Analytic methods for list-mode reconstruction

Abhinav K. Jha, Harrison H. Barrett, Eric Clarkson, Luca Caucci, and Matthew A. Kupinski

***Abstract*—List-mode (LM) acquisition of imaging-system data does not suffer from information loss due to data binning. To take advantage of this acquisition mechanism, efficient methods are required to perform object reconstruction using LM data. Current methods to perform reconstruction using LM data reconstruct discrete representation of the object, but since object functions are essentially defined on continuous domains, this leads to information loss. In this paper, we exploit the fact that LM data are defined on a continuous domain, and design analytic methods to reconstruct the object function from LM data. A general procedure to design analytic LM reconstruction algorithms is first formulated. We use this procedure to reconstruct the object function for a linear shift-invariant imaging system with a Gaussian point spread function. We then consider the problem of LM reconstruction in single-photon emission computed tomography (SPECT) imaging systems. We present an analytic method to perform LM reconstruction for a hypothetical SPECT system with an infinite object support and infinite angular sampling. We extend this method to finite angular sampling, and realize that due to the infinite support of the object, the reconstruction cannot be performed using our scheme, but with a finite support, such a reconstruction should be possible. The developed reconstruction schemes can aid in accurate comparison of LM and binned-data acquisition techniques from a task-based perspective.**

I. Introduction

List-mode (LM) acquisition and processing of data is gaining wide popularity for photon-counting imaging systems [1]–[4]. A major advantage of LM acquisition is that they do not suffer from information loss due to binning unlike the more conventional sinogram-based storage and processing of data. However, while LM data contain more information, in the absence of efficient information-retrieval algorithms, this extra information is not of much use. To retrieve this extra information from LM data, often the first step is to design methods to reconstruct the object from the LM data. Algorithms have been developed to reconstruct the object from LM data [1]–[3], but these methods reconstruct discrete representations of the object. The objects in imaging are functions defined on a continuous domain, and reconstructing a discrete representation of the object leads to information loss. A more appropriate methodology is to reconstruct the object as a function defined on a continuous domain. Our primary interest is in performing this object reconstruction for single-photon emission computed tomography (SPECT) imaging systems, but we will keep the problem general.

Interestingly, much literature on reconstruction in SPECT is based on a continuous-continuous (CC) formulation of the SPECT imaging system [5]. These analytic algorithms, such as the well-known analytical filtered-backprojection (FBP) algorithm, reconstruct the object function. However, most SPECT imaging systems bin the data, and therefore, the reconstruction methods developed for CC systems are modified to instead work with discrete image data, and reconstruct discrete object representations [5]–[7]. In this context, LM acquisition presents us with another advantage: It yields data that is defined on a continuous domain, and thus fits the CC formulation of the SPECT imaging system. The primary objective of this work is to design analytic algorithms that can exploit this advantage to reconstruct object functions from LM data defined on a continuous domain.

There are many motivations to design analytic reconstruction algorithms. Analytic algorithms can leverage the true potential of LM data by exploiting the CC nature of LM acquisition, and avoid information loss that would otherwise occur in algorithms that reconstruct discrete object representations. Another advantage of analytic algorithms is that they offer a method to compare information-retrieval techniques without being affected by the limitations of simulation studies, such as discretization requirement and floating-point issues. Analytical algorithms also offer insights on the information content and information-retrieval capacity from the data, which can help improve the design of the imaging system and the algorithm. There are also computational and economic advantages to using analytic approaches [5]. For example, although nonlinear iterative reconstruction algorithms can account for factors such as noise, they require significantly higher computation compared to the one-step analytic methods. Various assessment schemes can compare analytic algorithms with other reconstruction approaches, but these assessment schemes also require development of the analytic method.

We begin with deriving a general framework to reconstruct the object in any imaging system that acquires LM data, and then apply this framework to specific imaging systems.

II. General reconstruction approach

We assume that the object being imaged is a scalar-valued function of spatial position $\boldsymbol{r}$, where $\boldsymbol{r}$ is a vector with s components lying in $\mathbb{R}^s$. We will denote the object by the function $f(\boldsymbol{r})$ and assume that the object function lies in the Hilbert space $\mathbb{L}_2(\mathbb{R}^s)$. This object is viewed over a measurement time τ by some photon-counting imaging system, which detects the photons and then, for each detected photon, estimates attributes such as direction, energy, and the position of interaction of the photon with the detector. For the j^{th} event, these estimated attributes are grouped into a q-dimensional (q-D) vector $\hat{\boldsymbol{A}}_j$. The LM data can be described using the point

A. K. Jha, H. H. Barrett, E. Clarkson, and M. A. Kupinski are with the College of Optical Sciences, University of Arizona, Tucson AZ, USA.

