ECE 133A — Homework # 6 — Discussion 1B — Angel Jimenez — 604 667 905

Exercise A8.13

A is $\langle m \times n \rangle$

b is m-vector $\langle m, 1 \rangle$

A has linearly independent columns

$\hat{x}$ is the solution to minimize $\|Ax - b\|^2 = \sum_{i=1}^{m} (a_i^T x - b_i)^2$

a_i^T is the ith row of A

$\hat{y}_k$ is the solution to minimize $\sum_{i \neq k} (a_i^T y - b_i)^2 =$

row k is removed $\quad = \sum_{i=1}^{k-1} (a_i^T y - b_i)^2 + \sum_{i=k+1}^{m} (a_i^T y - b_i)^2$

* Assume for each $k = 1, \dots, m$, the matrix formed by removing row k from A has linearly independent columns. So solution $\hat{y}_k$ is unique

a)

$$\text{minimize} \quad \sum_{i=1}^{k-1} (a_i^T y - b_i)^2 + (a_k^T y + z - b_k)^2 + \sum_{i=k+1}^{m} (a_i^T y - b_i)^2$$

the segment in the middle is our k^{th} row, we want to remove that so we find the condition that makes this zero.

$(a_k^T y + z - b_k)^2 = 0 \Rightarrow a_k^T y + z - b_k = 0 \Rightarrow$ $\underline{z}$ is our outlier

$\therefore z = b_k - a_k^T y$, which aggrees with the problem statement

we're left with

$$\text{minimize} \quad \sum_{i=1}^{k-1} (a_i^T y - b_i)^2 + \sum_{i=k+1}^{m} (a_i^T y - b_i)^2$$

we know because A is linearly indep columns, $\hat{y}$ is a unique solution to equations of that form, which is what we have

b) $\|(A \; e_k)(y \; z)^T - b\|^2$

$\therefore$ normal equation $\begin{pmatrix} A^T \\ e_k^T \end{pmatrix} (A \; e_k) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} A^T \\ e_k^T \end{pmatrix} b \Rightarrow \begin{pmatrix} A^T A & a_k \\ a_k^T & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} A^T b \\ b_k \end{pmatrix}$

$A^T A y + a_k z = A^T b \rightarrow y = (A^T A)^{-1}(A^T b - a_k z) = \hat{x} - (A^T A)^{-1} a_k z$

$a_k^T y + z = b_k \rightarrow a_k^T(\hat{x} - (A^T A)^{-1} a_k z) + z = b \rightarrow z = \dfrac{b_k - a_k^T \hat{x}}{1 - a_k^T (A^T A)^{-1} a_k}$

$$\hat{y}_k = \hat{x} - \frac{b_k - a_k^T \hat{x}}{1 - a_k^T (A^T A)^{-1}} (A^T A)^{-1} a_k$$

c) $A = QR$, $\hat{y}_k = \hat{x} - \dfrac{b_k - a_k^T \hat{x}}{1 - a_k^T R^{-1} R^{-T}} R^{-1} R^{-T} a_k$

$A = QR$ — $2mn^2$ flops

$a^T b$ — n^2 flops

$\dfrac{b_k - a_k^T \hat{x}}{1 - u_k^T u_k}$ — $4mn$ flops

$R^{-1} u_k$ — mn^2 flops

$\dfrac{\hat{x} - b_k - a_k^T \hat{x} R^{-1} u_k}{1 - u_k^T u_k}$ — $2mn$ flops $\quad \therefore 3mn^2$ flops total

Problem A9.4

a)

$$\text{minimize} \quad \|Ax-y\|^2 + \lambda(\|D_v x\|^2 + \|D_h x\|^2)$$

$\hookrightarrow$ normal equation: $(A^TA + \lambda D_v^T D_v + \lambda D_h^T D_h)x = A^Ty$

Assume $A = T(B)$, $D_v = T(E)$, $D_h = T(E^T)$

$$T(U) = \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)\tilde{W}$$

Substitute into normal equations

$$T(U)^T T(U) = \left(\frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)\tilde{W}\right)^H\left(\frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)\tilde{W}\right)$$

$$= \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)^H \tilde{W} \cdot \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)\tilde{W}$$

$$= \frac{1}{n^4}\tilde{W}^H \operatorname{diag}(\tilde{W}u)^H \underbrace{\tilde{W}\tilde{W}^H}_{I n^2} \operatorname{diag}(\tilde{W}u)\tilde{W}$$

$$= \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}u)^H \operatorname{diag}(\tilde{W}u)\tilde{W}$$

