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Legendre Functions and Polynomials.

Legendre functions arise as solutions of the de

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

which is called Legendre's differential equation. The general soln of (1) in the case where $n = 0, 1, 2, 3, \cdots$ is given by $y = c_1 P_n(x) + C_2 Q_n(x)$

Where $P_n(x)$ are called Legendre Polynomials and $Q_n(x)$ are called Legendre functions of the second kind which are unbounded at $x = \pm 1$.

Legendre Polynomials

The Legendre polynomials are defined by

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} \cdots\right\} \quad \text{--- (2)}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \frac{n(n-1)\cdots(n-2r+1)}{2\cdot 4 \cdot 2r(2n-1)(2n-3)\cdots(2n-2r+1)} X^{n-2r}$$

$$= \sum_{r=0}^{n/2} (-1)^r \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^r r!\,(n-2r)!} X^{n-2r}$$

or More Consisely

$$P_n(x) = \frac{(2n)!}{2^n n!\, n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}\right]$$

Example 1:- Evaluate the values of

(1) $P_0(x) = \frac{1}{1} = 1$

(2) $P_1(x) = \frac{2!}{2^1\, 1!\, 1!} x^1 = x$

(3) $P_2(x) = \frac{4!}{4\cdot 2!\, 2!}\left[x^2 - \frac{2(1)}{2(4-1)}x\right]$

$$= \frac{4\times 3\times 2}{4\cdot 2\cdot 2}\left[x^2 - \frac{1}{3}\right] = \frac{3}{2}\left[x^2 - \frac{1}{3}\right] = \frac{1}{2}(3x^2-1)$$

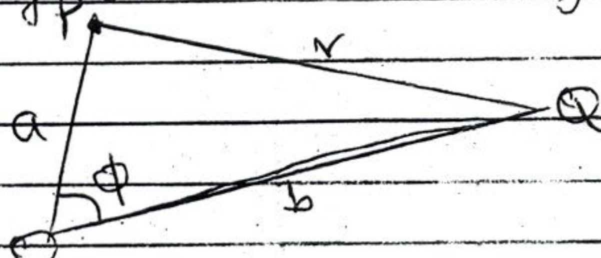
$$P_3(x) = \frac{6!}{2^3\,3!\,3!}\left[x^3 - \frac{3(2)}{2(5)}x + \cdots\right]$$

$$= \frac{\cancel{6}\times5\times\cancel{4}\times\cancel{3!}}{2\,\cancel{8}\times\cancel{3!}\,3!}\left[x^3 - \frac{3}{5}x\right]$$

$$= \frac{5}{2}\left[x^3 - \frac{3}{5}x\right] = \frac{1}{2}\left[5x^3 - 3x\right]$$

Generating Function For Legendre Polynomials

In order to obtain Legendre's results, let us suppose that a particle of mass m is located at point P, which is a units from the origin of our coordinate system (see fig below)



Let the pt Q represent a pt of free space r units from P and b units from the origin O. For the sake of definiteness, let us assume $b > a$. Then from the law of cosines, we find the relation

$$r^2 = a^2 + b^2 - 2ab\cos\phi \quad \text{——— (1)}$$

Where ϕ is the central angle btw the rays $\overline{OP}$ & $\overline{OQ}$. Rearranging the terms and factoring out b^2,

$$r^2 = b^2\left[1 - 2\frac{a}{b}\cos\phi + \left(\frac{a}{b}\right)^2\right], \quad a < b \quad (2)$$

For notational simplicity, we introduce the parameters

$$t = \frac{a}{b}, \quad x = \cos\phi \quad \text{——— (3)}$$

and thus, upon taking the square root,

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$$r = b(1-2xt+t^2)^{1/2} \quad (4)$$

