MTS 303 ABSTRACT ALGEBRA I DR OGHENERUEMU

Course Content

1 Review of set theory & algebra of sets.
2 Relations
3 Functions/maps
4 Binary Logic
5 Method of proof
6 Binary operations
7 Algebraic structures
8 Homomorphisms.

1.1.0 LECTURE I: REINTRODUCTION TO SET THEORY

Definition 1.1.1: A set is a well defined collection of distinct objects.

- The objects in a set are the elements or members of the sets
- Sets are usually denoted by capital letters such as A, B, S, T; or in some cases by special symbols such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, P(A).
- The elements of a set are usually enclosed in curly brackets: { }.

1.1.2 Describing ~~the element~~ a set

We can describe a set by

i listing the elements within curly brackets, or by
ii describing the properties the elements of the set satisfy.

Example 1.1.3

1 Let A be the set of factors of 12. Then
i $A = \{1, 2, 3, 4, 6, 12\}$, or

ii $A = \{x : x \text{ is a factor of } 12\}$
The symbol ":" reads as "such that" in the above description of set A.

2 Let B be the set of factors of 36 that are greater than or equal to 5. Then
i $B = \{6, 9, 12, 18, 36\}$, or
ii $B = \{y : y \text{ is a factor of } 36 \text{ and } y \geq 5\}$

3 Let C be the set of alphabets in Oghenertuemu. Then
i $C = \{o, g, h, e, n, r, u, m\}$
ii $= \{e, g, h, m, n, o, r, u\}$, or
iii $C = \{\alpha : \alpha \text{ is an english alphabet in "Oghenertuemu"}\}$.
α is the greek alphabet pronounced as "alpha".

Note
a We use a comma after each element listed within the curly brackets
b set C i & ii identify the same set (order of element does not matter as long as elements are completely listed).
c I did not write letters e and u more than once although they occur multiple times in my name. This is because a set consists of distinct elements.
d It ~~won't really~~ does change the set to write e & u (or any other element of C) more than once in the list, but if you want to count the elements in a set, each distinct element must be counted only once, then C has 8 elements not 11 elements that is the length of my name.
e I wrote the "o" in my name in small letter, because as member of set C it is not special in comparison to other members, and elements of a set are usually

written in small letters, except when a set is a collection of other sets, or its members are special characters.

If an element y is a member of a set D we denote that with $y \in D$, which reads as "y is an element of D". The y does not belong to the set D we write $y \notin D$, that is y is not an element of D.

Example 1.1.4
Considering sets A, B & C from Example 1.1.3 above
$1 \in A$, $2 \in A$ $8 \notin A$ ~~a~~ $a \notin A$, $b \notin$ ~~B~~
$6, 12, 18 \in B$, however $4, 5 \notin B$
$e \in C$, $g \in C$, $r \in C$, $n \in C$ e.t.c, but $x \notin C$, $y \notin C$, $\alpha \notin C$, $O \notin C$.
Although α is used as a variable to identify the element of C in the description of the set, α is not an element of the set, similarly $o \in C$ ~~is~~ ~~as~~ but $O \notin C$, although my name starts with capital O, we have not ~~dec~~ listed the elements of C as capital letters.

Exercise 1.1.5
1 Describe the following sets by listing the elements and also by specifying the property the members satisfy.
a The set A whose members are prime numbers greater than 7 but less than or equal to 37.
b The elements of B are factors of 35 that do not divide 14.
c The set C who elements are the roots of quadratic x^2+x-6.
d Set D consists of the items ~~in~~ ~~of~~ (symbols, alphabets, numbers) in your matric number. Example if my matric number is MTS/14/1234. Then $D = \{M, T, S, /, 1, 4, 2, 3\}$ in list form.

2. Mention 3 elements that are in each of the sets in (1) and 3 elements that do not belong to the sets.

1.2.1 Types of Sets.

a. Empty set: This is a set without any element or member. An empty set is denoted by { } or ϕ.
Example: i) Let F be the set of factors of 12 that are greater than 15.
F is an empty set since there is no factor of 12 greater than 15. Therefore
F = {x : x is a factor of 12 and x > 15}
 = { }

ii) Let M be the set of months with 27 days. Then
M = { }

b. Subset: A set T is called a subset of another set S if all the elements in T are elements S. We denote this by $T \subset S$.
Example: Let T be the set of all prime numbers less than 10 and S the set of numbers less than 10, then
T = {2, 3, 5, 7} and S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
Then $T \subset S$ since all the elements in T are in S.

c. Superset: A superset is a set that contains all the elements of another set. ~~Eg~~ V is a superset of W if V contains all the elements of W. We denote this by $V \supset W$.
Example: i) Let W = {a, b, c}, V = {a, b, c, d, f, g, k, l}. Since all the elements of W are in V, V is a superset of W.

ii) S is a superset for T in (b) above.

d. Equal sets: Two sets are equal if they have exactly the same elements. Equal sets are subsets of each other.

