A PREPARATORY LECTURE NOTE ON

MTS 305 (REAL ANALYSIS II)

Course Contents

Riemann integral of functions, continuous mono-positive functions, functions of bounded variation, the Riemann stieltjes integral, pointwise and uniform convergence of sequences and series of functions, effects on limit-sums (definite or Riemann integral) when functions are continuously differentiable or Riemann integrable, power series.

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Chapter One

This chapter sets at familiarising us with the concept of Riemann integration. It is already a well known fact that integration is the universe operation to differentiation. However, this is not the only way of introducing the concept of an integral. The definition of an integral as the area under a curve is another approach of introducing the concept. These two approaches lead, respectively, to definition of

a. the indefinite integral, and

b. the definite Riemann integral

1.1 Indefinite Integrals

If $F(x)$ and $f(x)$ are two functions of x such that

$$\frac{d}{dx}F(x) = f(x) \tag{1.1}$$

then $F(x)$ is said to be an indefinite integral of $f(x)$ and is written as

$$F(x) = \int f(x)\,dx + c \tag{1.2}$$

The function $f(x)$ is called the integrand and is said to be integrable if $F(x)$ exists, c is called the constant of integration.

Furthermore, suppose $f(x)$ is a function such that

$$\int f(x)\,dx = F(x) \tag{1.3}$$

We define the definite integral

$$\int_a^b f(x)\,dx \tag{1.4}$$

as

$$\int_a^b f(x)\,dx = \Big[g(x)\Big]_a^b = g(b) - g(a) \tag{1.5}$$

where a and b are two real numbers, and are called, respectively the <u>lower</u> and the <u>upper</u> limits of the integral.

1.2 Definite Riemann Integral

Definition 1.2.1. Consider a function (single-valued) $y = f(x)$ which is assumed to be positive, continuous and bounded over a closed finite interval $[a, b], a < b$. Consider a partition $p = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \tag{1.6}$$

Suppose further that s_r represents an arbitrary point of the interval $[x_{r-1}, x_r]$ and that

$$\delta_r = x_r - x_{r-1} \tag{1.7}$$

Then an approximation to the rth strip is given by

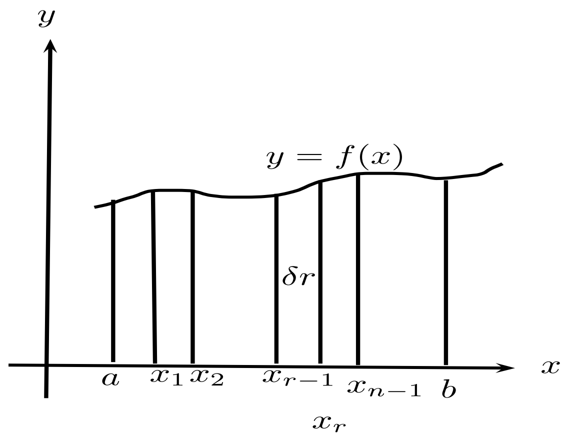

Figure 1.1

$$f(s_r)\,\delta r \tag{1.8}$$

Thus, the whole area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is approximately given by the sum

$$M_n = \sum_{r=1}^{n} f(s_r)\,\delta r \tag{1.9}$$

If the limit of M_n as $n \to \infty$ exists independent of the x_r and s_r and all $\delta r \to 0$, we say that $f(x)$ is Riemann integrable over $[a, b]$ and we write

$$I = \lim_{n\to\infty} \sum_{r=1}^{n} f(s_r)\,\delta r = \int_a^b f(x)\,dx \tag{1.10}$$

The limit-sum (1.10) is called the definite integral of $f(x)$ from $x = a$ to $x = b$ or in other words, Riemann integral.

Remarks

1. The limit (1.10) exists whenever $f(x)$ is continuous or sectionally continuous in $[a, b]$. When this limit exists, we say that $f(x)$ is Riemann integrable or simply integrable in $[a, b]$

2. Geometrically, the value of this definite integral represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ only if $f(x) \geq 0$.

Examples

1. If $f(x)$ is continuous in $[a, b]$, show that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left[a + r\left(\frac{b-a}{n}\right)\right]$$

Solution

Since $f(x)$ is continuous, the limit exists independent of the mode of subdivision. Choose the subdivision of $[a, b]$ into n equal parts of equal length

$$\delta_r = \frac{b-a}{n} \tag{1.11}$$

It is clearly seen that

$$s_r = a + r\frac{b-a}{n}, \quad r = 1, 2, \ldots, n \tag{1.12}$$

Consequently,

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{r=1}^{n} f(s_r)\delta r \\ &= \lim_{n\to\infty} \sum_{r=1}^{n} f\left[a + r\left(\frac{b-a}{n}\right)\right]\frac{b-a}{n} \\ &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left[a + r\left(\frac{b-a}{n}\right)\right] \end{aligned} \tag{1.13}$$

2. Express $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ as a definite integral.

Solution

By comparison, we have

$$a + r\frac{b-a}{n} = \frac{r}{n} \tag{1.14}$$

This implies that

$$\begin{aligned} a &= 0 \\ b - a &= 1 \\ \implies b &= 1 \end{aligned}$$

Therefore

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) \tag{1.15}$$

3. (a) Express $\int_0^1 x^2\,dx$ as limit of a sum, and use the result to evaluate the definite integral.

   (b) Interpret the result geometrically.

Solution

(a) If $f(x) = x^2$, then

$$f\left(\frac{r}{n}\right) = \left(\frac{r}{n}\right)^2 = \frac{r^2}{n^2} \tag{1.16}$$

Thus, by problem 2,

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{r^2}{n^2} \tag{1.17}$$

Consequently,

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty} \frac{1}{n}\Big(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \ldots + \frac{n^2}{n^2}\Big) && (1.18)\\ &= \lim_{n\to\infty} \frac{1^2 + 2^2 + \ldots + n^2}{n^3} && (1.19)\\ &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} && (1.20)\end{aligned}$$

(Since $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ by induction).

$$\begin{aligned} &= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} && (1.21)\\ \therefore \int_0^1 x^2\ dx &= \frac{1}{3} && (1.22)\end{aligned}$$

By using fundamental theorem of calculus, we observe that

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \tag{1.23}$$

(b) The area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$ is equal to $\frac{1}{3}$.

Example (Exercise)

Use the definition of Riemann integral given in Example 1 to evaluate

$$\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right\}$$

Remarks

Is a bounded function over a finite interval always integrable?

Answer

No

Consider the function

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational}\end{cases}$$

defined on the closed interval $[0,1]$

$$\begin{aligned}\int_a^b F(x)\,dx &= \lim_{n\to\infty}\sum_{r=1}^{n} f(s_r)\,\delta r \\ &= 0 \quad \text{when } s_r \text{ coincides with rational points} \\ &= b-a \quad \text{when } s_r \text{ coincides with irrational points}\end{aligned}$$

Since the limit is dependent on the choice of x_r and s_r, the function is not Riemann integrable.

Definition 1.2.2. If $a<b$, we define the integral

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{1.24}$$

As a consequence of the above definition, we have

$$\int_a^a f(x)\,dx = 0 \tag{1.25}$$

1.3 Properties of Riemann Integration

1. If $f(x)$ is integrable on $[a,b]$ and A is a constant, then $Af(x)$ is integrable on $[a,b]$ and

$$\int_a^b Af(x)\,dx = A\int_a^b f(x)\,dx$$

Proof.

$$\begin{aligned}\int_a^b Af(x)\,dx &= \lim_{n\to\infty}\sum_{r=1}^{n} Af(s_r)\,\delta r \\ &= A\lim_{n\to\infty}\sum_{r=1}^{n} f(s_r)\,\delta r \\ &= A\int_a^b f(x)\,dx\end{aligned}$$

□

2. If $f(x)$ and $g(x)$ are integrable over $[a,b]$, then $f(x)+g(x)$ is integrable over $[a,b]$ and

$$\int_a^b \Big[f(x)+g(x)\Big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{1.26}$$

Proof.

$$\begin{aligned}\int_a^b \Big[f(x)+g(x)\Big]\,dx &= \lim_{n\to\infty}\sum_{r=1}^{n}\Big[f(s_r)+g(s_r)\Big]\,\delta r \qquad (1.27)\\ &= \lim_{n\to\infty}\sum_{r=1}^{n} f(s_r)\,\delta r + \lim_{n\to\infty}\sum_{r=1}^{n} g(s_r)\,\delta r \qquad (1.28)\\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad (1.29)\end{aligned}$$

□

It follows from properties 1 and 2 that

$$\int_a^b \Big[Af(x) + Bg(x)\Big]\,dx = \int_a^b Af(x)\,dx + \int_a^b Bg(x)\,dx \tag{1.30}$$

$$= A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx \tag{1.31}$$

Remarks

From above properties, we deduce that integration is a linear transformation i.e. if $T(x, y) = T(x)+T(y)$ and $\alpha T(x) = T(\alpha x)$ T is a linear transformation. When $T(x, y) = T(x) + T(y)$, function is relative and when $\alpha T(x) = T(\alpha x)$ T, function is homogeneous.

Theorem 1.3.1. *If $f(x)$ is integrable over $[a, b]$ and m and M are, respectively, the greatest lower bound and lowest upper bound of $f(x)$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Proof.

$$m \leq f(s_r) \leq M \quad \text{where } s_r \in [x_{r-1}, x_r] \tag{1.32}$$

$$\implies \sum_{r=1}^{n} m\,\delta r \leq \sum_{r=1}^{n} f(s_r)\,\delta r \leq \sum_{r=1}^{n} M\,\delta r \tag{1.33}$$

Taking the limit as $n \to \infty$

$$\sum_{r=1}^{n} m\,\delta r \leq \lim_{n\to\infty} \sum_{r=1}^{n} f(s_r)\,\delta r \leq \sum_{r=1}^{n} M\,\delta r \tag{1.34}$$

$$\implies m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \tag{1.35}$$

□

Theorem 1.3.2. *If $f(x)$ is integrable and $f(x) \geq 0$, then*

$$\int_a^b f(x)\,dx \geq 0 \tag{1.36}$$

Proof. Clearly, $m =$ lower bound $= 0$. And from the previous theorem 1.3.1,

$$\int_a^b f(x)\,dx \geq m(b-a) = 0 \tag{1.37}$$

$$i.e. \int_a^b f(x)\,dx \geq 0 \tag{1.38}$$

□

Corollary

If $f(x)$ and $g(x)$ are integrable over $[a, b]$ and $f(x) \geq g(x)$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \tag{1.39}$$

Proof. Define $h(x) = f(x) - g(x)$. Now, $h(x)$ is integrable because $f(x)$ and $g(x)$ are integrable. Also, $h(x) \geq 0$ as $f(x) \geq g(x)$.

