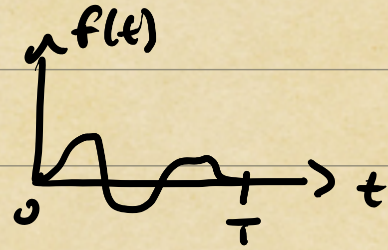
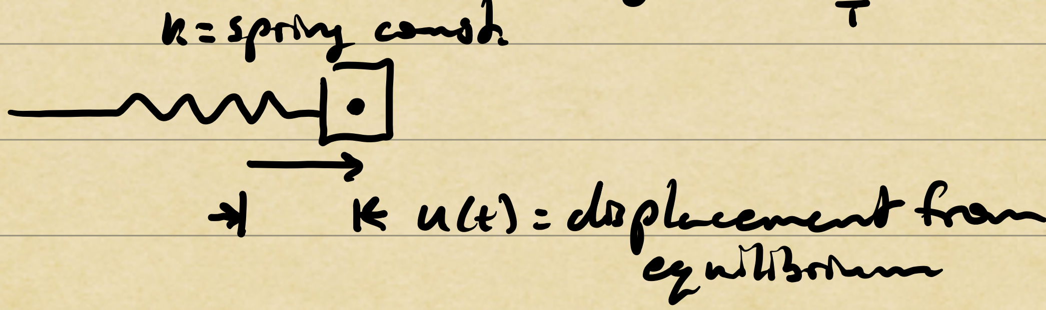
Boundary/Initial Value Problems

Consider the forced harmonic oscillator



$$(*) \quad u''(t) + k^2 u(t) = f(t)$$



The solutions to the homogeneous eqn. are

$$(**) \quad v''(t) + k^2 v(t) = 0 \quad \Rightarrow \quad \begin{aligned} v_1(t) &= c_1 \cos kt \\ v_2(t) &= c_2 \sin kt \end{aligned}$$

So, for any solution $u(t)$ of $(*)$, we also have

$$\tilde{u}(t) = c_1 \cos kt + c_2 \sin kt + u(t)$$

We can specify a unique solution by selecting boundary or initial values of $u(t)$ that fix the integration constants c_1, c_2

Q: How does the choice of particular solution $u(t)$ and constants c_1, c_2 show up in the Green's function?

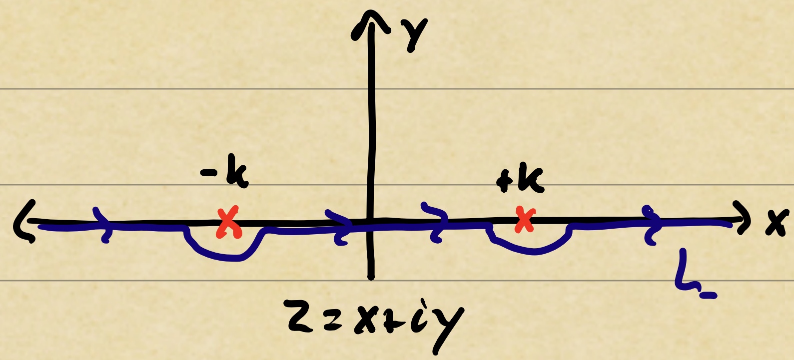
$$u(t) = \int_0^T G(t-\tau) f(\tau)\, d\tau$$