Finally, the substitution of (4) into (5) leads to the potential function V and note that the potential function V depends only upon the radial distance r given by

$$V(r) = \frac{K}{r}, \quad K = \text{constant} \quad (5)$$

$$\Rightarrow \quad V = \frac{K}{b}(1-2xt+t^2)^{-1/2}, \quad 0<t<1 \quad (6)$$

We refer to the function
$w(x,t) = (1-2xt+t^2)^{-1/2}$ as the generating function of the Legendre Polynomials.
Our task at this point is to develop $w(x,t)$ in a power series in the variable t.
Legendre Polynomials.
Recall the binomial series

$$(1-u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-1)^n u^n, \quad |u|<1 \quad (7)$$

Hence, by setting $u = t(2x-t)$, we find that (7) becomes

$$w(x,t) = (1-2xt+t^2)^{-1/2}$$
$$= \sum_{n=0}^{\infty} \binom{-1/2}{n}(-1)^n t^n (2x-t)^n \quad (8)$$

which is valid for $|2xt-t^2|<1$. For $|t|<1$, it follows that $|x| \le 1$. The factor $(2x-t)^n$ is simply a finite binomial series, and thus (8) can further be expressed as

$$w(x,t) = \sum_{n=0}^{\infty} \binom{-1/2}{n}(-1)^n t^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k (2x)^{n-k} t^k$$

or

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$$W(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{n}\binom{n}{k}(-1)^{n+k}(2x)^{n-k}t^{n+k} \quad (9)$$

Since our goal is to obtain a power series involving powers of t to a single index, the change of indice $n \to n-k$ is suggested. Thus recalling absolutely convergent series i.e

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n-k,k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n/2}A_{n-2k,k}$$

Then, (9) can be written in the equivalent form

$$W(x,t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n/2}\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k}(-1)^{n}(2x)^{n-2k}\right]t^{n} \quad (10)$$

The innermost summation in (10) is of finite length and therefore represents a polynomial in x, which happens to be of degree n. If we denote this polynomial by the symbol

$$P_n(x) = \sum_{k=0}^{n/2}\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k}(-1)^{n}(2x)^{n-2k} \quad (11)$$

Then, (10) leads to the intended result.

$$W(x,t) = \sum_{n=0}^{\infty}P_n(x)t^{n}. \quad |x| \leq 1, \ |t| < 1 \quad (12)$$

where $W(x,t) = (1-2xt+t^{2})^{-1/2}$.

Polynomials $P_n(x)$ are called the Legendre Polynomials.

$\Rightarrow$ We know that

$$\binom{-\frac{1}{2}}{n} = (-1)^{n}\binom{n-\frac{1}{2}}{n}$$

$$= (-1)^{n}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}$$

$$= \frac{(-1)^{n}(2n)!}{2^{2n}(n!)^{2}}$$

It follows that the product of binomial coeff in (11) is

$$\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k} = \frac{(-1)^{n-k}(2n-2k)!}{2^{2n-2k}(n-k)!\,k!\,(n-2k)!} \quad (13)$$

and hence (11) becomes

$$P_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!\,(n-k)!\,(n-2k)!} \quad (14)$$

D) Prove that $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

Using binomial theorem

$$(1+v)^p = 1 + pv + \frac{p(p-1)}{2!}v^2 + \frac{p(p-1)(p-2)}{3!}v^3 + \cdots$$

we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= 1 + \frac{1}{2}t(2x-t) + \frac{1.3}{2.4}t^2(2x-t)^2 + \frac{1.3.5}{2.4.6}t^3(2x-t)^3 \cdots$$

and the coeff of t^n in this expansion is

$$= \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}(2x)^n - \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)} \cdot \frac{(n-1)}{1!}(2x)^{n-2} + \frac{1.3.5\cdots(2n-5)}{2.4.6\cdots(2n-4)} \cdot \frac{(n-2)(n-3)}{2!}(2x)^{n-4} \cdots$$

which can be written as

$$= \frac{1.3.5\cdots(2n-1)}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

$$= P_n(x)$$

Hence, $\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt-t^2}}$

Rodrigue's Formula

A representation of the Legendre Polynomials involving differentiation is given by the Rodrigues formula.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right], \quad n=0,1,2\ldots$$