Consequently,

$$0 \leq \int_a^b h(x)\,dx = \int_a^b \Big[f(x) - g(x)\Big]\,dx \tag{1.40}$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \tag{1.41}$$

Therefore,

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \tag{1.42}$$

□

Theorem 1.3.3. ***Fundamental Theorem of Integral Calculus:*** *If $f(x)$ is integrable over $[a,b]$ and if there exists a differentiable function $\phi(x)$ whose derivative is $f(x)$ throughout $[a,b]$, then*

$$\int_a^b f(x)\,dx = \Big[\phi(x)\Big]_a^b = \phi(b) - \phi(a) \tag{1.43}$$

Proof. Since $\phi(x)$ is differentiable in $[a,b]$, therefore, in the sub-interval $[x_{r-1}, x_r]$, using mean value of differential calculus due to Lagrange

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(s_r), \qquad x_{r-1} < s_r < x_r \tag{1.44}$$

Also, we have

$$\phi'(x) = f(x) \qquad \text{for all } x \in [a,b] \tag{1.45}$$

Therefore

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})f(s_r) \tag{1.46}$$

$$= \delta r f(s_r) \tag{1.47}$$

$$\sum_{r=1}^{n} \Big[\phi(x_r) - \phi(x_{r-1})\Big] = \sum_{r=1}^{n} \delta r f(s_r) \tag{1.48}$$

Taking the limit as $n \to \infty$, we have

$$\phi(b) - \phi(a) = \int_a^b f(x)\,dx \tag{1.49}$$

□

1.4 Lower Sum and Upper Sum

Let $f(x)$ be bounded in $[a,b]$. Let $D = \{x_0, x_1, \ldots, x_n\}$ be a partition or a division of $[a,b]$. Also, let m_r be the greatest lower bound of $f(x)$ in the closed interval $[x_{r-1}, x_r]$. Then the sum

$$\sum_{r=1}^{n} m_r \delta r \tag{1.50}$$

is called the <u>lower sum</u> of $f(x)$ for the partition D and is denoted by $\mathscr{S}(D)$.

On the other hand, if M_r is the least upper bound of $f(x)$ in $[x_{r-1}, x_r]$, then the sum

$$\sum_{r=1}^{n} M_r \delta r \tag{1.51}$$

is called the <u>upper sum</u> of $f(x)$ for the division D and is denoted by $S(D)$.

Clearly, we see that $\mathscr{S}(D) \leq S(D)$ if we denote

$$\sum_{r=1}^{n} f(s_r)\delta r = \partial(D) \tag{1.52}$$

then $\mathscr{S}(D) \leq \partial(D) \leq S(D)$.

Theorem 1.4.1. *A necessary and sufficient condition for $f(x)$ to be integrable over $[a, b]$ is that*

$$S(D) - \mathscr{S}(D) \to 0 \text{ as } \partial \to 0 \tag{1.53}$$

where $\partial \geq S_r$ for all $r = 1, 2, \ldots, n$

Note: As $n \to \infty$, $\partial \to 0$ or $n \to \infty \implies \partial \to 0$.

Definition 1.4.1. Given a partition or a dissection D of $[a, b]$, if we form a new dissection by adding points to D, then the new dissection is called a <u>refinement</u> of D denoted by D_1.

If to D, only one point is added, then the new dissection is called a <u>simple refinement</u>.

Theorem 1.4.2. *If D_1 is a refinement of D, then $\mathscr{S}(D) \leq \mathscr{S}(D_1) \leq S(D) \leq S(D_1)$ and $S(D_1) - \mathscr{S}(D_1) \leq S(D) - \mathscr{S}(D)$*

Theorem 1.4.3. *Every lower sum is less than every upper upper sum unless $f(x)$ is a constant function.*

Theorem 1.4.4. *If $a \leq a' \leq b' \leq b$ and $f(x)$ is integrable over $[a, b]$, then $f(x)$ is integrable over $[a', b']$.*

Proof. Let D be a dissection of $[a, b]$ where $x = a^1$ and $y = b'$ are two points of division.

Let the upper and lower sums of $f(x)$ over $[a, b]$ be S and $\mathscr{S}$ respectively.

Let the upper and lower sums of $f(x)$ over $[a', b']$ be S' and $\mathscr{S}'$ respectively.

Then

$$\mathscr{S} \leq \mathscr{S}' \leq S' \leq S \tag{1.54}$$

Consequently

$$S' - \mathscr{S}' \leq S - \mathscr{S} \tag{1.55}$$

Since $f(x)$ is integrable over $[a, b]$, we have

$$\lim_{\partial \to 0} S - \mathscr{S} \to 0 \tag{1.56}$$

Hence, $S' - \mathscr{S}' \to 0$ as $\partial \to 0$ which implies $f(x)$ is integrable over $[a', b']$. $\square$

Theorem 1.4.5. *If $a < c < b$ and $f(x)$ is integrable over $[a, b]$, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{1.57}$$

Proof. Let D be a dissection of $[a, b]$where c is one of its points of division, say, x_r

$$\int_a^b f(x)\,dx = \lim_{\partial\to 0}\sum_{p=1}^{n} f(s_p)\,\delta p \tag{1.58}$$

$n \to \infty \implies \partial \geq \delta p \to 0.$

$$\begin{aligned} &= \lim_{\partial\to 0}\Big[\sum_{p=1}^{n} f(s_p)\,\delta p + \sum_{r+1}^{n} f(s_p)\,\delta p\Big] && (1.59)\\ &= \lim_{\partial\to 0}\sum_{p=1}^{r} f(s_p)\,\delta p + \lim_{\partial\to 0}\sum_{r+1}^{n} f(s_p)\,\delta p && (1.60)\end{aligned}$$

Since $f(x)$ is integrable over $[a, c]$ and $[c, b]$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{1.61}$$

□

1.5 Classes of Integrable Functions

Theorem 1.5.1. *If $f(x)$ is continuous in the closed interval $[a, b]$, then it is integrable in the interval*

Theorem 1.5.2. *If $f(x)$ is piecewise continuous in $[a, b]$, then it is integrable over $[a, b]$.*

Theorem 1.5.3. *A bounded monotone function is integrable*

Proof. Let $f(x)$ be monotonic increasing function in $[a, b]$. Since $f(x)$ is bounded in $[a, b]$, we have

$$m = f(a) = x_0,\ M = f(b) = x_n \tag{1.62}$$

Let D be a dissection of $[a, b]$, then

$$\begin{aligned} S(D) &= \sum_{r=1}^{n} M_r\,\delta r = \sum_{r=1}^{n} f(x_r)\,\delta r && (1.63)\\ \mathscr{S}(D) &= \sum_{r=1}^{n} m_r\,\delta r = \sum_{r=1}^{n} f(x_{r-1})\,\delta r && (1.64)\\ S(D) - \mathscr{S}(D) &= \sum_{r=1}^{n}\Big[f(x_r) - f(x_{r-1})\Big]\,\delta r && (1.65)\end{aligned}$$

Also, we have $\delta r \leq \partial$ for all $r = 1, 2, \ldots, n$

$$\begin{aligned} &\leq \sum_{r=1}^{n}\Big[f(x_r) - f(x_{r-1})\Big]\,\delta r && (1.66)\\ &= \partial\sum_{r=1}^{1}\Big[f(x_r) - f(x_{r-1})\Big] && (1.67)\\ &= \partial\Big[f(x_n) - f(x_0)\Big] && (1.68)\\ &= \partial(M - m) \to 0 \text{ as } \partial \to 0. && (1.69)\end{aligned}$$

Hence, $f(x)$ is integrable.

If $f(x)$ is monotonic decreasing function, then $-f(x)$ is monotonic increasing function.

Hence integrable.

Also, $(-1)[-f(x)] = f(x)$ is integrable. □

Theorem 1.5.4. *Let $f(x)$ be an integrable function over $[a, b]$, then*

(i.) $|f(x)|$ *is integrable*

(ii.) $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$

Proof. Let m, M be the bounds of $f(x)$ in $[a, b]$. Let m_r, M_r be the bounds of $f(x)$ in $[x_{r-1}, x_r]$. Let n, N be the bounds of $|f(x)|$ in $[a, b]$. Let n_r, N_r be the bounds of $|f(x)|$ in $[x_{r-1}, x_r]$, it is clear that

$$N - n \leq M - m \text{ and} \tag{1.70}$$

$$N_r - n_r \leq M_r - m_r \tag{1.71}$$

Let S_1 and $\mathscr{S}_1$ be the upper and lower sums of $f(x)$ w.r.t. D. Let S and $\mathscr{S}$ be the upper and lower sums of $f(x)$ w.r.t. D.

$$\begin{aligned}
S_1(D) - \mathscr{S}_1(D) &= \sum_{r=1}^{n} N_r\,\delta r - \sum_{r=1}^{n} n_r\,\delta r && (1.72)\\
&= \sum_{r=1}^{n} (N_r - n_r)\,\delta r && (1.73)\\
&\leq \sum_{r=1}^{n} (M_r - m_r)\,\delta r && (1.74)\\
&= \sum_{r=1}^{n} M_r\,\delta r - \sum_{r=1}^{n} m_r\,\delta r && (1.75)\\
&= S(D) - \mathscr{S}(D) \to 0 \text{ as } \partial \to 0 && (1.76)
\end{aligned}$$

because $f(x)$ is integrable.

