Randomization

Probability Review

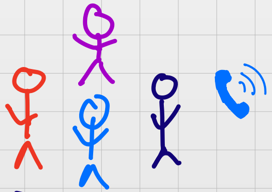
Shared Channel

Quick Select

Randomized Algorithms

- Execution depends on randomness
- Always output the correct answer eventually
- Running time depends on random choices

Shared Channel



Every one wants to send a message, but if >1 at a time tries, all fail.

A deterministic algorithm never sends a message!

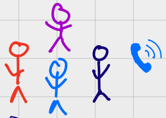
n clients

at each time t, a client can either try to send or not

no feedback

each client executes the same algorithm

Shared Channel



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A randomized algorithm:

each client sends their message at each timestep with prob p.

Probability

A probability space is a collection of outcomes, S, together with probabilities $p(s)$ for $s \in S$ such that

$$0 \leq p(s) \leq 1 \quad \forall s \in S$$
$$\sum_{s \in S} p(s) = 1$$

An event is any subset of a probability space.

Example:

Roll two dice:

Each outcome has prob $1/36$

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

What are some events?

1. any indiv. outcome, e.g., 3, 5 — $1/36$
2. roll a blue 2. — $6/36 = 1/6$
3. arb. subset
4. roll a blue 2 or a red 4 — $11/36$
   - or → union of events
5. roll a blue 2 and a red 4 — $1/36$
   - and → intersection of events
6. not roll a blue 2 — $30/36 = 5/6$
   - not → complement of event

Union Bound

If A and B are events, then

$$p(A \cup B) \leq p(A) + p(B)$$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (B \cap A)$$

$$p(A \cup B) = p(A \setminus B) + p(B \setminus A) + p(B \cap A)$$

$$p(A) = p(A \setminus B) + p(B \cap A) \qquad p(B) = p(B \setminus A) + p(B \cap A)$$

$$p(A \cup B) = p(A) + p(B) - p(B \cap A)$$

Independence & Events

Events A and B, what is $p(A \cap B)$?

	11	12	13	14	15	16	
A	21	22	23	24	25	26	$p(A) = 1/6$
	31	32	33	34	35	36	
B	41	42	43	44	45	46	$p(B) = 1/6$
	51	52	53	54	55	56	
	61	62	63	64	65	66	$p(A \cap B) = 0$

Independence & Events

Events A and B, what is $p(A \cap B)$?

	11	12	13	14	15	16	B
A	21	22	23	24	25	26	
	31	32	33	34	35	36	
	41	42	43	44	45	46	
	51	52	53	54	55	56	
	61	62	63	64	65	66	

$p(A) = 1/6$

$p(B) = 1/12$

$p(A \cap B) = 1/12$ $\quad A \cap B = B$

Independence & Events

Events A and B, what is $p(A \cap B)$?

				B		
	11	12	13	14	15	16
A	21	22	23	24	25	26
	31	32	33	34	35	36
	41	42	43	44	45	46
	51	52	53	54	55	56
	61	62	63	64	65	66

$p(A) = 1/6$

$p(B) = 1/6$

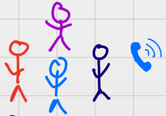
$p(A \cap B) = 1/36 = 1/6 \cdot 1/6 = p(A) \cdot p(B)$

Independence of Events

Events A and B, what is $p(A \cap B)$?

Say A and B are <u>independent</u> if $p(A \cap B) = p(A) \cdot p(B)$

Shared Channel



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How long before every client sends their message?

What is the probability that client i succeeds at time t?

1. client i sends their message
2. no one else does

$S[i,t]$ is the event that client i succeeds at time t

$A[j,t]$ is the event that client j sends at time t.

$$S[i,t] = A[i,t] \cap \left(\bigcap_{j \neq i} \overline{A[j,t]} \right)$$

$$p(S[i,t]) = p(A[i,t]) \cdot \prod_{j \neq i} p(\overline{A[j,t]}) = p(1-p)^{n-1}$$

Maximize $p(1-p)^{n-1}$

$$\frac{d}{dp}\, p(1-p)^{n-1} = 1\cdot(1-p)^{n-1} - p(n-1)(1-p)^{n-2} = 0$$

Solve for p: $(1-p)^{n-1} = p(n-1)(1-p)^{n-2}$

$$1-p = p(n-1) = pn - p$$

$$pn = 1$$

$$p = \frac{1}{n}.$$

Best choice for p is $\frac{1}{n}$

$$\Rightarrow S\{i,t\} = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

Best choice for p is $\frac{1}{n}$

$$\Rightarrow p(S\{i,t\}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

Fact

1. $\left(1-\frac{1}{n}\right)^{n-1}$ converges monotonically from $\frac{1}{2}$ to $\frac{1}{e}$.
2. $\left(1-\frac{1}{n}\right)^{n}$ converges monotonically from $\frac{1}{4}$ to $\frac{1}{e}$.

$$\Rightarrow \frac{1}{en} \leq p(S\{i,t\}) \leq \frac{1}{2n}$$

Define $F\{i,t\}$ to be the probability that client doesn't succeed in <u>any</u> round from $1, 2, \ldots, t$.

$$F\{i,t\} = \bigcap_{r=1}^{t} \overline{S\{i,r\}}$$

$$p(F\{i,t\}) = \prod_{r=1}^{t} p(\overline{S\{i,r\}})$$

$$\leq \left(1 - \frac{1}{en}\right)^{t}$$

Want this to look like $\left(1 - \frac{1}{n}\right)^{n} \ldots$

What if $t = \lceil en \rceil$?

$$= \left(1 - \frac{1}{en}\right)^{\lceil en \rceil}$$

$$\leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$$