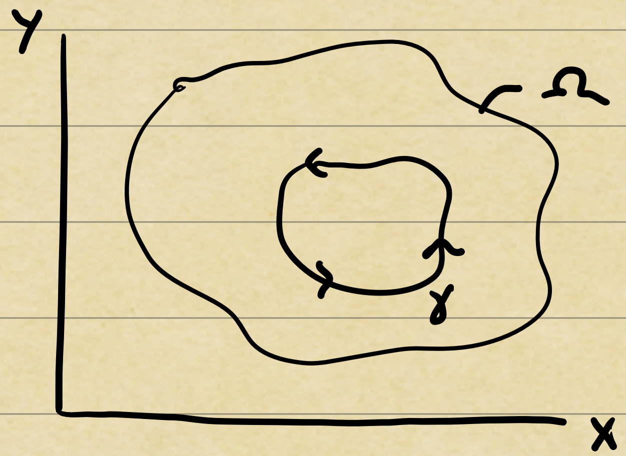
Cauchy's Theorem and Applications

Cauchy's Theorem: If $f:\Omega \to \mathbb{C}$ is holomorphic in a simply connected open set Ω, then for any (smooth) simple closed contour $\gamma \subset \Omega$, $\int_\gamma f(z)\,dz = 0$.



Sketch pf 1 | If we *also* assume that $f'(z)$ is continuous on Ω, there is a simple proof using Green's formula + Cauchy-Riemann.

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)$$

Green's Thm

$$= -\iint_{\mathrm{int}(\gamma)} (\partial_x v + \partial_y u)\,dx\,dy + i\iint_{\mathrm{int}(\gamma)} (\partial_x u - \partial_y v)\,dx\,dy$$

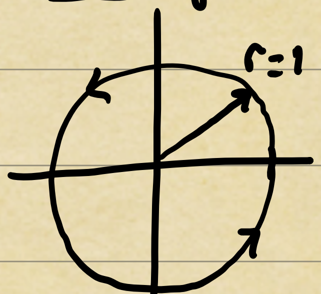
Cauchy-Riemann Eqn's

$$= 0$$

One can prove Cauchy's thm w/out assuming $f'(z)$ continuous, and Cauchy's thm actually *implies* that $f'(z)$ is continuous (Cauchy integral formulas).

The interior of γ <u>must</u> be in simply connected Ω.

Example: Consider $f(z) = z$, $g(z) = z^{-1}$



$\gamma = \{e^{i\theta} : \theta \in [0, 2\pi]\}$

$$\int_\gamma z\,dz = \int_0^{2\pi} \underbrace{(e^{i\theta})}_{z} \underbrace{(ie^{i\theta} d\theta)}_{dz}$$

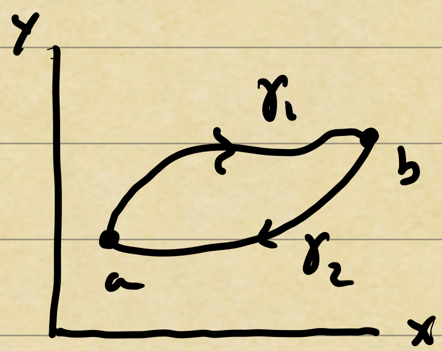
$$= i\int_0^{2\pi} e^{2i\theta} d\theta = 0$$

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} (e^{-i\theta})(ie^{i\theta}) d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

z^{-1} is holomorphic in $\mathbb{C} \setminus \{0\}$, so $\mathrm{int}(\gamma)$ <u>not</u> simp. conn.

Contour Deformation

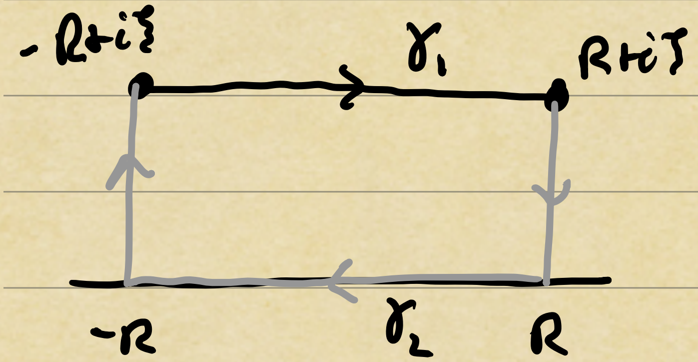
Cauchy's theorem allows us to evaluate tricky integrals by choosing "better" contours for the integration in the complex plane.