Example: i) Let A be the set of odd numbers from 3 to 7, ~~and~~ ~~let~~ B be the set of prime numbers from 3 to 7, while C is the set of odd numbers between 3 and 7. Then

$A = \{x : x \text{ is an odd number and } 3 \le x \le 7\}$
$= \{3, 5, 7\}$

$B = \{x : x \text{ is a prime and } 3 \le x \le 7\}$
$= \{3, 5, 7\}$

$C = \{x : x \text{ is an odd number and } 3 < x < 7\}$
$= \{5\}$

Since A and B (despite having different descriptions) have the same elements ($A \subseteq B$ & $B \subseteq A$), the two sets are equal, that is $A = B$.
Although the description of elements in C is similar to the description of elements in A, however, the two sets are not equal ($A \neq C$) since they do not have the same ~~set~~ elements. Similarly, $B \neq C$ (B is not equal to C).

ii) Let E be the set of prime factors of 6 and F the set of prime factors less than 5. Then
$E = \{2, 3\}$, $F = \{2, 3\}$, and $E = F$.

e Proper subset: Let P be a subset of Q (i.e all elements of P are in Q). P is a proper subset of Q if P is not equal to Q, that is, there is atleast one element in Q that is not in P. We denote propersubset by $P \subsetneq Q$.

Example: Let P be the set of factors of 6, ~~and~~ Q be the set of numbers less than or equal to 8 and R the set of factors of 12. Then

$P = \{1, 2, 3, 6\}$,
$Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$R = \{1, 2, 3, 4, 6, 12\}$

Clearly $P \subset Q$ and $P \subset R$, but $5 \in Q$, $5 \notin P$, therefore $P \neq Q$ and $P \subsetneq Q$. Similarly since R has element 4 but 4 is not in P but $P \subset R$, we have that $P \subsetneq R$.

Q is not contained in R since Q has some elements (e.g 5, 7) that are not in R, then $Q \not\subset R$, similarly R is not a subset of Q, since $12 \in R$ but $12 \notin Q$. ~~Since~~ The two sets are infact not equal, $Q \neq R$.

Note that the symbol $\subseteq$ can also ~~is same~~ be used for subset.

i(a) $A \subseteq B$ reads as A is a subset of B [A may or may not be equal to B].

$A \subset B$: A is a subset of B (we are not interested in any or absent equality between the sets).

$A \subsetneq B$: A is a subset of B that is not equal to B.

ii(b) The empty set { } is a subset of every set.

iii(c) Every set is a subset of itself.

f Improper subsets: The empty set and the set itself are the improper subsets of a set, all other subsets are proper subsets.

The empty set is improper subset because it is a subset that has no element, while the set itself is impro ~~sub~~ per because it is the only subset that is equal to itself.

g ~~Let~~ Power set: Let A be a set with n-distinct elements, where n is a positive whole number, then the powerset of A denoted by P(A) is ~~the~~ set that ~~that~~ contains all the possible ~~contains whose eleme~~ subsets of A.

The power set has 2^n subsets of A has its elements.

Example: i) Let $A = \{a, b\}$, then A has 2 distinct elements, $(n = 2)$ and P(A) has $2^2 = 4$ elements.

$P(A) = \{ \{\}, \{a\}, \{b\}, \{a, b\}\}$.

ii Let $B = \{1, 2, 3\}$. B has 3 distinct elements, $n = 3$, P(B) then has $2^3 = 8$ elements.

$P(B) = \{ \{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

The elements of a power set are ~~the~~ sets. The power set is our first example of a set whose elements are sets.

In (ii) above the proper subsets of B are the singletons ({1}, {2}, {3}), and the doubles ({1,2}, {1,3}, {2,3}), while { } and {1, 2, 3} are improper subsets of B.

h Set of sets: This is a set whose elements are themselves sets.

