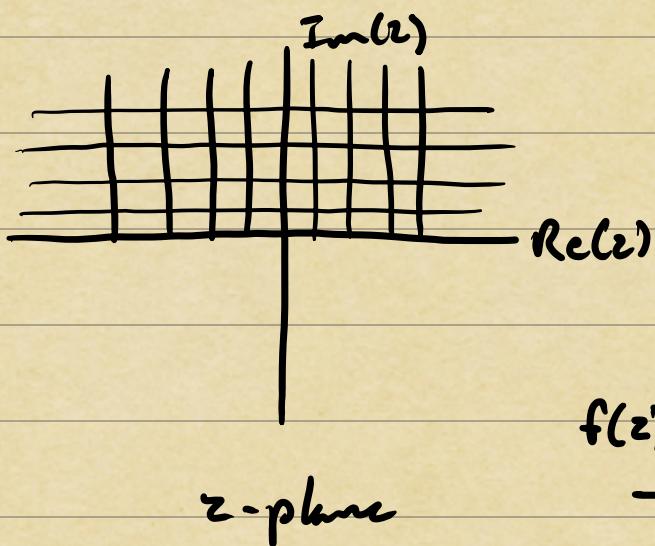


Cauchy's Integral Formulas

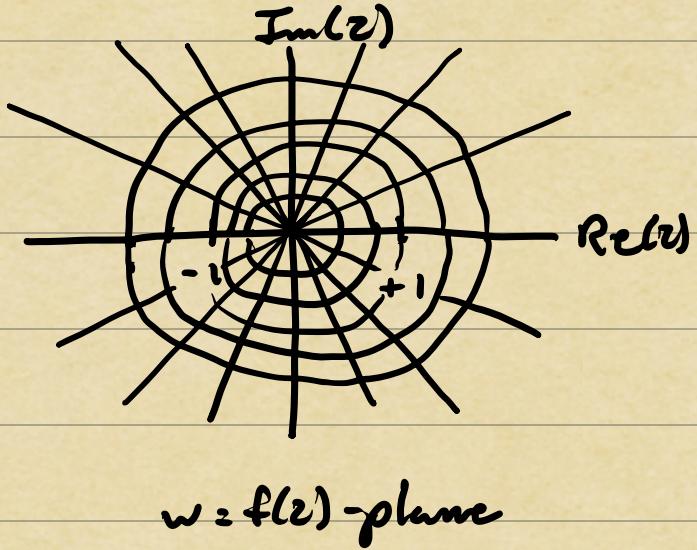
~~Recap~~

Complex-valued function $f: \Omega \rightarrow \mathbb{C}$



$$f(z) = e^z$$

→



⇒ $f: \Omega \rightarrow \mathbb{C}$ is holomorphic in $\Omega \subseteq \mathbb{C}$ if

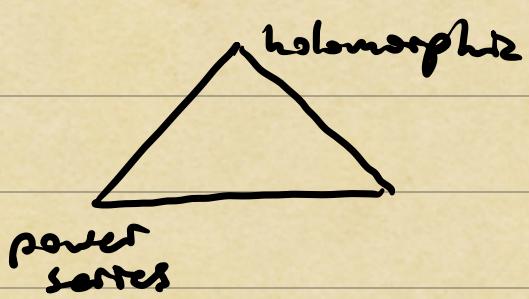
$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = w_0 \quad \text{for some } w_0 \in \mathbb{C},$$

at each point $z_0 \in \Omega$. We write $f'(z_0) = w_0$.

⇒ "Calculus" of complex differentiation is identical to real calculus: sum, product, quotient, and power rules all remain valid.

⇒ Holomorphic functions have additional structure: Cauchy-Riemann Equations

Power Series



Power series form an important class of holomorphic functions in complex analysis.

$$(*) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \{a_n\} \subset \mathbb{C}.$$

By "sticking" f into a series of simple powers, many computations can be simplified, at least when the series converges absolutely,

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

Thm Given $(*)$, there exists $0 \leq R \leq \infty$, s.t.

