ANALYTICAL DYNAMICS

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CHAPTER ONE

BASIC LAWS OF MOTION

Definition: PARTICLE: The term "particle" is a concept of the imagination and may be viewed as a bit of matter so small that its position in space is determined by the three coordinates of its "Centre", ~~above~~

Let $\underline{P} = m\underline{V}$ define a linear momentum of a given particle at the instant:

1. Every particle continues in its state of rest or of uniform velocity in a straight line unless compelled to do otherwise by a force acting on it. i.e force is that which changes motion.
2. The rate of change of momentum is proportional to the impressed force and takes place in the direction of action of the force. i.e we have given the magnitude of force.
3. To every action, there exists an equal and opposite reaction.

Newton's Second law above, applied to a particle of constant mass m, may be written as

$$\underline{F} = m\frac{d\underline{V}}{dt} = m\underline{a} \qquad (1)$$

where the force $\underline{F}$, Velocity $\underline{V}$ and acceleration $\underline{a}$ are vector quantities and the mass m and time t are scalars.

In a three Cartesian Coordinates System

the Component form of $\underline{F}$ is

$$\underline{F} = (F_x, F_y, F_z) \qquad (2)$$

Where

$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y} \qquad (3)$$
$$F_z = m\ddot{z}$$

x: spacial coordinate in the x-direction
y, z

For Convinience, we shall use the following notations through out this course

$$\frac{dx}{dt} = \dot{x} \qquad \frac{d^2x}{dt^2} = \ddot{x} \qquad etc$$

Equations (3), in the simple form shown, are by no means true under any and all conditions. We proceed to discuss the conditions under which they are valid

Definition: A frame of reference is some coordinate system which describes the location of the particle. A frame of reference w.r.t space is some rectangular axes X, Y, Z.

Equation (1) implies some frame of reference w.r.t. which $\frac{dv}{dt}$ is measured. Equations (3) indicates that the motion is refered to some rectangular axes X, Y, Z

INERTIAL AND NON INERTIAL FRAME OF REFERENCE

Definition: The term "inertial frame" may be defined abstractly, merely as one w.r.t which Newtons equations, in the simple form (3), are valid

In otherwords, it is a fact of experience that Newton's second law expressed in the simple form of (3) gives results in close

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Question ....

agreement with experience if the coordinate axes are fixed relative to the average position of the "fixed" stars or moving with uniform linear velocity and without rotation relative to the stars. In either case the frame of reference (the X, Y, Z axes) is referred to as an Inertial frame and corresponding coordinates as inertial coordinates. A frame of reference which has linear acceleration or is rotating in any manner is non-inertial

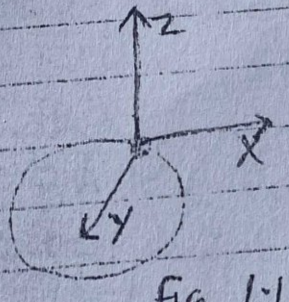


Fig. 1.1

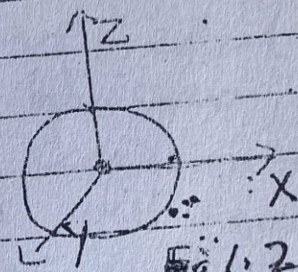


Fig 1.2

figures fig 1.1 and 1.2 show coordinate frames attached to the surface and centre of the earth respectively.

Due to annual and daily rotations and other motions of the earth, a coordinate frame attached to its surface is obviously non-inertial. Nevertheless, the acceleration of this frame is so slight that for many (but by no means all) purposes it may be regarded as inertial. A non-rotating frame (axes pointing always toward the same fixed stars) with origin attached to the Centre of the earth is more nearly inertial.

REMARKS

1. Non-rotating axes with origin fixed to the Centre of the sun constitutes an excellent (though perhaps not "perfect") inertial frame
2. The condition just stated must be regarded as one of the important foundation stones on which the super structure of dynamics rests, treatment of every dynamical problem begin with the consideration of an inertial frame.
3. The above statements, however, do not imply that non-inertial coordinates cannot be used. On the contrary, as will soon be seen they are employed perhaps just as frequently as inertial. How Newton's second law equation can be written for non inertial coordinates are demonstrated in example below. (As shown in chapter 2) the Lagrangian equations give correct equations of motion in inertial, non-inertial or mixed coordinates.

EXAMPLES

1 Consider a railroad car moving with a constant acceleration a_x along a straight horizontal track in an inertial frame X_1, Y_1. If a ball of mass m in the car is acted upon by some external force $\underset{\sim}{F}$ acting at an angle 30^c (say) to axis Y_1, considering

in a plane only and treating the ball as a particle, obtain the equations of motion relative to the earth and those relative to the car. Take the coordinate frame of the car to be X_2, Y_2.

Soln

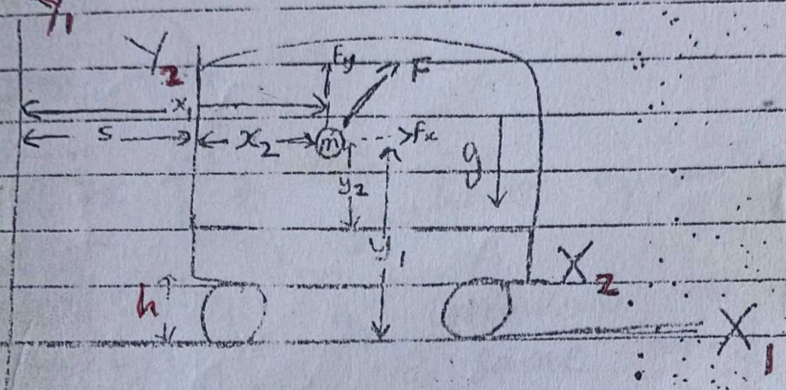


Fig 1.3

Let the position of the ball relative to the coordinate frames of the earth and those of the car be (x_1, y_1) and (x_2, y_2) respectively and s the distance moved by car at any time t.

Then, the equations of motion relative to the earth are

$$m\ddot{x}_1 = F_x \quad \text{(Component of F in } X_1 X_2 \text{ dir)} \quad (4)$$

$$m\ddot{y}_1 = F_y - mg \quad \text{(Component of F in } Y_1 Y_2 \text{ dir)} \quad (5)$$

where g is the acceleration due to gravity

Now the relations between the "earth coordinate" and "car coordinates" of m are clearly

$$x_1 = x_2 + s \quad (6)$$

$$= x_2 + v_1 t + \tfrac{1}{2} a_x t^2 \quad (7)$$

$$y_1 = y_2 + h \quad (8)$$

to obtain equations of motion relative to the car, we differentiate (7) and (8) twice w.r.t t i.e

$$\ddot{x}_1 = \ddot{x}_2 + a_x \quad (9)$$

and

$$\ddot{y}_1 = \ddot{y}_2 \quad (10)$$

Substituting (9) and (10) into (4) and (5) we have

$$m\ddot{x}_2 = F_x - ma_x \quad (11)$$

and

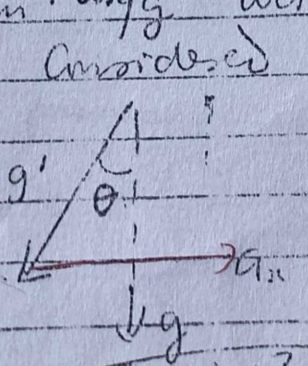
$$m\ddot{y}_2 = F_y - mg \quad (12)$$

which are the equations of motion of the ball relative to the car.

Clearly the y_2 coordinate is inertial since (5) and (12) have the same form. However x_2 is non-inertial since (4) and (11) are different. Equation (11) is a simple example of Newton's second law equation in terms of a non-inertial coordinate. (Note how incorrect it would be to write $m\ddot{x}_2 = F_x$)

Remarks

Notice that the effect of these non inertial condition on any mechanical system or on a person in the car is just as if g were increased to $(a_x^2 + g^2)^{1/2}$, acting downward at the angle $\theta = \tan^{-1} a_x/g$ with the vertical, and all coordinates considered as inertial (how?)



Car moves backwards as the g moves, acting on them, moves to g' Pythagoras $g' = \sqrt{a_x^2 + g^2}$

Fig 1.4

$$g' = \sqrt{a_x^2 + g^2} \quad (13)$$

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Question ....

2) Consider the motion of the particle of mass m acted upon by a force F acting at an angle (30 say) to Y_1 axes on an inertial frame X_1, Y_1, relative to the X_2, Y_2 axes which are rotating with constant angular velocity ω relative to the inertial X_1, Y_1 frame. Obtain the equations of motion in the inertial coordinates and those relative to the ~~...~~ X_2, Y_2 axes. Take the angle between X_1 and X_2 to be θ.

<u>Solution</u>

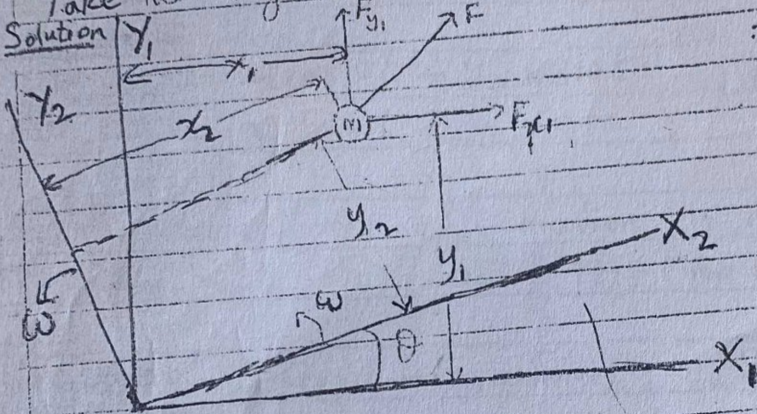

Fig (1.5)

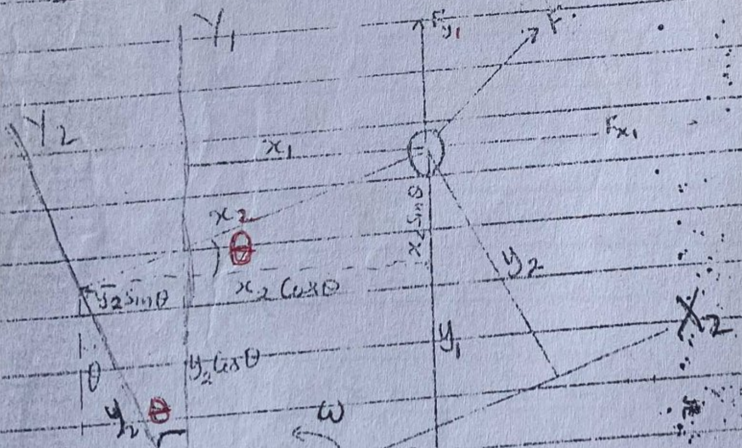

Fig 1.6

From Newton's law, the equations of motion relative to $X_1 Y_1$ frame are

$$m\ddot{x}_1 = F_{x_1}$$

$$m\ddot{y}_1 = F_{y_1}$$

14

15

there is no "-mg" in eq (15 was (15)) b/c the inertial frame $X_1 Y_1$ is not necessarily the earth

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~~whose~~ where F_{x_1} and F_{y_1} are Components of the applied force along the fixed axes.

Next, we obtain the Corresponding equations in the rotating Coordinates and hence prove that the rotating coordinates are non-inertial.

From figure (1.6), it is clearly seen that $\theta = \omega t$ and hence

$$x_1 = x_2 \cos\omega t - y_2 \sin\omega t \qquad (16)$$

$$y_1 = x_2 \sin\omega t + y_2 \cos\omega t \qquad (17)$$

To obtain equations of motion in rotating Coordinates, we differentiate (16) and (17) twice, and we have

$$\dot{x}_1 = \dot{x}_2 \cos\omega t - x_2 \omega \sin\omega t - \dot{y}_2 \sin\omega t - y_2 \omega \cos\omega t \qquad (18)$$

$$\ddot{x}_1 = \ddot{x}_2 \cos\omega t - \omega \dot{x}_2 \sin\omega t - \omega \dot{x}_2 \sin\omega t - \omega^2 x_2 \cos\omega t - \ddot{y}_2 \sin\omega t - \dot{y}_2 \omega \cos\omega t - \omega \dot{y}_2 \cos\omega t + \omega^2 y_2 \sin\omega t \qquad (19)$$

$$= \ddot{x}_2 \cos\omega t - 2\dot{x}_2 \omega \sin\omega t - 2\dot{y}_2 \omega \cos\omega t - \omega^2 x_2 \cos\omega t + \omega^2 y_2 \sin\omega t - \ddot{y}_2 \sin\omega t \qquad (20)$$

Similarly

$$\ddot{y}_1 = \ddot{x}_2 \sin\omega t + 2\dot{x}_2 \omega \cos\omega t - 2\dot{y}_2 \omega \sin\omega t - x_2 \omega^2 \sin\omega t - y_2 \omega^2 \cos\omega t + \ddot{y}_2 \cos\omega t \qquad (21)$$

Substituting (20) and (21) into (14) and (15), we obtain

$$F_{x_1} = m[\ddot{x}_2 \cos\omega t - 2\dot{x}_2 \omega \sin\omega t - 2\dot{y}_2 \omega \cos\omega t - \omega^2 x_2 \cos\omega t + \omega^2 y_2 \sin\omega t \qquad (22)$$

*How?

$$F_{y_1} = m[\ddot{x}_2 \sin\omega t + 2\dot{x}_2 \omega \cos\omega t - 2\dot{y}_2 \omega \sin\omega t - x_2 \omega^2 \sin\omega t - y_2 \omega^2 \cos\omega t + \ddot{y}_2 \text{[illegible]} \qquad (23)$$

From the figures above, it is seen that the components of F in the directions of X_2 and Y_2 are given by

$$F_{x_2} = F_{x_1}\cos\omega t + F_{y_1}\sin\omega t \quad (24)$$

$$F_{y_2} = F_{y_1}\cos\omega t - F_{x_1}\sin\omega t \quad (25)$$

Hence multiplying (22) and (23) through by $\cos\omega t$ and $\sin\omega t$ respectively and adding, the result is

$$F_{x_2} = m\ddot{x}_2 - m x_2\omega^2 - 2m\omega\dot{y}_2 \quad (26)$$

Similarly multiplying (22) and (23) through by $\sin\omega t$ and $\cos\omega t$ respectively and subtracting

$$F_{y_2} = m\ddot{y}_2 - m y_2\omega^2 + 2m\omega\dot{x}_2 \quad (27)$$

Equations (26) and (27) are the equations of motion relative to the non-inertial X_2, Y_2, axes

Remarks

1. It would indeed be a mistake to write

$$F_{x_2} = m\ddot{x}_2 \quad (28)$$

and

$$F_{y_2} = m\ddot{y}_2 \quad (29)$$

2. From the above example, it is evident that any rotating frame is non-inertial.

GENERAL REMARKS ON NEWTONIAN DYNAMICS

1. Equations (3) are valid only when m is constant.
In case m is variable, equation (1)

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(30)

~~must~~ must be replaced by

$$\underline{F} = \frac{d}{dt}(m\underline{v})$$

Mass of an object can vary with coordinates e.g a snowball rolling down a snow covered hill. It can vary with time e.g a tank car having a hole in one end from which liquid flows or a rocket during the burn-out period and It can vary with velocity e.g any object moving with a velocity approaching that of light. This course, however, shall not give a detailed treatment of Variable mass problems

2 For equations (3) to be valid, the masses of the system must be large compared with the masses of atoms and atomic particles.