Hence, $|f(x)|$ is integrable

$$\begin{aligned}
\left| \int_a^b f(x)\,dx \right| &= \left| \lim_{\partial \to 0} \sum_{r=1}^{n} f(s_r)\,\delta r \right| && (1.77)\\
&= \lim_{\partial \to 0} \left| \sum_{r=1}^{n} f(s_r)\,\delta r \right| && (1.78)\\
&= \lim_{\partial \to 0} \sum_{r=1}^{n} \left| f(s_r) \right| \delta r && (1.79)\\
&= \int_a^b \left| f(x)\,dx \right| && (1.80)
\end{aligned}$$

□

Remarks

Is the converse of $(i.)$ true? i.e if $|f(x)|$ is integrable, is $f(x)$ also integrable?

Solution

No

For illustration, consider

$$f(x) = \begin{cases} c \text{ when } x \text{ is rational} \\ -c \text{ when } x \text{ is irrational} \end{cases}$$

We observe that $f(x)$ is not integrable as $|f(x)| = c$ for all x.

Theorem 1.5.5. *If $f(x)$ is integrable over $[a, b]$, then $f^2(x)$ is integrable.*

Proof. Let n^2 and N^2 be the closed bounds of $f^2(x)$ in $[a, b]$. Let n_r^2 and N_r^2 be the closest bounds in $[x_{r-1}, x_r]$. Let S_2 and $\mathscr{S}_2$ be the upper and lower sums, respectively, of $f^2(x)$ w.r.t. D.

$$\begin{aligned} S_2 - \mathscr{S}_2 &= \sum (N_r^2 - n_r^2)\,\delta r & (1.81) \\ &\leq 2N(N_r - n_r)\,\delta r & (1.82) \\ &= 2N[S(D) - \mathscr{S}(D)] & (1.83) \end{aligned}$$

which tends to 0 as $\partial \to 0$. Consequently,

$$S_2 - \mathscr{S}_2 \to 0 \text{ as } \partial \to 0 \qquad (1.84)$$

Hence, $f^2(x)$ is integrable. □

Corollary

If $f(x)$ and $g(x)$ are integrable over $[a, b]$, then $f(x)g(x)$ is integrable over $[a, b]$.

Proof.

$$fg = \frac{1}{2}\Big[(f+g)^2 - f^2 - g^2\Big] \qquad (1.85)$$

Thus, since f and g are integrable, then $f + g$ is integrable.

Using the above theorem $(f + g)^2, f^2$ and g^2 are integrable.

Hence fg is integrable. □

Theorem 1.5.6. *If $f(x)$ is one signed and integrable, and if $f(x)$ is bounded, then $\frac{1}{f(x)}$ is integrable.*

Proof. Exercise □

Chapter Two

2.1 Mean Value

Definition 2.1.1. Mean value of $f(x)$ over (a, b) is defined as the real number

$$U = \frac{1}{b-a} \int_a^b f(x)\,dx \tag{2.1}$$

Theorem 2.1.1. *Let $f(x)$ be bounded and integrable. Let m and M be the bounds of $f(x)$ in $[a, b]$. Then there exits $U, m \leq U \leq M$ such that*

$$\int_a^b f(x)\,dx = U(b-a) \tag{2.2}$$

Proof. We know that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \tag{2.3}$$

$$m \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq M \tag{2.4}$$

or

$$U = \frac{1}{b-a} \int_a^b f(x)\,dx, \quad m \leq U \leq M \tag{2.5}$$

or

$$\int_a^b f(x)\,dx = U(b-a) \tag{2.6}$$

□

Theorem 2.1.2. *Let $f(x)$ satisfy the following conditions in $[a, b]$, $a < b$*

i) $f(x)$ is continuous

ii) $f(x) \geq 0$

iii) $f(x)$ is not zero identically

Then

$$\int_a^b f(x)\,dx > 0 \tag{2.7}$$

Proof. ii) and iii) implies there is a point $c \in [a, b]$ such that $f(c) > 0$.

Hence, by i), there exists a neighbourhood of $[c-h, c+h]$ of c such that $f(x) > 0$ in this neighbourhood.

$$\int_a^b f(x)\,dx = \int_a^{c-h} f(x)\,dx + \int_{c-h}^{c+h} f(x)\,dx + \int_{c+h}^{b} f(x)\,dx \tag{2.8}$$

$$\int_{c-h}^{c+h} f(x)\,dx = U(c+h-c+h) = U2h \tag{2.9}$$

where U is a number lying within the bounds of $f(x)$ over $[c-h, c+h]$.

$$U2h = 2f(s_1)h \quad c-h \le s_1 \le c+h \tag{2.10}$$

$$\implies U2h > 0 \tag{2.11}$$

$$\implies \int_a^b f(x)\,dx > 0 \text{ since } \int_a^{c-h} f(x)\,dx > 0, \text{ and} \tag{2.12}$$

$$\int_{c+h}^b f(x)\,dx > 0 \tag{2.13}$$

□

Theorem 2.1.3. *: **(Schwarz Inequality)***
Let $f(x)$ and $g(x)$ be integrable over $[a,b]$, $a<b$. Then

$$\int_a^b f(x)g(x)\,dx \le \Big(\int_a^b f^2(x)\,dx\Big)^{\frac{1}{2}}\Big(\int_a^b g^2(x)\,dx\Big)^{\frac{1}{2}} \tag{2.14}$$

or

$$\Big(\int_a^b f(x)g(x)\,dx\Big)^2 \le \Big(\int_a^b f^2(x)\,dx\Big)\Big(\int_a^b g^2(x)\,dx\Big) \tag{2.15}$$

Proof. If $f(x)$ and $g(x)$ are linearly independent, then

$$\lambda f(x) + \mu g(x) \tag{2.16}$$

cannot varnish unless both λ and μ are zero.

Also,

$$\Big(\lambda f + \mu g\Big)^2 \ge 0 \tag{2.17}$$

Hence,

$$\int_a^b \Big(\lambda f(x) + \mu g(x)\Big)^2 dx > 0 \tag{2.18}$$

or

$$\lambda^2 \int_a^b f^2(x)\,dx + 2\lambda\mu \int_a^b f(x)g(x)\,dx + \mu^2 \int_a^b g^2(x)\,dx > 0 \tag{2.19}$$

which can be written, for simplicity, as

$$A\lambda^2 + 2C\lambda\mu + B\mu^2 > 0 \tag{2.20}$$

Where

$$A = \int_a^b f^2(x)\,dx \; C = \int_a^b f(x)g(x)\,dx, \; B = \int_a^b g^2(x)\,dx$$

One of the conditions for the above quadratic (2.20) to be positive definite is that

$$AB - C^2 > 0 \tag{2.21}$$

i.e.

$$\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx - \Big(\int_a^b f(x)g(x)\,dx\Big)^2 > 0 \tag{2.22}$$

Next, we assume that $f(x)$ and $g(x)$ are linearly dependent. i.e.

$$lf(x) + mg(x) = 0 \tag{2.23}$$

implying at least one of l and m is non-zero.

Multiplying (2.23) by $f(x)$ and integrating w.r.t. x over $[a, b]$, we have

$$lA + mC = 0 \tag{2.24}$$

Next, multiplying (2.23) by $g(x)$ and integrate over $[a, b]$, we have

$$lC + mB = 0 \tag{2.25}$$

($Ax = 0$ has the only solution $x = 0$ if A is non-singular)

Equations (2.24) and (2.25) is a linear system of equations which has a non-zero solution. Hence

$$\begin{vmatrix} A & C \\ C & B \end{vmatrix} = 0 \tag{2.26}$$

or

$$AB = C^2 \tag{2.27}$$

$$\implies \Big(\int_a^b f(x)g(x)\,dx\Big)^2 = \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \tag{2.28}$$

□

2.2 Root Mean Square

Definition 2.2.1. Root mean square value K of an integrable function is defined as

$$K^2 = \frac{1}{b-a}\int_a^b f^2(x)\,dx \tag{2.29}$$

Remark:

$U = K$

2.3 Definite Integral as a Function of Its Limits

Let $f(x)$ be integrable over $[a, b]$ and define

$$F(x) = \int_a^x f(t)\,dt \tag{2.30}$$

$F(x)$ exists for all values of x in $[a, b]$.

Theorem 2.3.1. *The function $F(x)$ defined by (2.30) is continuous for all $x \in [a, b]$.*

Proof. Let x_0 be any point of $[a, b]$ and h be such that $x_0 + h \in [a, b]$.

Consider

$$F(x_0 + h) - F(x_0) = \int_a^{x_0+h} f(x)\,dx - \int_a^{x_0} f(x)\,dx \tag{2.31}$$

$$= \int_{x_0}^{x_0+h} f(x)\,dx \tag{2.32}$$

$$\Big|F(x_0 + h) - F(x_0)\Big| = \Big|\int_{x_0}^{x_0+h} f(x)\,dx\Big| \leq \int_{x_0}^{x_0+h} \Big|f(x)\Big|\,dx \tag{2.33}$$

Since $f(x)$ is bounded, there exists M such that

$$\Big|f(x)\Big| \leq M \tag{2.34}$$

$F(x)$ exists for all values of x in $[a, b]$.

Consequently,

$$\Big|F(x_0 + h) - F(x_0)\Big| \leq M \int_{x_0}^{x_0+h} dx \tag{2.35}$$

$$= M[x_0 + h - x_0] \tag{2.36}$$

$$= Mh < m\delta = \epsilon \tag{2.37}$$

Hence, $F(x)$ is continuous at x_0. But x_0 is any point of $[a, b]$.

