10-3

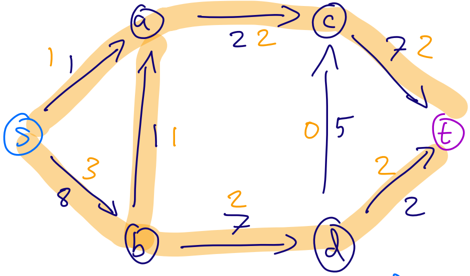
5112

Flow Networks

Ford-Fulkerson Algorithm

$G = (V, E)$

directed graph



Additional Assumptions:

1. $c_e \in \mathbb{N} \quad \forall e \in E$
2. No edges into s or out of t.
3. All $v \in V$ have an incident edge

$$f^{out}(v) = \sum_{e \text{ out of } v} f(e)$$

A flow f

$f: E \to \mathbb{R}$

2 constraints:

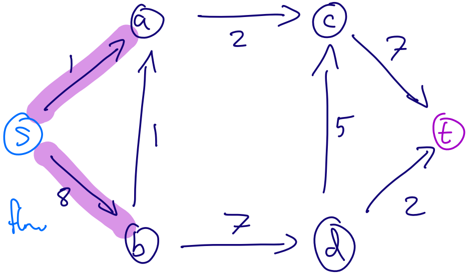
1. Capacity: $0 \le f(e) \le c_e$
2. Conservation: $f^{out}(v) = f^{in}(v)$ for $v \neq s, t$.

How do we find the maximum flow?

Is there a maximum flow? Yes, almost.

Define the value of a flow f:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$



We can upper bound the flow by $C(s) = \sum_{e \text{ out of } s} c_e$.

Start with zero flow f: $f(e) = 0 \ \forall e \in E$.
✓ a flow.

Suppose I have an s-t path P.
Let $\text{bottleneck}(P) = \min_{e \in P} c_e$

Augment f to get a flow f'.

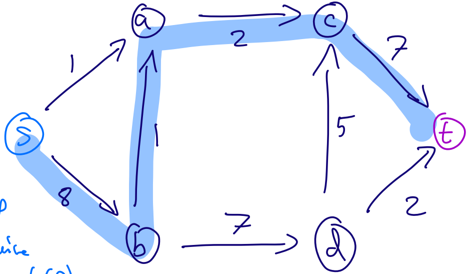
$$f'(e) = \begin{cases} \text{bottleneck}(P) & \text{if } e \in P \\ 0 & \text{otherwise} \end{cases}$$

Capacity: Case $e \in P$ $\quad f'(e) = \text{bottleneck}(P)$

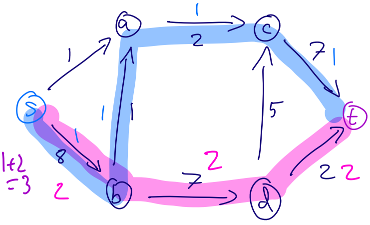
$$= \min_{g \in P} c_g$$

$$\leq c_e \checkmark$$

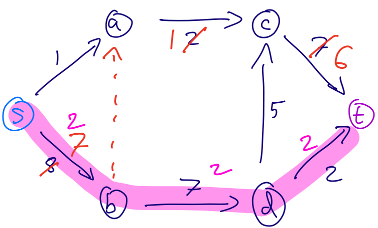
Case $e \notin P$ is easy



Conservation: Every vertex on P has one edge in and one edge out
$\Rightarrow$ same flow in and out. ✓



Starting with a flow f



"Residual" graph G_f

Choose a simple s-t path P in G_f.

$$\tilde{c}_e = c_e - f(e)$$

Augment f with P:

$$f'(e) = \begin{cases} f(e) + \text{bottleneck}(P) & \text{if } e \in P \\ f(e) & \text{otherwise} \end{cases}$$

Starting with a flow f

"Residual" graph G_f

$$\tilde{c}_e = c_e - f(e)$$

Augment f with P:

$$f'(e) = \begin{cases} f(e) + \text{bottleneck}(P) & \text{if } e \in P \\ f(e) & \text{otherwise} \end{cases}$$

Choose a simple s-t path P in G_f.