3 Also, whether a mass is large or small its velocity must be low compared with that of light to have a Newtonian problem

4 In case certain masses of the system are very large and/or long intervals of time are involved (whether less or more), the general theory of relativity agrees more closely with experiment than does Newtonian dynamics.

Two General Types of Dynamical Problems.

Almost every problem in classical dynamics is a special case of one of the following general types:

1 One could be given forces acting on a system of masses, given constraints, and the known position and velocity of each mass at a stated instant of time

1 In this case, one is required to find the "motion" of the system, that is,
(i) position
(ii) velocity
(iii) acceleration
of each of the masses as functions of time.

2 The second type of dynamical problems is to be given motions of a system and be required to find a possible set of forces which will produce such motions.
In general some or all of the forces may vary with time. ~~Of Course~~
Of course, considerations of work, energy, power, linear momentum and angular momentum may be an important part of either (1) or (2).

EXERCISE

1 State clearly what is meant by "inertial frame of reference".

2 Prove that any frame of reference moving with constant linear velocity (no rotation) relative to an inertial frame is itself inertial.

3 Can one recognize by inspection whether given coordinates are inertial or non-inertial?
Is it permissible, for the solution of certain problems, to use a combination of inertial and non-inertial coordinates?

4 A coordinate frame is attached to the inside of an automobile which is moving in the usual manner along a street with curves, bumps,

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stop lights and traffic cops. Is the frame inertial? Do occupants of the car feel forces other than gravity? Explain.

5 If the car, shown in Fig 1.3, were moving with constant speed around a level circular track which of the coordinates x_2, y_2, z_2 of m_1 (or of another point referred to the X_2, Y_2, Z_2 frame) would be non-inertial? Explain. (Assume Z_1 taken along the radius of curvature of track).

COORDINATE SYSTEMS AND TRANSFORMATION EQUATIONS.

The various topics under this heading will be treated, to a large extent, by specific examples

1 RECTANGULAR SYSTEMS

Consider first the two-dimensional rectangular systems shown below

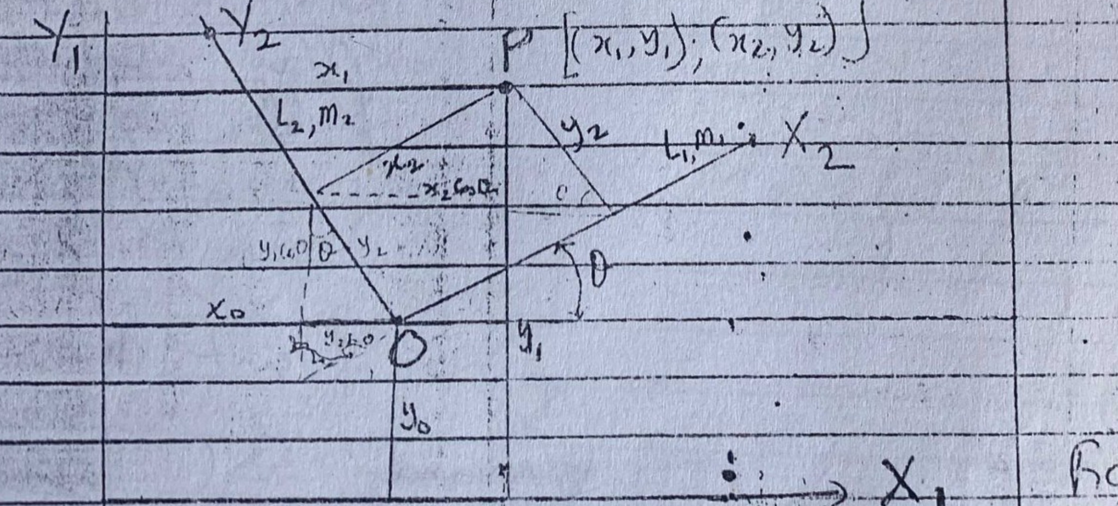

Fig 1.7

The lengths x_1, y_1 locate the point P relative to the X_1, Y_1 frame of reference. Likewise x_2, y_2 locate the same point relative to X_2, Y_2

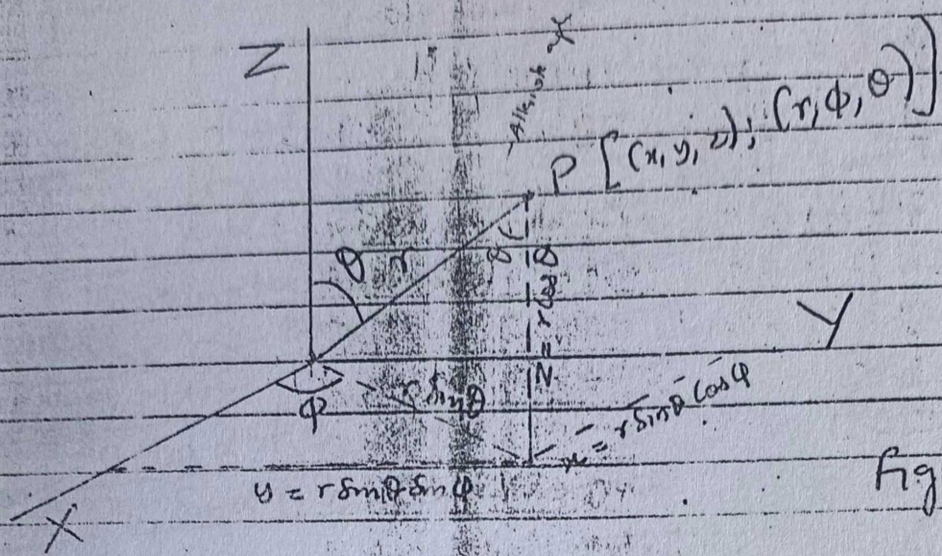

Fig 1.9

It is seen that the equations relating the coordinates (x, y, z) to the spherical coordinates (r, ϕ, θ) are

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \qquad (38)$$

It is observed that x and y are each fns of r, ϕ, θ. It happens that z is a function of r and θ only.

OTHER COORDINATE SYSTEMS

Consider the two sets of axes X, Y and Q_1, Q_2 as shown in the figures below, let the oblique axes Q_1, Q_2 make known angles α and β respectively.

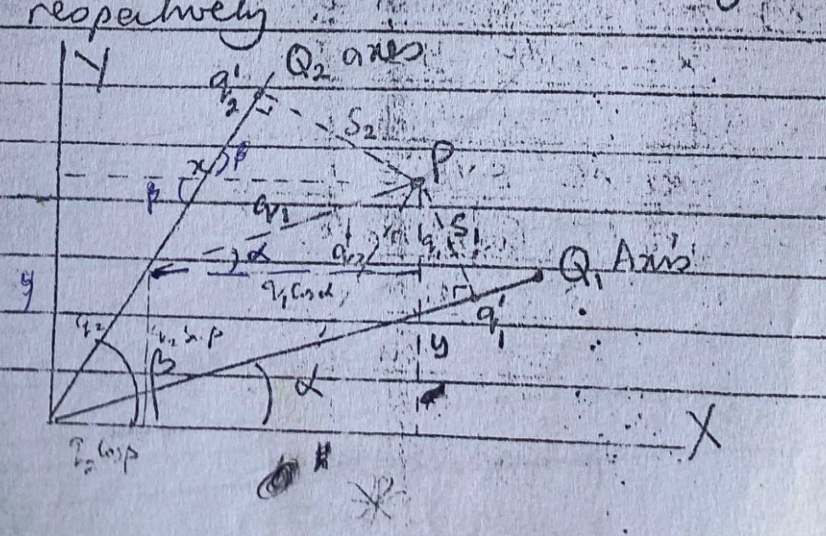

Fig 1.10

(14)

(2)

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It is observed that the point P may be located by several pairs of quantities such as (x, y), (q_1, q_2), (q_1', q_2'), (S_1, S_2), $(S, \theta x)$ etc. Each pair constitutes a set of coordinates. Transformation equations relating some of these are

$$x = q_1 \cos\alpha + q_2 \cos\beta$$
$$y = q_1 \sin\alpha + q_2 \sin\beta \qquad (39)$$

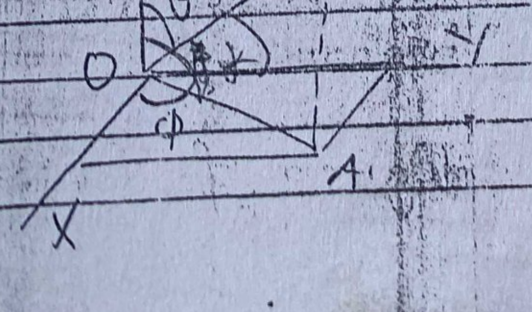
$$q_1' = q_1 + q_2 \cos(\beta - \alpha)$$
$$q_2' = q_2 + q_1 \cos(\beta - \alpha) \qquad (40)$$

$$S_2 = x \sin\beta - y \cos\beta$$
$$S_1 = y \cos\alpha - x \sin\alpha \qquad (41)$$

EXERCISE

1 Consider three-dimensional form of Fig 1.7 sketch it and obtain the transformation equations relating the (x_1, y_1, z_1) coordinates of a point to the x_2, y_2, z_2 coordinates of the same point. What are the corresponding transformation equations if the frame X_2, Y_2, Z_2 is moving

2



Assignment.

Fig 1.11

Using the above diagram (Fig. 1.11), show that the rectangular coordinates x, y, z of m are related to the θ, Φ, α coordinates by

$$x = (R\sin\theta - r\cos\alpha\cos\theta)\cos\phi + r\sin\alpha\sin\phi$$
$$y = (R\sin\theta - r\cos\alpha\cos\theta)\sin\phi - r\sin\alpha\cos\phi$$
$$z = R\cos\theta + r\cos\alpha\sin\theta$$

where $R = Oa$

CHAPTER TWO.

CONSTRAINTS. HOLONOMIC AND NONHOLONOMIC CONSTRAINTS

Often in practice, the motion of a particle or system of particles is restricted in some way. For example, in rigid bodies the motion must be such that the distance between any two particular particles of the rigid body is always the same. As another example, the motion of particles may be restricted to curves or surfaces.

The limitations in the motion are often called constraints. If the constraint condition can be expressed as an equation

$$\phi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N, t) = 0 \qquad (42)$$

connecting the position vectors of the particles and the time, then the constraint is called holonomic. If it cannot be so expressed it is called non-holonomic.

Definition:

The number of degrees of freedom of

a given system is given by the number independent coordinates (not including time) required to specify completely the position of each and every particle of the system

~~Examples~~
Examples
A particle constrained to move along a straight line (bead on a wire) the equation of which is

$y = a + bx$

is a system having one degree of freedom. If either x or y is given the other is known.
A particle free to move in contact with a plane or any known surface: spherical, cylindrical, etc. is a system having ~~for~~ two degrees of freedom

A particle free to move in space has (x, y, z) or (r, ϕ, θ) etc as possible coordinates. This is a system having 3 degrees of freedom.
Another example of a system having 3 degrees of freedom is a board or any lamina free to slide in contact with a plane. Two coordinates are required for translation and one for rotation.

GENERALIZED COORDINATES
A dynamical system is one comprised of ~~a moving system~~ several particles. Suppose a system of N particles of masses M_i $(i = 1, 2, \cdots N)$ moves subject to possible constraints e.g. a particle moving along a circular wire or a rigid body

moving along an inclined plane. Then there will be a minimum number of independent coordinates needed to specify the position of each particle at any time t. These coordinates denoted by q_α $(\alpha = 1, 2, \ldots, n)$ are called generalized coordinates.

The generalized coordinates can be distances, angles or quantities relating to them. The number n of generalized coordinates is the number of degrees of freedom.

Many sets of generalized coordinates may be possible in a given problem, but a strategic choice can simplify the analysis considerably.