H. H. Barrett, E. Clarkson, L. Caucci, and M. A. Kupinski are with the Department of Medical Imaging, University of Arizona, Tucson AZ, USA.

This work was supported by National Institute of Biomedical Imaging and Bioengineering of National Institute of Health under grant number RC1-EB010974, R37-EB000803 and P41-EB002035.

process $u(\hat{\boldsymbol{A}})$ given by

$$u(\hat{\boldsymbol{A}}) = \sum_{j=1}^{J} \delta(\hat{\boldsymbol{A}} - \hat{\boldsymbol{A}}_j), \tag{1}$$

where $\delta(\ldots)$ denotes the Dirac delta function. Taking the mean of this point process gives [4]

$$\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau) = \int_{\mathbb{S}_f} d^s r \ \tau \mathrm{pr}(\hat{\boldsymbol{A}}|\boldsymbol{r}) s(\boldsymbol{r}) f(\boldsymbol{r}), \tag{2}$$

where $s(\boldsymbol{r})$ denotes the sensitivity of the detector to activity occurring at location $\boldsymbol{r}$ and $\mathbb{S}_f$ denotes the support of the functions in object space. Eq. (2) can be written in operator form as

$$\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau) = [\mathcal{L}\boldsymbol{f}](\hat{\boldsymbol{A}}). \tag{3}$$

where $\mathcal{L}$ denotes the linear LM imaging operator. The kernel for the operator $\mathcal{L}$ is given by

$$l(\hat{\boldsymbol{A}}, \boldsymbol{r}) = \tau \mathrm{pr}(\hat{\boldsymbol{A}}|\boldsymbol{r}) s(\boldsymbol{r}). \tag{4}$$

In LM acquisition, the function $\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau)$ lies in $\mathbb{L}_2(\mathbb{R}^{q-t}) \times \mathbb{E}^t$, where $\mathbb{E}^t$ denotes the t-D Eucledian space, and where $q > t$. Thus, the operator $\mathcal{L}$ maps from the set of functions $f(\boldsymbol{r})$ that lie in $\mathbb{L}_2(\mathbb{R}^s)$ to the set of functions that lie in $\mathbb{L}_2(\mathbb{R}^{q-t}) \times \mathbb{E}^t$. If $\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau)$ lies in $\mathbb{L}_2(\mathbb{R}^q)$, i.e. if $t = 0$, and if $q \geq s$, then it is possible that the operator $\mathcal{L}$ has no null space. This is unlike in binned-data acquisition, where we map from the set of functions lying in $\mathbb{L}_2(\mathbb{R}^s)$ to vectors in the $\mathbb{E}^q$, a mapping that definitely has null space. Even if $\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau)$ lies in $\mathbb{L}_2(\mathbb{R}^{q-t}) \times \mathbb{E}^t$ where $q > t$, the operator $\mathcal{L}$ should have a smaller null space compared to a system in which data is binned. Since the operator $\mathcal{L}$ might have no null space or a reduced null space, LM acquisition provides an avenue to reconstruct the object with lesser information loss compared to binned-data acquisition.

To derive the reconstruction technique, we determine the expression for the pseudoinverse of the LM operator, which requires performing a singular value decomposition (SVD) of the $\mathcal{L}$ operator. Let us denote the singular values and the singular vectors of $\mathcal{L}$ in object and data space by $\mu_i, \boldsymbol{w}_i$, and $\boldsymbol{v}_i$, respectively. The pseudoinverse of the $\mathcal{L}$ operator, which we denote by $\mathcal{L}^+$, can be represented as

$$\mathcal{L}^+ = \sum_{i=1}^{R} \frac{1}{\sqrt{\mu_i}} \boldsymbol{w}_i \boldsymbol{v}_i^\dagger. \tag{5}$$