- if $|z| < R$, $(*)$ converges absolutely.
- if $|z| > R$, $(*)$ diverges.

Pf If $(*)$ converges (diverges) for some $z_0 \in \mathbb{C}$ then it converges (diverges) for all $|z| \leq |z_0|$ ($|z| \geq |z_0|$) by the comparison test.

R is called the radius of convergence and the region $|z| < R$ is called the disc of convergence.

"Hadamard's formula"

$$R^{-1} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

If limit exists

$$= \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

"Ratio Test"

Note: On the boundary $|z| = R$, the series may or may not converge.

Then The power series (f) is a holomorphic function in its disk of convergence and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

where f' and f have same radius of convergence.

Pf Classic " ε " argument, see Stein & Shakarchi.

Corollary Power series (f) is infinitely complex differentiable in its disk of convergence, and higher derivatives are power series obtained by term-by-term differentiation.

In short, power series are infinitely complex-differentiable holomorphic functions on the disc of convergence, where they converge absolutely. They are easy to manipulate term-wise.

Example: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Disk of convergence is \mathbb{C} ($R=\infty$): for any $z \in \mathbb{C}$

$$\left| \frac{z^n}{n!} \right| = \frac{|z|^n}{n!} \Rightarrow \sum_{n=0}^{\infty} \left| \frac{z^n}{n!} \right| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$$

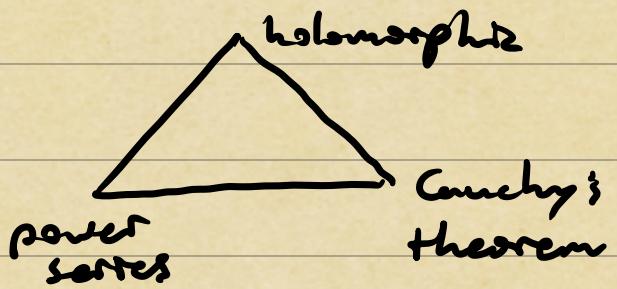
Derivative: $(e^z)' = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \quad \checkmark$

Remarkably, all holomorphic functions behave locally like power series, which makes power series an indispensable tool.

"All happy families are similar to one another..."

- Leo Tolstoy

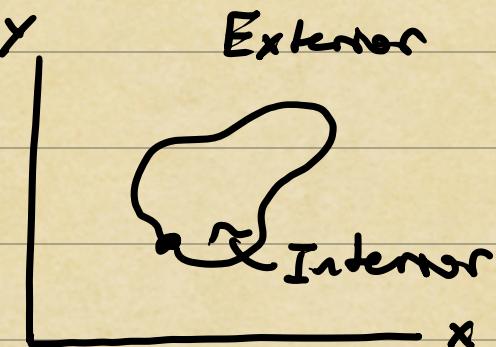
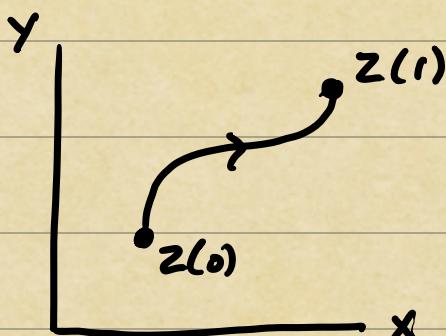
Complex Integration



A simple smooth Jordan arc is a curve

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b$$

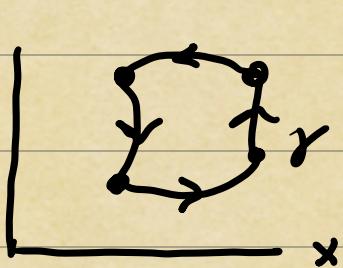
with $x'(t), y'(t)$ continuous and $x'(t)^2 + y'(t)^2 \neq 0$,
and $t_1 \neq t_2$ implying $z(t_1) \neq z(t_2)$.