VELOCITY EXPRESSED IN GENERALIZED COORDINATES

The expressions for the velocity of a point or particle are developed below from the fundamental definition of velocity and the basic physical and geometrical ideas involved.

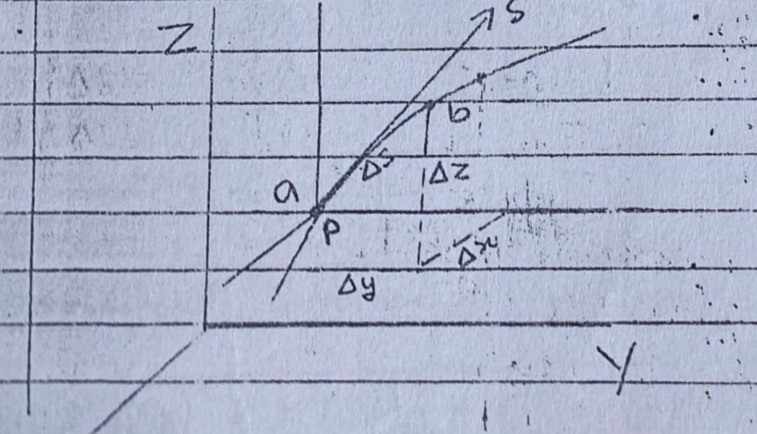


Fig 1.17

(18) (T) Vol 1

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Let the point P in the figure above move the distance Δs from a to b in time Δt. Its average Velocity over the interval is

$$\frac{\Delta s}{\Delta t}$$

Thus,

$$\text{Velocity} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \dot{\underset{\sim}{s}} \qquad (43)$$

where $\dot{\underset{\sim}{s}}$ is a vector quantity of magnitude $\left|\frac{ds}{dt}\right|$, pointing in the direction of the tangent to the path at a.

As an aid in appreciating the physics and geometry involved, we may think of a particle as having a velocity "in the direction of its path at any position in the path.

The above definition of Velocity makes no reference to any particular coordinate System

EXAMPLES

1 In rectangular Coordinates (figure (1.12) above)

Displacement $\Delta \underset{\sim}{s}$ is the Vector $\vec{ab}$:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (44)$$

Thus,

$$\lim_{\Delta t \to 0} \frac{(\Delta s)^2}{(\Delta t)^2} = \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t}\right)^2 + \lim_{\Delta t \to 0} \left(\frac{\Delta y}{\Delta t}\right)^2 + \lim_{\Delta t \to 0} \left(\frac{\Delta z}{\Delta t}\right)^2 \qquad (45)$$

which implies

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \qquad (46)$$

2. In Spherical Coordinates, (See figure below).

$$(\Delta s)^2 = (\Delta r)^2 + r^2 (\Delta \theta)^2 + r^2 \sin^2\theta \, (\Delta \phi)^2 \qquad (47)$$

Considering element of length Δs:

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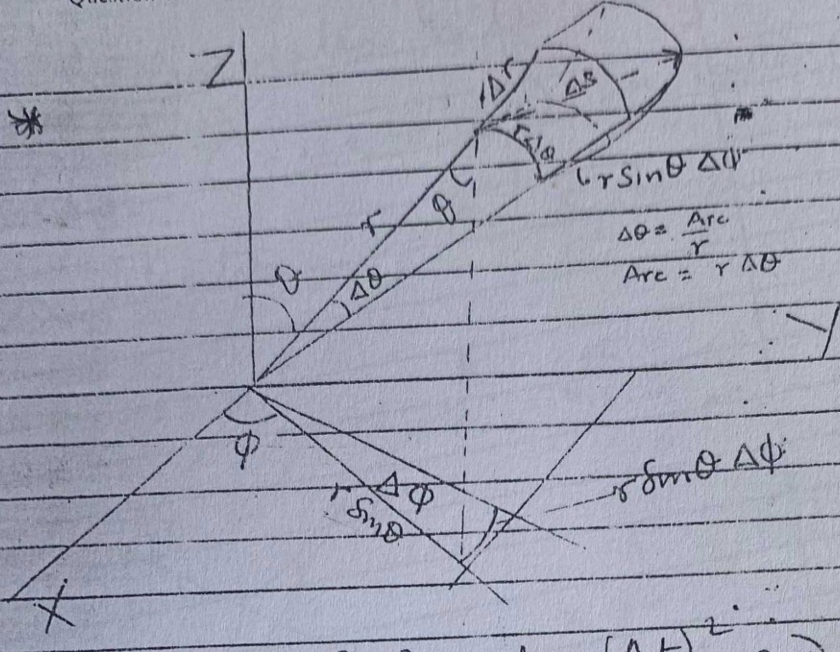

Fig 1.13

Dividing (47) by $(\Delta t)^2$ and taking the limit as $\Delta t \to 0$,

$$\dot{s}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \qquad (48)$$

In the two-dimensional oblique system, Fig 1.10, let P be given a small general displacement Δs. It is seen that

$$(\Delta s)^2 = (\Delta q_1)^2 + (\Delta q_2)^2 + 2(\Delta q_1)(\Delta q_2)\cos(\beta-\alpha)$$

thus,

$$\dot{s}^2 = \dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos(\beta-\alpha)$$

GENERALIZED VELOCITIES

Let the dynamical system be comprised of N particles of masses m_i $(i = 1, 2, \ldots N)$ and at time t.

Suppose each particle is specified

Write on both sides of the paper

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generalized Coordinates q_j $(j = 1, 2, \dots, n)$.

The n quantities

$$\dot{q}_j = \frac{dq_j}{dt} \quad (j = 1, 2, \dots, n)$$

are called generalized velocities of the system. (51)

WORK

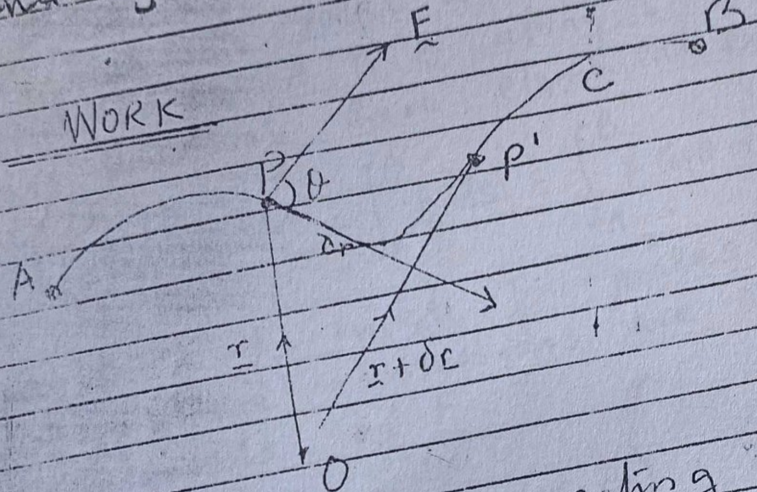

Fig 1.14

Let $\underset{\sim}{F}$ denote a force acting on a particle constrained to move along a curve C joining points A and B. Consider two neighbouring points P, P' on C with (52)

$$\underset{\sim}{OP} = \underline{r}$$
$$\underset{\sim}{OP'} = \underline{r} + \delta \underline{r}$$
$$PP' = \delta \underline{r} \equiv \delta \underline{s}$$

where $\underline{s}$ is the arc length of C. The work done by $\underset{\sim}{F}$ in moving its point of application from P to P' is defined to be the scalar product of $\underset{\sim}{F}$ and δr (53)

$$\text{i.e } \underline{F} \cdot \delta \underline{r} \text{ or } \underline{F} \cdot \delta \underline{s} = \delta W.$$

It is the resolved part of F along the

(21)

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uestion

gent at P directed towards P'

(a)plied by the element of arc, length

Thus the total work done in moving

particle from A to B along C is the

niting Sum of there elemental expressions (54)

i.e $W = \int_A^B \underset{\sim}{F} \cdot d\underline{r} = \int_A^B \underset{\sim}{F} \cdot ds = \int_A^B F \cos\theta \, ds$

where θ is the angle $\underset{\sim}{F}$ makes with $\delta \underline{r}$

at P. There expressions are termed

line integrals of $\underline{F}$

KINETIC ENERGY (K.E)

If the particle P in fig 1.14 has Constant

mass m and Velocity V at P (55)

$\underline{F} \cdot d\underline{r} = m\left(\frac{d\underline{v}}{dt}\right)(\underline{v}\, dt)$

$d(\underline{v}\cdot\underline{v}) = d\underline{v}\cdot\underline{v} + \underline{v}\cdot d\underline{v}$

$= 2 d\dot{v}\cdot v$

$= m(d\underset{\sim}{V} \cdot \underline{V})$

$\Rightarrow \frac{1}{2} d(v \cdot v) = d\dot{v} \cdot v$ (56)

$= \frac{1}{2} m\, d(\underline{V} \cdot \underline{V})$

$= \frac{1}{2} m\, d(\underset{\sim}{V}^2)$

Hence, the total work done on the particle is

$W = \int_A^B \underset{\sim}{F} \cdot d\underline{r} = \frac{1}{2} m \int_A^B d(\underset{\sim}{V}^2)$

$= \frac{1}{2} m \left[\underline{V}^2\right]_A^B$ (57)

$= \frac{1}{2} m V_B^2 - \frac{1}{2} m V_A^2$

(21)

Write on both sides of the paper

Question ....

gent at P directed towards P'

(ti)plied by the element of arc length

Thus the total work done in moving

particle from A to B along C is the

limiting Sum of these elemental expressions

i.e

$$W = \int_A^B \underline{F} \cdot d\underline{r} = \int_A^B \underline{F} \cdot d\underline{s} = \int_A^B F \cos\theta \, ds \quad (54)$$

where θ is the angle $\underline{F}$ makes with $\delta\underline{r}$

at P. these expressions are termed

line integrals of $\underline{F}$

KINETIC ENERGY (K E)

If the particle P in fig 1.14 has constant

mass m and velocity $\underline{V}$ at P :

$$\underline{F} \cdot d\underline{r} = m\left(\frac{d\underline{v}}{dt}\right) \cdot \underline{v}\,dt \quad (55)$$

$$d(\underline{v} \cdot \underline{v}) = d\underline{v} \cdot \underline{v} + \underline{v} \cdot d\underline{v} = 2 d\underline{v} \cdot \underline{v}$$

$$= m(d\underline{V} \cdot \underline{V})$$

$$\frac{1}{2} d(\underline{v} \cdot \underline{v}) = d\underline{v} \cdot \underline{v}$$

$$= \tfrac{1}{2} m \, d(\underline{V} \cdot \underline{V}) \quad (56)$$

$$= \frac{1}{2} m \, d(\underline{V}^2)$$

Hence, the total work done on the particle

is

$$W = \int_A^B \underline{F} \cdot d\underline{r} = \frac{1}{2} m \int_A^B d(\underline{V}^2)$$

$$= \frac{1}{2} m \left[\underline{V}^2\right]_A^B$$

$$= \frac{1}{2} m V_B^2 - \frac{1}{2} m V_A^2 \quad (57)$$

Thus the quantity $T = \frac{1}{2}mv^2$ has the same dimensions as work and is termed the K·E of the particle.

Thus, we have shown that the total work done by the external force F in carrying the particle from A to B along C is equal to the K·E gained in the process.

POWER

Since $\underset{\sim}{F} \cdot \delta \underset{\sim}{r}$ is the work done by the force in moving the particle from P to P', if the motion takes place in time δt, the rate of working ~~of the~~ force at P is

$$P = \lim_{\delta t \to 0} \left(\underset{\sim}{F} \cdot \frac{\delta \underset{\sim}{r}}{\delta t} \right) = \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}$$

$$= \underset{\sim}{F} \cdot \underset{\sim}{V} \qquad (58)$$

This is known as power or activity of the force. Denoting it by P, we see that the work done is

$$W = \int_{t_A}^{t_B} P\,dt = \int_{t_A}^{t_B} \underset{\sim}{F} \cdot \underset{\sim}{V}\,dt \qquad (59)$$

REMARKS

If we denote the K·E by T, then

$$\frac{dT}{dt} = \frac{d}{dt}\left[\frac{1}{2}mV^2\right] \qquad \frac{d}{dt}V^2 = \frac{d}{dt}V \cdot V = 2V\frac{dv}{dt}$$

$$= m\underset{\sim}{V}\left(\frac{d\underset{\sim}{V}}{dt}\right)$$

$$= \underset{\sim}{F} \cdot \underset{\sim}{V} = P = \text{Power} \qquad (60)$$

(23)

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CONSERVATIVE FIELDS.

When W depends on the positions of A and B and not at all on the curve C, such forces are referred to as Conservative

THEOREM

A force field F is Conservative iff there exists a Continuously differentiable scalar field V such that $F = -\nabla V$ or, $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ equivalently, if and only if

$$\nabla \times F = \text{Curl } F = 0 \quad \text{identically} \quad (6.1)$$

THEOREM

A Continuously differentiable force field F is Conservative iff for any closed non-intersecting curve C (simple closed curve)

$$\oint_C F \cdot dr = 0 \quad (6.2)$$

i.e the total work done in moving a particle around any closed path is zero.

DEFINITION (Force field)

If F is uniquely defined at every point of a region of space, then the sum total of all such vector forces F throughout the space is referred to as field of force

EXAMPLES

1 The earth's gravitational field:

Except over large distances, the acceleration g due to gravity is constant.

Define conservative field and give three examples to justify your defination.

