$\boxed{5112}$

10-12

Max Flow

Min Cut

Applications

Ford-Fulkerson Algorithm

Start with zero flow f

while ($\exists$ an s-t path P in R_f)

Augment f with P

Output f.

Find P: BFS or DFS. $O(m+n) = O(m)$

Construct R_f: Create two edges for each edge in G: $O(m)$

$O(m)$

Cost of an iteration: $O(m)$

Why does FF terminate?

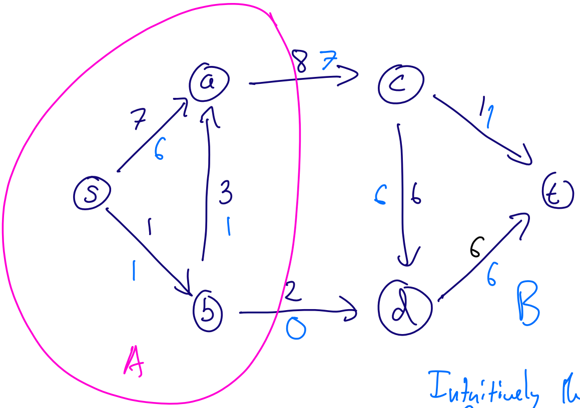
What happens to the flow in a given step?
The flow goes up.

Lemma FF produces integer flows at every time step.

Lemma Max flow is bounded by
$$C = \sum_{e \text{ out of } s} c_e .$$

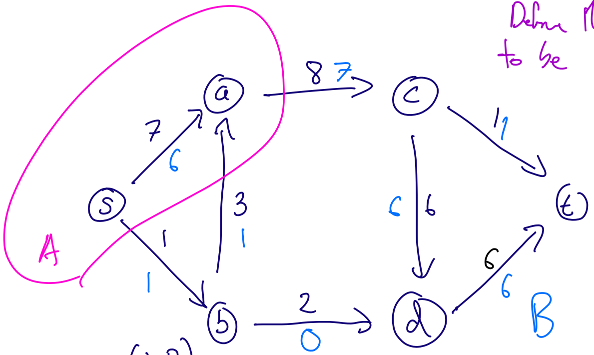
Theorem F terminates after C iterations.

Theorem: FF has runtime $O(mC)$



Intuitively the flow out of A
= the flow out of s
= the flow into t.

Define the flow out of A to be $f^{out}(A) - f^{in}(A)$



An s-t cut (A, B) is a partition of V such that $s \in A$ and $t \in B$.

Intuitively the flow out of A
= the flow out of s
= the flow into t.

Lemma Let (A, B) be an s-t cut. Then

$$v(f) = f^{out}(A) - f^{in}(A).$$

Proof.

0 b/c no flow into s.

$$v(f) = \sum_{e \text{ out of } s} f(e) = f^{out}(\{s\}) - f^{in}(\{s\})$$

$$= \sum_{a \in A} f^{out}(a) - f^{in}(a)$$

$$= f^{out}(A) - f^{in}(A) \quad \square$$

Lemma Let (A, B) be an s-t cut. Then

$$v(f) = f^{out}(A) - f^{in}(A).$$

Lemma Let (A, B) be an s-t cut. Then

$$v(f) \leq \sum_{e \text{ out of } A} c_e = C(A, B)$$

the capacity of the cut.

Lemma Let f be a s-t flow such that there is no s-t path in R_f. Then $\exists$ an s-t cut (A,B) such that $v(f) = c(A,B)$.

Corollary FF produces a max flow.

Corollary For a flow network, the max flow value is equal to the min cut capacity.

Lemma Let f be a s-t flow such that there is no s-t path in R_f. Then $\exists$ an s-t cut (A,B) such that $v(f) = c(A,B)$.

Proof. Let A be the set of all vertices v such that $\exists$ an s-v path in R_f. Let $B = V \setminus A$.
(A,B) is an s-t cut.

Let $e=(u,v)$ be an edge from A to B in G.
WTS $f(e) = c_e$.

If not, then e is a forward edge in R_f.

But we know there is a $s-u$ path P b/c $u \in A$.

But then P together with e is

an $s-v$ path. $\times$

Suppose $e = (u, v)$ is an edge from B into A.

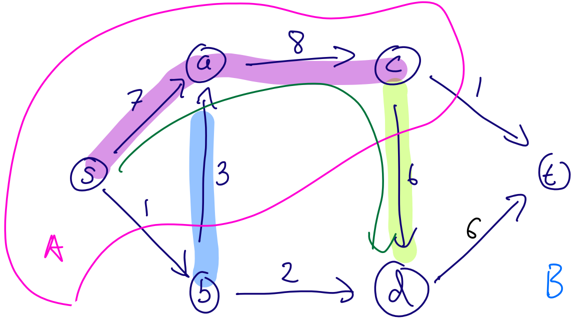
WTS $f(e) = 0$.

If not, then e gives rise to a backward edge in R_f, $e' = (v, u)$

Same thing as above. $\times$

By the earlier lemma $v(f) = f^{out}(A) - f^{in}(A)$

$= C(A, B) - 0$ $\square$.



a
c
s
t
b
d
8
7
1
3
6
1
6
2
A
B