$\therefore (A^TA + \lambda D_v^T D_v + \lambda D_h^T D_h)x = A^Ty$

$$\hookrightarrow \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}_b)^H \operatorname{diag}(\tilde{W}_b)\tilde{W} + \frac{1}{n^2}\tilde{W}^H \lambda \operatorname{diag}(\tilde{W}_e)^H \operatorname{diag}(\tilde{W}_e)\tilde{W} + \frac{1}{n^2}\tilde{W}^H \lambda \operatorname{diag}(\tilde{W}_e)^H \operatorname{diag}(\tilde{W}_e)\tilde{W}$$
$$= \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}b)^H \tilde{W} y$$

$$\hookrightarrow \frac{1}{n^2}\tilde{W}^H\left[\operatorname{diag}(\tilde{w}_b)^H \operatorname{diag}(\tilde{w}_b) + \lambda \operatorname{diag}(\tilde{w}_e)^H \operatorname{diag}(\tilde{w}_e) + \lambda \operatorname{diag}(\tilde{w}_{\hat{e}})^H \operatorname{diag}(\tilde{w}_{\hat{e}})\right]\tilde{W} = \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{w}b)^H \tilde{W} y$$

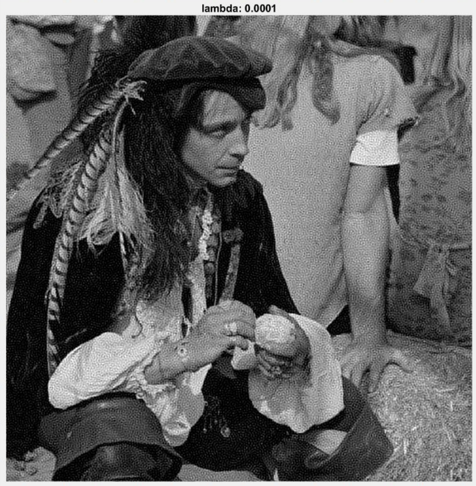
$$D = \operatorname{diag}(\tilde{w}_b)^H \operatorname{diag}(\tilde{w}_b) + \lambda \operatorname{diag}(\tilde{w}_e)^H \operatorname{diag}(\tilde{w}_e) + \lambda \operatorname{diag}(\tilde{w}_{\hat{e}})^H \operatorname{diag}(\tilde{w}_{\hat{e}})$$

$$\hookrightarrow \frac{1}{n^2}\tilde{W}^H D \tilde{W}x = \frac{1}{n^2}\tilde{W}^H \operatorname{diag}(\tilde{W}b)^H \tilde{W} y \rightarrow D\tilde{W}x = (\tilde{W}^H)^{-1}(\tilde{W}^H)\operatorname{diag}(\tilde{w}b)^H \tilde{w} y$$

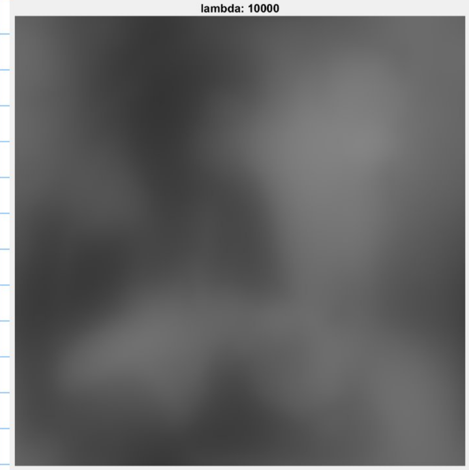
$$x = \tilde{W}^{-1} D^{-1} \operatorname{diag}(\tilde{W}b)^H \tilde{W} y$$

```
%Homework 6, Problem 2 (A9.4)

clear all;
load('deblur.mat');

lambda = 10^(-5);
[m,n] = size(Y);
E = zeros(m,m);
E(1,1) = 1;
E(m,1) = -1;
Afourier = fft2(B);
dQ = abs(fft2(E)).^2 + abs(fft2(E')).^2;
dW = abs(Afourier).^2 + lambda * dQ ;
dE = conj(Afourier).* fft2(Y) ./ dW;
X = real(ifft2(dE));
imshow(X);
title(['lambda: ' num2str(lambda) ]);
```

