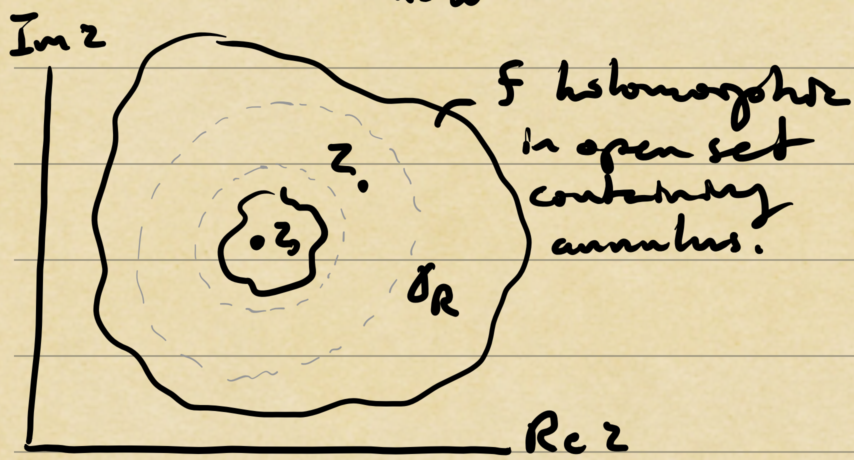
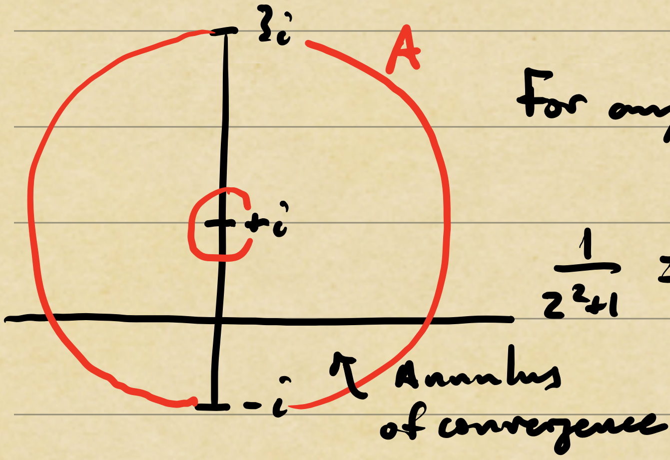
Residue Calculus

Recap: To study the behavior of $f(z)$ near a point singularity, we look at the Laurent series,

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k, \qquad a_k = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\xi)}{(\xi-z)^{k+1}}\,d\xi.$$



Example Expand $f(z) = \dfrac{1}{z^2+1}$ around $z = +i$.



For any $0 < |z-i| < 2$, we have

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{(z-i)(2i+z-i)}$$

expand in powers of $(z-i)$

$$= \frac{1}{2i(z-i)\left(1+\frac{z-i}{2i}\right)} = \frac{1}{2i(z-i)}\sum_{k\geq 0}\left(-\frac{z-i}{2i}\right)^k = \sum_{k\geq 0}\frac{(-1)^k}{(2i)^{k+1}}(z-i)^{k-1}$$

$$\Rightarrow \quad f(z) = \sum_{k=-1}^{\infty} \frac{(-1)^{k+1}}{(2i)^{k+2}} (z-i)^k, \qquad 0 < |z-i| < 2.$$

Example Expand $g(z) = e^{1/z}$ around $z=0$.

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \Rightarrow \quad e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{0} \frac{z^k}{k!}$$

$f(z) = \frac{1}{z^2+1}$ has only one "negative" power of z in its expansion around $z=i$, while $g(z) = e^{1/z}$ has infinitely many negative powers of z at $z=0$.

This leads to very different behavior of $f(z)$ and $g(z)$ in the vicinity of their singular pts.

Classification of Isolated Singularities

We say that $f(z)$ has an isolated singularity at z_0 if there is some $R>0$ s.t. $f(z)$ is holomorphic in the "punctured disk" $D_0^R = \{z : 0 < |z-z_0| < R\}$.