Example: i) Power set.

ii FUTA is a set of sets. For instance the university can be described as

1) FUTA = {{students}, {academic staff}, {non-academic staff}},

or as

2) = {{students, academic and non-academic staff of each department}, {health centre staff}, {FUTA business subsidiaries staff}, {administrative staff that are not attached to a department}, {other non-academic and non-admin staff}}

iii) MTS 303 25/26 is a set of sets = {{MTS 303 MTS registered students}, {MTS 303 CSC registered students}, {MTS 303 CYS registered students}}

i) Universal set: This is the set that contains all the elements under consideration in a particular problem or discussion. It is the super set (not power set) that encompasses every element belonging to all other sets being discussed within the context of a given problem. The universal set is denoted by $\mathcal{U}$

Example: Let A be the set of first ten English alphabets

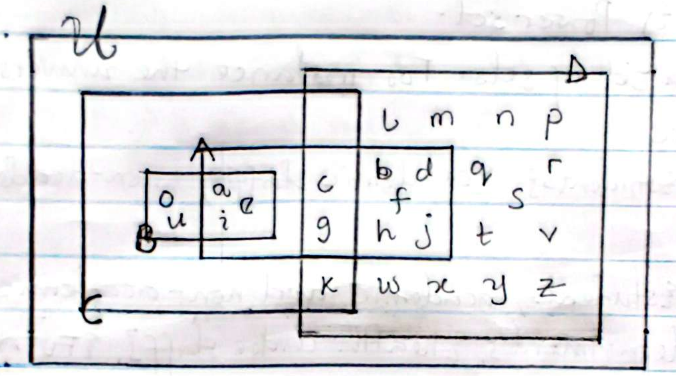
$A = \{a, b, c, d, e, f, g, h, i, j\}$

B is the set of English vowels

$B = \{a, e, i, o, u\}$

C is the set of odd numbered English alphabets.

$C = \{a, c, e, g, i, k, m, o, q, s, u, w, y\}$
and D is the set of English consonants.
$D = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.
Then the universal set $\mathcal{U}$ in this case is the set of English alphabets.
$\mathcal{U} = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$.



In this example $\mathcal{U}$ is superset for sets A, B, C & D.
The universal set in ~~another~~ this example is a set of sets.
The universal set is not always a set of sets, its members can consist of both sets and ordinary elements.

j Finite set: This is a set with a ~~countable~~ ~~or~~ limited number of elements.
Example
i $A = \{1, 2, 3, 4, 6, 7\}$ is a ~~countable~~ finite set with 6 elements
ii $B = \{x : x \text{ is a number from } 1 \text{ to } 100\}$ is a ~~countable~~ finite set with 100 elements.
iii $C = \{x : x \text{ is an alphabet in Oghenerucemu}\}$ is a finite set with 8 elements.

k Infite set: A set whose elements go on indefinitely

Example: Let $\mathbb{N}$ be the set of counting numbers. Then $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots\}$ is an infinite set.

1.2.2 Cardinality of a set

Let A be a set. The cardinality of A is the number of ~~ele~~ distinct elements in A. It is denoted by $|A|$.

Examples

i In the universal set example in 1.2.1(i).

A has 10 elements therefore $|A| = 10$
B " ___ " " $|B| = 5$
C " ___ " " $|C| = 13$
D " ___ " " $|D| = 21$
$\mathcal{U}$ " ___ " " $|\mathcal{U}| = 26$

ii The cardinality of an infinite set is denoted by the infinity symbol ∞.
In the example above $|\mathbb{N}| = \infty$

Exercise 1.2.3

1 Which of the following sets is an empty set.

a $A = \{x : x$ is a prime number less than $2\}$
b $B = \{x : x$ is a prime number greater than $17\}$
c $C = \{y : y$ is a multiple of $7\}$
d $D = \{n \in \mathbb{N} : n^2 < 0\}$

2 Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $\subseteq$ denotes subset without ~~any regard or disregard for~~ possibility of equality. Which of the following is true?

a $A \subseteq B$. b) $B \subseteq A$ c) $B \supseteq A$

b

3 Suppose $A = \{x : x$ is an even positive number$\}$ and $B = \{x : x$ is a multiple of $4\}$. Which of the following is true? Justify your answers. 1

a) $B \subseteq A$ b) $A \supseteq B$ c) $B \supseteq A$.

