Real numbers

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1 Field and Order Axioms

1.1 Field Axioms

$(\mathbb{R}, +, \cdot)$ with operations $+(addition)$, $\cdot(multiplication) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a field. *i.e.*,

(F1) Associativity of operations $+$ and $\cdot$, *i.e.*, for $x, y, z \in \mathbb{R}$, we have $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(F2) Commutativity of operations $+$ and $\cdot$, *i.e.*, for $x, y \in \mathbb{R}$, we have $x + y = y + x$ and $x \cdot y = y \cdot x$.

(F3) Existence of unique multiplicative and additive identities, *i.e.*, $\exists! 0, 1 \in F$ s.t. $x + 0 = 0 + x = x$ and $x \cdot 1 = 1 \cdot x = x$, $\forall x \in \mathbb{R}$.

(F4) Existence of additive inverse, *i.e.*, for every $x \in \mathbb{R}$, $\exists! -x \in \mathbb{R}$ with $x + (-x) = (-x) + x = 0$.

(F5) Existence of multiplicative inverse, *i.e.*, for every $x \in \mathbb{R}$, $\exists! x^{-1} \in \mathbb{R}$ with $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

(F6) Distributivity of $\cdot$ over $+$, *i.e.*, for $x, y, z \in \mathbb{R}$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$.

fi $\exists!$ means 'there exists a unique'.

Remark 1. *A field is an integral domain. Therefore, for $x, y \in \mathbb{R}$, if $x \cdot y = 0$, then $x = 0$ or $y = 0$.*

Also $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$. Proof: $(x \cdot y) \cdot (x^{-1} \cdot y^{-1}) = (y \cdot x) \cdot (x^{-1} \cdot y^{-1})$ (from F2) $= y \cdot (x \cdot x^{-1}) \cdot y^{-1}$ (from repeated application of (F1)) $= y \cdot y^{-1}$ (from (F5)) $= 1$. Therefore $x^{-1} \cdot y^{-1}$ is the multiplicative inverse of $(x \cdot y)$.

1.2 Order Axioms

There exists a non-empty subset $P \subset \mathbb{R}$ which satisfies the following:

(O1) $x, y \in P \Rightarrow x + y, x \cdot y \in P$.

(O2) For $x \in \mathbb{R}$, only of the following is true: either $x = 0$ or $x \in P$ or $-x \in P$.

The set P is called the set of positive numbers of $\mathbb{R}$ and $-P = \{-x|x \in P\}$ is called the set of negative numbers. Therefore, $\mathbb{R} = P \uplus -P \uplus \{0\}$ (where $\uplus$ is the disjoint union).

Definition: (order " $<$ ") We define the order relation ' $<'$ as follows: for $x, y \in \mathbb{R}$, $x < y$, iff $y - x \in P$. Similarly, we define " $>$ " as $x > y$ iff $x - y \in P$. Further, $x \geq y$ implies $x > y$ or $x = y$. Similarly, we define " $\leq$ ".

Also, for $x, y \in \mathbb{R}$, from (O1) either $x - y \in P$ or $y - x \in P$ or $x - y = 0$. This implies that either $x > y$ or $y > x$ or $x = y$.

Thus, R **is an ordered field**.

And for $x \in P$, we have $x - 0 = x \in P$ and therefore, $x > 0$. Conversely, if $x > 0$, then $x - 0 \in P$ (by definition of ' $>'$) which implies that $x \in P$. This implies that the set P is equivalent of $\{x \in \mathbb{R}|x > 0\}$. Similarly, $-P = \{x \in \mathbb{R}|x < 0\}$.

Theorem 1. *For $x, y, z \in \mathbb{R}$, we have*

1. $-(-x) = x$.
2. $-x = (-1) \cdot x$.
3. $(-x) \cdot y = x \cdot (-y) \; = -(xy)$.
4. $(-x) \cdot (-y) = x \cdot$.

Proof.

1. From F4, we have $(-x) + (-(-x)) = (-(-x)) + (-x) = 0$. Also, $x + (-x) = (-x) + x = 0$. From F4, additive inverse is unique which implies that $x = -(-x)$.
2. $0 = 0 \cdot x = (-1 + 1) \cdot x = (-1) \cdot x + x$. Therefore from (F4), we get $-x = (-1) \cdot x$.
3. $(-x) \cdot y = ((-1) \cdot x) \cdot y = (x \cdot (-1)) \cdot y$(from F2) $= x \cdot ((-1) \cdot y)$ (from F1) $= x \cdot (-y)$. Also, $(-x) \cdot y = ((-1) \cdot x) \cdot y = (-1)(xy) = -(xy)$.
4. From the repeated application of (3), we get $(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = xy$ (from (1)).

□

Corollary 1. $(-1)^{-1} = -1$.

Proof. From Theorem 1 (1-2), we get $1 = -(-1) = (-1) \cdot (-1)$. Therefore, $(-1)^{-1} = -1$. $\square$

Theorem 2. $x \neq 0 \Leftrightarrow x \cdot x > 0$.

Proof. ($\Rightarrow$) If $x < 0$, then $-x \in P$. Then from (O1), we get $(-x) \cdot (-x) \in P \Rightarrow (-1) \cdot (-1) x \cdot x \in P \Rightarrow x \cdot x \in P$. If $x > 0$, then from (O1), we get $x \cdot x \in P$.
($\Leftarrow$) We just need to show that if $x = 0$, then $0 \cdot 0 = 0$. This is trivial. $\square$

Theorem 3. *For $x, y, z \in \mathbb{R}$, we have*

1. $x < y \Leftrightarrow x + z < y + z$.
2. *If $x < y$ and $y < z$, then $x < z$.*
3. *If $x < y$ and $z > 0$, then $xz < yz$.*
4. *If $x < y$ and $z < 0$, then $yz < xz$.*
5. *If $x < y$ and $w < z$, then $x + w < y + z$.*
6. $x < y \Leftrightarrow -y < -x$.
7. *If $x > 0, y > 0, z > 0, w > 0$ and $x < y$ and $w < z$, then $wx < yz$.*
8. *If $x \in P$ and $y > x$, then $y \in P$.*

Proof.

1. $x < y \Rightarrow y - x \in P \Rightarrow y - x + z - z \in P \Rightarrow (y + z) - (x + z) \in P \Rightarrow x + z < y + z$.
2. If $x < y$ and $y < z$, then $y - x \in P$ and $z - y \in P$. Further, by (O1), we get $y - x + z - y \in P \Rightarrow z - x \in P \Rightarrow x < z$.
3. If $x < y$ and $z > 0$, then $y - x, z \in P$. Therefore, by (O1), we have $(y - x) \cdot z \in P \Rightarrow yz - xz \in P \Rightarrow xz < yz$.
4. If $x < y$ and $z < 0$, then $y - x, -z \in P$. Therefore, by (O1), we have $(y - x) \cdot (-z) \in P \Rightarrow y \cdot (-z) - x \cdot (-z) \in P \Rightarrow -yz + xz \in P \Rightarrow yz < xz$.
5. $x < y \Leftrightarrow y - x \in P$. Also, $w < z \Leftrightarrow z - w \in P$. Now, from (O1), we get $(y - x) + (z - w) \in P \Leftrightarrow (y + z) - (x + w) \in P \Leftrightarrow x + w < y + z$.
6. We know that $x < y \Leftrightarrow y - x \in P$. Now consider $-x - (-y) = y - x \in P$. Therefore, $-y < -x$.
7. Proof left as exercise.
8. Since $y > x$, we have $y - x \in P$. Also since $x \in P$, we have from $(y - x) + x \in P$ (from (O1)) $\Rightarrow y \in P$.

$\square$

Exercise 1.

1. *Prove the following: if $x > 0, y > 0$ and $x < y$, then $x^{-1} > y^{-1}$.*

Theorem 4. *For $x, y, z \in \mathbb{R}$, if $xy = z$ and either two of x, y, z is positive, then the other is also positive.*

Proof. There are 3 cases to consider.
Case 1: If x, y are positive. Then by (O1), we have $xy \in P$ and therefore $z \in P$.
Case 2: If $x, z \in P$. We will prove that $y \in P$ by proof by contradiction. For that we assume the contrary, *i.e.*, $y \notin P \Rightarrow y \leq 0$. Therefore, from (O1) and Lemma 1(3), we have $0 \leq x \cdot (-y) = -(x \cdot y) = -z < 0$. This is a contradiction. Therefore, $y \in P$.
Case 3: If $y, z \in P$. This is similar to Case 2. $\square$

Corollary 2. *$1 \in P$.*

Proof. Since P is non-empty, there exists an $x \in P$. Now from (F3), we have $x \cdot 1 = x$. Since $x \in P$, we get $1 \in P$ (follows from Theorem 4). $\square$

Corollary 3. *For $x \in \mathbb{R}$, if $x \in P$, then $x^{-1} \in P$. Similarly, for $x \in -P$, we have $x^{-1} \in -P$.*

Proof. For $x \in P$, from (F5) we get $x \cdot x^{-1} = 1$. Since $1 \in P$ (from Corollary 2), we get $x^{-1} \in P$ (follows from Theorem 4).

Now if $-x \in P$, we get from the above paragraph that $(-x)^{-1} \in P$. Further, from Corollary 1, we get $P \ni (-x)^{-1} = ((-1) \cdot x)^{-1} = (-1)^{-1} \cdot x^{-1} = -(x)^{-1}$. Therefore, $x^{-1} \in -P$. $\square$

2 Subsets of $\mathbb{R}$

Definition: (Inductive set) A set $A \subseteq \mathbb{R}$ is called an *inductive set*, if it satisfies the following properties:

1. $1 \in A$.
2. If $a \in A$, then $a + 1 \in A$.

Example: $\mathbb{R}$ is an inductive set.

2.1 Natural numbers ($\mathbb{N}$)

Definition (Natural numbers $\mathbb{N}$) The set of natural numbers $\mathbb{N}$ ($\subset \mathbb{R}$) is defined as the smallest inductive set contained in $\mathbb{R}$. This implies that if $A \subseteq \mathbb{R}$ is an inductive set, then $\mathbb{N} \subseteq A$.

Remark 2. *$\mathbb{N}$ is nothing but the intersection of all the inductive sets in $\mathbb{R}$.*

Definition: (Smallest element of a set) Given a set $A \subseteq \mathbb{R}$, we say s is the *smallest element* of A, if $s \in A$ and $s \leq a$, $\forall a \in A$.

