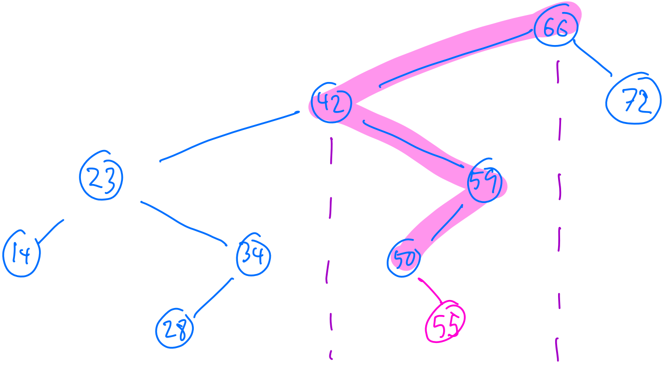
5112

10-3

Treaps

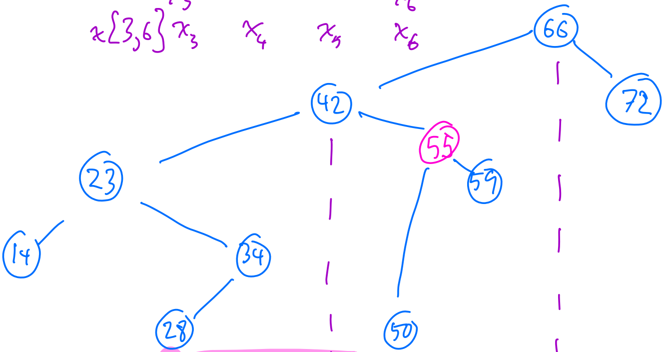
Flow Networks



Key	14	23	28	34	42	50	55	59	66	72
Priority	81	26	95	51	15	70	25	66	3	58

x_3 $\qquad\qquad\qquad\qquad$ x_6

$x\{3,6\}$ x_3 $\quad$ x_4 $\quad$ x_5 $\quad$ x_6



	1	2	3	4	5	6	7	8	9	10
Key	14	23	28	34	42	50	55	59	66	72
Priority	81	26	95	51	15	70	25	66	3	58

Theorem: $E\{\text{depth of a node}\} = O(\log n)$

Define $Y_{i,k} = \begin{cases} 1 & \text{if } x_i \text{ is an ancestor of } x_k \\ 0 & \text{otherwise} \end{cases}$

Lemma: $E\{\text{depth of } x_k\} = E\left[\sum_{i=1}^{n} Y_{i,k}\right] = \sum_{i=1}^{n} E\{Y_{i,k}\}$

Define x_i to be the ith smallest item.

$$x\{i, k\} = \begin{cases} \{x_i, x_{i+1}, \ldots, x_k\} & \text{if } i < k \\ \{x_k, x_{k+1}, \ldots, x_i\} & \text{otherwise} \end{cases}$$

Lemma $Y_{i,k} = 1$ if and only if x_i has the lowest priority among $x\{i, k\}$.

Lemma: $\varphi_{i,k} = 1$ if and only if x_i has the lowest priority among $x\{i,k\}$.

Proof. Assume w.l.o.g. $i < k$.

Assume x_i has the lowest priority among $x\{i,k\}$.

If we consider the recursive tree structure, the first item from $x\{i,k\}$ we pick as a node will be x_i.

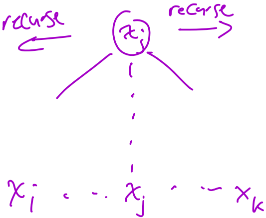
So x_i will be the root of the subtree that includes $x\{i,k\}$.

Thus x_i is an ancestor of x_k.

On the other hand if x_j has lowest priority among $x\{i,k\}$, where $j \neq i$. Then x_j is the root of the subtree containing $x\{i,k\}$.

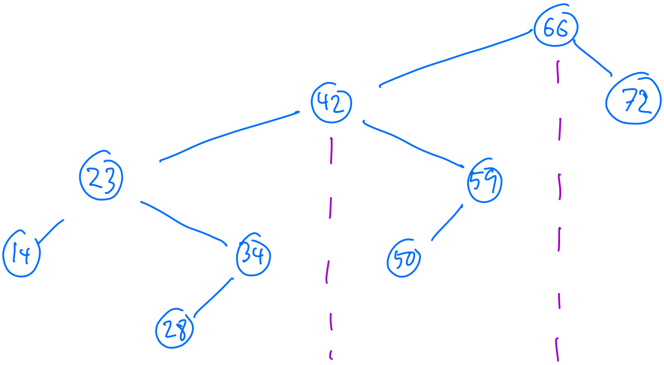
Proof (cont.)

If $i < j < k$, then $\qquad i < j \leq k$



In particular, x_i is not an ancestor of x_k.

$\square$



Key	14	23	28	34	42	50	59	66	72
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What's the probability that

x_i has the lowest priority among $x_i, x_{i+1}, x_{i+2}, \ldots, x_k$?

Answer: $\frac{1}{|k-i|+1}$, because each x_j is equally likely.

$$E\{Y_{i,k}\} = \frac{1}{|k-i|+1}$$

$$E\{\text{depth of } x_k\} = \sum_{i=1}^{n} E\{I_{i,k}\} = \sum_{i=1}^{k-1} \frac{1}{k-i+1} + \sum_{i=k+1}^{n} \frac{1}{i-k+1}$$

subs $j = k-i+1$

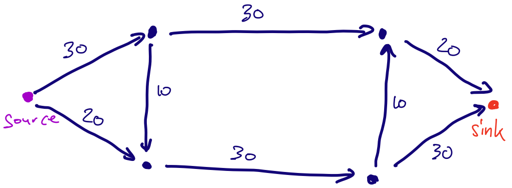
$$= \sum_{j=2}^{k+1} \frac{1}{j} + \sum_{\ell=2}^{n-k+1} \frac{1}{\ell}$$

subs $\ell = i-k+1$

$$\leq 2 \sum_{j=2}^{n} \frac{1}{j} \leq 2H_n = O(\log n)$$

$\square$

Flow Networks

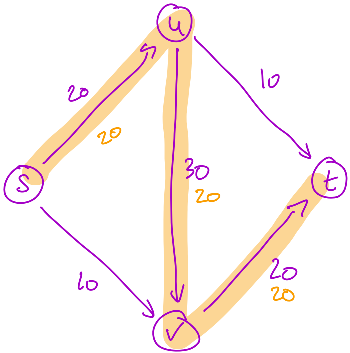


source s For edge e, c_e the capacity.   sink t.

A flow F is assignment $f : E \longrightarrow \mathbb{N}$ satisfying two constraints:

1. Capacity conditions: For each $e \in E$ $\quad 0 \leq f(e) \leq c_e$
2. Conservation conditions: For each $v \neq s, t$,

$$f^{in}(v) = \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e) = f^{out}(v)$$



Can create a flow f.