A simple closed Jordan curve has $z(a) = z(b)$.

Jordan Curve Thm. Every simple closed
Jordan curve in the complex plane divides
the plane into two disjoint sets: interior
of the curve is b'd'l, exterior is un'b'd'l and
the curve is the boundary of each.

Simple piecewise smooth curves are simple Jordan curves w/piecewise continuous derivatives.



The integral of $f: \Omega \rightarrow \mathbb{C}$ along $\gamma: [a, b] \rightarrow \Omega$ is

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

$z'(t) = x'(t) + iy'(t)$

\Rightarrow The integral is independent of parametrization of curve γ

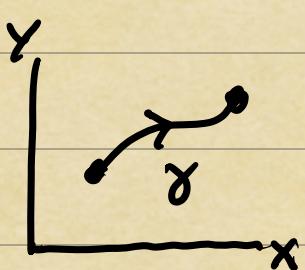
\Rightarrow The length of γ is

$$|\gamma| = \int_a^b |z'(t)| dt$$

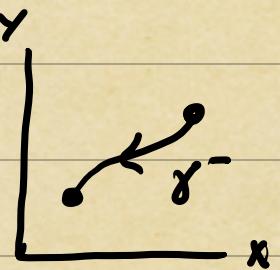
Thm If $f, g: \Omega \rightarrow \mathbb{C}$ is continuous and $\gamma: [a, b] \rightarrow \Omega$ is simple piecewise smooth:

$$(a) \int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz$$

(b) If γ^- is γ with reverse orientation,



$$\int_{\gamma} f(z) dz = - \int_{\gamma^-} f(z) dz$$

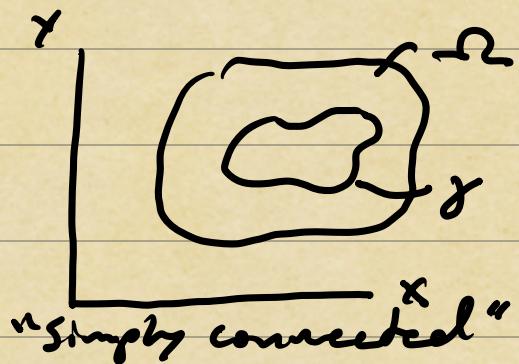
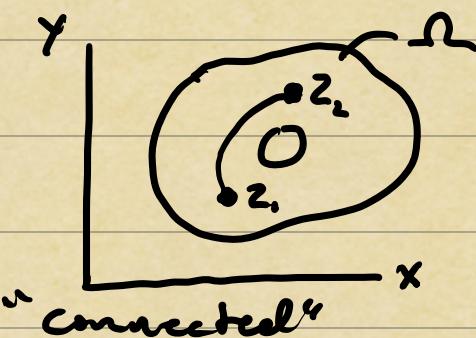


(c) $|\int_{\gamma} f(z) dz| \leq |\gamma| \sup_{z \in \gamma} |f(z)|$

To state our main result, we need a special notion of connectedness.

A nonempty open set $\Omega \subset \mathbb{C}$ is **connected** if any two points in Ω can be joined by a simple Jordan arc in Ω .

If every simple closed Jordan curve γ in Ω has $\text{int}(\gamma) \subset \Omega$, then Ω is called **simply connected**.



Cauchy's Theorem

Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic on simply connected Ω . Then, for any simple closed Jordan curve $\gamma \subset \Omega$,

$$\int_{\gamma} f(z) dz = 0.$$

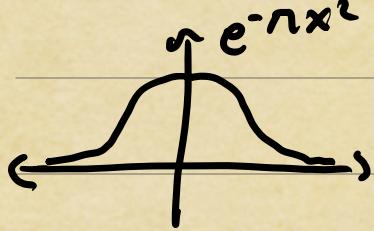
[PF] Show for triangle contours (Goursat) and then work up to more complicated. See Shakarchi, Stein or Deitman.