We assume the following axiom about $\mathbb{N}$.

Well-ordering Axiom: Every non-empty subset of $\mathbb{N}$ has a smallest element.

Theorem 5. *1. 1 is the smallest element of* $\mathbb{N}$.

2. If $n, m \in \mathbb{N}$*, then* $n + m \in \mathbb{N}$ *and* $n \cdot m \in \mathbb{N}$.

3. For every $n \in \mathbb{N} \setminus \{1\}$*, we have* $n - 1 \in \mathbb{N}$.

4. For every $n \in \mathbb{N}$*, we have* $n \in P$.

Proof. 1. By the well-ordering axiom, every subset of $\mathbb{N}$ has a smallest element. Since $\mathbb{N} \subseteq \mathbb{N}$, the set $\mathbb{N}$ itself has a smallest element. Let it be $a \in \mathbb{N}$. Further assume that $a \neq 1$. Then by the definition of the smallest element, we get $a < 1$. This further implies that $a^2 < a$ (follows from Theorem 3(3)). This is a contradiction since a is assumed to be the smallest element of $\mathbb{N}$.

2. Let $n \in \mathbb{N}$. Now consider the set $S := \{n \in \mathbb{N} | n + m \in \mathbb{N}, \forall m \in \mathbb{N}\}$. If we could prove that $S = \mathbb{N}$, then we are done. For that purpose, we will first prove that S is inductive.

   Note that for every $m \in \mathbb{N}$, we have $1 + m = m + 1 \in \mathbb{N}$. This follows since $\mathbb{N}$ is inductive. Therefore $1 \in S$.

   Now let $n \in S$. This implies that $\forall m \in \mathbb{N}$, $n + m \in \mathbb{N}$. Now consider the case for $n + 1$. In this case, we have, for any $m \in \mathbb{N}$,

$$(n + 1) + m = \underbrace{n + \underbrace{(m + 1)}_{\in \mathbb{N} \text{ since } m \in \mathbb{N}}}_{\in \mathbb{N} \text{ since } n \in S}$$

   Therefore S is an inductive set. And by the definition of $\mathbb{N}$, we know that $\mathbb{N}$ is the smallest inductive set. Therefore $\mathbb{N} \subseteq S$. From the definition of S we have $S \subseteq \mathbb{N}$. Thus $S = \mathbb{N}$.

   For the case for $\cdot$, one can use similar proof.

3. Define $S := \{n \in \mathbb{N} | n - 1 \in \mathbb{N} \cup \{0\}\}$. Note that $1 - 1 = 0 \in \mathbb{N} \cup \{0\}$. Therefore, $1 \in S$. Now assume that $n \in S$. Then if we could show that $n + 1 \in S$, then S is an inductive set which implies that $S = \mathbb{N}$. For that purpose, note that $(n + 1) - 1 = n \in S \subseteq \mathbb{N}$. Therefore, $n + 1 \in S$. Thus S is an inductive set. Therefore $S = \mathbb{N}$. But if $n - 1 = 0 \Rightarrow n = 1$. Therefore, the claim follows.

4. Define $S := \{n \in \mathbb{N} | n \in P\}$. From Corollary 2, we have $1 \in S$. Now assume $m \in S$. Now note that since $(m + 1) - m = 1 \in P$, we have $m + 1 > m$. Now from Theorem 3(8), we have $m + 1 \in P$. Therefore $m + 1 \in S$. Thus S is an inductive set. Therefore $S = \mathbb{N}$. Hence the claim follows.

$\square$

Theorem 6. *There is no* $m \in \mathbb{N}$ *s.t.* $1 < m < 2$.

Proof. Let $S := \{k \in \mathbb{N} | 1 < k < 2\}$. Assume that $S \neq \emptyset$. By the well-ordering axiom there exists a smallest element of S. Let ℓ be the smallest element of S. This implies that $\ell \in S$ which implies that $1 < \ell < 2$. Therefore by Theorem 5(3), we have $\ell - 1 \in \mathbb{N}$. Since ℓ is the smallest element of S, and $\ell - 1 < \ell$, we have $\ell - 1 \notin S$ and $\ell - 1 \ngeq 2$. This implies that $\ell - 1 \leq 1$. However, $\ell - 1 \nless 1$ since 1 is the smallest element of $\mathbb{N}$. The only other possibility is $\ell - 1 = 1 \Rightarrow \ell = 2$ which is a contradiction since $1 < \ell < 2$. Therefore $S = \emptyset$. □

Corollary 4. *For every $n \in \mathbb{N}$, there is no $m \in \mathbb{N}$ s.t. $n < m < n + 1$.*

Proof. Proof left as exercise. □

2.2 Integers ($\mathbb{Z}$)

We define $-\mathbb{N} := \{-n | n \in \mathbb{N}\}$. Now we define the set of integers as follows:

$$\mathbb{Z} := \mathbb{N} \cup \{0\} \cup -\mathbb{N} \tag{1}$$

Theorem 7. *For $a, b \in \mathbb{Z}$, we have $a + b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{Z}$.*

Proof. Proof left as exercise. □

2.3 Metric properties

Definition: The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}_+$ is defined as follows:

$$|x| = \begin{cases} x & if\ x \geq 0, \\ -x & otherwise \end{cases} \tag{2}$$

Theorem 8. *For $xy \in \mathbb{R}$,*

1. $|x| \geq 0.$ *and* $|x| = 0 \Leftrightarrow x = 0.$
2. $|x| = |-x|.$
3. $-|x| \leq x \leq |x|.$
4. $|x| \leq y \Leftrightarrow -y \leq x \leq y.$
5. $|x + y| \leq |x| + |y|.$ *(Triangle Inequality)*
6. $||x| - |y|| \leq |x - y|.$

Proof. 1. Follows from definition.

2. If $x > 0$, then from Eq. (2), we get $|x| = x$. Now, since $-x < 0$, we get $|-x| = -(-x) = x$ (from Theorem 1(1)). Therefore, $|x| = |-x|$. Now, if $x < 0$, then from Eq. (2), we get $|x| = -x$. Since, $-x > 0$, we get $|-x| = -x$. Therefore, $|x| = |-x|$. Finally, for $x = 0$, we get $|x| = |-x|$.

3. If $x < 0$ and $|x| > 0$ (from Theorem 8(1)), then $x < |x|$ (follows from Theorem 3(2)). Also, since $x < 0$, we get $|x| = -x$ (follows from Eq.(2)). Thus, $-|x| = x$. Therefore, $-|x| \leq x \leq |x|$.

   If $x > 0$, then $-x < 0$. Now from the above part, we get $-|-x| \leq -x \leq |-x| \Leftrightarrow -|x| \leq -x \leq |x|$ (from Theorem 8(2)). Therefore, $-|x| \leq x \leq |x|$ (follows from Theorem 3(6)).

   For $x = 0$, the claim follows trivially.

4. From part (3), we have $-|x| \leq x \leq |x|$. From this and $|x| \leq y$ imply (using Theorem 3(2)) that $x \leq y$. Also, $|x| \leq y \Rightarrow -y \leq -|x|$ (follows from Theorem 3(6)) $\Rightarrow -y \leq x$ (follows from Theorem 3(2)).

5. From part (3), we get $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Now from Theorem 3, we get $-(|x| + |y|) \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$ (follows from part (4)).

6. $|x| = |x-y+y| \leq |x-y|+|y|$(by triangle inequality) $\Rightarrow |x|-|y| \leq |x-y|$. Similarly starting from $|y|$, we get $|y| - |x| \leq |x - y|$. Combining these two, we get $-|x - y| \leq |x| - |y| \leq |x - y|$. The result then follows from Case 4.

□

Therefore, $(\mathbb{R}, d)$ with $d(x, y) := |x - y|$ is a **metric space**.

Definition: A *metric space* is an ordered pair (M, d), where M is a set and $d : M \times M \to \mathbb{R}$ is the metric which satisfies the following properties:
For $x, y, z \in M$,

1. $d(x, y) \geq 0$. *(non-negativity)*
2. $d(x, y) = 0 \Leftrightarrow x = y$.
3. $d(x, y) = d(y, x)$. (*symmetry*)
4. $d(x, z) \leq d(x, y) + d(y, z)$. (*triangle inequality*)

2.4 Completeness Axiom

Definition: For a set $A \subseteq \mathbb{R}$, we say $M \in \mathbb{R}$ is an upper bound of A if $x \leq M$, $\forall x \in A$. Similarly, we call $L \in \mathbb{R}$ as a lower bound of A if $x \geq L, \forall x \in A$. We say a set is bounded above if there exists an upper bound (similarly bounded below). A set is called *bounded* if it is both bounded below and bounded above.

A bounded set A satisfies $|x| \leq K, \forall x \in A$, for some $K \geq 0$.

Definition: For a set $A \subseteq \mathbb{R}$ which is bounded above, we define $\alpha \in \mathbb{R}$ as the *supremum (least upper bound)* of A, if α is an upper bound of A and further if M is an upper bound of A, then $\alpha \leq M$. We denote the supremum of A as *supA*.

Definition: For a set $A \subseteq \mathbb{R}$ which is bounded below, we define $\beta \in \mathbb{R}$ as the *infimum (greatest lower bound)* of A, if β is a lower bound of A and further if L is a lower bound of A, then $\beta \geq L$. We denote the infimum of A as $infA$.

Theorem 9. *For a set $A \subseteq \mathbb{R}$ which is bounded above, if $supA$ exists, then it is unique. Similarly, for a set $A \subseteq \mathbb{R}$ which is bounded below, if $infA$ exists then it is unique.*

Proof. Let $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$ be two distinct supremums of A. Then by the definition of supremum we have $\alpha_1 \leq \alpha_2$. Also, $\alpha_2 \leq \alpha_1$. Hence $\alpha_1 = \alpha_2$. Similarly, we can show that the infimum is also unique. □

Theorem 10. *1. For a set $A \subseteq \mathbb{R}$ which is bounded above, let $\alpha := supA$ exists. For any $\epsilon > 0$, there exists an $x \in A$ s.t. $x > \alpha - \epsilon$.*

2. For a set $A \subseteq \mathbb{R}$ which is bounded below, let $\beta := infA$ exists. For any $\epsilon > 0$, there exists an $x \in A$ s.t. $x < \beta + \epsilon$.

