MAPS (FUNCTIONS)

1 Definition: A map f from set A to set B denoted by $f: A \longrightarrow B$ is a rule that assigns to each element in A exactly one ~~(a unique)~~ element in B.

- A is called the domain of f. B its codomain.
- If $c \in A$ is assigned to $d \in B$, under the map f, then $f(c) = d$, we say the image of c is d, and call c a pre-image for d.
- d is a dependent variable whose value is determined by the independent variable c and the rule f specifies.

Since a map is a single-valued relation, we have the ~~follow~~ following alternative definition.

2 Definition: A map $f: A \longrightarrow B$ is a relation contained in the cartesian product $A \times B$ such that ~~if~~

i ~~c~~ if $c \in A$, then there exists an element $d \in B$ such that $(c, d) \in f$ [image of each element in A exists in B],

ii if $(c, d), (c, e) \in f$, then $d = e$ [the image of an element (in A) is unique (in B)].

Example

Determine if the following are functions.

a $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x + 2$

b $f: \mathbb{Z} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2$

c $f: \mathbb{Z} \longrightarrow \mathbb{R}$ $\quad z \longmapsto \sqrt{z}$ (z is mapped to $\sqrt{z}$)

d $f: \mathbb{Z} \longrightarrow \mathbb{C}$ $\quad z \longmapsto \sqrt{z}$ ($f(z) = \sqrt{z}$).

Solution

A rule is a function/map if ~~for any x in the do~~

i for any x in the domain, $f(x)$ exists in the codomain, ii if $x_1 = x_2$ in the ~~codomain~~ domain implies $f(x_1) = f(x_2)$ [identical elements in the domain have identical images in the codomain]. A rule that satisfies condition (ii) is said to be well defined.

a i Given $x \in \mathbb{Z}$, $f(x) = x+2 \in \mathbb{R}$ (This point is so obvious, I don't need to add $x+2 \in \mathbb{Z} \subseteq \mathbb{R}$).

ii Suppose $x_1 = x_2 \in \mathbb{Z}$, then

$$x_1 + 2 = x_2 + 2,$$

therefore $f(x_1) = f(x_2)$

Since the image of each element in the domain exists uniquely in the codomain, f is a map.

Exercise. What is f of x when x is i) 3, ii) -2, iii) 1, iv) 0 v) π vii) -7

b i ~~f(x)~~ $f: \mathbb{Z} \to \mathbb{R}$ ~~f(x)~~ $x \mapsto x^2$

Given $x \in \mathbb{Z}$, $f(x) = x^2 \in \mathbb{R}$ since $\mathbb{R}$ contains the square of all integers.

ii Suppose $x_1 = x_2 \in \mathbb{Z}$, then

$$x_1^2 = x_2^2$$

and $f(x_1) = f(x_2)$

f is a well defined map. ~~wf~~

4 Note that $f(2) = 2^2 = 4 \in \mathbb{R}$.

If $x_1 = 2$ and $x_2 = 2$, then $x_1^2 = x_2^2 = 4 = f(x_1) = f(x_2)$. We cannot test well defined here with $x_1 = 2$, $x_2 = -2$ since $2 \neq -2$.

c) $f: \mathbb{Z} \to \mathbb{R}$ $z \mapsto \sqrt{z}$

Given $z \in \mathbb{R}$, does $f(z) = \sqrt{z}$ always exist in $\mathbb{R}$

No. Counter example: $z = -1 \in \mathbb{Z}$, but $f(-1) = \sqrt{-1} = i$ $\notin \mathbb{R}$.

Since f fails one of the two necessary conditions of a function, it is not a function.

d $f: \mathbb{Z} \to \mathbb{C}$ $f(z) = \sqrt{z}$

i Given $z \in \mathbb{Z}$, then $f(z) = \sqrt{z} \in \mathbb{C}$, since $\sqrt{z} \in \mathbb{R} \subseteq \mathbb{C}$ if z is positive, and $\sqrt{z} \in i\mathbb{R} \subseteq \mathbb{C}$ if z is negative.

~~Suppose $z_1 = z_2$ in $\mathbb{Z}$, if $f(z_1) = f(z_2)$ for all~~

Is $f(z_1) = f(z_2)$ for all $z_1 = z_2 \in \mathbb{Z}$?

