Approximation Algorithms

Network Flow

Weighted Vertex Cover

Input: graph $G=(V,E)$, $\{w_x \mid x\in V\}$.

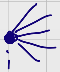
Output: minimal weight set $S\subseteq V$ such that every $e\in E$ is adjacent to some $s\in S$.

Pricing Algorithm

Associate prize p_e to each edge $e \in E$. "Cost of VC distributed among the edges"

Say prizes p_e are fair if $\sum_{e=(i,j)} p_e \leq w_i$.

"No edge is overpaying for its covering vertex"



If S^* is a vertex cover, and p_e is fair,

then $\sum_{e \in E} p_e \leq w(S^*)$

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Know that $\sum_{e=(i,j)} p_e \leq w_i$ ~~for~~ for all nodes $i \in S^*$.

$$\Rightarrow \sum_{i \in S^*} \sum_{e=(i,j)} p_e \leq w(S^*)$$

b/c S^* is a vertex cover,
every edge appears in this sum

$$\Rightarrow \sum_{e \in E} p_e \leq \sum_{i \in S^*} \sum_{e=(i,j)} p_e$$

A vertex i is <u>tight</u> if $\sum_{e=(i,j)} p_e = w_i$

Algorithm:

Set $p_e = 0$ for all $e \in E$

while $\exists e = (i,j)$ s.t. neither i nor j is tight:

    increase p_e until one of i or j is tight

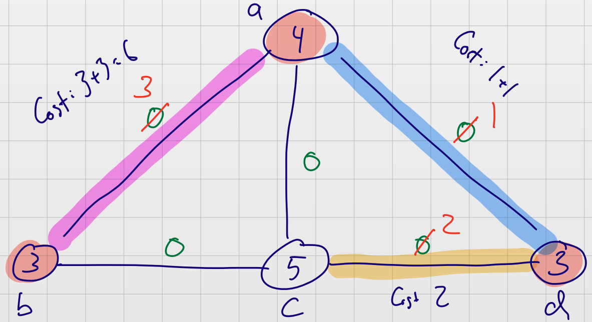
return $S = \{$all tight nodes$\}$

Running time: $O(|E|)$

$= O(|V|^2)$

(w) weight

prizes



$S = \{a, b, d\}$ $w(S) = 10$

S and $\{p_e\}$ returned by the algorithm satisfy

$$w(S) \leq 2 \sum_{e \in E} p_e .$$

All nodes in S are tight, so $\sum_{e=(i,j)} p_e = w_i$

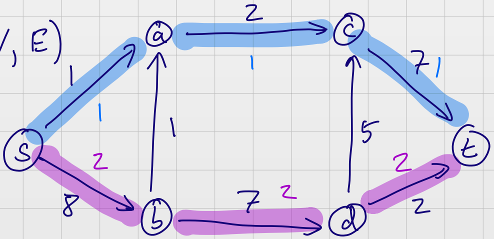
each edge appears at most twice

Summing across S: $w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \leq 2 \sum_{e \in E} p_e$

$$\sum_{e \in E} p_e \leq w(S^*) \leq w(S) \leq 2 \sum_{e \in E} p_e \implies \text{2-approximation}$$

Network Flow

$G = (V, E)$



The value of a flow is

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

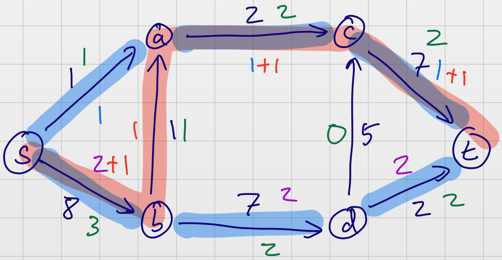
$$f^{out}(v) = \sum_{e \text{ out of } v} f(e)$$

$$f^{in}(v) = \sum_{e \text{ into } v} f(e)$$

A flow $f : E \to \mathbb{R}$

2 constraints: 1. Capacity $0 \le f(e) \le c_e \quad \forall e \in E$

2. Conservation $f^{out}(v) = f^{in}(v) \quad \forall v \in V$ except s, t.

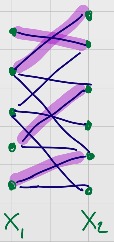


Flow f is the green assignments

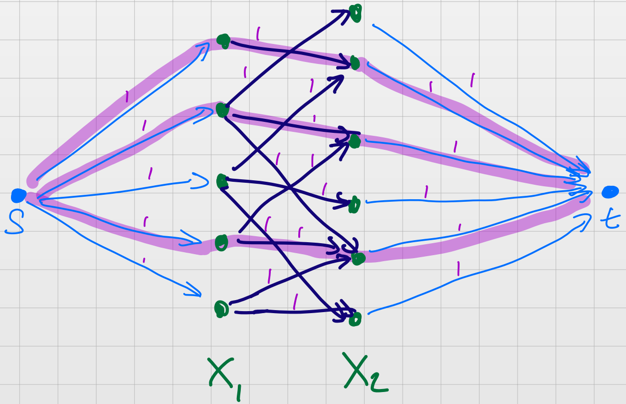
Bipartite Matching Problem

Have a bipartite graph $G = (V, E)$.

<u>Bipartite</u> : $V = X_1 \sqcup X_2$, every edge goes from X_1 to X_2.



A match is a subset $M \subseteq E$ such that no vertex is adjacent to more than 1 edge in M



S
t
X1
X2