Our reconstruction approach is to apply this pseudoinverse to the acquired noisy image data:

$$\hat{f}(\boldsymbol{r}) = [\mathcal{L}^+ u](\boldsymbol{r}). \tag{6}$$

The reconstruction in the noise-free case is given by $\hat{f}_{\mathrm{nfree}}(\boldsymbol{r}) = [\mathcal{L}^+ \bar{u}](\boldsymbol{r})$. We can verify that Eq. (6) will lead to this solution in a mean sense, i.e. $\langle \hat{f}(\boldsymbol{r}) \rangle_{u|f} = \hat{f}_{\mathrm{nfree}}(\boldsymbol{r})$, where $\langle \cdots \rangle$ denotes the mean of the quantity inside the parenthesis.

To compute the SVD of $\mathcal{L}$, we must determine the expression for $\mathcal{L}^\dagger \mathcal{L}$, where $\mathcal{L}^\dagger$ denotes the adjoint of $\mathcal{L}$. We can derive the kernel for the $\mathcal{L}^\dagger \mathcal{L}$ operator, which we denote by $k(\boldsymbol{r}, \boldsymbol{r}')$ to be

$$k(\boldsymbol{r}', \boldsymbol{r}) = \tau^2 \int_{\mathbb{S}_u} d^q \hat{A} \ s(\boldsymbol{r}) s(\boldsymbol{r}') \mathrm{pr}(\hat{\boldsymbol{A}}|\boldsymbol{r}) \mathrm{pr}(\hat{\boldsymbol{A}}|\boldsymbol{r}'), \tag{7}$$

where $\mathbb{S}_u$ denotes the LM-data support. We now investigate whether, for specific imaging systems, we can perform the SVD of the $\mathcal{L}$ operator.

III. A SYSTEM WITH GAUSSIAN POINT SPREAD FUNCTION

Let us consider a simple imaging system that satisfies similar assumptions as made in Caucci et al. [4]. The system consists of a 2-D object $f(\boldsymbol{r})$ imaged to a 2-D detector. The LM attributes acquired are the x and y coodinates of the detection. Thus $\hat{\boldsymbol{A}}$ is a 2-D vector, which we henceforth denote by $\hat{\boldsymbol{R}}$. The optics of the imaging system and the detector is assumed to be linear and shift invariant (LSIV), and characterized by Gaussian point spread functions (PSFs). With these assumptions, we can obtain that $\hat{\boldsymbol{R}}$ conditioned on $\boldsymbol{r}$ is normally distributed:

$$\mathrm{pr}(\hat{\boldsymbol{R}}|\boldsymbol{r}) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(\hat{\boldsymbol{R}} - \boldsymbol{r})^2}{2\sigma^2}\right], \tag{8}$$

where σ^2 is the sum of the variances due to the optics of the imaging system and the detector. We also assume that $s(\boldsymbol{r})$ is equal to unity for all $\boldsymbol{r}$. Using Eqs. (4) and (8), we find that

$$l(\hat{\boldsymbol{R}}, \boldsymbol{r}) = \frac{\tau}{2\pi\sigma^2} \exp\left[-\frac{(\hat{\boldsymbol{R}} - \boldsymbol{r})^2}{2\sigma^2}\right], \tag{9}$$

and thus $\mathcal{L}$ resembles a convolution operator. Therefore, the pseudoinverse of $\mathcal{L}$ can be derived to be represented as

$$\mathcal{L}^+ = \frac{1}{\tau} \int d^2\rho \ \exp(2\pi^2\sigma^2\rho^2) \exp\{2\pi i \boldsymbol{\rho} \cdot (\boldsymbol{r} - \hat{\boldsymbol{R}})\}. \tag{10}$$

where $\rho = |\boldsymbol{\rho}|$. Thus, for this imaging system, the object function can be reconstructed by taking the Fourier transform of the acquired data, dividing it by the Fourier transform of kernel of the LM operator at that frequency, and then taking the inverse Fourier transform of the result. In theory, the $\mathcal{L}^+$ operator should be applied to $\bar{u}(\hat{\boldsymbol{A}}|\boldsymbol{f},\tau)$, but in practice, it can be used as in Eq. (6) to obtain $\hat{f}(\boldsymbol{r})$ from $u(\hat{\boldsymbol{A}})$.

