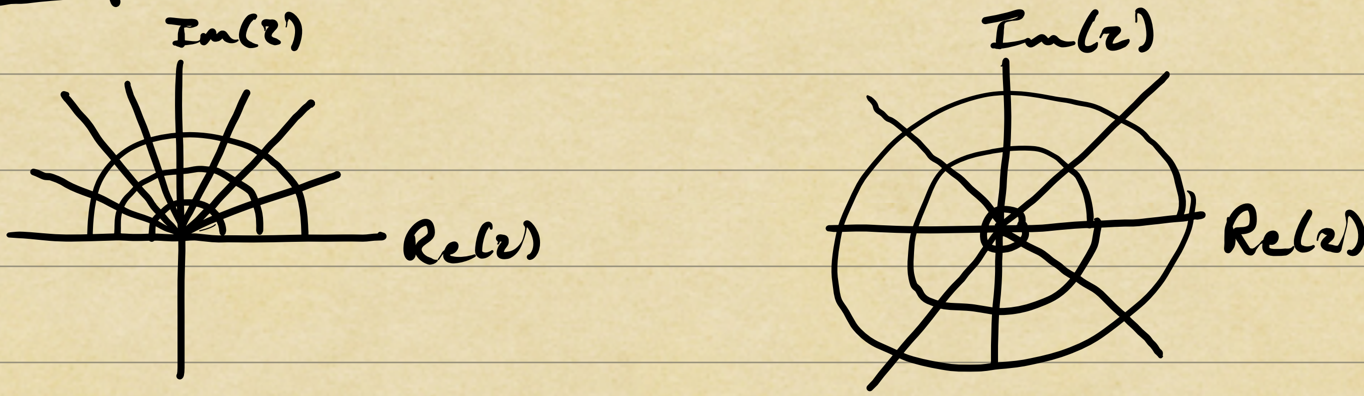
Functions of a Complex Variable

Let $f : \Omega \to \mathbb{C}$ be a function defined on $\Omega \subseteq \mathbb{C}$.

$$f(z) = u(x,y) + i v(x,y) = u(r,\theta) + i v(r,\theta)$$
$$= R(x,y)\, e^{i\theta(x,y)} = R(r,\theta)\, e^{i\theta(r,\theta)}$$

Note that $f : \Omega \to \mathbb{C}$ can always be identified with $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (u(x,y), v(x,y))$. Consequently, trickier to visualize than $f : [-1,1] \to \mathbb{R}$.

Example 1: $f(z) = z^2$ defined on $\Omega = \{\operatorname{Im} z > 0\}$

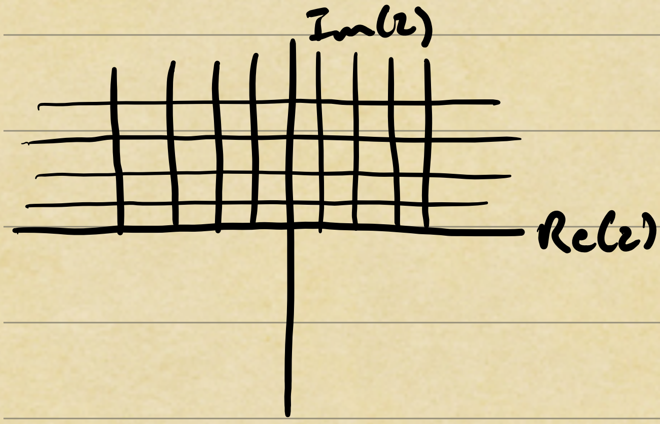


z-plane $\qquad\qquad$ $w = f(z)$-plane

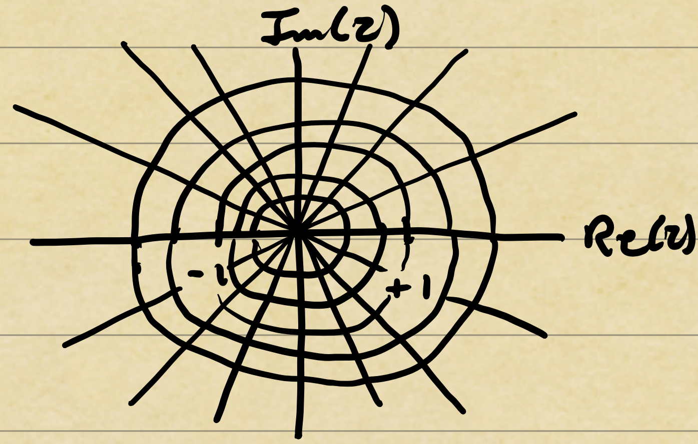
$z = re^{i\theta} \quad \to \quad z^2 = r^2 e^{2i\theta}$

Quadratic map "wraps" the upper half z-plane onto the full complex w-plane.