Proof. 1. Assume to the contrary, *i.e.*, there does not exist $x \in A$ *s.t.* $x > \alpha - \epsilon \Rightarrow x \leq \alpha - \epsilon$, $\forall x \in A$. Therefore, $\alpha - \epsilon$ is an upper bound of A which further implies that $\alpha = supA \leq \alpha - \epsilon$. This is a contradiction.

2. Proof is similar. Left as exercise.

□

Theorem 11. *Let the bounded sets A and B ($A, B \subseteq \mathbb{R}$) satisfy the following: for every $x \in A$, $\exists y \in B$ s.t. $x \leq y$. Then if $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.*

Proof. $A \ni x \leq y \in B \Rightarrow supA \leq y \leq supB$.

□

A further few properties of sup and inf:

Proposition 1. *If $A \subseteq B \subseteq \mathbb{R}$ and $\sup A$, $\sup B, \inf A$ and $\inf B$ exist, then*

1. $\sup A \geq \inf A$.
2. $\sup A \leq \sup B$.
3. $\inf A \geq \inf B$.
4. $\sup A \geq \inf B$.
5. $\inf -A = -\sup A$. *(where $-A$ is defined as $-A := \{-a | a \in A\}$)*
6. $\sup -A = -\inf A$.

Proof. 1. Trivial. Follows from definition.

2. By defintion, $b \leq \sup B, \forall b \in B \Rightarrow a \leq \sup B, \forall a \in A$ (since $A \subseteq B$). Thus $\sup B$ is a upper bound of A. Therefore $\sup A \leq \sup B$ (since $\sup A$ is the smallest of the upper bounds).

3. By defintion, $b \geq \inf B, \forall b \in B \Rightarrow a \geq \inf B, \forall a \in A$ (since $A \subseteq B$). Thus $\inf B$ is a lower bound of A. Therefore $\inf A \geq \inf B$ (since $\inf A$ is the greatest of the lower bounds).

4. All the rest are easy to prove.

□

We state the completeness axiom here:

Completeness Axiom: Any nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Now we will prove some results using this axiom.

Theorem 12. *$\mathbb{N}$ is not bounded above.*

Proof. Assume to the contrary, *i.e.*, assume $\mathbb{N}$ is bounded above. By the completeness axiom, $sup\mathbb{N}$ exists in $\mathbb{R}$. Let $\alpha := sup\mathbb{N}$. Also, let $0 < \epsilon < 1$. Then by Theorem 10, there exists an $n \in \mathbb{N}$ *s.t.* $n > \alpha - \epsilon$. This implies that $n + 1 > \alpha$. This contradicts the supremum property of α. □

Theorem 13. ***(Archimedean property)*** *For $x \in \mathbb{R}$, there exists an $n_x \in \mathbb{N}$ s.t. $n_x > x$.*

Proof. Assume the contrary. This implies that $n \leq x$, $\forall n \in \mathbb{N}$. This further implies that x is an upper bound of $\mathbb{N}$. Then by the completeness axiom, $sup\mathbb{N}$ exists in $\mathbb{R}$. Let $\alpha := sup\mathbb{N}$. Also, let $0 < \epsilon < 1$. Then by Theorem 10, there exists an $n \in \mathbb{N}$ *s.t.* $n > \alpha - \epsilon$. This implies that $n + 1 > \alpha$. This contradicts the supremum property of α. □

Corollary 5. *For $x, y \in \mathbb{R}$, with $x > 0$,*

1. *there exists an $n \in \mathbb{N}$ s.t. $nx > y$.*

2. *there exists an $n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$. ($\frac{1}{n}$ denotes n^{-1})*

3. *there exists an $n \in \mathbb{N}$ s.t. $n - 1 \leq x < n$.*

Proof.

1. From the Archimedean property (Theorem 13), there exists an $n \in \mathbb{N}$ *s.t.* $n > yx^{-1}$. This implies $nx > y$.

2. For x and 1, apply Archimedean property to obtain an $n \in \mathbb{N}$ *s.t.* $nx > 1$. This implies that $1/n < x$.

3. By applying the Archimedean property on x and 1, we obtain an $n \in \mathbb{N}$ *s.t.* $n > x$. Let $S := \{m \in \mathbb{N} \cup \{0\}$ *s.t.* $m > x\}$. S is non-empty since $n \in S$ and $0 \notin S$ (since $x > 0$). Then by the well-ordering axiom, there exists a $p \in S$ *s.t.* $p \leq m$, $\forall m \in S$. Also, $p \in S$ implies that $p > x$. Further, $p - 1 \in \mathbb{N} \cup \{0\}$ and $p - 1 \notin S$ (follows since $p - 1 < p$). So $p - 1 \leq x$. Therefore, $p - 1 \leq x < p$.

$\square$

2.5 Intervals

Definition: A set $J \subseteq \mathbb{R}$ is called an *interval* if satisfies the following: if $x, y \in J$ and c be *s.t.* $x < c < y$, then $c \in J$.

The following are all intervals: For $a, b \in \mathbb{R}$,

$$\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} : a < x < b\} : \text{ open interval }. \\
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} : \text{ closed interval }. \\
[a,b) &:= \{x \in \mathbb{R} : a \leq x < b\} : \text{ half-closed, half-open interval }. \\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\} : \text{ half-closed, half-open interval }. \\
(-\infty,b] &:= \{x \in \mathbb{R} : x \leq b\} : \text{ closed interval }. \\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\} : \text{ open interval.} \\
[a,\infty) &:= \{x \in \mathbb{R} : x \geq a\} : \text{ closed interval.} \\
(a,\infty) &:= \{x \in \mathbb{R} : x > a\} : \text{ open interval.} \\
(-\infty,\infty) &= \mathbb{R} : \text{open interval.}
\end{aligned} \tag{3}$$

Exercise 2. *1. Show that any interval in $\mathbb{R}$ is in one of the above form.*

2.6 Even and Odd numbers

Definition: An integer x is called an *even number* if it is of the form $x = 2k$, where $k \in \mathbb{Z}$. An integer x is called an *odd number* if it is of the form $x = 2k+1$, where $k \in \mathbb{Z}$. Therefore the set of even numbers is given by $\{\ldots, -2, 0, 2, 4, \ldots\}$ and the odd numbers is given by $\{\ldots, -3, -1, 1, 3, \ldots\}$.

2.7 Rational numbers

Definition: A real number r is called a *rational number* if it is of the form $r = \frac{a}{b}$ (which denotes $a \cdot b^{-1}$), where $a, b \in \mathbb{Z}$ and $b \neq 0$. The set of rational numbers is denoted as follows:

$$\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}.$$

Theorem 14. *(Density of rationals) For every $x, y \in \mathbb{R}$ with $x < y$, there exists an $r \in \mathbb{Q}$ s.t. $x < r < y$.*

Proof. Consider $x > 0$. Then by the Archimedean property, there exists and $n \in \mathbb{N}$ *s.t.* $0 < \frac{1}{y-x} < n$. This implies that $ny - nx > 1$. Also, by Corollary 5(3), there exists an $m \in \mathbb{N}$ *s.t.* $m - 1 \leq nx < m$. This implies that $x < \frac{m}{n}$. Further, $m \leq nx + 1 < nx + (ny - nx) = ny \Rightarrow \frac{m}{n} < y$. Thus the rational number $\frac{m}{n}$ satisfies $x < \frac{m}{n} < y$.

Consider $x < 0$ and $y < 0$. Then $x < y$ implies that $-y > 0$, $-x > 0$ and $-y < -x$. Then from the earlier case, we obtain an $r \in \mathbb{Q}$ *s.t.* $-y < r < -x$. This implies that $x < -r < y$. Thus $-r$ is the required rational number.

Consider $x < 0$ and $y > 0$. Then by Corollary 5(2), there exists an $n \in \mathbb{N}$ *s.t.* $\frac{1}{n} < y$. Then by the first case, we obtain an $r \in \mathbb{Q}$ *s.t.* $\frac{1}{n} < r < y$ which implies that $x < r < y$. $\square$

2.8 Countable and Uncountable sets

Definition: A function f from a set X to Y is called *one-to-one (bijective)* if it satisfies the following:

1. f is one-one (injective), *i.e.*, $f(x) = f(y) \Leftrightarrow x = y$
2. f is onto (surjective), *i.e.*, for every $y \in Y$, $\exists x \in X$ *s.t.* $f(x) = y$.

Proposition 2. *If $f : X \to Y$ is a bijective map, then $f^{-1} : Y \to X$ exists and is defined as, for $y \in Y$, $f^{-1}(y) := x$, where $x \in X$ such that $f(x) = y$ (x exists since f is onto and is unique since f is one-one). Also, f^{-1} is bijective.*

Definition: A set A is called *finite*, if there exists an $n \in \mathbb{N}$ with a bijective map between $\{1, 2, \dots, n\}$ and A. Here, n is called the *cardinality* of set A denoted as $card(A)$. A set A is called *countably infinite* if there exists a bijective map between the set of natural numbers $\mathbb{N}$ and A. A set is called *countable* if it is either finite or countably infinite.If a set is not countable (*i.e.*, there does not exist such a map), then it is called an *uncountable* set.

Examples:

1. A singleton set $\{a\}$ is finite. The trivial map f is $f(1) = a$.
2. $\mathbb{N}$ is countably infinite. Here, the bijective map is the identity map.
3. The set of positive even numbers is countably infinite. The map is given by $f(n) = 2n$.

Remark 3. *If a set A is countable, then there exists a bijective map f between $\mathbb{N}$ and A. Hence, its elements can be enumerated as $\{a_1, a_2, \dots\}$, where $a_n := f(n), n \in \mathbb{N}$.*

Remark 4. *If a set A is finite with $card(A) = n$, then there exists a bijective map $f : \{1, 2, \dots, n\} \to A$. If $n > 2$ and $a \in A$, then there exists a bijective map*

$g : \{1, 2, \ldots, n-1\} \to A \setminus \{a\}$. *Indeed, g can be defined as:*
For $k \in \mathbb{N}$ and $1 \le k \le n-1$,

$$g(k) := \begin{cases} f(k), & \text{if } k < m \\ f(k+1), & \text{if } m < k \le n-1. \end{cases} \tag{4}$$

This implies that $card(A \setminus \{a\}) = n - 1$.

