Solution Operators for ODEs

The Fourier Transform diagonalizes constant coefficient linear differential operators and allows us to construct solution operators for a large class of ODE/PDE.

$$[Lu](x) = \left[a_n \frac{d^n}{dx^n} + \cdots + a_1 \frac{d}{dx} + a_0\right] u(x)$$

Fourier Transform

$$\widehat{[Lu]}(\xi) = [\underbrace{a_n(i\xi)^n + \cdots + a_1(i\xi) + a_0}_{\text{degree } n \text{ poly in } \xi}]\,\hat{u}(\xi)$$

The polynomial $P(\xi) = a_n(i\xi)^n + \cdots + a_1(i\xi) + a_0$ is called the characteristic polynomial and its behavior in the complex plane is intimately linked to ODE/PDE involving L:

$$[Lu](x) = 0 \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = 0$$

$$[Lu](x) = \lambda u(x) \quad \Rightarrow \quad [P(\xi) - \lambda]\hat{u}(\xi) = 0$$

$$[Lu](x) = f(x) \quad \Rightarrow \quad P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$$

$$[\partial_t u](x) = [Lu](x) \quad \Rightarrow \quad \partial_t \hat{u}(\xi) = P(\xi)\hat{u}(\xi)$$

The solution of these problems boils down to solving for $\hat{u}(\xi)$ and inverting the transform. E.g., formally

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{f}(\xi)}{P(\xi)} e^{i\xi x}\, d\xi$$

inserting $\hat{f}(\xi)$ and swapping integration limits

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y) \underbrace{\left[\int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)}}{P(\xi)}\, d\xi\right]}_{G(x,y)} dy$$

The kernel $G(x,y)$ acts as an operator, mapping the "data" f to the solution u. Depending on the context, it is known as a Green's function, a fundamental solution, or a propagator (for time-dependent problems).

Let's take a look at an example to

a) Rigorously construct u and G w/tools from complex analysis.

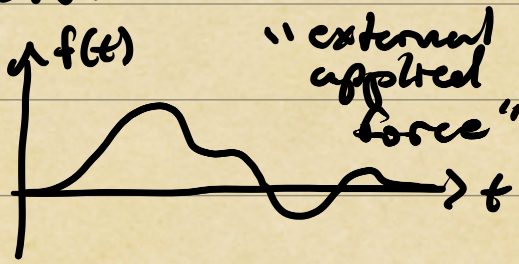
b) Examine how properties of G & $\hat{G}$ in complex plane influence soln's u.

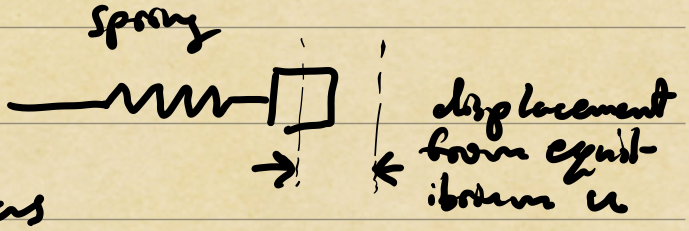
Example: Forced Oscillator

$$u''(t) + k^2 u(t) = f(t)$$

k: spring const.

$f(t)$: "forcing"

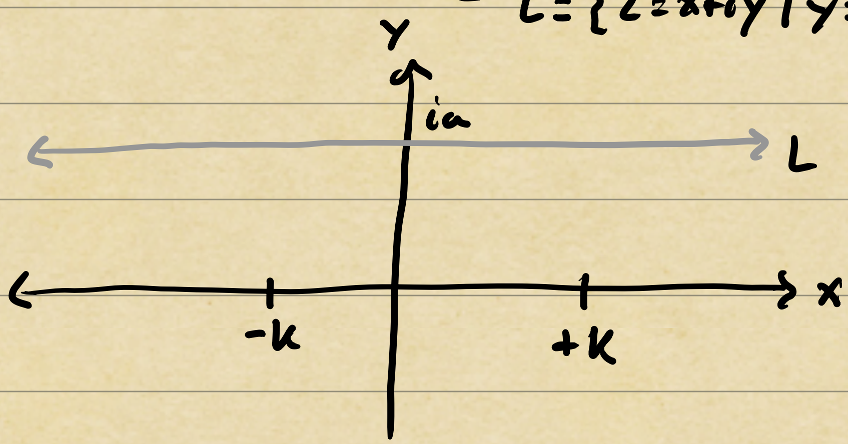




We'll assume that $f(t)$ has compact support in $[0,T]$ and f, f'' are cont.

Fourier Domain
$$(-\xi^2 + k^2)\hat{u}(\xi) = \hat{f}(\xi)$$
$\hat{f}$ well-defined since f has compact supp.