To verify (*), we start with the binomial series

$$\cancel{P_n(x) =}\ (x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}$$

and differentiate n times. Noting that

$$\frac{d^n}{dx^n} x^m = \begin{cases} \frac{m}{(m-n)!} x^{m-n}, & n \le m \\ 0, & n > m \end{cases}$$

We infer

$$\frac{d^n}{dx^n}\left[(x^2-1)^n\right] = \sum_{k=0}^{n/2} \frac{(-1)^k n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$$

$$\frac{d^n}{dx^n}\left[(x^2-1)^n\right] = 2^n n! P_n(x)$$

$$\Rightarrow \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

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Orthogonal Properties of Legendre's Polynomial

Although we have seen and derived many identities associated with the Legendre polynomials, none of these is so fundamental and far-reaching in practice as is the Orthogonality Property

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad m \neq n \qquad (1)$$

Proof:

Since $P_m(x)$, $P_n(x)$ satisfy Legendre's eqn (1)

then,

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

Multiplying the first eqn by P_n, the 2nd eqn by P_m and subtracting, we have

$$(1-x^2)\left[P_nP_m'' - P_mP_n''\right] - 2x\left[P_nP_m' - P_mP_n'\right] = \left[n(n+1) - m(m+1)\right]P_mP_n$$

which can be written

$$(1-x^2)\frac{d}{dx}\left[P_nP_m' - P_mP_n'\right] - 2x\left[P_nP_m' - P_mP_n'\right] = \left[n(n+1) - m(m+1)\right]P_mP_n$$

or

$$\frac{d}{dx}\left\{(1-x^2)\left[P_nP_m' - P_mP_n'\right]\right\} = \left[n(n+1) - m(m+1)\right]P_mP_n$$

Thus by integrating we have

$$\left[n(n+1) - m(m+1)\right]\int_{-1}^{1} P_m(x)P_n(x)\,dx = (1-x^2)\left[P_nP_m' - P_mP_n'\right]\Big|_{-1}^{1} = 0$$

Then since $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

We also verify ~~the~~ another result that

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$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

Proof

From the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Squaring both sides, we have

$$\frac{1}{1-2xt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n}$$

Then by integrating from -1 to 1 we have

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_m(x)P_n(x)\,dx\right\}t^{m+n} \qquad \boxed{m=n}$$

$$-\frac{1}{2t}\ln(1-2xt+t^2)\Big|_{-1}^{1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

OR

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

i.e

$$\sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

Equating Coeffs of t^{2n} we have as required

$$\int_{-1}^{1}P_n^2(x)\,dx = \frac{2}{2n+1}$$

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Hypergeometric Function, Equation

The function $F(a, b; c; x)$ defined by the series $1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots =$

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad \text{---} (1)$$

is called the hypergeometric series. It gets its name from the fact that for $a = 1$ and $c = b$ the series reduces to the elementary geometric series

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{---} (2)$$

Denoting the general term of (1) by $U_n(x)$ and applying the ratio test. We see that

$$\lim_{n\to\infty}\left|\frac{U_{n+1}(x)}{U_n(x)}\right| = \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}x^{n+1}}{(c)_{n+1}(n+1)!} \cdot \frac{(c)_n n!}{(a)_n (b)_n x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}\right|$$

Completing the limit process reveals that

$$\lim_{n\to\infty}\left|\frac{U_{n+1}(x)}{U_n(x)}\right| = |x| .$$

The function

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| <$$

defined by the hypergeometric series is

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Called the hypergeometric function. It is also commonly denoted by the symbol

$$_2F_1(a, b; c; x) \equiv F(a, b; c; x) \quad \text{—} \quad (4)$$

where the 2 and 1 refer to the number of numerator and denominator parameters respectively, in its series representation. The semicolons separate the numerator parameters a and b (which are themselves separated by a comma), the denominator parameter c, and the argument x.

