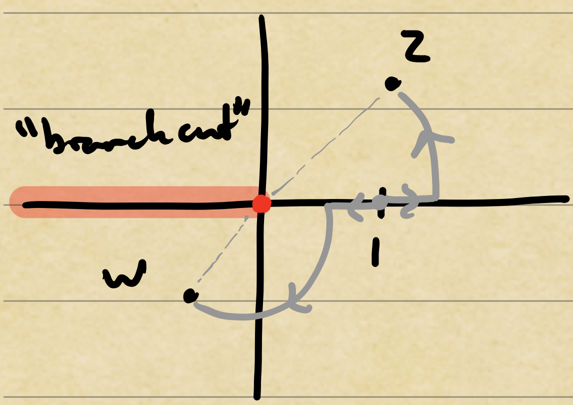
The Argument Principle

Recap: The complex logarithm has the form:

$$\log z = \log|z| + i(\arg z + 2\pi k), \quad k = 0, \pm 1, \pm 2, \ldots$$

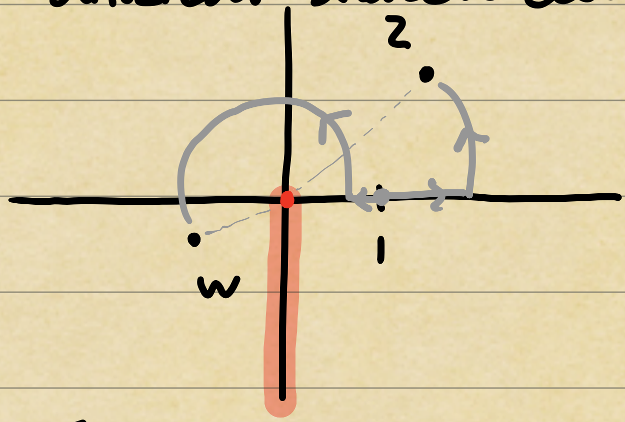
In general, it is multivalued. To obtain a single-valued logarithm, we choose a branch:



$$\text{Log}\, z = \log|z| + i \arg z,$$

where $-\pi < \arg z < \pi$.

This is called the principle branch of $\log z$. We can select different branches by making different branch cuts, e.g.,



$$\log z = \log|z| + i \arg z,$$

where $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

The branch cut must connect branch pts $z = 0, \infty$.

The Argument Principle

[might want to illustrate with phase plots.]

For functions w/only pole singularities, the number and location of poles & zeros is intimately connected w/changes in argument.

A function f on an open set Ω is called meromorphic if there is a sequence $z_1, z_2, z_3, \ldots \subset \Omega$ w/no limit pts in Ω, s.t.

(i) f is holomorphic in $\Omega \setminus \{z_1, z_2, z_3, \ldots\}$
(ii) f has poles at $z_1, z_2, z_3, \ldots$

The argument principle relates the number of zeros & poles of f enclosed by a simple closed contour to the change in argument of $f(z)$ as z traverses the contour.

The key ingredients are the residue theorem and the complex logarithm.

Noting that $\log f(z) = \log|f(z)| + i\arg(f(z))$ and that $(\log f(z))' = f'(z)/f(z)$ $(f(z) \neq 0)$, we can track the change in argument of f over a closed curve γ by

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi k i.$$

Here, k is the # of times $f(z)$ crosses a branch cut of the logarithm as z traverses γ. This is precisely the # of times $f(z)$ "winds" around the origin as z traverses γ.

