# The Complex Logarithm

**Recap**

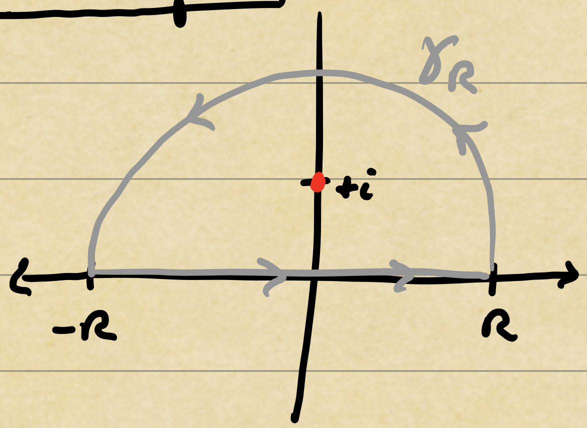
Residue Thm $\Rightarrow$ $\displaystyle\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} c_k$ ← residue at $z_k$

↑ holomorphic except at $z_1, \ldots, z_n \in \text{int}(\gamma)$

Residue at $z_k$ $\Rightarrow$ $\displaystyle c_k = \int_{|z-z_k|=\varepsilon} f(z)\,dz$ = coeff. of $(z-z_0)^{-1}$ in Laurent series

For pole of order $m$ $\Rightarrow$ $\displaystyle c_k = \lim_{z\to z_k} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_k)^m f(z)\right]$

**Example/** Show that $\displaystyle\int_{\gamma_R} \frac{1}{z^2+1}\,dz = \pi$




$$\frac{1}{z^2+1} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

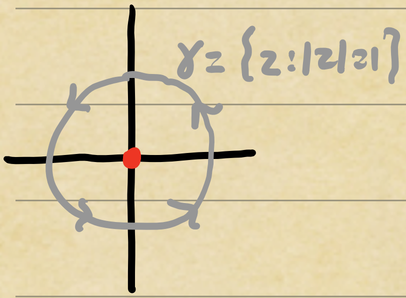
$$\Rightarrow \text{residue} = \lim_{z\to i} (z-i)(z^2+1)^{-1} = 1/2i$$

$$\Rightarrow \int_{\gamma_R} \frac{1}{z^2+1}\,dz = 2\pi\left(1/2i\right) = \pi \checkmark$$

↑ residue at $+i$

Alternatively, read off $a_{-1}$ from Laurent series.

**Example/** Compute $\int_\gamma e^{1/z}\,dz$ where $\gamma = \{|z|=1\}$.




$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k} \rightarrow a_{-1} = 1$$

$$\Rightarrow \int_\gamma e^{1/z}\,dz = 2\pi i$$

## Complex Antiderivatives

To develop the theory further, we must consider the question: why does only the $k=-1$ term in the Laurent series contribute to the residue $\int_{|z-z_0|=\varepsilon} f(z)\,dz$ at $z_0$?

$$\Rightarrow \int_{|z-z_0|=\varepsilon} z^k\,dz = \int_0^{2\pi} (\varepsilon e^{i\theta})^k (i\varepsilon e^{i\theta})\,d\theta = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases}$$

The explanation lies in the properties of complex antiderivatives of $z^k$ $(k=1,2,3,\dots)$.

A complex antiderivative (or "primitive") for $f:\Omega \to \mathbb{C}$ is a function $F:\Omega\to\mathbb{C}$ holomorphic in $\Omega$ and s.t. $F'(z) = f(z)$ for all $z \in \Omega$.

**Thm (FTC)** / If $f$ is continuous on $\Omega$ with complex antiderivative $F$, and $\gamma:[0,1]\to\Omega$ is a Jordan curve, then

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)).$$

In particular, if $\gamma$ is closed $(\gamma(1)=\gamma(0))$

$$\int_\gamma f(z)\,dz = 0.$$

**Example** / first, consider $z^k$ for $k\neq -1$.

$$f(z) = z^k \quad \text{and} \quad F(z) = \frac{z^{k+1}}{k+1}$$

$F(z)$ is holomorphic in $\mathbb{C}\setminus\{0\}$ so for any closed curve $\gamma$ <u>not</u> passing through $z=0$,

$$\int_\gamma z^k\,dz = \frac{\gamma(1)^{k+1}}{k+1} - \frac{\gamma(0)^{k+1}}{k+1} = 0.$$

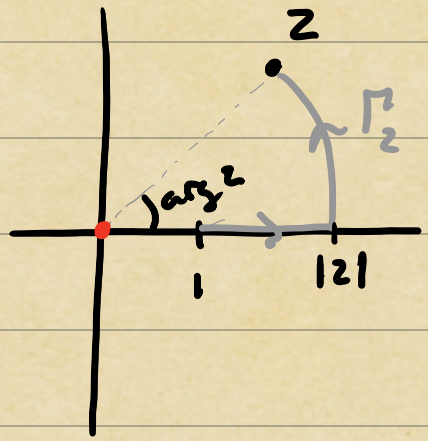
What about $k=-1$? For real valued functions we know that the antiderivative of $1/x$ is $\log x$, for $x>0$:

$$(*) \qquad \log x = \int_1^x \frac{dy}{y}, \qquad x>0.$$

The properties of $\log x$ can be derived from $(*)$.