lambda: 0.0001

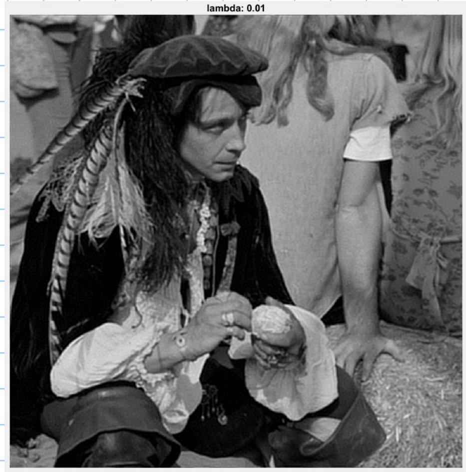


lambda: 10000



From the samples to the right, $\lambda > 1$ made the image worse, between $10^{-4} < \lambda < 10^{-1}$ were good clarity

lambda: 0.01



lambda: 1

Problem 10.1

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \quad t = 0, 1, 2, \dots \qquad s_1(0) = s_2(0) = 0$$

$\therefore\ s(0) = 0$

$s(t+1) = A\,s(t) + Bu(t)$

$t=0 \rightarrow s(1) = As(0) + Bu(0) \qquad s(1) = Bu(0)$

$t=1 \rightarrow s(2) = As(1) + Bu(1) \qquad s(2) = ABu(0) + Bu(1)$

$t=2 \rightarrow s(3) = As(2) + Bu(2) \qquad s(3) = A^2Bu(0) + ABu(1) + Bu(2)$

$t=3 \rightarrow s(4) = As(3) + Bu(3) \qquad s(4) = A^3Bu(0) + A^2Bu(1) + ABu(2) + Bu(3)$

$t=5 \rightarrow s(5) = As(4) + Bu(4) \qquad s(5) = A^4Bu(0) + A^3Bu(1) + A^2Bu(2) + ABu(3) + Bu(4)$

$\vdots$

$t=N-1 \rightarrow s(N) = A^{N-1}Bu(0) + A^{N-2}Bu(1) + \dots + A^1Bu(N-2) + Bu(N-1)$

$$\therefore s(N) = \begin{bmatrix} A^{N-1}B & A^{N-2} & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-2) \\ u(N-1) \end{bmatrix}$$

a) minimize $\|x\|^2$ $\qquad s_1(N) = 10$

subject to $Cx = d$ $\qquad s_2(N) = 0$

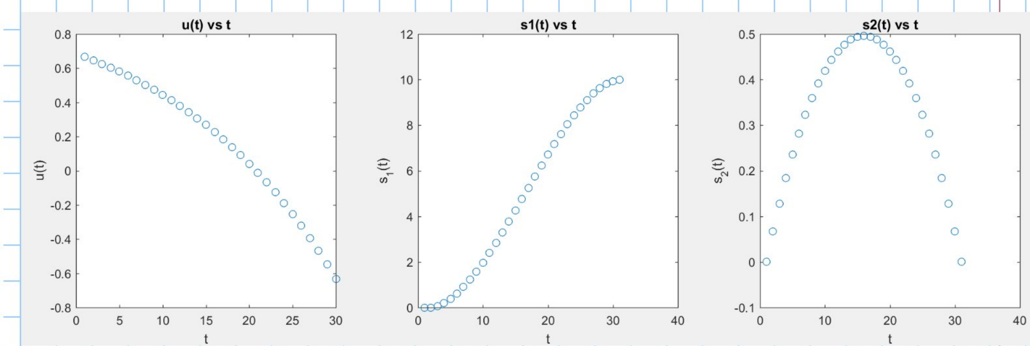
$$E = \sum_{t=0}^{N-1} u(t)^2 \qquad C = A, \qquad d = s(N) = \begin{bmatrix} s_1(N) = 10 \\ s_2(N) = 0 \end{bmatrix} \qquad \therefore x\ u(t) = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-2) \\ u(N-1) \end{bmatrix}$$

