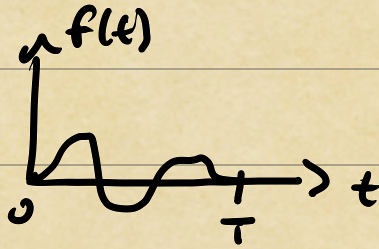
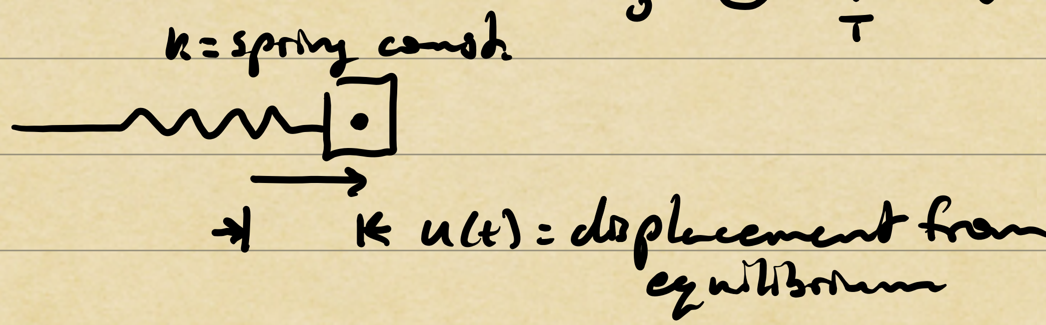
More on ODE Solution Ops

Consider the forced harmonic oscillator

(*) $u''(t) + k^2 u(t) = f(t)$





The solutions to the homogeneous eqn. are

(**) $v''(t) + k^2 v(t) = 0 \quad \Rightarrow \quad v_1(t) = c_1 \cos kt$
$v_2(t) = c_2 \sin kt$

So, for any solution $u(t)$ of (*), we also have

$$\tilde{u}(t) = c_1 \cos kt + c_2 \sin kt + u(t)$$

is a solution of (*) for any $c_1, c_2 \in \mathbb{C}$.
This leads to a whole family of solution operators with different behavior as $t \to \pm\infty$.

$$u(t) = \int_0^{T} G(t-\tau) f(\tau)\, d\tau$$

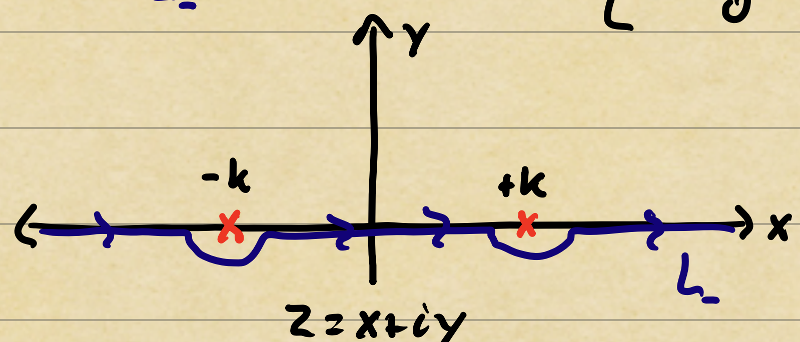
$\uparrow$

Green's function
= Kernel of soln op.

Causal Green's Function

The causal Green's function is

$$G_-(t-\tau) = \frac{1}{2\pi} \int_{L_-} \frac{e^{iz(t-\tau)}}{k^2 - z^2}\, dz = \begin{cases} \frac{1}{k} \sin k(t-\tau) & \tau < t \\ 0 & \tau > t \end{cases}$$



The solution is "causal" because

$$u(t) = \frac{1}{k} \int_0^t \sin k(t-\tau)\, f(\tau)\, d\tau$$

only depends on $f(\tau)$ for $\tau < t$. In particular, $u(t) = 0$ for $t < 0$, which is before the forcing turns on.

