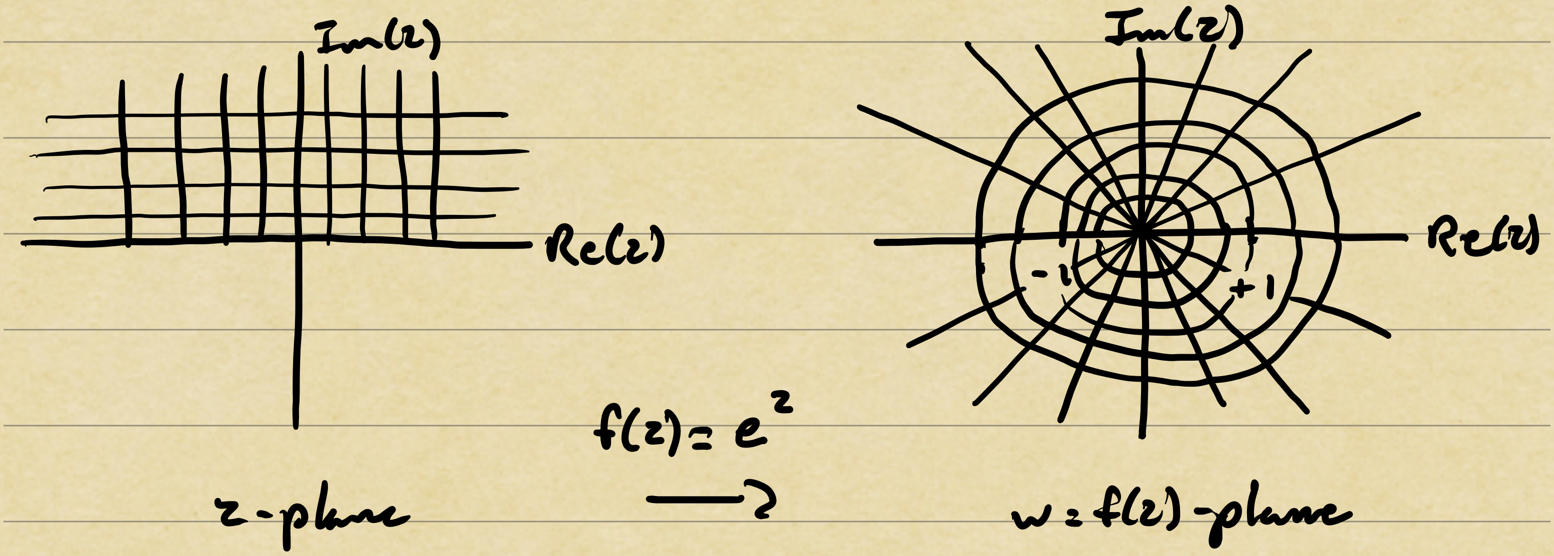
Cauchy's Integral Formulas

Recap Complex-valued function $f : \Omega \to \mathbb{C}$



$\Rightarrow$ $f : \Omega \to \mathbb{C}$ is holomorphic in $\Omega \subseteq \mathbb{C}$ if

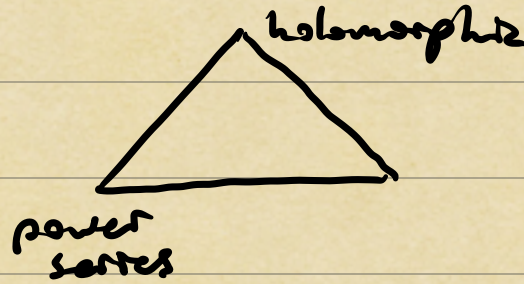
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = w_0 \quad \text{for some} \quad w_0 \in \mathbb{C},$$

at each point $z_0 \in \Omega$. We write $f'(z_0) = w_0$.

$\Rightarrow$ "Calculus" of complex differentiation is identical to real calculus: sum, product, quotient, and power rules all remain valid.

