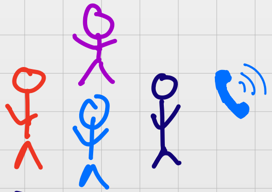
Randomization

Shared Channel

Quick Select

Shared Channel



Every one wants to send a message, but if >1 at a time tries, all fail.

A deterministic algorithm never sends a message!

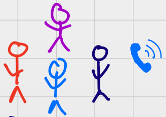
n clients

at each time t, a client can either try to send or not

no feedback

each client executes the same algorithm

Shared Channel



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A randomized algorithm:

each client sends their message at each timestep with prob p.

How long before every client sends their message?

What is the probability that client i succeeds at time t?

1. client i sends their message
2. no one else does

$S\{i,t\}$ is the event that client i succeeds at time t

$A\{j,t\}$ is the event that client j sends at time t.

$$S\{i,t\} = A\{i,t\} \cap \left(\bigcap_{j \neq i} \overline{A\{j,t\}} \right)$$

$$p(S\{i,t\}) = p(A\{i,t\}) \cdot \prod_{j \neq i} p(\overline{A\{j,t\}}) = p(1-p)^{n-1}$$

Maximize $p(1-p)^{n-1}$

$$\frac{d}{dp} p(1-p)^{n-1} = 1\cdot(1-p)^{n-1} - p(n-1)(1-p)^{n-2} = 0$$

Solve for p: $(1-p)^{n-1} = p(n-1)(1-p)^{n-2}$

$$1-p = p(n-1) = pn - p$$

$$pn = 1$$

$$p = \frac{1}{n}.$$

Best choice for p is $\frac{1}{n}$

$$\Rightarrow S\{i,t\} = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

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$$\Rightarrow \quad p(S\{i,t\}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

Fact

1. $\left(1-\frac{1}{n}\right)^{n-1}$ converges monotonically from $\frac{1}{2}$ to $\frac{1}{e}$.
2. $\left(1-\frac{1}{n}\right)^{n}$ converges monotonically from $\frac{1}{4}$ to $\frac{1}{e}$.

$$\Rightarrow \quad \frac{1}{en} \leq p(S\{i,t\}) \leq \frac{1}{2n}$$

Define $F\{i,t\}$ to be the probability that client doesn't succeed in <u>any</u> round from $1, 2, \ldots, t$.

$$F\{i,t\} = \bigcap_{r=1}^{t} \overline{S\{i,r\}}$$

$$p(F\{i,t\}) = \prod_{r=1}^{t} p\left(\overline{S\{i,r\}}\right)$$

$$\leq \left(1 - \frac{1}{en}\right)^{t}$$

Want this to look like $\left(1 - \frac{1}{n}\right)^{n} \ldots$

What if $t = \lceil en \rceil$?

$$= \left(1 - \frac{1}{en}\right)^{\lceil en \rceil}$$

$$\leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$$

Define $F\{i,t\}$ to be the probability that client doesn't succeed in <u>any</u> round from $1, 2, \ldots, t$.

$$F\{i,t\} \leq \left(1 - \frac{1}{en}\right)^t$$

Want $< f$

If $t = xen$

$$= \left(1 - \frac{1}{en}\right)^{x \cdot en}$$

Set $x = \ln\left(\frac{1}{f}\right)$

$$= \left\{\left(1 - \frac{1}{en}\right)^{en}\right\}^x$$

$$\leq \left(\frac{1}{e}\right)^x < \left(\frac{1}{e}\right)^{\ln\left(\frac{1}{f}\right)} = e^{-\ln\left(\frac{1}{f}\right)}$$

$$= \left(\frac{1}{f}\right)^{-1} = f.$$

$\Rightarrow$ Go $t = \ln\left(\frac{1}{f}\right) en$ time steps then succeed w/ prob $\geq 1 - f$.

Define $F_t = \bigcup_{i=1}^{n} F\{i, t\}$

Prob that any client hasn't succeeded in any round $1, \ldots, t$.

Apply Union Bound: $p(F_t) \leq \sum_{i=1}^{n} p(F\{i, t\})$

$$= n\, p(F\{1, t\})$$

If we want this to be less than f, want $p(F[1, t]) < \frac{f}{n}$

Set $f' = \frac{f}{n}$ on last slide

$\Rightarrow$ If $t = \ln\left(\frac{n}{f}\right) en$, then

$p(F\{1, t\}) < \frac{f}{n} \Rightarrow p(F_t) < f.$

If we want any particular $\Theta(1)$ success, go $O(n \log n)$ time steps.

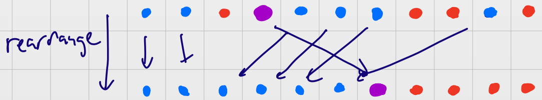
Selection

Input : array of n numbers, $x_1, \ldots, x_n$
Output : the kth smallest.

Simple algorithm : sort, then return kth index $O(n \log n)$

QuickSelect is a randomized algorithm w/
expected $O(n)$ runtime
worst case $O(n)$ runtime.

Idea: Choose a random pivot.
Use divide and conquer.
Only going to recurse on one side.



If $k=4$, then only need to look in the blue numbers.

Random Variables / Expectation

Probability Space S.

A RV associates a value to each outcome.

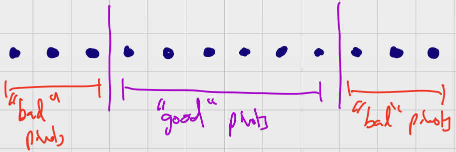
If X is a RV, it's expected value is:

$$E\{X\} = \sum_{s \in S} p(s) X_s$$

Important Property: $E[aX + bY\} = aE\{X\} + bE\{Y\}$

What's the expected running time of QuickSelect?

Define $T\{n, k\}$ to be the expected runtime of k-QS.
Define $T\{n\}$ to be the max over k.

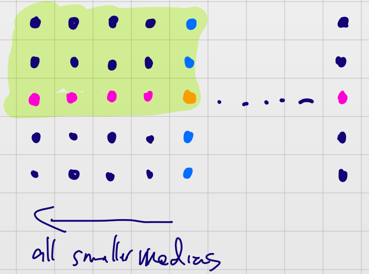


The idea is that good pivots always reduce the problem size to $\leq \frac{3}{4}n$, regardless of k.

$$T\{n\} \leq n + \frac{1}{2}T\left\{\frac{3}{4}n\right\} + \frac{1}{2}T\{n\}$$

$$\Rightarrow \frac{1}{2}T\{n\} \leq n + \frac{1}{2}T\left\{\frac{3}{4}n\right\} \Rightarrow T\{n\} \leq 2n + T\left\{\frac{3}{4}n\right\} \Rightarrow T\{n\} = O(n)$$

QuickSelect in worst case $O(n)$.



Partitioned into groups of 5

Find the median of each group.

Find the median of medians (recursively).