Therefore, $F(x)$ is continuous in $[a, b]$. □

Theorem 2.3.2. *The function $F(x) = \int_a^x f(t)\,dt$ is differentiable at any point where $f(x)$ is continuous, and $F'(x) = f(x)$ at any such point.*

Proof.

$$F'(x) = \lim_{h\to 0} \frac{F(x + h) - F(x)}{h} \tag{2.38}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \tag{2.39}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \tag{2.40}$$

Since $f(x)$ is continuous at x,

$$= \lim_{h\to 0} \frac{1}{h} \Big[f(x) + \eta \int_x^{x+h} dt\Big] \text{ where } \eta \to 0 \text{ as } h \to 0 \tag{2.41}$$

$$= \lim_{h\to 0} \frac{1}{h} \Big[f(x) + \eta\Big] h \tag{2.42}$$

$$= \lim_{h\to 0} \Big[f(x) + \eta\Big] \tag{2.43}$$

$$= f(x) \tag{2.44}$$

□

Theorem 2.3.3. *If $f(x)$ is continuous and $Q(x)$ is any function such that $Q'(x) = f(x)$ throughout $[a, b]$, then*

$$\int_a^b f(x)\,dx = Q(x)\Big|_a^b = Q(b) - Q(a) \tag{2.45}$$

Application

1. Evaluate the integral using definition

$$I = \int_0^{\frac{\pi}{2}} \cos x \, dx \tag{2.46}$$

Solution

Let $x_0 = 0, x_1 = \theta, \ldots, x_{r-1} = (r-1)\theta, x_r = r\theta, \ldots, x_n = n\theta$ and let s_r coincide with x_{r-1} i.e. $s_r = (r-1)\theta$.

Therefore

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos x \, dx &= \lim_{\theta \to \infty} \left[\sum_{r=1}^{n} f(s_r)\, \delta r \right], \;\; \delta r = x_r - x_{r-1} = \theta & (2.47)\\
&= \lim_{\theta \to 0} \left[\sum_{r=1}^{n} \cos(r-1)\theta \right] \theta & (2.48)\\
&= \lim_{\theta \to 0} \theta c \qquad \text{where } c = \sum_{r=1}^{n} \cos(r-1)\theta & (2.49)
\end{aligned}$$

Let $\theta = \frac{\pi}{2n}, 0 \leq \theta \leq \frac{\pi}{2}$.

Thus, $c = 1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta$.

Now, let $s = \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta$

Then

$$c + is = 1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{i(n-1)\theta} \tag{2.50}$$

and

$$(c + is)e^{i\theta} = e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta} \tag{2.51}$$

Subtracting (2.51) from (2.50), and we have

$$\begin{aligned}
c + is - (c + is)e^{i\theta} &= (c + is)(1 - e^{i\theta}) & (2.52)\\
&= 1 - e^{in\theta} & (2.53)
\end{aligned}$$

Consequently,

$$\begin{aligned}
c + is &= \frac{1 - e^{in\theta}}{1 - e^{i\theta}} & (2.54)\\
&= \frac{1 - e^{in\theta}}{1 - e^{i\theta}} \times \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}} & (2.55)\\
&= \frac{1 - e^{in\theta} - e^{i\theta} + e^{i(n-1)\theta}}{1 - (e^{i\theta} + e^{-i\theta}) + 1} & (2.56)\\
&= \frac{1 - i - [\cos\theta - i\sin\theta] + [i\cos\theta + \sin\theta]}{2 - 2\cos\theta} \qquad \text{since } \theta = \frac{\pi}{2n} & (2.57)
\end{aligned}$$

Therefore

$$c = \frac{1 - \cos\theta + \sin\theta}{2 - 2\cos\theta} \tag{2.58}$$

But

$$\cos 2\theta = 1 - 2\sin^2\theta \implies \cos\theta = 1 - 2\sin^2\frac{\theta}{2} \tag{2.59}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \implies \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{2.60}$$

Thus

$$c = \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2.2\sin^2\frac{\theta}{2}} \tag{2.61}$$

$$= \frac{1}{2}\frac{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \tag{2.62}$$

Consequently,

$$\lim_{\theta\to 0}\theta c = \lim_{\theta\to 0}\frac{\theta}{2}\frac{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \tag{2.63}$$

$$= \lim_{\theta\to 0}\frac{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}{\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}} \tag{2.64}$$

$$= \lim_{\theta\to 0}\Big(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\Big) \text{ since } \lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{2.65}$$

$$= 1 \tag{2.66}$$

2. Compute $\int_1^2 \frac{dx}{x}$ using fundamental definition of Riemann integration.

Solution

$$f(x) = \frac{1}{x} \tag{2.67}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{r=1}^{n} f\Big(a + \frac{r(b-a)}{n}\Big) \tag{2.68}$$

Consequently,

$$\int_1^2 \frac{1}{x}\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\Big(1 + \frac{r}{n}\Big) \tag{2.69}$$

Since $f(x) = \frac{1}{x}$, then

$$f\Big(\frac{n+r}{r}\Big) = \frac{1}{\frac{n+r}{n}} = \frac{n}{n+r} = \frac{1}{1+\frac{r}{n}} \tag{2.70}$$

Therefore,

$$\int_1^2 \frac{dx}{x} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{(1+\frac{r}{n})}\Big(\frac{1}{n}\Big) \tag{2.71}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} \tag{2.72}$$

$$= \lim_{n\to\infty}\Big[\frac{1}{1+n} + \frac{1}{2+n} + \ldots + \frac{1}{n+n}\Big] \tag{2.73}$$

$$= \log_e 2 \tag{2.74}$$

2.4 Fundamental Theorems on Integral Calculus

1. $$\int_a^b f(x)\,dx = \phi(x)\Big|_a^b = \phi(b) - \phi(a) \tag{2.75}$$

2. Integration by parts:

   Let $f(x)$ and $g(x)$ be differentiable whose derivatives $f'(x)$ and $g'(x)$ are integrable.

   Then
   $$\int_a^b \Big(fg' + gf'\Big)\,dx = \Big[fg\Big]_a^b \tag{2.76}$$

3. Integration by substitution:

   Let $x = \phi(t)$ be a continuous function of t with a continuous derivative $\phi'(t)$ from $t = \alpha$ to $t = \beta$.

   Let $f(x)$ be a continuous function of x from $x = a$ to $x = b$.

   Then
   $$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\phi(t)]\phi'(t)\,dt \tag{2.77}$$

Example

Prove that
$$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases} \tag{2.78}$$

Solution

$$\begin{aligned}
\int_{-a}^a f(x)\,dx &= \int_0^a f(x)\,dx + \int_{-a}^0 f(x)\,dx && (2.79)\\
&= \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx && (2.80)\\
&= \int_0^a [f(x) + f(-x)]\,dx && (2.81)
\end{aligned}$$

Thus
$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases} \tag{2.82}$$

e.g.

$$\begin{aligned}
\int_0^{2\pi} \cos x\,dx &= 2\int_0^\pi \cos x\,dx = 0 && (2.83)\\
\int_{-\pi}^\pi \sin mx \cos mx\,dx &= \frac{1}{2}\int_{-\pi}^\pi 2\sin mx\cos mx\,dx && (2.84)\\
&= \frac{1}{2}\int_{-\pi}^\pi \sin 2mx\,dx && (2.85)\\
&= 2\times\frac{1}{2}\int_0^\pi \sin 2mx\,dx && (2.86)
\end{aligned}$$

2.5 Differentiation of Definite Integral

The definite integral

$$\int_a^b f(x)\, dx \tag{2.87}$$

depends on a, b and y. Suppose a and b are fixed, then we may put

$$G(y) = \int_a^b f(x, y)\, dx \tag{2.88}$$

Let $f(x, y)$ be defined on the rectangle

$$R = \left\{ \begin{array}{ll} (x, y): & a \leq x \leq b \\ & \alpha \leq y \leq \beta \end{array} \right. \tag{2.89}$$

Theorem 2.5.1. *If $f(x, y)$ is a continuous function in R, then*

$$G(y) = \int_a^b f(x, y)\, dx \tag{2.90}$$

is a continuous function of y in $[\alpha, \beta]$.

Proof. Let y_0 be any point in $[\alpha, \beta]$.

$$\begin{aligned} G(y) - G(y_0) &= \int_a^b f(x, y)\, dx - \int_a^b f(x, y_0)\, dx && (2.91) \\ &= \int_a^b [f(x, y) - f(x, y_0)]\, dx && (2.92) \\ \Big| G(y) - G(y_0) \Big| &\leq \int_a^b \Big| f(x, y) - f(x, y_0) \Big|\, dx && (2.93) \\ & && (2.94) \end{aligned}$$

Since $f(x, y)$ is continuous,

$$\Big| f(x, y) - f(x, y_0) \Big| < \epsilon \quad (y - y_0)^2 < \delta^2 \tag{2.95}$$

Consequently,

$$\Big| G(y) - G(y_0) \Big| < \int_a^b \epsilon\, dx = \epsilon(b - a) \tag{2.96}$$

Hence, $G(y)$ is continuous at $y = y_0$. But y_0 is any point. Hence, $G(y)$ is continuous in $[\alpha, \beta]$. □

Theorem 2.5.2. *If $f(x, y)$ and $f_y(x, y)$ are continuous in R, then*

$$G(y) = \int_a^b f(x, y)\, dx \tag{2.97}$$

is differentiable in $[\alpha, \beta]$ and

$$\begin{aligned} G'(y) &= \frac{d}{dy} \int_a^b f(x, y)\, dx && (2.98) \\ &= \int_a^b f_y(x, y)\, dx && (2.99) \end{aligned}$$

Where $f_y(x, y)$ denotes the partial derivations of $f(x, y)$ w.r.t. y and $G'(y)$ denotes the derivation of $G(y)$ w.r.t. y in $[\alpha, \beta]$.

