Applications of P-W Theory

Recap

Given $f:\mathbb{R}\to\mathbb{R}$ with appropriate regularity & decay:

$$(1) \qquad \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i \xi x}\,dx$$ "Fourier Transform"

is the Fourier Transform of f, which satisfies

$$(2) \qquad f(x) = \int_{-\infty}^{+\infty} \hat{f}(\xi)\,e^{2\pi i \xi x}\,d\xi.$$ "Inverse Fourier Transform"

Broadly speaking, we have examined how

$\Rightarrow$ Smoothness in $f(x)$ translates to decay in $\hat{f}(\xi)$ as $\xi \to \pm\infty$.

$\Rightarrow$ Decay in $\hat{f}(\xi)$ translates to Smoothness in $f(x)$ for $x \in \mathbb{R}$.

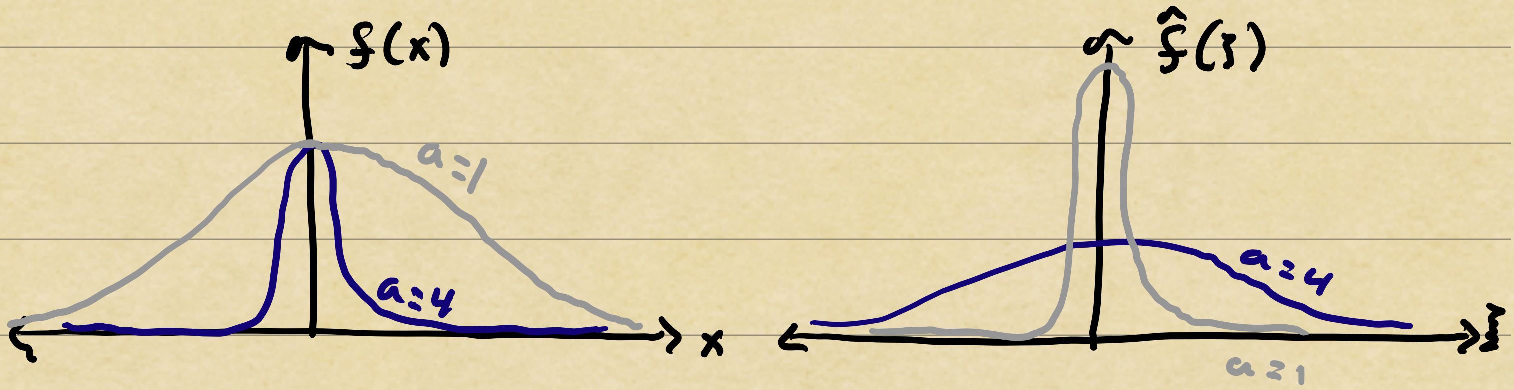
Today, we'll focus on a few implications of these ideas to problems in applied math.

Hardy's Uncertainty Principle

Remarkably, the smoothness/decay connection implies that $f(x)$ and $\hat{f}(\xi)$ cannot be simultaneously localized.

Example: Given $a>0$, we have the pair

$$f(x) = e^{-a\pi x^2} \qquad \hat{f}(\xi) = \frac{1}{\sqrt{a}} e^{-\pi \xi^2/a}$$



Intuitively, the Gaussian grows faster off the real axis as a increases, which decreases the decay rate of $\hat{f}(\xi)$ exponentially, since

$$f(iy) = e^{-a(iy)^2} = e^{ay^2}$$

grows exponentially as $z = iy \to \pm i\infty$.

Thm Suppose that $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$(*) \quad |f(x)| \leq C_1 e^{-a\pi x^2} \text{ and } |\hat{f}(\xi)| \leq C_2 e^{-b\pi \xi^2},$$

with $ab \geq 1$ and $a, b > 0$. Then $f(x) = 0$ for $x \in \mathbb{R}$.

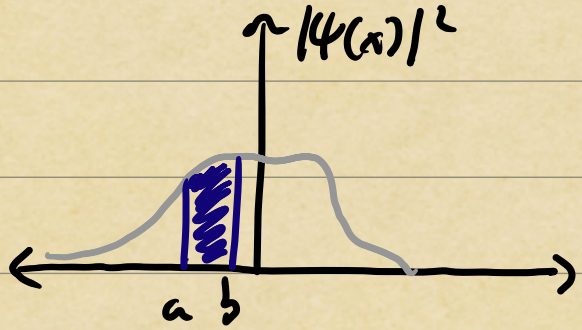
Proof Idea: Show that if $a = b = 1$, then $(*)$ implies that $f(x) = (\text{const.}) e^{-\pi x^2}$. By a change of variables $x \to \sqrt{a}\, x$ in F.T., then $(*)$ with $a, b = 1/a$ implies $f(x) = (\text{const.}) e^{-a\pi x^2}$. If $b > 1/a$, then $(*)$ with $a, b = 1/a$ is satisfied and $f(x) = (\text{const.}) e^{-a\pi x^2}$, $\hat{f}(\xi) = (\text{const.}) e^{-\pi x^2/a}$ so the only way $\hat{f}(\xi) \leq (\text{const.}) e^{-b\pi x^2}$ w/ $b > 1/a$ is if $f(x) = 0$.

(See Ch. 4 Problem 12 in Stein/Shak.)

Quantum Uncertainty

In QMech., position of particle (1D) is described by a "wave function" $\psi: \mathbb{R} \to \mathbb{R}$ with

$$P(a \leq X \leq b) = \int_a^b |\psi(x)|^2 \, dx$$



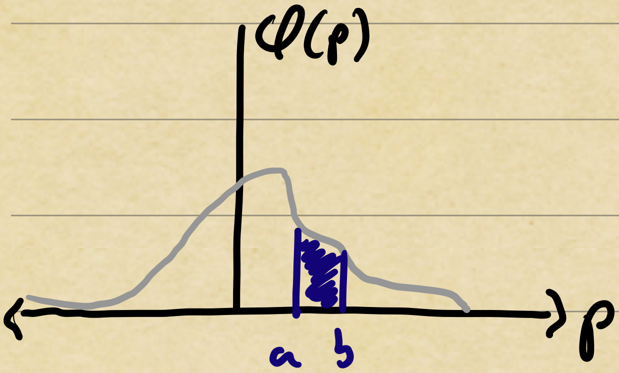
$|\psi(x)|^2$ acts as a prob. density for position.

Similarly, the momentum is described by a related wave function $\phi(p)$, with

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(p)\, e^{ipx}\, dp$$

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x)\, e^{-ipx}\, dx$$

i.e., $\psi(x)$ and $\phi(p)$ are Fourier Transforms.



Again, momentum has

$$P(a<P<b) = \int_a^b |\phi(p)|^2\, dp$$

aka $|\phi(p)|^2$ is prob. density for momentum.

Hardy Uncertainty Principle implies that Position and Momentum cannot be both known with a high degree of uncertainty at once.