$$0 = \int_{\gamma_1 + \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = -\int_{\gamma_2} f(z)\,dz$$

For example, last lecture we computed



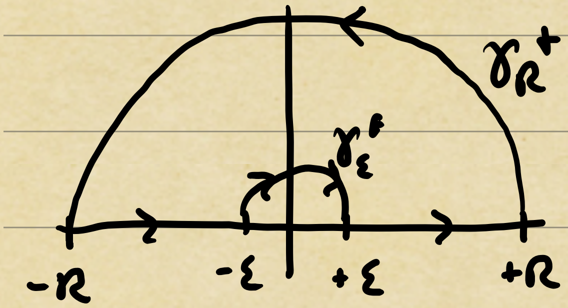
$$\int_{-R}^{+R} e^{-\pi x^2} e^{-2\pi i x \xi}\, dx$$

$$= e^{-\pi \xi^2} \int_{\gamma_1} e^{-\pi z^2}\, dz$$

by "deforming" the contour from γ_1 to γ_2.

Because $e^{-\pi z^2}$ is holomorphic in $\mathbb{C}$, integration between points $-R+i\xi$ and $R+i\xi$ is independent of contour.

<u>Example:</u> $\displaystyle\int_0^{+\infty} \frac{1-\cos x}{x^2} = \lim_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_\varepsilon^R \frac{1-\cos x}{x^2}\, dx$



Take $f(z) = \dfrac{1-e^{iz}}{z^2}$ and consider

$\displaystyle\int_\gamma f(z)\,dz = 0$ by Cauchy Thm.

$$\int_\gamma f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz + \int_\varepsilon^R \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz$$

Along γ_R^+, we have $|f(z)| = \left|\dfrac{1-e^{iz}}{z^2}\right| \le \dfrac{2}{|z|^2} \to 0$ as $R\to\infty$.

So that $\left|\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz\right| \le (\pi R)\left(\frac{2}{R^2}\right) = \frac{2\pi}{R} \to 0$

as $R \to \infty$.

We are left with

$$\int_{|x|\ge\varepsilon} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz.$$

Now we need to compute $\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz$

and we have that $\frac{1-e^{iz}}{z^2} = \frac{-1}{z^2}\Big(iz \underbrace{- \frac{z^2}{2} - i\frac{z^3}{6} + \cdots}_{g(z)\,\sim\,O(z^2) \text{ as } z\to 0}\Big).$

so $$\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_\varepsilon^+} \frac{-i}{z}\,dz + \int_{\gamma_\varepsilon^+} \frac{-g(z)}{z^2}\,dz.$$

$= -\frac{1}{2} + i\frac{z}{6} \cdots$

Now, $\left|\int_{\gamma_\varepsilon^+} \frac{-g(z)}{z^2}\,dz\right| \le \pi\varepsilon \sup_{z\in\gamma_\varepsilon^+} \left|\frac{g(z)}{z^2}\right| \to 0$ as $\varepsilon \to 0$.

and $$\int_{\gamma_\varepsilon^+} \frac{-i}{z}\,dz = \int_0^\pi \left(\frac{-i}{e^{i\theta}}\right) ie^{i\theta}\,d\theta = \int_0^\pi d\theta = \pi.$$

$z = e^{i\theta},\ dz = ie^{i\theta}\,d\theta$

$$\lim_{\varepsilon\to 0} \int_{|x|\ge\varepsilon} \frac{1-e^{ix}}{x^2}\,dx = \pi$$

Take real part and use even integrand: $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$.

In general, when choosing how to "deform" contour:

⇒ look to exploit regions where integrand decays and contribution to integral becomes negligible.

⇒ Watch out for singularities near or on contour, which contribute to integral.

We'll make this strategy systematic w/ "residue calculus."

Cauchy's Integral Formula(s)

From Cauchy's theorem, we can derive one of the most important representations of holo. $f(z)$.

Thm | Suppose f is holomorphic in simply connected open set $\Omega \subseteq \mathbb{C}$ that contains a (smooth) simple closed Jordan curve γ. Then for $z \in \operatorname{int}(\gamma)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$