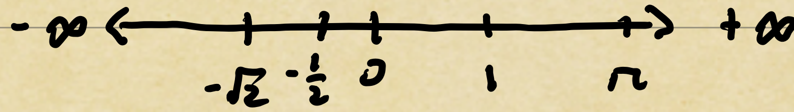
Complex Variables & Integral Transforms

$\mathbb{R}$ = Real Numbers



Arise naturally to model "continuum"

⇒ Trajectories of Rigid Bodies

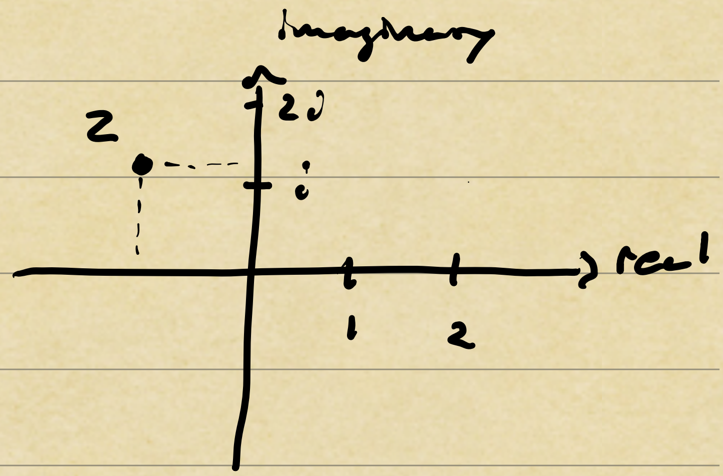
⇒ Mechanics of deformable bodies

⇒ Fluid mechanics

⋮

$\mathbb{C}$ = Complex Numbers

$$z = \underbrace{x}_{\text{real}} + \underbrace{i}_{\substack{\text{imaginary} \\ \text{unit}}} y \qquad i = \sqrt{-1}$$



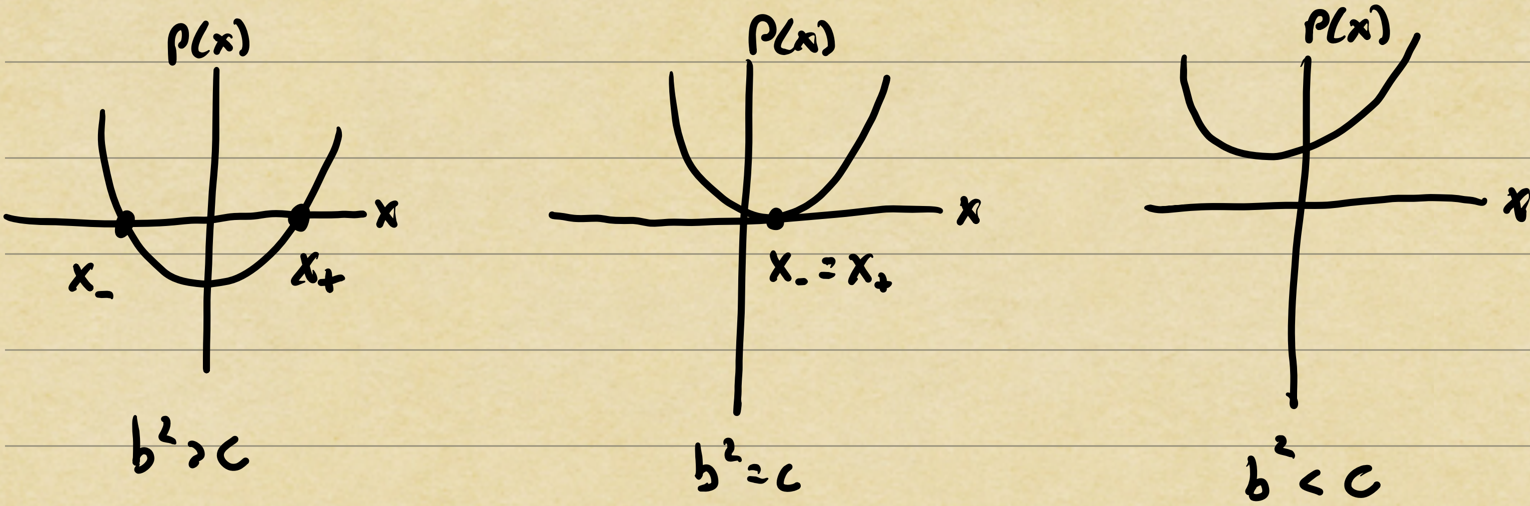
Why do we need complex #'s?