Theorem 15. *Every subset of a countable set is countable.*

Proof. Let A be the countable set and $\emptyset \ne B \subseteq A$. Assume A is finite. Let $card(A) = n$. Then there exists a bijective map $f : \{1, 2, \ldots, n\} \to A$. Let $A_1 := A$ and $B_1 := B$. Define $g(1) := \min\{f^{-1}(b) : b \in B_1\}$. By the well-ordering axiom, $g(1)$ is well-defined and $f(g(1)) \in B_1$. Now, set $B_2 := B_1 \setminus \{f(g(1))\}$ and $A_2 := A_1 \setminus \{f(g(1))\}$. Repeat this procedure as follows: for $k \in \mathbb{N}$, if $B_k = \emptyset$ then stop, else $g(k) := \min\{f^{-1}(b) : b \in B_k\}$ and $B_{k+1} := B_k \setminus \{f(g(k))\}$ and $A_{k+1} := A_k \setminus \{f(g(k))\}$. The procedure has to stop after a finite number of steps. If not, then g is defined for every $k \in \mathbb{N}$ and hence $B_k \ne \emptyset$, $\forall k \in \mathbb{N}$. However, $B_k \subseteq A_k$ and by the Remark 4, we have $card(A_k) = n - k + 1$ and therefore $card(A_{n+1}) = 0 \Rightarrow A_{n+1} = \emptyset \Rightarrow B_{n+1} = \emptyset$ which is a contradiction. Now define $m := \min\{p \in \mathbb{N} : B_p = \emptyset\}$. By the well-ordering axiom, m exists and $B_{m-1} \ne \emptyset$. Also, since $B_1 \ne \emptyset$, we have $m > 1$ and $m - 1 \in \mathbb{N}$. Thus the above procedure stops at m. Thus $f \circ g$ is a function which maps $\{1, \ldots, m-1\}$ to B. It is easy to verify that $f \circ g$ is bijective. Thus B is finite.

If A is countably infinite, then there exists a bijective map $f : \mathbb{N} \to A$. Now define g as above. If the procedure stops after finite steps, then B is finite. Else, g is defined for every $k \in \mathbb{N}$ and $f(g(k)) \in B$. We will now show that $f \circ g : \mathbb{N} \to B$ is bijective. By construction g is one-one (since f^{-1} is bijective and we remove the mapped element $f(g(k))$ from B_k to obtain B_{k+1} at each iteration). Therefore $f \circ g$ is one-one. Assume $f \circ g$ is not onto, *i.e.*, $\exists b \in B$ *s.t.* $b \notin f(g(\mathbb{N}))$. Since $B \subseteq A$, there exists an $n \in \mathbb{N}$ *s.t.* $f(n) = b \Rightarrow n = f^{-1}(b)$. Also since $b \in B$ is not removed in any iteration, $b \in B_k$, $\forall k \in \mathbb{N}$. Therefore, $g(k) < n$, $\forall k \in \mathbb{N}$. In particular, $g(\{1, \ldots, n\}) \subseteq \{1, \ldots, n-1\}$. Then by the pigeonhole principle, $\exists p, q \in \{1, \ldots, n\}$ *s.t.* $p \ne q$ and $g(p) = g(q)$ which is contradiction since g is one-one by construction. Thus $f \circ g$ is the required bijective map from $\mathbb{N}$ to B. $\square$

Theorem 16. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof. Define the function $g : \mathbb{N} \times \mathbb{N}$ as $g(m, k) := (m + k)^2 + k$. We show that g is one-one. Let $(m, k), (m', k') \in \mathbb{N} \times \mathbb{N}$ be *s.t.* $g(m, k) = g(m', k')$. Then

$$|(m+k) + ((m'+k')||(m+k) - (m'+k')| = |(m+k)^2 - (m'+k')^2| = \\ |g(m,k) - k - g(m',k') + k'| = |k' - k|. \tag{5}$$

If $k' \ne k$, then both $|(m+k) + (m'+k')|$ and $|(m+k) - (m'+k')|$ are greater than 0 and belong to $\mathbb{N}$. However, $|(m+k) + (m'+k')| > |k - k'|$ and then by Theorem 3(7), we have $|(m+k)+((m'+k')||(m+k)-(m'+k')|>|k-k'|$ which is a contradiction. Therefore $k=k'$. Then from Eq. (5), we get $|(m+k)-(m'+k')|=0 \Rightarrow m=m'$. Therefore, g is one-one and in particular g is a bijective map from $\mathbb{N}\times\mathbb{N}$ to $g(\mathbb{N}\times\mathbb{N})$. Now since $g(\mathbb{N}\times\mathbb{N})\subseteq\mathbb{N}$, we have $g(\mathbb{N}\times\mathbb{N})$ is countable (follows from Theorem 15, in particular, countably infinite), *i.e.*, there exists a bijective map h from $\mathbb{N}$ to $g(\mathbb{N}\times\mathbb{N})$. Thus $g^{-1}\circ h$ is the required bijective map from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$. □

Theorem 17. *For any $n\in\mathbb{N}$, if each $E_i, 1\leq i\leq n$ is countably infinite, then $E_1\times E_2\times\cdots\times E_n$ is countably infinite.*

Proof. We prove for case when $n=2$. One can then apply induction to show for arbitrary $n\in\mathbb{N}$. Since E_1 and E_2 are countably infinite, there exists bijective maps $f_1:\mathbb{N}\to E_1$ and $f_2:\mathbb{N}\to E_2$. Now define $h:\mathbb{N}\times\mathbb{N}\to E_1\times E_2$ as follows:

$$h(n,m):=(f_1(n),f_2(m)),\quad n,m\in\mathbb{N}$$

It is easy to verify that h is bijective. Now by Theorem 16, we have a bijective map $g:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$. Therefore, $h\circ g$ is the required bijective map from $\mathbb{N}$ to $E_1\times E_2$. □

An immediate corollary is the following:

Corollary 6. $\mathbb{Q}$ *is countably infinite.*

Proof. $\mathbb{Q}$ can be identified with $\mathbb{Z}\times\mathbb{Z}$ which is countably infinite (follows from Theorem 17). □

Definition: (Countable union) Let $\{A_i : i\in\mathbb{N}, A_i \text{ be any set}\}$ be a countable collection of sets. We define $A=\cup_{i\in\mathbb{N}}A_i$ as its countable union as follows: $a\in A\Leftrightarrow \exists i\in\mathbb{N}$ *s.t.* $a\in A_i$.

Theorem 18. *Countable union of countable sets is countable.*

Proof. Let $\{A_i : i\in\mathbb{N}\}$ be the countable collection with each A_i a countable set. Let $A:=\cup_{i\in\mathbb{N}}A_i$. Define a new collection of sets $\{B_i : i\in\mathbb{N}\}$, where $B_1:=A_1$, $B_2:=A_2\setminus A_1$ and $B_{n+1}:=A_{n+1}\setminus(\cup_{j=1}^{n}A_j)$. Note that $\{B_i : i\in\mathbb{N}\}$ is a disjoint collection, *i.e.*, $B_i\cap B_j=\emptyset,\ i,j\in\mathbb{N}, i\neq j$. Also, $A=\cup_{i\in\mathbb{N}}A_i$ $=\cup_{i\in\mathbb{N}}B_i$. Further, $B_i\subseteq A_i$, $\forall i\in\mathbb{N}$. Since A_i is countable, B_i is also countable (from Theorem 15) for every $i\in\mathbb{N}$. This implies that for each i, there exists a bijective map $f_i:\mathbb{N}\to B_i$. Now, we define the map $g:\mathbb{N}\times\mathbb{N}\to A$ as follows: for $(n,m)\in\mathbb{N}\times\mathbb{N}$, set $g(n,m):=f_n(m)$. It is easy to verify that g is bijective. Now the theorem follows from Theorem 16. □

Exercise 3. *1. Prove that* $\underbrace{\mathbb{N}\times\mathbb{N}\times\ldots\ldots}_{\mathbb{N}\ times}$ *is not countable.*

3 Real sequences

Definition: A real sequence is a map from $\mathbb{N}$ to $\mathbb{R}$. Since a function on $\mathbb{N}$ can be identified by the value it takes for each input n, we can denote the real sequence associated with the map f by its enumerated form $(a_1, a_2, \ldots, a_n, \ldots)$, where $a_n = f(n)$. It is shortened as (a_n).

Definition: We say a real sequence (a_n) converges to $a \in \mathbb{R}$, if it satisfies the following:

$$\text{For a given } \epsilon > 0, \text{ there exists an } N \in \mathbb{N} \text{ s.t. } |a_k - a| < \epsilon, \forall k \geq N. \tag{6}$$

We call a the limit of the sequence (a_n) and denote it as $\lim_{n\to\infty} a_n = a$.

Ex: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Let $\epsilon > 0$. Now, by Corollary 5(2), we obtain an $N \in \mathbb{N}$ *s.t.* $\frac{1}{N} < \epsilon$. Also, for $k > N$, $\frac{1}{k} < \frac{1}{N} < \epsilon$ (from Exercise 1). Therefore, $\lim_{n\to\infty} \frac{1}{n} = 0$. $\square$

To make the definition well defined, we have to prove that the limit is unique:

Theorem 19. *If a real sequence converges, then its limit is unique.*

Proof. Let (a_n) be the convergent real sequence. Let $\epsilon > 0$. Let there exists $a, b \in \mathbb{R}$ s.t. $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} a_n$. These imply that (from Eq. (6)), there exist an $N, M \in \mathbb{N}$ s.t. $|a_k - a| \leq \frac{\epsilon}{2}$, $\forall n \geq N$ and $|a_k - b| \leq \frac{\epsilon}{2}$, $\forall n \geq M$. Now, for $k \geq \max\{N, M\}$, we have $|a - b| = |a - a_k + a_k - b| \leq |a_k - a| + |a_k - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This is true for every $\epsilon > 0$. This implies that $|a - b| = 0 \Rightarrow a = b$. $\square$

3.1 Algebra of real sequences

Theorem 20. *If a sequence (a_n) converges, then (a_n) is bounded, i.e., $\exists K > 0$ s.t. $|a_n| \leq K$, $\forall n \in \mathbb{N}$.*

Proof. Let $a := \lim_{n\to\infty} a_n$. This further implies that exists an $N \in \mathbb{N}$, *s.t.* $|a_k - a| < 1$, $\forall k \geq N$. Further for $k \geq N$, $|a_k| = |a_k - a + a| \leq |a_k - a| + |a| < 1 + |a|$. Therefore $|a_k| \leq K$, $\forall k \in \mathbb{N}$, where $K := \max\{\{a_1, a_2, \ldots, a_{N-1}\}, 1 + |a|\}$. $\square$

Theorem 21. *For real sequences (a_n), (b_n) and (c_n), we have*

1. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then $\lim_{n\to\infty}(a_n + b_n) = L + M$.*
2. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then $\lim_{n\to\infty}(a_n b_n) = LM$.*
3. *If $\lim_{n\to\infty} a_n = L \neq 0$ and $a_n \neq 0$, $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.*
4. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} |a_n - b_n| = 0$, then $\lim_{n\to\infty} b_n = L$.*
5. *If $a_n \leq b_n$, $\forall n \geq T \in \mathbb{N}$, $L := \lim_{n\to\infty} a_n$ and $M := \lim_{n\to\infty} b_n$, then $L \leq M$.*

6. *If* $a_n \le b_n \le c_n, \forall n \in \mathbb{N}$, $L := \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, *then* $\lim_{n\to\infty} b_n = L$. *(squeeze theorem)*

7. *If* $\lim_{n\to\infty} a_n = L$, *then* $\lim_{n\to\infty} |a_n| = |L|$.