Let $z_1 = 4$ and $z_2 = 4$, Then ~~$z_1 = z_2$~~ and

$f(z_1) = \sqrt{4}$ $= 2$, $f(z_2) = \sqrt{4}$ $= -2$

Since $z_1 = z_2$ but $f(z_1) \neq f(z_2)$, f is not a map since it is not well defined.

Exercise.

Determine if the following are maps

a) $f: \mathbb{R}^{\geq 0} \to \mathbb{R}$: $f(x) = \sqrt{x}$

b) $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto |x|$

c) $f: \mathbb{Z} \to \mathbb{Z}$ $f(z) = z^2 - z - 6$ shown

i Given $z \in \mathbb{Z}$ $f(z) = z^2 - z - 6 \in \mathbb{Z}$, since $z^2, -z, -6 \in \mathbb{Z}$ and $\mathbb{Z}$ contains the addition of all its elements

ii Suppose $z_1 = z_2 \in \mathbb{Z}$, then

$$z_1^2 = z_2^2 \quad \text{and}$$
$$z_1^2 - z_1 = z_2^2 - z_2 \quad \text{and}$$
$$z_1^2 - z_1 - 6 = z_2^2 - z_2 - 6.$$

Therefore $f(z_1) = f(z_2)$. f is a map.

d $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^3 + 3x - 7$.

e $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \ln x$ f) $f: \mathbb{Q} \to \mathbb{Q}$ $f(x) = \frac{3}{2x-5}$

(Range)

Definition in words: Image of a function

The image/range of $f: A \to B$ is a subset of B consisting of elements that are images of elements in A.

~~Image $f = \{y \in B : x \in A\}$~~

Symbol: Image $f = \{y \in B : f(x) = y,$ when $x \in A\}$.

* I like definitions in words & in symbols. Symbolic representations show there is some understanding of how the maths should go. I'll be more inclined to give someone who writes ~~the~~ a definition in symbol full marks than someone who just writes it in words.

Examples

Find the range of

a) $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x+2$

b) $f: \mathbb{Z} \to \mathbb{R}$ " " $f(x) = x^2$

c) $f: \mathbb{Z} \to \mathbb{Z}$ " " $f(z) = z^2 - z - 6$.

Solution

$w \in$ Codomain of f belong to the range of f if there exist $v \in$ domain f and $w = f(v)$.

a) $f: \mathbb{R} \to \mathbb{R}$ $\quad x \mapsto x+2$.

Let $w \in \mathbb{R}$ as codomain

Suppose $f(x) = w$ (we do not yet know if any such x exist in $\mathbb{R}$ (as domain) or not).

then $x + 2 = w$

then $x = w - 2$.

Since $w - 2 \in \mathbb{R}$ whenever $w \in \mathbb{R}$, every w in the codomain of f has a preimage $(w-2)$ in the domain of f.

Therefore Image/Range of $f = \mathbb{R}$ (the entire codomain).

Assume & check for existence of x in $\mathbb{R}$ or for a contradiction.

b) Let $w \in \mathbb{R}$, suppose $f(x) = w$, then

$x^2 = w$ ~~for $x \in \mathbb{Z}$~~, and

$x = \sqrt{w}$.

$x = \sqrt{w} \in \mathbb{Z}$ if and only if w is a square of an integer. If w is not an integer square in $\mathbb{R}$, there would be no preimage $\sqrt{w}$ in $\mathbb{Z}$ for w. Therefore the range of $f \neq \mathbb{R}$; rather

Image of f / Range of f $= \{ w \in \mathbb{R} : \sqrt{w} \in \mathbb{Z} \}$ or

$= \{ w \in \mathbb{R} : x^2 = w,\ x \in \mathbb{Z} \}$

c $f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(z) = z^2 - z - 6$

Let $y \in \mathbb{Z}$ (as codomain).

Suppose $f(z) = y$, then

$$z^2 - z - 6 = y$$

$$\cancel{z^2 = y + 6 + z} \qquad z^2 - z = y + 6.$$

$$\cancel{z = \sqrt{y + 6 + z}} \qquad \text{Therefore}$$

Image $f = \{ y \in \mathbb{Z} : y + 6 = z^2 - z,\ z \in \mathbb{Z} \}$ or

$= \{ y \in \mathbb{Z} : y = z^2 - z - 6,\ z \in \mathbb{Z} \}$

Note that Image $f \neq$ Codomain $\mathbb{Z}$. ~~since~~

Image of $f \subseteq [-6, \infty) \cap \mathbb{Z}$ since

$z^2 - z \geq 0$ for all $z \in \mathbb{Z}$ ~~and~~

therefore $z^2 - z - 6 \geq -6$ for all $z \in \mathbb{Z}$.