Proof. Let y_0 be any point of $[\alpha, \beta]$. Then

$$\frac{G(y_0+k)-G(y_0)}{k} = \frac{\int_a^b f(x,y_0+k)\,dx - \int_a^b f(x,y_0)\,dx}{k} \tag{2.100}$$

$$= \int_a^b \left[\frac{f(x,y_0+k)\,dx - f(x,y_0)}{k}\right] dx \tag{2.101}$$

is the conditions for mean value theorem are satisfied w.r.t. y, we have

$$= \int_a^b \frac{1}{k} k f_y(x, y_0+k)\,dx \tag{2.102}$$

$$= \int_a^b f_y(x, y_0+\theta k)\,dx \quad 0<\theta<1 \tag{2.103}$$

Taking limit as $k \to 0$, we have

$$G'(y_0) = \int_a^b f_y(x,y_0)\,dx \tag{2.104}$$

Since y_0 is any point of $[\alpha,\beta]$, we have

$$G'(y) = \int_a^b f_y(x,y)\,dx \;\forall\; y \in [\alpha,\beta] \tag{2.105}$$

□

Theorem 2.5.3. *Let the limits of integration a, b be function of y, then G is a function of three independent variables a, b, y. i.e.*

$$G(y,a,b) = \int_{a(y)}^{b(y)} f(x,y)\,dx \tag{2.106}$$

Then, if $f(x,y)$ and $f_y(x,y)$ are continuous in R, the integral (2.106) *is a differentiable of a, b, y regarded as independent variables involving continuous partial derivatives.*

2.6 First Mean Value Theorem of Integral Calculus

Theorem 2.6.1. *In the interval $[a,b]$, $a<b$, let $f(x)$ and $g(x)$ be bounded and integrable and $g(x)>0$. Also, let m and M be the bounds of $f(x)$ in $[a,b]$.*

Then

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx \tag{2.107}$$

Proof.

$$\begin{aligned} m \le f(x) &\le M \text{ for all } x\in[a,b],\ mg(x)\le f(x)g(x)\le Mg(x) \\ g(x) &> 0 \\ m\int_a^b g(x)\,dx &\le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx \end{aligned}$$

Corollary 1

If $g(x)$ does not change sign in $[a, b]$, a number U exists $(m \leq U \leq M)$ such that

$$\int_a^b f(x)g(x)\,dx = U \int_a^b g(x)\,dx \tag{2.108}$$

Corollary 2

If $f(x)$ is monotonic, the number U lies between $f(a)$ and $f(b)$.

Corollary 3

If $f(x)$ is continuous at $x = a$, we can put

$$\int_a^b f(x)g(x)\,dx = [f(a) + \eta] \int_a^b g(x)\,dx \;\text{ where } \eta \to 0 \text{ as } b \to a. \tag{2.109}$$

Corollary 4

If $f(x)$ is continuous throughout $[a, b]$, a number η exists $a \leq \eta \leq b$ such that

$$\int_a^b f(x)g(x)\,dx = f(\eta) \int_a^b g(x)\,dx \tag{2.110}$$

Illustrative Example of Corollary 4

Evaluate

$$I = \lim_{n\to\infty} \int_{\frac{1}{n+a}}^{\frac{1}{n}} \frac{1}{x^4} \sin x^2\,dx \;\; a > 0 \tag{2.111}$$

Solution

$$I \;=\; \lim_{n\to\infty} \int_{\frac{1}{n+a}}^{\frac{1}{n}} \left(\frac{\sin x^2}{x^2}\right)\left(\frac{1}{x^2}\right) dx \tag{2.112}$$

Thus, $g(x) = \frac{1}{x^2} \;> 0$, $f(x) = \frac{\sin x^2}{x^2}$, $g(x)$ and $f(x)$ are continuous in $[\frac{1}{n+a}, \frac{1}{n}]$.
Hence, M.V.T. implies

$$\begin{aligned}
I &= \lim_{n\to\infty} \frac{\sin \eta^2}{\eta^2} \int_{\frac{1}{n+a}}^{\frac{1}{n}} \frac{1}{x^2}\,dx, \;\; \frac{1}{n+a} \leq \eta \leq \frac{1}{n} && (2.113)\\
&= \lim_{n\to\infty} \frac{\sin \eta^2}{\eta^2} \Big[-\frac{1}{x}\Big]_{\frac{1}{n+a}}^{\frac{1}{n}} && (2.114)\\
&= a \lim_{n\to\infty} \frac{\sin \eta^2}{\eta^2} && (2.115)\\
&= a \lim_{\eta\to 0} \frac{\sin \eta^2}{\eta^2} && (2.116)\\
&= a && (2.117)
\end{aligned}$$

□

2.7 Second Mean Value Theorem for Integrals

Theorem 2.7.1. *Let $f(x)$be continuous and monotonic so that $f'(x)$ does not change sign in $[a, b]$ and $f'(x)$ is integrable.*

Let $g(x)$ be continuous in $[a, b]$. Then, a number η exists, $a \leq \eta \leq b$ such that

$$\int_a^b f(x)g(x)\,dx = f(a) \int_a^\eta g(t)\,dt + f(b) \int_\eta^b g(t)\,dt \tag{2.118}$$

Proof. Define
$$G(x) = \int_a^x g(t)\,dt \tag{2.119}$$
Then, since $g(x)$ is continuous
$$G'(x) = g(x) \tag{2.120}$$

Now,

Since $G'(x)$ and $f'(x)$ are integrable and $G(a) = 0$ from (2.119), we have

$$\begin{aligned}
\int_a^b [f(x)G'(x) + f'(x)G(x)]\,dx &= \Big[fG\Big]_a^b && (2.121)\\
&= f(b)G(b) - f(a)G(a) && (2.122)\\
&= f(b)G(b), \text{ since } G(a) = 0 && (2.123)
\end{aligned}$$

or
$$\int_a^b fG'\,dx + \int_a^b f'G\,dx = f(b)G(b) \tag{2.124}$$
Since $f'(x)$ is integrable and has same sign in $[a, b]$, and $G(x)$ is continuous in $[a, b]$ a number η, $a \le \eta \le b$ exists such that

$$\begin{aligned}
\int_a^b G(x)f'(x)\,dx &= G(\eta)\int_a^b f'(x)\,dx \text{ from first M.V.T.} && (2.125)\\
\int_a^b G(x)f'(x)\,dx &= G(\eta)\Big[f(b) - f(a)\Big] && (2.126)
\end{aligned}$$

From (2.125) and (2.126), we have

$$\begin{aligned}
\int_a^b fG'\,dx &= f(b)G(b) - G(\eta)\Big[f(b) - f(a)\Big] && (2.127)\\
&= f(b)\Big[G(b) - G(\eta)\Big] + f(a)G(\eta) && (2.128)\\
&= f(b)\left[\int_a^b g(t)\,dt - \int_a^\eta g(t)\,dt\right] + f(a)\int_a^\eta g(t)\,dt && (2.129)\\
&= f(b)\left[\int_a^\eta g(t)\,dt + \int_\eta^b g(t)\,dt - \int_a^\eta g(t)\,dt\right] + f(a)\int_a^\eta g(t)\,dt && (2.130)\\
&= f(b)\int_\eta^b g(t)\,dt + f(a)\int_a^\eta g(t)\,dt && (2.131)
\end{aligned}$$

□

2.8 Function of Bounded Variations

Definition 2.8.1. Let $[a, b]$ be a bounded and closed interval. Let $D = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$ and $\triangle f_k = f_k - f_{k-1}$, $k = 1, 2, \ldots, n$ if there exists a positive number M such that
$$\sum_{k=1}^{n} \Big|\triangle f_k\Big| \le M \tag{2.132}$$
for all partitions of $[a, b]$, then $f(x)$ is said to be of bounded variations on $[a, b]$.

2.8.1 Total Variation

Definition 2.8.2. The total variation of f on $[a, b]$ is defined as

$$V_f[a,b] = \sup_{\text{all } D} \left\{ \sum_{k=1}^{n} \Big|\triangle f_k\Big| \right\}, \;\; D = \{x_0, x_1, \ldots, x_n\} \tag{2.133}$$

Theorem 2.8.1. *If $f(x)$ and $g(x)$ are of bounded variations on $[a, b]$, then $f(x)+g(x)$ is of bounded variations on $[a, b]$ and $V_{f+g}[a, b] \leq V_f[a, b] + V_g[a, b]$.*

Proof. Since $f(x)$ is of bounded variations on $[a, b]$, there exists $M_1 > 0$ such that

$$\sum_{k=1}^{n} \Big|\triangle f_k\Big| \leq M_1 \tag{2.134}$$

Also, $g(x)$ is of bounded variations on $[a, b]$, there exists $M_2 > 0$ such that

$$\sum_{k=1}^{n} \Big|\triangle g_k\Big| \leq M_2 \tag{2.135}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} \Big|\triangle (f_k + g_k)\Big| &= \sum_{k=1}^{n} \Big|f_k + g_k - f_{k-1} - g_{k-1}\Big| && (2.136)\\
&= \sum_{k=1}^{n} \Big|(f_k - f_{k-1}) + (g_k - g_{k-1})\Big| && (2.137)\\
&= \sum_{k=1}^{n} \Big|\triangle f_k + \triangle g_k\Big| && (2.138)\\
&\leq \sum_{k=1}^{n} \Big[\triangle f_k + \triangle g_k\Big] && (2.139)\\
&= \sum_{k=1}^{n} \Big|\triangle f_k\Big| + \sum_{k=1}^{n} \Big|\triangle g_k\Big| && (2.140)\\
&\leq M_1 + M_2 = M && (2.141)
\end{aligned}$$

Hence, $f + g$ is of bounded variations on $[a, b]$

$$\begin{aligned}
V_{f+g}[a,b] &= \sup_{\forall D} \left[\sum_{k=1}^{n} \Big|\triangle (f_k + g_k)\Big| \right], \;\; D = \{x_0, x_1, \ldots, x_n\} && (2.142)\\
&\leq \sup_{\forall D} \left[\sum_{k=1}^{n} \Big|\triangle f_k\Big| + \sum_{k=1}^{n} \Big|\triangle g_k\Big| \right], \;\; D = \{x_0, x_1, \ldots, x_n\} && (2.143)\\
&\leq \sup_{\forall D} \sum_{k=1}^{n} \Big|\triangle f_k\Big| + \sup_{\forall D} \sum_{k=1}^{n} \Big|\triangle g_k\Big| && (2.144)\\
&= V_f[a,b] + V_g[a,b] && (2.145)
\end{aligned}$$

□

Remarks

A continuous function is not necessarily of bounded variations.