Proof.
1. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there exists an $N_a \in \mathbb{N}$ *s.t.* $\forall k \ge N_a$, $|a_k - L| < \frac{\epsilon}{2}$. Also since $\lim_{n\to\infty} b_n = M$, there exists an $N_b \in \mathbb{N}$ *s.t.* $\forall k \ge N_b$, $|b_k - M| < \frac{\epsilon}{2}$. Therefore, $\forall k \ge \max\{N_a, N_b\}$, we have $|a_k + b_k - L - M| \le |a_k - L| + |b_k - M|$ (by triangle inequality) $< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$. Thus the claim follows.

2. Let $\epsilon > 0$. Since (b_n) converges, then by Theorem 20, we have $|b_n| \le K \ge 0$, $\forall n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_n = L$, there exists an $N_a \in \mathbb{N}$ *s.t.* $\forall k \ge N_a$, $|a_k - L| < \frac{\epsilon}{2K}$. Also since $\lim_{n\to\infty} b_n = M$, there exists an $N_b \in \mathbb{N}$ *s.t.* $\forall k \ge N_b$, $|b_k - M| < \frac{\epsilon}{2|L|}$. Therefore, $\forall k \ge \max\{N_a, N_b\}$, we have $|LM - a_k b_k| = |LM - Lb_k + Lb_k - a_k b_k| \le |L||M - b_k| + |L - a_k||b_k| < |L|\frac{\epsilon}{2|L|} + K\frac{\epsilon}{2K} < \epsilon$. Thus the claim follows.

3. Proof left as exercise.

4. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there exists an $N \in \mathbb{N}$ *s.t.* $\forall k \ge N$, $|a_k - L| < \frac{\epsilon}{2}$. Also since $\lim_{n\to\infty} |a_n - b_n| = 0$, there exists an $T \in \mathbb{N}$ *s.t.* $\forall k \ge T$, $|a_k - b_k| < \frac{\epsilon}{2}$. Therefore, $\forall k \ge \max\{N, T\}$, we have $|b_k - L| = |b_k - a_k + a_k - L| \le |b_k - a_k| + |a_k - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus the claim follows.

5. Since $L = \lim_{n\to\infty}(a_n)$, there exists an $N_a \in \mathbb{N}$ *s.t.* $\forall k \ge N_a$, $|a_k - L| < \frac{\epsilon}{2}$. Also, since $\lim_{n\to\infty} b_n = M$, there exists an $N_b \in \mathbb{N}$ *s.t.* $\forall k \ge N_b$, $|b_k - M| < \frac{\epsilon}{2}$. Then, for some $k' \ge \max\{N_a, N_b, T\}$, we have $L - M < a_{k'} + \frac{\epsilon}{2} - b_{k'} + \frac{\epsilon}{2} = a_{k'} - b_{k'} + \epsilon < \epsilon$ (since $a_{k'} - b_{k'} \le 0$). This implies that $L \le M$. (The last deduction used the fact that if $x < \epsilon$, for every $\epsilon > 0$, then $x \le 0$. Prove this.)

6. Note that

$$\begin{aligned} & a_n \le b_n \le c_n, \forall n \in \mathbb{N} \\ & \Rightarrow 0 \le b_n - a_n \le c_n - a_n, \forall n \in \mathbb{N} \\ & \Rightarrow |b_n - a_n| \le |c_n - a_n|. \end{aligned} \tag{7}$$

Now since $\lim a_n = \lim c_n$, we have $\lim(c_n - a_n) = 0$. This implies that for $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|c_n - a_n| < \epsilon$. Now from Eq. (7), we get $|b_n - a_n| < \epsilon$. This implies that $\lim_{n\to\infty}(b_n - a_n) = 0$. Therefore

$$\begin{aligned} \lim_{n\to\infty} b_n = \lim_{n\to\infty}(b_n - a_n + a_n) = \lim_{n\to\infty}(b_n - a_n) + \lim_{n\to\infty} a_n \\ = 0 + L = L. \end{aligned}$$

7. Let $\lim_{n\to\infty} a_n = L$. Now note that, for $\epsilon > 0$, we have

$$||a_n| - |L|| \le |a_n - L| < \epsilon, \forall n \ge N \in \mathbb{N}.$$

The first inequality in the above equation follows from Case 6 of Theorem 8. Note that the converse of this claim is note true. Example: the sequence $(-1)^n$.

□

Exercise 4. 1. *Prove that* $\lim_{n\to\infty} \frac{1}{2^n} = 0$.
Solution: Note that $0 < \frac{1}{2^n} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. *The result then follows from squeeze theorem.*

2. *Prove that* $\lim_{n\to\infty} \frac{1}{n^2} = 0$.
Solution: Note that $\frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n}$, $\forall n \in \mathbb{N}$. *The result then follows from Case 2 of Theorem 21.*

3. *Prove that if* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.
Solution: Note that since $|a_n|$ *converges to* 0, *we have* $||a_n|| < \epsilon$ *for all* $n \geq N \in \mathbb{N}$. *But* $||a_n|| = |a_n|$. *Hence the claim follows.*

Definition: A sub-sequence (a_{n_k}) of (a_n) is nothing but one which satisfies $\{a_{n_k}; k \in \mathbb{N}\} \subseteq \{a_n; n \in \mathbb{N}\}$ with $n_1 < n_2 < \cdots < n_k < n_{k+1} < \ldots$.

Theorem 22. *A real sequence converges iff every sub-sequence of it converges to the same point.*

Proof. ($\Rightarrow$) Let (a_n) be the convergent sequence and (a_{n_k}) be a sub-sequence of (a_n). Let $a := \lim_{n\to\infty} a_n$. Then, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ *s.t.* $|a_k - a| < \epsilon$, $\forall k \geq N$. Since $n_k \geq k$, $|a - a_{n_k}| < \epsilon$, $\forall n_k \geq N$. This implies that $\lim_{k\to\infty} a_{n_k} = a$.
($\Leftarrow$) In the class, I mentioned that (a_n) can be treated as a sub-sequence itself. Since every sub-sequence converges, we have (a_n) converges.

(OR)

We can prove by contradiction. The hypothesis is that every sub-sequence converges to the same point. Let that point be L. Now assume the contrary to what we have to prove, *i.e*, the sequence (a_n) does not converge. Now we will create a sub-sequence (a_{n_k}) which does not converge which becomes the contradiction.

Fix $k \in \mathbb{N}$. Since (a_n) does not converge, we have for $\epsilon = 1$ and for every $N \in \mathbb{N}$

$$\exists n' \geq N \text{ s.t. } |a_{n'} - L| \geq 1. \tag{8}$$

Set $n_k = n'$.

Now for $k+1$, repeat this step, but choose $N > n_k$.

Thus we create the sub-sequence (a_{n_k}) by following the procedure starting from $k = 1$. The sub-sequence (a_{n_k}) so generated satisfies

$$\begin{aligned} |a_{n_k} - L| \geq 1, \forall k \in \mathbb{N} &\Rightarrow a_{n_k} \leq L - 1 \text{ or } a_{n_k} \geq L + 1 \\ &\Rightarrow a_{n_k} \notin (L-1, L+1), \forall k \in \mathbb{N}. \end{aligned}$$

Therefore (a_{n_k}) does not converge to L which is a contradiction. $\square$

Definition: A sequence (a_n) is called *monotonically increasing sequence* if $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$. Similarly, a sequence (a_n) is called *monotonically decreasing sequence* if $a_n \geq a_{n+1}$, $\forall n \in \mathbb{N}$. A sequence is simply called *monotone* if it is either monotonically increasing or monotonically decreasing.

Theorem 23. *Every bounded monotone real sequence converges.*

Proof. Assume the sequence (a_n) is monotonically increasing (*i.e.* $a_{n+1} \geq a_n$, $\forall n \in \mathbb{N}$) and bounded, *i.e.*, $a_n \leq |a_n| \leq K, \forall n \in \mathbb{N}$, for some $K \geq 0$. By the Completeness axiom, $\alpha := \sup_{n\in\mathbb{N}} a_n$ exists in $\mathbb{R}$. Then by Theorem 10, $\exists N \in \mathbb{N}$ *s.t.* $a_N > \alpha - \epsilon$. Since (a_n) is monotonically increasing, we have $a_k \geq a_N$, $\forall k \geq N$. Also, $a_k \leq \alpha$, $\forall k \in \mathbb{N}$. These imply that $|\alpha - a_k| < \epsilon$, $\forall k \geq N$. Therefore $\lim_{n\to\infty} a_n = \alpha$.

Similarly for a monotonically decreasing bounded sequence (a_n), we have $\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n$. Proof left as exercise. $\square$

Exercise 5.