Types of Map:

1 Words: A function $f: A \to B$ is surjective (onto) if its range is the entire codomain. That is, ~~given b~~ each element in the codomain has a preimage in the domain.

Symbols: $f: A \to B$ is surjective if and only if given $b \in B$, there exist $a \in A$ such that $f(a) = b$.

Example

a) $f: \mathbb{R} \to \mathbb{R}$ $(x \mapsto x + 2)$ is surjective, since for any $w \in$ codomain $\mathbb{R}$, there exist preimage $w - 2 \in$ domain $\mathbb{R}$ and $f(w-2) = w$.

b $f: \mathbb{Z} \to \mathbb{R}$ $(f(x) = x^2)$ is not surjective. Counterexam

$5 \in \mathbb{R}$ has no preimage $x \in \mathbb{Z}$. No integer x, for which $x^2 = 5$.

c $f: \mathbb{Z} \to \mathbb{Z}$ $(z \mapsto z^2 - z - 6)$ is not surjective since $m < -6$ $\in \mathbb{Z}$ (codomain) is not in the range of ~~Z~~ f.

d Determine if the map $f: \mathbb{R} \to [0, \infty)$ defined as $f(x) = x^2$ is surjective.

Take $y \in [0, \infty)$, suppose $f(x) = y$, then

$x^2 = y$,

$\cancel{y =}$ $\sqrt{x^2} = \sqrt{y}$

$= \pm x = \sqrt{y}$.

Since $\pm x \in \mathbb{R}$ are preimages ~~for~~ for y $(f(\pm x) = (\pm x)^2 = x^2 = y)$, $y \in$ range of f.

~~Since~~ Therefore codomain $[0, \infty) \subseteq$ range of f.

And f is surjective.

Note: Given any map $f: A \to B$, the image/range of f is always a subset of the codomain B, but if the codomain is ~~also~~ a subset of the range, then range of f = codomain of f and f is surjective.

Exercises.

1 Find the range of the following functions

i $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin x$

ii $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 3\cos x$

iii $T: \mathbb{R}^2 \to \mathbb{R}$ $T(x, y) = x$.

iv $f: \mathbb{C} \to \mathbb{C}$ $f(x) = x^2 - 3x - 10$.

2. Which of the functions above is surjective?

3 Which of the following functions is surjective.

i $f: \{1, 2, 3\} \to \{a, b\}$, $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$.

ii $g: \{1, 2, 3, 4\} \to \{x, y, z\}$, $g(1) = x$, $g(2) = y$, $g(3) = y$, $g(4) = x$.

iii $f: \mathbb{R} \to \mathbb{R} \quad x \mapsto x^2+1$

iv $g: \mathbb{R} \to [1, \infty) \quad x \mapsto x^2+1$

v $h: \mathbb{R} \to \mathbb{R} \quad h(x) = x^3$

vi $f: \mathbb{Z} \to \mathbb{Z} \quad f(z) = 2z$

vii $g: \mathbb{Z} \to \mathbb{Z} \quad g(z) = n+1$

viii $h: \mathbb{N} \to \mathbb{N} \quad h(n) = n+1$

2 Injective Maps (One to One): A map $f: A \to B$ is injective if each element in the range of f has exactly one preimage in the domain of f.

If $f(m) = f(n)$, then $m = n$.

Examples: Determine if the following functions are injective or not.

1. $f: \mathbb{R} \to \mathbb{R} \quad f(x) = 3x - 7$
2. $f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$
3. $g: [0, \infty) \to \mathbb{R}, \quad g(x) = x^2$
4. $h: \mathbb{R} \to \mathbb{R} \quad h(x) = x^2 - x - 6$
5. ~~$g: \mathbb{R} \to \mathbb{R} \quad g(x) = \sin x$~~ ✓
6. $g: \mathbb{R} \setminus \{1\} \to \mathbb{R} \quad g(x) = \dfrac{x-2}{x-1}$

Solution

1 f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Suppose $f(x_1) = f(x_2)$, then

$$3x_1 - 7 = 3x_2 - 7$$

Add 7 to both sides

$$3x_1 = 3x_2$$

divide by 3

$$x_1 = x_2.$$

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, f is injective.

