

Probability Theory

March 14, 2019

1 Notation

\mathbb{R}	: Real line.
\mathbb{R}^*	: Extended real line, <i>i.e.</i> , $\mathbb{R}^* := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.
\mathbb{R}_+^*	: Non-negative extended real line, <i>i.e.</i> , $\mathbb{R}_+^* := \{r \in \mathbb{R}^*; r \geq 0\}$.
$(a_n) \uparrow a$, for $a_n, a \in \mathbb{R}^*$: (a_n) is a monotonically increasing real (extended) sequence (<i>i.e.</i> , $a_{n+1} \geq a_n, \forall n$) and (a_n) converges to a .
$(f_n) \uparrow f$, for $f, f_n : \Omega \rightarrow \mathbb{R}^*$: (f_n) is a monotonically increasing real (extended) valued function sequence (<i>i.e.</i> , $f_{n+1}(\omega) \geq f_n(\omega), \omega \in \Omega$) and (f_n) converges to f , <i>i.e.</i> , $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega), \forall \omega \in \Omega$.
I_A	: Indicator function, <i>i.e.</i> , $I_A = 1$ if $\omega \in A$ and $I_A = 0$ otherwise.
$f_1 \wedge f_2$, for $f_1, f_2 : \Omega \rightarrow \mathbb{R}^*$: $f \wedge f_2$ is a function from Ω to \mathbb{R}^* defined as $(f_1 \wedge f_2)(\omega) = \min \{f_1(\omega), f_2(\omega)\}$.
$f_1 \vee f_2$, for $f_1, f_2 : \Omega \rightarrow \mathbb{R}^*$: $f \vee f_2$ is a function from Ω to \mathbb{R}^* defined as $(f_1 \vee f_2)(\omega) = \max \{f_1(\omega), f_2(\omega)\}$.

2 Probability space

Definition: The 3-tuple (Ω, \mathcal{F}, P) is called a probability space, where

1. Ω is a set called the sample space.
2. \mathcal{F} is a σ -field.

Definition of σ -field: \mathcal{F} is a non-empty collection of subsets of Ω which satisfies

- (S1) $\Omega \in \mathcal{F}$.
- (S2) If $A \subseteq \Omega$ and $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
- (S3) If each set in the collection $\{A_n; n \in \mathbb{N}\}$ belongs to \mathcal{F} , *i.e.*, $A_n \in \mathcal{F}$, $\forall n \in \mathbb{N}$ (not necessarily disjoint), then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Note that $A \subseteq \Omega$ is called \mathcal{F} -set if $A \in \mathcal{F}$.

3. P is a probability measure.

Definition of probability measure: $P : \mathcal{F} \rightarrow [0, 1]$ is called a probability measure if it satisfies:

(M1) $P(\Omega) = 1$ and $P(\emptyset) = 0$.

(M2) If $\{A_n\}_{n \in \mathbb{N}}$ is a disjoint collection of \mathcal{F} -sets, i.e., $A_k \cap A_j = \emptyset$, for $k \neq j$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n). \quad (1)$$

This property is called the *countable additivity of the probability measure*.

In other words, P is a set function (i.e., P takes sets in \mathcal{F} to real values in $[0, 1]$) which satisfies M1 and M2.

Remark 1. A similar concept to countable additivity is the finite additivity which is defined as follows: If $\{A_i; 1 \leq i \leq n\}$ is a finite collection of disjoint \mathcal{F} -sets, then $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$. Note that countable additivity implies finite additivity. Indeed, by considering the countable collection $\{B_i; i \in \mathbb{N}\}$, where $B_1 = A_1, \dots, B_n = A_n$, and $B_k = \emptyset$, for $k > n$, the claim follows.

Remark 2. A more generalized set function is the notion of measure. A measure $\mu : \mathcal{F} \rightarrow \mathbb{R}_+^*$ (contrary to the probability measure where the range of P is contained in $[0, 1]$) which satisfies $\mu(\emptyset) = 0$ (need not satisfy $\mu(\Omega) = 1$) and countable additivity (M2). Thus, probability measure is a measure with the additional condition that $P(\Omega) = 1$.

Lemma 1. If A and B are \mathcal{F} -sets with $A \subseteq B$, then $P(A) \leq P(B)$. Also, $P(B \setminus A) = P(B) - P(A)$.

Proof. Note that since $A \subseteq B$, we have $B = A \cup (B \setminus A)$ and, A and $B \setminus A$ are disjoint. Now, by the finite additivity of P , we have

$$\begin{aligned} P(B) &= P(A) + \underbrace{P(B \setminus A)}_{\geq 0} \\ &\Rightarrow P(B) \geq P(A). \end{aligned} \quad (2)$$

This proves the first part. The second part follows from Eq. (2). \square

Lemma 2. If A is an \mathcal{F} -set, then $P(A^c) = 1 - P(A)$.