1. *Prove that* $\lim_{n\to\infty} c^n = 0$*, for* $0 < c < 1$.
   Solution: Note that since $0 < c < 1$*, we have* $c^{n+1} < c^n$*. Therefore the sequence* (c^n) *is monotonically decreasing. Also,* $0 < c^n < 1$*,* $\forall n \in \mathbb{N}$*. Therefore the sequence is bounded also. Then by Theorem 23, we know that the sequence* (c^n) *converges. Let* $\lim_{n\to\infty} c^n = L$*. Now consider the sequence* (c^{n+1})*. Note that* $\lim_{n\to\infty} c^{n+1} = L$ *(put* $m = n + 1$*, we get* $\lim_{m\to\infty} c^m = L$*). Hence*

$$\begin{aligned} L &= \lim_{n\to\infty} c^{n+1} = \lim_{n\to\infty} c \cdot c^n = c \cdot \lim_{n\to\infty} c^n = cL \\ &\Rightarrow (c-1)L = 0 \Rightarrow L = 0 (\textit{since } 0 < c < 1). \end{aligned}$$

2. *Prove that Prove that* $\lim_{n\to\infty} c^n = 0$*, for* $|c| < 1$.
   Solution: Use the above solution and Problem 3 of Exercise 4.

3. *Analyze the convergence of the sequence* $a_n = (1 + \frac{1}{n})^n$.
   Solution: Claim 1: (a_n) *is monotonically decreasing.*
   Proof: Recall that Arithmetic-mean $\geq$ *Geometric-mean,* i.e., *for* $x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}$*, we have*

$$\frac{x_1 + x_2 \cdots + x_{n+1}}{n+1} \geq (x_1 \cdot x_2 \cdots x_{n+1})^{\frac{1}{n+1}}. \tag{9}$$

   Now put $x_1 = 1, x_2 = x_3 = \cdots = x_{n+1} = 1 + \frac{1}{n}$*. Therefore from Eq. (9), we have*

$$\begin{aligned} \frac{1 + (1+\frac{1}{n}) + \cdots + (1+\frac{1}{n})}{n+1} &\geq \left(1 \cdot (1+\frac{1}{n}) \ldots (1+\frac{1}{n})\right)^{\frac{1}{n+1}} \\ \left(1 + \frac{1}{n+1}\right) &\geq \left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}} \\ \left(1 + \frac{1}{n+1}\right)^{n+1} &\geq \left(1 + \frac{1}{n}\right)^{n} \end{aligned}$$

Claim 2: The sequence (a_n) is bounded.
Proof: By binomial series we have

$$(1+\frac{1}{n})^n = 1+1+\sum_{k=2}^{n} nC_k \frac{1}{n^k}$$
$$= 1+1+\sum_{k=2}^{n} \frac{(n-k+1)!}{k!}\frac{1}{n^k}$$
$$\leq 1+1+\sum_{k=2}^{n} \frac{1}{k!}$$
$$\leq 1+1+\sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k}$$
$$= 3-\frac{1}{n} \leq 3.$$

Therefore by Theorem 23, the sequence $(1+\frac{1}{n})^n$ converges. We call that limit e (Euler's constant). Therefore

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e. \tag{10}$$

3.2 Liminf and Limsup

Here we consider the extended real line $\mathbb{R}^*$, were

$$\mathbb{R}^* := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}. \tag{11}$$

with the following properties: for $a \in \mathbb{R}$,

$$\begin{aligned} a < +\infty \text{ and } a > -\infty. \\ a + +\infty = +\infty + a = +\infty. \\ a - \infty = -\infty + a = -\infty. \\ a\cdot(+\infty) = +\infty \text{ if } a>0; \quad a\cdot(+\infty) = -\infty \text{ if } a<0. \\ (+\infty)\cdot(+\infty) = +\infty; (-\infty)\cdot(+\infty) = +\infty; (-\infty)\cdot(-\infty) = +\infty. \\ +\infty + -\infty = \text{ undefined }; -\infty + +\infty = \text{ undefined} \end{aligned}$$

Definition: (Liminf of a real sequence): Given a real sequence (a_n), where $a_n \in \mathbb{R}$, we define

$$\liminf_n a_n = \sup_n \inf_{k\geq n} a_k. \tag{12}$$

Definition: (Limsup of a real sequence): Given a real sequence (a_n), where $a_n \in \mathbb{R}$, we define

$$\limsup_n a_n = \inf_n \sup_{k\geq n} a_k. \tag{13}$$

The definition needs some clarification. In Eq. (12), when we write $\sup_{k\geq n} a_n$, we mean $\sup\{a_n; k \geq n\}$. Note that in the extended real line, every set is lower bounded by $-\infty$ and upper bounded by $+\infty$, therefore every set has an infimum and a supremum by the completeness axiom. Thus the definitions of liminf and limsup are well-defined. Thus every sequence has a liminf and limsup in the extended real line, however it make take $+\infty$ and $-\infty$.

Further clarification: Please go through Proposition 1 for more on inf and sup. Now, consider

$$b_n := \inf_{k\geq n} a_k \qquad \bar{b}_n := \sup_{k\geq n} a_k \tag{14}$$

Note that b_n, $\bar{b}_n \in \mathbb{R}^*$ (however $a_n \in \mathbb{R}$) and the existence of b_n and $\bar{b}_n$ is justified in the previous paragraph. Also,

$$b_{n+1} \geq b_n, \forall n \in \mathbb{N}. \tag{15}$$
$$\bar{b}_{n+1} \leq \bar{b}_n, \forall n \in \mathbb{N}. \tag{16}$$
$$\bar{b}_n \geq b_n, \forall n \in \mathbb{N}. \tag{17}$$
$$\tag{18}$$

Also, for fixed m, we have $\bar{b}_n \geq b_m, \forall n \in \mathbb{N}$. (please verify). Therefore,

$$\begin{aligned} \inf_n \bar{b}_n &\geq b_m \\ \Rightarrow \inf_n \bar{b}_n &\geq \sup_m b_m \\ \Rightarrow \inf_n \sup_{k\geq n} a_k &\geq \sup_m \inf_{k\geq m} a_k \\ \Rightarrow \limsup_n a_n &\geq \liminf_n a_n. \end{aligned}$$

Also, from Eq. (15) we have b_n is a monotonically increasing sequence in $\mathbb{R}^*$ and similarly from Eq. (16), we have $\bar{b}_n$ is a monotonically decreasing sequence in $\mathbb{R}^*$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} b_n = \sup_n b_n = \sup_n \inf_{k\geq n} a_k &\text{ (by simply substituting the definition of } b_n) \\ = \liminf_n a_n. \end{aligned} \tag{19}$$

$$\text{Similarly, } \lim_{n\to\infty} \bar{b}_n = \inf_n \bar{b}_n = \inf_n \sup_{k\geq n} a_k = \limsup_n a_n. \tag{20}$$

Example:

1. For the real sequence $(-1)^n$, we have $b_n = -1$ and $\bar{b}_n = 1$, $\forall n \in \mathbb{N}$. Therefore $\liminf_n a_n = -1$ and $\limsup_n a_n = 1$.

2. For the real sequence $((-1)^n \frac{1}{n})$, we have

$$b_n = \begin{cases} -\frac{1}{n} \text{ if n is odd} \\ -\frac{1}{n+1} \text{ if n is even} \end{cases}$$

$$b_n = \begin{cases} \frac{1}{n} \text{ if n is even} \\ \frac{1}{n+1} \text{ if n is odd} \end{cases}$$

Therefore, $\liminf_n a_n = \limsup_n a_n = 0$.

Theorem 24. *Given a real sequence (a_n), if (a_{n_k}) is a sub-sequence of (a_n) and $\lim_{k\to\infty} a_{n_k} = L$ exists , then $\liminf a_n \leq L \leq \limsup a_n$.*

Proof. Note that $b_{n_k} \leq a_{n_k} \leq \bar{b}_{n_k}$, $\forall k \in \mathbb{N}$. Now from Eqs. (19-20) and from Case 6 of Theorem 21, the claim follows. □

Lemma 1. *Given a real sequence (a_n), if $\liminf_n a_n \in \mathbb{R}$, then there exists a sub-sequence (a_{n_k}) s.t. $\lim_{k\to\infty} a_{n_k} = \liminf_n a_n$.*

Proof. Let $L = \liminf_n a_n$. We will create a sub-sequence (a_{n_k}) as follows.

Fix $k \in \mathbb{N}$. Now note that since $L = \lim_{n\to\infty} b_n$, we have

$$\exists n' \in \mathbb{N} \text{ s.t. } |b_{n'} - L| < \frac{1}{2k}. \tag{21}$$

Also since b_n is the inf of the tail of a_n starting at n (by definition in Eq. (14)), we have

$$\exists n'' \geq n' \text{ s.t. } |a_{n''} - b_{n'}| < \frac{1}{2k}. \tag{22}$$

From Eqs: (21-22), we have

$$|a_{n''} - L| < |a_{n''} - b_{n'}| + |b_{n'} - L| < \frac{1}{k}. \tag{23}$$

Now set $n_k = n''$.

Repeat this procedure for $k + 1$. But in this case, choose n' in such a way that $n' > n_k$. This is possible since b_k is a monotonically increasing sequence.

The sub-sequence (a_{n_k}) created in this manner satisfies

$$|a_{n_k} - L| < \frac{1}{k}. \tag{24}$$

Therefore, $\lim_{k\to\infty} a_{n_k} = L$. □

Similarly, we have

Lemma 2. *Given a real sequence (a_n), if $\limsup_n a_n \in \mathbb{R}$, then there exists a sub-sequence (a_{n_k}) s.t. $\lim_{k\to\infty} a_{n_k} = \limsup_n a_n$.*

Theorem 25. *Given a real sequence (a_n), we have $\liminf_n a_n = \limsup_n a_n \in \mathbb{R}$ iff $\lim_{n\to\infty} a_n$ exists (and $\liminf_n a_n = \limsup_n a_n = \lim_{n\to\infty} a_n$).*

Proof. ($\Rightarrow$) Note that $b_n \le a_n \le \bar{b}_n$, $\forall n \in \mathbb{N}$. Now from Eqs. (19-20) and from Case 6 of Theorem 21, we have the sequence (a_n) converges and $\lim_{n\to\infty} a_n = \liminf_n a_n = \limsup_n a_n$.
($\Leftarrow$) Let $L := \lim_{n\to\infty} a_n$. Again since $b_n \le a_n \le \bar{b}_n$, $\forall n \in \mathbb{N}$, we have

$$\liminf_n a_n \le \lim_{n\to\infty} a_n = L.$$

This implies that $\liminf_n a_n$ is finite. Thus from Lemma 1, we have a sub-sequence (a_{n_k}) converging to $\liminf_n a_n$. However, by Theorem 22, we have

$$\liminf_n a_n = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n. \tag{25}$$

Similarly, from Lemma 2, we have

$$\limsup_n a_n = \lim_{n\to\infty} a_n. \tag{26}$$

Thus the claim follows. $\square$

Another important observation is

Proposition 3. $-\liminf_n a_n = \limsup_n -a_n$.