2 Let $f(x_1) = f(x_2) = 9$

Then $x_1^2 = x_2^2 = 9$,

Take square root of the equation

$$\pm\sqrt{x_1^2} = \pm\sqrt{x_2^2} = \pm\sqrt{9}$$

$$\pm x_1 = \pm x_2 = \pm 3$$

If $x_1 = 3$ and $x_2 = -3$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Therefore $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ is not injective (some elements in its range do not have exactly one (a unique) preimage).

3 $g: [0, \infty) \to \mathbb{R}$ defined as $g(x) = x^2$.

Suppose $g(x_1) = g(x_2)$,

then $x_1^2 = x_2^2$

Taking square root of both sides to find the preimage of the left & right hand sides of the equation in $[0, \infty)$

$$\cancel{\pm}\left|\sqrt{x_1^2}\right| = \cancel{\pm}\left|\sqrt{x_1^2}\right|$$

$$\cancel{\pm} x_1 = x_2 \qquad \text{for } x_1, x_2 \in [0, \infty).$$

Therefore g is injective.

4 Let $h(x_1) = h(x_2) = \overset{=h(x)}{0}$. Then

$$x^2 - x - 6 = 0$$

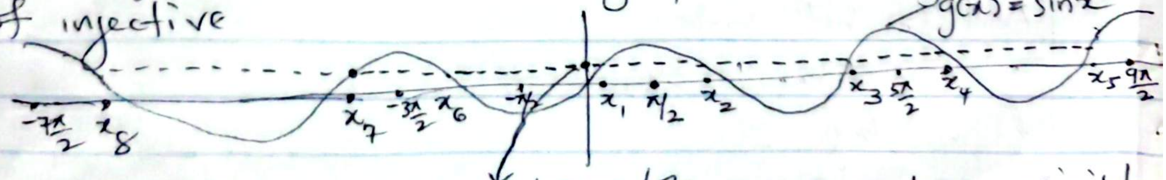
$$(x-3)(x+2) = 0$$

Then $x = 3$ or $x = 2$.

Take $x_1 = 3$, $x_2 = 2$.

Since $h(x_1) = h(x_2)$ but $x_1 \neq x_2$, h is not injective.

5 The sine function has a wave graph and therefore it is not injective



$y \in$ Image/Range of g has infinitely many preimages of different values on the x-axis including $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$.

So it is not injective.

Proper proof: Let $g(x_1) = g(x_2) = 1 = g(x)$

Then $\sin x = 1$

$x = \sin^{-1}(1)$

$x \in \{ m(\frac{\pi}{2}) : \cancel{m \in (\mathbb{Z}^+ \cap 4\mathbb{Z}+1)} \ m \in 4\mathbb{Z}+1 \}$

Take $x_1 = \frac{\pi}{2}$ $\quad x_2 = -\frac{3\pi}{2}$

Then $h(x_1) = h(x_2) = 1$, but $x_1 \neq x_2$. ~~g(x) i~~

Therefore $g(x)$ is not injective.

6 Suppose $g(x_1) = g(x_2)$.

Then $\frac{x_1 - 2}{x_1 - 1} = \frac{x_2 - 2}{x_2 - 1}$

Cross multiply

$(x_1 - 2)(x_2 - 1) = (x_1 - 1)(x_2 - 2)$

$x_1x_2 - x_1 - 2x_2 + 2 = x_1x_2 - 2x_1 - x_2 + 2$

Subtract x_1x_2 & 2 from both sides

$-x_1 - 2x_2 = -2x_1 - x_2$

Collect like terms

$-x_1 + 2x_1 = -x_2 + 2x_2$

$x_1 = x_2$

Therefore g is injective.

Exercise: Determine if the following functions are injective

a $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 5x - 2$

b $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 + 1$

c $f: (0, \infty) \to \mathbb{R}$ $\quad f(x) = \ln(x)$

d $f: \cancel{(0, \infty)}\ \mathbb{R} \to (0, \infty)$ $\quad f(x) = e^x$

e $f: \mathbb{R} \setminus \{2\} \to \mathbb{R}$ $\quad f(x) = \frac{x+1}{x-2}$

3 Identity function: A function $f_{id}: A \to A$ defined as $f(x) = x$ is the identity function on A.

Exercise: Show that the identity function is injective & surjective.

4 Composite function: Given functions $f: A \to B$, $g: B \to C$. The composite $g \circ f: A \to C$ is such that $(g \circ f)(x) = g(f(x))$.

