

Assignment 2

April 9, 2019

Here we consider the probability space (Ω, \mathcal{F}, P) , where \mathcal{F} is the σ -field, Ω is the sample space and P is the probability measure.

We will prove the strong law of large numbers. We state the theorem first

Theorem 1. *If $\{X_i\}_{i \in \mathbb{N}}$ is an IID sequence and $\mathbb{E}[|X_1|] < \infty$. Then*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \mathbb{E}[X_1] \text{ a.s.,}$$

where $S_n = \sum_{i=1}^n X_i$.

We will use the following fact in this proof.

Lemma 1. *Given $p > 0$ and X a non-negative r.v., i.e., $X \geq 0$, we have*

$$\mathbb{E}[X^p] = \int_0^\infty px^{p-1}P(X > x)dx. \quad (1)$$

Proof of Theorem 1:

Proof. We may assume $\mathbb{E}[X_i] = 0$. Otherwise, one can replace X_i with $X_i - \mathbb{E}[X_i]$. This implies that we have to prove that $\frac{1}{n} \sum_{i=1}^n X_i$ converges to 0.

Let

$$Y_n = X_n I_{\{|X_n| \leq n\}} \text{ and } Z_n = Y_n - \mathbb{E}[Y_n]. \quad (2)$$

We will break up the proof into parts.

Part 1: We first prove $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0$.

Define

$$M_n = \sum_{i=1}^n \frac{Z_i}{i} \quad (3)$$

with $M_0 = 0$. Define the filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ where $\mathcal{F}_n := \sigma(X_1, X_2 \dots X_n)$.

Q1: Prove that $\{M_n\}$ is a martingale w.r.t. the filtration \mathcal{F}_n .

Now we will show that $\mathbb{E}[M_n^2]$ is bounded by a constant not dependent on n . For that, consider

$$\begin{aligned} \mathbb{E}[M_n^2] &= \text{Var}(M_n) = \sum_{i=1}^n \frac{\text{Var}(Z_i)}{i^2} = \sum_{i=1}^n \frac{1}{i^2} \text{Var}(Y_i) \\ &\leq \sum_{i=1}^n \frac{1}{i^2} \mathbb{E}[Y_i^2] \leq \sum_{i=1}^n \frac{1}{i^2} \int_0^\infty 2yP(|X_i| \geq y)dy \quad (\text{from Lemma 1}) \\ &= 2 \sum_{i=1}^\infty \frac{1}{i^2} \int_0^\infty I_{\{y \leq i\}} y P(|X_i| \geq y) dy \\ &\quad \dots \\ &\quad \dots \\ &\leq c \mathbb{E}[|X_1|], \quad \text{where } c > 0. \end{aligned} \tag{4}$$

Q2: Fill in the missing details (First use the fact that $\{X_i\}$ is IID which implies that X_i and X_1 have the same distribution. Second use Dominated Convergence Theorem to take the summation inside the integral. Then use the fact that $\sum_{i=1}^\infty \frac{1}{i^2} I_{\{y \leq i\}} \leq \frac{1}{y}$).

Q3: Claim that $\mathbb{E}[M_n^2]$ is bounded by a constant not dependent on n using Eq. (4).

Q4: Further claim that the martingale $\{M_n\}$ converges.

Now note that from Eqs. (2) and (3), we have

$$Z_i = i(M_i - M_{i-1}). \tag{5}$$

This implies that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n Z_i &= \frac{1}{n} \sum_{i=1}^n i(M_i - M_{i-1}) = \frac{1}{n} \left(\sum_{i=1}^n i M_i - \sum_{i=1}^{n-1} (i+1) M_i \right) \\ &= M_n - \frac{n-1}{n} \left(\frac{1}{n-1} \sum_{i=1}^{n-1} M_i \right) \end{aligned} \tag{6}$$

Q5: Now finally show that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0$.

Part 2: We next show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Y_i = 0. \tag{7}$$

Q6: Using dominated convergence theorem and the fact that X_n are identically distributed show that

$$\lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = \mathbb{E}[X_1] = 0. \quad (8)$$

This implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i] = 0$. Now observe that $Y_i = Z_i + \mathbb{E}[Y_i]$.

Q7: Finally, show (7).

Part 3: We next show that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n Y_i \right| = 0 \text{ a.s.} \quad (9)$$

First show that

$$\begin{aligned} \sum_{i=1}^{\infty} P(|X_i| \geq i) &= \sum_{i=1}^{\infty} \int_{i-1}^i P(|X_i| \geq i) dx \\ &\dots \\ &\dots \\ &= \mathbb{E}[|X_1|] < \infty. \end{aligned} \quad (10)$$

Q8: Fill the missing part in the above proof. Hint: Use Lemma 1.

(It is important to note that in Eq. (10) we are bounding the summation $\sum_{i=1}^{\infty} P(|X_i| \geq i)$ by the expectation of the modulus of X_1 , the first r.v. of the sequence.)

Q9: Now using Eq. (10) show that

$$\sum_{i=1}^{\infty} P(X_i \neq Y_i) < \infty. \quad (11)$$

Hint: Rewrite the above summation using the summation in Eq. (10).

Therefore by the Borel–Cantelli lemma, except for a set of probability zero, $X_i = Y_i$, for all i greater than some positive integer N (N depends on ω). Henceforth,

Q10: Using Eq. (7) and the above claim (Borel-Cantelli claim), show (9).

Q11: Finally, conclude that the proof is complete. \square