Angel Jimenez — 604 667 905 — EE 133A — HW #1 — Dis 1B

1.17 *Linear combinations of cash flows.* We consider cash flow vectors over T time periods, with a positive entry meaning a payment received, and negative meaning a payment made. A (unit) *single period loan*, at time period t, is the T-vector l_t that corresponds to a payment received of \$1 in period t and a payment made of $\$(1+r)$ in period $t+1$, with all other payments zero. Here $r>0$ is the interest rate (over one period).

Let c be a \$1 $T-1$ period loan, starting at period 1. This means that \$1 is received in period 1, $\$(1+r)^{T-1}$ is paid in period T, and all other payments (*i.e.*, $c_2, \ldots, c_{T-1}$) are zero. Express c as a linear combination of single period loans.

"+" payment received
"-" payment made

Unit: Single period loan @ t is l_t

$$l_t = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1+r \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow t \\ \leftarrow t+1 \\ \\ \\ \end{matrix}$$

$r>0$ is interest rate over on period

$t = 1 \rightarrow T-1$

$$c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -(1+r)^{T-1} \end{pmatrix} \begin{matrix} \leftarrow T-1 \\ \\ \\ \\ \leftarrow T \end{matrix}$$

$$\therefore c = c_1 l_{t_1} + c_2 l_{t_2} + c_3 l_{t_3} + \cdots + c_{T-1} l_{T-1}$$

$$c_1 = c @ T=1 \quad \therefore c_1 = \begin{pmatrix} 1 \\ -(1+r)^{1-1} = -1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \leftarrow T-1=0 \\ \leftarrow T=1 \\ \\ \\ \end{matrix}$$

$$c_2 = \begin{pmatrix} 0 \\ 1 \\ -(1+r)^{2-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \leftarrow T-1=1 \\ \leftarrow T=2 \\ \\ \\ \\ \end{matrix} \qquad c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -(1+r)^{3-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow T-1=2 \\ \leftarrow T=3 \\ \\ \\ \\ \end{matrix}$$

$$\overset{c}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -(1+r)^{T-1} \end{pmatrix}} = (1)\begin{pmatrix} 1 \\ -(1+r) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (1+r)\begin{pmatrix} 0 \\ 1 \\ -(1+r) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (1+r)^2\begin{pmatrix} 0 \\ 0 \\ 1 \\ -(1+r) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (1+r)^{T-2}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -(1+r) \end{pmatrix}$$

Q: techically positive?

$$c = c_1 l_{t_1} + c_2 l_{t_2} + \cdots + c_{T-1} l_{T-1}$$

$$= (1) l_{t_1} + (1+r) l_{t_2} + \cdots + (1+r)^{T-1-1} l_{T-1}$$

Q: how do I expand on $c = c_1 l_{t=1} + c_2 l_{t=2} + \cdots + c_3 l_{T-1}$
I think $c_1 = 1$ because it say it's always one in the problem statement.

but what is c_2? is it $(1+r)^{2-1}$? isn't techically $l_t(c_{T-1} + c_T)$?

2.1 *Linear or not?* Determine whether each of the following scalar-valued functions of n-vectors is linear. If it is a linear function, give its inner product representation, *i.e.*, an n-vector a for which $f(x) = a^T x$ for all x. If it is not linear, give specific x, y, α, and β for which superposition fails, *i.e.*,

$$f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y).$$

(a) The spread of values of the vector, defined as $f(x) = \max_k x_k - \min_k x_k$.

(b) The difference of the last element and the first, $f(x) = x_n - x_1$.

(c) The median of an n-vector, where we will assume $n = 2k + 1$ is odd. The median of the vector x is defined as the $(k+1)$st largest number among the entries of x. For example, the median of $(-7.1, 3.2, -1.5)$ is -1.5.

(d) The average of the entries with odd indices, minus the average of the entries with even indices. You can assume that $n = 2k$ is even.