$\Rightarrow$ Holomorphic functions have additional structure: Cauchy-Riemann Equations

Power Series



Power series form an important class of holomorphic functions in complex analysis.

$$(*) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n; \qquad \{a_n\} \subset \mathbb{C}.$$

By "slicing" f into a series of simple powers, many computations can be simplified, at least when the series converges absolutely,

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

<u>Thm</u> Given $(*)$, there exists $0 \leq R \leq \infty$, s.t.

i) if $|z| < R$, $(*)$ converges absolutely.

ii) if $|z| > R$, $(*)$ diverges.

<u>Pf</u> If $(*)$ converges (diverges) for some $z_0 \in \mathbb{C}$ then it converges (diverges) for all $|z| \leq |z_0|$ $(|z| \geq |z_0|)$ by the comparison test.

R is called the radius of convergence and the region $|z|<R$ is called the disc of convergence.

"Hadamard's formula"

$$R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n} \overset{\text{If lim exists}}{=} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$ "Ratio Test"

Note: On the boundary $|z|=R$, the series may or may not converge.

<u>Thm</u> The power series (*) is a holomorphic function in it's disc of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1},$$

where f' and f have same radius of convergence.

<u>Pf</u> Classic "3ε" argument, see Stein & Shakarchi.

<u>Corollary</u> Power series (*) is infinitely complex differentiable in its disk of convergence, and higher derivatives are power series obtained by term-by-term differentiation.

In short, power series are infinitely complex-differentiable holomorphic functions on the disc of convergence, where they converge absolutely. They are easy to manipulate term-wise.

Example: $e^z = \sum_{n\geq 0}^{\infty} \frac{z^n}{n!}$

Disk of convergence is $\mathbb{C}$ ($R=\infty$): for any $z \in \mathbb{C}$

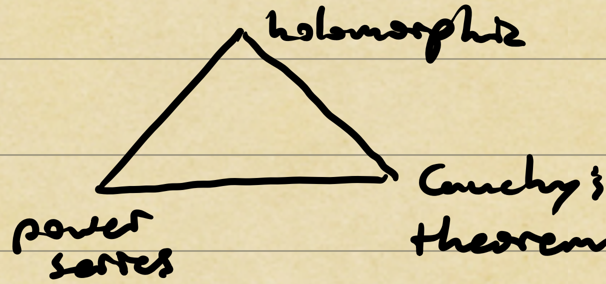
$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!} \Rightarrow \sum_{n\geq 0}^{\infty} \left|\frac{z^n}{n!}\right| \leq \sum_{n\geq 0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$$

Derivative: $(e^z)' = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{m\geq 0}^{\infty} \frac{z^m}{m!} = e^z$ ✓

Remarkably, all holomorphic functions behave locally like power series, which makes power series an indispensible tool.

"All happy families are similar to one another, ..."

- Leo Tolstoy

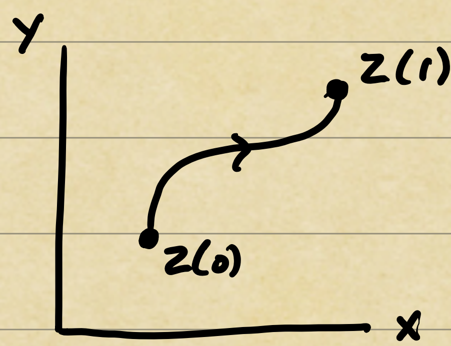


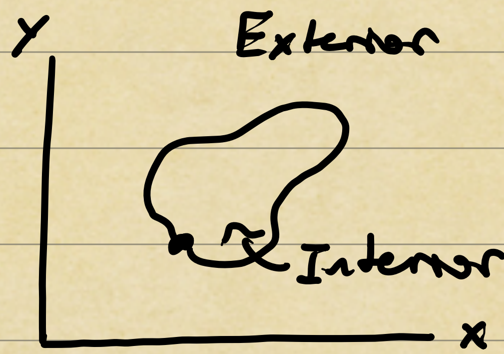
Complex Integration

A **simple smooth Jordan arc** is a curve

$$z(t) = x(t) + i y(t), \qquad a \le t \le b$$

with $x'(t), y'(t)$ continuous and $x'(t)^2 + y'(t)^2 \neq 0$, and $t_1 \neq t_2$ implying $z(t_1) \neq z(t_2)$.