If $f(z)$ has an isolated singularity at z_0, then there is an $R>0$ such that $f(z)$ has a Laurent expansion at z_0 that converges for all $z: 0<|z|<R$.

We call $\sum_{k=-\infty}^{-1} a_k(z-z_0)^k$ the principle part of $f(z)$.

Isolated singularities fall into 3 categories:

Removable singularities have $a_k = 0$ for $k<0$.

e.g., $h(z) = \frac{\sin(z)}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$

$\Rightarrow$ Removable singularities are "artificial" in the sense that defining $f(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{z\to z_0} f(z) & z = z_0 \end{cases}$ makes $f(z)$ holomorphic at z_0, and therefore, there is a disk $D_R = \{z: |z-z_0|<R\}$ for some $R>0$ s.t. $f(z)$ is holomorphic in D_R.

Poles have only finitely many $a_k \neq 0$ for $k<0$.

e.g., $f(z) = \frac{\sin(z)}{z^2} = \frac{1}{z} - \frac{z}{6} + \frac{z^3}{5!} - \frac{z^5}{7!} + \cdots$

$\Rightarrow$ The principle part can be written

$$\sum_{k=-m}^{-1} a_k(z-z_0)^k \quad \text{with } a_{-m} \neq 0,$$

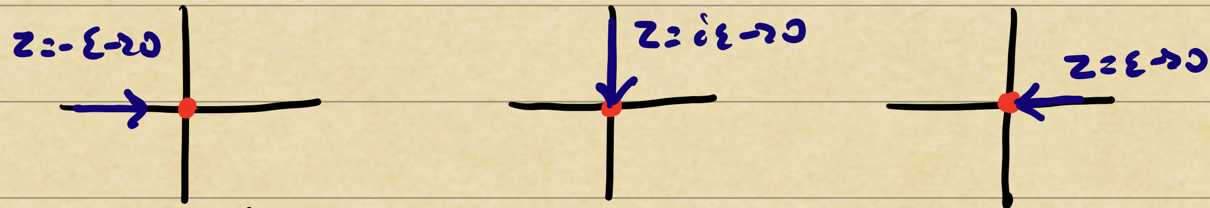
and m is called the order of the pole.

$\Rightarrow$ As $z \to z_0$, $f(z) = \dfrac{a_{-m}}{(z-z_0)^m} + g(z-z_0)$ where $(z-z_0)^m g(z) \to 0$. In particular $|f(z)| \to \infty$ as $z \to z_0$.

Essential singularities have infinitely many $a_k \neq 0$ for $k<0$.

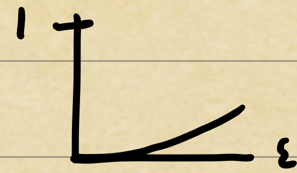
e.g., $g(z) = e^{1/z} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$

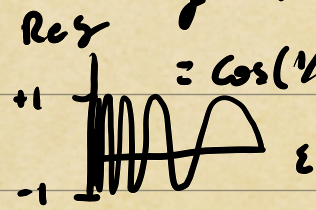
$\varepsilon > 0$



$g(-\varepsilon) = e^{-1/\varepsilon} \to 0$ $\quad$ $g(i\varepsilon) = e^{-i/\varepsilon}$ $\quad$ $g(\varepsilon) = e^{1/\varepsilon} \to \infty$

$= \cos(1/\varepsilon) - i\sin(1/\varepsilon)$





$\Rightarrow$ If $f(z)$ has an essential singularity at z_0, then it takes on every value (in $\mathbb{C}$) with one possible exception in every ε-neighborhood ($D_\varepsilon = \{z : |z-z_0| < \varepsilon\}$) of z_0.

Residue Calculus

The residue calculus gives a systematic

way to reduce many integrals to a sum of contributions from singularities of the integrand.

Let $f(z)$ have an isolated singularity at z_0, with f holomorphic in $D_0^R = \{z : 0<|z|<R\}$. Then the residue of f at z_0 is

$$a_{-1} = \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\,d\zeta \qquad (0<r<R),$$

i.e., the first negative coefficient in the Laurent series of $f(z)$ at the point z_0.