(e) Vector extrapolation, defined as $x_n + (x_n - x_{n-1})$, for $n \geq 2$. (This is a simple prediction of what x_{n+1} would be, based on a straight line drawn through x_n and x_{n-1}.)

a) $f(x) = \max_k x_k - \min_k x_k$, let $\alpha = \beta = 1$

Suppose $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\therefore f(x) = 2 - 1 = 1$

Suppose $y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\therefore f(y) = 2 - 1 = 1$

$x + y = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\therefore f(x+y) = 3 - 3 = 0$

$f(x) + f(y) = 1 + 1 = 2$

$\therefore f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$

not linear.

b) $f(x) = x_n - x_1 \Rightarrow -x_1 + x_n = a^T x = a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n$

linear if $a = (-1, 0, \cdots, 0, 1)$ $\therefore$ Linear

c) Suppose $\alpha = \beta = 1$ $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\therefore f(x) = 0$, $f(y) = 0$ $x + y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $f(x+y) = 1$

not linear

d) $(\frac{1}{n} \mathbb{1})^T a = \dfrac{a_1 + a_2 + \dots + a_n}{n}$

if vector is even $n = 2k$ $k_{even} = k_{odd}$

$\therefore \dfrac{a_1 + a_3 + \dots a_{k-1}}{k} - \dfrac{a_2 + a_4 + \dots + a_k}{k}$ $\therefore a = (1, -1, 1, -1, \cdots, -1)\frac{1}{k}$

if vector is odd $n = 2k + 1$ $\therefore$ k even, k+1 odd

$\dfrac{a_1 + a_3 + \dots a_{k+1}}{k+1} - \dfrac{a_2 + a_4 + \dots a_k}{k}$

$\therefore$ Since $n = 2k$ is even

Linear

e) $x_n + (x_n - x_{n-1}) \Rightarrow 2x_n - x_{n-1}$

$\Leftrightarrow -x_{n-1} + 2x_n$

$\therefore f(x) = a^T x = (0, \cdots, 0, -1, 2)x$

$\therefore$ Linear

3.25 *Leveraging.* Consider an asset with return time series over T periods given by the T-vector r. This asset has mean return μ and risk σ, which we assume is positive. We also consider cash as an asset, with return vector $\mu^{\mathrm{rf}}\mathbf{1}$, where μ^{rf} is the cash interest rate per period. Thus, we model cash as an asset with return μ^{rf} and zero risk. (The superscript in μ^{rf} stands for 'risk-free'.) We will create a simple portfolio consisting of the asset and cash. If we invest a fraction θ in the asset, and $1-\theta$ in cash, our portfolio return is given by the time series

$$p = \theta r + (1-\theta)\mu^{\mathrm{rf}}\mathbf{1}.$$

We interpret θ as the fraction of our portfolio we hold in the asset. We allow the choices $\theta > 1$, or $\theta < 0$. In the first case we are *borrowing* cash and using the proceeds to buy more of the asset, which is called *leveraging*. In the second case we are *shorting* the asset. When θ is between 0 and 1 we are blending our investment in the asset and cash, which is a form of *hedging*.

(a) Derive a formula for the return and risk of the portfolio, *i.e.*, the mean and standard deviation of p. These should be expressed in terms of μ, σ, μ^{rf}, and θ. Check your formulas for the special cases $\theta = 0$ and $\theta = 1$.

(b) Explain how to choose θ so the portfolio has a given target risk level σ^{tar} (which is positive). If there are multiple values of θ that give the target risk, choose the one that results in the highest portfolio return.

(c) Assume we choose the value of θ as in part (b). When do we use leverage? When do we short the asset? When do we hedge? Your answers should be in English.

return vector $\mu^{rf}\mathbb{1}$, r

$\theta > 1$ borrowing "leveraging"

$\theta < 1$ "shorting"

$0 < \theta < 1$ hedging

$p = \theta r + (1-\theta)\mu^{rf}\mathbb{1}$

θ & $(1-\theta)$ are constants

μ^{rf} is a rate $\therefore$ constant

a) $\operatorname{avg}(p) = \operatorname{avg}(\theta r + (1-\theta)\mu^{rf}\mathbb{1}) = \operatorname{avg}(\theta r) + \operatorname{avg}((1-\theta)\mu^{rf}\mathbb{1})$

$\Rightarrow \theta \operatorname{avg}(r) + (1-\theta)\mu^{rf}\operatorname{avg}(\mathbb{1})$

$\mu = \operatorname{avg}(r)$ (arbitrary) $\quad \operatorname{avg}(\mathbb{1}) = 1$

$\therefore \operatorname{avg}(p) = \theta\mu + (1-\theta)\mu^{rf}$

$\operatorname{std}(p) = \operatorname{rms}(p - \operatorname{avg}(p)\mathbb{1}) = \dfrac{\|p - (\overbrace{(\mathbb{1}^T p)/n}^{\text{avg}})\mathbb{1}\|}{\sqrt{n}}$