How does $u(t)$ behave for $t > T$?

$$u(t) = \frac{1}{k}\int_0^T \underbrace{\sin k(t-\tau)}_{\sin kt \cos k\tau \,-\, \cos kt \sin k\tau} f(\tau)\, d\tau$$

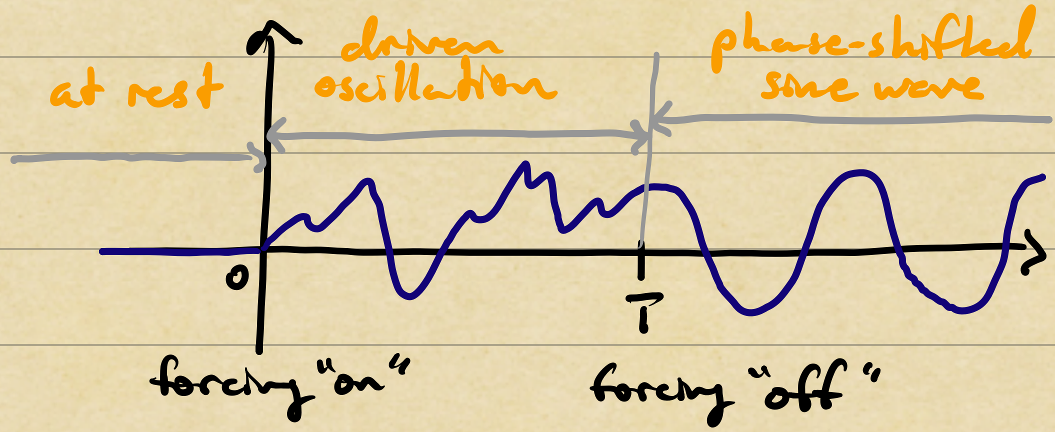
$$= \sin kt \left[\frac{1}{k}\int_0^T f(\tau)\cos k\tau\, d\tau\right] - \cos kt \left[\frac{1}{k}\int_0^T f(\tau)\sin k\tau\, d\tau\right]$$

$$= c_1 \sin kt + c_2 \cos kt, \qquad t > T.$$

phase shift = atan(b/a)

$$= A \sin(kt + \varphi)$$

amplitude = $\sqrt{c_1^2 + c_2^2}$



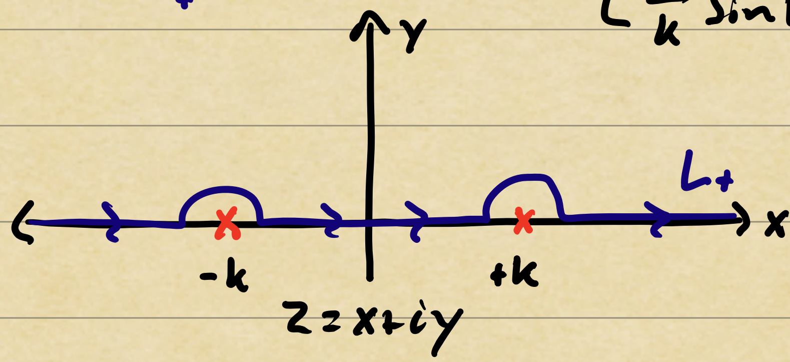
Q1: How are the causal & anti-causal Green's functions related?

Q2: How are the corresponding solutions related?

Anti-Causal Green's Function

The anti-causal Green's function is

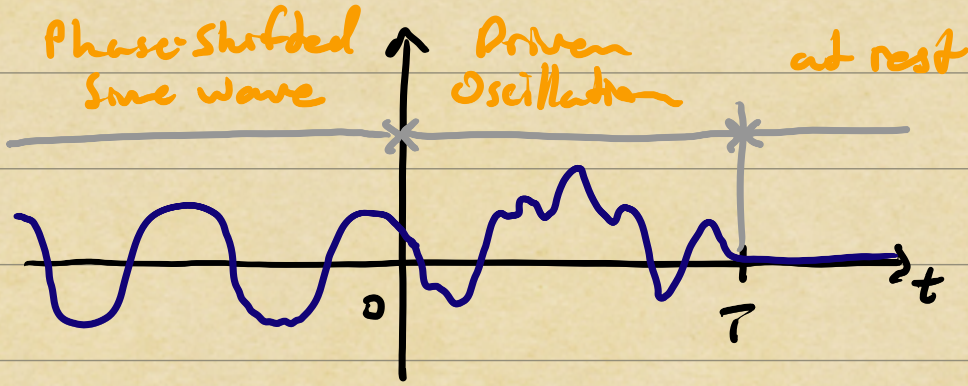
$$G_+(t-\tau) = \frac{1}{2\pi}\int_{L_+} \frac{e^{iz(t-\tau)}}{k^2 - z^2}\,dz = \begin{cases} 0 & \tau < t \\ \frac{1}{k}\sin k(t-\tau) & \tau > t \end{cases}$$