Causal Green's Function

The causal Green's function is

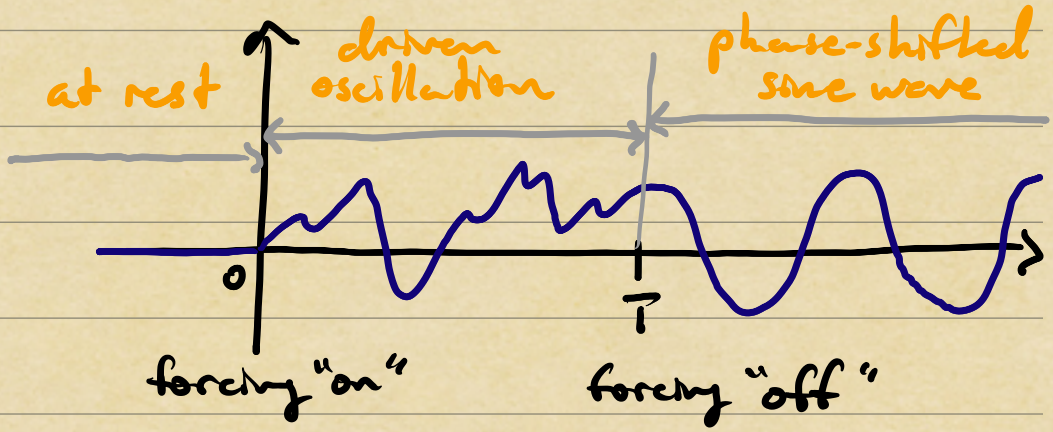
$$G_-(t-\tau) = \frac{1}{2\pi} \int_{L_-} \frac{e^{iz(t-\tau)}}{k^2 - z^2}\, dz = \begin{cases} \frac{1}{k} \sin k(t-\tau) & \tau < t \\ 0 & \tau > t \end{cases}$$

$\Rightarrow$ $u(t)$ depends only on $f(\tau)$, $\tau < t$



$\Rightarrow$ $u(t) = 0$, $t < 0$.

$\Rightarrow$ $u(t) = A \sin(kt + \varphi)$ for $t \geq 0$.

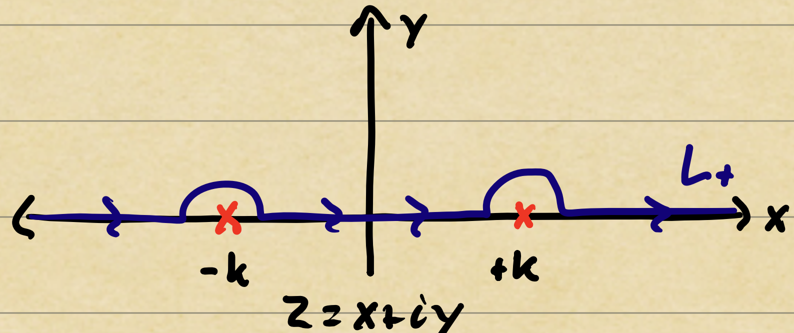


Anti-Causal Green's Function

The anti-causal Green's function is

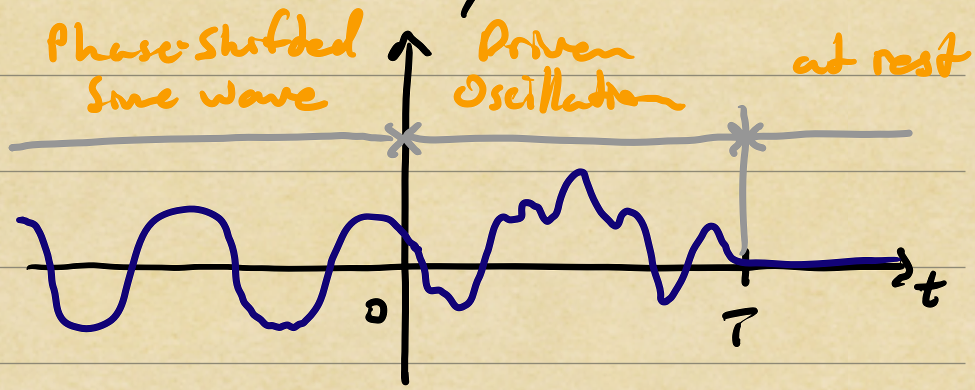
$$G_+(t-\tau) = \frac{1}{2\pi}\int_{L_+} \frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz = \begin{cases} 0 & \tau < t \\ \frac{1}{k}\sin k(t-\tau) & \tau > t \end{cases}$$

$\Rightarrow u(t)$ depends on $f(\tau)$ for $\tau \geq t$.



$\Rightarrow u(t) = 0, \quad t > T.$

$\Rightarrow u(t) = A\sin(kt+\varphi)$ for $t < 0$.



