Assignment I

January 27, 2019

Notation

$\mathbb{N}$: Set of natural numbers, *i.e.*, $\mathbb{N} := \{1, 2, 3, \dots\}$.
$\mathbb{Z}$: Set of intergers, *i.e.*, $\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
$\mathbb{R}$: Set of real numbers.
P: Set of positive reals, *i.e.*, $P := \{x \in \mathbb{R} | x > 0\}$.

Problems

1. Prove the following:
   (a) If $x \in P$, then $x^{-1} \in P$. (Also show this for $x \in -P$, *i.e.*, If $x \in -P$, then $x^{-1} \in -P$).
   (b) If $x, y \in P$ or $x, y \in -P$, then we have $x > y$ iff $y^{-1} > x^{-1}$.
2. Prove that if $a, b \in \mathbb{N}$ and $a > b$, then $a - b \in \mathbb{N}$.
3. Prove the following:
   (a) if $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{Z}$.
   (b) $a \in \mathbb{N}$ iff $a \in \mathbb{Z}$ and $a > 0$.
4. For $x > 0$, there exists an $n \in \mathbb{N}$ s.t. $n - 1 \leq x < n$.
5. Prove the following:
   (a) $|x| = |-x|$.
   (b) For $y > 0$, we have $-y \leq x \leq y$ iff $|x| \leq y$.
6. Prove that a subset of a countable set is countable. (Reading assignment. The solution is provided in the lecture notes)
7. Prove that for $x > 0$, we have $\sqrt{x}$ exists. ($\sqrt{x}$ is defined as $\sqrt{x} = y \in \mathbb{R}$ s.t. $y^2 = x$)
8. Prove that for $A \subset \mathbb{R}$ with A bounded above, we have $\sup A = -\inf(-A)$.

9. Find the supremum and infimum of the set

$$A := \{\frac{m}{n} : m, n \in \mathbb{N}, m < 2n\}.$$

10. Find the limit of the sequence $a_n = \frac{n}{2^n}$, $n \in \mathbb{N}$.

11. Show the following:

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}.$$

Hint: Note that $1 - \frac{1}{n} = \frac{n-1}{n}$. Now set $m = n + 1$ and rewrite the limit equation in terms of m. Then use the result from Exercise 5(3) from the lecture notes.