Write on both sides of the paper Question (24) Do not write in either margin

Thus, the force on a particle falling freely under gravity, i.e its weight $\underset{\sim}{w}$, is (63)

$$\underset{\sim}{w} = m\underline{g}$$

$$\therefore W = \int_A^B \underline{w}\cdot d\underline{r} = m\int_A^B \underline{g}\cdot d\underline{r} = m\int_A^B d(\underline{g}\cdot\underline{r})$$ (64)

$$= m\left[\underline{g}\cdot\underline{r}\right]_A^B$$

$d(\underline{g}\cdot\underline{r}) = \underline{g}\cdot d\underline{r} + \underline{r}\cdot d\underline{g}$
$= \underline{g}\cdot d\underline{r} + 0$
s/c $d\underline{g} = 0$ ($\underline{g}$ const)

$$= m\left[\underline{g}\cdot\underline{r}_B - \underline{g}\cdot\underline{r}_A\right]$$ (65)

$$= m\underline{g}\cdot(\underline{r}_B - \underline{r}_A)$$

This last expression only involves the position vectors of A and B. Thus, the earth's gravitational field is conservative.

$\hat{r} = \frac{\underline{r}}{|\underline{r}|}$

A Central force field.

Let $\underline{F} = F(r)\,\hat{\underline{r}}$ (66)

The work done by $\underline{F}$ is

$r^2 = \underline{r}\cdot\underline{r}$, $r\,dr = \underline{r}\cdot d\underline{r}$, $\hat{\underline{r}}\cdot d\underline{r} = dr$

$$W = \int_A^B \underline{F}\cdot d\underline{r} = \int_A^B F(r)\,\hat{\underline{r}}\cdot d\underline{r} = \int_A^B F(r)\,dr$$ (67)

This integral is dependent only on the functional form of $f(r)$ and on the endpoints A and B. Hence the ~~force~~ field of Force F is Conservative.

Force exerted by an elastic Spring.

$F = K\hat{\underline{S}}$, where K is a Constant (spring const)

$\hat{\underline{r}}\cdot d\underline{r} = dr$

$$W = K\int_A^B \hat{\underline{S}}\cdot d\underline{S} = K\int_A^B dS$$ (68)

$$= K(S_B - S_A)$$ (69)

$S_B - S_A$ is the length

But $S_B - S_A$ is the length of the arc C and this will vary as C is varied. Hence the field of force is non-conservative.

POTENTIAL ENERGY

We define the potential energy V for the case of a conservative field of force to be the work done by the force moving the particle from its existing position to some standard position

ie if the existing position be given by vectors $\underset{\sim}{r} \equiv \underset{\sim}{OP}$ and the standard position by $\underset{\sim}{r_0} \equiv \underset{\sim}{OP_0}$, the P.E is

$$V = \int_{r}^{r_0} \underset{\sim}{F} \cdot d\underline{r} \qquad (70)$$

Define

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \qquad (71)$$

Then

$$\underset{\sim}{V} = V(x, y, z) \qquad (72)$$

Let

$$\underset{\sim}{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k} \qquad (73)$$

Since

$$d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k} \qquad (74)$$

then from (70)

$$V = -\int_{r_0}^{r} (F_1 dx + F_2 dy + F_3 dz) \qquad (75)$$

In a conservative force field

$$F_1 = -\frac{\partial V}{\partial x}, \quad F_2 = -\frac{\partial V}{\partial y}, \quad F_3 = -\frac{\partial V}{\partial z} \qquad (76)$$

ie

$$\underset{\sim}{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\underline{i} + \frac{\partial V}{\partial y}\underline{j} + \frac{\partial V}{\partial z}\underline{k}\right) \qquad (77)$$

(26)

$$V = \int_{r_0}^{r} dV = \int_{r_0}^{r} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) \quad (75*)$$

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from (75) and (75*)

$$F_1 = -\frac{\partial V}{\partial x}, \quad F_2 = -\frac{\partial V}{\partial y}, \quad F_3 = -\frac{\partial V}{\partial z}. \quad (76)$$

i.e

$$\underset{\sim}{F} = -\left[\frac{\partial V}{\partial x} \underline{i} + \frac{\partial V}{\partial y} \underline{j} + \frac{\partial V}{\partial z} \underline{k} \right] \quad (77)$$

$$= -\nabla V \quad (78)$$

It thus follows that for such a Conservative field of force,

$$\text{Curl} \underset{\sim}{F} = \nabla \times \underline{F} = 0 = -\nabla \times (\nabla V) \quad (79)$$

Remark

$$\text{Curl} \underset{\sim}{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0 \quad (80)$$

Using (77)

$$\underline{F} \cdot \delta \underline{r} = -\left\{ \frac{\partial V}{\partial x} \partial x + \frac{\partial V}{\partial y} \partial y + \frac{\partial V}{\partial z} \partial z \right\} \quad (81)$$

$$= -\partial V$$

$$\int_A^B \underline{F} \cdot d\underline{r} = -\int_A^B dV = V_A - V_B \quad (82)$$

where V is termed the P.E

from the previous lecture Work done is

$$W = \int_A^B \underset{\sim}{F} \cdot d\underline{r} = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 \quad (83)$$

from (82) and (83), for Conservative force we have

$$\frac{1}{2} m v_A^2 + V_A = \frac{1}{2} m v_B^2 + V_B \quad (84)$$

Go to next page

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a Scalar function V such that $F =$ (24)
the P.E or simply potential

Hence from

$$V = \int_{r_0}^{r} dV = \int_{r_0}^{r} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) \qquad (78)$$

~~from (75) and (78)~~ $\cancel{F = -\left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k\right) = -\nabla V}$

It follows from fig 1.14 that

$$F \cdot \partial r = -\left\{ \frac{\partial V}{\partial x} \partial x + \frac{\partial V}{\partial y} \partial y + \frac{\partial V}{\partial z} \partial z \right\} \qquad (79)$$

$$= -\partial V \qquad (80)$$

$$\therefore \int_A^B F \cdot dr = -\int_A^B dV = \int_B^A dV \qquad (81)$$

$$= V_A - V_B \qquad (82)$$

From the previous lecture ~~total~~ Work done is

$$W = \int_A^B F \cdot dr = \frac{1}{2} m V_B^2 - \frac{1}{2} m V_A^2 \qquad (83)$$

from (82) and (83), for Conservative force we have

$$\frac{1}{2} m V_A^2 + V_A = \frac{1}{2} m V_B^2 + V_B \qquad (84)$$

The quantity $E = \frac{1}{2} m v^2 + V$, which is the sum of the K.E and P.E, is called the total Energy. From (84) we see that the total energy at A is the same as the total energy at B.

Thus, in Conservative force field, the total energy is a Constant. This is called the principle of Conservation of energy.

80/1

REMARKS

1. In rectangular coordinates the K.E of a particle is

$$K \cdot E = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), \qquad (85)$$

In spherical coordinates

$$K \cdot E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) \qquad (86)$$

and in two-dimensional oblique system

$$K \cdot E = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos(\beta - \alpha)\right) \qquad (87)$$

2. The principle of conservation of energy holds for a single particle as well as a system of particles moving in a conservative field of force.

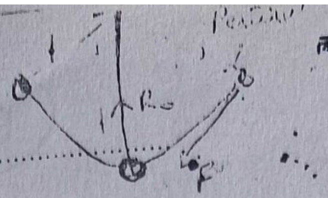
EXAMPLES

1. A smooth wire bent in the form of a parabola is fixed with its axis vertical and vertex downwards. A particle of mass m oscillates on the wire coming to rest at the extremities of the latus rectum. Show that the reaction, R_o, of the wire on the particle when passing through the vertex is $2mg$. (Hint: Assume the equation of the wire is $X^2 = 4ay$ and the radius of curvature ρ_o at O is $2a$)

Curvature $\rho_o = \frac{1}{2}$ Lactus Rectum

radius of

(29)



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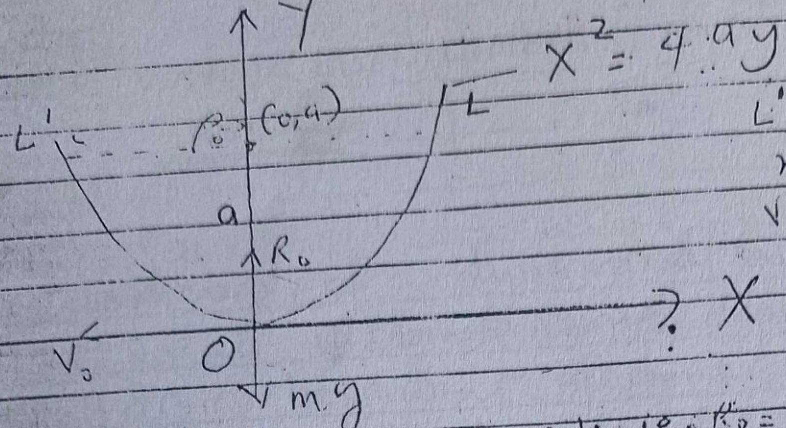
Soln



$x^2 = 4ay$

$L'O$ = radius of Curv P_0

$L'L$ = Latus Rectum

$x = 0$ = axis

Vertex = (0, 0)

Directrix = $y = -a$

Focus = (0, a)

Action and reaction are equal and opposite ie $R_0 = mg$.
but b/c motion is circular, $R_0 - mv^2 = mg$

The motion is Simple harmonic

$F = ma$

Reaction $R_0 - mg = \frac{mV_0^2}{P_0}$ since motion is circular

= centripetal force $\frac{mv^2}{r}$ since

$V = \omega r \Rightarrow \omega = \frac{V}{r}$

$a = \omega V \Rightarrow a = \frac{V^2}{r}$ angular ~~acceleration~~

P_0 = radius of Curvature

(88)

$$R_0 = mg + \frac{mV_0^2}{2a} = m\left(g + \frac{V_0^2}{2a}\right)$$

(89)

At L KE = 0 but at O = $\frac{1}{2}mV_0^2$

At L PE = mga but at O = 0

Total Energy at L = mga

Total Energy at O = $\frac{1}{2}mV_0^2$

Since Total energy is constant as Force producing S.h.m is conservative ($F = -kx$)

$$\frac{1}{2}mV_0^2 = mga$$

(90)

$$\Rightarrow V_0^2 = 2ga$$

$$\therefore R_0 = m\left(g + \frac{2ga}{2a}\right)$$

(91)

$$\Rightarrow R_0 = 2mg \text{ as reqd.}$$

(92)

PRINCIPLES OF VIRTUAL DISPLACEMENTS AND VIRTUAL WORK

Definition:

If the particles of a dynamical System undergo a small instantaneous displacement denoted by $\delta \underset{\sim}{r_i}$ independent of time in Conformity with the Constraints of the System and such that all internal and external forces remain unchanged in magnitude and direction during the displacement then such a displacement is said to be Virtual. This is a fictitious motion and so we say it is hypothetical. True displacement denoted by $d\underset{\sim}{r_i}$ occurs in a time interval where forces and constraints could be changing.

* Define virtual displacement

VIRTUAL WORK

Let the i^{th} particle m_i at position $\underset{\sim}{r_i}$ at time t undergo a virtual displacement to position $\underline{r_i + \delta r_i}$. Let $\underline{F}_i$, $\underline{F}_i'$ be the external and internal forces respectively acting on m_i, we define the Virtual work done on m_i in the displacement by

$$\delta W_v = (\underline{F}_i + \underline{F}_i') \cdot \delta \underset{\sim}{r_i} \qquad (93)$$

The total Virtual work done on

particles of the system when similar displacements are made is

$$\delta W_v = \sum_{i=1}^{N} (F_i + F_i') \cdot \delta r_i = \sum_{i=1}^{N} F_i \cdot \delta r_i + \sum_{i=1}^{N} F_i' \cdot \delta r_i \quad (94)$$

The total virtual work done by the internal forces $\sum_{i=1}^{N} F_i' \cdot \delta r_i$ of the system is zero in many cases. In such cases, for an example, when the particles of the system are connected by rigid constraints,

$$\delta W_v = \sum_{i=1}^{N} F_i \cdot \delta r_i \quad (95)$$

$$= \sum_{i=1}^{N} (F_{1i}\delta x_i + F_{2i}\delta y_i + F_{3i}\delta z_i) \quad (96)$$

where

$$F_i = (F_{1i}, F_{2i}, F_{3i}) \quad (97)$$

$$\delta r_i = (\delta x_i, \delta y_i, \delta z_i) \quad (98)$$

δW_v is referred to as the virtual work function

The coefficients in δW_v are F_{1i}, F_{2i}, F_{3i}.

VIRTUAL WORK IN GENERALIZED COORDINATES.

Consider a dynamical system specified by the n generalized coordinates q_j $(j = 1, 2, \dots, n)$

Such that we can change q_j to $q_j + \delta q_j$ without making corresponding changes in the other $(n-1)$ coordinates the independent

Let this virtual displacement take effect instantaneously and suppose the corresponding work done on the dynamical system to be $Q_j \delta q_j$

Then

$$Q_j \delta q_j = \sum_{i=1}^{N} F_i \cdot \delta r_i \qquad (99)$$

Consider now similar changes in each q_j $(j = 1, 2, \ldots n)$ then

$$\delta W_v = \sum_{j=1}^{n} Q_j \delta q_j = \sum_{j=1}^{n} \sum_{i=1}^{N} F_i \cdot \delta r_i \qquad (100)$$

The virtual work function δW_v has been constructed from the external forces alone acting on the system and we call the generalized force Q_j associated with coordinate q_j $(j = 1, 2, \ldots, n)$ the coefficient of the generalized virtual displacement δq_j.

REMARKS

1. Theorem: A system of particles is in equilibrium iff the total virtual work of the actual forces is zero i.e

$$\sum_{i=1}^{N} F_i \cdot \delta r_i = 0 \qquad (101)$$

This is often called the principle of virtual work.

F_i is also called the actual force while F_i' is called the constraint force ~~....~~, acting on the i^{th} particle.