Example 2: $f(z) = \exp(z)$ on $\Omega = \{\operatorname{Im} z > 0\}$



z-plane



$w = f(z)$-plane

$$x + iy \quad \rightarrow \quad e^{x+iy} = e^x e^{iy}$$

Horizontal lines map to rays and vertical lines map to arcs of constant radius.

In general one can visualize $f(z)$

$\Rightarrow$ Landscape, plot $|f(z)|$
$\Rightarrow$ Can also add color for $\arg(z)$
$\Rightarrow$ Phase plots $f/|f| = e^{i\theta}$

I recommend Elias Wegert's phase plotting software in MATLAB and accompanying book/article.

Smooth Complex Functions (Ω = open set $\subseteq \mathbb{C}$)

A central theme in applied complex analysis is the interplay between regions of complex differentiability ("smoothness") and points of non-differentiability ("singularity").

Limit: We say that $\lim_{z \to z_0} f(z) = w_0$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $z \in \Omega$ and $|z - z_0| < \delta$, then $|f(z) - w_0| < \varepsilon$.

Continuity: $f : \Omega \to \mathbb{C}$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$ and f is continuous on Ω if f is continuous at each point in Ω.

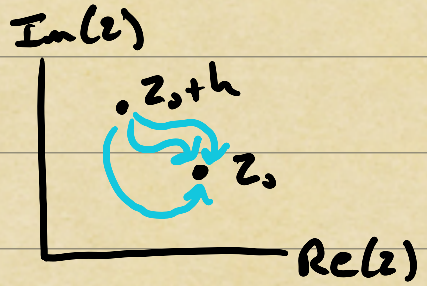
Note: this is equivalent to continuity of the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x,y) = (u(x,y), v(x,y))$.

Differentiability: f is holomorphic at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = w_0 \qquad \text{for some } w_0 \in \mathbb{C}.$$

If f is holomorphic at every point in Ω, we say that f is **holomorphic on Ω**.

<u>Note:</u> The parameter h is complex and the limit $h \to 0$ can be taken along any "direction."



The existance of the limit w_0 regardless of path is a <u>strong</u> requirement, and

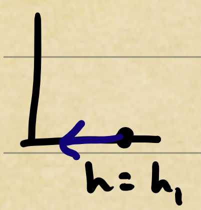
$$\begin{array}{ccc} \text{complex differentiability} & \not\Leftrightarrow & \text{real differentiability} \\ \text{of } f:\Omega \to \mathbb{C} & & \text{of } F:\mathbb{R}^2 \to \mathbb{R}^2 \end{array}$$

<u>Example:</u> Real differentiable, but not holomorphic

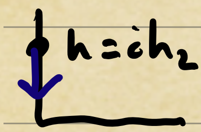
$$f(z) = \bar{z} \iff (x,y) \to (x,-y)$$

continuously diff. everywhere on $\mathbb{R}^2$.

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\bar{h}}{h}$$



$$\lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h_1 \to 0} \frac{h_1}{h_1} = 1$$



$$\lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h_2 \to 0} \frac{-ih_2}{ih_2} = -1$$

$f(z) = \bar{z}$ not holomorph. at any $z \in \mathbb{C}$.

However, mechanics are similar to 1D diff.

If $f, g: \Omega \to \mathbb{C}$ are holomorphic on Ω,

"Sum" $\Rightarrow$ $(f+g)' = f' + g'$

"product" $\Rightarrow$ $(fg)' = f'g + fg'$

"Chain" $\Rightarrow$ $(f \circ h)'(z) = f'(h(z))h'(z)$

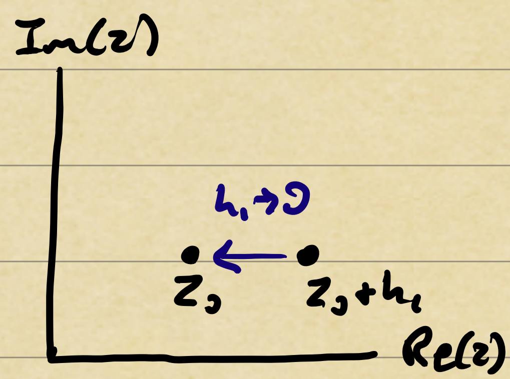
"power" $\Rightarrow$ $(z^n)' = nz^{n-1}$ ($n \geq 1$ integer)