2

4 Let $P = \{x : x$ is a vowel in the English alphabet$\}$ and $Q = \{a, e, i, o, u\}$.

a Is $Q \subseteq P$? b) Is $Q \subseteq P$ c) Are Q & P equal?

3a

5 After identifying the subset and superset in questions 2, 3 & 4 above, determine if the subset is a proper or improper subset of its superset.

6 Write out the powerset of the following sets.

$A = \{a\}$

$B = \{\ \}$

$C = \{1, 2, 3, 4\}$ 4

$D = \{1, f, g\}$

7a What is the cardinality of each of the sets above.

b What is the cardinality of ~~each~~ each of their power set.

8 Let $P = \{\alpha : \alpha$ is a prime number less than $1000\}$ and $S = \{\alpha : \alpha$ is a prime number$\}$. Which of P & S is finite, which infinite?

9 Determine the cardinality of the following 5

$A = \{y : y$ is a factor of $100\}$

$B = \{y : y$ is a multiple of $100\}$

6

1.2.4 Special Sets of Numbers.

1 The natural numbers denoted by $\mathbb{N}$ is the set of counting numbers.

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$

2 The integers denoted by $\mathbb{Z}$ is the set of positive and negative whole numbers, with zero included.

$\mathbb{Z} = \{\dots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$

3a The set of rational numbers $\mathbb{Q}$ comprises of all fractions of whole numbers with non-zero denominator }.

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$

Observe that $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{Q}$, ~~where~~ and $\mathbb{Z} \subset \mathbb{Q}$ (any whole number can be expressed as a fraction, e.g $-2 \in \mathbb{Z}$, $-2 = \frac{-2}{1} = \frac{6}{-3} \in \mathbb{Q}$).

4 An irrational number is a number that ~~is~~ cannot be expressed as a fraction of integers.

Unlike rational numbers the decimal expansion of an irrational number is infinite, it does not repeat nor terminates.

Examples are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{p}$ where p is a prime, $n\in\mathbb{Z}$ times $\sqrt{p}$ $n(\sqrt{p})$

Other examples include ~~E~~ e (Euler's number), φ (the golden ratio approximated to 1.61803) and π approximated to 3.1415 or $22/7$ (approximations are not exact values).

5 The set of real numbers $\mathbb{R}$ consist of all rational and irrational number, they correspond to the points on the x-axis (number line) in the positive & negative direction.

6 ~~Comp~~ Imaginary numbers: Consider the imaginary unit i when $i^2 = -1$. The imaginary numbers are formed by multiplying a real number and the imaginary unit.

$\mathbb{I} = \{ai : a \in \mathbb{R} \text{ and } i^2 = -1\}$

7 The complex numbers $\mathbb{C}$ is the set of the sums (and differences) of real & imaginary numbers.
$\mathbb{C} = \{a+bi : a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$.
The complex numbers consist of real numbers, when $b=0$, therefore $\mathbb{R} \subset \mathbb{C}$, $\mathbb{C}$ also contains the imaginary numbers when $a = 0$, $\mathbb{I} \subset \mathbb{C}$

Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Exercise 1.2.5

1 The set of irrational numbers is a subset of which of the sets above (1–7 in 1.2.4)

2 The set of imaginary numbers is a subset of which of the sets above.

1.2.6 Some Subsets of Integers

i The set of positive integers denoted by $\mathbb{Z}^{>0}$ or $\mathbb{Z}^+$ is the set $\mathbb{Z}^{>0} = \{1, 2, 3, 4, \ldots\}$
$= \{m : m \in \mathbb{Z} \text{ and } m > 0\}$

ii The set of non-negative integers denoted by $\mathbb{Z}^{\geq 0}$ is $\mathbb{Z}^{\geq 0} = \{0, 1, 2, 3, 4, \ldots\}$.

Exercise 1.2.7

1 Which of the set in 1.2.6 above is equal to $\mathbb{N}$.

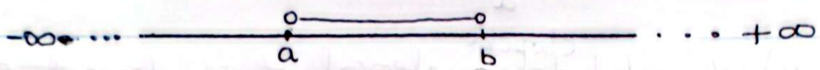
2. Describe the set of negative integers and the set of non-positive integers by listing their elements and also by specifying the property(ies) they satisfy. Choose the appropriate symbol to denote each of the set.

3 What is the cardinality of each of the set in 1·2·4, 1·2·6 and above in 1·2·7?

1·2·8 Some Subsets of Real Numbers.