$\sum F_i' \cdot \delta r_i = 0$ is true for rigid bodies and for motion on curves and surfaces without friction.

EXAMPLE

Suppose that the internal forces of a system of particles are conservative and are derived from a potential

$$V_{jk}(r_{jk}) = V_{kj}(r_{kj}) \qquad (102)$$

$$\text{where } r_{jk} = r_{kj} = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2} \qquad (103)$$

is the distance between particles j and k of the system.

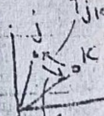
(a) Prove that

$$\sum_j \sum_k \underset{\sim}{f}_{jk} \cdot d\underset{\sim}{r}_k = -\frac{1}{2} \sum_j \sum_k dV_{jk} \qquad (104)$$

force on j due to k

where f_{jk} is the internal force on particle j due to particle k.

(b) Sketch a diagram of the above discription and indicate r_{kj}.



Soln

The force acting on particle k is

See (102) $V_{jk} = V_{kj}$

$$\underset{\sim}{f}_{kj} = -\frac{\partial V_{jk}}{\partial x_k}\underline{i} - \frac{\partial V_{jk}}{\partial y_k}\underline{j} - \frac{\partial V_{jk}}{\partial z_k}\underline{k} \qquad (105)$$

$$= -\text{grad}_k V_{jk}$$

$$= -\nabla_k V_{jk} \qquad (106)$$

Since the system is conservative

(34)

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The force acting on particle j is

$$\underset{\sim}{f}_{jk} = -\frac{\partial V_{jk}}{\partial x_j}\underline{i} - \frac{\partial V_{jk}}{\partial y_j}\underline{j} - \frac{\partial V_{jk}}{\partial z_j}\underline{k} \quad (107)$$

$$= -\text{grad}_j V_{jk} \quad (108)$$

$$= -\nabla_j V_{jk} = -\underset{\sim}{f}_{kj} \quad (109)$$

The work done by these forces in producing the displacements dr_k and dr_j of particles k and j respectively is

$$\underset{\sim}{f}_{kj}\cdot d\underset{\sim}{r}_k + \underset{\sim}{f}_{jk}\cdot d\underset{\sim}{r}_j = -\left\{\frac{\partial V_{jk}}{\partial x_k}dx_k + \frac{\partial V_{jk}}{\partial y_k}dy_k + \frac{\partial V_{jk}}{\partial z_k}dz_k\right.$$

$$\left.+\frac{\partial V_{jk}}{\partial x_j}dx_j + \frac{\partial V_{jk}}{\partial y_j}dy_j + \frac{\partial V_{jk}}{\partial z_j}dz_j\right\} \quad (110)$$

$$= -dV_{jk} \quad (111)$$

Then the total work done by the internal forces is

$$\sum_k \sum_j \underset{\sim}{f}_{jk}\cdot d\underset{\sim}{r}_k = -\frac{1}{2}\sum_k \sum_j dV_{jk} \quad (112)$$

the factor 1/2 on the right being introduced because otherwise the terms in the summation would enter twice.

$$\sum_k^2 \sum_j^2 dV_{jk} = \sum_k^2 \{dV_{1k} + dV_{2k}\}$$

$$= dV_{11} + dV_{12} + dV_{21} + dV_{22}$$

$$= dV_{12} + dV_{21} \quad \text{b/c } dV_{12} = dV_{21}$$

$$= 2dV_{12}$$

It is not reasonable to say work done is 2 × P.E. ie $2dV_{12}$ ∴ we divide by 1/2

Sum $\sum_k \sum_j \underset{\sim}{f}_{jk}\cdot d\underset{\sim}{r}_k$ ... double sum

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CHAPTER THREE

HOLONOMIC AND NON-HOLONOMIC SYSTEMS.

Defn:

· A holonomic System is one for which each separate q_r of the n generalized Coordinates q_j $(j = 1, 2, \ldots n)$ Can be changed from q_r to $q_r + dq_r$ without any changes in the remaining $(n-1)$ Coordinates. Otherwise it is a non-holonomic System.

In other words, in a holonomic System, the generalized Coordinates q_r and Velocities $\dot{q}_r$ $(r = 1, \ldots n)$ are independent of each other while in non-holonomic system they are dependent on each other.

Hence, in a holonomic Systems, to which we shall limit our Consideration, it is possible to Consider a displacement, keeping time fixed consistent with geometrical Constraints, so that q_k is unchanged for $k \neq r$ and q_r becomes

$$q_r + \delta q_r \qquad (113)$$

Such a displacement is known as a Virtual or instantaneous displacement (Since time is Supposed to be unchanged during such displacement).

The position Vector $\underline{r}_i$ of an arbitrary point P_i of the System is then changed from

$$\underline{r}_i = \underline{r}_i(q_r, t) \qquad (114)$$

to

36

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$$\underline{r}_i + \delta \underline{r}_i = \underline{r}_i + \sum_{r=1}^{m} \frac{\partial \underline{r}_i}{\partial q_r} \delta q_r \quad (115)$$

$$\left(\text{Since} \quad \delta \underline{r}_i = \sum_{r=1}^{m} \frac{\partial \underline{r}_i}{\partial q_r} \delta q_r + \frac{\partial \underline{r}_i}{\partial t} \delta t \right) \quad (116)$$

but for Virtual displacement, $\delta t = 0$

Hence, the work done by the force acting at P_i is proportional to δq_r, so that the total work done by the forces acting at P_i is of the form $Q_r \delta q_r$. The constant of proportionality is known as the generalized force Q_r corresponding to the generalized coordinate q_r

GENERALIZED COMPONENT OF MOMENTUM.

Consider a translation of a rigid body without rotation whose centroid is at (x, y, z) at time t. If the mass is M, and we denote the K.E by T

$$T = \frac{1}{2} M \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) \quad \text{(from the origin)} \quad (117)$$

Let $P_x = \dfrac{\partial T}{\partial \dot{x}} = M\dot{x}$

$P_y = M\dot{y} \quad (118)$

$P_z = M\dot{z}$

then (P_x, P_y, P_z) are the components of linear momentums in the X, Y, Z directions. Similarly, if the rigid body rotates about an axis L which is fixed through its centroid then

$$T = \frac{1}{2} I \dot{\theta}^2 \qquad (119)$$

and

$$P_\theta = \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} \qquad (120)$$

where P_θ = angular momentums about L
I = moment of Inertia about L
θ = the angle turned through in time t.

If we now have a holonomic system specified by generalized coordinates q_j $(j = 1, 2, \ldots n)$ at time t, we can evaluate the K.E denoted by T

$$T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) \qquad (121)$$

and then form the quantities

$$P_j = \frac{\partial T}{\partial \dot{q}_j} \quad (j = 1, 2, \ldots, n). \qquad (122)$$

These are called the generalized components of momentum of the systems.
We often call P_j the momentums conjugate to q_j or the conjugate momentum.

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Question ....

LAGRANGE'S EQUATIONS OF MOTION

We shall consider a holonomic dynamical system and establish the following equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \; (j = 1, 2, \ldots, n) \quad (123)$$

Called the Lagrange's equations of motion where T is the K.E of the system at time t when the system is specified by the n generalized coordinates q_j $(j = 1, 2, \ldots, n)$ and Q_j $(j = 1, 2, \ldots n)$ are the generalized forces.

Note that T is a function of the $(2n+1)$ variables
$(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$
This is the reason why T is differentiated w.r.t q_j and $\dot{q}_j$

DERIVATION

Let the dynamical system be comprised of N particles of masses M_i $(i = 1, 2, \ldots N)$ and at time t. Let $\underline{r}_i$ be the position vector of the i^{th} particle M_i at time t so that

$$\underline{r}_i = \underline{r}_i (q_1, q_2, \ldots, q_n; t) \quad (124)$$

and $\underline{F}_i$ be the net external force acting on the i^{th} particle of the system then

$$\frac{d\underline{r}_i}{dt} = \dot{\underline{r}}_i = \frac{\partial \underline{r}_i}{\partial q_1}\dot{q}_1 + \dots + \frac{\partial \underline{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \underline{r}_i}{\partial t} \quad (125)$$

Differentiating the above term by term w.r.t $\dot{q}_j$ we have

$$\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_1} + \dots + \frac{\partial \underline{r}_i}{\partial q_n} \quad (126)$$

$$= \frac{\partial \underline{r}_i}{\partial q_j} \quad (127)$$

Differentiating (125) w.r.t q_j we have

$$\frac{\partial \dot{\underline{r}}_i}{\partial q_j} = \frac{\partial^2 \underline{r}_i}{\partial q_j \partial q_1}\dot{q}_1 + \dots + \frac{\partial^2 \underline{r}_i}{\partial q_j \partial q_n}\dot{q}_n + \frac{\partial^2 \underline{r}_i}{\partial q_j \partial t} \quad (128)$$

Now

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)\frac{dq_1}{dt} + \dots + \frac{\partial}{\partial q_n}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)\frac{dq_n}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)$$

$$= \frac{\partial^2 \underline{r}_i}{\partial q_1 \partial q_j}\dot{q}_1 + \dots + \frac{\partial^2 \underline{r}_i}{\partial q_n \partial q_j}\dot{q}_n + \frac{\partial^2 \underline{r}_i}{\partial t \partial q_j} \quad (129)$$

Since $\underline{r}_i$ is assumed to have continuous second order partial derivates, the order of differentiation does not matter.

Therefore from (128) and (129)

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\underline{r}}_i}{\partial q_j} \quad (130)$$

The result can be interpreted as an interchange of order of the operators i.e

$$\frac{d}{dt}\left(\frac{\partial}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d}{dt}\right) \quad (131)$$

Suppose that the system undergoes increment $dq_1, dq_2, \ldots, dq_n$ of the generalized coordinate, then the i^{th} particle undergoes a displacement

$$d\underline{r}_i = \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} dq_j \quad (132)$$

Thus the total work done is

$$dW = \sum_{i=1}^{N} \underline{F}_i \cdot d\underline{r}_i \quad (133)$$

$$= \sum_{i=1}^{N} \left\{ \sum_{j=1}^{n} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right\} dq_j$$

$$= \sum_{j=1}^{n} \phi_j \, dq_j \quad * \quad (134)$$

where $$\phi_j = \sum_{i=1}^{n} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \quad (135)$$

We call ϕ_j the generalized force associated with the generalized coordinate q_j

Furthermore,

$$dW = \sum \frac{\partial W}{\partial q_j} dq_j \quad * \quad (136)$$

Using (134) and (136)

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Question (41)

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$$\sum \left(\Phi_j - \frac{\partial W}{\partial q_j} \right) dq_j = 0 \quad (137)$$

Thus since the dq_j are independent, all coefficients of dq_j must be zero, so that

$$\Phi_j = \frac{\partial W}{\partial q_j} \quad (138)$$

Newton's Second law applied to the i^{th} particle ~~gives~~ being acted upon by $\underset{\sim}{F}_i$ gives

$$m_i \ddot{\underset{\sim}{r}}_i = \underset{\sim}{F}_i \quad (139)$$

Then

$$m_i \ddot{\underset{\sim}{r}}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} = \underset{\sim}{F}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} \quad (140)$$

In view of (129) and (130)

$$\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} \right) = \ddot{\underset{\sim}{r}}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} + \dot{\underset{\sim}{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \underset{\sim}{r}_i}{\partial q_j} \right)$$

$$= \ddot{\underset{\sim}{r}}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} + \dot{\underset{\sim}{r}}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \quad (141)$$

Thus

$$\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} \right) - \dot{\underset{\sim}{r}}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \quad (142)$$

Hence from (140) we have, since m_i is a constant,

$$\frac{d}{dt}\left(m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}\right) - m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_j} = \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \qquad (143)$$

Summing both sides from $i=1$ to $i=N$, we have

$$\frac{d}{dt}\left\{\sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}\right\} - \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_j} = \sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \qquad (144)$$

The K.E is $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\underline{r}}_i^2$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$$

$$\frac{d(\dot{r}_i \cdot \dot{r}_i)}{dq} = \dot{r}_i \cdot \frac{d\dot{r}_i}{dq} + \frac{d\dot{r}_i}{dq} \cdot \dot{r}_i = 2\dot{r}_i \cdot \frac{d\dot{r}_i}{dq}$$

Thus

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_j} \qquad (145)$$

Using (126) and (127)

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}$$

for 127 $\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$

$$= \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \qquad (146)$$

Substituting (145) and (146) into (144), we find

see 180

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \Phi_j \qquad (147)$$

The quantity

$$p_j = \frac{\partial T}{\partial \dot{q}_j} \qquad (148)$$

is called the generalized momentum or conjugate momentum associated with the generalized coordinate q_j.