Consider an example;

$$f(x) = \begin{cases} x\cos\frac{\pi x}{2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{2.146}$$

Clearly, the function is continuous on $[0,1]$. Consider the partitions $D = \left[0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right]$

$$\sum_{k=1}^{2n} \left|\triangle f_k\right| = \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \ldots + \frac{1}{2} + \frac{1}{2} \tag{2.147}$$

$$= 1 + \frac{1}{2} + \ldots + \frac{1}{n} \tag{2.148}$$

Since the sequence $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, this is not bounded for all n.

Chapter Three

3.1 Improper Integrals

In defining the Riemann definite integral and showing its equivalence with an area under a curve, we have assumed that the limits of integration a and b are finite and that the integrand remains finite in the range of integration. If either or both of these conditions are not satisfied, the Riemann integral may not exist.

Definition 3.1.1. The integral

$$\int_a^b f(x)\,dx \tag{3.1}$$

is called an improper integral if

a) one or both integration limits is infinite

b) $f(x)$ is unbounded at one or more points of $a \leq x \leq b$. Such points are called singularities of $f(x)$.

Integrals corresponding to a) and b) above are called improper integrals of the first and second kinds respectively. Integrals with both conditions are called improper integrals of the third kind.

Example

Indicate which of the following integrals are of the first, second and third kind.

(a) $\int_0^\infty \sin x^2\,dx$ (b) $\int_0^\infty \frac{dx}{x-3}$ (c) $\int_0^\infty \frac{e^{-x}}{x}\,dx$ (d) $\int_0^1 \frac{\sin x}{x}\,dx$

Solution

(a), (b) and (c) are improper integrals of the first, second and third kind respectively while (d) is a proper integral since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{3.2}$$

3.1.1 Convergence of Improper Integrals of the First Kind

Let $f(x)$ be a bounded and integrable in every finite interval $a \leq x \leq b$. Then, we define

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{3.3}$$

The integral on the left is called convergent if the limit on the right exists and divergent if it does not.

Remarks

We note that $\int_0^\infty f(x)\,dx$ bears close analogy to the infinite series $\sum_{n=1}^\infty U_n$ where $U_n =$ while $\int_a^b f(x)\,dx$ corresponds to the partial sums of such infinite series.

As in the definition above, we also have

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \tag{3.4}$$

where the integral on the left is called convergent if the limit on the right exists and divergent if it does not.

Examples

1.

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} && (3.5)\\ &= \lim_{b\to\infty}\left[1-\frac{1}{b}\right] && (3.6)\\ &= 1\end{aligned}$$

which implies $\int_1^\infty \frac{dx}{x^2}$ converges to 1.

2.

$$\begin{aligned}\int_{-\infty}^{u} \cos x\,dx &= \lim_{a\to-\infty}\int_a^u \cos x\,dx && (3.7)\\ &= \lim_{u\to\infty}(\sin u-\sin a) && (3.8)\\ &= \infty\end{aligned}$$

Thus, $\int_{-\infty}^{u} \cos x\,dx$ is divergent.

Theorem 3.1.1.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{x_0} f(x)\,dx + \int_{x_0}^{\infty} f(x)\,dx \tag{3.9}$$

Remarks

Exponential integral

$$\int_a^\infty e^{-tx}\,dx \text{ where } t \text{ is a constant} \tag{3.10}$$

converges if $t > 0$ and diverges if $t \leq 0$.

$$\int_a^\infty \frac{dx}{x^p}, \text{ where } p \text{ is a constant and } a > 0 \tag{3.11}$$

converges if $p > 1$ and diverges if $p \leq 1$. This is called p-integral of the first kind.

3.2 Convergence Tests

3.2.1 Comparison Test

Let $f(x)$ be continuous and thus integrable in every finite interval $a \leq x \leq b$ and let $g(x) \geq 0$ for all $x \geq a$. Then if

a) $\int_a^\infty g(x)\,dx$ converges and $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then $\int_a^\infty f(x)\,dx$ converges.

b) $\int_a^\infty g(x)\,dx$ diverges and $f(x) \geq g(x)$ for all $x \geq a$, then $\int_a^\infty f(x)\,dx$ diverges.

Examples

1. Examine the convergence of $\int_0^\infty \frac{dx}{e^x+1}$.

   Solution

$$\frac{1}{e^x+1} \leq \frac{1}{e^x}$$

and since $\int_0^\infty e^{-x}\,dx$ converges, $\int_0^\infty \frac{dx}{e^x+1}$ also converges.

2. Examine the convergence of $\int_2^\infty \frac{dx}{\ln x}$.

   Solution

$$\frac{1}{\ln x} > \frac{1}{x}$$

and since $\int_2^\infty \frac{dx}{x}$ diverges by p integral test, $\int_2^\infty \frac{dx}{\ln x}$ also diverges.

3.2.2 Quotient Test (For Integrals with Non-Negative Integrands)

If $f(x) > 0$ and $g(x) > 0$, then if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} A \neq 0 & \text{and } \int_a^b g(x)\,dx \text{ converges, then } \int_a^b f(x)\,dx \text{ converges} \\ \infty & \text{and } \int_a^b g(x)\,dx \text{ diverges, then } \int_a^b f(x)\,dx \text{ diverges} \end{cases}$$

Exercise: Prove this.

Theorem 3.2.1. *The integral $\int_a^\infty f(x)\,dx$ converges if we can find $g(x) = x^p$, $p > 1$ such that*

$$\lim_{x\to\infty} x^p f(x) = A \text{ (finite)} \tag{3.12}$$

If $p \leq 1$ and $A \neq 0$ (A may be finite), then integral is divergent.

Examples

Examine the convergence of $\int_1^\infty \frac{x}{3x^4+5x^2+1}\,dx$ using quotient test and theorem 3.2.1.

Solution

Let $f(x) = \frac{x}{3x^4+5x^2+1}$, $g(x) = \frac{1}{x^3}$.

Now,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{3} = A \tag{3.13}$$

and $\int_1^\infty g(x)\,dx$ converges, $\int_1^\infty \frac{x}{3x^4+5x^2+1}\,dx$ also converges.

Next, we use theorem 3.2.1. We can find x^3 such that

$$\lim_{x\to\infty} x^3 \left(\frac{x}{3x^4+5x^2+1}\right) = \frac{1}{3} = A \tag{3.14}$$

Since A is finite and $p > 1$, $\int_1^\infty g(x)\,dx$ converges.

3.2.3 Series Test (For Integrals with Non-Negative Integrands)

$\int_a^\infty f(x)\,dx$ converges or diverges according as $\sum U_n$ where $U_n = f(n)$ converges or diverges.

3.2.4 Absolute and Conditional Convergence

$\int_a^\infty f(x)\,dx$ is called absolutely convergent if $\int_a^\infty |f(x)|\,dx$ converges. If $\int_a^\infty f(x)\,dx$ converges but $\int_a^\infty |f(x)|\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ is called **conditionally convergent**.

Theorem 3.2.2. *If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges. i.e an absolutely convergent integral is convergent.*

Examples

Examine the absolute convergence of $\int_1^\infty \frac{\cos x}{x^2}\,dx$

Solution

$$\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2} \text{ for } x \ge 1 \tag{3.15}$$

By comparison test, since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, it follows that $\int_1^\infty \left|\frac{\cos x}{x^2}\right|\,dx$ converges.

Hence, $\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges absolutely.

3.2.5 Convergence of Improper Integrals of the Second Kind

If $f(x)$ becomes unbounded only at the end point $x = a$ of the interval $a \le x \le b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)\,dx \tag{3.16}$$

If the limit on the right of (3.16) exists, we call the integral on the left convergent, otherwise, it is divergent.

Similarly, if $f(x)$ becomes unbounded only at the end point $x = b$ of the interval $a \le x \le b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx \tag{3.17}$$

In such case, the integral on the left of (3.17) is called convergent or divergent according as the limit on the right exists or does not exist.

If $f(x)$ becomes unbounded only at an interior point $x = x_0$ of the interval $a \leq x \leq b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0^+} \int_a^{x_0+\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_{x_0+\epsilon_2}^b f(x)\,dx \tag{3.18}$$

The integral on the left of (3.18) converges or diverges according as the limits on the right exists or do not exist.

Extension of these definitions can be made in case $f(x)$ becomes unbounded at two or more points of the interval $a \leq x \leq b$.

3.2.6 Cauchy Principal Value

If it is possible that by choosing $\epsilon_1 = \epsilon_2 = \epsilon$ in (3.18) by just putting

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \left[\int_a^{x_0+\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^b f(x)\,dx \right] \tag{3.19}$$

when it does exist. If it does, we call this limiting value the **principal value** of the integral on the left.

Remarks

a) $\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$

b) $\int_a^b \frac{dx}{(b-x)^p}$ converges if $p < 1$ and diverges if $p \geq 1$

These are called p integrals of the second kind. When $p \leq 0$, the integrals are proper.

3.3 Convergence Tests

3.3.1 Comparison Test

Let $f(x)$ be continuous and thus integrable in the interval $a \leq x \leq b$, unbounded only at $x = a$ and let $g(x) \geq 0$ for $a \leq x \leq b$. Then if

a) $\int_a^b g(x)\,dx$ converges and $0 \leq f(x) \leq g(x)$ for $a < x \leq b$, then $\int_a^b f(x)\,dx$ also converges,

b) $\int_a^b g(x)\,dx$ diverges and $f(x) \geq g(x)$ for $a < x \leq b$, then $\int_a^b f(x)\,dx$ also diverges.

Examples

1. Examine the convergence of $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$

   Solution

   $$\frac{1}{\sqrt{x^4-1}} < \frac{1}{\sqrt{x-1}}, \text{ for } x > 1$$

   and since $\int_1^5 \frac{dx}{\sqrt{x-1}}$ converges, $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges using the P–integral test.

   $p = \frac{1}{2}$.