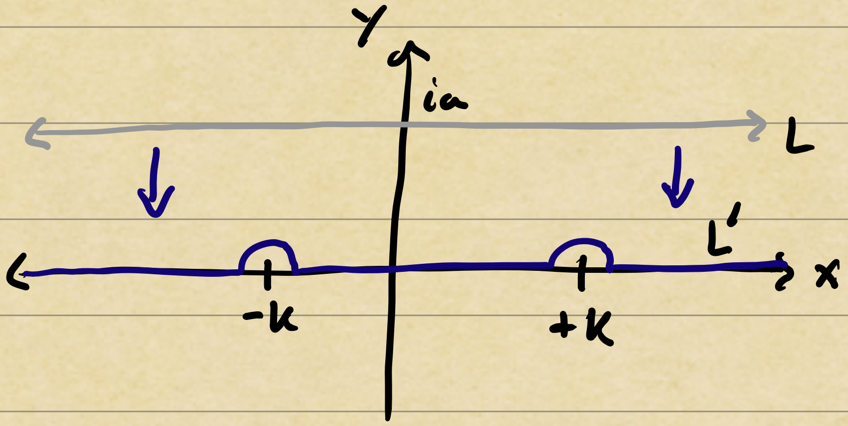
Solution
$$u(t) = \frac{1}{2\pi}\int_{L_\tau} \frac{e^{izt}}{k^2 - z^2}\hat{f}(z)\,dz \qquad t \geq 0$$
$$L = \{z = x+iy \mid y = a \geq 0, x \in \mathbb{R}\}$$



To calculate the solution operator, we swap

$$u(t) = \int_0^T f(\tau)\underbrace{\left[\frac{1}{2\pi}\int_L \frac{e^{iz(t-\tau)}}{k^2 - z^2}\,dz\right]}_{G(t-\tau)}d\tau$$

Q: Can we calculate the kernel, $G(t-\tau)$, of the solution operator explicitly?



By Cauchy's theorem, we can deform L to L'

$$u(t) = \frac{1}{2\pi}\int_{L'} \frac{e^{izt}}{k^2-z^2}\hat{f}(z)\,dz$$

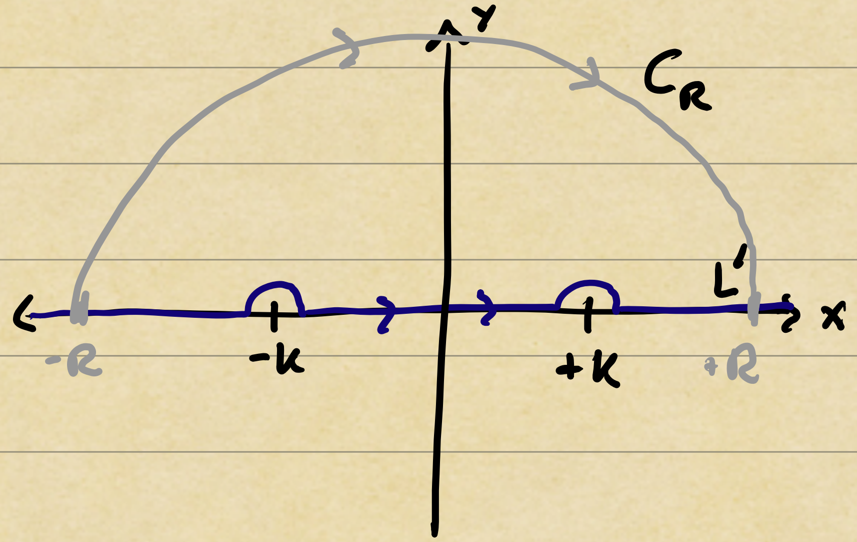
$$= \int_0^T f(\tau)\left[\frac{1}{2\pi}\int_{L'}\frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz\right]d\tau$$

Now, $e^{iz(t-\tau)}$ is bounded on L' and the inner integral is well-defined. The solution op. is

$$G(t-\tau) = \frac{1}{2\pi}\int_{L'}\frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz$$

and we can calculate $G(t-\tau)$ explicitly using the residue theorem.

