The Fourier Transform

Fourier series provide a powerful tool for analysis on periodic, bounded intervals.

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ik\theta}, \qquad \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ik\theta}\, d\theta.$$

They arise naturally, e.g., in solution of BVPs:

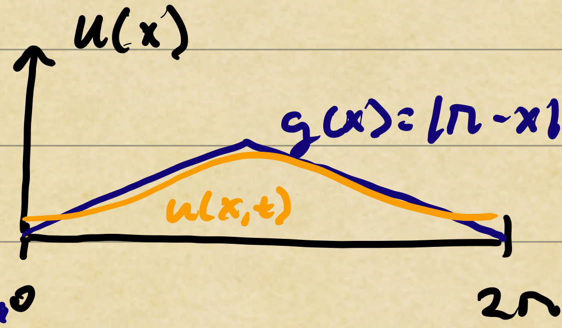
$$-u''(\theta) = \lambda u(\theta), \qquad u(0) = u(2\pi) = 0$$

$$\Rightarrow \quad u(\theta) = e^{ik\theta}, \qquad \lambda = k^2, \qquad \overset{\text{to satisfy b.c.'s}}{k = 0, \pm 1, \pm 2, \ldots}$$

Differentiation, integration, and multiplication are all straightforward in Fourier Basis (Sec. 12).

The Fourier Transform arises naturally in a similar way for problems posed on unbounded Euclidean domains, e.g., on $\mathbb{R}$.

Example: Heated Ring



$\partial_t u = \partial_x^2 u$

$u(x,0) = g(x)$ "initial data"

$u(0,t) = u(2\pi,t)$ "boundary data" $\partial_x u(0,t) = \partial_x u(2\pi,t)$

B.C. $\Rightarrow$ Expand $u(x,t) = \sum_{k=-\infty}^{+\infty} c_k(t)\, e^{ikx}$ $\leftarrow$ find $c_k(t)$

$\Rightarrow$ $\partial_t u = \sum_{k=-\infty}^{+\infty} c_k'(t)\, e^{ikx}$, $\partial_x^2 u = \sum_{k=-\infty}^{+\infty} -k^2 c_k(t)\, e^{ikx}$

PDE + I.C. $\Rightarrow$ $c_k'(t) = -k^2 c_k(t)$ $\quad c_k(0) = \hat{g}_k$

$\Rightarrow$ $c_k(t) = \tilde{g}_k e^{-k^2 t}$

$\Rightarrow$ $u(x,t) = \sum_{k=-\infty}^{+\infty} \hat{g}_k e^{-k^2 t} e^{ikx}$

By construction, $u(x,t)$ satisfies heat equation w/ boundary & initial conditions.

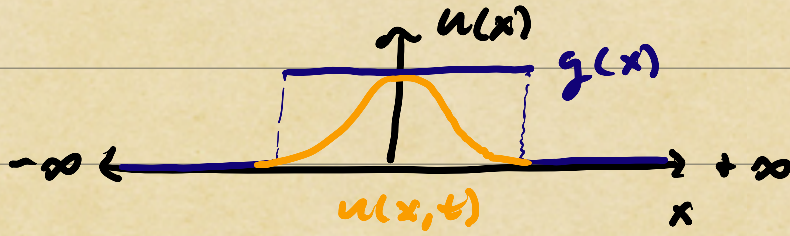
Q: What changes if $[0,2\pi]_{per} \to \mathbb{R}$?

Example: Heated Wire

$$\partial_t u = \partial_x^2 u$$

$$u(x,0) = g(x)$$

$$\lim_{x \to \pm\infty} u(x,t) = 0$$



How should we expand $u(x,t)$?

$$\Rightarrow \quad \partial_x^2 e^{i\xi x} = -\xi^2 e^{i\xi x} \qquad \xi \in \mathbb{R}$$

Idea: replace Fourier sum, by Fourier integral.

$$u(x,t) = \int_{-\infty}^{+\infty} \hat{u}(\xi,t)\, e^{i\xi x}\, d\xi$$

$$\partial_x^2 u(x,t) = \int_{-\infty}^{+\infty} -\xi^2 \hat{u}(\xi,t)\, e^{i\xi x}\, d\xi$$

$$\partial_t u(x,t) = \int_{-\infty}^{+\infty} \partial_t u(\xi,t)\, e^{i\xi x}\, d\xi$$

$$\Rightarrow \quad \partial_t \hat{u}(\xi,t) = -\xi^2 \hat{u}(\xi,t)$$

$$\Rightarrow \quad \hat{u}(\xi,t) = \overset{\hat{u}(\xi,0)}{\overset{\shortparallel}{\hat{g}(\xi)}}\, e^{-\xi^2 t}$$

$$\Rightarrow \quad u(x,t) = \int_{-\infty}^{+\infty} \hat{g}(\xi)\, e^{-\xi^2 t} e^{i\xi x}\, d\xi$$

How do we find $\hat{g}(\xi)$ from $g(x)$?

need $\Rightarrow$ $$g(x) = \int_{-\infty}^{+\infty} \hat{g}(\xi) e^{i\xi x}\, d\xi$$

<u>Fourier Transform ! Inverse Transform</u>

The Fourier Transform of $f: \mathbb{R} \to \mathbb{R}$ is

(1) $$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-i\xi x}\, dx, \text{ and}$$

The inverse transform of $\hat{f}(\xi)$ is

(2) $$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

under some mild regularity + integrability requirements on $f(x)$ and $\hat{f}(\xi)$.

We'll use complex analysis to examine

(1) - (2) for an important class of functions that appear in many practical situations, including the heat equation.

Consider functions $f: \Omega \to \mathbb{C}$ that

(i) are holomorphic in strip

$$S_a = \{ z \in \mathbb{C} : |\mathrm{Im}(z)| < a \}$$

(ii) satisfy, for some $A > 0$,

$$|f(x+iy)| \leq \frac{A}{1+x^2} \qquad x \in \mathbb{R},\ |y| < a.$$

We say that f belongs to $\mathfrak{F}_A$.

Thm) If f belongs to $\mathfrak{F}_A$ for some $a > 0$, then $|\hat{f}(\xi)| \leq B e^{-2\pi b|\xi|}$ for all $0 \leq b < a$.

$\Rightarrow$ Analogue of exp. decaying Fourier coeffs.