2. Investigate the convergence of $\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$.

   Solution
   $$\frac{\ln x}{(x-3)^4} < \frac{1}{(x-3)^4} \text{ for } x > 3$$
   Then, since $\int_3^6 \frac{dx}{(x-3)^4}$ diverges, $\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$ diverges using p–integral test.

   $p = 4$

3.3.2 Quotient Test (For Integrals with Non-Negative Integrands)

If $f(x) \geq 0$ and $g(x) \geq 0$, then if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} A \neq 0 & \text{and } \int_a^b g(x)\,dx \text{ converges, then } \int_a^b f(x)\,dx \text{ converges} \\ \infty & \text{and } \int_a^b g(x)\,dx \text{ diverges, then } \int_a^b f(x)\,dx \text{ diverges} \end{cases}$$

Theorem 3.3.1. *The integral $\int_a^b f(x)\,dx$ converges if we can find $g(x) = (x-a)^p$, $p < 1$ such that*

$$\lim_{x\to a^+} (x-a)^p f(x) = A \text{ (finite)}$$

If $p \geq 1$ and $A \neq 0$ (A may be finite), then the integral is divergent.

Theorem 3.3.2. *The integral $\int_a^b f(x)\,dx$ converges if we can find $g(x) = (b-x)^p$, $p < 1$ such that*

$$\lim_{x\to b^-} (b-x)^p f(x) = B \text{ (finite)}$$

If $p \geq 1$ and $B \neq 0$ (B may be infinite), then the integral is divergent.

Examples

1. Examine the convergence of $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$

   Solution

   $$\lim_{x\to 1^+} (x-1)^{\frac{1}{2}} \frac{1}{(x^4-1)^{\frac{1}{2}}} = \lim_{x\to 1^+} \sqrt{\frac{x-1}{x^4-1}} = \frac{1}{2} \tag{3.20}$$

   Hence, $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ is convergent.

2. Examine the convergence of $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$

   Solution

   $$\lim_{x\to 3^-} (3-x) \frac{1}{(3-x)\sqrt{x^2+1}} = \frac{1}{\sqrt{10}} \tag{3.21}$$

   Hence, integral diverges as $p = 1$.

3.3.3 Absolute and Conditional Convergence

$\int_a^b f(x)\,dx$ is called absolutely convergent if $\int_a^b |f(x)|\,dx$ converges. If $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ is called conditionally convergent.

Theorem 3.3.3. *A absolutely convergent integral converges.*

Example

Investigate the convergence of $\int_{\pi}^{4\pi} \frac{\sin x}{(x-\pi)^{\frac{1}{3}}}\, dx$

Solution

$$\left| \frac{\sin x}{(x-\pi)^{\frac{1}{3}}} \right| \leq \frac{1}{(x-\pi)^{\frac{1}{3}}} \tag{3.22}$$

Since $\int_{\pi}^{4\pi} \frac{dx}{(x-\pi)^{\frac{1}{3}}}\, dx$ converges, $\int_{\pi}^{4\pi} \left| \frac{\sin x}{(x-\pi)^{\frac{1}{3}}} \right| dx$ converges and thus $\int_{\pi}^{4\pi} \frac{\sin x}{(x-\pi)^{\frac{1}{3}}}\, dx$ converges absolutely.

3.3.4 Convergence of the Improper Integrals of the Third Kind

Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kinds and hence the question of their convergence or divergence is answered by using results already established.

Chapter Four

4.1 Convergence of a Sequence of Real-Valued Functions

Definition 4.1.1. Let $D \subset R$. Suppose that for each $n \in N$, there is a function f_n with D as domain and with range contained in R. Then, we say that $\{f_n\}$ is a sequence of real-valued functions on D.

4.1.1 Pointwise Convergence

Taking a fixed point $x \in D$, we may consider $\{f_n(x)\}$ of real numbers and ask whether or not this sequence converges to a real number. At some points, the sequence $\{f_n(x)\}$ may converge while at others, it may diverge. Let $D_0 \subset D$ be the set of all points x at which

$$\lim_{n\to\infty} f_n(x) \text{ exists} \tag{4.1}$$

Then, we define a new function f on D_0 by setting

$$f(x) = \lim_{n\to\infty} f_n(x), \ x \in D_0 \tag{4.2}$$

Definition 4.1.2. Let $\{f_n\}$ be a sequence of real-valued functions on $D \subset R$ and let D_0 be a subset of D. Suppose that there exists a real-valued function f with domain containing D_0, we say that the sequence $\{f_n\}$ converges pointwise to f on D_0 if for each $x \in D_0$, the sequence $\{f_n(x)\}$ converges to $f(x)$.

Remarks

f_n converges pointwise to f on D_0 is usually written

$$f_n(x) \to f(x), \ x \in D_0 \tag{4.3}$$

Or

$$\lim f_n(x) = f(x), \ x \in D_0 \tag{4.4}$$

and we say f is the pointwise limit function of $\{f_n\}$.

Definition 4.1.3. Let $\{U_n(x)\}, n = 1, 2, 3, \ldots$ be a sequence of functions defined in $[a, b]$. The sequence is said to converge to have the limit $F(x)$ in $[a, b]$, if for each $\epsilon > 0$ and each $[a, b]$ we can find $N > 0$ such that

$$\Big|U_n(x) - F(x)\Big| < \epsilon \tag{4.5}$$

for all $n > N$.

In such case, we write

$$\lim_{n\to\infty} U_n(x) = F(x) \tag{4.6}$$

The number N may depend on x as well as ϵ.

Definition 4.1.4. The infinite series of functions

$$\sum_{n=1}^{\infty} U_n(x) = U_1(x) + U_2(x) + \dots \tag{4.7}$$

is said to be convergent in $[a, b]$, if the sequence of partial sums $\{S_n(x)\}, n = 1, 2, 3, \dots$ where

$$S_n(x) = U_1(x) + U_2(x) + \dots + U_n(x) \tag{4.8}$$

is convergent in $[a, b]$ and we write

$$\lim_{n\to\infty} S_n(x) = S(x) \tag{4.9}$$

where $S(x)$ is the sum of the series.

4.1.2 Uniform Convergence

Definition 4.1.5. A series $\sum_{n=1}^{\infty} U_n(x)$ is called uniformly convergent in $[a, b]$ if for each $\epsilon > 0$ and each x in $[a, b]$, we can find $N > 0$ which depends only on ϵ and not on x such that

$$\Big|S_n(x) - S(x)\Big| < \epsilon \tag{4.10}$$

for all $n > N$.

Remarks

Since $S(x) - S_n(x) = R_n(x)$, the remainder after n terms, we can equivalently say that $\sum U_n(x)$ is uniformly convergent in $[a, b]$ if for each $\epsilon > 0$, we can find N depending on ϵ but not on x such that

$$\Big|R_n(x)\Big| < \epsilon \tag{4.11}$$

for all $n > N$ and all x in $[a, b]$.

Let us define uniform convergence on $D_0 \subseteq D \subseteq R$.

Definition 4.1.6. A sequence $\{f_n\}$ of function on $D \subset R$ converges uniformly on a subset D_0 of D to a function f if for each $\epsilon > 0$, there is a positive integer N depending on ϵ but not on $x \in D_0$ such that if $n > N$ and $x \in D_0$, then

$$\Big|f_n(x) - f(x)\Big| < \epsilon \tag{4.12}$$

which equivalently can be written as

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon \tag{4.13}$$

Example

Using definition, examine the uniform convergence of $f_n(x) = \frac{1}{n^2+x^2}$

Solution

$$\lim_{n\to\infty} \frac{1}{n^2+x^2} = 0 \tag{4.14}$$

Thus

$$\Big|f_n(x) - 0\Big| = \frac{1}{n^2+x^2} \leq \frac{1}{n^2} < \epsilon \tag{4.15}$$

for all $n > n(\epsilon)$ must obtain for uniform convergence choose $\epsilon = \frac{1}{2}$, after the first term

$$\frac{1}{2^2} < \frac{1}{2}, \quad \frac{1}{3^2} < \frac{1}{2}, \quad \frac{1}{4^2} < \frac{1}{2}$$

Thus $n(\epsilon) = 1$.

Thus for $\epsilon = \frac{1}{2} > 0$, we can find $n(\epsilon) = N = 1$ such that for all x

$$\Big|f_n(x) - 0\Big| < \epsilon \tag{4.16}$$

Theorem 4.1.1. *Let $\{f_n\}$ be a sequence of functions on $D \subset R$. Then $\{f_n\}$ is uniformly convergent on D to some function f if and only if, given $\epsilon > 0$, there exists $n_0(\epsilon) \in N$ (independent of x) such that for all $x \in D$*

$$\Big|f_m(x) - f_n(x)\Big| < \epsilon, \qquad m > n_0(\epsilon),\ n > n_0(\epsilon) \tag{4.17}$$

Proof. First suppose $\{f_n\}$ converges uniformly to f on D. Then for every $\epsilon > 0$, there exists $n_0(\epsilon)$ such that for all $x \in D$,

$$\Big|f_n(x) - f(x)\Big| < \frac{\epsilon}{2} \text{ for } n > n_0(\epsilon) \tag{4.18}$$

Thus, if $m > n_0(\epsilon)$, we have for every $x \in D$

$$\Big|f_m(x) - f_n(x)\Big| \leq \Big|f_m(x) - f(x)\Big| + \Big|f(x) - f_n(x)\Big| \tag{4.19}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \tag{4.20}$$

and so (4.17) holds for this $n_0(\epsilon)$.

Conversely, let $\{f_n\}$ be such that given $\epsilon > 0$,

$$\Big|f_m(x) - f_n(x)\Big| < \epsilon \text{ for every } x \in D \tag{4.21}$$

We are required to obtain a function f on D such that $\{f_n\}$ converges uniformly to f on D.

Now, for each fixed x, (4.17) implies that $\{f_n\}$ is cauchy sequence of real numbers which therefore converges to a real number (see MTS 206).