Proof. Note that $A \cup A^c = \Omega$. Also, A and A^c are disjoint. Therefore by the finite additivity property of P and M1, we have

$$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c).$$

Hence, the claim follows. \square

2.1 Limit of sets

Definition: (Liminf of a sequence of sets) Given a sequence of sets $(A_n)_{n \in \mathbb{N}}$, where $A_n \subseteq \Omega$, we define

$$\liminf_n A_n = \bigcup_n \bigcap_{k \geq n} A_k. \quad (3)$$

Definition: (Limsup of a sequence of sets) Given a sequence of sets $(A_n)_{n \in \mathbb{N}}$, where $A_n \subseteq \Omega$, we define

$$\limsup_n A_n = \bigcap_n \bigcup_{k \geq n} A_k. \quad (4)$$

Definition: (Limit of a sequence of sets) We say the limit of the sequence of sets $(A_n)_{n \in \mathbb{N}}$ exists if $\liminf_n A_n = \limsup_n A_n$ and the $\lim_n A_n$ is that common set.

We will consider specific sequences here

2.1.1 Monotonically increasing sequence of sets

Definition: A sequence $(A_n)_{n \in \mathbb{N}}$ is called *monotonically increasing* sequence if $A_n \subseteq A_{n+1}$, $\forall n \in \mathbb{N}$.

In this case, note that for $n \in \mathbb{N}$,

$$\bigcap_{k \geq n} A_k = A_n, \text{ since } A_n \subseteq A_{n+1} \subseteq A_{n+2} \dots$$

Therefore,

$$\liminf_n A_n = \bigcup_n \bigcap_{k \geq n} A_k = \bigcup_n A_n. \quad (5)$$

Note that for $n > 1$, since $A_1 \subseteq A_2 \dots \subseteq A_{n-1} \subseteq A_n$, we have

$$\bigcup_{k=1}^n A_k = A_n \Rightarrow \bigcup_{k \geq n} A_k = \bigcup_{k \geq 1} A_k \quad (6)$$

Therefore,

$$\limsup_n A_n = \bigcap_n \bigcup_{k \geq n} A_k = \bigcap_n \bigcup_{k \geq 1} A_k = \bigcup_{k \geq 1} A_k. \quad (7)$$

Therefore, by the definition of $\lim_n A_n$, we have

$$\lim_n A_n = \bigcup_n A_n. \quad (8)$$

The next question is what happens to the probability of the monotonically increasing sets A_n when each A_n is an \mathcal{F} -set. Indeed, we are considering the

real sequence $(P(A_n))_{n \in \mathbb{N}}$. The real sequence $(P(A_n))_{n \in \mathbb{N}}$ is bounded since $0 \leq P(A_n) \leq 1$, $\forall n \in \mathbb{N}$. Also since the set sequence $(A_n)_{n \in \mathbb{N}}$ is monotonically increasing, we have, for $n \in \mathbb{N}$,

$$A_{n+1} \supseteq A_n \Rightarrow P(A_{n+1}) \geq P(A_n), \text{ (follows from Lemma 1)}$$

Therefore, the real sequence $(P(A_n))_{n \in \mathbb{N}}$ is a monotonically increasing bounded sequence. Hence it should converge. But where does it converges to?

Theorem 1. *If $(A_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence of \mathcal{F} -sets, then*

$$\lim_{n \rightarrow \infty} P(A_n) = P(\lim_n A_n) = P\left(\bigcup_n A_n\right). \quad (9)$$

Proof. Let $A_0 = \emptyset$. Now set

$$\begin{aligned} B_1 &:= A_1 \setminus A_0; \\ B_2 &:= A_2 \setminus A_1; \\ &\vdots \\ B_n &:= A_n \setminus A_{n-1}; \\ &\vdots \end{aligned}$$

Now note that the set sequence $(B_n)_{n \in \mathbb{N}}$ is a disjoint sequence, *i.e.*, $B_i \cap B_j = \emptyset$, for $i \neq j$. Also,

$$\bigcup_n B_n = \bigcup_n A_n. \quad (10)$$

Therefore, from Eq. (10) and the fact that the set sequence $(A_n)_{n \in \mathbb{N}}$ is monotonically increasing, we have

$$\begin{aligned} P(\lim_n A_n) &= P\left(\bigcup_n A_n\right) = P\left(\bigcup_n B_n\right) \\ &= \sum_{n \in \mathbb{N}} P(B_n) \text{ (follows from M2)} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(B_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(A_i) - P(A_{i-1}) \text{ (follows from Lemma 1)} \\ &= \lim_{n \rightarrow \infty} P(A_n) - \underbrace{P(A_0)}_{=0} \\ &= \lim_{n \rightarrow \infty} P(A_n). \end{aligned}$$

□

Remark 3. *Note that in the proof of the above theorem, we never used the condition $P(\Omega) = 1$ of the probability measure. This implies that the above result also holds for any measure on Ω .*

2.1.2 Monotonically decreasing sequence of sets

Definition: A sequence $(A_n)_{n \in \mathbb{N}}$ is called *monotonically decreasing* sequence if $A_{n+1} \subseteq A_n, \forall n \in \mathbb{N}$.