$\Rightarrow \dfrac{\|\theta r + (1-\theta)\mu^{rf}\mathbb{1} - (\theta\mu + (1-\theta)\mu^{rf})\mathbb{1}\|}{\sqrt{T}}$ ← over a period T

$\Rightarrow \dfrac{\|\theta r + \cancel{(1-\theta)\mu^{rf}\mathbb{1}} - \theta\mu\mathbb{1} - \cancel{(1-\theta)\mu^{rf}\mathbb{1}}\|}{\sqrt{T}} = \dfrac{\|\theta(r-\mu\mathbb{1})\|}{\sqrt{T}} \Rightarrow |\theta| \overbrace{\dfrac{\|r-\mu\mathbb{1}\|}{\sqrt{T}}}^{\operatorname{std}(r) = \sigma} = |\theta|\sigma$

b) $|\theta| = \sigma^{tar}/\sigma \Rightarrow \theta = \pm \dfrac{\sigma^{tar}}{\sigma}$

to maximize return... $\theta\mu + (1-\theta)\mu^{rf} \Rightarrow \theta\mu + \mu^{rf} - \theta\mu^{rf} \Rightarrow \mu^{rf} + \theta(\mu - \mu^{rf})$

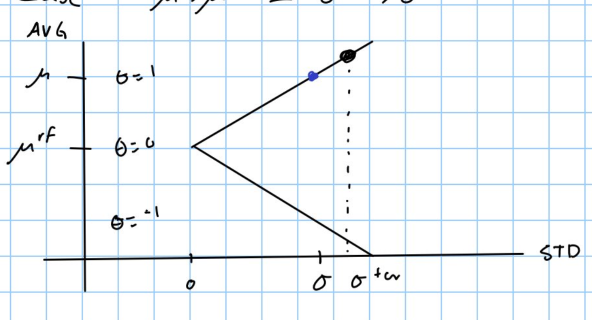
$\theta > 0$ if $\mu > \mu^{rf}$ (leveraging)

$\theta < 0$ if $\mu < \mu^{rf}$ (shorting)

c) risk, return graphs from book to visualize, $(|\theta|\sigma, \mu^{rf} + \theta(\mu - \mu^{rf}))$

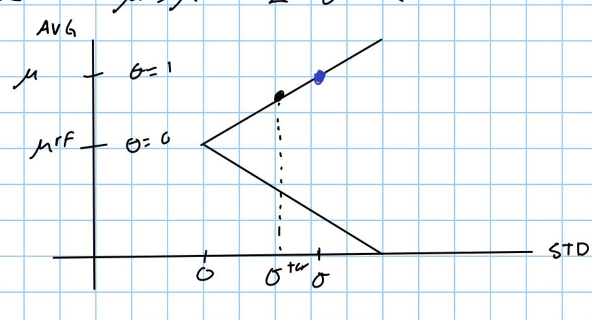
"leverage" $\theta > 1$

Case 1: $\mu > \mu^{rf}$ & $\sigma^{tar} > \sigma$



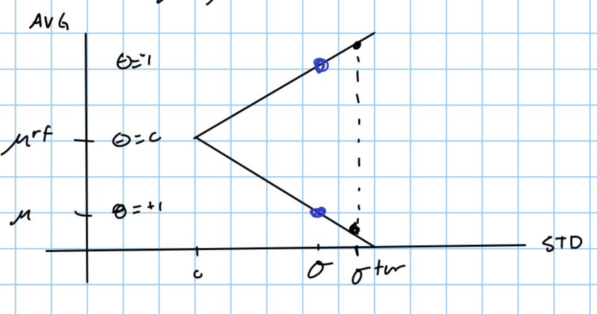
"Hedge" $0 < \theta < 1$

Case 2: $\mu > \mu^{rf}$ & $\sigma^{tar} < \sigma$



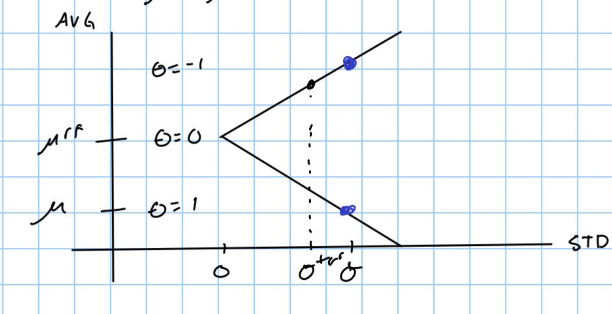
"Short" $\theta < 0$

Case 3: $\mu < \mu^{rf}$ & $\sigma^{tar} > \sigma$



"Short" $\theta < 0$

Case 4: $\mu < \mu^{rf}$ & $\sigma^{tar} < \sigma$



3.27 *Another measure of the spread of the entries of a vector.* The standard deviation is a measure of how much the entries of a vector differ from their mean value. Another measure of how much the entries of an n-vector x differ from each other, called the *mean square difference*, is defined as