IV. TOMOGRAPHIC IMAGING SYSTEM

Consider a tomographic 2-D SPECT imaging system in the x-y plane. The SPECT imaging system consists of a parallel-hole collimator with bores on a regular grid, followed by a 1-D detector. The system rotates about the z axis to acquire data at multiple angles θ, where θ denotes the angle that the detector makes with the x-axis. Let us assume that the parallel-hole collimator accepts photons only normal to the detector surface. Also, let us ignore attenuation in this analysis. In each LM event, the coordinate of the position of interaction of the gamma ray photon with the scintillation crystal is estimated and recorded. We denote the estimated position of interaction by $\hat{p}$, and its corresponding true value by p. The angular orientation of the detector θ is also recorded. To derive the expression for the $\mathcal{L}$ operator for this system, we have to determine the expression for $\mathrm{pr}(\hat{p}, \theta|\boldsymbol{r})$. Using marginal probabilities, $\mathrm{pr}(\hat{p}, \theta|\boldsymbol{r})$ can be written as

$$\mathrm{pr}(\hat{p}, \theta|\boldsymbol{r}) = \mathrm{pr}(\theta|\boldsymbol{r}) \int dp \ \mathrm{pr}(\hat{p}|p, \theta, \boldsymbol{r}) \mathrm{pr}(p|\theta, \boldsymbol{r}). \tag{11}$$

The probability of the position of interaction p given a particular value of the detector angle θ and object location $\boldsymbol{r}$ is simply the delta function $\delta(p-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_\theta)$, where $\hat{\boldsymbol{n}}_\theta$ is the normal to the detector face when the detector is aligned at an angle θ. Thus

$$\mathrm{pr}(p|\theta,\boldsymbol{r})=\delta(p-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_\theta). \tag{12}$$

Let us assume that $\hat{p}$ was estimated using a maximum-likelihood (ML) scheme, where all the scintillation photons were used to estimate the attribute. Then using the asymptotic properties of ML estimates, it can be shown that $\mathrm{pr}(\hat{p}|p,\theta,\boldsymbol{r})$ is normally distributed with the mean given by the true value p and the variance σ_p^2 given by the Cramér-Rao lower bound for the estimate on p [4]:

$$\mathrm{pr}(\hat{p}|p,\theta,\boldsymbol{r})=\frac{1}{\sqrt{2\pi}\sigma_p}\exp\left[-\frac{(\hat{p}-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_\theta)^2}{2\sigma_p^2}\right]. \tag{13}$$

Under the assumption that $s(\boldsymbol{r})$ is unity for all values of $\boldsymbol{r}$, using Eqs. (4), (11)-(13), we can obtain the kernel of the LM operator $\mathcal{L}$ to be

$$l(\hat{p},\theta,\boldsymbol{r})=\tau\mathrm{pr}(\theta|\boldsymbol{r})\frac{1}{\sqrt{2\pi}\sigma_p}\exp\left[-\frac{(\hat{p}-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_\theta)^2}{2\sigma_p^2}\right]. \tag{14}$$

Having derived the general form for the kernel of the LM operator, we now analyze the possibility of the pseudoinverse of the LM operator for some specific cases.

A. Infinite angular sampling and infinite object support

Consider a SPECT imaging system with infinite object support and infinite angular sampling. Due to the isotropic emission of photons, for this system

$$\mathrm{pr}(\theta|\boldsymbol{r})=\frac{1}{2\pi}. \tag{15}$$

Inserting this expression into Eq. (14), the kernel of the LM operator is given by

$$l(\hat{p},\theta,\boldsymbol{r})=\frac{\tau}{2\pi}\frac{1}{\sqrt{2\pi}\sigma_p}\exp\left[-\frac{(\hat{p}-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_\theta)^2}{2\sigma_p^2}\right]. \tag{16}$$

Using Eq. (7), the kernel $k(\boldsymbol{r}',\boldsymbol{r})$ for the $\mathcal{L}^\dagger\mathcal{L}$ operator can be derived to be

$$k(\boldsymbol{r}',\boldsymbol{r})=\left[\frac{\tau}{2\pi}\right]^2\frac{1}{2\sqrt{\pi}\sigma_p}\int d\theta\ \exp\left[\frac{-\{(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}_\theta\}^2}{4\sigma_p^2}\right]. \tag{17}$$