* If c is zero or a negative integer, the series (3) generally does not exist and hence the function $F(a, b; c; x)$ is not defined. However, if either a or b (or both) ~~and hence the~~ is zero or negative ~~function~~ integer, the series is finite and thus converges for all x. When $n > m$,

$= -m$ $(m = 0, 1, 2)$
en $(-m)_n = 0$

and in this case (3) reduces to the hypergeometric polynomial

$$F(-m, b; c; x) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

$$-\infty < x < \infty$$

Example: Find the solution of hypergeometric equation

$$x(1-x)y'' + 4(1-x)y' - 2y = 0$$

Sol:- Recall the hypergeometric equation

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Hypergeometric Equation

The linear second-order DE

$$x(1-x)y'' + [c-(a+b+1)x]y' = aby = 0 \quad (6)$$

is called the HGE of Gauss.

Elementary Properties of HGE

(1) Symmetry property of the parameter a & b

i.e $F(a,b;c;x) = F(b,a;c,x)$ — (7)

(2) The general formula

$$\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k(b)_k}{(c)_k}F(a+k, b+k; c+k; x) \quad k=1,2,3$$

P.T.O

It is so named because the function

$y_1 = F(a,b;c;x)$, $c \neq 0,-1,-2,\ldots$ (10)

is a solution of (6). To verify that (9) is indeed a soln, we can substitute the series for $F(a,b;c;x)$ directly into (6)

Also

$$y_2 = x^{1-c}F(1+a-c, 1+b-c; 2-c; x) \quad c \neq 2,3,4$$

(11)

(5) is a second soln of (6). For $c = 2,3,4,\ldots$, the HGF in (11) are linearly independent of (10) and

$$y = C_1F(a,b;c;x) + C_2x^{1-c}F(1+a-c, 1+b-c; 2-c; x)$$

(12)

is a general soln of (6)

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Second, by differentiating the series (3) termwise, we find that

$$\frac{d}{dx}F(a,b;c;x) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^{n-1}}{(n-1)!}$$

If $n = n+1$

$$\frac{d}{dx}F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \frac{x^n}{n!}$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n} \frac{x^n}{n!}$$

and Hence

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1, b+1; c+1; x)$$

Repeated application of (8) leads to the general formula

$$\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k (b)_k}{(c)_k} F(a+k, b+k; c+k; x)$$

Example:- Find the solution of hypergeometric equation $x(1-x)y'' + 4(1-x)y' - 2y = 0$

Soln

Recall hypergeometric equation

$$x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0$$

Comparing (1) and (2), we get

$a+b+1 = 4$, $c = 4$, $ab = 2$

Solving eqn (3), we have

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$a=1,\ b=2,\ c=4$ or $a=2,\ b=1,\ c=4$

For the first choice the soln is

$F(1,2,4,x)$ i.e

$y_1 = F(1,2,4,x) = F(2,1,4,x)$ by symmetry ppty.

$= 1 + \frac{x}{2} + \frac{3}{10}x^2 + \frac{1}{5}x^3 + \frac{1}{7}x^4 + \cdots$ from (1)

For the second choice, the soln is

$y_2 = x^{-3}F(-2,-1,-2,x) = x^{-3}(1-x)$

(8)

$\therefore y = AF(a,b;c;x) + Bx^{1-c}F(1+a-c, 1+b-c; 2-c; x)$

$y = AF(1,2,4,x) + B\left(\frac{1-x}{x^3}\right)$ where A & B are arbitrary constants.