Capacity:

Case $e \in P$ $\quad f(e) + \text{bottleneck}(P) = f(e) + \min_{g \in P} \tilde{c}_g$

$$\leq f(e) + \tilde{c}_e$$

$$= f(e) + c_e - f(e)$$

$$= c_e \quad \checkmark$$

Case $e \notin P$ $\quad f(e) \checkmark$

Conservation: f is already a flow, so $f^{in}(v) = f^{out}(v)$.

What about f'?

Case $v \in P$:

$$f^{in}(v) = \sum_{e \text{ into } v} f(e) = \sum_{\substack{e \text{ into } v \\ e \notin P}} f(e) + \sum_{\substack{e \text{ into } v \\ e \in P}} f(e)$$

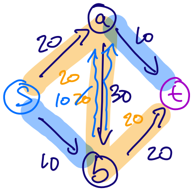
g is the only edge in this sum

$$= \sum_{\substack{e \text{ into } v \\ e \notin P}} f'(e) + f(g)$$

$$f'^{in}(v) = f^{in}(v) + \text{bottleneck}(P).$$

$$= \sum_{\substack{e \text{ into } v \\ e \notin P}} f'(e) + f'(g) - \text{bottleneck}(P)$$

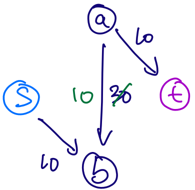
$$= f'^{in}(v) - \text{bottleneck}(P).$$



Use this path P with bottleneck(P) = 20.

to get a flow f.

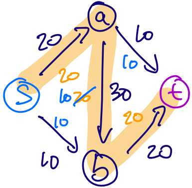
G_f



In the residual graph, no path from s to t.

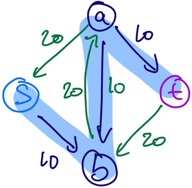
The residual graph G_f:

The vertices of G_f are the same as G.



The edges of G_f are

1. forward edges: edges of G with remaining capacity (i.e. $f(e) \neq c_e$)
   - labelled with $c_e - f(e)$



residual capacity

2. backward edges: reversed edges of G with non-zero flow.
   - labelled with $f(e)$.

Have a flow f, and a simple s-t path P in G_f.

Define f' : $f'(e) = \begin{cases} f(e) + \text{bottleneck}(P) & e \text{ is a forward edge in } P \\ f(e) - \text{bottleneck}(P) & e \text{ is a backward edge in } P \\ f(e) & \text{otherwise.} \end{cases}$

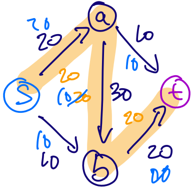
Define bottleneck(P) to be the minimal residual capacity among $g \in P$.

Capacity: Case e is a forward edge.

$$f'(e) = f(e) + \text{bottleneck}(P)$$
$$\leq f(e) + c_e - f(e) = c_e$$

Case e is a backward edge

$$f'(e) = f(e) - \text{bottleneck}(P)$$
$$\geq f(e) - f(e) = 0.$$



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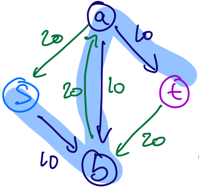

residual capacity

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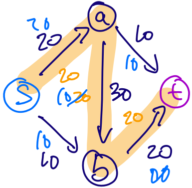

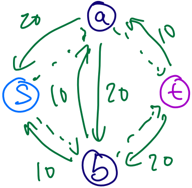

The residual graph G_f:

The vertices of G_f are the same as G.

The edges of G_f are

1. forward edges: edges of G with remaining capacity (i.e. $f(e) \neq c_e$)
   - labelled with $c_e - f(e)$
2. backward edges: reversed edges of G with non-zero flow.
   - labelled with $f(e)$.

residual capacity