LAGRANGE'S EQUATION FOR CONSERVATIVE FORCES

When the forces are conservative and the system is specified by the generalized coordinates q_j ($j = 1, 2, \ldots n$), we ~~showed~~ know that

$$\text{Curl } \underset{\sim}{F} = 0 \qquad (149)$$

which implies there exists a potential function V dependent on position only i.e

$$V = V(q_1, \ldots, q_n) \text{ Satisfying} \qquad (150)$$

$\underline{r} = \underline{r}(q_1, q_2 \ldots q_n)$ $\therefore d\underline{r} = (dq_1, dq_2, \ldots dq_n)$ $\delta \underline{r} = (\delta q_1, \ldots \delta q_n) \hat{r}$

$$\underline{F} = -\nabla V \qquad (151)$$

$$\underset{\sim}{F} \cdot \delta \underline{r} = \left(-\sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \hat{r}_j \right) \cdot \left(\sum_{j=1} \delta q_j \hat{r}_j \right)$$

(A vector is the magnitude of the vector $\times$ the unit vector in its direction. eg $\hat{a} = \frac{a}{|a|}$)

$$= -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \qquad (152)$$

From (82) we know that

$$\underline{F} \cdot \delta \underline{r} = -\delta V \qquad (153)$$

$\therefore$ from 152 and 153

$$\delta V = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \qquad (154)$$

But from (150)

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j = \sum_{j=1}^{n} \sum_{i=1}^{N} \underset{\sim}{F_i} \cdot \delta r_i \qquad (155)$$

$$= \underset{\sim}{F} \cdot \delta \underline{r} \qquad (156)$$

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$$= -\delta V$$

$$= -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j \qquad (157)$$

Equating Coefficients we obtain

$$Q_j = -\frac{\partial V}{\partial q_j} \qquad j = 1, \ldots, n \qquad (158)$$

So the Lagrange's equations for a Conservative holonomic dynamical System become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \qquad j = 1, \ldots, n \qquad (159)$$

it is not actually a fn of $\dot{q}_j$ in eqn (159)

If V is not explicitly a function of generalized velocities $\dot{q}_j$, then it is normal to define the Lagrangian function

$$L = T - V \qquad (160)$$

$$\Rightarrow T = L + V \qquad (161)$$

Thus (147) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - \cancel{\frac{\partial V}{\partial q_j}} = -\cancel{\frac{\partial V}{\partial q_j}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (162)$$

it depends only on position

Since V is not a function of $\dot{q}_j$, then

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \qquad (163)$$

(the potential or potential energy is a function of only the q_j's (and possibly the time t))

Eqn (162) then becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, \ldots, n \qquad (164)$$

This is the most Common form of the Lagrange's equations and L is called "the Lagrangian" or "the Kinetic potential" or the Lagrangian function".

We have derived Lagrange's equations from Newton's laws of motion based on the Concept of Virtual work and expressing the results by means of generalized Coordinates and forces.

We have seen that a system having n-degrees of freedom will in general be described by n-second order ordinary differential equations. These equations are equivalent to the equation of motion which would have been obtained by a direct application of Newton's laws. Hence they don't Contain new and independent physical principles.

But nevertheless the Lagrangian method of obtaining the equations of motion is more systematic and frequently easier to apply than Newton's laws. Only Velocities and displacements (46) enter into the Lagrangian formula. Once L is found, the procedure for obtaining the equations of motion are quite straight forward.

REMARKS

1 Lagrange's equation for a holonomic dynamical system specified by a generalized coordinates q_j $(j = 1, 2, \ldots, n)$ are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad (j = 1, 2, \ldots n) \qquad (65)$$

where T is the K.E of the system and Q_j is the coefficient of ∂q_j in the virtual work function

$$\partial W = \sum_{j=1}^{n} Q_j \partial q_j \qquad (66)$$

constructed solely from the forces which do work in displacement when we allow the generalized coordinates q_j $(j = 1, 2, \ldots n)$ to undergo instantaneous virtual increment ∂q_j $(j = 1, 2, \ldots n)$

2 The physical meaning of Lagrange's equation can best be seen from

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = Q_j + \frac{\partial T}{\partial q_j} ; \quad (j = 1, 2, \ldots n) \qquad (67)$$

which states that "the time rate of change of the scalar generalized component of momentum of the system is equal to the generalized force Q_j due to the applied forces plus the term $\frac{\partial T}{\partial q_j}$ which is an initial generalized force due to motion in the other generalized coordinates."

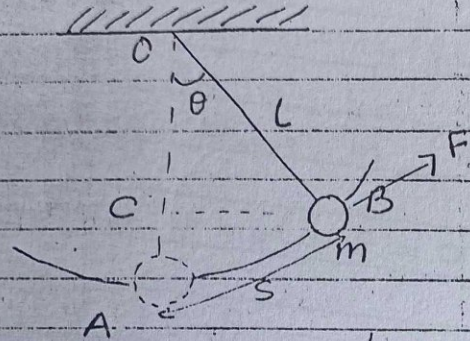
(47) Vol 2

ANALYTICAL DYNAMICS MAT342

AMPLES

Set up the Lagrangian for a simple pendulum and obtain an equation describing its motion

S/n



$\theta = \dfrac{\text{Arc}}{\text{radius}} = \dfrac{S}{L}$

$\Rightarrow S = \theta L$

We choose as generalized coordinate the angle θ

se θ is the angle between OB and OA

$$K.E = T = \tfrac{1}{2} m v^2 \quad , \; m = \text{mass of the bob} \tag{168}$$

But $S = \theta L$ = displacement of pendulum from A to B (169)

$\therefore \dot{S} = \dot{\theta} L = v$ (170)

$$\therefore T = \tfrac{1}{2} m (\dot{\theta} L)^2 = \tfrac{1}{2} m L^2 \dot{\theta}^2 \tag{171}$$

e P.E of m (taking as reference level a zontal plane through the lowest point A) is en by

$$V = mg(OA - OC) = mg(L - L\cos\theta)$$
$$= mgL(1 - \cos\theta) \tag{172}$$

s the Lagrangian is

$$L = T - V = \tfrac{1}{2} m L^2 \dot{\theta}^2 - mgL(1 - \cos\theta) \tag{173}$$

(b) Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (174)$$

From (173)

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \,, \quad \frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta$$

Substituting these in (174), we find

$$ml^2\ddot\theta + mgl\sin\theta = 0 \qquad (175)$$

or

$$\ddot\theta + \frac{g}{l}\sin\theta = 0 \qquad (176)$$

REMARKS (Essential)

In the diagram above

From Newton's 2nd law $F = -mg\sin\theta$ (177)

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

Since $s = l\theta$ and since l does not change with time. (178)

$$m\frac{d^2\theta}{dt^2} = -\frac{mg\sin\theta}{l} \qquad (179)$$

when θ is small ($\sin\theta = \theta$)

$$\ddot\theta = -\frac{g\theta}{l} \qquad (180)$$

It is observed that (176) is same as (180)

mass M_2 hangs at one end of a string which passes over a fixed frictionless non-slating pulley. At the other end of this string there is a non-rotating pulley of mass M_1 over which there is a string carrying masses m_1 and m_2.
(a) Set up the Lagrangian of the system
(b) Find the acceleration of mass M_2.

Ln

Let X_1 and X_2 be the distances of masses M_1 and M_2 respectively below the centre of the fixed pulley as in figure below

Let x_1 and x_2 be the distances of masses m_1 and m_2 respectively below the centre of the movable pulley M_1

Since the strings are fixed in length

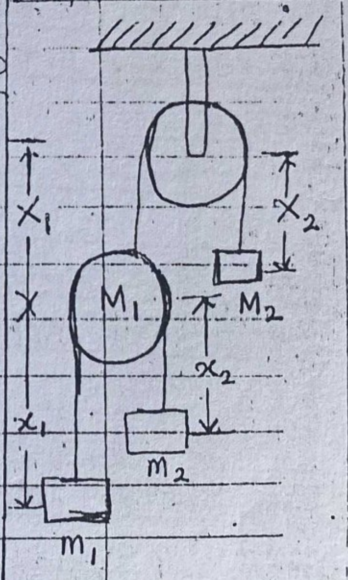


Fig.

$$X_1 + X_2 = \text{Constant} = a$$
$$x_1 + x_2 = \text{Constant} = b$$

Then by differentiating w.r.t t

$$\dot{X}_1 + \dot{X}_2 = 0 \text{ or } \dot{X}_2 = -\dot{X}_1$$

and $\quad \dot{x}_1 + \dot{x}_2 = 0$ or $\dot{x}_2 = -\dot{x}_1$

Thus we have

Velocity of $M_1 = \dot{X}_1$ (181)

Velocity of $M_2 = \dot{X}_2 = -\dot{X}_1$ (182)

Velocity of $m_1 = \frac{d}{dt}(X_1 + x_1) = \dot{X}_1 + \dot{x}_1$ (183)

Velocity of $m_2 = \frac{d}{dt}(X_1 + x_2) = \dot{X}_1 + \dot{x}_2 = \dot{X}_1 - \dot{x}_1$ (184)

Then the total kinetic energy of the system is

$$T = \frac{1}{2}M_1\dot{X}_1^2 + \frac{1}{2}M_2\dot{X}_1^2 + \frac{1}{2}m_1(\dot{X}_1+\dot{x}_1)^2 + \frac{1}{2}m_2(\dot{X}_1-\dot{x}_1)^2 \quad (185)$$

The total potential energy of the system measured from a horizontal plane through the Centre of the fixed pulley as reference is

$$V = -M_1 g X_1 - M_2 g X_2 - m_1 g(X_1+x_1) - m_2 g(X_1+x_2)$$

$$= -M_1 g X_1 - M_2 g(a-X_1) - m_1 g(X_1+x_1) - m_2 g(X_1+b-x_1) \quad (186)$$

then the Lagrangian is

$$L = T - V$$

$$= \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2\dot{X}_1^2 + \tfrac{1}{2}m_1(\dot{X}_1+\dot{x}_1)^2 + \tfrac{1}{2}m_2(\dot{X}_1-\dot{x}_1)^2$$
$$+ M_1 g X_1 + M_2 g(a-X_1) + m_1 g(X_1+x_1) + m_2 g(X_1+b-x_1) \quad (187)$$

Lagrange's equations corresponding to X_1 and x_1 are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \quad (188)$$

From (187), we have

$$\frac{\partial L}{\partial X_1} = M_1 g - M_2 g + m_1 g + m_2 g = (M_1 - M_2 + m_1 + m_2)g \quad (189)$$

$$\frac{\partial L}{\partial \dot{X}_1} = M_1\dot{X}_1 + M_2\dot{X}_1 + m_1(\dot{X}_1+\dot{x}_1) + m_2(\dot{X}_1-\dot{x}_1)$$

$$= (M_1+M_2+m_1+m_2)\dot{X}_1 + (m_1-m_2)\dot{x}_1 \quad (190)$$

$$\frac{\partial L}{\partial x_1} = m_1 g - m_2 g = (m_1-m_2)g \quad (191)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1(\dot{X}_1+\dot{x}_1) - m_2(\dot{X}_1-\dot{x}_1) = (m_1-m_2)\dot{X}_1 + (m_1+m_2)\dot{x}_1 \quad (192)$$

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so equations (188) become

$$(M_1+M_2+m_1+m_2)\ddot{X}_1 + (m_1-m_2)\ddot{x}_1 = (M_1-M_2+m_1+m_2)g \quad (193)$$

$$(m_1-m_2)\ddot{X}_1 + (m_1+m_2)\ddot{x}_1 = (m_1-m_2)g \quad (194)$$

Solving simultaneously, we find

$$\ddot{X}_1 = \frac{(M_1-M_2)(m_1+m_2)+4m_1m_2}{(M_1+M_2)(m_1+m_2)+4m_1m_2}\, g \quad (195)$$

$$\ddot{x}_1 = \frac{2M_2(m_1-m_2)}{(M_1+M_2)(m_1+m_2)+4m_1m_2}\, g \quad (196)$$

Then the downward acceleration of mass M_2 is constant and equal to

$$\ddot{X}_2 = -\ddot{X}_1 = \frac{(M_2-M_1)(m_1+m_2)-4m_1m_2}{(M_1+M_2)(m_1+m_2)+4m_1m_2}\, g \quad (197)$$

A particle of mass m moves in a conservative force field. Find

(a) the Lagrangian function in cylindrical coordinates

(b) the equations of motion in cylindrical coordinates (ρ, ϕ, z)

Soln

In cylindrical coordinates

$$\underset{\sim}{r} = \rho\cos\phi\, \underline{i} + \rho\sin\phi\, \underline{j} + z\,\underline{k} \quad (198)$$

$$\underset{\sim}{v} = \dot{\rho}\,\rho_1 + \rho\dot{\phi}\,\phi_1 + \dot{z}\,\underline{k}$$

$$\underset{\sim}{a} = (\ddot{\rho}-\rho\dot{\phi}^2)\underset{\sim}{\rho_1} + (\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})\underset{\sim}{\phi_1} + \ddot{z}\,\underline{k}$$

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Question ....

Write on both sides of the paper

Thus, the total K.E

$$T = \tfrac{1}{2} m(\underset{\sim}{v} \cdot \underset{\sim}{v}) \qquad (199)$$

$$= \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) \qquad (200)$$

The P.E $= V = V(\rho, \phi, z)$ (201)

Then the Lagrangian function is

$$L = T - V = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho,\phi,z) \qquad (202)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0 \text{ ie } \frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$$

$$\text{or} \quad m(\ddot{\rho} - \rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho} \qquad (203)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \text{ ie } \frac{d}{dt}(m\rho^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0$$

$$\text{or} \quad m\frac{d}{dt}(\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi} \qquad (204)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \text{ ie } \frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z}$$

$$\text{or} \quad m\ddot{z} = -\frac{\partial V}{\partial z} \qquad (205)$$

PROBLEM

What will be the equations of motion if the particle moves in the xy plane and if the potential depends only on the distance from ~~the~~ origin

Soln

In this Case V, depends only on ρ and $z = 0$.