Let a, b be real numbers such that $a \leq b$. We define ~~the~~ some bounded intervals in $\mathbb{R}$ as follows.

i $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

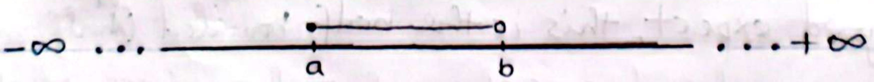


~~This, into is the set~~ This open interval is the set of real numbers between a and b. The interval is open because its members do not include the real numbers a & b at its ends.

ii $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

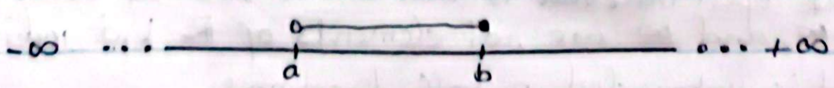
This closed interval represents the range of interval from a to b, it is ~~called~~ closed because it includes its end points a & b.

iii $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$



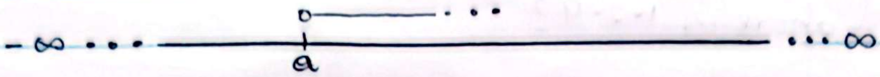
This clopen interval consists of all the elements in (a, b) with the endpoint a included, that is $[a, b) = (a, b) \cup \{a\}$; ~~wher~~ where $\cup$ is the union symbol.

iv $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$



As you'd expect this is the clopen interval with all the elements of sets (a, b) and $\{b\}$.

v $(a, \infty) = \{x \in \mathbb{R} : a < x\}$

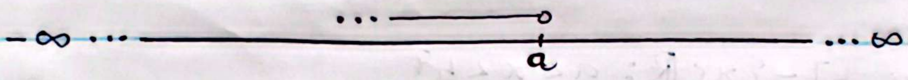


Is the (half) unbounded open interval consisting of all real numbers strictly greater than a.

vi $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

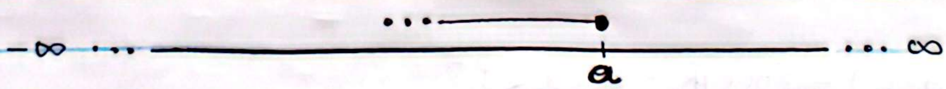
The (half) unbounded clopen ~~closed~~ interval whose elements are the real numbers greater or equal to a.

vii $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$



The half ~~bo~~ unbounded open interval which represents the range of real numbers strictly less than a.

viii $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$



As you'd expect, this is the half bounded (or half unbounded) ~~ba~~ clopen interval consisting of real numbers less than or equal to a.

ix $(-\infty, \infty) = \mathbb{R}$

This open interval is the entire set of real numbers. $-\infty$, and ∞ are not elements of $\mathbb{R}$ but indicate the set is unbounded in both directions.

NOTE! NOTE! NOTE!

i When $a < b$ all the intervals above are infinite sets.

ii Since the intervals are subsets of $\mathbb{R}$, these ~~intervals~~ do not just consist of ~~consist~~ whole numbers, but uncountable numbers of reals when $a \neq b$.

iii When $a=b$, the closed interval $[a, b] = \{a\}$ and $(a, b) = [a, b) = (a, b] = \{\}$. These are the only ~~finite~~ scenarios the intervals are finite.

Exercise 1.2.9

1 Describe each of the following sets by the properties their elements satisfy.

2 Determine which of the sets are ~~is~~ empty, finite or infinite.

3 Show each of the sets on a number line where possible.

4 ~~Mention~~ Give an example one natural number, one integer, one rational number, one irritional number, in each of the sets (possible for most of the sets).

i $(2, 7)$ ii) $(-2, 7)$, iii) $(-7, -2)$ iv) $(0, 1)$

v $[0, 1]$ vi) $(0, 0)$ vii) $[0, 0]$ viii) $[-100, -200]$

ix $[-100, 200)$ x) $(0, 2]$ xi) $[0, 3)$ xii) $(0, +\infty)$

xiii $(4, +\infty)$ xiv) $[\pi, \infty)$ xv) $(-\infty, -3)$ xvi) $(-\infty, 0)$

xvii $(-\infty, 7)$ xviii) $(-\infty, 0]$ xix) $(-\infty, \infty)$.