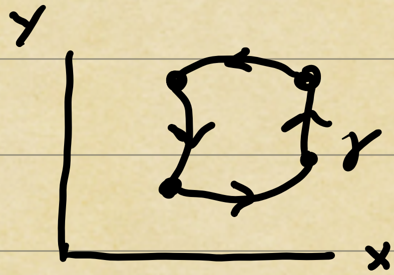




A simple closed Jordan curve has $z(a) = z(b)$.

Jordan Curve Thm. Every simple closed Jordan curve in the complex plane divides the plane into two disjoint sets: interior of the curve is b'dd, exterior is unb'dd and the curve is the boundary of each.

Simple piecewise smooth curves are simple Jordan curves w/piecewise continuous derivatives.



The **integral** of $f:\Omega\to\mathbb{C}$ along $\gamma:[a,b]\to\Omega$ is

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\underbrace{z'(t)}_{z'(t) = x'(t) + i y'(t)}\,dt$$

=> The integral is independent of parametrization of curve γ

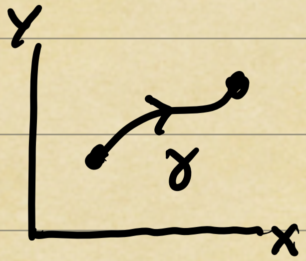
=> The **length** of γ is

$$|\gamma| = \int_a^b |z'(t)|\,dt$$

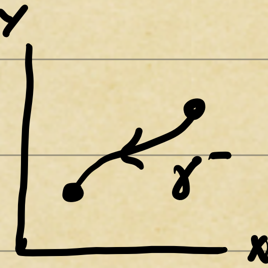
<u>Thm</u> If $f,g:\Omega\to\mathbb{C}$ is continuous and $\gamma:[a,b]\to\Omega$ is simple peicewise smooth:

(a) $$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz \qquad \alpha,\beta\in\mathbb{C}$$

(b) If γ^- is γ with reverse orientation,



$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz$$

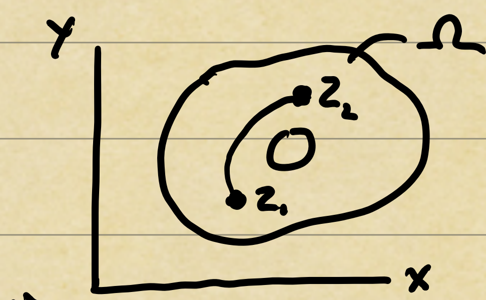


(c) $$\left|\int_\gamma f(z)\,dz\right| \leq |\gamma| \sup_{z\in\gamma} |f(z)|$$

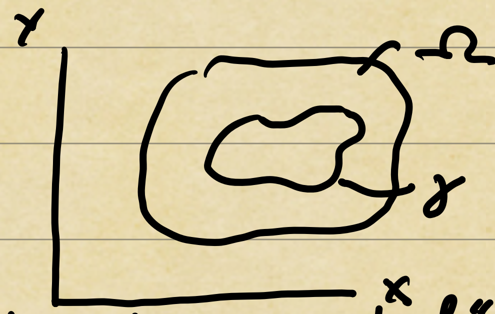
To state our main result, we need a special notion of connectedness.

A nonempty open set $\Omega \subset \mathbb{C}$ is **connected** if any two points in Ω can be joined by a simple Jordan arc in Ω.

If every simple closed Jordan curve γ in Ω has $\mathrm{int}(\gamma) \subset \Omega$, then Ω is called **simply connected**.



"connected"



"simply connected"

Cauchy's Theorem

Let $f : \Omega \to \mathbb{C}$ be holomorphic on simply connected Ω. Then, for any simple closed Jordan curve $\gamma \subset \Omega$,

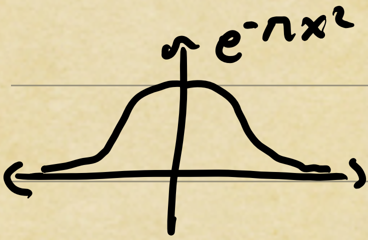
$$\int_\gamma f(z)\,dz = 0.$$

Pf.| Show for triangle contours (Goursat) and then work up to more complicated. See Shakarchi & Stein or Dettman.

