Assignment 2

February 6, 2019

Here we consider the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-field, Ω is the sample space and P is the probability measure.

We will prove the strong law of large numbers. We state the theorem first

Theorem 1. *If $\{X_i\}_{i\in\mathbb{N}}$ is an IID sequence and $\mathbb{E}[|X_1|] < \infty$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1] \ \ a.s.,$$

where $S_n = \sum_{i=1}^{n} X_i$.

We will use the following fact in this proof.

Lemma 1. *Given $p > 0$ and X a non-negative r.v.,* i.e., *$X \geq 0$, we have*

$$\mathbb{E}[X^p] = \int_0^\infty px^{p-1} P(X > x)dx. \tag{1}$$

Proof of Theorem 1:

Proof. We may assume $\mathbb{E}[X_i] = 0$. Otherwise, one can replace X_i with $X_i - \mathbb{E}[X_i]$. This implies that we have to prove that $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges to 0.

Let

$$Y_n = X_n I_{\{|X_n|\leq n\}} \text{ and } Z_n = Y_n - \mathbb{E}[Y_n]. \tag{2}$$

Further, define

$$M_n = \sum_{i=1}^{n} \frac{Z_i}{i} \tag{3}$$

with $M_0 = 0$. Define the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ where $\mathcal{F}_n := \sigma(X_1, X_2 \ldots X_n)$.

Q1: Prove that $\{M_n\}$ is a martingale w.r.t. the filtration $\mathcal{F}_n$.

Now we will show that $\mathbb{E}[|M_n|]$ is bounded by a constant not dependent on n. For that, consider

$$\begin{aligned}
\mathbb{E}\left[M_n^2\right] &= Var(M_n) = \sum_{i=1}^{n} \frac{Var(Z_i)}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} Var(Y_i) \\
&\leq \sum_{i=1}^{n} \frac{1}{i^2} \mathbb{E}\left[Y_i^2\right] \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^i 2yP(|X_i| \geq y)dy \text{ (from Lemma 1)} \\
&= 2\sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^{\infty} I_{\{y \leq i\}} P(|X_i| \geq y)dy \\
&\dots \\
&\dots \\
&\leq c\mathbb{E}\left[|X_1|\right], \text{ where } c > 0. \qquad (4)
\end{aligned}$$

Q2: Fill the missing details.

Q3: Claim that $\mathbb{E}[|M_n|]$ is bounded by a constant not dependent on n using Jensen's inequality and Eq. (4).

Q4: Further claim that the martingale $\{M_n\}$ converges.

Now note that from Eqs. (2) and (3), we have

$$Z_i = i(M_i - M_{i-1}). \qquad (5)$$

This implies that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} Z_i &= \frac{1}{n}\sum_{i=1}^{n} i(M_i - M_{i-1}) = \frac{1}{n}\left(\sum_{i=1}^{n} iM_i - \sum_{i=1}^{n-1}(i+1)M_i\right) \\
&= M_n - \frac{n-1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} M_i\right) \qquad (6)
\end{aligned}$$

Q5: Now claim that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Z_i = 0$.

Q6: Using dominated convergence theorem and the fact that X_n are identically distributed show that

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[X_1] = 0. \qquad (7)$$

This implies that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[Y_i] = 0$. Now observe that $Y_i = Z_i + \mathbb{E}[Y_i]$. Therefore,

Q7: Claim that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = 0. \qquad (8)$$

Now we will show that

$$
\begin{aligned}
\sum_{i=1}^{\infty} P(|X_i| \geq i) &= \sum_{i=1}^{\infty} \int_{i-1}^{i} P(|X_i| \geq i) dx \\
&\dots \\
&\dots \\
&= \mathbb{E}\left[|X_1|\right] < \infty. \qquad (9)
\end{aligned}
$$

Q8: Fill the missing part in the above proof. Hint: Use Lemma 1.

(It is important to note that in Eq. (9) we are bounding the summation $\sum_{i=1}^{\infty} P(|X_i| \geq i)$ by the expectation of the modulus of X_1, the first r.v. of the sequence.)

Q9: Now using Eq. (9) show that

$$
\sum_{i=1}^{\infty} P(X_i \neq Y_i) < \infty. \qquad (10)
$$

Hint: Rewrite the above summation using the summation in Eq. (9).

Therefore by the Borel–Cantelli lemma, except for a set of probability zero, $X_i = Y_i$, for all i greater than some positive integer N (N depends on ω). Henceforth,

Q10: Using Eq. (8) and the above claim (Borel-Cantelli claim), show that

$$
\lim_{n\to\infty} \left| \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} Y_i \right| = 0 \text{ a.s.} \qquad (11)
$$

Q11: Conclude that the proof is complete. □