The Periodic Trapezoid Rule (Applications)

In a practical setting, the residue theorem provides elegant error bounds for quadrature.

$$I = \int_0^{2\pi} f(\theta)\,d\theta \qquad\qquad I_N = \frac{2\pi}{N}\sum_{k=1}^{N} f(\theta_k)$$

$\theta_k = 2\pi\frac{k}{N}$

<u>Example</u> Fourier coeffs of periodic signal.

Fourier series $\quad f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}, \qquad \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta$

In practice, $\quad \hat{f}_k \approx \frac{1}{N}\sum_{k=1}^{N} f(\theta_k)e^{-ik\theta_k}$

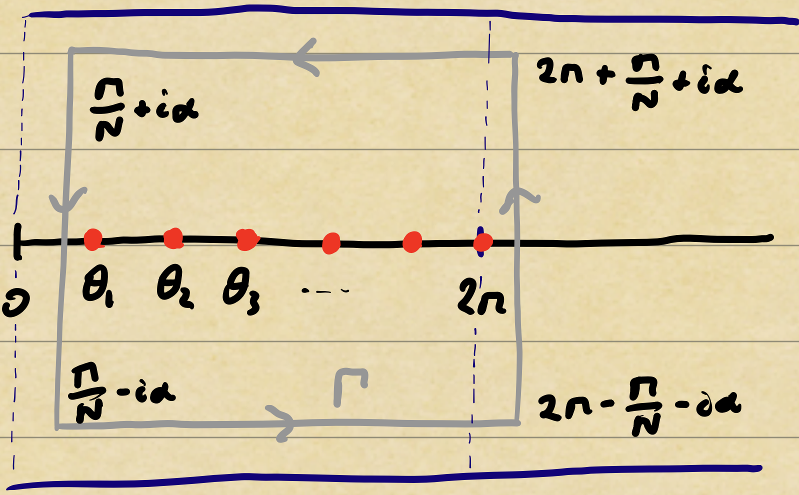
equispaced samples

Q1: How accurate are the computed coeffs?

<u>Thm</u> Suppose f is 2π-periodic and holomorphic in the strip $S:\{-a < \operatorname{Im}\theta < a\}$ $(a>0)$, and $|f(\theta)| \leq M$ for all $\theta \in S$. Then for any $N \geq 1$,

$$|I - I_N| \leq \frac{4\pi M}{e^{\alpha N}-1} \quad \text{for any} \quad 0<\alpha<a.$$

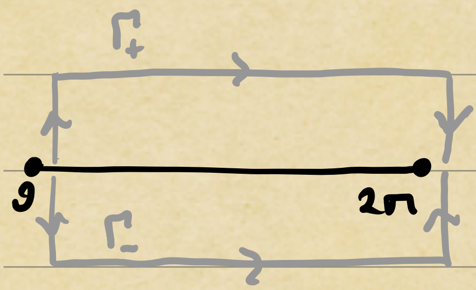
Note that error decreases $\sim Me^{-\alpha N}$ as $N \to \infty$.



f holomorphic in periodic strip $\{-a < \operatorname{Im} z < a\}$ with $|f(\theta)| \le M$.

Pf/ The first key idea is to rewrite both I and I_N as integrals over Γ by using contour deformation & residue thm.

To write I as an integral over Γ,



$$I = \frac{1}{2}\int_{\Gamma_+} f(\theta)\,d\theta + \frac{1}{2}\int_{\Gamma_-} f(\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\Gamma_+} f(\theta)\,d\theta - \frac{1}{2}\int_{\Gamma^-} f(\theta)\,d\theta = \frac{1}{2}\int_{\Gamma} \underbrace{\operatorname{sign}}_{=\begin{cases}+1 & \operatorname{Im}(\theta)>0\\ -1 & \operatorname{Im}(\theta)<0\end{cases}}(\operatorname{Im}(\theta))\, f(\theta)\,d\theta$$

To write I_N as a "comparable" integrand over Γ, we look for a function that

a) has simple poles at $\theta_1, \dots, \theta_N$

b) approximates $\text{sign}(\text{Im}(\theta))$ as $N \to \infty$

To satisfy a)-b), consider "characteristic"

$$m(\theta) = -\frac{i}{2}\cot\left(\frac{N\theta}{2}\right) = \frac{1}{2}\,\frac{1+e^{-iN\theta}}{1-e^{-iN\theta}}$$

which has poles at $\theta_k = \frac{2\pi}{N}$ (roots of $\sin(\frac{N\theta}{2})$) with corresponding residues $c_k = {}^{i}/N$.

