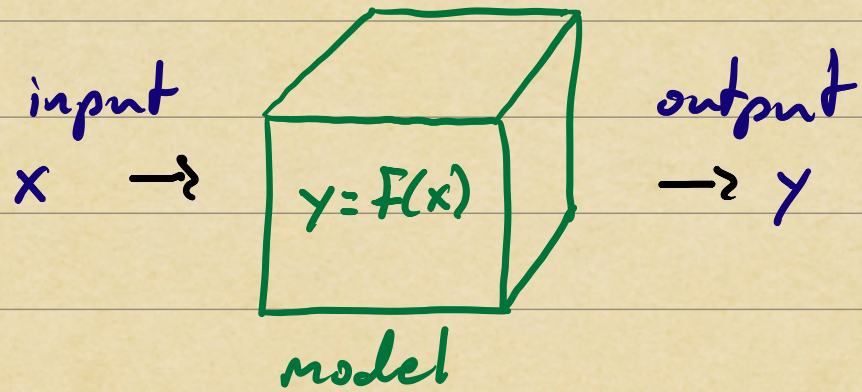
1D "Warm Up" - Approximation Theory

Aim: use data to develop mathematical models of complex phenomena.



Idea 1: Model Regression

Q: How well does a model fit data?

Idea 2: Model Accuracy

Q: How well does a model predict new data?

Idea 3: Model Generalization

Q: How accurate is model "far" from data?

Problem 1. Interpolating "clean" data

Given input-output data from continuous function $G:[0,1]\to\mathbb{R}$,

$$y_0 = G(x_0),\ y_1 = G(x_1),\ \ldots,\ y_n = G(x_n),$$

construct a degree $\leq n$ poly. interpolant:

$$F_n(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

s.t. $\quad F_n(x_j) = G(x_j) \quad$ for $\quad j = 0, \ldots, n.$

Q: How accurate is the interpolant?

"Model Error" $\qquad E_n = \sup_{x\in[0,1]} |G(x) - F_n(x)|$

$\Rightarrow$ E_n depends on "smoothness" of G and the distribution of $\{x_j\}_{j=0}^{n}$.

$\Rightarrow$ In practice, also need stable algorithm.

Algorithm 1. Dictionary Fitting

Let $\text{span}\{e_0, e_1, \ldots, e_3\} = \mathbb{P}_n$, so that any degree $\leq n$ polynomial can be written

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = b_0 e_0 + \cdots + b_n e_n,$$

for some unique coeffs $\{b_0, \ldots, b_n\}$.

In particular, there are unique coeffs s.t.

$$F_n(x) = c_0' e_0(x) + \cdots + c_n' e_n(x),$$

and we can compute them via

$$\begin{bmatrix} e_0(x_0) & \cdots & e_n(x_0) \\ e_0(x_1) & \cdots & e_n(x_1) \\ \vdots & & \\ e_n(x_n) & \cdots & e_n(x_n) \end{bmatrix} \begin{bmatrix} c_0' \\ c_1' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

"Generalized Vandermonde System"

$\Rightarrow$ Sensitivity of solution depends on data.

$\Rightarrow$ Stability of algorithm depends on dictionary.

See demo01.m on course repository for numerical experiments w/ interpolation.

Analysis of Interpolation

To analyze the accuracy of the interpolant, we follow a strategy that is common in approximation theory.

Step 1. Best Approximation in $\mathbb{P}_n$.

Q: How well is G approximated by polynomials in $\mathbb{P}_n$?

Step 2. Suboptimality of $\mathbb{P}_n$-interpolant.

Q: How "far from best" is the interpolant?

This 2-step analysis separates the influence of the ground truth and model from the influence of the data.

Best Approximation

We want to find $p_* \in \mathbb{P}_n$ that solves

$$(*) \qquad p_* = \operatorname*{argmin}_{p \in \mathbb{P}_n} \|p - G\|,$$

where $\|p\| = \sup_{x \in [0,1]} |p(x)|$ is the "sup" norm.

Key Fact 1. Existance & Uniqueness

If $G \in C[0,1]$ (continuous on $[0,1]$), then there is a unique best approximant $p_* \in \mathbb{P}_n$ that satisfies $(*)$.

pf| Existence: the space $\mathbb{P}_n$ is a complete n-dimensional vector space and the subset

$$B = \{p \in \mathbb{P}_n \text{ s.t. } \|p - G\| \le \|G\|\}$$

is closed & bounded, hence, compact.

The error function $p \to \|p - G\|$ is continuous w.r.t. $p \in B$ and, therefore, achieves its minimum on B.

Uniqueness: Chebyshev Equioscillation Thm.

Key Fact 2. Estimates for Smooth Functions

The error in the best polynomial approximation to G is tightly linked to the "smoothness" or "regularity" of G.

Jackson Thm. Let $G \in C^k[0,1]$ $(k \geq 1)$ and $n \geq k-1$. Then,

$$\inf_{p \in \mathbb{P}_n} \|p - G\| \leq \left(\frac{\pi}{2}\right)^k \frac{1}{(n+1)\cdots(n-k+2)} \|G\|$$

Roughly, the error decreases algebraically

$$\inf_{p \in \mathbb{P}_n} \|p - G\| \leq C_k \frac{\|G\|}{n^k}$$