Then Lagrange's equations in part (b) of the above problem become,

$$m(\ddot{\rho} - \rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho} \qquad (206)$$

$$M\frac{d}{dt}(\rho^2\dot{\phi}) = 0 \qquad (207)$$

These are the equations of motion in a Central force field obtained in MAT 241 i.e.

$$m(\ddot{r} - r\dot{\phi}^2) = f(r) \qquad r \equiv \rho$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (208)$$

Conservative

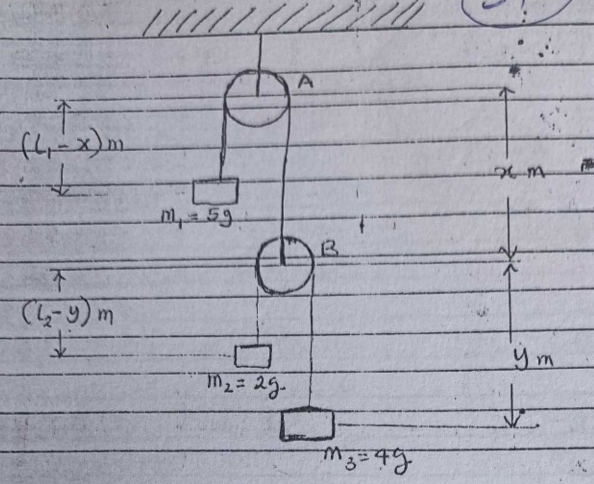
Consider the motion of a 3 particle system of pulley wheels having negligible masses and moments of inertia and their axles being frictionless. Let the axle of the movable pulley be x metres below that of the fixed one A, and let the 4 kg mass be y metres below the axle of the movable pulley. L_1 and L_2 are the constant lengths of two strings minus their overlaps round the wheels. Derive the downward accelerations of the particles of masses 5 kg, 2 kg and 4 kg respectively.

Soln

The generalized Coordinates of this 3-particle system are x, y. Since each can be varied independently, the system is holonomic.)

Question .... Write on both sides of the paper

(54)



A is fixed and B is movable.

The masses $m_1 = 5$ kg, $m_2 = 2$ kg, $m_3 = 4$ kg at respective distances $(l_1 - x)$; $(x + l_2 - y)$, $(x+y)$ metres below the axles of the upper pulley.

Downwards Velocities are

$$\therefore -\dot{x},\ (\dot{x} - \dot{y}),\ (\dot{x} + \dot{y})\ \text{m/sec.} \qquad (209)$$

$$\therefore K.E = \tfrac{1}{2}\left[5\dot{x}^2 + 2(\dot{x} - \dot{y})^2 + 4(\dot{x} + \dot{y})^2\right] \qquad (210)$$

allow x, y to have Virtual displacements δx, δy respectively

$$\begin{aligned} \delta W &= 5g\,\delta(l_1 - x) + 2g\,\delta(x + l_2 - y) + 4g\,\delta(x+y) \qquad (211) \\ &= 5g\,\delta l_1 - 5g\,\delta x + 2g\,\delta x + 2g\,\delta l_2 - 2g\,\delta y + 4g\,\delta x + 4g\,\delta y \\ &= 5g\,\delta l_1 + g\,\delta x + 2g\,\delta l_2 + 2g\,\delta y \\ &= g\,\delta x + 2g\,\delta y \quad \text{Since } l_1, l_2 = \text{Constants} \qquad (212) \end{aligned}$$

$\therefore$ the generalized forces are

$$Q_x = g \qquad Q_y = 2g \qquad (213)$$

Using the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_x \qquad (214)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = Q_y \qquad (215)$$

Using (210) in (214) and (215) respectively, we have

$$11\ddot{x} + 2\ddot{y} = g \qquad (216)$$

$$2\ddot{x} + 6\ddot{y} = 2g \qquad (217)$$

Simultaneously

$$\ddot{x} = \frac{9}{31} \text{ m/sec}^2 \qquad \ddot{y} = \frac{109}{31} \text{ m/s}^2 \qquad (218)$$

Thus the downward accelerations of the particles of masses 5, 2, 4 kgs are respectively,

$$-\ddot{x} = -\frac{9}{31} \text{ m/s}^2$$

$$\ddot{x} - \ddot{y} = -\frac{99}{31} \text{ m/s}^2 \qquad (219)$$

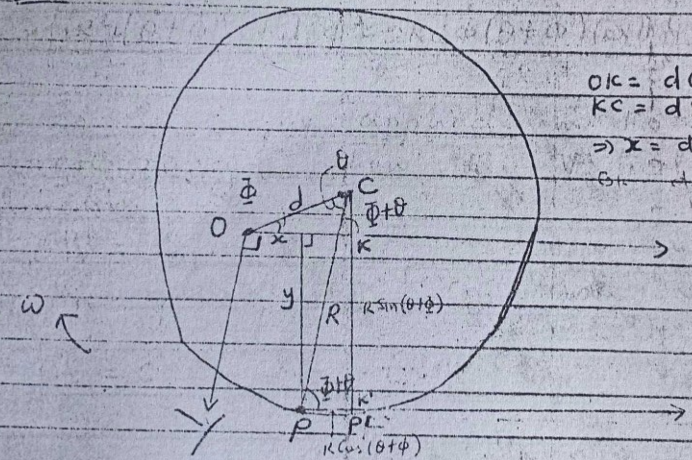
$$\ddot{x} + \ddot{y} = \frac{119}{31} \text{ m/s}^2$$

5 A horizontal circular wire has radius R, centre C and is free to rotate about a vertical axis through a point O in its plane distance d from C. The wire carries a smooth particle P and angle $O\hat{C}P = \theta$ at time t.

If ω is the angular velocity of the wire, show that

$$R\ddot{\theta} + \dot{\omega}(R - d\cos\theta) = d\omega^2 \sin\theta$$

Sol^n



$OK = d\cos\Phi = x + R\cos(\theta+\phi)$
$KC = d\sin\Phi$
$\Rightarrow x = d\cos\Phi - R\cos(\theta+\phi)$

At time t, let OC make an angle Φ with the fixed line OX so that PC makes an angle $(\theta+\Phi)$ with OX.

The generalized coordinates are θ, Φ.

Let P have coordinates x, y referred to OX, OY, respectively, then

$$x = d\cos\Phi - R\cos(\theta+\Phi)$$
$$y = R\sin(\theta+\Phi) - d\sin\Phi$$

Therefore

$$\dot{x} = -d\dot{\Phi}\sin\Phi + R(\dot{\theta}+\dot{\Phi})\sin(\theta+\Phi)$$
$$\dot{y} = R(\dot{\theta}+\dot{\Phi})\cos(\theta+\Phi) - d\dot{\Phi}\cos\Phi \qquad 221$$

If m is the mass of the particle at P

$$K \cdot E = T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) \qquad 222$$

$$= \tfrac{1}{2}m\{R^2(\dot{\theta}+\dot{\Phi})^2 + d^2\dot{\Phi}^2 - 2Rd\dot{\Phi}(\dot{\theta}+\dot{\Phi})[\sin\Phi \cdot$$
$$\cdot \sin(\theta+\Phi) + \cos\Phi\cos(\theta+\Phi)]\} \qquad 223$$

$\therefore$

$$T = \tfrac{1}{2}m\{R^2(\dot{\theta}+\dot{\Phi})^2 + d^2\dot{\Phi}^2 - 2Rd\dot{\Phi}(\dot{\theta}+\dot{\Phi})\cos\theta\} \qquad 224$$

Let G be the couple applied in the direction Φ increasing. Then the virtual work function is

$$\delta W = G\,\delta\Phi \qquad 225$$

So the generalized forces are

$$Q_\theta = 0 \qquad Q_\Phi = G \qquad 226$$

Lagrange's equations for coordinate θ is

$$\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \qquad 227$$

$$\{R^2(\dot{\theta}+\dot{\Phi}) - Rd\dot{\Phi}\cos\theta\} - Rd\dot{\Phi}(\dot{\theta}+\dot{\Phi})\sin\theta = 0 \qquad 228$$

$$\ddot{\theta}+\ddot{\Phi}) - d\ddot{\Phi}\cos\theta + d\dot{\Phi}\dot{\theta}\sin\theta - d\dot{\Phi}(\dot{\theta}+\dot{\Phi})\sin\theta \qquad 229$$

...cing $\dot{\Phi}$ by ω and $\ddot{\Phi}$ by $\dot{\omega}$ gives the ...ed result.

REMARKS

A couple is a set of two equal unlike parallel forces whose lines of action are not the same. Since in any resolution of forces the constituent forces of a couple will balance, they may be neglected, but their moment about any point will not be zero and must be taken into account.

Defn: The moment of a couple about any point in the plane of the forces is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces.

EXERCISE

1 In example (5) show that by using the 2nd Lagrange's equation of motion for the system we obtain i.e for Φ

$$R\ddot{\theta} + (R^2+\phi^2)\dot{\omega} - Rd(\ddot{\theta}+2\omega)\cos\theta + Rd\dot{\theta}\sin\theta(\dot{\theta}+2\omega) = G/M \cdot \text{w.r}$$

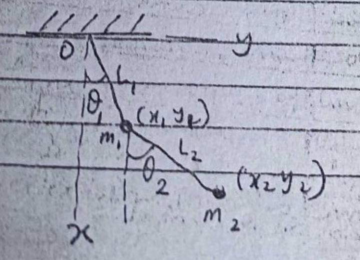
Constant

Hint:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\Phi}}\right) - \frac{\partial T}{\partial \Phi} = Q_{\Phi} = G. \quad \text{see note}$$

2 Write the differential equations of motion for a particle in a uniform gravitational field using spherical coordinates. see note

3 Consider a double pendulum shown below



...ides of the paper Question (59)

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Show that the transformation equations for the System are

$$x_1 = l_1 \cos\theta_1 \qquad y_1 = l_1 \sin\theta_1$$

$$x_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \quad y_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$

Hence

(a) Write the Lagrangian of the System

(b) Obtain equations for the motions — PP 299, 286 T.M.

Hint: $T = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2)$

$V = m_1 g(l_1 + l_2 - l_1\cos\theta_1) + m_2 g\,(l_1 + l_2 - (l_1\cos\theta_1 + l_2\cos\theta_2))$

The System is conservative

REMARKS

If System is not conservative use

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad (230)$$

but if conservative

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (231)$$

Gravitational force acting is derivable from potential e.g. A sphere rolling down from the top of a fixed sphere then the System is conservative (non holonomic)

A cylinder rolling without slipping down a rough inclined plane of angle α is conservative since equation of constraint is that of line or plane (holonomic)

A particle moving on a very long frictionless wire which rotates with constant angular speed about horizontal axis is ~~non~~ conservative. B/c ~~force due to friction is not derivable from a potential~~

~~is holonomic~~ eqn of constraint is that of a line which involves t explicitly.

A particle sliding down the inner surface, with coeff of friction μ, of a paraboloid of revolution having its axis vertical vertex downward is non conservative b/c force due to friction is not derivable from a potential.

CHAPTER FOUR

IMPULSIVE MOTION

Defn 1

The impulse of forces acting on a particle in any interval of time is defined to be the momentum change produced.

Thus if a particle of constant mass m has a velocity changed from $\underset{\sim}{v}_1$ to $\underset{\sim}{v}_2$ in a time t by a force $\underset{\sim}{F}$ acting, then the impulses $\underset{\sim}{I}$ is given by

$$\underset{\sim}{I} = m\underset{\sim}{v}_2 - m\underset{\sim}{v}_1 = m(\underset{\sim}{v}_2 - \underset{\sim}{v}_1) \qquad (232)$$

$$= m\int_{t_1}^{t_2}\left(\frac{dv}{dt}\right)dt = \int_{t_1}^{t_2} \underset{\sim}{F}\,dt \quad , \underset{\sim}{F} = m\frac{dv}{dt} \qquad (233)$$

The impulse of the force F is the line-integral of the force.

Defn 2

Impulsive forces are forces for which F grows very large and over the interval $(t_2 - t_1)$ very small, with line integral I remaining finite.

i.e

If $$I = \lim_{\substack{F\to\infty \\ t_2-t_1\to 0}} \int_{t_1}^{t_2} \underset{\sim}{F}\,dt = \text{finite}, \text{ then } I \text{ is} \qquad (234)$$

an impulsive force.

Their measurement as forces is impracticable but one can measure the momentum change they produce.

EXAMPLE

An impulse I changes the velocity of a particle of mass m from v_1 to v_2. Show that the K.E gained is $\frac{1}{2} I (v_1 + v_2)$.

Write on both sides of the paper Question (61)

Do not write in either margin

SOLUTION

$I = m(v_2 - v_1)$

K.E gained $= \frac{1}{2} m (v_2^2 - v_1^2)$ (235)

$= \frac{1}{2} m (v_2 - v_1)(v_2 + v_1)$ (236)

$= \frac{1}{2} I (v_2 + v_1)$ as reqd.

GENERALIZED IMPULSES AND IMPULSIVE VIRTUAL WORK (237)

Now suppose (F_1, F_2, F_3) are components of force F acting at a point of a body. Then the impulses of these forces over a time interval $0 < t < \tau$ are

$$\int_0^\tau F_1 dt, \quad \int_0^\tau F_2 dt, \quad \int_0^\tau F_3 dt$$

Furthermore, suppose the forces F_1, F_2, F_3 become very large and the time interval τ very small in such a way that

$$\lim_{\substack{F_1 \to \infty \\ \tau \to 0}} \int_0^{t=\tau} F_1 dt = J_1 = \text{finite}$$ (238)

$$\lim_{\substack{F_2 \to \infty \\ \tau \to 0}} \int_0^{t=\tau} F_2 dt = J_2 = \text{finite}$$ (239)

$$\lim_{\substack{\to \infty \\ \to 0}} \int_0^{t=\tau} F_2 dt = J_3 = \text{finite}.$$ (240)

J_1, J_2, J_3 are each an impulse or are impulses or ... forces.

... analogy, if for generalized forces Q_j $(j = 1, \dots n)$ ... system, we have

Write on both sides of the paper

$$\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \int_0^{\tau} Q_j\,dt = \mathcal{J}_j = \text{finite} \qquad (j = 1, 2, \dots n)$$

then the quantities $\mathcal{J}_j$ are called generalized impulses.