Examples

1 We Define functions $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$ and $g: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^2$.
Then $f \circ g(x) = f(g(x)) = f(x^2) = 2x^2 + 3$.
and $g \circ f(x) = g(f(x)) = g(2x+3) = (2x+3)^2 = 4x^2 + 6x + 9$.
Both $f \circ g$ & $g \circ f$ have domain & codomain $\mathbb{R}$.

2 Let $f: [0, \infty) \to [0, \infty)$ $x \mapsto \sqrt{x}$ and $g: \mathbb{R} \to [1, \infty)$ $x \mapsto x+1$ be functions. Then
~~$f \circ g$~~ $g \circ f: [0, \infty) \to [1, \infty)$ is defined as

$$g \circ f(x) = g(\sqrt{x}) = \sqrt{x} + 1$$

While $f \circ g: \mathbb{R} \to [0, \infty)$ is defined as

$$f \circ g(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$$

Note that codomain of g is a subset of the domain of g, therefore Image $g \subseteq$ domain

Note that $f \circ g$ & $g \circ f$ are defined on their domain & codomain (i.e each element in the domain of each function has (exactly one image in its codomain.

~~Exercises~~: ~~Find the composites~~

3 Suppose $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $f(x) = \frac{1}{x}$ and $g: \mathbb{R} \to \mathbb{R}$ $g(x) = x^2 - 4$ are functions. Find the domain & codomain for which each of $f \circ g$ & $g \circ f$ are defined.

Solution

$$f \circ g(x) = f(g(x)) = f(x^2 - 4) = \frac{1}{x^2 - 4}.$$

This rule would be undefined when $x^2 - 4 = 0$, i.e when $x^2 = 4$,

$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

Therefore $f \circ g$ has domain $\mathbb{R} \setminus \{\pm 2\}$. Since $y = \frac{1}{x^2 - 4} \in \mathbb{R}$ for all $x \in \mathbb{R} \setminus \{\pm 2\}$. ~~$f \circ g$~~ We can choose $\mathbb{R}$ has codomain

for $f \circ g$. Therefore
$f \circ g : \mathbb{R} \setminus \{\pm 2\} \longrightarrow \mathbb{R}$.

ii) $g \circ f(x) = g(f(x))$

$$= g\left(\frac{1}{x}\right)$$
$$= \left(\frac{1}{x}\right)^2 - 4$$
$$= \frac{1}{x^2} - 4$$
$$= \frac{1-4x^2}{x^2}$$

Since $g \circ f(x)$ is undefined when $x^2 = 0$ ($\Longleftrightarrow x = 0$), $g \circ f$ has domain $\mathbb{R} \setminus \{0\}$.
Since $y = \frac{1-4x^2}{x^2} \in \mathbb{R}$ for all $x \in \mathbb{R} \setminus \{0\}$;
~~$f \circ g$~~ $g \circ f$ has codomain contained in $\mathbb{R}$. Therefore
$g \circ f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$.

Exercises
Find the domain & codomain, and rule for which $f \circ g$ and $g \circ f$, in the following are defined

a) $f : \mathbb{R} \to [0, \infty)$, $f(x) = x^2$, $g : \mathbb{R} \to \mathbb{R}$ $g(x) = x - 1$
b) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$ $g : \mathbb{R} \to \mathbb{R}$ $g(x) = e^{2x}$

c) Suppose $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{1\}$, $f(x) = \frac{x-1}{x+1}$. What domain & codomain is $f^2 = f \circ f$ defined.

5 Inverse of a function
Given a function $f : A \to B$, if the inverse f^{-1} of f denoted by $f^{-1} : B \to A$ exists, then $f^{-1} \circ f = f \circ f^{-1} = \mathrm{Id}$, where Id is the identity function.

Examples: Find the inverse of the following functions if they exist

1 $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 2x + 3$

2 $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$

3 $g: [0, \infty) \to [0, \infty)$ $g(x) = x^2$

4 $h: [0, \pi] \to [-1, 1]$ $h(x) = \sin x$

5 $f: \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{3\}$ $f(x) = \dfrac{3x-1}{x+2}$

Solution

Suppose $f(x) = 2x+3$

$= y \in \mathbb{R}$ as codomain of f

then making x the subject

$y = 2x+3$

$y-3 = 2x$

$\dfrac{y-3}{2} = x \in \mathbb{R}$ as domain

x or y are just representing variables.

gives us $\dfrac{y-3}{2}$ is the preimage of y ~~.~~

Then, take $f^{-1}(y) = \dfrac{y-3}{2}$. ~~Is f~~ defined ~~f~~ f^{-1} as $f^{-1}: \mathbb{R} \to \mathbb{R}$.