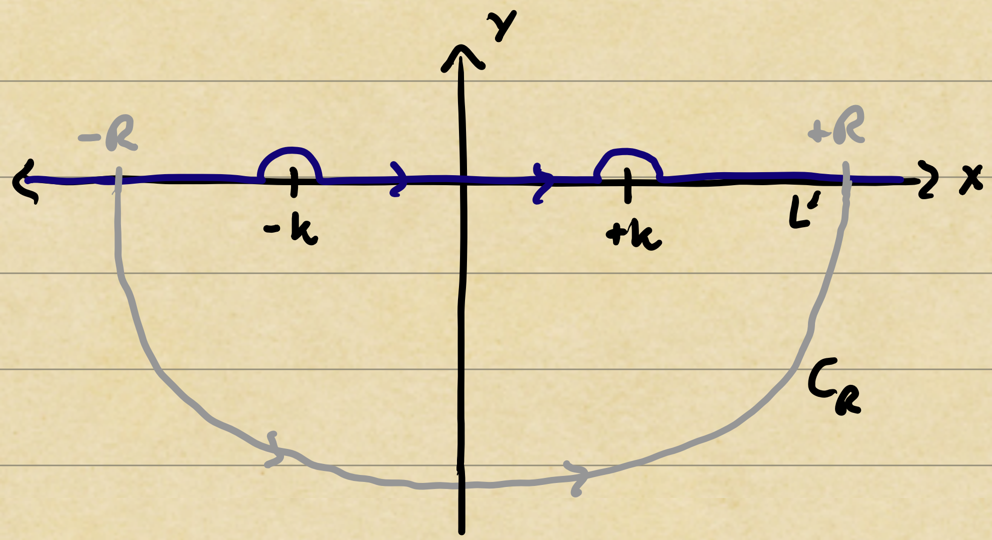
$\underline{\tau < t}$



By Cauchy's Theorem we calculate

$$G(t-\tau) = \frac{1}{2\pi}\int_{L_-} \frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz = \lim_{R\to\infty}\frac{1}{2\pi}\int_{C_R} \frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz = 0$$

$\underline{\tau > t}$



By the residue theorem, we calculate

$$G(t-\tau) = 2\pi i \operatorname*{Res}_{z=-k} \frac{1}{2\pi}\frac{e^{iz(t-\tau)}}{k^2-z^2} + 2\pi i \operatorname*{Res}_{z=k} \frac{1}{2\pi}\frac{e^{iz(t-\tau)}}{k^2-z^2} + \underbrace{\cancel{\lim_{R\to\infty}\frac{1}{2\pi}\int_{C_R}\frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz}}_{=0}$$

Recall
$(z-k)\,f(z)$
in res. thm.

$$= i\lim_{z\to -k}\frac{e^{iz(t-\tau)}}{(k-z)} + i\lim_{z\to k} -\frac{e^{iz(t-\tau)}}{(k+z)}$$

$$= i\,\frac{e^{-ik(t-\tau)}}{2k} - i\,\frac{e^{ik(t-\tau)}}{2k}$$

$$= \frac{i}{2k}\left(-2i\sin k(t-\tau)\right) = \frac{1}{k}\sin k(t-\tau)$$

Therefore, the solution operator can be written

$$G(t-\tau) = \begin{cases} 0 & \tau < t \\ \dfrac{\sin k(t-\tau)}{k} & \tau > t \end{cases}$$

and $$u(t) = \int_0^T G(t-\tau)\, f(\tau)\, d\tau .$$

A Family of Solution Operators

The homogeneous equation has nontrivial sol'ns:

$$[Lu](x) = u''(x) + k^2 u(x) = 0 \qquad x \in \mathbb{R}$$

So the solution to $Lu = f$ is not unique. However, one can specify conditions at $t = \pm\infty$ that determine the solution uniquely. The corresponding solution operator is then unique.

Anti-Causal Green's Function

The solution operator derived above:

$$G(t-\tau) = \begin{cases} 0 & \tau < t \\ \frac{\sin k(t-\tau)}{k} & \tau > t, \end{cases}$$

with $$u(t) = \int_0^T G(t-\tau) f(\tau)\, d\tau,$$

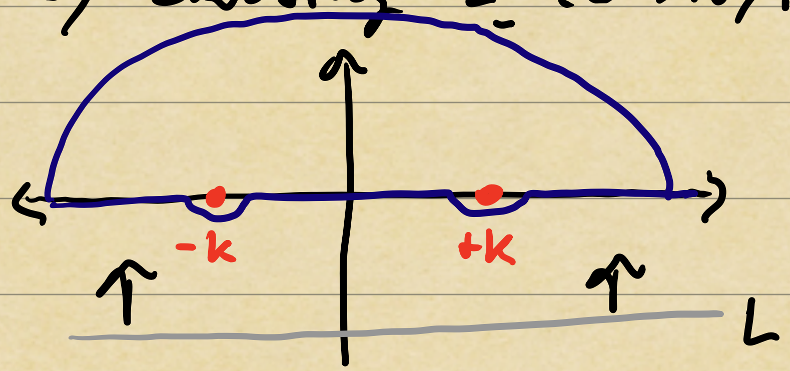
is called the anti-causal Green's function b/c $u(t)$ is determined by forcing behavior $f(\tau)$ with $\tau > t$! We can check that

$$u(t) = 0 \qquad \text{for} \quad t > T$$

and in general, $u(t)$ is not identically zero for $t < T$.

Causal Green's Function

We can obtain a "causal" Green's function by choosing $L_- = \{z = x + iy \mid y = \text{const} < 0,\ x \in \mathbb{R}\}$



$$u(t) = \frac{1}{2\pi} \int_{L_-} \frac{e^{izt}}{k^2 - z^2} \hat{f}(z)\, dz$$

The Green's function becomes

$$G(t-\tau) = \frac{1}{2\pi} \int_{L_-} \frac{e^{iz(t-\tau)}}{k^2 - z^2}\, dz$$

$$= \begin{cases} \dfrac{\sin k(t-\tau)}{k} & \tau < t \\ 0 & \tau > t \end{cases}$$

Now, the solution is "physical" in that $u(t)$ depends on the forcing $f(\tau)$ only for values $0 < \tau < t$, $t > 0$, and $u(t) = 0$ for $t < 0$.