Example] Compute the Fourier transform

$$\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx \quad \xi \in \mathbb{R}$$

We'll do the case $\xi > 0$. The case $\xi < 0$ is similar.

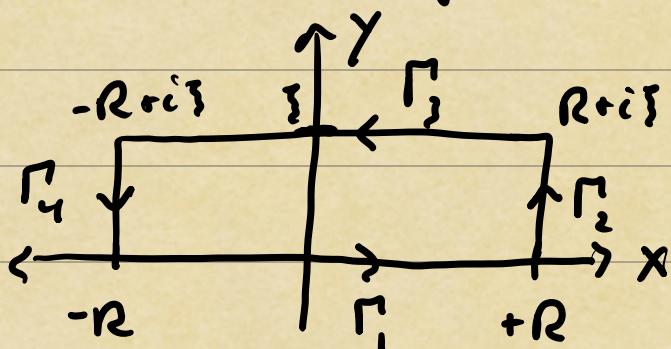
Note that for $\xi = 0$, we have



$$\int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$$

$e^{-\pi x^2}$ is the standard normal probability density

Consider the complex function $f(z) = e^{-\pi z^2}$:



By Cauchy's theorem

$$\int_{\Gamma_R} f(z) dz = 0$$

$$\sum_{k=1}^4 \int_{\Gamma_k} f(z) dz = 0$$

Consider the 4 contours $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ each:

$$\int_{\Gamma_1} f(z) dz = \int_{-R}^R e^{-\pi x^2} dx$$

$$\int_{\Gamma_2} f(z) dz = \int_0^R f(R+iy) i dy = \int_0^R e^{-\pi(R+iy)^2} i dy$$

$$\int_{\Gamma_3} f(z) dz = \int_R^{-R} f(x+iy) dx = - \int_{-R}^R e^{-\pi(x+iy)^2} dx$$

$$= -e^{\pi y^2} \int_{-R}^R e^{-\pi x^2} e^{-2\pi i xy} dx$$

$$\int_{\Gamma_4} f(z) dz = \int_0^R f(-R+iy) i dy = \int_0^R e^{-\pi(-R+iy)^2} i dy$$

Note that $\lim_{R \rightarrow \infty} \int_{\Gamma_3} f(z) dz = -e^{\pi i s^2} \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x s} dx$,
 ↑ scaling factor

which is precisely the integral we want to compute. On the other hand, we know that

$$\lim_{R \rightarrow \infty} \int_{\Gamma_1} f(z) dz = \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1.$$

Finally, the vertical contours satisfy the bound

$$\left| \int_{\Gamma_2} f(z) dz \right| \leq \underbrace{\sup_{|\Gamma_2|} |e^{-\pi(R+iy)^2}|}_{\leq 3} \leq 3 e^{-\pi R^2 + \pi s^2}$$

$$\left| \int_{\Gamma_4} f(z) dz \right| \leq \underbrace{\sup_{|\Gamma_4|} |e^{-\pi(-R+iy)^2}|}_{\leq 3} \leq 3 e^{-\pi R^2 + \pi s^2}$$

Therefore $\lim_{R \rightarrow \infty} \int_{\Gamma_2} f(z) dz = \lim_{R \rightarrow \infty} \int_{\Gamma_4} f(z) dz = 0$.

By Cauchy's theorem, for any $R > 0$, we have

$$0 = \int_{\Gamma_R} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz + \int_{\Gamma_4} f(z) dz.$$

Taking the limit $R \rightarrow \infty$ on both sides, we find

$$J = \frac{1}{2} - e^{\pi \beta^2} \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \beta} dx$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \beta} dx = e^{-\pi \beta^2}$$

So the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi \beta^2}$.

We'll discuss Fourier Transforms, applications, and techniques for computing them in detail later in this course.