Example| Compute the Fourier transform

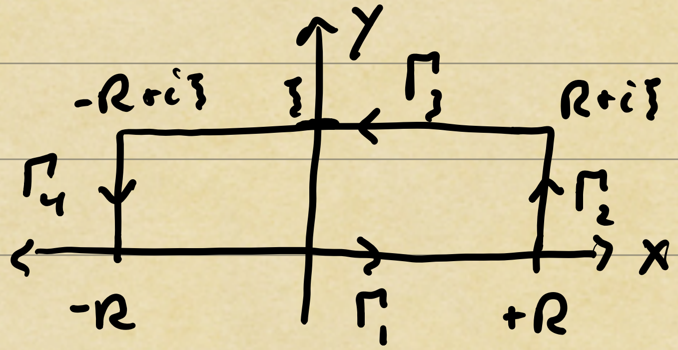
$$\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\,dx \qquad \xi \in \mathbb{R}$$

We'll do the case $\xi > 0$. The case $\xi < 0$ is similar. Note that for $\xi = 0$, we have



$$\int_{-\infty}^{+\infty} e^{-\pi x^2}\,dx = 1 \qquad \left(\begin{array}{c} e^{-\pi x^2} \text{ is the} \\ \text{standard normal} \\ \text{probability density} \end{array} \right)$$

Consider the complex function $f(z) = e^{-\pi z^2}$:



By Cauchy's theorem

$$\int_{\Gamma_R} f(z)\,dz = 0$$

$$\Gamma_R = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

$$\parallel$$

$$\sum_{k=1}^{4} \int_{\Gamma_k} f(z)\,dz = 0$$

Consider the 4 contours $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ each:

$$\int_{\Gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-\pi x^2}\,dx$$

$$\int_{\Gamma_2} f(z)\,dz = \int_0^{\xi} f(R+iy)\,i\,dy = \int_0^{\xi} e^{-\pi(R+iy)^2}\,i\,dy$$

$$\int_{\Gamma_3} f(z)\,dz = \int_{R}^{-R} f(x+i\xi)\,dx = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\,dx$$

$$= -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx$$

$$\int_{\Gamma_4} f(z)\,dz = \int_0^{\xi} f(-R+iy)\,i\,dy = \int_0^{\xi} e^{-\pi(-R+iy)^2}\,i\,dy$$

Note that $\lim_{R\to\infty}\int_{\Gamma_3} f(z)\,dz = -\underbrace{e^{\pi\xi^2}}_{\text{scaling factor}}\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx$,

which is precisely the integral we want to compute. On the other hand, we know that

$$\lim_{R\to\infty}\int_{\Gamma_1} f(z)\,dz = \int_{-\infty}^{+\infty} e^{-\pi x^2}\,dx = 1.$$

Finally, the vertical contours satisfy the bound

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \le \underbrace{\xi}_{|\Gamma_2|} \sup_{y\in\xi}\left|e^{-\pi(R+iy)^2}\right| \le \xi e^{-\pi R^2}$$

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \le \underbrace{\xi}_{|\Gamma_4|} \sup\left|e^{-\pi(-R+iy)^2}\right| \le \xi e^{-\pi R^2}$$

Therefore $\lim_{R\to\infty}\int_{\Gamma_2} f(z)\,dz = \lim_{R\to\infty}\int_{\Gamma_4} f(z)\,dz = 0$.

By Cauchy's theorem, for any $R>0$, we have

$$0 = \int_{\Gamma_R} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz.$$

Taking the limit $R\to\infty$ on both sides, we find

$$0 = 1 - e^{\pi \xi^2} \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx = e^{-\pi \xi^2}$$

So the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi \xi^2}$.

We'll discuss Fourier Transforms, applications, and techniques for computing them in detail later in this course.