## The Complex Logarithm

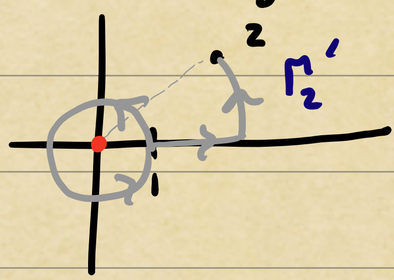
To construct an antiderivative of $1/z$, consider the following contour:




$$F(z) = \int_{\Gamma_z} \frac{dz}{z} = \int_1^{|z|} \frac{dx}{x} + \int_0^{\arg z} \frac{i|z|e^{i\varphi}}{|z|e^{i\varphi}}\, d\varphi$$

$$= \log|z| + i\arg z$$

Formally, $F(z)$ is the complex logarithm. However note that the value of $F(z)$ depends heavily on the choice of contour connecting 1 to z. For example,

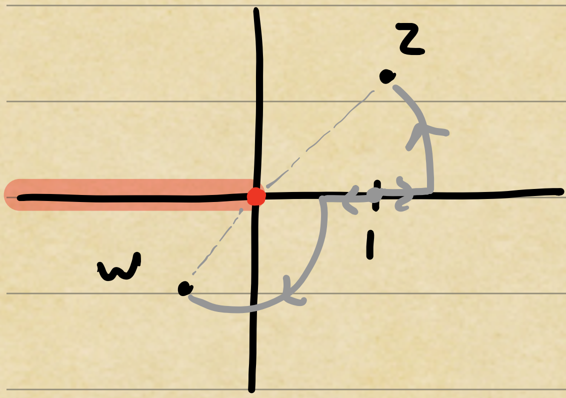



$$F(z) = \int_{\Gamma_z'} \frac{dz}{z} = \log|z| + i(\arg z + 2\pi)$$

In general, the complex logarithm is multi-valued with possible values at $z$,

$$F(z) = \log|z| + i(\arg z + 2\pi k), \quad k = 0, \pm 1, \pm 2, \ldots$$

To obtain a single-valued function, we can restrict which contours connecting 1 to $z$ are permissible for our construction.




If we prohibit contours that pass through the negative real axis, $\mathbb{R}_{\leq 0}$, we obtain

$$F_p(z) = \log|z| + i \arg z, \quad -\pi < \arg z < \pi.$$

This is called the principal branch of the complex logarithm. One can show that

$\Rightarrow F_p(z)$ is holomorphic in $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with $F_p'(z) = 1/z$ for all $z \in \Omega$.

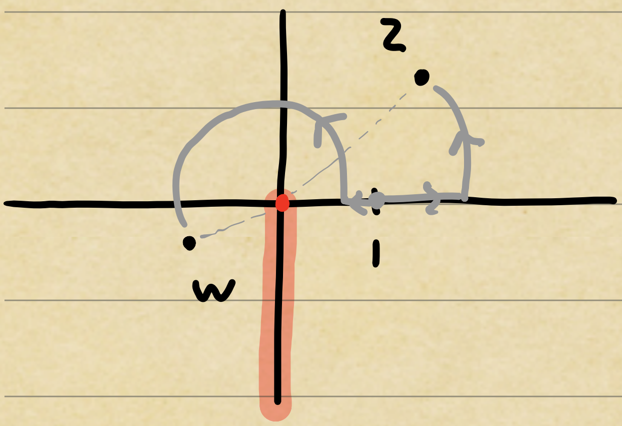
$\Rightarrow F_p(x) = \log x$ when $x \in \mathbb{R}$ and $x \geq 0$.

$\Rightarrow \exp(F_p(z)) = z$ for all $z \in \Omega$.

Note that $F_p(z)$ has a "jump" of $2\pi i$ on $\mathbb{R}_{<0}$,

$$\lim_{\varepsilon \to 0^+} F_p(x+i\varepsilon) - F_p(x-i\varepsilon) = 2\pi i, \quad x<0.$$

The choice of $\mathbb{R}_{<0}$ is called a branch cut. We can obtain other "versions" of the complex logarithm, called branches, by selecting different branch cuts. For example,




$$F(z) = \log|z| + i\arg z,$$
where $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

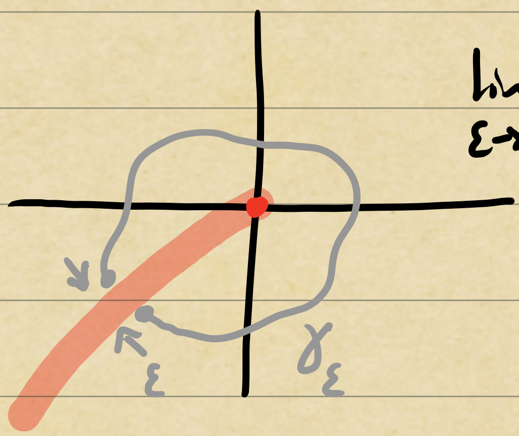
Note that after a branch cut is chosen, $F(z)$ is single-valued and the value of $F(z)$ is the same for all contours connecting 1 to z and not passing through the branch cut.

While the branch cut is "arbitrary", it must connect the points $0$ and $\infty$ to ensure that the resulting logarithm is single-valued.

The points $0$ and $\infty$ are called branch points.

## Antiderivative of $z^{-1}$

No matter which branch of the complex logarithm we choose, any closed curve winding enclosing the origin will have to intersect the branch cut.




$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{dz}{z} = \lim_{\varepsilon \to 0} \left[\log \gamma_\varepsilon(1) - \log \gamma_\varepsilon(0)\right]$$

$$= 2\pi i$$

So the residue of an isolated singularity only depends on the coefficient of $(z-z_0)^{-1}$ in the Laurent series b/c the antiderivative of $(z-z_0)^{-1}$, i.e., the complex logarithm $\log(z-z_0)$, has a branch cut – a jump discontinuity – connecting $z_0$ to $\infty$. While the singularities of antiderivatives for $z^k$ ($k \neq -1$) are local in character, the branch cut of $\log z$ means that $1/z$ contributes to closed contour integrals no matter how far the contour is from the origin (or, more generally, from $z_0$).