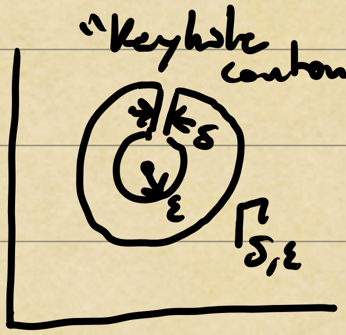
Cauchy's Integral Formula(s)

From Cauchy's theorem, we can derive one of the most important representations of holo. $f(z)$.

<u>Thm</u> | Suppose f is holomorphic in simply connected open set $\Omega \subseteq \mathbb{C}$ that contains a (smooth) simple closed Jordan curve γ. Then for $z \in \operatorname{int}(\gamma)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi .$$

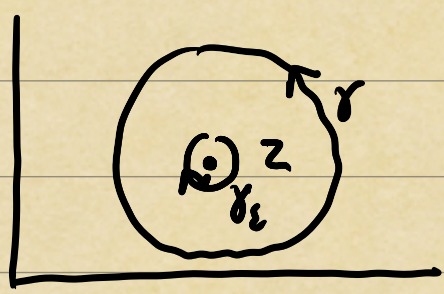
<u>Pf</u> | We'll sketch the proof for γ = circle



By Cauchy Thm, $\displaystyle\int_{\Gamma_{\delta,\varepsilon}} \overbrace{\frac{f(\xi - z)}{\xi - z}}^{F(z)} d\xi = 0$

By continuity, verticals cancel as $\delta \to 0$.

$\downarrow \delta \to 0$



$$\int_\gamma F(z)\,dz = -\int_{\gamma_\varepsilon} F(z)\,dz$$

What happens to $\displaystyle\int_{\gamma_\varepsilon} F(z)\,dz$?

$$\int_{\gamma_\varepsilon} F(\xi) = \int_{\gamma_\varepsilon} \frac{f(\xi)-f(z)}{\xi - z}\,d\xi + \int_{\gamma_\varepsilon} \frac{f(z)}{\xi - z}\,d\xi$$

We have $\dfrac{f(\xi)-f(z)}{\xi-z} \to f'(z)$ as $\xi \to z$, so

$$\sup_{\xi\in\gamma_\varepsilon} \left|\frac{f(\xi)-f(z)}{\xi-z}\right| \le |f'(z)| + |\underbrace{g(\xi,z)}_{\to 0 \text{ as } \varepsilon\to 0}|$$

and $\left|\int_{\gamma_\varepsilon} \dfrac{f(\xi)-f(z)}{\xi-z}\,d\xi\right| \le 2\pi\varepsilon\left(|f'(z)|+|g(\xi,z)|\right) \to 0$ as $\varepsilon \to 0$.

$$\int_{\gamma_\varepsilon} \frac{f(z)}{\xi-z}\,d\xi = f(z)\int_{\gamma_\varepsilon} \frac{1}{\xi-z}\,d\xi = f(z)\int_0^{2\pi} \frac{1}{e^{i\theta}}\, i e^{i\theta}\, d\theta = 2\pi i f(z)$$

$$\Rightarrow \int_\gamma \frac{f(\xi)}{\xi-z}\,d\xi = 2\pi i f(z) \quad \checkmark$$

Cauchy's Integral Formula tells us that the values of a holomorphic $f(z)$ in simply connected $\mathrm{int}(\gamma)\subset\Omega$ can be <u>recovered</u> exactly from values of $f(z)$ along the contour γ.

It gives us a rich set of insights into the behavior of holomorphic functions and, moreover, the "influence" of nearby singularities

The first major insight is that $f(z)$ holomorphic is infinitely complex differentiable in Ω.