$f^{-1}(x) = \dfrac{x-3}{2}$

Is f^{-1} ~~be~~ a function?

Given $y \in \mathbb{R}$ (domain of f^{-1}) ~~is~~ does $f^{-1}(y)$ exist in $\mathbb{R}$ (codomain of f^{-1})? Yes, for all $y \in \mathbb{R}$, $y-3 \in \mathbb{R}$ & $\frac{y-3}{2} \in \mathbb{R}$

Is this image unique to each element in this domain?

Suppose $y_1 = y_2$, ~~then~~

then $y_1 - 3 = y_2 - 3$,

then $\dfrac{y_1 - 3}{2} = \dfrac{y_2 - 3}{2}$

That is $f^{-1}(y_1) = f^{-1}(y_2)$.

Therefore f^{-1} is a well defined function from $\mathbb{R} \to \mathbb{R}$.

To show that f^{-1} is indeed f inverse

$f^{-1} \circ f(x) = f^{-1}(f(x))$

$= f^{-1}(2x+3)$

$= \dfrac{(2x+3)-3}{2}$

$= 2x/2$

$= x$

Similarly

$f \circ f^{-1}(x) = f\left(\frac{x-3}{2}\right)$

$= 2\left(\frac{x-3}{2}\right) + 3$

$= x - 3 + 3$

$= x.$

So ~~f^{-1} is well de~~ $f^{-1} \circ f = f \circ f^{-1} = Id$

2. Suppose ~~$f: \mathbb{R}$~~ $f(x) = x^2 = y$

Then $x = \sqrt{y}$

$f^{-1}(y) = \sqrt{y}$ has been shown to not ~~been~~ a map. From $\mathbb{R}$. ~~f^{-1} does~~

In particular, if f is not injective, f^{-1} would not satisfy the well defined property of a map.

f^{-1} ~~does~~ not exist (because it is not well defined)

3 $g(x) = x^2$ $g(x): [0, \infty) \longrightarrow [0, \infty)$

If $y = x^2$. Then

$x = |\sqrt{y}| \in [0, \infty)$

Suppose $g^{-1}(y) = |\sqrt{y}| \iff g^{-1}(x) = |\sqrt{x}|$.

i Is g^{-1} a map? $g^{-1}: [0, \infty) \longrightarrow [0, \infty)$

ii Is g^{-1} indeed inverse of g?

i Given $x \in [0, \infty)$ as domain of g^{-1}, ~~is~~ then

$g^{-1}(x) = |\sqrt{x}|$ ~~&~~ exists in $[0, \infty)$ as codomain of g^{-1}.

Let $x_1 = x_2$, then

$|\sqrt{x_1}| = |\sqrt{x_2}|$

i.e $g^{-1}(x_1) = g^{-1}(x_2)$

g^{-1} is well defined.

$g^{-1} \circ g(x) = g^{-1}(x^2)$, when $x \in [0, \infty)$

$= |\sqrt{x^2}|$

$= x$ ~~Since~~ ~~x~~ $\in [0, \infty)$.

Similarly $g \circ g^{-1}(x) = g(|\sqrt{x}|)$ where $x \in [0, \infty)$.
$$= |\sqrt{x}|^2$$
$$= x \in [0, \infty)$$

4 $h : [0, \pi] \longrightarrow [-1, 1]$.
The inverse of h exists as $h^{-1} : [-1, 1] \longrightarrow [0, \pi]$.
~~Thus~~ $h^{-1}(x) = \sin^{-1} x$.
Note that for the inverse of a function ~~has images to~~ ~~only for preimages of~~ to satisfy the first condition of a function the domain ~~of~~ the inverse must be a subset of the range of the function (any element in the codomain of $f \setminus$ range of f would not have an image under f^{-1}), also for the inverse function to be well defined (the 2nd function condition), the function itself must be injective.