(9) Example:- If $F(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(c)_m m!}$, show that

(1) $F(1,b;b;x) = (1-x)^{-1}$

Sol $F(1,b;b;x) = \sum_{m=0}^{\infty} \frac{(1)_m \cancel{(b)_m} x^m}{\cancel{(b)_m} m!} = \sum_{m=0}^{\infty} (1)_m \frac{x^m}{m!}$

(1)

$= \sum \frac{\cancel{m!} x^m}{\cancel{m!}} = \sum_{m=0}^{\infty} x^m = (1-x)^{-1}$

(2) (2) $\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k (b)_k}{(c)_k} F(a+k, b+k; c+k; x)$

Sol

$= \frac{ab}{c} \sum \frac{(a+1)_n (b+1)_n}{(c+1)_n} \frac{x^n}{n!} = \frac{ab}{c} F(a+1, b+1; c+1; x)$

(3) $\frac{d^2}{dx^2}F(a,b;c;x) = \frac{(a)_2 (b)_2}{(c)_2} F(a+2, b+2; c+2; x)$

$\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k (b)_k}{(c)_k} F(a+k, b+k; c+k; x)$

Ass (3) $\arcsin x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$

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Sturm - Liouville Systems.

Defn :- A second ordered d.e of the form

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \quad x \in [a,b], \; a \le x \le b$$

$a_1 y(a) + a_2 y'(a) = 0, \; b_1 y(b) + b_2 y'(b) = 0$

with $p(x)$, $q(x)$ and $r(x)$ specified given function such that and a_1, a_2, b_1 and b_2 are given constants and assume to be differentiable and λ is an unspecified parameter indept of x, is called a S.L bvp. or SL system.

A non-trivial soln of this system, i.e one which is not identically zero, exists in general only for a particular set of values of the parameter λ. These values are called xtic values, or eigenvalues of the system. In general to each eigenvalue there is one eigen function, although exceptions can occur.

If $p(x)$ and $q(x)$ are real, then the eigenvalue are real. Also the eigenfunctns form an orthogonal set wrt the density function $r(x)$ generally taken as non-negative i.e $r(x) \ge 0$.

Defn: The functions of a sequence $\{\phi_n(x)\}$ are said to be orthogonal wrt the density function $r(x)$ on the interval $[a,b]$ if

$$\int_a^b \phi_m(x)\,\phi_n(x)\,r(x)\,dx = 0 \quad \forall \; m \ne n.$$

Defn: The norm of sequence of function $\phi_n(x)$ wrt the density function $r(x)$ on the interval $[a,b]$ is

$$\|\phi_n(x)\| = \left[\int_a^b \phi_n^2(x)\,r(x)\,dx\right]^{1/2} \quad \forall \; m \ne n.$$

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Properties of SL systems.

1. Theorem 1: Eigenvalues of SL problem are real

Proof:-

We have $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0$ — ①

$\alpha_1 y(a)+\alpha_2 y'(a)=0,\ \beta_1 y(b)+\beta_2 y'(b)=0$ — ②

Taking the complex conjugate of λ & y

$\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right]+\left[q(x)+\bar{\lambda} r(x)\right]\bar{y}=0$ — ③

$\alpha_1 \bar{y}(a)+\alpha_2 \bar{y}'(a)=0,\ \beta_1 \bar{y}(b)+\beta_2 \bar{y}'(b)=0$ — ④

Multiplying eqn ① by $\bar{y}$, ③ by y and subtracting, we have

$\frac{d}{dx}\left[p(x)(y\bar{y}'-\bar{y}y')\right]=(\lambda-\bar{\lambda})r(x)y\bar{y}dx$

Integrating from a & b, we have

$(\lambda-\bar{\lambda})\int_a^b r(x)|y|^2dx=p(x)(y\bar{y}'-\bar{y}y')\Big|_a^b=0$ — ⑤

Using the conditions ② & ④. Since $r(x)\geq 0$ and is not identically zero in (a,b), the integral on the left of ⑤ is positive and distinct eigen value then, $\int_a^b r(x)|y|^2dx=0$

$\therefore \lambda-\bar{\lambda}=0 \Rightarrow \lambda=\bar{\lambda}$

2. Theorem 2:- If λ_m and λ_n are two distinct eigenvalues of a SL system, with corresponding eigenfunctions y_m and y_n, then y_m and y_n are orthogonal.