<u>Thm</u> Suppose f is holomorphic in simply connected open set $\Omega \subseteq \mathbb{C}$ that contains a (smooth) simple closed Jordan curve γ. Then for $z \in \operatorname{int}(\gamma)$, $f(z)$ has derivatives of all orders given by

"Cauchy Integral Formulas"
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi, \qquad n = 0, 1, 2, \ldots$$

<u>"Sketch" Pf</u> The basic idea is to differentiate under the integral. Suppose it holds for $n-1$, then

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^n} d\xi$$

$$f^{(n)}(z) = \lim_{h \to 0} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$$

$$= \lim_{h \to 0} \frac{(n-1)!}{2\pi i} \int_\gamma \frac{1}{h} \left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n} \right] f(\xi) d\xi$$

why can we exchange limit & integral?

$$= \frac{(n-1)!}{2\pi i} \int_\gamma \frac{n}{(\xi - z)^{n+1}} f(\xi) d\xi$$

derivative of $(\xi - z)^{-n}$

Not only is $f(z)$ infinitely differentiable in Ω, its derivatives cannot grow "too" fast.

Thm Let $f(z)$ be holomorphic in Ω, with disk $\overline{D}_R(z_0) = \{ z : |z-z_0| \le R \} \subset \Omega$. Then

"Cauchy Inequalities"

$$|f^{(n)}(z_0)| \le n! \frac{M}{R^n} \qquad n = 0, 1, 2, \ldots$$

where $M = \max_{|z-z_0|=R} |f(z)|$.

$\Rightarrow |f^{(n)}(z_0)|$ is controlled by $\max_{|z-z_0|=R} |f(z)|$

Pf $|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right|$

max arc length bound

$$\le \frac{n!}{2\pi} \underbrace{|2\pi R|}_{|\gamma_R|} \max_{|z-z_0|=R} \frac{|f(z)|}{|z-z_0|^{n+1}}$$

$$= n! \frac{M}{R^n}$$

Cauchy's inequalities allow us to establish convergence of Taylor series for $f(z)$ in disks of holomorphicity.

Thm | Suppose $f(z)$ is holomorphic in open set Ω. If $D_R = \{z : |z - z_0| < R\} \subset \Omega$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ for $n \geq 0$.

Pf | For any *fixed* $z \in D$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{\xi - z} \, d\xi$$

Idea! $\dfrac{1}{\xi - z} = \dfrac{1}{(\xi - z_0) - (z - z_0)} = \dfrac{1}{\xi - z_0} \left[\dfrac{1}{1 - \left(\frac{z - z_0}{\xi - z_0} \right)} \right]$

There is $0 < r < 1$ such that $\left| \frac{z - z_0}{\xi - z_0} \right| < r$ for $\xi \in \gamma_r$

$\Rightarrow \dfrac{1}{1 - \left(\frac{z - z_0}{\xi - z_0} \right)} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n$ converges uniformly for $\xi \in \gamma_r$.

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi) \, d\xi}{(\xi - z_0)^{n+1}} \right)}_{a_n} (z - z_0)^n$$

By C-F, $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. ✓

The fact that $f(z)$ always has power series expansions in disks $D_R(z_0) \subset \Omega$ has far reaching implications for $f(z)$ behavior.

$\Rightarrow$ Liouville's theorem: if f is entire and b'dd, it is constant.

$\Rightarrow$ Fundamental Thm of Algebra: every polynomial of degree n (w/ coeff $\in \mathbb{C}$) has precisely n roots in $\mathbb{C}$.

Perhaps the most remarkable of these is that $f(z)$ holomorphic is determined by it's values on any subset of Ω containing a limit point ("cluster pt") in Ω.

<u>Thm</u> Suppose $f(z)$ is holomorphic in an open connected set Ω and $f(z)$ vanishes on a sequence of distinct points w/ limit point in Ω. Then $f(z) = 0$ for all $z \in \Omega$.

$\Rightarrow$ "If zeros of $f(z)$ accumulate in Ω, $f(z) = 0$."

pf] Let z_0 be limit point of sequence $\{w_k\}_{k=1}^{\infty}$ s.t. $f(w_k)=0$. Choose disc D centered at z_0 contained in Ω:

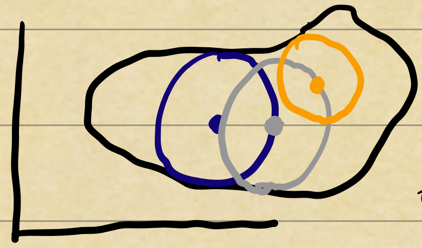
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad z \in D.$$

If $f(z) \neq 0$, then $\exists$ smallest m s.t. $a_m \neq 0$ and

$$f(z) = a_m (z-z_0)^m (1 + g(z-z_0))$$

where $g(z-z_0) \to 0$ as $z \to z_0$. For suff. large k, we have $|g(w_k - z_0)| < 1$ so $(1+g(w_k - z_0)) \neq 0$ and $a_m (w_k - z_0)^m \neq 0$ for $w_k \neq z_0$, but this contradicts $f(w_k)=0$.

