The Paley-Wiener Thm

In signal processing and inverse problems, a bandlimited function is composed of freq. $\xi \in [-B, B]$

$$f(x) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

In other words, $\hat{f}(\xi) = 0$ when $|\xi| \geq B$.

Paley-Wiener Thm characterizes bandlimited f.

Thm| Suppose f is continuous and of moderate decrease ($\leq \frac{A}{1+x^2}$, $x \in \mathbb{R}$). Then f has an extension to the complex plane that is entire with $|f(z)| \leq Ae^{B|z|}$ for some $A \geq 0$, IFF $\hat{f}(\xi) = 0$ for $|\xi| \geq B$.

Pf| ($\Uparrow$) $\quad g(z) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi) e^{i\xi z}\, d\xi.$

last time $\Rightarrow$ $g(x) = f(x)$, $g(z)$ entire, $|g(z)| \leq Ae^{B|y|} \leq Ae^{B|z|}$

($\Downarrow$) To prove the converse, start with simpler class of functions: "bootstrap."

<u>Step 1:</u> Show that $\hat{f}(\xi)=0$ for $|\xi|>B$ when

$$|f(x+iy)| \le A' \frac{e^{B|y|}}{1+x^2}$$

← growth of real axis

← integrability

and f is entire. Idea is to take

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-i\xi x}\, dx$$

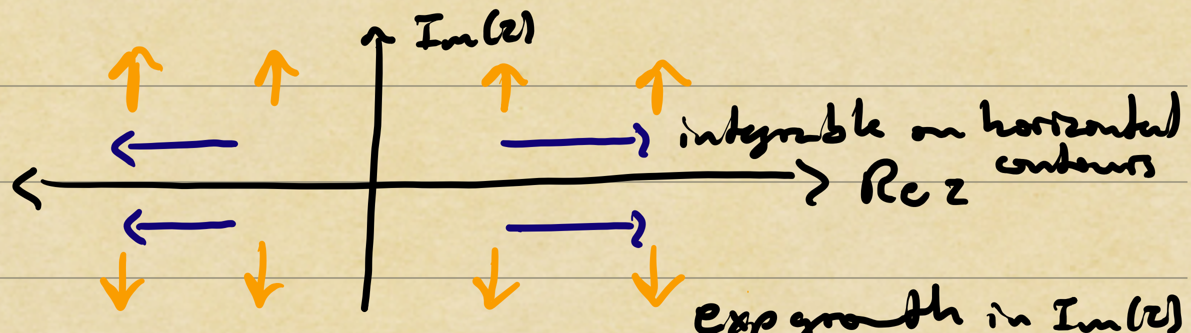
and deform contour from $x \to x \pm iy$ when

$$|\hat{f}(\xi)| \le \int_{-\infty}^{+\infty} A' \frac{e^{B|y|} e^{\pm \xi y}}{1+x^2}\, dx \le C e^{(B-|\xi|)y}$$

$\to 0$ as $y \to \infty$

when $|\xi|>B$.

This establishes $\hat{f}(\xi)=0$ for $|\xi|>B$ when



<u>Step 2:</u> Remove decay on horizontal lines. Let f be entire with growth bound

$$|f(x+iy)| \leq A e^{B|y|}.$$

Take $\underline{\xi > B}$ and $\varepsilon > 0$, consider "regularized"

$$f_\varepsilon(z) = \frac{f(z)}{(1+i\varepsilon z)^2},$$

The regularized function satisfies

$\Rightarrow$ $f_\varepsilon(z)$ holomorphic for $\mathrm{Im}(z) \leq 1/\varepsilon$,

$\Rightarrow$ $|f_\varepsilon(z)| \leq |f(z)|$ for $\mathrm{Im}(z) \leq 0$,

$\Rightarrow$ $f_\varepsilon(z) \to f(z)$ as $\varepsilon \to 0$.

Moreover, for each fixed $\varepsilon > 0$, we have

$$|f_\varepsilon(x+iy)| \leq A_\varepsilon \frac{e^{B|y|}}{1+x^2}.$$

By step 1 argument, $\hat{f}_\varepsilon(\xi) = 0$ for $\xi > B$.

Now, idea is to show that $\hat{f}_\varepsilon(\xi) \to \hat{f}(\xi)$ for each $\xi \in \mathbb{R}$, so that $\hat{f}(\xi) = 0$ for $\xi > B$ also.

$$|\hat{f}_\varepsilon(\xi) - \hat{f}(\xi)| \leq \int_{-\infty}^{+\infty} |f(x)| \left[\frac{1}{|1+i\varepsilon x|^2} - 1 \right] dx$$

Since $|f(x)| \leq \frac{A}{1+x^2}$ $(x \in \mathbb{R})$ by hypothesis,

$$\int_{-\infty}^{+\infty} |f(x)| \left[\frac{1}{|1+i\varepsilon x|^2} - 1 \right] dx \leq \int_{|x|>L} \frac{A}{1+x^2} \underbrace{\left[\frac{1}{|1+i\varepsilon x|^2} - 1 \right]}_{\leq 2} dx$$

indep. of ε

so pick L s.t. $\leq \delta/2$

$$+ \int_{|x|\leq L} \frac{A}{1+x^2} \left[\underbrace{\frac{1}{|1+i\varepsilon x|^2}}_{\to 0} - 1 \right] dx$$

uniformly as $\varepsilon \to 0$ on $|x| \leq L$, so pick $\varepsilon(L)$ s.t. $\leq \delta/2$

$\to 0$ as $\varepsilon \to 0$

Similar argument for $\xi < -B$ establishes that

$$\hat{f}(\xi) = 0 \text{ for } |\xi| > B.$$

Step 3: The last step is to then show that the growth condition in the thm implies step 2 hyp.