$$\mathrm{MSD}(x) = \frac{1}{n^2}\sum_{i,j=1}^{n}(x_i - x_j)^2.$$

(The sum means that you should add up the n^2 terms, as the indices i and j each range from 1 to n.) Show that $\mathrm{MSD}(x) = 2\,\mathbf{std}(x)^2$. *Hint.* First observe that $\mathrm{MSD}(\tilde{x}) = \mathrm{MSD}(x)$, where $\tilde{x} = x - \mathbf{avg}(x)\mathbf{1}$ is the de-meaned vector. Expand the sum and recall that $\sum_{i=1}^{n}\tilde{x}_i = 0$.

Show $\mathrm{MSD}(x) = 2\,\mathrm{std}(x)^2$

$\mathrm{MSD}(\tilde{x}) = \mathrm{MSD}(x)$

$\tilde{x} = x - \mathrm{avg}(x)\mathbb{1}$

$\sum_{i=0}^{n}\tilde{x}_i = 0$

$$\mathrm{MSD}(x) = \frac{1}{n^2}\sum_{i,j=1}^{n}(x_i - x_j)^2 = \mathrm{MSD}(\tilde{x})\ \frac{1}{n^2}\sum_{i,j=1}^{n}(\tilde{x}_i - \tilde{x}_j)^2$$

$$\hookrightarrow \frac{1}{n^2}\sum_{i=1}\sum_{j=1}(x_i^2 - 2\tilde{x}_i x_j + x_j^2)$$

$$\hookrightarrow \frac{1}{n^2}\left[\underbrace{\sum_{i=1}\sum_{j=1}(\tilde{x}_i^2)}_{n\|\tilde{x}\|^2} - \underbrace{2\sum_{i=1}\sum_{j=1}(\tilde{x}_i\tilde{x}_j)}_{-2\times 0\times 0=0} + \underbrace{\sum_{i=1}\sum_{j=1}(\tilde{x}_j^2)}_{n\|\tilde{x}\|^2}\right]$$

$$\hookrightarrow \frac{1}{n^2}(2n\|\tilde{x}\|^2) \Rightarrow \frac{2\|\tilde{x}\|^2}{n} = 2\frac{\|\tilde{x}\|}{\sqrt{n}}\frac{\|\tilde{x}\|}{\sqrt{n}} = 2\,\mathrm{std}(x)^2$$

1.2 Use the Cauchy-Schwarz inequality to prove that

$$\frac{1}{n}\sum_{k=1}^{n}x_k \geq \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)^{-1}$$

for all n-vectors x with positive elements x_k. The left-hand side of the inequality is the arithmetic mean (average) of the numbers x_k; the right-hand side is called the harmonic mean.

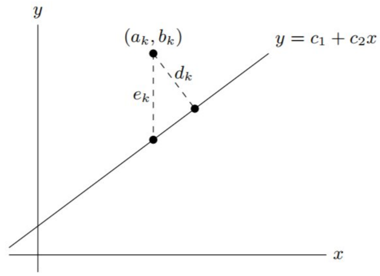
rearrange the inequality $\frac{1}{n}\sum_{k=1}^{n}x_k \geq \frac{1}{\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)}$

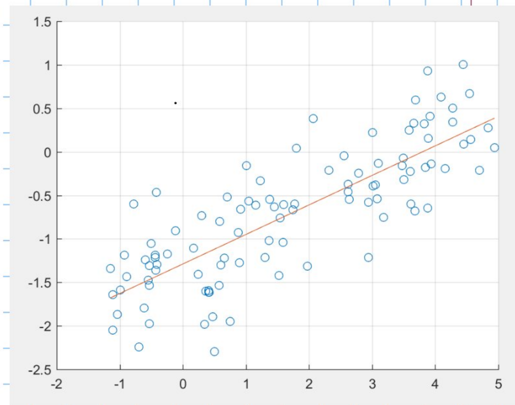
$$\Rightarrow \sum_{k=1}^{n}x_k\sum_{k=1}^{n}\frac{1}{x_k} \geq n^2 \Rightarrow \left[\sum_{k=1}^{n}x_k\sum_{k=1}^{n}\frac{1}{x_k}\right]^{1/2} \geq n$$

recall Cauchy-Schwarz inequality is $a^Tb \leq \|a\|\|b\|$

$$\|a\| = \sqrt{\sum_{k=1}^{n}x_k} \quad \|b\| = \sqrt{\sum_{k=1}^{n}\frac{1}{x_k}}$$

1.8 *Orthogonal distance regression.* We use the same notation as in exercise 1.7: a, b are non-constant n-vectors, with means m_a, m_b, standard deviations s_a, s_b, and correlation coefficient ρ.