Here, the solution depends on $f(\tau)$ for $\tau > t$,

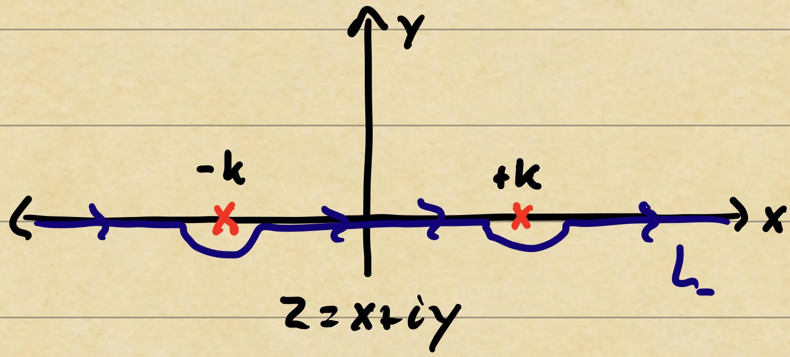
$$u(t) = \frac{1}{k}\int_t^T \sin k(t-\tau)\, f(\tau)\,d\tau.$$

A similar calculation from above shows

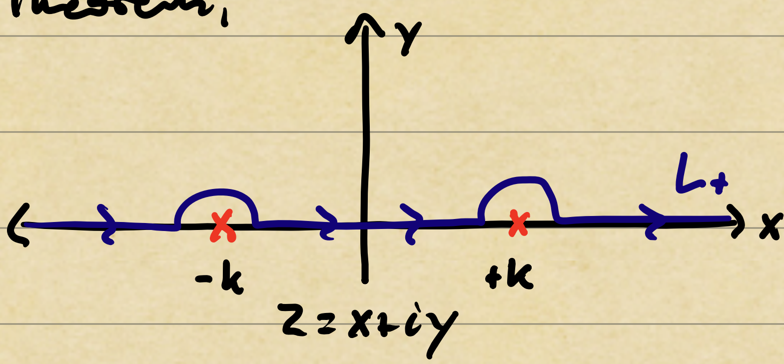


It's as if we set up an unforced oscillation with just the right amplitude and phase so that the forcing between $t=0$ and $t=T$ has the net effect of eliminating all oscillation and leaving the oscillator at rest.

How are the two Green's functions related?



Residue Theorem!



$$G_+(t-\tau) - G_-(t-\tau) = \frac{1}{2\pi}\int_{L_+} \frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz - \frac{1}{2\pi}\int_{L_-} \frac{e^{iz(t-\tau)}}{k^2-z^2}\,dz$$

$$= i \operatorname*{res}_{z=-k} \frac{e^{iz(t-\tau)}}{k^2-z^2} + i \operatorname*{res}_{z=k} \frac{e^{iz(t-\tau)}}{k^2-z^2}$$

$$= i \lim_{z \to -k} (z+k) \frac{e^{iz(t-\tau)}}{(k+z)(k-z)} + i \lim_{z \to k} (z-k) \frac{e^{iz(t-\tau)}}{(k+z)(k-z)}$$

$$= i \frac{e^{-ik(t-\tau)}}{2k} - i \frac{e^{ik(t-\tau)}}{2k}$$

$$= \frac{1}{k} \left[\frac{e^{ik(t-\tau)}}{2i} - \frac{e^{-ik(t-\tau)}}{2i} \right]$$

$$= \frac{1}{k} \sin k(t-\tau)$$

Much of this goes through for more general problems if we replace

$1/p(z)$ by $\hat{G}(z)$