrum and can develop a resonance expansion

$$u(x,t) = \sum_j e^{\lambda_j t} \langle w_k, h \rangle v_k + \frac{1}{2\pi i} \int_{\gamma' - i\infty}^{\gamma' + i\infty} e^{zt} (L - zI)^{-1} h \, dz$$

resonances "background"