<u>Example:</u> $p(z) = a_0 + a_1 z + \cdots + a_n z^n$

Applying the power rule termwise, we get

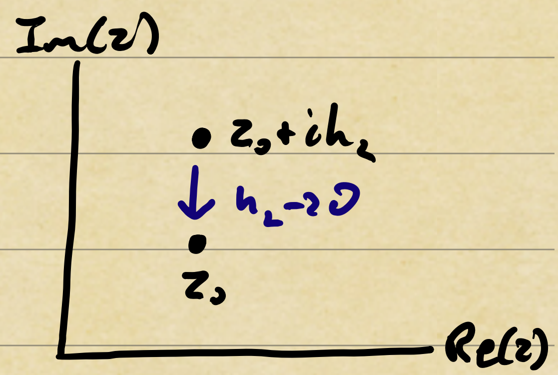
$$p'(z) = a_1 + a_2 z + \cdots + n a_n z^{n-1}$$

<u>Cauchy-Riemann Equations</u>



$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1}$$

$$= \partial_x f(x_0, y_0)$$



$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2}$$

$$= \frac{1}{i} \partial_y f(x_0, y_0)$$

Therefore, if f is holomorphic at $z_0 = (x_0, y_0)$

$$\partial_x \underbrace{f(x_0, y_0)}_{u+iv} = \frac{1}{i} \partial_y \underbrace{f(x_0, y_0)}_{u+iv}$$

Equating real & imaginary parts separately,

(CR) $\quad \partial_x u = \partial_y v \quad$ and $\quad \partial_y u = -\partial_x v.$

A converse of this statement also holds.

<u>Thm</u> Suppose $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subseteq \mathbb{C}$. If u, v are continuously differentiable and satisfy (CR) on Ω, then f is holomorphic on Ω.

(CR) does not hold in general for continuously differentiable maps $F : \mathbb{R}^2 \to \mathbb{R}^2$. It is a special consequence of complex differentiability and is a first sign of the rich structure of the family of holomorphic functions on $\Omega \subseteq \mathbb{C}$.

(CR) also provides our first application link.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

By CR, $\left.\frac{\partial f}{\partial z}\right|_{z_0} = f'(z_0)$ and $\left.\frac{\partial f}{\partial \bar{z}}\right|_{z_0} = 0$ $\quad f:\Omega\to\mathbb{C}$ holomorphic

"Laplacian" $\quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$

Solutions to Laplace equation, $\Delta u = 0$, are precisely the real & imaginary parts of holomorphic (complex differentiable) functions.

$\Rightarrow$ We can study solutions to Laplace's equation on open domain Ω by studying the behavior of holomorphic func's.

This link leads to the rich field "potential theory."

We'll return to potential theory after we have gathered more tools/insights for holomorphic functions.

Power Series

Power series form an important class of holomorphic functions in complex analysis.

$$(*) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n; \qquad \{a_n\} \subset \mathbb{C}.$$

By "slicing" f into a series of simple powers, many computations can be simplified, at least when the series converges absolutely,

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

__Thm__ Given $(*)$, there exists $0 \leq R \leq \infty$, s.t.

i) if $|z| < R$, $(*)$ converges absolutely.

ii) if $|z| > R$, $(*)$ diverges.

__Pf__ If $(*)$ converges (diverges) for some $z_0 \in \mathbb{C}$ then it converges (diverges) for all $|z| \leq |z_0|$ $(|z| \geq |z_0|)$ by the comparison test.

R is called the **radius of convergence** and the region $|z|<R$ is called the **disc of convergence**.

"Hadamard's formula"

$$R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n} \underbrace{=}_{\text{If lim exists}} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

"Ratio Test"

Note: On the boundary $|z|=R$, the series may or may not converge.

<u>Thm</u> The power series $(*)$ is a holomorphic function in it's disc of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1},$$

where f' and f have same radius of convergence.

<u>Pf</u> Classic "3ε" argument, see Stein & Shakarchi.

<u>Corollary</u> Power series $(*)$ is infinitely complex differentiable in its disk of convergence, and higher derivatives are power series obtained by term-by-term differentiation.

In short, power series are infinitely complex-differentiable holomorphic functions on the disc of convergence, where they converge absolutely. They are easy to manipulate term-wise.

Remarkably, <u>all</u> holomorphic functions behave locally like power series, which makes power series an indispensible tool.

"All happy families are similar to one another, ..."

-Leo Tolstoy