By the residue calculus, we have

$$I_N = \frac{2\pi}{N}\sum_{k=1}^{N} f(\theta_k) = \int_\Gamma m(\theta)\, f(\theta)\, d\theta.$$

Since I and I_N are now written as integrals over Γ, we can compare the integrands directly. We find that

$$I - I_N = \int_\Gamma \left[\frac{1}{2}\text{sign}(\text{Im}(\theta)) - m(\theta)\right] f(\theta)\, d\theta.$$

Note that, by 2π-periodicity of the integrand, the contributions from the vertical contours cancel each other out. We are left with

$$I - I_N = \frac{1}{2}\int_{\frac{\pi}{N}-i\alpha}^{2\pi+\frac{\pi}{N}-i\alpha}\left(1 - \frac{1+e^{-iN\theta}}{1-e^{-iN\theta}}\right) f(\theta)\,d\theta - \frac{1}{2}\int_{\frac{\pi}{N}+i\alpha}^{2\pi+\frac{\pi}{N}+i\alpha}\left(1 + \frac{1+e^{-iN\theta}}{1-e^{-iN\theta}}\right) f(\theta)\,d\theta$$

$$= \underbrace{\int_{\frac{\pi}{N}-i\alpha}^{2\pi+\frac{\pi}{N}-i\alpha} \frac{-1}{e^{iN\theta}-1} f(\theta)\,d\theta}_{I_1} + \underbrace{\int_{\frac{\pi}{N}+i\alpha}^{2\pi+\frac{\pi}{N}+i\alpha} \frac{-1}{1-e^{iN\theta}} f(\theta)\,d\theta}_{I_2}$$

$$|I_1| \leq 2\pi M |e^{N\alpha}-1|^{-1} \qquad |I_2| \leq 2\pi M |e^{N\alpha}-1|$$

$$\Rightarrow \quad |I - I_N| \leq \frac{4\pi M}{e^{\alpha N}-1} \qquad \checkmark$$

Exponential convergence makes the trapezoidal rule the gold standard for smooth periodic functions.

$\boxed{\text{Example}}$ Periodic signal from coeffs.

Fourier series

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}, \qquad \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-ik\theta}\,d\theta$$

Given $\hat{f}_{-N}, \ldots, \hat{f}_N$, approximate $f(\theta) \approx \sum_{k=-N}^{+N} \hat{f}_k e^{ik\theta}$

Q2: How accurate is the computed signal?

The truncation error for $f_N(\theta) = \sum_{k=-N}^{N} \hat{f}_k e^{ik\theta}$ is

$$E_N = \sup_{0 \leq \theta \leq 2\pi} |f(\theta) - f_N(\theta)| \leq \sum_{|k|>N} |\hat{f}_k|.$$

How do the Fourier coefficients behave?

Thm) Suppose that f is 2π-periodic, holomorphic in the strip $S = \{-a < \operatorname{Im}\theta < a\}$, and bounded by $M>0$ in S. Then, for any $0<\alpha<a$,

$$|\hat{f}_k| \leq M e^{-\alpha|k|}, \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