In this case, note that for $n \in \mathbb{N}$,

$$\bigcup_{k \geq n} A_k = A_n, \text{ since } A_n \supseteq A_{n+1} \supseteq A_{n+2} \dots$$

Therefore,

$$\limsup_n A_n = \bigcap_n \bigcup_{k \geq n} A_k = \bigcap_n A_n. \quad (11)$$

Note that for $n > 1$, since $A_1 \supseteq A_2 \dots \supseteq A_{n-1} \supseteq A_n$, we have

$$\bigcap_{k=1}^n A_k = A_n \Rightarrow \bigcap_{k \geq n} A_k = \bigcap_{k \geq 1} A_k \quad (12)$$

Therefore,

$$\liminf_n A_n = \bigcup_n \bigcap_{k \geq n} A_k = \bigcup_n \bigcap_{k \geq 1} A_k = \bigcap_{k \geq 1} A_k. \quad (13)$$

Therefore, by the definition of $\lim_n A_n$, we have

$$\lim_n A_n = \bigcap_{n \geq 1} A_n. \quad (14)$$

What happens to the probability of the monotonically decreasing sets A_n when each A_n is an \mathcal{F} -set. Here also, the real sequence $(P(A_n))_{n \in \mathbb{N}}$ is bounded since $0 \leq P(A_n) \leq 1, \forall n \in \mathbb{N}$. Also since the set sequence $(A_n)_{n \in \mathbb{N}}$ is monotonically decreasing, we have, for $n \in \mathbb{N}$,

$$A_{n+1} \subseteq A_n \Rightarrow P(A_{n+1}) \leq P(A_n), \text{ (follows from Lemma 1)}$$

Therefore, the real sequence $(P(A_n))_{n \in \mathbb{N}}$ is a monotonically decreasing bounded sequence. Hence it should converge.

Theorem 2. If $(A_n)_{n \in \mathbb{N}}$ is a monotonically decreasing sequence of \mathcal{F} -sets, then

$$\lim_{n \rightarrow \infty} P(A_n) = P(\lim_n A_n) = P\left(\bigcap_{n \geq 1} A_n\right). \quad (15)$$

Proof. Since $(A_n)_{n \in \mathbb{N}}$ is a monotonically decreasing sequence of \mathcal{F} -sets, we have $(A_n^c)_{n \in \mathbb{N}}$ to be a monotonically increasing sequence of \mathcal{F} -sets. This follows from S2.

Now from Theorem 1, we know that

$$\lim_{n \rightarrow \infty} P(A_n^c) = P(\lim_n A_n^c) = P(\bigcup_n A_n^c) \quad (16)$$

However, note that $\bigcup_n A_n^c = (\bigcap_n A_n)^c$. Therefore from Lemma 2 and Eq. (16), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} 1 - P(A_n) &= 1 - P(\bigcap_n A_n) \\ \Leftrightarrow 1 - \lim_{n \rightarrow \infty} P(A_n) &= 1 - P(\bigcap_n A_n) \\ \Leftrightarrow \lim_{n \rightarrow \infty} P(A_n) &= P(\bigcap_n A_n). \end{aligned}$$

□

3 Random variables

Definition: (Borel σ -field) The smallest σ -field on \mathbb{R}^* containing intervals. Recall that intervals are of the form (a, b) , $[a, b]$, $[a, b)$, $(a, b]$, where $a, b \in \mathbb{R}^*$ and $a \leq b$.

Remark 4. *The definition is indeed well-defined. Note that given a collection C of subsets of \mathbb{R}^* , one can ask what is the smallest σ -field containing C . We denote such a sigma field as $\sigma(C)$. Indeed, one can obtain $\sigma(C)$ as follows. Consider the new collection $\mathcal{G} := \{\mathcal{H} \text{ s.t. } \mathcal{H} \text{ is a } \sigma\text{-field and } C \subseteq \mathcal{H}\}$. Note that this is a collection of σ -fields. Is \mathcal{G} non-empty? YES - since the power set of \mathbb{R}^* itself is a σ -field and it contains C . Hence the power set belongs to \mathcal{G} . Now it is easy to verify that*

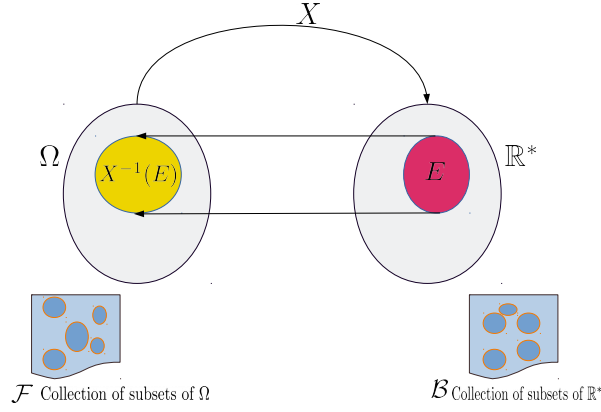
$$\sigma(C) = \bigcap_{\mathcal{H} \in \mathcal{G}} \mathcal{H}. \quad (17)$$