Proof. Follows from Case 5 of Proposition 1. $\square$

3.3 Cauchy sequence (Not an important topic)

Theorem 26. *Every bounded real sequence has a converging sub-sequence.*

Proof. Let the real sequence (a_n) be bounded, *i.e.*, $|a_n| \le K, \forall n \in \mathbb{N}$, for some $K \ge 0$. Now consider the sequence (b_n), where $b_n := \frac{a_n}{K}$. Now $|b_n| \le 1, \forall n \in \mathbb{N}$. If the set $\{b_n; n \in \mathbb{N}\}$ is finite, then at least one element repeats infinitely in (a_n) which gives us the constant sub-sequence (hence converging). If it is not finite, then either $\{b_n; n \in \mathbb{N}\} \cap [0,1]$ or $\{b_n; n \in \mathbb{N}\} \cap [-1,0]$ is countably infinite. Without loss of generality, assume that $\{b_n; n \in \mathbb{N}\} \cap [0,1]$ is countably infinite. Let $I_1 := [0,1]$ and $c_1 := b_1$ and $n_1 := 1$. Now again $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1\}) \cap [0, \frac{1}{2}]$ or $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1\}) \cap [\frac{1}{2}, 1]$ is countably infinite. Set $I_2 := [0, \frac{1}{2}]$ if $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1\}) \cap [0, \frac{1}{2}]$ is countably infinite or set $I_2 := [\frac{1}{2}, 1]$ if $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1\}) \cap [\frac{1}{2}, 1]$ is countably infinite. Choose one of those in case both are countably infinite. Also choose c_2 from $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1\}) \cap I_2$ and let n_2 be the index of c_2 in the sequence (b_n). Repeat this procedure to obtain the interval sequence $\{I_k; k \in \mathbb{N}\}$ and $\{c_k; k \in \mathbb{N}\}$, where c_k is chosen from $(\{b_n; n \in \mathbb{N}\} \setminus \{c_1, c_2, \ldots, c_{k-1}\}) \cap I_k$. Let $I_k := [p_k, q_k]$ and the index sequence be (n_k) . It is also easy to verify that (p_k) is a monotonically increasing sequence and (q_k) is a monotonically decreasing sequence. Therefore, from Theorem 23, we have $\sup_k p_k = \lim_k p_k$ and $\inf kq_k = \lim_k q_k$. Also $|p_k - q_k| = \frac{1}{2^k}$, $\forall k \in \urcorner$ which implies that $\lim_k |p_k - q_k| = 0$ (from Exercise 4(1)). Therefore from Theorem 21(4), we have $\sup_k p_k = \inf_k q_k$. Also, since $c_k \in I_k$, we have $p_k \le c_k \le q_k$. Therefore, by squeeze theorem, we have $\lim_{k\to\infty} c_k = \sup_k p_k = \inf_k q_k$. Thus (b_{n_k}) is the convergent sub-sequence of (b_n) and (Kb_{n_k}) is the required convergent sub-sequence of (a_n). $\square$

Exercise 6. 1. *Use the above proof technique to show that* $[0,1]$ *is uncountable.*

3.4 Cauchy sequence

Definition: A real sequence (a_n) is called a *Cauchy sequence*, if for a given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ *s.t.* $|a_n - a_m| < \epsilon$, $\forall n, m \geq N$.

Theorem 27. *Every convergent real sequence is a Cauchy sequence.*

Proof. Let (a_n) be the convergent real sequence and $L := \lim_n(a_n)$. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ *s.t.* $\forall k \geq N$, $|a_k - L| < \frac{\epsilon}{2}$. This further implies that for $n, m \geq N$, we have $|a_n - a_m| \leq |a_n - L| + |a_m - L| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Lemma 3. *Every Cauchy sequence in* $\mathbb{R}$ *is bounded.*

Proof. Proof left as an exercise. □

Lemma 4. *For a real sequence* (a_n)*, if there exists a convergent sub-sequence, then* (a_n) *converges.*

Proof. Let (a_{n_k}) be the convergent sub-sequence of (a_n) and $L := \lim_{k\to\infty} a_{n_k}$. Let $\epsilon > 0$. This implies that there exists an $N \in \mathbb{N}$ *s.t.* $\forall k \geq N$, $|a_{n_k} - L| < \frac{\epsilon}{2}$. Also, since a_n is a Cauchy sequence, there exists a $T \in \mathbb{N}$ *s.t.* $|a_p - a_q| \leq \frac{\epsilon}{2}$, $\forall p, q \geq T$. Now, for $k \geq \max\{T, N\}$, we have $|a_k - L| = |a_k - a_{n_k} + a_{n_k} - L| \leq |a_k - a_{n_k}| + |a_{n_k} - L| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $\lim_{k\to\infty} a_k = L$. □

Theorem 28. *Every Cauchy sequence in* $\mathbb{R}$ *converges.*

Proof. The claim follows from Theorem 26, Lemma 3 and Lemma 4. □

4 Topology on $\mathbb{R}$

Intervals (defined in Eq. (3)) are basic, intuitive sets of R. Let

$$\mathcal{C} := \{J \subseteq \mathbb{R} : J \text{ is an open interval }\} \tag{27}$$

(recall that open intervals are of the form (a, b), $(-\infty, b)$ and (a, ∞)). They satisfy the following prorpery.

Theorem 29. *If* $J_1, J_2 \in \mathcal{C}$ *and* $J_1 \cap J_2 \neq \emptyset$*, then for every* $x \in J_1 \cap J_2$*, there exists a* $J_3 \in \mathcal{C}$ *s.t.* $x \in J_3 \subseteq J_2 \cap J_2$.

Proof. Intervals are one of the kinds defined in Eq. (3). Let $J_1 = (a, b)$ amd $J_2 = (c, d)$. We consider the various cases here:

1. For $a \leq c \leq d \leq b$, we have $J_3 = (c, d)$.
2. For $c \leq a \leq b \leq d$, we have $J_3 = (a, b)$.

3. For $a \leq c \leq b \leq d$, we have $J_3 = (c, b)$.
4. For $c \leq a \leq d \leq b$, we have $J_3 = (a, d)$.

One can similarly prove for cases when J_1 or J_2 is unbounded. $\square$

Definition: (Arbitrary union of sets) Let $\{A_t : t \in T\}$ (where T may be countable or uncountable) be an arbitrary collection of sets. Then their union of the collection $\bigcup_{t \in T} A_t$ is defined as follows:

$$x \in \bigcup_{t \in T} A_t \Leftrightarrow \exists t' \in T \text{ s.t. } x \in A_{t'}.$$

Definition: (Standard topology on $\mathbb{R}$) The *standard topology* on $\mathbb{R}$ is a collection $\mathcal{T}^{\mathbb{R}}$ of subsets of $\mathbb{R}$ which is defined as follows:

$$\mathcal{T}^{\mathbb{R}} := \{B \subseteq \mathbb{R} : B = \bigcup_{t \in \Lambda} J_t, \text{ where } J \in \mathcal{C} \text{ and } \Lambda \text{ an arbitrary index set}\}. \quad (28)$$

Theorem 30. *$\mathcal{T}^{\mathbb{R}}$ satisfies the following:*

1. *If $A_t \in \mathcal{T}^{\mathbb{R}}$, $\forall t \in \Lambda$, where Λ some arbitrary index set. Then $\bigcup_{t \in \Lambda} A_t \in \mathcal{T}^{\mathbb{R}}$.*
2. *If $A_t \in \mathcal{T}^{\mathbb{R}}$, $\forall t \in \{1, 2, ..n\}$, then $\bigcap_{t \in 1}^{n} A_t \in \mathcal{T}^{\mathbb{R}}$.*
3. *$\emptyset$, $\mathbb{R} \in \mathcal{T}^{\mathbb{R}}$.*

Proof. 1. Since $A_t \in \mathcal{T}^{\mathbb{R}}$, for $t \in \Lambda$, we have $A_t = \bigcup_{u \in H_t} J_u^t$, where $J_u^t \in \mathcal{C}$ and H_t is some index set. Therefore $\bigcup_{t \in \Lambda} A_t = \bigcup_{t \in \Lambda} \bigcup_{u \in H_t} J_u^t$. Thus the claim follows.

2. We prove for the case $n = 2$. For a general n, one can then use induction. For $n = 2$, we have Since $A_t \in \mathcal{T}^{\mathbb{R}}$, for $t \in \{1, 2\}$, we have $A_t = \bigcup_{u \in H_t} J_u^t$, where $J_u^t \in \mathcal{C}$ and H_t is some index set. Then $\bigcap_{t=1}^{t=2} A_t = \bigcup_{u \in H_1, s \in H_2} J_u^1 \bigcap J_s^2$. Now from Theorem 29, the claim follows.

3. Trivial.

$\square$

Remark 5. *Alternatively, for a given set X, one can define a topology $\mathcal{T}^X$ as the collection of subsets of X which satisfies the following:*

1. *$\emptyset, X \in \mathcal{T}^X$.*
2. *If $A_t \in \mathcal{T}^X$, $t \in \Lambda$ (Λ be any index set), then $\bigcup_{t \in \Lambda} A_t \in \mathcal{T}^X$.*

3. *If* $A_t \in \mathcal{T}^X$, $t \in \{1, 2, \ldots, n\}$, *then* $\bigcap\limits_{t=1}^{t=n} A_t \in \mathcal{T}^X$.