We note that $k(\boldsymbol{r}',\boldsymbol{r})$ is a function of $\boldsymbol{r}-\boldsymbol{r}'$ and therefore, the eigenanalysis of the $\mathcal{L}^\dagger\mathcal{L}$ operator can be performed via Fourier analysis. The eigenvectors of $\mathcal{L}^\dagger\mathcal{L}$ are the complex exponentials given by

$$w(\boldsymbol{\rho}_0)(\boldsymbol{r})=\exp(2\pi i\boldsymbol{\rho}_0\cdot\boldsymbol{r}). \tag{18}$$

The corresponding eigenvalues for these eigenvectors are determined by computing the Fourier transform of the convolution kernel (Eq. (17)). Denoting the Fourier transform of this kernel at frequency $\boldsymbol{\rho}$ by $K(\boldsymbol{\rho})$, we can derive that

$$K(\boldsymbol{\rho})=\frac{1}{\rho}\left[\frac{\tau}{2\pi}\right]^2\exp(-4\pi^2\sigma_p^2\rho^2). \tag{19}$$

Expressing the vectors $\boldsymbol{r}$ and $\boldsymbol{\rho}$ in terms of the basis vectors $\hat{\boldsymbol{n}}_\theta$ and $\hat{\boldsymbol{n}}_{\perp,\theta}$ as $\boldsymbol{r}=r_1(\theta)\hat{\boldsymbol{n}}_\theta+r_2(\theta)\hat{\boldsymbol{n}}_{\perp,\theta}$ and $\boldsymbol{\rho}=\rho_1(\theta)\hat{\boldsymbol{n}}_\theta+\rho_2(\theta)\hat{\boldsymbol{n}}_{\perp,\theta}$, the singular vectors for the $\mathcal{L}$ operator in data space can be derived to be

$$v(\hat{p},\theta)=\sqrt{\rho}\exp(-2\pi i\hat{p}\rho_1(\theta))\delta(\rho_2(\theta)). \tag{20}$$

Using Eqs. (5), (18)-(20) the pseudoinverse of the $\mathcal{L}$ operator is represented as

$$\mathcal{L}^+=\frac{2\pi}{\tau}\int d\rho_1(\theta)\rho_1\exp[2\pi i\rho_1(\theta)(r_1(\theta)+\hat{p})]\exp(2\pi^2\sigma_p^2\rho_1^2). \tag{21}$$

Using Eq. (6), the reconstructed object $\hat{f}(\boldsymbol{r})$ is given by

$$\hat{f}(\boldsymbol{r})=\frac{2\pi}{\tau}\int d\theta\int d\hat{p}\int d\rho_1(\theta)\rho_1(\theta)\times \exp[2\pi i\rho_1(\theta)(r_1(\theta)+\hat{p})]\exp(2\pi^2\sigma_p^2\rho_1^2)u(\hat{p},\theta). \tag{22}$$

B. Finite angular sampling and infinite object support

We now consider a more conventional SPECT system that acquires data at multiple angles θ_j, where the index j varies from 1 to J. For this system, each LM event consists of the position estimate $\hat{p}$ and the detector angle index j. Thus this system maps from a set of functions that lie in the space $\mathbb{L}_2(\mathbb{R}^2)$ to a set of functions that lie in the space $\mathbb{L}_2(\mathbb{R})\times\mathbb{E}$, and the LM operator for this system has the kernel given by $l(\hat{p},j,\boldsymbol{r})$. To determine the expression for this kernel, we consider the general expression given by Eq. (14). The expression for $\mathrm{Pr}(\theta|\boldsymbol{r})$ in this case is equal to $\frac{1}{J}$. Thus the expression for the kernel of the LM operator is given by

$$l(\hat{p},j,\boldsymbol{r})=\frac{\tau}{\sqrt{2\pi}\sigma_pJ}\exp\left[-\frac{(\hat{p}-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_j)^2}{2\sigma_p^2}\right], \tag{23}$$

where $\hat{\boldsymbol{n}}_j$ denotes the normal to the detector surface, when the detector is aligned at angle θ_j. We can derive the kernel of the $\mathcal{L}^\dagger\mathcal{L}$ operator to be

$$k(\boldsymbol{r}',\boldsymbol{r})=\left[\frac{\tau}{J}\right]^2\frac{1}{2\sqrt{\pi}\sigma_p}\sum_{j=1}^{J}\exp\left[-\frac{\{(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}_j\}^2}{4\sigma_p^2}\right]. \tag{24}$$