Definition: (Random variable) A function $X : \Omega \rightarrow \mathbb{R}^*$ is called a *random variable (r.v.)* if $X^{-1}(B) \in \mathcal{F}$, for every $B \in \mathcal{B}$. Here, $X^{-1}(B)$ is defined as follows: for $B \subseteq \mathbb{R}^*$,

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}. \quad (18)$$

By the above it is hard to verify whether a function $X : \omega \rightarrow \mathbb{R}^*$ is a r.v. since we don't know the sets inside \mathcal{B} . However, we do know that the intervals are inside \mathcal{B} . However, the following claim reduces this effort by providing a sufficient condition.

Theorem 3. *If $X^{-1}([-\infty, a]) \in \mathcal{F}$, $\forall a \in \mathbb{R}^*$, then X is a r.v.*



Proof. Given that $X^{-1}([-\infty, a]) \in \mathcal{F}$, $\forall a \in \mathbb{R}^*$, we have to show that X is a r.v. Define

$$\mathcal{C} := \{B \subseteq \mathbb{R}^* | X^{-1}(B) \in \mathcal{F}\} \quad (19)$$

If we can show that $\mathcal{B} \subseteq \mathcal{C}$ we are done. Because if so then for every $E \in \mathcal{B}$, we have $X^{-1}(E) \in \mathcal{F}$ (by definition of \mathcal{C}). To do so we show that \mathcal{C} is a σ -field containing intervals. Since \mathcal{B} (the Borel σ -field) is the smallest σ -field containing intervals, we have $\mathcal{B} \subseteq \mathcal{C}$.

Part 1: To show that \mathcal{C} contains intervals

From the hypothesis we know that $[-\infty, a] \in \mathcal{C}$, $\forall a \in \mathbb{R}$. Now note that for $b \in \mathbb{R}^*$, we have

$$[-\infty, b) = \bigcup_{n \in \mathbb{N}} [-\infty, b - \frac{1}{n}]. \quad (20)$$

Therefore,

$$\begin{aligned} X^{-1}([-\infty, b)) &= X^{-1}\left(\bigcup_{n \in \mathbb{N}} [-\infty, b - \frac{1}{n}]\right) \\ &= \bigcup_{n \in \mathbb{N}} \underbrace{X^{-1}([-\infty, b - \frac{1}{n}])}_{\substack{\in \mathcal{F} \text{ by hypothesis} \\ \in \mathcal{F} \text{ by countable union}}} \end{aligned} \quad (21)$$

• This implies that $[-\infty, b) \in \mathcal{C}$, $\forall b \in \mathbb{R}^*$.

Now note that

$$X^{-1}((b, +\infty]) = X^{-1}([-\infty, b]^c) = \underbrace{(X^{-1}([-\infty, b]))^c}_{\in \mathcal{F} \text{ since } X^{-1}([-\infty, b]) \in \mathcal{F} \text{ by hypothesis}} \quad (22)$$

• This implies that $(b, +\infty] \in \mathcal{C}$, $\forall b \in \mathbb{R}^*$.

Also, note that

$$X^{-1}([b, +\infty]) = X^{-1}([-\infty, b)^c) = \underbrace{(X^{-1}([-\infty, b]))^c}_{\in \mathcal{F} \text{ since } X^{-1}([-\infty, b]) \in \mathcal{F} \text{ by Eq. (21)}} \quad (23)$$

• This implies that $(b, +\infty] \in \mathcal{C}, \forall b \in \mathbb{R}^*$.

Further, for $a, b \in \mathbb{R}^*, a < b$, we have

$$(a, b) = (a, +\infty] \cap [-\infty, b) \Rightarrow X^{-1}((a, b)) = \underbrace{X^{-1}((a, +\infty])}_{\in \mathcal{F} \text{ Eq. (22)}} \cap \underbrace{X^{-1}([-\infty, b))}_{\in \mathcal{F} \text{ Eq. (21)}}. \quad (24)$$

• This implies that $(a, b) \in \mathcal{C}, \forall a, \forall b \in \mathbb{R}^*, a < b$. (24)

• Similarly, $[a, b), [a, b], (a, b] \in \mathcal{C}, \forall a, \forall b \in \mathbb{R}^*, a < b$. (25)

