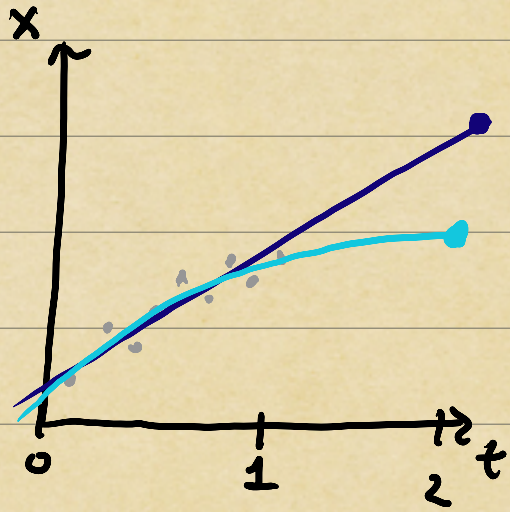
Analytic Continuation

Extrapolating beyond known data is a notoriously ill-posed problem.



Goal: given "data" $(t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)$, for $0 \le t_k \le 1$, $k = 1, \ldots, n$, predict $x(t)$ for $1 < t \le 2$.

Q: When can one "stably" extrapolate?

$\Rightarrow$ If parametrized model is known to be valid for all $0 \le t \le 2$, e.g.

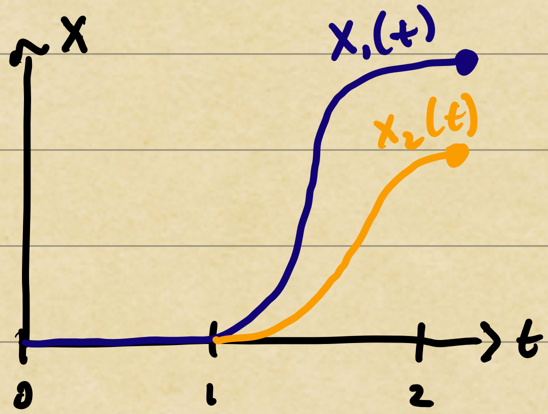
$$x(t) = \underbrace{a}_{\text{parameters}} t + \underbrace{b}_{} \quad \text{or} \quad x(t) = \exp(\underbrace{a}_{\text{parameters}} t + \underbrace{b}_{})$$

$\Rightarrow$ If parametrized model is not known,

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots,$$

more flexible model can capture a wide variety of simple & complex behavior.

$\Rightarrow$ However, w/out knowledge of underlying model, extrapolation is ill-posed b/c there may be no unique extrapolant matching data.



Two C^∞ extrapolants may agree perfectly on $[0,1]$ and differ by an arbitrary amount on $[3/2, 2]$!

<u>Example</u>

$$x_1(t) = \begin{cases} 0 & 0 \le t \le 1 \\ \exp\left(-\frac{1}{t-1}\right) & 1 < t \le 2 \end{cases}$$

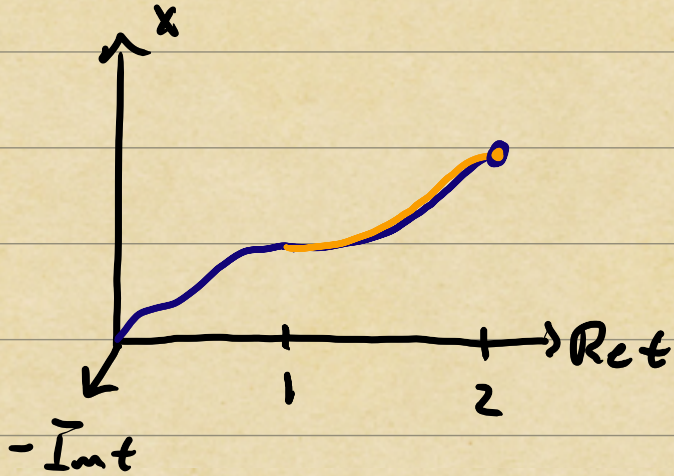
$$x_2(t) = \underbrace{(\text{const.})}_{\text{arbitrary constant}}\, x_1(t)$$

For any $f:[0,1] \to \mathbb{R}$, $f(t) + x_1(t)$ and $f(t) + x_2(t)$ are valid extrapolations differing by an arbitrary amount for any $1 < t \le 2$.

Extrapolating Analytic Functions

Remarkably, analytic continuation suggests extrapolation is possible for holomorphic f.

Recall] If $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are holomorphic in an open connected set Ω and agree on any subset $E \subset \Omega$ containing a limit point of E, then $f(z) = g(z)$ for all $z \in \Omega$



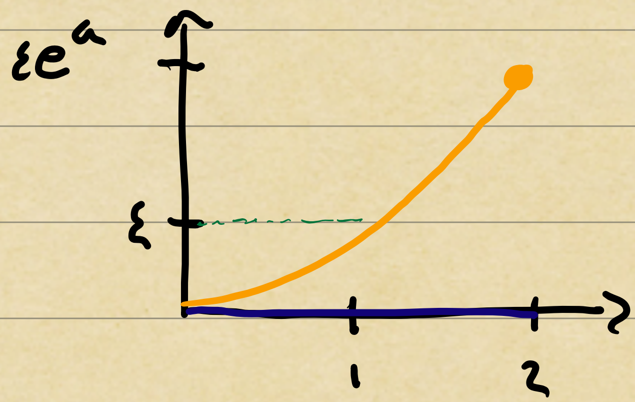
If $[0,2] \subset \Omega$ and $f(t) = g(t)$ for $0 \leq t \leq 1$, then $f(t) = g(t)$ for all $0 \leq t \leq 2$.

However, we typically only know f on $[0,1]$ within some tolerance $\|\hat{f} - f\| \leq \varepsilon$.

Q] When is extrapolation of analytic functions stable?

Example: Suppose $f = 0$ for $t \in [0,1]$ but we only know that $|\hat{f}| \leq \varepsilon$. How far apart can f and $\hat{f}$ be at $t = 2$?

$$\hat{f}(t) = \varepsilon \exp(-a(1-t))$$



They may differ exponentially!

In practice, analytic continuation from data may be ill-posed.

Q] Under what circumstances on f can the problem be well-posed, stable?

Conditions for Stable Analytic Continuation

To control the growth of the error

$$E(2) = f(2) - \hat{f}(2)$$

as we extrapolate further from [0,1], we need a bound on E (or f and $\hat{f}$) in the region of holomorphicity around [0,2].

Hadamard Three Lines Lemma

Let E be holomorphic in $S = \{z : 0 < \operatorname{Re} z < 1\}$ with $\sup_{z \in S} |E(z)| \le 1$ and $\lim_{x \to 0^+} \sup_{y \in \mathbb{R}} |E(x+iy)| < \varepsilon$ for some $0 < \varepsilon < 1$. Then, for all $z \in S$

$$(**) \qquad |E(z)| \le \varepsilon^{1 - \operatorname{Re}(z)}$$

and this bound is tight, meaning there is a function satisfying hyp. of theorem and achieving equality in $(**)$.