The residue of a simple pole is

$$a_{-1} = \lim_{z\to z_0}(z-z_0)f(z),$$

while the residue of a pole of order m is

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m\sum_{k=-m}^{\infty}a_k(z-z_0)^k\right] = \frac{d^{m-1}}{dz^{m-1}}\sum_{k=-m}^{\infty}a_k(z-z_0)^{k+m}$$

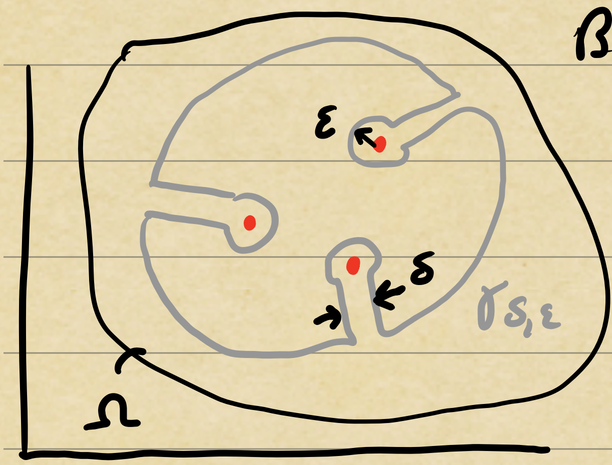
(first $k=-m,\dots,-2$ terms annihilated) $= (m-1)!\,a_{-1} + O(z-z_0)$

Thm Let f be holomorphic in a simply connected open set $\Omega \subseteq \mathbb{C}$ except at finitely many isolated singular points $z_1, \ldots, z_n$. If γ is a smooth simple closed Jordan curve in Ω with $\{z_1, \ldots, z_n\} \subset \mathrm{int}(\gamma)$,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} c_k$$

where c_k is the residue of f at z_k.

Pf First consider the contour $\gamma_{\delta,\varepsilon}$. By Cauchy's theorem,



$$\int_{\gamma_{\delta,\varepsilon}} f(z)\,dz = 0.$$

As $\delta \to 0$, the contribution from the parallel contours vanish, b/c $f(z)$ is continuous in Ω. We are left with

$$\int_\gamma f(z) = \sum_{k=1}^{n} \int_{\gamma_{k,\varepsilon}} f(z)\,dz$$

where $\gamma_{k,\varepsilon}$ is the counterclockwise circular contour of radius $\varepsilon > 0$ centered at z_0.
These integrals are precisely $2\pi i c_k$, so

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} c_k . \qquad \checkmark$$

Notice that the residue theorem implies that only the $j=-1$ term in the Laurent series contributes to integrals winding around singularities.

It's instructive to integrate the Laurent series term-by-term and find out why $j=-1$ is special.

For $0 < |z - z_k| < \varepsilon$ (ε sufficiently small), we have

$$f(z) = \sum_{j=-\infty}^{+\infty} a_j (z - z_k)^j$$

near the singularity at z_k. Integrating term-by-term (Laurent series converges uniformly on $\{\xi : |\xi - z_k| = \varepsilon\}$), we find

$$\int_{\gamma_{k,\varepsilon}} f(z)\,dz = \sum_{j=-\infty}^{+\infty} a_j^{(k)} \int_{\gamma_{k,\varepsilon}} (z-z_k)^j\,dz.$$

Parametrizing with $z = z_k + \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta} d\theta$

$$\int_{\gamma_{k,\varepsilon}} (z-z_k)^j\,dz = \int_0^{2\pi} (\varepsilon e^{i\theta})^j (i\varepsilon e^{i\theta})\,d\theta$$

$$= i\varepsilon^{j+1} \int_0^{2\pi} e^{i(j+1)\theta}\,d\theta$$

$$= \begin{cases} 2\pi i & j=-1 \\ 0 & j\neq -1 \end{cases}$$

The $(z-z_k)^{-1}$ term may not dominate the growth or "behavior" of f locally, but it is the term w/global significance in the sense that it influences the integral of f over any contour winding around z_k.

Soon, we'll see that this is related to the absence of a holomorphic antiderivative for f in any punctured disk $\{0 < |z-z_k| < \varepsilon\}$.