"The shortest path between two truths of the real domain often passes through the complex one."

– Paul Painlevé

Example 1: If $\overbrace{x^2 + 2bx + c}^{p(x)} = 0$, find x.

$$x_\pm = -b \pm \sqrt{b^2 - c}$$



For $b^2 < c$, the quadratic equation has no real solutions - the roots are complex.

$$x_\pm = -b \pm i\sqrt{c - b^2}$$

Polynomials w/complex roots play a key role in many areas of applied math!

The distinction between real, imaginary, and complex roots often demarcates qualitatively different physical behavior.

__Example 2:__ If $x^2u'' + 3xu' + 2u = 0$, find $u(x)$.

Enter ansatz: $u(x) = x^r$

$$\Rightarrow \quad (\underbrace{r(r-1) + 3r + 2}_{\text{characteristic polynomial}})\, x^r = 0$$

$$\Rightarrow \quad r^2 + 2r + 2 = 0 \text{ if } r = -1 \pm i$$

$$\Rightarrow \quad u(x) = c_+ x^{-1+i} + c_- x^{-1-i}$$

$$= \frac{1}{x}\Big(\underbrace{c_+ e^{i\log x} + c_- e^{-i\log x}}_{\text{oscillatory}}\Big)$$

$$= \frac{1}{x}\Big[(c_+ + c_-)\cos(\log x) + i(c_+ - c_-)\sin(\log x)\Big]$$

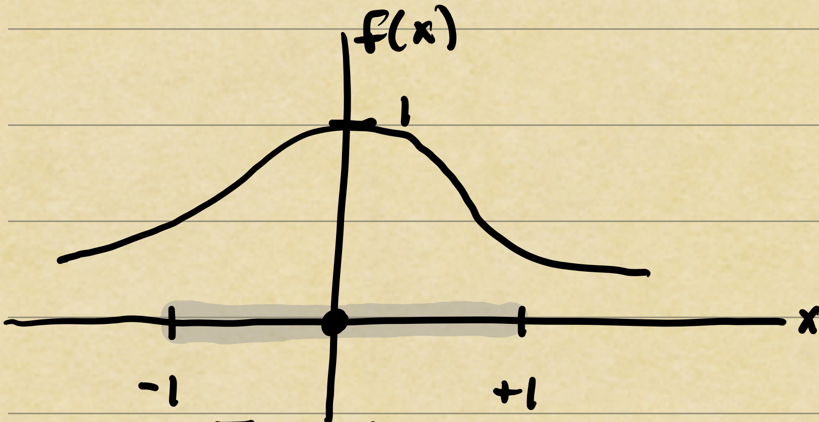
using
- $\exp(\log(x)) = x$
- $\log x^a = a\log x$
- $e^{ix} = \cos x + i\sin x$

Real part of root governs "blow-up" at $x = 0$

Complex part of root governs oscillatory "frequency."

$\Rightarrow$ Similar themes in dynamics, signals, time-series analysis

Example 3: Taylor series of $f(x) = \frac{1}{1+x^2}$ at $x=0$.



Taylor series converges here

$f(x)$ is "smooth:" infinitely differentiable at every real x.

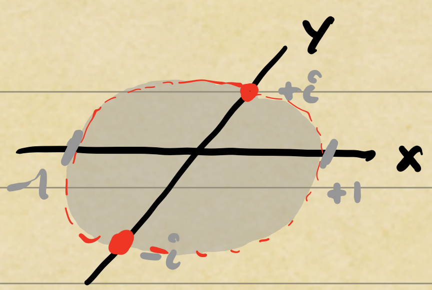
$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

Converges absolutely if $|x| < 1$

What obstructs the convergence for $|x| \geq 1$?

$$f(z) = \frac{1}{1+z^2} \to \infty \quad \text{as} \quad z \to \pm i \quad (\text{"poles"})$$

In the complex plane, $f(z)$ is not smooth at $z = \pm i$



$z = x + iy$

By comparison test, Taylor series converges in disk of radius 1.