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Proof:

$$\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]+\left[q(x)+\lambda_m r(x)\right]y_m=0 \quad (1)$$

$$\alpha_1 y_m(a)+\alpha_2 y_m'(a)=0,\ \beta_1 y_m(b)+\beta_2 y_m'(b)=0 \quad (2)$$

$$\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right]+\left[q(x)+\lambda_n r(x)\right]y_n=0 \quad (3)$$

$$\alpha_1 y_n(a)+\alpha_2 y_n'(a)=0,\ \beta_1 y_n(b)+\beta_2 y_n'(b)=0 \quad (4)$$

Multiplying (1) by y_n, (3) by y_m and subtracting,

$$\frac{d}{dx}\left[p(x)(y_m y_n'-y_n y_m')\right]=(\lambda_m-\lambda_n)r(x)y_m y_n$$

Integrating from a to b, and using (2) & (4)

$$(\lambda_m-\lambda_n)\int_a^b r(x)y_m y_n\,dx = p(x)(y_m y_n'-y_n y_m')\Big|_a^b$$

$$=0$$

and since $\lambda_m\neq\lambda_n$ we have the required

$$\int_a^b r(x)y_m y_n\,dx=0.$$

Prof Chukwu 21/4/18

$(\overset{a}{0},\overset{b}{1})$

Example

(1) Verify that the system $y''+\lambda y=0$, $y(\overset{a}{0})=0$, $y'(\overset{b}{1})=0$ is a Sturm-Liouville System

Sol

Considering the SL de system

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0$$

$\therefore$ $p(x)=1$, $q(x)=0$, $r(x)=1$, $a=0$, $b=1$, $\alpha_1=1$, $\alpha_2=0$, $\beta_1=1$, $\beta_2=0$ and thus (1) is a SL System

(2) Determine the eigenvalues and eigenfunctions of the periodic SL equation $y'' + \lambda y = 0$, — (1)
$-\pi < x < \pi$, with conditions $y(-\pi) = y(\pi)$, — (2)
$y'(-\pi) = y'(\pi)$.

Sol

Let $y = e^{mx}$, the eqn (1) becomes
$(m^2 + \lambda) e^{mx} = 0$ —— (3)

Since $e^{mx} \neq 0$, then
$m^2 + \lambda = 0 \Rightarrow m = \pm i\sqrt{\lambda}$ —— (4)

Consequently, $y(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x$ — (5)

Using the periodic end conditions, we have

$$\left.\begin{aligned}(2\sin\sqrt{\lambda}\, x) B &= 0\\ (2\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi) A &= 0\end{aligned}\right\} \quad (6)$$

Thus to obtain a non-trivial soln, we must have
$\sin\sqrt{\lambda}\,\pi = 0$, $A \neq 0$, $B \neq 0$ —— (7)
$\Rightarrow \lambda_n = n^2$, $n = 1, 2, \ldots$

Since $\sin\sqrt{\lambda}\,\pi = 0$ is satisfied for arbitrary A and B, we obtain two linearly indept eigenfunctions, $\cos nx$, $\sin nx$ corresponding to the same eigenvalue n^2.

It can be readily shown that if $\lambda = 0$, the soln of the SL eqn does not satisfy the periodic end conditions.

However, when $\lambda = 0$, the corresponding eigenfunction is 1.

Thus, the eigenvalues of the periodic SL eqns are (0), $\{n^2\}$, and the corresponding eigenfunctions are 1, $\{\cos nx\}$; $\{\sin nx\}$, when n is a +ve integer.

Question.....................................
write on both sides of the paper

Ass: Find the eigenvalues and the corresponding
① eigen functions $y'' + \lambda y = 0$, $y(0) = 0$
$y(1) = 0$.

② Prove that the eigen functions are orthogonal in $(0,1)$.