Now let $\mathcal{A}$ *be any arbitrary collection of subsets of* X. *Then* $\mathcal{T}^X(\mathcal{A})$ *is called the topology generated by* $\mathcal{A}$ *if* $\mathcal{T}^X(\mathcal{A})$ *is the minimal topology containing* $\mathcal{A}$. *The "minimal" implies that if* $\mathcal{T}'$ *be any topology containing* $\mathcal{A}$, *then* $\mathcal{T}^X(\mathcal{A}) \subseteq \mathcal{T}'$. *For a given collection* $\mathcal{A}$, *the existence of* $\mathcal{T}^X(\mathcal{A})$ *can be shown as follow: Consider the collection* $\mathcal{F} := \{\mathcal{T},$ *where* $\mathcal{T}$ *is a topology on* X *and* $\mathcal{A} \subseteq \mathcal{T}\}$. *Note that* $\mathcal{F} \neq \emptyset$ *since the power set of* X *is in* $\mathcal{F}$. *Now, it is easy to verify that* $\mathcal{T}^X(\mathcal{A}) = \bigcap\limits_{\mathcal{T} \in \mathcal{F}} \mathcal{T}$.

In the context of the above remark, one view the standard topology $\mathcal{T}^{\mathbb{R}}$ as the minimum topology containing all the open intervals of $\mathbb{R}$. This follows from Theorem 30 and the fact that $\mathcal{T}^{\mathbb{R}}$ defined in Eq. (28) is contained in every topology containing the open intervals.

Definition: (Open set and closed set in $\mathbb{R}$) A set $A \subseteq \mathbb{R}$ is called *open* if $A \in \mathcal{T}^{\mathbb{R}}$. A set $E \subseteq \mathbb{R}$ is called *closed* if E^c is open.

Remark 6. *Now by Eq. (28), every open set in* $\mathbb{R}$ *is an arbitrary union of open intervals. Thus, for an open set* O *with* $x \in O$, *there exists an open interval* J s.t. $x \in J$ *and* $J \subseteq O$. *In particular, there exists an* $\epsilon > 0$ s.t. *the open interval* $(x - \epsilon, x + \epsilon) \subseteq O$. *Equivalently, we can say that a set* $G \subseteq \mathbb{R}$ *is open, if for every* $x \in G$, *there exists an open set* O *with* $x \in O$ *and* $O \subseteq G$.

For $x \in \mathbb{R}$ and $\epsilon > 0$, we call $(x - \epsilon, x + \epsilon)$ the ϵ-neighbourhood of x.

Alternate definition of sequence convergence: By the definition of convergence of the real sequence (a_n) given in Eq. (6), there exists an $N \in \mathbb{N}$ s.t. $a_n \in (L - \epsilon, L + \epsilon)$, $\forall n \geq N$. One can indeed rewrite this definition in terms of the standard topology as follows: for a given open set O with $L \in O$, there exists an $N \in \mathbb{N}$ s.t. $a_n \in O$, $\forall n \geq N$. (Proof left as exercise)

Exercise 7. *Prove the following:*

1. *If* $E_t \subseteq \mathbb{R}, t \in \Lambda$ *(Λ be any index set) are closed sets, then* $\bigcap\limits_{t \in \Lambda} E_t$ *is also a closed set.*
2. *If* $E_t \subseteq \mathbb{R}$, $t \in \{1, 2, \ldots, n\}$ *are closed sets, then* $\bigcup\limits_{t=1}^{t=n} E_t$ *is also a closed set.*
3. *The only sets which are both open and closed in* $\mathcal{T}^{\mathbb{R}}$ *are* $\mathbb{R}$ *and* $\emptyset$.

4.1 Limit point of a set of reals

Definition: (Limit point of a set) For a given set $A \subseteq \mathbb{R}$, we say x is a limit point of A if for every open set O containing x (*i.e.*, $x \in O$), there exists an element $a \in A$ s.t. $a \neq x$ and $a \in O$. The set of limit points of A is denoted as A'.

Remark 7. *The above definition says that every open set containing the limit point should contain an element (different from the limit point) from the set, however, it is both sufficient (and necessary) to only consider every ϵ-neighbourhood ($\epsilon > 0$) of x (follows from Remark 6). One can even further relax and say it is sufficient to consider only the $\frac{1}{n}$-neighbourhood ($n \in \mathbb{N}$) of x (follows from Corollary 5).*

Note that there exists many limit points for a set. For ex: for the set $(0,1)$, the limits points are every element in $[0,1]$.

Theorem 31. *For a given set $A \subseteq \mathbb{R}$, if $x \in A'$, then there exists a sequence (x_n) with $x_n \in A$, $x_n \neq x$ and $\lim_n x_n = x$.*

Proof. □

Lemma 5. *If $A \subseteq \mathbb{R}$ is closed, then $A' \subseteq A$. In particular, $A = A \cup A'$. Also, the converse is also true,* i.e., *if $A = A \cup A'$, then A is closed.*

Proof. Let $x \in A'$. Assume $x \notin A$. Since A is closed, A^c is open. This implies that there exists an $\epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \subseteq A^c$. This implies that $(x-\epsilon, x+\epsilon) \cap A = \emptyset$. This is a contradiction since $x \in A'$.

To prove the second part, note that (from the first part) $A \cup A' \subseteq A$. However, $A \subseteq A \cup A'$. Therefore the claim follows.

To prove the third part, assume that A is not closed. This implies that A^c is not open. This further implies that there exists and $x \in A^c$ s.t. for every $n \in \mathbb{N}$, the open interval $(x - \frac{1}{n}, x + \frac{1}{n})$ contains $y_n \in A$. Hence x is a limit point of A (follows from Remark 7) which implies that $x \in A$ (follows from the hypothesis that $A\prime \subseteq A$.). This is a contradiction since $x \in A^c$. Therefore A is closed. □

Another interpretation of the above lemma is that for any set A, $A \cup A'$ is a closed set.

Definition: (Closure and interior) For a given set $A \subseteq \mathbb{R}$, we define the *closure* of A as the smallest closed set containing A and is denoted as $\bar{A}$. Also, we define the *interior* of A as the largest open set contained inside A and is denoted as $\mathring{A}$.

A trivial observation is that $A \subseteq \bar{A}$ and $\mathring{A} \subseteq A$. Another observation is that if A is open, then $\mathring{A} = A$ (follows from definition). And if A is closed, then $\bar{A} = A$ (follows from definition).

Theorem 32.

1. *For $A \subseteq \mathbb{R}$, we have $\bar{A} = A \cup A'$.*
2. *For $A, B \subseteq \mathbb{R}$, we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

Proof. 1. From Lemma 5, we know that $A \cup A'$ is a closed set. Therefore, $\bar{A} \subseteq A \cup A'$. However, since $\bar{A}$ is closed and contains A, we have $A' \subseteq \bar{A}$ (again follows from Lemma 5). This implies that $A \cup A' \subseteq \bar{A}$. Therefore, $\bar{A} = A \cup A'$.

2. Note that $\overline{A \cup B}$ is a closed set which contains A. Therefore $\bar{A} \subseteq \overline{A \cup B}$. Similarly, $\bar{B} \subseteq \overline{A \cup B}$. Therefore, $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Now to prove the reverse inclusion, note that $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$. Therefore, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Therefore the claims follows.

$\square$

Note that the equality in Theorem 32(2) does not hold for countable union, *i.e.*, $\overline{\bigcup\limits_{n\in\mathbb{N}} A_n}$ may not be equal to $\bigcup\limits_{n\in\mathbb{N}} \bar{A}_n$. For example, consider $A_n := (0, 1-\frac{1}{n})$. In this case, $\overline{\bigcup\limits_{n\in\mathbb{N}} A_n} = \overline{(0,1)} = [0,1]$. However, $\bigcup\limits_{n\in\mathbb{N}} \bar{A}_n = \bigcup\limits_{n\in\mathbb{N}} [0, 1-\frac{1}{n}] = (0,1)$.

Also $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$. For example, consider $A := (0,1)$ and $B := (1,2)$. In this case, $\overline{A \cap B} = \emptyset$ and $\bar{A} \cap \bar{B} = \{1\}$.

Definition: For a given set $A \subseteq \mathbb{R}$, we call the set $E \subseteq A$ a *dense subset* of A if $\bar{E} = A$.

Remark 8. *An equivalent statement of the above definition is that for a given $x \in A$ and every open set O containing x, we have $E \cap (O \setminus \{x\}) \neq \emptyset$. (or) Equivalently, for a given $x \in A$ and $\epsilon > 0$, we have $(x-\epsilon, x+\epsilon) \setminus \{x\} \cap E \neq \emptyset$.*

Theorem 33. *$\mathbb{Q}$ is a dense subset of $\mathbb{R}$.*

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$. By Theorem 14, there exists a $q \in \mathbb{Q}$ *s.t.* $x < q < x + \epsilon$. Therefore, $\bar{\mathbb{Q}} = \mathbb{R}$. $\square$

Theorem 34. *$\mathbb{Q}^c$ (set of irrational numbers) is dense in $\mathbb{R}$.*

Proof. Proof left as exercise. $\square$

5 Compact sets in $\mathbb{R}$

Definition: (Open cover of a set) For a given set $A \subseteq \mathbb{R}$, we call a collection of open sets $\{U_t; t \in \Lambda\}$ (Λ an index set and U_t an open set for every $t \in \Lambda$) an *open cover* if $A \subseteq \bigcup\limits_{t\in\Lambda} U_t$.

Definition: (Compact set) A set $A \subseteq \mathbb{R}$ is called a *compact set* if every open cover $\{U_t; t \in \Lambda\}$ of A has a finite sub-cover, *i.e.*, there exist $t_1, t_2, \ldots, t_n$, for some $n \in \mathbb{N}$ *s.t.* $A \subseteq \bigcup\limits_{i=1}^{i=n} U_{t_i}$.

Theorem 35. *If $A \subseteq \mathbb{R}$ is compact, then A is bounded and closed.*

Proof. Let $\epsilon > 0$. For every $x \in A$, define $U_x := (x-\epsilon, x+\epsilon)$. Then $\mathcal{G} := \{U_x; x \in A\}$ is an open cover of A. Since A is compact, $\mathcal{G}$ has a finite sub-cover, *i.e.*, there exists $\{x_1, x_2, \ldots, x_n\}$ s.t. $A \subseteq \bigcup_{i=1}^{i=n} U_{x_i}$. Now define $K := max\{|x_i|, 1 \leq i \leq n\}$. For any $a \in A$, we know that since $\{U_{x_i}; 1 \leq i \leq n\}$ coverts A, there exists a k, $1 \leq k \leq n$ *s.t.* $a \in U_{x_k}$. Therefore, $|a| = |a - x_k + x_k| \leq |a - x_k| + |x_k| \leq \epsilon + K$. Therefore, A is bounded. $\square$