Points $\pm i$ where $f(z)$ is not differentiable restrict "disc of convergence."

$$f(z) = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

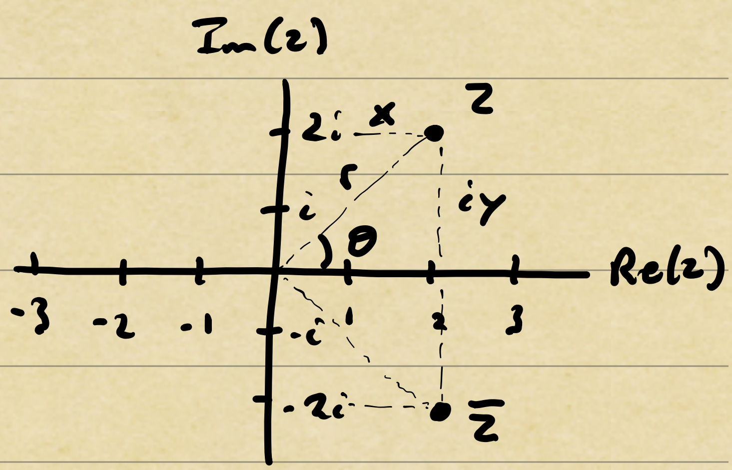
⇒ Examining functions in the complex plane often clarifies/reveals their behavior.

⇒ Tension between regions of differentiability and points of non-differentiability (singularities).

⇒ Compact representations via singularities.

⇒ Remarkably beautiful and simple-to-use tools for analyzing, representing, computing with functions in the complex plane.

Complex Numbers



imaginary unit $i^2 = -1$

$$z = \underset{\substack{\text{real}\\ \text{part}}}{x} + i\,\underset{\substack{\text{imaginary}\\ \text{part}}}{y}$$

$\mathrm{Re}(z) = x$ $\qquad$ $\mathrm{Re}(z) = y$

"polar form"

$$= \underset{\text{"modulus"}}{r}\,\overbrace{(\cos\theta + i\sin\theta)}^{e^{i\theta}}$$

"argument"

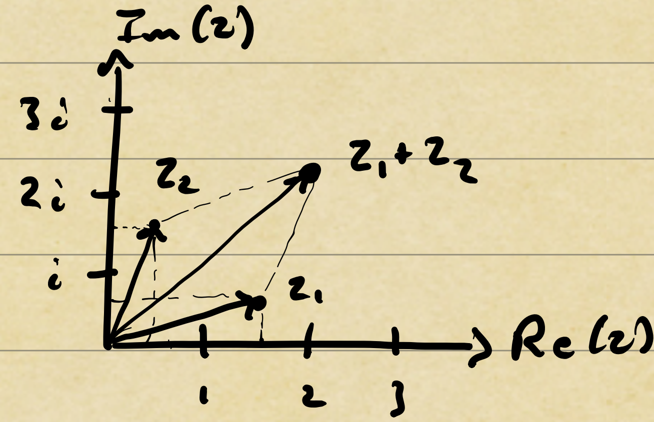
$|z| = \sqrt{x^2+y^2}$ $\qquad$ $\arg(z) = \theta + 2\pi n$

"complex conjugate"

$\bar{z} = x - iy$

$= re^{-i\theta}$

Addition:



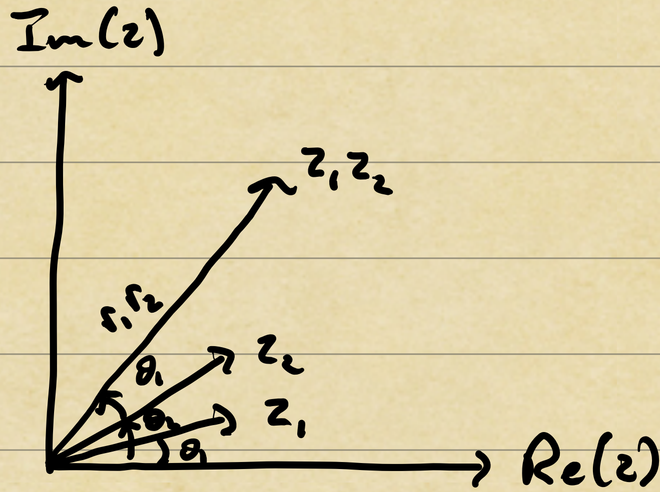
$$z_1 + z_2 = (x_1 + i y_1) + (x_2 + i y_2)$$

$$= (x_1 + x_2) + i(y_1 + y_2)$$

$\Rightarrow$ identical to vector addition in $\mathbb{R}^2$

$\Rightarrow \quad \big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ "triangle inequality"

Multiplication:



$$z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2)$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$= r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$\Rightarrow$ moduli multiply

$\Rightarrow$ arguments add

Useful to note that

$$|z|^2 = z\bar{z}$$

$$1/z = \bar{z}/|z|^2 \qquad (z \neq 0)$$