$$= \begin{bmatrix} A^{N-1}B & A^{N-2} & \cdots & AB & B \end{bmatrix}$$

$= \text{minimize } \|x\|^2$

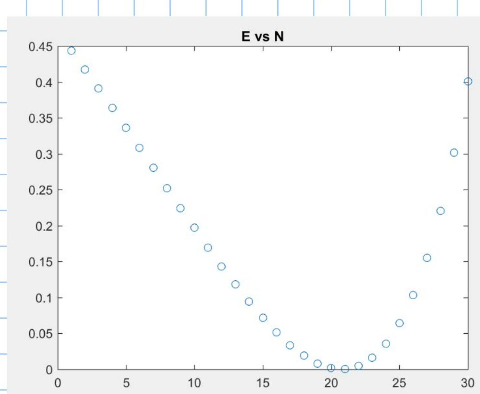
b)

```
% Homework 6, Problem 3
clear all;

N = 30;
A = [1 1; 0 0.95];
B = [0;0.1];
C = zeros(2,N);
C(:,N) = B;
for t= N-1:-1:1
    C(:,t)=A*C(:,t+1);
end;
d=[10;0];
u = C'*((C*C') \ d);

s = zeros(2,N+1); s(1,1)
for t=1:N
    s(:,t+1) = A*s(:,t)+B*u(t);
end;
subplot(1,3,1);
plot(u,'o'); title('u(t) vs t');
ylabel('u(t)'); xlabel('t');
subplot(1,3,2);
plot(s(1,:),'o'); title('s1(t) vs t');
ylabel('s_1(t)'); xlabel('t');
subplot(1,3,3);
plot(s(2,:),'o'); title('s2(t) vs t');
ylabel('s_2(t)'); xlabel('t');
```



c)

```
for k = 1:N
    E(k)= u(k).^2;
end;
plot(E,'o');
title('E vs N');

```



Problem A10.11

$$\text{minimize} \quad \sum_{i=1}^{50} (p(t_i) - y_i)^2 + \sum_{i=51}^{100} (q(t_i) - y_i)^2 \qquad p(0) = q(0), \quad p'(0) = q'(0)$$

$$p(x) = \theta_1 + \theta_2 x + \theta_3 x^2 + \theta_4 x^3 \qquad q(x) = \theta_5 + \theta_6 x + \theta_7 x^2 + \theta_8 x^3$$

$$x = \theta_1 \cdots \theta_8$$

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & & & & \\ \vdots & & & & & 0 & & \\ 1 & t_m & t_m^2 & t_m^3 & & & & \\ & & & & 1 & t_{m+1} & t_{m+1}^2 & t_{m+1}^3 \\ & 0 & & & \vdots & & & \\ & & & & 1 & t_{2m} & t_{2m}^2 & t_{2m}^3 \end{bmatrix} \qquad b = \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_{2m} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & \hat{t} & \hat{t}^2 & \hat{t}^3 & -1 & -\hat{t} & -\hat{t}^2 & -\hat{t}^3 \\ 0 & 1 & 2\hat{t} & 3\hat{t}^2 & 0 & -1 & 2\hat{t} & -3\hat{t} \end{bmatrix}$$

$$d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

```matlab
%Homework 6, Problem 4

clear all;
splinefit;
m = 50;
n = 4;
y1 = y(1:m);
y2 = y(m+1:2*m);
t1 = t(1:m);
t2 = t(m+1:2*m);
A1 = fliplr(vander(t1));
A1 = A1(:, 1:n);
A2 = fliplr(vander(t2));
A2 = A2(:, 1:n);

A = [ A1, zeros(m,n); zeros(m,n) , A2];
b = [y1; y2];
c = [1, zeros(1, n-1), -1, zeros(1, n-1);
    0, 1, zeros(1, n-2), 0, -1, zeros(1,n-2)];
d = zeros(2,1);
x = [A'*A, c'; c, zeros(2,2)] \ [A'*b; d];
pyC1 = x(1:n);
pyC2 = x(n+1:2*n);

disp(pyC1);disp(pyC2);
```

```
>> A10_11
    0.2841
   -0.9025
   -1.2503
   -0.1317
```

PyC1

```
    0.2841
   -0.9025
   -1.8160
    2.5004
```

PyC2