The range of the sine function is $[-1, 1]$, therefor for any $x \in [-1, 1]$, $\sin^{-1} x$ is defined.
The sine function completes a cycle in exactly $[0, \pi]$. $\sin x$ is therefore defined in $[0, \pi]$, infact well defined in $[0, \pi]$ since $\sin x$ is injective in $[0, \pi]$.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is $f(x) = \sin x$.
Then f^{-1} is not defined for $\mathbb{R} \setminus [-1, 1]$ and not well defined outside the range $\theta \le x$ $\theta + \pi \cdot n$ the domain of f^{-1} contained in $\mathbb{R}$.

5 $f : \mathbb{R} \setminus \{-2\} \rightarrow \mathbb{R} \setminus \{3\}$ $f(x) = \frac{3x-1}{x+2}$
If ~~$f(x) =$~~ $y = \frac{3x-1}{x+2}$
Then $(x+2)y = 3x - 1$
$$xy + 2y = 3x - 1$$
$$xy - 3x = -1 - 2y$$
$$3x - xy = 2y + 1$$
$$x(3-y) = 2y + 1$$

$$x = \frac{2y+1}{3-y}$$

If we take $f^{-1}(y) = \frac{2y+1}{3-y}$, $f^{-1}(y)$ is defined when

$3-y \neq 0, \iff y \neq 3.$

Therefore domain f^{-1} is $\mathbb{R} \setminus \{3\}$.

~~Ch~~ f^{-1} is well defined (proof skipped).

a Exercise: Show that $f \circ f^{-1} = f^{-1} \circ f$ in the above example.

b Define the appropriate domain and codomain in $\mathbb{R}$ for which h & h^{-1} exist if $h(x) = \frac{x-1}{x+1}$.

ii) Verify that h & h^{-1} are indeed functions.

iii) Verify $h \circ h^{-1} = h^{-1} \circ h = Id$.

6 A function $f: A \to B$ is called a bijection if it is both injective and surjective.

* The inverse of a bijection alway exist.

* If $f: A \to B$ is a bijection, then $|A| \to |B|$.

* A bijection establishes 1-1 correspondence between elements of A & B. ~~of A is~~

Example

Show that $f: (0,1) \to \mathbb{R}^{>0}$ defined as $\frac{1-x}{x}$ is a bijection. Q3bii (8 marks) in 24/25 Exams.

Solution

Is f injective? Suppose $f(x_1) = f(x_2)$

Then $\frac{1-x_1}{x_1} = \frac{1-x_2}{x_2}$,

then $x_2(1-x_1) = x_1(1-x_2)$

$$x_2 - x_1x_2 = x_1 - x_1x_2$$

add x_1x_2 to both sides

$x_1 = x_2$.

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. f is injective. 2½ marks

Is f: surjective? Given $y \in \mathbb{R}^{>0}$, does y have a pre-image in $(0,1)$

Let $y = \frac{1-x}{x}$,

then $xy = 1 - x$

$xy + x = 1$

$x(y+1) = 1$

$x = \frac{1}{y+1}$

2½ marks
~~The first~~ Many students that attempted the question correctly stopped here.

Is $\frac{1}{y+1} \in (0,1)$?

Since $y \in \mathbb{R}^{>0}$, $y+1, \frac{1}{y+1} \in \mathbb{R}^{>0}$

Suppose $\frac{1}{y+1} > 1$

then $1 > y+1$ (~~since~~ cross multiplying above inequality since $y+1 > 0$)

subtracting 1 from both sides

$1-1 > y+1-1$

$0 > y$

this contradicts $y \in \mathbb{R}^{>0}$,

therefore $\frac{1}{y+1} < 1$ ~~and then~~

and then $\frac{1}{y+1} \in (0,1)$, the domain of f is the preimage of $y \in \mathbb{R}^{>0}$

f is indeed surjective. 3mks.

Exercise.

1a Show that f above is indeed a function — 4 mks 3bi 24/25 Exam

b ~~Find~~ Show that f^{-1} ~~and show it~~ is a function.

c Show that $f \circ f^{-1} = f^{-1} \circ f = Id$.

2 Which of the following is a bijection

a $f: \mathbb{C} \to \mathbb{C}$ $a+ib \mapsto a \mp ib$.

b $f: (0, \infty) \longrightarrow \mathbb{R}$ $f(x) = \ln x$

c $h: (0, 1) \longrightarrow \mathbb{R}^{>0}$ $h(x) = \frac{x}{1-x}$

3 Suggest suitable domain and codomain in $\mathbb{R}$ over which the following are byjections

a $f(x) = \frac{2x-1}{x-1}$

b) $g(x) = x^2 + 1$

c) $h(x) = \cos x$.