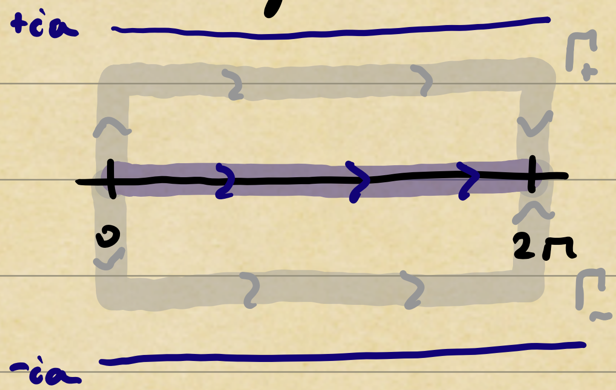
Note that $\sum_{|k|>N} |\hat{f}_k| \leq M \sum_{|k|>N} (e^{-\alpha})^{|k|}$

$$= 2M \frac{e^{-\alpha(N+1)}}{1 - e^{-\alpha}}$$

tail of double geometric series

Therefore, $E_N \leq \dfrac{2Me^{-\alpha(N+1)}}{1-e^{-\alpha}}$ for $N = 1, 2, 3, \ldots$

<u>Pf</u>/ The idea is to deform the contour of integration to bound the Fourier coefficients:



$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta$$

If $k \geq 0$, the integrand decays exponentially in the lower half-plane. We have

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\int_{\Gamma_-} f(z)e^{-ikz}\,dz$$

contributions from vertical segments cancel by periodicity

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta - i\alpha)e^{-ik(\theta - i\alpha)}\,d\theta$$

$$\Rightarrow |\hat{f}_k| \leq \sup_{0\leq\theta\leq 2\pi} |f(\theta - i\alpha)|\, e^{-k\alpha} \leq Me^{-k\alpha}$$

If $k < 0$, the integrand decays exponentially in the upper half-plane and the argument is essentially the same:

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ik\theta}\, d\theta = \frac{1}{2\pi}\int_{\Gamma_+} f(z)\, e^{-ikz}\, dz$$

contributions from vertical segments cancel by periodicity

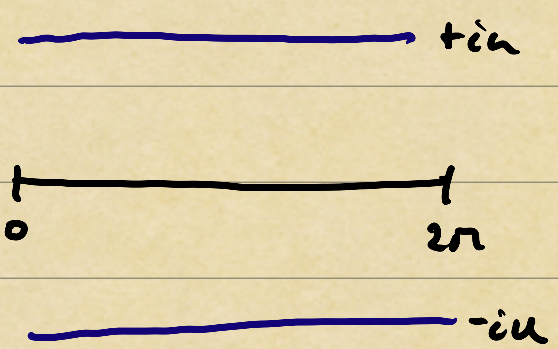
$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta+i\alpha)\, e^{-ik(\theta+i\alpha)}\, d\theta$$

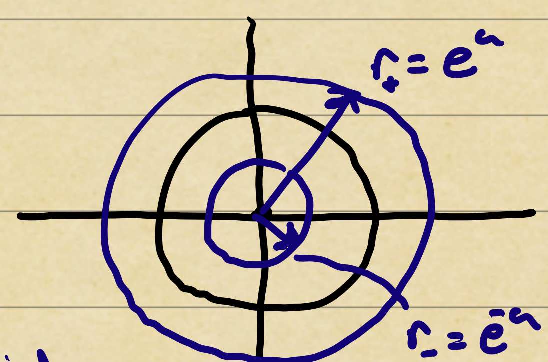
$$\Rightarrow \quad |\hat{f}_k| \leq \sup_{0\leq\theta\leq 2\pi} |f(\theta+i\alpha)|\, e^{k\alpha} \leq M e^{-|k|\alpha}$$

Fourier Series : Laurent Series

The geometrically decaying Fourier coeffs of holomorphic periodic signals can be understood as an analogue of Cauchy's inequalities for Laurent series.

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta} \quad \overset{e^{i\theta} \leftrightarrow z}{\Longleftrightarrow} \quad g(z) = \sum_{k=-\infty}^{+\infty} \hat{g}_k z^k$$





$e^{i(ia)} = e^{-a}$

$e^{i(-ia)} = e^{a}$

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ik\theta}\, d\theta \iff \hat{g}_k = \frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)}{z^{k+1}}\, dz$$

$$f(\theta) = g(e^{i\theta}) \iff \hat{f}_k = \hat{g}_k$$