Part 2: To show that \mathcal{C} is a σ -field over \mathbb{R}^*

Note that $X^{-1}(\mathbb{R}^*) = \Omega \in \mathcal{F}$. Therefore,

$$\mathbb{R}^* \in \mathcal{C}. \quad (26)$$

If $A \in \mathcal{C}$, then $X^{-1}(A) \in \mathcal{F}$. Therefore,

$$\begin{aligned} X^{-1}(A^c) &= (X^{-1}(A))^c \in \mathcal{F} \\ \Rightarrow A^c &\in \mathcal{C}. \end{aligned} \quad (27)$$

Given a countable collection $\{A_n\}_{n \in \mathbb{N}}$ with $A_n \in \mathcal{C}, \forall n \in \mathbb{N}$ (which implies that $X^{-1}(A_n) \in \mathcal{F}, \forall n$ by the definition of \mathcal{C}), we have

$$\begin{aligned} X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \bigcup_{n=1}^{\infty} \underbrace{X^{-1}(A_n)}_{\in \mathcal{F}} \\ &\underbrace{\hspace{10em}}_{\in \mathcal{F}} \\ \Rightarrow \bigcup_{n=1}^{\infty} A_n &\in \mathcal{C}. \end{aligned} \quad (28)$$

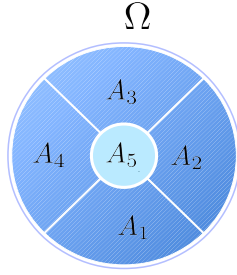
Therefore, \mathcal{C} is a σ -field over \mathbb{R}^* . □

We now consider the simplest of random variables.

3.1 Non-negative simple functions

Definition: We call a finite collection $\{A_i\}_{i=1}^n$ an \mathcal{F} -partition of Ω if

1. Each $A_i \in \mathcal{F}$.
2. A_i 's are disjoint (i.e., $A_k \cap A_t = \emptyset$, if $k \neq t$) and
3. $\bigcup_{i=1}^n A_i = \Omega$ (i.e. their union gives the entire set Ω).



Definition: A function $s : \Omega \rightarrow \mathbb{R}_+^*$ is called a non-negative simple function if it has the form

$$s(\omega) = \sum_{i=1}^n a_i I_{A_i}(\omega), \text{ where } a_i \in \mathbb{R}_+^*, 1 \leq \forall i \leq n. \quad (29)$$

Note that s is a *r.v.* To see that, lets assume that $a_1 < a_2 < a_3 < \dots < a_n$ (if not, then re-number). Then

$$s^{-1}([-\infty, a]) = \begin{cases} \emptyset, & \text{if } a < a_1. \\ A_1, & \text{if } a_1 \leq a < a_2. \\ A_1 \cup A_2, & \text{if } a_2 \leq a < a_3. \\ A_1 \cup A_2 \cup A_3, & \text{if } a_3 \leq a < a_4. \\ \vdots & \\ \Omega, & \text{if } a \geq a_n. \end{cases}$$

Thus $s^{-1}([-\infty, a]) \in \mathcal{F}, \forall a \in \mathbb{R}^*$. Therefore s is a *r.v.*

We denote by \mathbb{L}_0^+ the collection of non-negative simple functions.

$$\mathbb{L}_0^+ := \{s : \Omega \rightarrow \mathbb{R}_+^* | s \text{ is a non-negative simple function}\}. \quad (30)$$

Properties:

Proposition 1. If $s_1, s_2 \in \mathbb{L}_0^+$, then

1. $s_1 + s_2 \in \mathbb{L}_0^+$ and $s_1 s_2 \in \mathbb{L}_0^+$.
2. $cs_1 \in \mathbb{L}_0^+$, for $c \in \mathbb{R}_+^*$.
3. $s_1 \wedge s_2 \in \mathbb{L}_0^+$.
4. $s_1 \vee s_2 \in \mathbb{L}_0^+$.

Proof. Let

$$s_1 = \sum_{i=1}^n a_i I_{A_i} \text{ and } s_2 = \sum_{j=1}^m b_j I_{B_j}.$$

1. It is easy to verify that $\{A_i \cap B_j; 1 \leq i \leq n, 1 \leq j \leq m\}$ is a \mathcal{F} -partition. Then

$$s_1 + s_2 = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) I_{A_i \cap B_j}. \quad (31)$$

To justify this claim, note that

For $\omega \in \Omega \Rightarrow \omega \in A_i$ and $\omega \in B_j$, for some $i, j, 1 \leq i \leq n, 1 \leq j \leq m$,

since $\{A_i\}, \{B_j\}$ are \mathcal{F} -partitions.

$$\Leftrightarrow \omega \in A_i \cap B_j$$

$$\Leftrightarrow s_1(\omega) = a_i \text{ and } s_2(\omega) = b_j \text{ with } \omega \in A_i \cap B_j$$

$$\Leftrightarrow (s_1 + s_2)(\omega) = s_1(\omega) + s_2(\omega) = a_i + b_j, \text{ with } \omega \in A_i \cap B_j$$

$$\Leftrightarrow s_1 + s_2 = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) I_{A_i \cap B_j}.$$

Therefore $s_1 + s_2 \in \mathbb{L}_0^+$.

