

# Assignment 2

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Here we consider the probability space  $(\Omega, \mathcal{F}, P)$ , where  $\mathcal{F}$  is the  $\sigma$ -field,  $\Omega$  is the sample space and  $P$  is the probability measure.

We will prove the strong law of large numbers. We state the theorem first

**Theorem 1.** *If  $\{X_i\}_{i \in \mathbb{N}}$  is an IID sequence and  $\mathbb{E}[|X_1|] < \infty$ . Then*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \mathbb{E}[X_1] \text{ a.s.,}$$

where  $S_n = \sum_{i=1}^n X_i$ .

We will use the following fact in this proof.

**Lemma 1.** *Given  $p > 0$  and  $X$  a non-negative r.v., i.e.,  $X \geq 0$ , we have*

$$\mathbb{E}[X^p] = \int_0^\infty px^{p-1}P(X > x)dx. \quad (1)$$

**Proof of Theorem 1:**

*Proof.* We may assume  $\mathbb{E}[X_i] = 0$ . Otherwise, one can replace  $X_i$  with  $X_i - \mathbb{E}[X_i]$ . This implies that we have to prove that  $\frac{1}{n} \sum_{i=1}^n X_i$  converges to 0.

Let

$$Y_n = X_n I_{\{|X_n| \leq n\}} \text{ and } Z_n = Y_n - \mathbb{E}[Y_n]. \quad (2)$$

We will break up the proof into parts.

**Part 1:** We first prove  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0$ .

Define

$$M_n = \sum_{i=1}^n \frac{Z_i}{i} \quad (3)$$

with  $M_0 = 0$ . Define the filtration  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  where  $\mathcal{F}_n := \sigma(X_1, X_2 \dots X_n)$ .

**Q1:** Prove that  $\{M_n\}$  is a martingale w.r.t. the filtration  $\mathcal{F}_n$ .

Now we will show that  $\mathbb{E}[M_n^2]$  is bounded by a constant not dependent on  $n$ . For that, consider

$$\begin{aligned} \mathbb{E}[M_n^2] &= \text{Var}(M_n) = \sum_{i=1}^n \frac{\text{Var}(Z_i)}{i^2} = \sum_{i=1}^n \frac{1}{i^2} \text{Var}(Y_i) \\ &\leq \sum_{i=1}^n \frac{1}{i^2} \mathbb{E}[Y_i^2] \leq \sum_{i=1}^n \frac{1}{i^2} \int_0^i 2yP(|X_i| \geq y)dy \quad (\text{from Lemma 1}) \\ &= 2 \sum_{i=1}^n \frac{1}{i^2} \int_0^\infty I_{\{y \leq i\}} y P(|X_i| \geq y) dy \\ &\quad \dots \\ &\quad \dots \\ &\leq c \mathbb{E}[|X_1|], \quad \text{where } c > 0. \end{aligned} \tag{4}$$

**Q2:** Fill in the missing details (First use the fact that  $\{X_i\}$  is IID which implies that  $X_i$  and  $X_1$  have the same distribution. Second use Monotone Convergence Theorem (MCT) to take the summation inside the integral (Note that  $\int dy$  is the expectation w.r.t. the Lebesgue measure. So MCT can be applied). Then use the fact that  $\sum_{i=1}^\infty \frac{1}{i^2} I_{\{y \leq i\}} \leq \frac{1}{y}$ ).

**Q3:** Claim that  $\mathbb{E}[M_n^2]$  is bounded by a constant not dependent on  $n$  using Eq. (4).

**Q4:** Further claim that the martingale  $\{M_n\}$  converges.

Now note that from Eqs. (2) and (3), we have

$$Z_i = i(M_i - M_{i-1}). \tag{5}$$

This implies that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n Z_i &= \frac{1}{n} \sum_{i=1}^n i(M_i - M_{i-1}) = \frac{1}{n} \left( \sum_{i=1}^n i M_i - \sum_{i=1}^{n-1} (i+1) M_i \right) \\ &= M_n - \frac{n-1}{n} \left( \frac{1}{n-1} \sum_{i=1}^{n-1} M_i \right) \end{aligned} \tag{6}$$

**Q5:** Now finally show that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0$ .

**Part 2:** We next show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Y_i = 0. \tag{7}$$

**Q6:** Using dominated convergence theorem and the fact that  $X_n$  are identically distributed show that

$$\lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = \mathbb{E}[X_1] = 0. \quad (8)$$

This implies that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i] = 0$ . Now observe that  $Y_i = Z_i + \mathbb{E}[Y_i]$ .

**Q7:** Finally, show (7).

**Part 3:** We next show that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n Y_i \right| = 0 \text{ a.s.} \quad (9)$$

First show that

$$\begin{aligned} \sum_{i=1}^{\infty} P(|X_i| \geq i) &= \sum_{i=1}^{\infty} \int_{i-1}^i P(|X_i| \geq i) dx \\ &\dots \\ &\dots \\ &= \mathbb{E}[|X_1|] < \infty. \end{aligned} \quad (10)$$

**Q8:** Fill the missing part in the above proof. Hint: Use Lemma 1.

(It is important to note that in Eq. (10) we are bounding the summation  $\sum_{i=1}^{\infty} P(|X_i| \geq i)$  by the expectation of the modulus of  $X_1$ , the first r.v. of the sequence.)

**Q9:** Now using Eq. (10) show that

$$\sum_{i=1}^{\infty} P(X_i \neq Y_i) < \infty. \quad (11)$$

Hint: Rewrite the above summation using the summation in Eq. (10).

Therefore by the Borel–Cantelli lemma, except for a set of probability zero,  $X_i = Y_i$ , for all  $i$  greater than some positive integer  $N$  ( $N$  depends on  $\omega$ ). Henceforth,

**Q10:** Using Eq. (7) and the above claim (Borel-Cantelli claim), show (9).

**Q11:** Finally, conclude that the proof is complete.  $\square$