Periodic Signals : Fourier Series

Recap

Consider a 2π-periodic signal $f:[0,2\pi] \to \mathbb{C}$.

Fourier series

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}$$

Fourier coefficients

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$$

In many applications, we access either

samples $\quad f(\theta_1), \ldots, f(\theta_N), \qquad \theta_k = 2\pi\frac{k}{N} \quad (k=1,\ldots,N)$

coefficients $\quad \hat{f}_{-N}, \ldots, \hat{f}_N$

For smooth periodic signals, trapezoidal rule efficiently approximates coeffs from samples:

$$\hat{f}_k = \frac{1}{N}\sum_{j=1}^{N} f(\theta_k) e^{-ij\theta_k}$$

The first N coeffs can be computed in $\mathcal{O}(N\log N)$ time using the Fast Fourier Transform (FFT).

Today

Given $\hat{f}_{-N}, \dots, \hat{f}_N$, approximate $f(\theta) \approx \sum_{k=-N}^{+N} \hat{f}_k e^{ik\theta}$

Q2: How accurate is the computed signal?

The truncation error for $f_N(\theta) = \sum_{k=-N}^{N} \hat{f}_k e^{ik\theta}$ is

$$E_N = \sup_{0 \le \theta \le 2\pi} |f(\theta) - f_N(\theta)| \le \sum_{k=-N}^{N} |\hat{f}_k|.$$

How do the Fourier coefficients behave?

Thm/ Suppose that f is 2π-periodic, holomorphic in the strip $S = \{-a < \operatorname{Im}\theta < a\}$, and bounded by $M > 0$ in S. Then, for any $0 < \alpha < a$,

$$|\hat{f}_k| \le M e^{-\alpha|k|}, \quad \text{for } k = 0, \pm 1, \pm 2, \dots$$

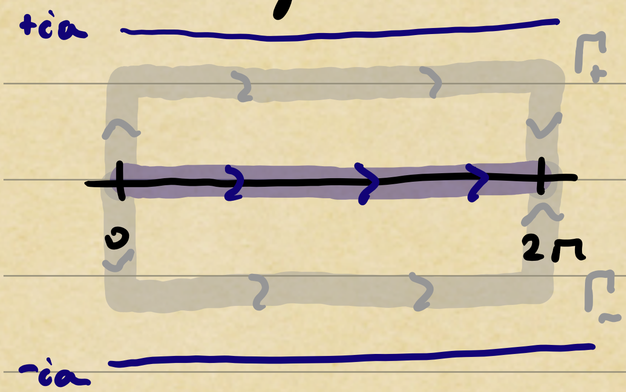
Note that $\displaystyle\sum_{|k|>N} |\hat{f}_k| \le M \sum_{|k|>N} (e^{-\alpha})^{|k|}$

tail of double geometric series

$$= 2M \frac{e^{-\alpha(N+1)}}{1 - e^{-\alpha}}$$

Therefore, $E_N \leq \dfrac{2M e^{-\alpha(N+1)}}{1-e^{-\alpha}}$ for $N = 1, 2, 3, \ldots$

Pf/ The idea is to deform the contour of integration to bound the Fourier coefficients:



$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta$$

If $k \geq 0$, the integrand decays exponentially in the lower half-plane. We have

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\int_{\Gamma_-} f(z)e^{-ikz}\,dz$$

contributions from vertical segments cancel by periodicity

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta - i\alpha)e^{-ik(\theta - i\alpha)}\,d\theta$$

$$\Rightarrow |\hat{f}_k| \leq \sup_{0\leq\theta\leq 2\pi} |f(\theta - i\alpha)|\, e^{-k\alpha} \leq M e^{-k\alpha}$$

If $k < 0$, the integrand decays exponentially in the upper half-plane and the argument is essentially the same:

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\int_{\Gamma_+} f(z)e^{-ikz}\,dz$$

contributions from vertical segments cancel by periodicity

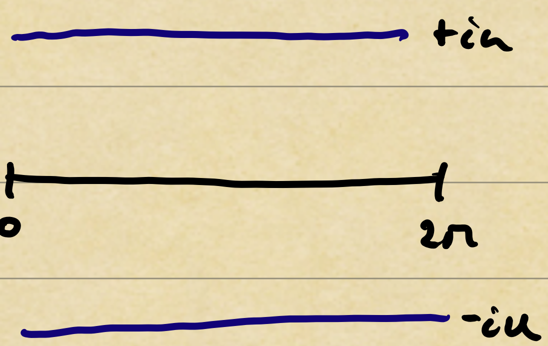
$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta+i\alpha)e^{-ik(\theta+i\alpha)}\,d\theta$$

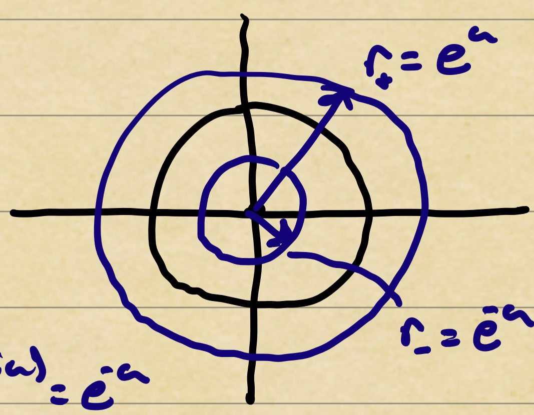
$$\Rightarrow |\hat{f}_k| \le \sup_{0\le\theta\le 2\pi} |f(\theta+i\alpha)|\,e^{k\alpha} \le Me^{-|k|\alpha}$$

Fourier Series : Laurent Series

The geometrically decaying Fourier coeffs of holomorphic periodic signals can be understood as an analogue of Cauchy's inequalities for Laurent series.

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta} \quad \overset{e^{i\theta} \leftrightarrow z}{\Longleftrightarrow} \quad g(z) = \sum_{k=-\infty}^{+\infty} \hat{g}_k z^k$$





$e^{i(ia)} = e^{-a}$

$e^{i(-ia)} = e^{a}$

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta \iff \hat{g}_k = \frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)}{z^{k+1}} dz$$

$$f(\theta) = g(e^{i\theta}) \iff \hat{f}_k = \hat{g}_k$$

Q: At what rate do the $k<0$ coeffs of Laurent series decay?

Example: Solve $-u''(x) = f(x)$, $x \in [0, 2\pi]$

where $\int_0^{2\pi} u(x) dx = 0$ and f is periodic + smooth.

Note that if f is smooth, so must be u!

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx} \quad \text{and} \quad f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx}$$

$$\Rightarrow \quad -(e^{ikx})'' = k^2 e^{ikx}$$

$$\Rightarrow \quad k^2 \hat{u}_k = \hat{f}_k$$

$k=0$ be careful!

$$\Rightarrow \quad u(x) = \sum_{k=-\infty}^{+\infty} \frac{\hat{f}_k}{k^2} e^{ikx}$$

$\int_0^{2\pi} u(x) dx = 0$

$(\hat{f}_k = 0)$

"Pseudo"
Fourier Spectral Method

1) Compute $\hat{f}_k \approx \frac{1}{N} \sum_{j=1}^{N} f(2\pi \frac{j}{N}) e^{-ik\frac{2\pi j}{N}} = \tilde{f}_k$

for $k = -N, \dots, N$

2) Compute $u(2\pi j/N) \approx \sum_{j=-N}^{N} \frac{\tilde{f}_k}{k^2} e^{ik 2\pi j/N}$

for $j = 1, \dots, N$

⇒ Output is approximate samples of $u(x)$ on grid $2\pi 1/N, \dots, 2\pi$

⇒ Exponentially accurate for smooth periodic right-hand sides.

<u>Example:</u> Compute spectral projector of $\overset{\text{real symm}}{A}$ onto interval $[a,b]$.