Therefore, $f(z)=0$ in D. Can extend to all of Ω using connectedness (see Stein & Shakarchi). ✓



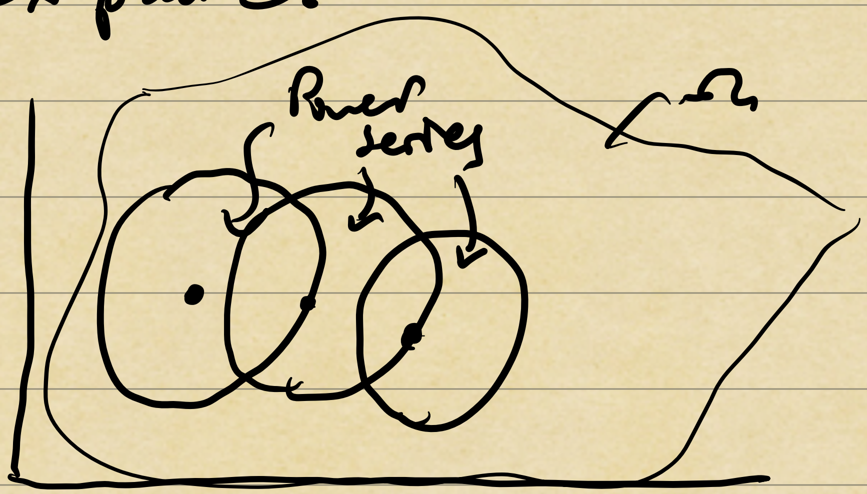
Fill in Ω by recursively applying result for D.

Consequently, any holomorphic function vanishing on an open set is zero.

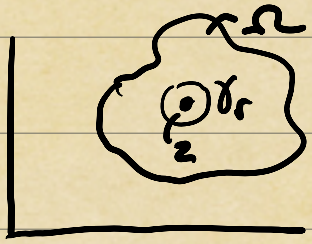
<u>Corollary</u> | Suppose $f(z)$ and $g(z)$ are holomorphic in an open connected $\Omega \subseteq \mathbb{C}$ and $f(z) = g(z)$ for all z in some non-empty open set $D \subset \Omega$ (or sequence of distinct pts w/ limit point in Ω). Then $f(z) = g(z)$ throughout Ω.

This is the basic idea of "analytic continuation," which allows us to <u>uniquely</u> continue analytic/holomorphic functions into new regions of complex plane.

Hadamard chain of disks.



Maximum Modulus Principle



$$= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\xi)}{\xi - z}\, d\xi$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\, d\theta$$

$=$ "average of $f(z)$ on γ_r"

<u>Thm</u> If $f(z)$ is holomorphic in b'dd $\Omega \subset \mathbb{C}$, and continuous in $\bar{\Omega}$. Then, $|f(z)|$ attains its maximum M on the boundary $\partial\Omega$ and either

(a) $|f(z)| < M$ for all $z \in \Omega$, or

(b) $f(z) = M$ for all $z \in \Omega$.

<u>Pf</u> $$|f(z)| \le \underbrace{\frac{1}{2\pi}\left|\int_0^{2\pi} f(z+re^{i\theta})\, d\theta\right|}_{\text{"average"}} \le \underbrace{\max_{\xi \in \gamma_r} |f(\xi)|}_{\text{"max"}} = M$$

The only way "average" = "max" is if $f(\xi) = M$ on γ_r.

Then $$f(z) = \frac{M}{2\pi i}\int_{\gamma_r} \frac{d\xi}{\xi - z} = M \quad \text{for all } z \in \operatorname{int}(\gamma_r)$$

Since $f(z) = M$ on open set $\operatorname{int}(\gamma_r) \subset \Omega$, then $f(z) = M$ for all $z \in \Omega$. Alternatively, $|f(z)| < M$ in Ω, and therefore $|f(z)|$ must attain max only on boundary (cont. f on compact $\partial\Omega$).