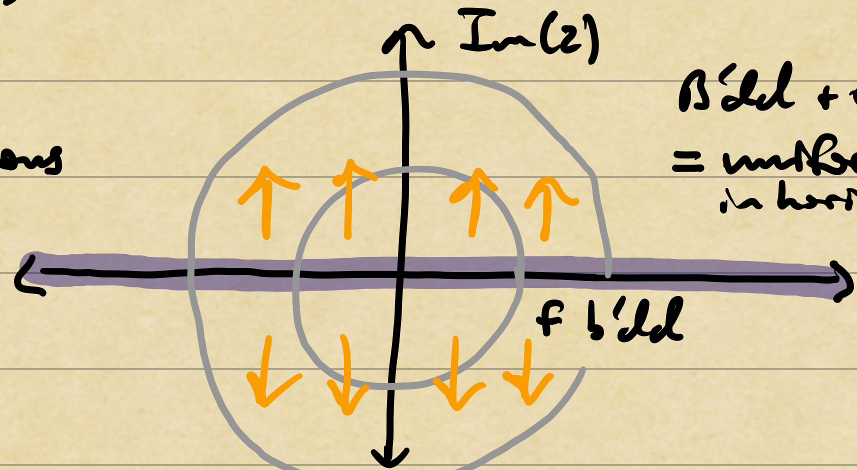
$$|f(x)| \leq \frac{A}{1+x^2} \quad x \in \mathbb{R}$$

$$\Rightarrow \quad |f(x+iy)| \leq A e^{B|y|}$$

$$|f(z)| \leq A e^{B|z|} \quad z \in \mathbb{C}$$

In fact, it suffices that f is b'dd on $\mathbb{R}$.

"All such functions behave like exp(z)"



B'dd + exp growth = uniformly b'dd in horizontal lines.

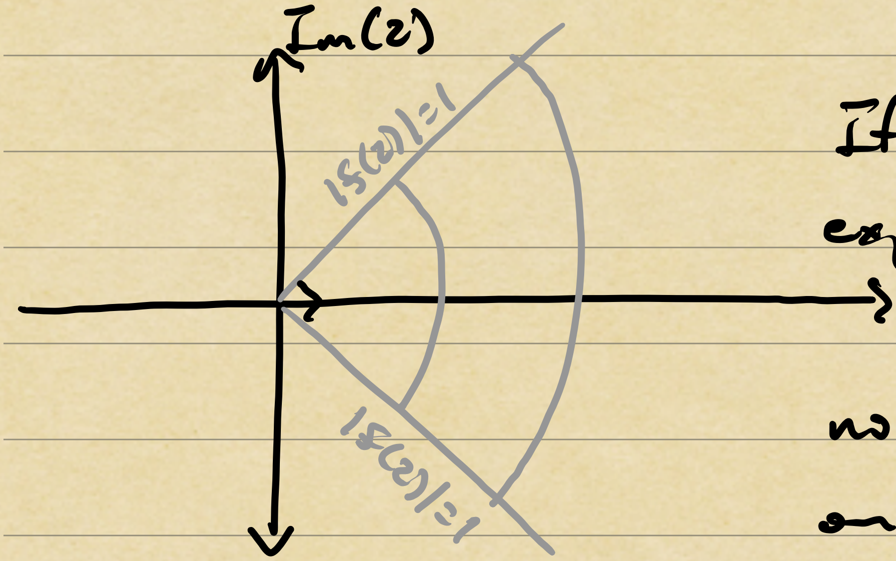
Re(z)

<u>Thm</u> (Phragmén & Lindelöf)

Suppose that F is a holomorphic function in the sector $S = \{z : -\pi/4 < \arg z < \pi/4\}$, where F continuous & $|F(z)| \leq 1$ on ∂S, and there are constants $C, c \geq 0$ s.t. $|F(z)| \leq C e^{c^2}$ for all $z \in S$. Then

$$|F(z)| \leq 1 \quad \text{for all } z \in S.$$

<u>Intuition</u>: Function $g(z) = e^{z^2}$ is b'dd on ∂S but blows up on positive real axis. P-L says that the <u>only</u> functions that do this grow faster than the exponential.



If growth is at most exponential, then f is no larger in S than on the boundary.

This is a generalization of the maximum principle to unbounded domains.

<u>Pf of Step 3</u>: Result of P-L holds in first quadrant after rotation $e^{i\pi/4}S$. Now take

$$F(z) = f(z) e^{iBz},$$

which has $|F(z)| \leq 1$ on real axis and $|F(x+iy)| \leq e^{By} e^{-By} = 1$ for $y \geq 0$. Moreover,

$$|F(z)| \leq A e^{B|z|} |e^{iBx} e^{-By}| \leq A e^{B|z|}, \qquad x, y \geq 0.$$

Therefore, P-L implies that $|F(z)| \leq 1$ in the first quadrant, so that

$$|f(z)| \leq A e^{-iB(x+iy)} = A e^{By} \qquad x, y \geq 0.$$

Similar argument for other quadrants.