Choose in (4.17), an arbitrary but fixed $n_1 > n_0(\epsilon)$. Also, for each $x \in D$ let m tend to infinity. Then for all $x \in D$

$$\Big|f(x) - f_{n_1}(x)\Big| < \epsilon \tag{4.22}$$

Since n_1 is any integer greater than $n_0(\epsilon)$, we have

$$\Big|f(x) - f_n(x)\Big| < \epsilon \tag{4.23}$$

for all $n > n_0(\epsilon)$ and all $x \in D$.

Hence, $\{f_n\}$ converges uniformly to f on D. $\square$

Theorem 4.1.2. *Let f_n be a sequence of continuous functions on $D \subset R$ and suppose that $\{f_n\}$ converges uniformly to f on D. Then, f is continuous on D.*

Proof. Let x_0 be an arbitrary point of D. We want to show that f is continuous at x_0.

Since $\{f_n\}$ converges uniformly to f on D, given $\epsilon > 0$, there is a positive integer $N(\frac{\epsilon}{3})$ such that

$$\Big|f_n(x) - f(x)\Big| < \frac{\epsilon}{3} \text{ for all } x \in D \text{ and } n > N(\tfrac{\epsilon}{3}) \tag{4.24}$$

Furthermore, since f_n is continuous at x_0, there exists $\delta > 0$ depending on ϵ, x_0 and f_n such that for all x in D satisfying

$$\Big|x - x_0\Big| < \delta \tag{4.25}$$

we have

$$\Big|f_n(x) - f_n(x_0)\Big| < \frac{\epsilon}{3} \tag{4.26}$$

For such x, we have

$$\Big|f(x) - f(x_0)\Big| \leq \Big|f(x) - f_n(x)\Big| + \Big|f_n(x) - f_n(x_0)\Big| + \Big|f(x) - f(x_0)\Big| \leq \epsilon \tag{4.27}$$

This shows that f is continuous at x_0 and since x_0 was arbitrary, f is continuous on D. $\square$

Theorem 4.1.3. *Suppose that $\{f_n\}$ is a sequence of bounded functions each of which is integrable in the closed interval $[a, b]$ and suppose that $\{f_n\}$ converges uniformly to f on $[a, b]$. Then f is integrable on $[a, b]$.*

Furthermore, if

$$g_n(x) = \int_a^x f_n(t)\, dt \text{ and } g(x) = \int_a^x f(t)\, dt,\ x \in [a, b]$$

then $\{f_n\}$ converges uniformly to g on $[a, b]$.

Proof. The proof is left to students as exercise. $\square$

Corollary

If $\{f_n\}$ and f are as in above theorem, then

$$\int_a^b f(x)\, dx = \int_a^b \Big[\lim_{n\to\infty} f_n(x)\Big]\, dx = \lim_{n\to\infty} \int_a^b f(x)\, dx \tag{4.28}$$

4.1.3 Weierstrass M-Test

If a sequence of positive constants $M_1, M_2, M_3, \ldots$ can be found such that is some interval

a) $\Big|U_n(x)\Big| \leq M_n \;\; n = 1, 2, 3, \ldots$

b) $\sum M_n$ converges

then $\sum U_n$ is uniformly and absolutely convergent.

Example

Which of the series is uniformly and absolutely convergent.

$$\text{a) } \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \qquad\qquad \text{b) } \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Solution

$$\text{a) } \sum \left|\frac{\cos nx}{n^2}\right| \leq \sum \frac{1}{n^2} \quad \text{which converges}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \quad \text{converges uniformly and absolutely.}$$

$$\text{b) } \sum \left|\frac{\sin nx}{n}\right| \leq \sum \frac{1}{n} \quad \text{which diverges}$$

Theorem 4.1.4. *If $\{U_n(x)\}, n = 1, 2, \dots$ are continuous and have continuous derivatives in $[a, b]$ and if $\sum U_n(x)$ converges to $S(x)$ while $\sum U_n'(x)$ is uniformly convergent in $[a, b]$, then in $[a, b]$*

$$\frac{d}{dx}\Big[\sum_{n=1}^{\infty} U_n(x)\Big] = \sum_{n=1}^{\infty} \frac{d}{dx} U_n(x) \tag{4.29}$$

This is the condition under which a series can be differentiated term by term.

Remarks

Under the above conditions too;

$$\lim_{n\to\infty} \int_a^b U_n(x)\,dx = \int_a^b \lim_{n\to\infty} U_n(x)\,dx \tag{4.30}$$

Theorem 4.1.5. *If $\{U_n(x)\}, n = 1, 2, \dots$ are continuous in $[a, b]$ and if $\sum U_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$.*

Remarks

The above theorem implie a uniformly convergent series of continuous functions is a continuous function.

If $x_0 \in [a, b]$, then the above theorem states that

$$\lim_{x\to x_0} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \lim_{x\to x_0} U_n(x) \tag{4.31}$$

$$= \sum_{n=1}^{\infty} U_n(x_0) \tag{4.32}$$

4.1.4 Power Series

We shall now consider a special but important class of series of functions, called power series in this subsection.

Definition 4.1.7. Let c be a real number and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. A series of functions $\sum_{n=0}^{\infty} f_n$ is said to be a **power series** with $x = c$ as centre if the functions f_n are of the form

$$f_n(x) = a_n(x-c)^n \; x \in R \tag{4.33}$$

i.e. a power series is a series having the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n \tag{4.34}$$

In the special case when $c = 0$, the series takes the form

$$\sum_{n=0}^{\infty} a_n x^n \tag{4.35}$$

For simplicity of notation, we shall consider a power series to be of the form of the latter, i.e. (4.35)

4.1.5 Convergence of a Power Series

The values of x for which (4.35) converges may be found using d'Alembert's ratio test. Hence, for the series to be absolutely convergent, we must have

$$\lim_{r\to\infty}\left|\frac{a_{r+1}x^{r+1}}{a_r x^r}\right| = |x|\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| \tag{4.36}$$

$$= k < 1 \tag{4.37}$$

The above condition may be more conviniently expressed as

$$|x| < R \tag{4.38}$$

where R, the radius of convergence, is given by

$$R = \lim_{r\to\infty}\left|\frac{a_r}{a_{r+1}}\right| \tag{4.39}$$

provided this limit exists.

Equation (4.38), when written in full, takes the form

$$-R < x < R \tag{4.40}$$

However, the inclusion of end points is possible.

Although, the ratio test is often successful in obtaining this interval, it may fail and in such cases other tests may be used.

Remarks

Two special cases, $R = 0$ and $R = \infty$ can arise. When $R = 0$, the series converges only for $x = 0$ and when $R = \infty$, it converges to all x. The latter is often written $-\infty < x < \infty$.

In this course, when we speak of a convergent power series, we shall assume unless otherwise indicated, that $R > 0$.

Theorem 4.1.6. *The power series* $\sum_{n=0}^{\infty} a_n x^n$ *and the corresponding series of derivatives* $\sum_{n=0}^{\infty} n a_n x^{n-1}$ *have the same radius of convergence.*

Proof. Let $R > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Let $0 < |x_0| < R$. Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges $\lim_{n\to\infty} a_n x_0^n = 0$.

Thus, for every large $n > N, N > 0$,

$$|a_n x_0^n| < 1$$

That is,

$$|a_n| < \frac{1}{|x_0|^n}$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} |a_n||x|^n && (4.41) \\ &< \sum_{n=1}^{\infty} \frac{|x|^n}{|x_0|^n} && (4.42) \end{aligned}$$

The last series converges for all

$$|x| < |x_0| < R \tag{4.43}$$

It follows by comparison test that

$$\sum_{n=1}^{\infty} a_n x^n \quad \text{converges}$$

and so,

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{converges absolutely}$$

Similarly,

$$\begin{aligned} \sum \left| n a_n x^{n-1} \right| &= \sum n |a_n||x|^{n-1} && (4.44) \\ &< \sum n \frac{|x|^{n-1}}{|x_0|^n} \text{ for } n > N && (4.45) \end{aligned}$$

The last series also converges by the ratio test for $|x| < |x_0| < R$.

Hence, $\sum n a_n x^{n-1}$ converges absolutely for all points x_0 no matter how close $|x_0|$ is to R. Thus $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. □

Example

Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n} \tag{4.46}$$

and the corresponding series of derivatives

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n 3^n} \tag{4.47}$$

have the same radius of convergence.

Solution

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 3^{n+1}}.\frac{n^2 3^n}{x^n}\right| \tag{4.48}$$

$$= \lim_{n\to\infty}\frac{n^2}{3(n+1)^2}|x| \tag{4.49}$$

$$= \frac{|x|}{3} \tag{4.50}$$

which converges for $|x| < 3$.

At $x = \pm 3$, the series also converges so that the interval of convergence is

$$-3 \leq x \leq 3 \tag{4.51}$$

The series of derivatives is

$$\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n3^n} \tag{4.52}$$

which implies

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{x^n}{(n+1)3^{n+1}}.\frac{n3^n}{x^{n-1}}\right| \tag{4.53}$$

$$= \lim_{n\to\infty}\frac{n}{3(n+1)}|x| \tag{4.54}$$

$$= \frac{|x|}{3} \tag{4.55}$$

Reference

[1] Walter, R. (1976), *Principles of Mathematical Analysis*, 2nd Edition, McGraw-Hill, Inc, New York

[2] Bartle, R. G. (1964), *Elements of Real Analysis*, John Wiley and Sons, Inc, New York

[3] Bartle, R. G. and Sherbert, D. R. (2000), *Introduction to Real Analysis*, 2nd Edition, John Wiley and Sons, Inc, U.S.A.

[4] Adegoke, O. (1979), *Introduction to Real Analysis*, Hebn Publishers Plc

[5] Bridger, M. (2007), *Real Analysis-A Constructive Approach*, John Wiley and Sons, Inc, U.S.A

[6] Goffman, C. (1966), *Introduction to Real Analysis*, Harper & Row Publishers, Inc, New York

[7] Stoll, M. (2001), *Introduction to Real Analysis*, 2nd Edition, Addison Wesley Longman, Inc.