We again note that $\mathcal{L}^\dagger\mathcal{L}$ resembles a convolution operator, so its eigenvectors are the complex exponentials. The eigenvalues corresponding to these eigenvectors are determined by taking the Fourier transform of the convolution kernel in Eq. (24). These eigenvalues are given by

$$K(\boldsymbol{\rho})=\left[\frac{\tau}{J}\right]^2\sum_{j=1}^{J}\exp(-4\pi^2\sigma_p^2(\boldsymbol{\rho}\cdot\hat{\boldsymbol{n}}_j)^2)\delta(\boldsymbol{\rho}\cdot\hat{\boldsymbol{n}}_{\perp,j}), \tag{25}$$

where we have expressed $\boldsymbol{r}$ and $\boldsymbol{\rho}$ in terms of the basis vectors $\hat{\boldsymbol{n}}_j$ and $\hat{\boldsymbol{n}}_{\perp,j}$ as previously. Due to the delta function, the sum over j exists only when $\boldsymbol{\rho}$ is parallel to $\hat{\boldsymbol{n}}_j$. The delta function complicates further analysis. For example, to find the singular vectors in data space or to determine the expression for $\mathcal{L}^\dagger$, we must divide by the square root of the eigenvalues $K(\boldsymbol{\rho})$, which requires taking the square root of the delta function. We can avoid this issue when determining the singular vectors in data space by obtaining the SVD representation of $\mathcal{L}$ and then using that to compute the data space singular vectors. However, the issue cannot be avoided when computing the pseudoinverse. Thus, it seems unlikely that the pseudoinverse of $\mathcal{L}$ can be determined for this case. A physical interpretation of the absence of the pseudoinverse is the following: The finite angular sampling leads to a set of null functions. Also, the infinite support leads to an infinite number of solutions when $\boldsymbol{\rho}$ is parallel to $\hat{\boldsymbol{n}}_j$, and thus the absence of the pseudo-inverse. Having a finite support for the object might cause this problem to disappear. We consider this case now.

C. Finite support and finite angular sampling

Following a similar treatment that led to Eq. (23) but constraining the object support $\mathbb{S}_f$ to be finite, we obtain the transformation from the object to the image space as:

$$\bar{u}(\hat{p},j) = \frac{\tau}{J}\frac{1}{\sqrt{2\pi}\sigma_p}\int_{\mathbb{S}_f} d^2r \exp\left[-\frac{(\hat{p}-\boldsymbol{r}\cdot\hat{\boldsymbol{n}}_j)^2}{2\sigma_p^2}\right] f(\boldsymbol{r}). \tag{26}$$

Fourier analysis of the $\mathcal{L}^\dagger\mathcal{L}$ operator is not useful in this case due to the finite object support. However, if the detector and the collimator are aligned at equally spaced angles, then this system has a discrete rotational symmetry. This property can be used to obtain the singular vectors, and thus determine the pseudoinverse operator of this system. The basic idea behind the approach is to evaluate the singular vectors of the system for one particular detector orientation, and then use this rotational symmetry to determine the singular vectors of the complete system. For this system, rotating the detector and collimator by θ_j is equivalent to rotating the object by $-\theta_j$. Let $\mathcal{T}_j$ be a functional transform corresponding to the geometric rotation $\mathcal{R}_j$. Then

$$\mathcal{T}_j t(\boldsymbol{r}) = t(\mathcal{R}_j^{-1}\boldsymbol{r}), \tag{27}$$

for an arbitrary function $t(\boldsymbol{r})$. Let us now denote the LM operator at detector orientation of $\theta = 0$, by $\mathcal{L}_0$, and at $\theta = \theta_j$ by $\mathcal{L}_j$. Also, let us denote the complete system matrix, which includes the LM operators at all the angles, by $\mathcal{L}$. Then, we can show that [5]

$$\mathcal{L}_j = \mathcal{L}_0\mathcal{T}_j^\dagger. \tag{28}$$

Therefore, the adjoint of the $\mathcal{L}_j$ operator is given by

$$\mathcal{L}_j^\dagger = \mathcal{T}_j\mathcal{L}_0^\dagger. \tag{29}$$

The adjoint is a back-projection operation that smears the 1-D projection data acquired by the detector back into the 2-D space described by the object support. Therefore, performing the backprojection operation for the data acquired at all the angles amounts to summing up all the backprojections. This leads to an easy representation for the backprojection operator:

$$\mathcal{L}^\dagger = \sum_j \mathcal{L}_j^\dagger = \sum_j \mathcal{T}_j\mathcal{L}_0^\dagger. \tag{30}$$