Now, we know that

$$\delta W = \sum_{j=1}^{n} Q_j\,\delta q_j \qquad (242)$$

$$\therefore \int_0^{\tau} \delta W\,dt = \sum_{j=1}^{n} \left\{ \delta q_j \int_0^{\tau} Q_j\,dt \right\} \qquad (243)$$

Thus

$$\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \int_0^{\tau} \delta W\,dt = \sum_{j=1}^{n} \left\{ \delta q_j \left(\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \int_0^{\tau} Q_j\,dt \right) \right\} \qquad (244)$$

and hence we have

$$\int_0^{\tau} \left(\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \delta W \right) dt = \sum_{j=1}^{n} \left\{ \mathcal{J}_j\,\delta q_j \right\} \qquad (245)$$

If the above double limits $Q_j \to \infty$, $\tau \to 0$ exist, then

$$\delta u = \lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \delta W \qquad j = 1, 2, \dots n \qquad (246)$$

exist and is finite.

δu is constructed from those impulses which do impulsive virtual work in a virtual displacement. Impulses which do no such impulsive work need not be included in δu.

LAGRANGE'S EQUATION FOR IMPULSIVE FORCES

The generalized momenta is written as

$$p_j = \frac{\partial T}{\partial \dot{q}_j} \quad (j = 1, 2, \ldots n) \qquad (247)$$

where $T = T(q_1, \ldots q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ (248)

It follows that the Lagrange's equations for a holonomic system are

$$\frac{d}{dt} p_j - \frac{\partial T}{\partial q_j} = Q_j \quad (j = 1, 2, \ldots, n) \qquad (249)$$

Integrating these equations from $t = 0$ to $t = \tau$ we obtain

$$(p_j)_{t=\tau} - (p_j)_{t=0} = \int_0^\tau \frac{\partial T}{\partial q_j} dt + \int_0^\tau Q_j dt \qquad (250)$$

Now let $Q_j \to \infty$, $\tau \to 0$ in such a way that

$$\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \int_0^\tau Q_j dt \to J_j = \text{finite} \qquad (251)$$

As the coordinates q_j do not change abruptly and do not change with τ

$$\lim_{\tau \to 0} \int_0^\tau \frac{\partial T}{\partial q_j} dt = 0 \quad (j = 1, 2, 3, \ldots) \qquad (252)$$

Defining $\Delta p_j = \lim_{\tau \to 0} \left[(p_j)_{t=\tau} - (p_j)_{t=0} \right]$ and using (253)

equations (251) and (252) in (250)

Question... Write on both sides of the paper

we obtain

$$\Delta P_j = \mathcal{F}_j \qquad j = (1, 2, \ldots n) \quad (25)$$

which is the Lagrange's equations for impulsive forces. acting from $t=0$ to $t=\tau$

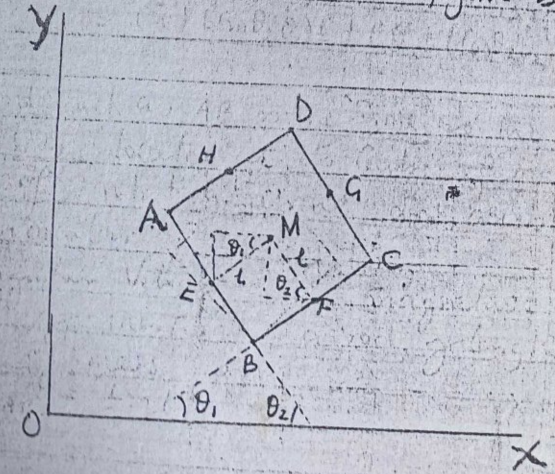
The Statement of these equations is:- " the generalized momentum increment is equal to the generalized impulsive force associated with each generalized coordinate"

EXAMPLES

1 A Square ABCD formed by four rods of length $2l$ and mass m hinged at their ends, rests on a horizontal frictionless table. An impulse of magnitude $\mathcal{J}$ is applied to the Vertex A in the direction AD. Find the equations of motion.

Sol

After the square is struck, its shape will in general be a rhombus as in figure below



Write on both sides of the paper Question .... Do not write in either margin

Suppose that at any time t, the angles made by sides AD (or BC) and AB (or CD) with the x axis are θ_1 and θ_2 respectively.

Let the coordinates of the Centre M be (x, y). It follows that x, y, θ_1, θ_2 are the generalized coordinates.

From the figure above, the position vectors of the Centres E, F, G, H of the rods are

$$\underset{\sim}{r}_E = (x - l\cos\theta_1)\underline{i} + (y - l\sin\theta_1)\underline{j}$$
$$\underset{\sim}{r}_F = (x + l\cos\theta_2)\underline{i} + (y - l\sin\theta_2)\underline{j}$$
$$\underset{\sim}{r}_G = (x + l\cos\theta_1)\underline{i} + (y + l\sin\theta_1)\underline{j} \qquad (254)$$
$$\underset{\sim}{r}_H = (x - l\cos\theta_2)\underline{i} + (y + l\sin\theta_2)\underline{j}$$

The Velocities of E, F, G, and H at any time are given by

$$\underline{V}_E = \dot{\underset{\sim}{r}}_E = (\dot{x} + l\sin\theta_1\dot{\theta}_1)\underline{i} + (\dot{y} - l\cos\theta_1\dot{\theta}_1)\underline{j}$$
$$\underline{V}_F = \dot{r}_F = (\dot{x} - l\sin\theta_2\dot{\theta}_2)\underline{i} + (\dot{y} - l\cos\theta_2\dot{\theta}_2)\underline{j}$$
$$\underline{V}_G = \dot{r}_G = (\dot{x} - l\sin\theta_1\dot{\theta}_1)\underline{i} + (\dot{y} + l\cos\theta_1\dot{\theta}_1)\underline{j} \qquad (255)$$
$$\underline{V}_H = \dot{r}_H = (\dot{x} + l\sin\theta_2\dot{\theta}_2)\underline{i} + (\dot{y} + l\cos\theta_2\dot{\theta}_2)\underline{j}$$

The K.E of a rod such as AB is the same as the K.E of a particle of mass m located at its Centre of mass E plus the K.E of rotation about an axis through E perpendicular to the xy plane.

Now, the angular Velocity has magnitude $\dot{\theta}_2$. Also, the moment of inertia of a rod of length 2L about its Centre of mass is

$$I_{AB} = \frac{1}{3}ml^2 \quad ?? \qquad (256)$$

Therefore, the total energy of rod AB is

$$T_{AB} = \frac{1}{2} m \dot{\underset{\sim}{r}}_E^2 + \frac{1}{2} I_{AB} \dot{\theta}_2^2 \qquad (257)$$

Similarly, the total kinetic energies of rods BC, CD ~~and AD~~ are

$$T_{BC} = \tfrac{1}{2} m \dot{\underset{\sim}{r}}_F^2 + \frac{1}{2} I_{BC} \dot{\theta}_1^2 \qquad (258)$$

$$T_{CD} = \tfrac{1}{2} m \dot{\underset{\sim}{r}}_G^2 + \tfrac{1}{2} I_{CD} \dot{\theta}_2^2 \qquad (259)$$

$$T_{AD} = \tfrac{1}{2} m \dot{\underset{\sim}{r}}_H^2 + \tfrac{1}{2} I_{AD} \dot{\theta}_1^2 \qquad (260)$$

Thus the total K.E is (using the fact that

$$I = I_{AB} = I_{BC} = I_{CD} = \frac{1}{3} m l^2 \qquad (261)$$

$$\begin{aligned} T &= T_{AB} + T_{BC} + T_{CD} + T_{AD} \qquad (262) \\ &= \tfrac{1}{2} m (\dot{\underset{\sim}{r}}_E^2 + \dot{\underset{\sim}{r}}_F^2 + \dot{\underset{\sim}{r}}_G^2 + \dot{\underset{\sim}{r}}_H^2) + I(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\ &= \tfrac{1}{2} m (4\dot{x}^2 + 4\dot{y}^2 + 2l^2\dot{\theta}_1^2 + 2l^2\dot{\theta}_2^2) + \tfrac{1}{3} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) \\ &= 2m(\dot{x}^2 + \dot{y}^2) + \frac{4}{3} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) \qquad (263) \end{aligned}$$

Let us assume that initially the rhombus is a square at rest with its sides parallel to the coordinate axes and its centre located at the origin.

Then we have

$$x = 0,\ y = 0,\ \theta_1 = \pi/2,\ \theta_2 = 0,$$
$$\dot{x} = 0,\ \dot{y} = 0,\ \dot{\theta}_1 = 0,\ \dot{\theta}_2 = 0 \qquad (264)$$

Let us use $(\ )_1$ and $(\ )_2$ to denote quantities before and after the impulse is applied, we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 = (4m\dot{x})_1 = 0 \,;\; \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = (4m\dot{y})_1 = 0 \qquad (265)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1 = \left(\tfrac{8}{3} m l^2 \dot{\theta}_1\right)_1 = 0 \,;\; \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_1 = \left(\tfrac{8}{3} m l^2 \dot{\theta}_2\right)_1 = 0 \qquad (266)$$

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 = (4m\dot{x})_2 = 4m\dot{x} \,;\; \left(\frac{\partial T}{\partial \dot{y}}\right)_2 = (4m\dot{y})_2 = 4m\dot{y} \qquad (267)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2 = \left(\tfrac{8}{3} m l^2 \dot{\theta}_1\right)_2 = \tfrac{8}{3} m l^2 \dot{\theta}_1 \,;\; \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2 = \tfrac{8}{3} m l^2 \dot{\theta}_2 = \tfrac{8}{3} m l^2 \dot{\theta}_2 \qquad (268)$$

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 - \left(\frac{\partial T}{\partial \dot{x}}\right)_1 = F_x \quad \text{or} \quad 4m\dot{x} = F_x \qquad (269)$$

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2 - \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = F_y \quad \text{or} \quad 4m\dot{y} = F_y \qquad (270)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2 - \left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1 = F_{\theta_1} \quad \text{or} \quad \tfrac{8}{3} m l^2 \dot{\theta}_1 = F_{\theta_1} \qquad (271)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2 - \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_1 = F_{\theta_2} \quad \text{or} \quad \tfrac{8}{3} m l^2 \dot{\theta}_2 = F_{\theta_2} \qquad (272)$$

for simplicity we have now removed subscript $(\;)_2$.

Question.............................. Write on both sides of the paper

We proceed to obtain $F_x, F_y, F_{\theta_1}, F_{\theta_2}$. To this end we note

$$F_i = \sum_j \underset{\sim}{J}_j \cdot \frac{\partial \underset{\sim}{r}_i}{\partial q_j} \qquad (273)$$

where J_j are the impulsive forces.
We thus have

$$F_x = \underset{\sim}{J}_A \cdot \frac{\partial \underset{\sim}{r}_A}{\partial x} + \underset{\sim}{J}_B \cdot \frac{\partial \underset{\sim}{r}_B}{\partial x} + \underset{\sim}{J}_C \cdot \frac{\partial \underset{\sim}{r}_C}{\partial x} + \underset{\sim}{J}_D \cdot \frac{\partial \underset{\sim}{r}_D}{\partial x} \qquad (274)$$

$$F_y = \underset{\sim}{J}_A \cdot \frac{\partial \underset{\sim}{r}_A}{\partial y} + \underset{\sim}{J}_B \cdot \frac{\partial \underset{\sim}{r}_B}{\partial y} + \underset{\sim}{J}_C \cdot \frac{\partial \underset{\sim}{r}_C}{\partial y} + \underset{\sim}{J}_D \cdot \frac{\partial \underset{\sim}{r}_D}{\partial y} \qquad (275)$$

$$F_{\theta_1} = \underset{\sim}{J}_A \cdot \frac{\partial \underset{\sim}{r}_A}{\partial \theta_1} + \underset{\sim}{J}_B \cdot \frac{\partial \underset{\sim}{r}_B}{\partial \theta_1} + \underset{\sim}{J}_C \cdot \frac{\partial \underset{\sim}{r}_C}{\partial \theta_1} + \underset{\sim}{J}_D \cdot \frac{\partial \underset{\sim}{r}_D}{\partial \theta_1} \qquad (276)$$

$$F_{\theta_2} = \underset{\sim}{J}_A \cdot \frac{\partial \underset{\sim}{r}_A}{\partial \theta_2} + \underset{\sim}{J}_B \cdot \frac{\partial \underset{\sim}{r}_B}{\partial \theta_2} + \underset{\sim}{J}_C \cdot \frac{\partial \underset{\sim}{r}_C}{\partial \theta_2} + \underset{\sim}{J}_D \cdot \frac{\partial \underset{\sim}{r}_D}{\partial \theta_2} \qquad (277)$$

Now from Figure above, we find the position vectors of A, B, C, D given by

$$\begin{aligned}
\underset{\sim}{r}_A &= (x - l\cos\theta_1 - l\cos\theta_2)\underset{\sim}{i} + (y - l\sin\theta_1 + l\sin\theta_2)\underset{\sim}{j} \\
\underset{\sim}{r}_B &= (x - l\cos\theta_1 + l\cos\theta_2)\underset{\sim}{i} + (y - l\sin\theta_1 - l\sin\theta_2)\underset{\sim}{j} \\
\underset{\sim}{r}_C &= (x + l\cos\theta_1 + l\cos\theta_2)\underset{\sim}{i} + (y + l\sin\theta_1 - l\sin\theta_2)\underset{\sim}{j} \\
\underset{\sim}{r}_D &= (x + l\cos\theta_1 - l\cos\theta_2)\underset{\sim}{i} + (y + l\sin\theta_1 + l\sin\theta_2)\underset{\sim}{j}
\end{aligned} \qquad (278)$$

Since the impulsive force at A is initially in the direction of the positive y axis, we have

$$\underset{\sim}{J}_A = J\,\underset{\sim}{j} \qquad (279)$$