Network Flow

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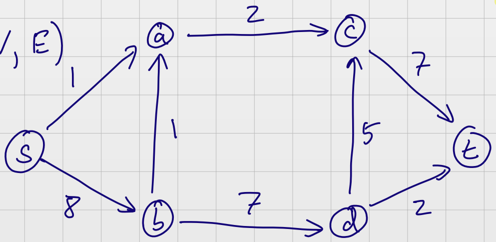
$G=(V, E)$



The value of a flow is

$$v(f) = \sum_{e \text{ out of } s} f(e) = f^{out}(s)$$

$$f^{out}(v) = \sum_{e \text{ out of } v} f(e)$$

$$f^{in}(v) = \sum_{e \text{ into } v} f(e)$$

A flow $f : E \to \mathbb{R}$

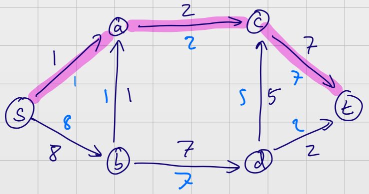
2 constraints: 1. Capacity $0 \le f(e) \le c_e \quad \forall e \in E$

2. Conservation $f^{out}(v) = f^{in}(v) \quad \forall v \in V$ except s, t.

A Greedy Algorithm

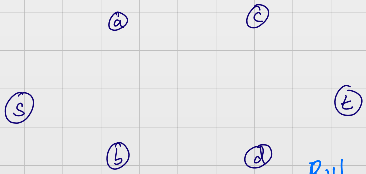
Choose path P in the residual graph and add a flow along the path

Add Bottleneck(P) flow.



Define Bottleneck(P) to be the minimum edge capacity on P in the residual graph.

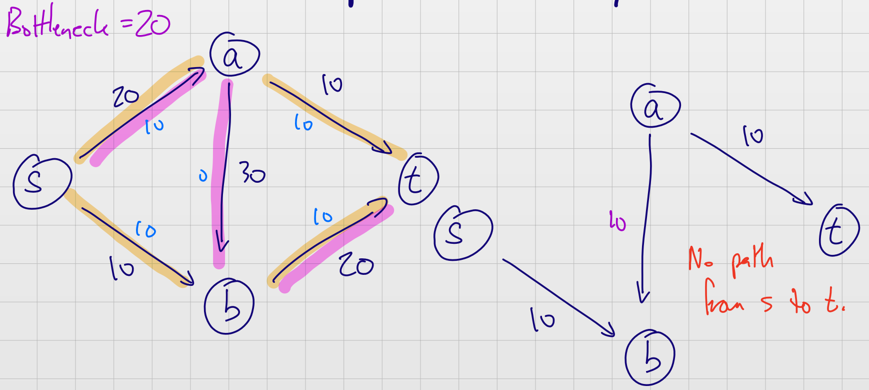
Residual Graph



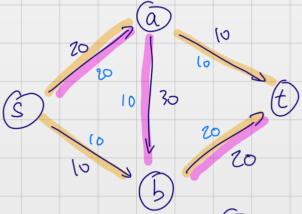
Bottleneck = 2

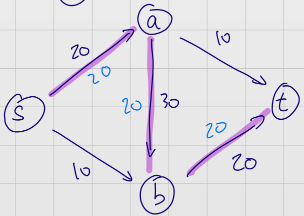
Remaining capacity of each edge

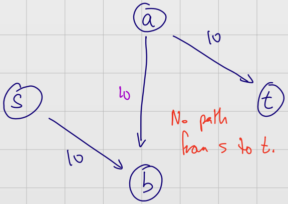
An example where Greedy fails



An example where Greedy fails







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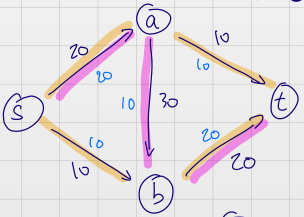
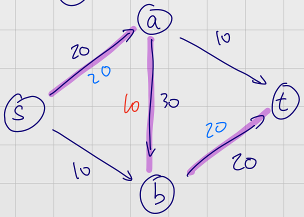
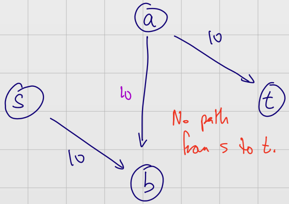






Ford-Fulkerson Algorithm

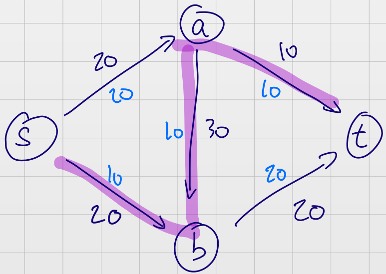
The residual graph G_f for a flow f on G.

Two types of edges:

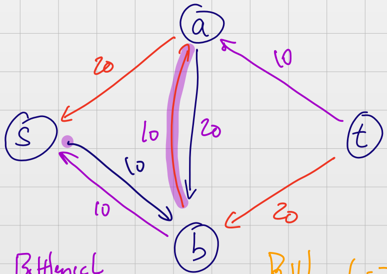
forward edges: edges of G with remaining capacity
- labelled with $c_e - f(e)$.

backward edges: reversed edges of G with non-zero flow
- labeled with $f(e)$

G



Residual graph $G(f)$



Bottleneck = 10

Bottleneck = 20

Ford-Fulkerson Algorithm

Start with 0 flow f.

while $\exists$ an s-t path P in G_f:
    Augment f with Bottleneck(P) flow

return f.

How many times does the loop execute?

Does the loop terminate?

Is the returned flow maximal?

Ford-Fulkerson Algorithm

(n vertices, m edges)

Find P: Use BFS or DFS
$O(n+m) = O(m)$

Start with 0 flow f.

while $\exists$ an s-t path P in G_f:
    Augment f with Bottleneck(P) flow

return f.

$O(m)$

Update G_f:
≤ 2 modifications to each edge in P
$= O(m)$

Each loop iteration is $O(m)$.

Ford-Fulkerson Algorithm

Start with 0 flow f.

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    Augment f with Bottleneck(P) flow

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Upper bound any flow by $C = \sum_{e \text{ out of } s} c_e$.

How many times do we execute the loop? $\boxed{\le C \text{ iterations}}$

Assume all capacities are integers.

$\Rightarrow$ flow in FF is always integer-valued.

Each iteration adds flow (must take a forward edge out of s)

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while $\exists$ an s-t path P in G_f:
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How many times do we execute the loop? $\boxed{\leq C \text{ iterations}}$

$\boxed{\text{Runtime: } O(mC)}$

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