2. Similarly, $s_1 s_2 \in \mathbb{L}_0^+$ with

$$s_1 s_2 = \sum_{i=1}^n \sum_{j=1}^m a_i b_j I_{A_i \cap B_j}. \quad (32)$$

3. Also, for $c \in \mathbb{R}_+^*$, $cs_1 \in \mathbb{L}_0^+$ with

$$cs_1 = \sum_{i=1}^n \sum_{j=1}^m ca_i I_{A_i}. \quad (33)$$

4. $s_1 \wedge s_2 \in \mathbb{L}_0^+$ with

$$s_1 \wedge s_2 = \sum_{i=1}^n \sum_{j=1}^m \min\{a_i, b_j\} I_{A_i \cap B_j}. \quad (34)$$

5. $s_1 \vee s_2 \in \mathbb{L}_0^+$ with

$$s_1 \vee s_2 = \sum_{i=1}^n \sum_{j=1}^m \max\{a_i, b_j\} I_{A_i \cap B_j}. \quad (35)$$

□

The simple functions even though are simple are not that simple. They are strong enough to approximate any non-negative $r.v.$

Theorem 4. *If X is a non-negative r.v., then there exists a sequence (s_n) , where $s_n \in \mathbb{L}_0^+$ s.t. $s_n \uparrow X$. This means that for each $\omega \in \Omega$, we have $(s_n(\omega))$ is a monotonically increasing sequence and $\lim_{n \rightarrow \infty} s_n(\omega) = X(\omega)$.*

Proof. We will create the sequence (s_n) as follows: Let

$$E_{n,k} := \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right), 1 \leq k \leq n2^n \text{ and } E_{n,\infty} = [n, +\infty]. \quad (36)$$

Also, let

$$A_{n,k} := X^{-1}(E_{n,k}), 1 \leq k \leq n2^n \text{ and } A_{n,\infty} = X^{-1}(E_{n,\infty}). \quad (37)$$

Define

$$s_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{A_{n,k}} + n I_{A_{n,\infty}}. \quad (38)$$

It is easy to verify that $s_n \in \mathbb{L}_0^+$ since $\{A_{n,k}, 1 \leq k \leq n2^n; A_{n,\infty}\}$ is an \mathcal{F} -partition.

It is also easy to verify from Fig. 2 that

$$s_{n+1}(\omega) \geq s_n(\omega), \forall \omega \in \Omega. \quad (39)$$

Now we will verify that $\lim_{n \rightarrow \infty} s_n(\omega) = X(\omega)$, $\forall \omega \in \Omega$.

For $\omega \in \Omega$, there are two cases possible

1) Either $\omega \in A_{n,k}$ for some $1 \leq k \leq n2^n$. In this case,

$$\begin{aligned} s_n(\omega) &= \frac{k-1}{2^n} \text{ and } X(\omega) \in E_{n,k} \\ \Rightarrow \frac{k-1}{2^n} &\leq X(\omega) < \frac{k}{2^n} \\ \Rightarrow \frac{k-1}{2^n} - \frac{k-1}{2^n} &\leq X(\omega) - s_n(\omega) < \frac{k}{2^n} - \frac{k-1}{2^n} \\ \Rightarrow 0 &\leq X(\omega) - s_n(\omega) < \frac{1}{2^n}. \\ \Rightarrow \lim_{n \rightarrow \infty} s_n(\omega) &= X(\omega) \text{ (by squeeze theorem).} \end{aligned}$$

2) Or $\omega \in A_{n,\infty}$. In this case, we have

$$\begin{aligned} s_n(\omega) &= n \text{ and } X(\omega) \in [n, +\infty] \\ \Rightarrow s_n(\omega) &= n \text{ and } X(\omega) \geq n. \end{aligned}$$

Hence, we cannot obtain the bound similar to the earlier case. However, one can consider two sub-cases here: 1) If $X(\omega) < +\infty$. In this case, by the Archimedean theorem, there exists an $N \in \mathbb{N}$ s.t. $N > X(\omega)$. Therefore, $\forall n \geq N$, we have the bound