We know that causal and anti-causal solutions can differ only by homogeneous solutions. Therefore, causal solution is uniquely specified by, e.g.,

initial value $\Rightarrow$ $u(0)=0$, $u'(0)=0$.

Large-time behavior $\Rightarrow$ amplitude A, phase φ.

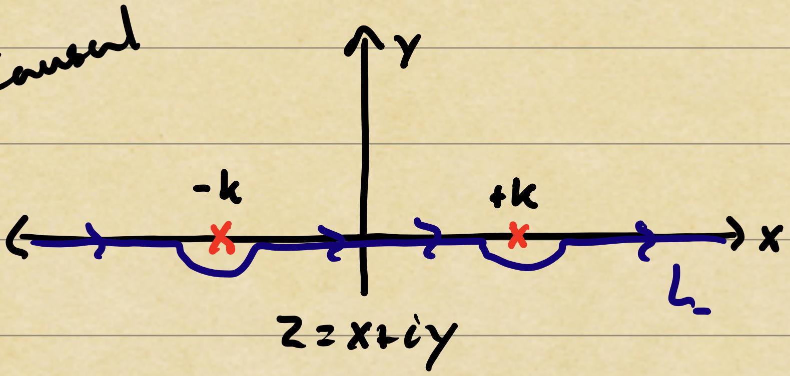
Similarly, anti-causal solution is uniquely determined by

initial-value ⇒ amplitude A, phase φ

Large-time ⇒ $u(T)=0$, $u'(T)=0$.

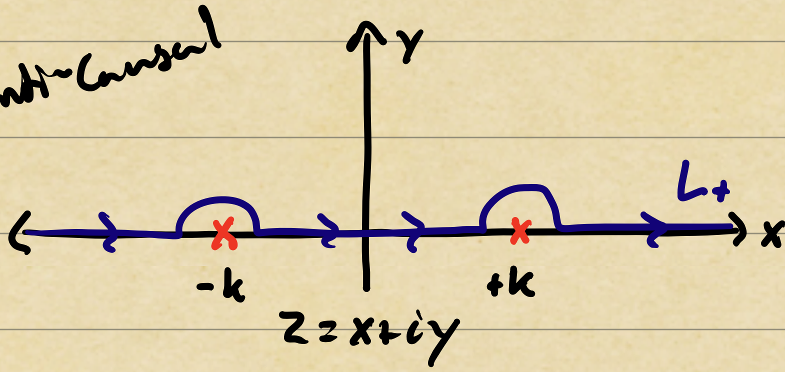
Q| How does changing contour in calculation of $G(t-\tau)$ lead to selecting different combo of homogeneous solutions?

Causal



Contour is deformed through Residues of $\frac{1}{2\pi}\frac{e^{iz(t-\tau)}}{k^2-z^2}$

Anti-Causal



$$\text{Residues are } \mp\frac{1}{2\pi}\frac{e^{\pm ik(t-\tau)}}{2k}$$

⇒ Different choices of contour differ only by residues.

⇒ Residues are solutions of homogeneous eqn.

To select solution satisfying boundary/initial value data, select contour so that $G(t-\tau)$ satisfies boundary/initial data.

$\Rightarrow$ The solutions obtained this way are equivalent to variation of params.

Constant-Coefficient Differential Ops

$$[Lu](x) = \left[a_n \frac{d^n}{dx^n} + \cdots + a_1 \frac{d}{dx} + a_0\right] u(x)$$

Fourier Transform $\widehat{[Lu]}(\xi) = [\underbrace{a_n(i\xi)^n + \cdots + a_1(i\xi) + a_0}_{\text{degree } n \text{ poly in } \xi}]\hat{u}(\xi)$

The polynomial $P(\xi) = a_n(i\xi)^n + \cdots + a_1(i\xi) + a_0$ is called the characteristic polynomial and its behavior in the complex plane is intimately linked to ODE/PDE involving L:

$[Lu](x) = 0 \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = 0$

$[Lu](x) = f(x) \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$