<u>Thm</u>/ Suppose f is meromorphic in an open set Ω containing smooth simple closed Jordan curve γ and its interior. If f has no poles or roots on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = R - P$$

where $R =$ "# roots c int(γ)" and $P =$ "# roots c int(γ)."

including multiplicity

Pf Since f is meromorphic, it is holomorphic except at isolated poles in Ω and there are at most finitely many poles and zeros in $\mathrm{int}(\gamma)$. In a suff. small disk of radius $\varepsilon>0$ around a root of order n at $z_0 \in \mathrm{int}(\gamma)$,

$$f(z) = (z-z_0)^n g(z)$$

where $g(z)$ is holomorphic and nonzero in $D_\varepsilon(z_0) = \{z : |z-z_0| \leq \varepsilon\}$. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}, \qquad z \in D_\varepsilon(z_0).$$

Note that $g'(z)/g(z)$ is holomorphic in $D_\varepsilon(z_0)$ since $g(z)$ is nonzero in $D_\varepsilon(z_0)$ and

$$\int_{|z-z_0|=\varepsilon} \frac{f'(z)}{f(z)}\,dz = 2\pi n i$$

by the residue theorem. Similarly, at a pole of order m at w_0, we have

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-w_0} + \frac{h'(z)}{h(z)} \qquad z \in D_\varepsilon(w_0)$$

↑ holomorphic

and apply the residue theorem to find

$$\int_{|z-w_0|=\varepsilon} \frac{f'(z)}{f(z)}\,dz = -2\pi m i.$$

In other words, a zero of f with order n leads to a pole of $f'(z)/f(z)$ w/residue n while a pole of f with order m leads to a pole of $f'(z)/f(z)$ w/residue $-m$.

Applying the residue theorem w/ γ,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(R-P)$$

where $R = \sum$ (order of zeros in $\mathrm{int}(\gamma)$) and $P = \sum$ (order of poles in $\mathrm{int}(\gamma)$).

The argument principle gives us a useful handle on the behavior of roots and poles of meromorphic functions.

Rouché's

Thm/ Suppose that f and g are holomorphic in an open set Ω containing a smooth simple closed Jordan curve γ and its interior. If $|f(z)| > |g(z)|$ for $z \in \gamma$, then f and $f+g$ have the same # of zeros inside γ.

HW: Rouché implies continuity of roots of f under small perturbations of f.

Pf/ Define $f_t(z) = f(z) + t\,g(z)$, $0 \le t \le 1$. Then $f_0 = f$ and $f_1 = f+g$. Let n_t denote # of zeros of $f_t(z)$ in $\mathrm{int}(\gamma)$. The idea is to show that $n_t = \text{constant}$.

Since $|f(z)| > |g(z)|$, $f_t(z)$ cannot have a root on the curve γ. By argument princ.

$$n_t = \frac{1}{2\pi i} \int_\gamma \frac{f_t'(z)}{f_t(z)}\, dz$$

We'll show that n_t is continuous, which implies it is constant as that is the only way an integer-valued function can be continuous. We have that for any $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|t - t'| < \delta$ implies $|n_t - n_{t'}| < \varepsilon$. We have that

$$|n_t - n_{t'}| = \frac{1}{2\pi} \left| \int_\gamma \left[\frac{f_t'(z)}{f_t(z)} - \frac{f_{t'}'(z)}{f_{t'}(z)} \right] dz \right|$$

$$\leq \frac{|\gamma|}{2\pi} \sup_{z \in \gamma} \left| \frac{f_t'(z)}{f_t(z)} - \frac{f_{t'}'(z)}{f_{t'}(z)} \right|$$

Since $f_t(z)$ and $f_t'(z)$ are both jointly continuous for $t \in [0,1]$, $z \in \gamma$ and $f_t(z)$ is not zero on γ, then $f_t(z)/f_t'(z)$ is jointly continuous on $[0,1] \times \gamma$. This ensures that we can find $\delta > 0$ to make

$$\sup_{z \in \gamma} \left| \frac{f_t'(z)}{f_t(z)} - \frac{f_{t'}'(z)}{f_{t'}(z)} \right| < \varepsilon \quad \text{when } |t - t'| < \delta.$$

Therefore, n_t is continuous and integer-valued for $t \in [0,1]$ and must be constant.

Beyond continuity of roots and poles under appropriate perturbations, Rouché's theorem has deep implications for the theory of holomorphic functions:

⇒ Nonconstant holomorphic functions map open sets to open sets.

⇒ Nonconstant holomorphic functions cannot attain their maximum in an open set (Maximum Principle).