The Fourier Transform (Pt 2)

Recap

Given $f: \mathbb{R} \to \mathbb{R}$ with appropriate regularity & decay:

(1) $$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-i\xi x}\, dx$$ "Fourier Transform"

is the Fourier Transform of f, which satisfies

(2) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi.$$ "Inverse Fourier Transform"

Bandlimited Functions

In signal processing and inverse problems, a bandlimited function is composed of freq. $\xi \in [-B, B]$

$$f(x) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

In other words, $\hat{f}(\xi) = 0$ when $|\xi| \geq B$.

What are the properties of bandlimited f?

The Paley-Wiener Theorem provides a precise characterization of band-limited functions in terms of regularity & growth of f. It is the "crowning" result of our investigation into smoothness & decay results for FT.

f		$\hat{f}$
$\int_{-\infty}^{+\infty} \lvert f \rvert \, dx, \dots, \int_{-\infty}^{+\infty} \lvert f^{(k)} \rvert \, dx$	$\Rightarrow$	$\lvert \hat{f}(\xi) \rvert \le C \lvert \xi \rvert^{-k}$
f hol & b'dd in strip	$\Rightarrow$	$\lvert \hat{f}(\xi) \rvert \le M e^{-a\lvert \xi \rvert}$
f entire with $\lvert f(z) \rvert \le A e^{B\lvert z \rvert}$	$\Rightarrow$	$\hat{f}(\xi) = 0, \quad \xi \in [-B, B]$

When solving ODE/PDE, also useful to go $\Leftarrow$

f		$\hat{f}$
f hol in strip (half width $< a$)	$\Leftarrow$	$\lvert \hat{f} \rvert < M e^{-a\lvert \xi \rvert}$
f entire w/ $\lvert f(z) \rvert \le A e^{B\lvert z \rvert}$	$\Leftarrow$	$\hat{f}(\xi) = 0 \quad \xi \in [-B, B]$

To establish regularity/smoothness of f from the decay of its Fourier transform, we need two results that allow us to construct holomorphic functions via integrals and as uniform limits of holomorphic functions.

1 **Lemma** (ch 2 Thm 5.2, Stein)

If $\{f_n\}_{n\geq 1}^{\infty}$ is a sequence of holo. functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

2 **Lemma** (ch 2 Thm 5.4) Let $F(z,s) : \Omega \times [0,1] \to \mathbb{C}$ satisfy

i) $F(z,s)$ holo. in z for each s

ii) F continuous on $\Omega \times [0,1]$

Then, $f(z) = \int_0^1 F(z,s)\,ds$ is holo. on Ω

To illustrate, we start w/a partial converse to our earlier theorem about criteria for functions w/exponentially decaying Fourier T.

Theorem 4 Suppose $\hat{f}$ satisfies the decay condition $|\hat{f}(\xi)| \leq A e^{-a|\xi|}$ for consts $a, A > 0$. Then $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ holomorphic in the strip S_b for any $0 < b < a$.

Pf Define $$f_n(z) = \frac{1}{2\pi} \int_{-n}^{n} \hat{f}(\xi) e^{i\xi z} \, d\xi$$

By Lem 2, each f_n is entire. Then also

$$f(z) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-n}^{+n} \hat{f}(\xi) e^{i\xi z} \, d\xi$$

converges absolutely for each fixed $z \in S_b$ by assumption on $\hat{f}$, since

$$|f(z)| \leq \frac{A}{2\pi} \int_{-\infty}^{+\infty} e^{-a|\xi|} e^{b|\xi|} \, d\xi < \infty \quad (b < a).$$

Also, $$|f_n(z) - f(z)| \leq \frac{A}{2\pi} \int_{|\xi| \geq n} e^{-a|\xi|} e^{b|\xi|} \, d\xi \to 0 \text{ as } n \to \infty.$$

Since $|f_n(z) - f(z)| \to 0$ uniformly for $z \in S_b$, Lemma 1 establishes that $f(z)$ is holomorphic in S_b.

Corollary | If $\hat{f}(\xi) = O(e^{-a|\xi|})$ for some $a>0$ and f vanishes in nonempty open interval, then $f=0$.

Paley-Wiener extends this characterization of exponentially decaying Fourier Transforms to Fourier Transforms supported in $[-B, B]$.

Thm | Suppose f is continuous and of moderate decrease $(\leq \frac{A}{1+x^2}, x \in \mathbb{R})$. Then f has an extension to the complex plane that is entire with $|f(z)| \leq Ae^{B|z|}$ for some $A>0$, if and only if $\hat{f}(\xi)=0$ for $|\xi| \geq B$.

Pf |

$\Leftarrow$ Since f has moderate decrease + continuous and $\hat{f}$ compactly supported, Fourier inversion holds

$$f(x) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi) e^{i\xi x} d\xi$$

and we can extend to complex plane by

$$g(z) = \frac{1}{2\pi}\int_{-B}^{+B} \hat{f}(\xi) e^{i\xi z}\, d\xi.$$

Clearly $f(x) = g(x)$ $(x \in \mathbb{R})$ and $g(z)$ is entire by Lemma 2. Moreover, $e^{i\xi(x+iy)} = e^{i\xi x} e^{-\xi y}$

$$|g(z)| \leq \int_{-B}^{+B} |\hat{f}(\xi)| e^{-\xi y}\, d\xi$$

$$\leq A e^{B|y|} \leq A e^{B|z|} \quad ✓$$

$\Rightarrow$) Converse is more involved. Break into 3 steps, starting w/ "nicer" functions.

<u>Step 1</u>| Take f entire and satisfying

$$|f(x+iy)| \leq A' \frac{e^{B|y|}}{1+x^2} \quad \begin{matrix} \leftarrow \text{growth of real axis} \\ \leftarrow \text{integrability} \end{matrix}$$

Need to show $\hat{f}(\xi) = 0$ for $|\xi| > B$.

<u>$\xi > B$</u> $\quad \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-ix\xi}\, dx = \int_{-\infty}^{+\infty} f(x-iy) e^{-i\xi(x-iy)}\, dx$

shift contour down

$$|\hat{f}(\xi)| \le A' \int_{-\infty}^{+\infty} \frac{e^{By} e^{-\xi y}}{1+x^2} \, dx$$

$$\le (\text{Const}) \, e^{-y(\xi - B)} \to 0 \text{ as } y \to \infty$$

b/c $\xi > B$

$\Rightarrow \; |\hat{f}(\xi)| = 0$ for $\xi > B$.

$\underline{\xi < -B}$ Same argument w/contour shifted up.

Steps 2-3 extend result to entire f, continuous and of moderate decrease on $\mathbb{R}$, s.t.

$$|f(z)| \le A e^{B|z|} \qquad z \in \mathbb{C}.$$

Will complete proof in next lecture.