$$\Rightarrow 0 \leq X(\omega) - s_n(\omega) < \frac{1}{2^n}. \quad (40)$$

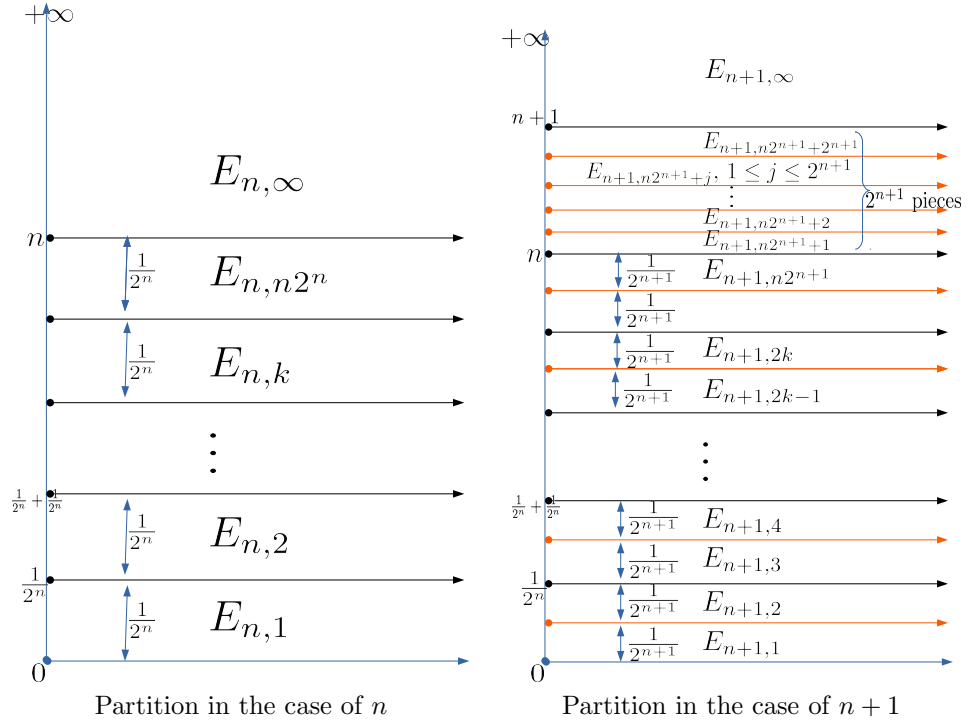


Figure 1: Partitions

Therefore, $\lim_{n \rightarrow \infty} s_n(\omega) = X(\omega)$, by squeeze theorem.

2) If $X(\omega) = +\infty$. In this case, we have $s_n(\omega) = n$. Therefore,

$$\lim_{n \rightarrow \infty} s_n(\omega) = +\infty = X(\omega).$$

Thus, we have addressed every possible scenario. Therefore,

$$\lim_{n \rightarrow \infty} s_n(\omega) = X(\omega), \forall \omega \in \Omega. \quad (41)$$

□

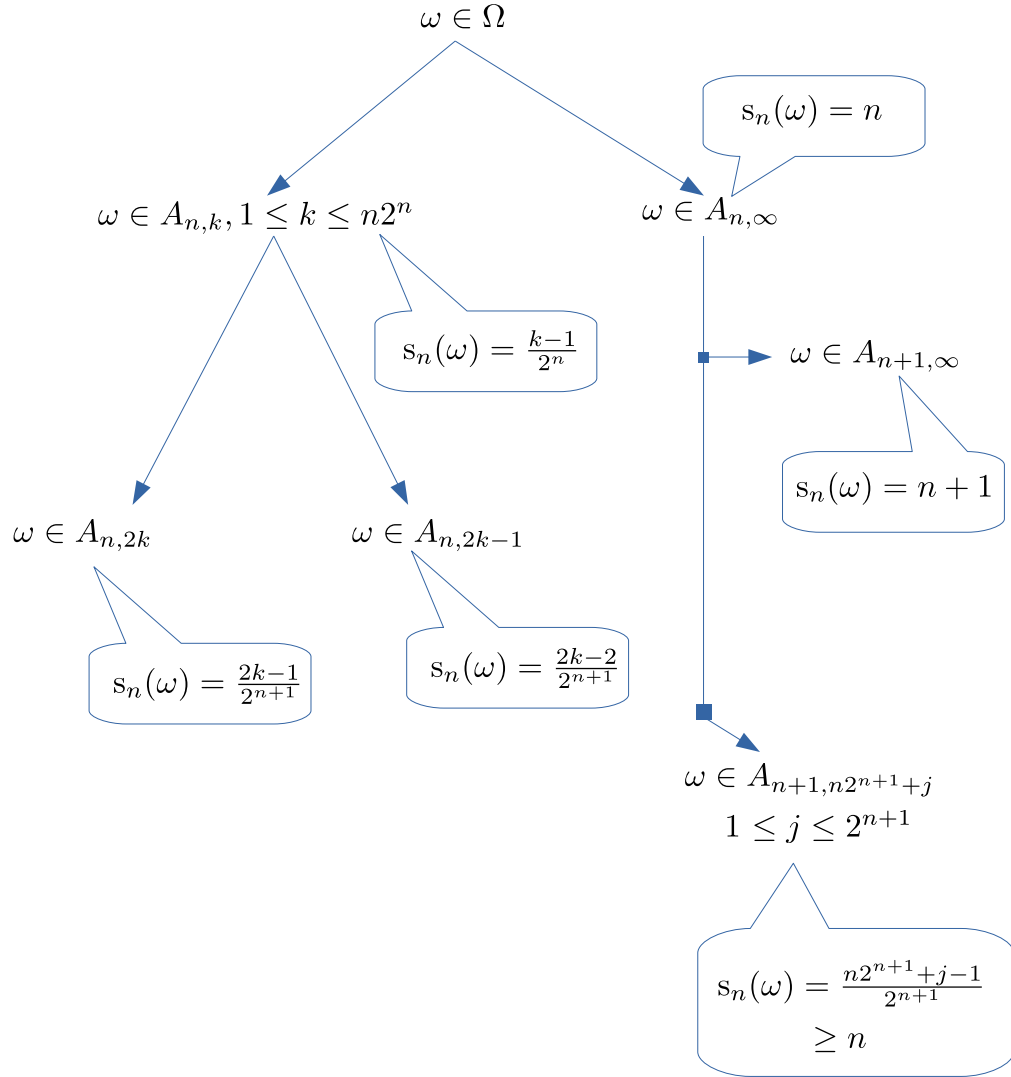
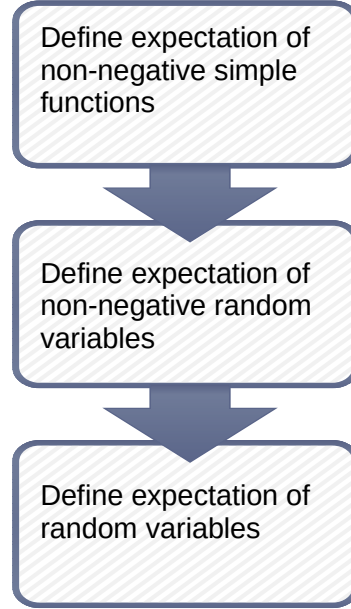