The expression for the $\mathcal{L}^\dagger\mathcal{L}$ operator is then given by [5]

$$\mathcal{L}^\dagger\mathcal{L} = \sum_j \mathcal{L}_j^\dagger\mathcal{L}_j = \sum_j \mathcal{T}_j\mathcal{L}_0^\dagger\mathcal{L}_0\mathcal{T}_j^\dagger. \tag{31}$$

Therefore, we observe that the $\mathcal{L}^\dagger\mathcal{L}$ operator can be expressed in terms of the $\mathcal{L}_0$ operator. The $\mathcal{L}_0$ operator can also be thought of as the planar imaging system operator. Currently, we are investigating that given this relation, how the singular vectors of $\mathcal{L}_0$ and $\mathcal{L}$ operator are related. We can show that when $\hat{p} = p$, i.e. we estimate the true value of the position of interaction, the singular vectors, and thus the pseudoinverse of the $\mathcal{L}$ operator can be found by following a similar approach as in Davison et al. [8]. However, we need to perform further investigation to derive the reconstruction approach when we account for estimation statistics.

V. Conclusions

In this paper, we have investigated the problem of reconstructing object functions from LM data. We have first suggested a general framework to perform this reconstruction, and then applied this framework to an LSIV imaging system with Gaussian PSF, and to SPECT imaging systems. We have presented the reconstruction solution for a SPECT system with infinite object support and infinite angular sampling. We have also shown that for finite angular sampling but infinite support, the reconstruction cannot be performed using the proposed framework. Finally, we have considered a SPECT system with finite support and finite angular sampling, and shown that for this system, the tomographic LM operator is related to the planar LM operator. We are currently investigating the use of this property to perform the reconstruction for this system. As one of the first investigations on analytic LM reconstruction, we have begun with problems in 2-D tomography, but we are also interested in developing these approaches for 3-D tomography, where this work will be very useful. We are also interested in using the developed reconstruction methods to compare systems that acquire LM data to systems that instead bin the data, by evaluating these systems based on objective measures of image quality. These studies will highlight the usefulness of the information that is not lost when data are stored in LM format.

References

[1] A. K. Jha, E. Clarkson, M. A. Kupinski, and H. H. Barrett, "Joint reconstruction of activity and attenuation map using LM SPECT emission data," in *Proc. SPIE Medical Imaging*, vol. 8668, 2013, pp. 86 681W1–9.

[2] L. Bouwens, R. Van de Walle, H. Gifford, M. King, I. Lemahieu, and R. Dierckx, "LMIRA: list-mode iterative reconstruction algorithm for SPECT," *IEEE Trans. Nucl. Sc.*, vol. 48, no. 4, pp. 1364 –1370, Aug 2001.

[3] L. Parra and H. H. Barrett, "List-mode likelihood: EM algorithm and image quality estimation demonstrated on 2-D PET," *IEEE Trans. Med. Imaging*, vol. 17, pp. 228–235, Apr 1998.

[4] L. Caucci and H. H. Barrett, "Objective assessment of image quality. V. Photon-counting detectors and list-mode data," *J. Opt. Soc. Am. A*, vol. 29, no. 6, pp. 1003–1016, Jun 2012.

[5] H. H. Barrett and K. J. Myers, *Foundations of Image Science*. Wiley, 2004.

[6] J. Qi and R. M. Leahy, "Iterative reconstruction techniques in emission computed tomography," *Phys. Med. Biol.*, vol. 51, no. 15, pp. R541–578, Aug 2006.

[7] P. P. Bruyant, "Analytic and iterative reconstruction algorithms in SPECT," *J. Nucl. Med.*, vol. 43, no. 10, pp. 1343–1358, Oct 2002.

[8] M. Davison and F. Grunbaum, "Tomographic reconstruction with arbitrary directions," *Comm. Pure Appl. Math.*, vol. 34, no. 1, pp. 77–119, 1981.