For each point (a_k, b_k), the vertical deviation from the straight line defined by $y = c_1 + c_2 x$ is given by

$$e_k = |c_1 + c_2 a_k - b_k|.$$

The least squares regression method of the lecture minimizes the sum $\sum_k e_k^2$ of the squared vertical deviations. The orthogonal (shortest) distance of (a_k, b_k) to the line is

$$d_k = \frac{|c_1 + c_2 a_k - b_k|}{\sqrt{1 + c_2^2}}.$$

As an alternative to the least squares method, we can find the straight line that minimizes the sum of the squared orthogonal distances $\sum_k d_k^2$. Define

$$J = \frac{1}{n} \sum_{k=1}^{n} d_k^2 = \frac{\|c_1 \mathbf{1} + c_2 a - b\|^2}{n(1 + c_2^2)}.$$

(a) Show that the optimal value of c_1 is $c_1 = m_b - m_a c_2$, as for the least squares fit.

(b) If we substitute $c_1 = m_b - m_a c_2$ in the expression for J, we obtain

$$J = \frac{\|c_2(a - m_a \mathbf{1}) - (b - m_b \mathbf{1})\|^2}{n(1 + c_2^2)}.$$

Simplify this expression and show that it is equal to

$$J = \frac{s_a^2 c_2^2 + s_b^2 - 2\rho s_a s_b c_2}{1 + c_2^2}.$$

Set the derivative of J with respect to c_2 to zero, to derive a quadratic equation for c_2:

$$\rho c_2^2 + \left(\frac{s_a}{s_b} - \frac{s_b}{s_a}\right) c_2 - \rho = 0.$$

If $\rho = 0$ and $s_a = s_b$, any value of c_2 is optimal. If $\rho = 0$ and $s_a \neq s_b$ the quadratic equation has a unique solution $c_2 = 0$. If $\rho \neq 0$, the quadratic equation has a positive and a negative root. Show that the solution that minimizes J is the root c_2 with the same sign as ρ.

(c) Download the file `orthregdata.m` and execute it in MATLAB to create two arrays `a`, `b` of length 100. Fit a straight line to the data points (a_k, b_k) using orthogonal distance regression and compare with the least squares solution. Make a MATLAB plot of the two lines and the data points.

a) $J = \dfrac{\| c_1 \mathbb{1} + c_2 a - b \|}{n(1 + c_2^2)} = \dfrac{(c_1 \mathbb{1} + c_2 a - b)^T (c_1 \mathbb{1} + c_2 a - b)}{n(1 + c_2^2)} \leq \dfrac{n c_1^2 + 2(c_1 \mathbb{1})^T (c_2 a - b) + (c_2 a - b)^T (c_2 a - b)}{n(1 + c_2^2)}$

$\hookrightarrow d/dc_1 \Rightarrow \dfrac{2n(c_1 + c_2 m_a - m_b)}{n(1 + c_2^2)} = 0 \Rightarrow c_1 + c_2 m_a - m_b = 0$

$\therefore c_1 = m_b - m_a c_2$

b) $\| c_2 (a - m_a \mathbb{1}) - (b - m_b \mathbb{1}) \|^2 = c_2^2 \| a - m_a \mathbb{1} \| + \| b - m_b \mathbb{1} \| - 2 c_2 (a - m_a \mathbb{1})^T (b - m_b \mathbb{1}) =$

$n(c_2^2 s_a^2 + s_b^2 - 2 \rho s_a s_b)$

take derivative w/r to c_2 and set to zero

$\rho c_2^2 + \left(\frac{s_a}{s_b} - \frac{s_a}{s_b}\right) c_2 - \rho = 0$

$c_2 = 0$ & $\begin{cases} \rho > 0 & J' = \text{"}-\text{"} \\ \rho < 0 & J' = \text{"}+\text{"} \end{cases}$ ← minimum