Figure 2: Illustration to show that $s_{n+1} \geq s_n$

4 Expectation of a random variable

Goal:



4.1 Expectation of non-negative simple functions

We first define the expectation of the non-negative simple functions as follows: For $s \in \mathbb{L}_0^+$ with $s = \sum_{i=1}^n a_i I_{A_i}$, ($\{A_i\}$ is an \mathcal{F} -partition and $a_i \in \mathbb{R}_+^*$), we define

$$\mathbb{E}[s] = \sum_{i=1}^n a_i P(A_i). \quad (42)$$

Properties of expectation of non-negative simple functions

Theorem 5. For $s_1, s_2 \in \mathbb{L}_0^+$ with $s_1 = \sum_{i=1}^n a_i I_{A_i}$ and $s_2 = \sum_{j=1}^m b_j I_{B_j}$, ($\{A_i; 1 \leq i \leq n\}$ and $\{B_j; 1 \leq j \leq m\}$ are \mathcal{F} -partitions and $a_i, b_j \in \mathbb{R}_+^*$), we have

1. $\mathbb{E}[s_1] \geq 0$.
2. $\mathbb{E}[s_1 + s_2] = \mathbb{E}[s_1] + \mathbb{E}[s_2]$.
3. For $c \in \mathbb{R}_+^*$, $\mathbb{E}[cs_1] = c\mathbb{E}[s_1]$.
4. If $s_1 \geq s_2$, then $\mathbb{E}[s_1] \geq \mathbb{E}[s_2]$. (Note that $s_1 \geq s_2$ means that $s_1(\omega) \geq s_2(\omega), \forall \omega \in \Omega$)

Proof. 1.

$$\begin{aligned}\mathbb{E}[s_1] &= \sum_{i=1}^n \underbrace{a_i}_{\geq 0} \underbrace{P(A_i)}_{\geq 0} \\ &\geq 0.\end{aligned}$$

2. We know that

$$\begin{aligned}s_1 + s_2 &= \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) I_{A_i \cap B_j} \\ \Rightarrow \mathbb{E}[s_1 + s_2] &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) I_{A_i \cap B_j} \right] \\ &= \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) P(A_i \cap B_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i P(A_i \cap B_j) + \sum_{i=1}^n \sum_{j=1}^m b_j P(A_i \cap B_j) \\ &= \sum_{i=1}^n a_i P(A_i \cap (\cup_{j=1}^m B_j)) + \sum_{j=1}^m b_j P((\cup_{i=1}^n A_i) \cap B_j) \quad (\text{by M2}) \\ &= \sum_{i=1}^n a_i P(A_i \cap \Omega) + \sum_{j=1}^m b_j P(\Omega \cap B_j) \\ &= \sum_{i=1}^n a_i P(A_i) + \sum_{j=1}^m b_j P(B_j) \\ &= \mathbb{E}[s_1] + \mathbb{E}[s_2].\end{aligned}$$

3. Again,

$$\begin{aligned}cs_1 &= \sum_{i=1}^n ca_i I_{A_i} \\ \Rightarrow \mathbb{E}[cs_1] &= \sum_{i=1}^n ca_i P(A_i) = c \sum_{i=1}^n a_i P(A_i) = c\mathbb{E}[s_1].\end{aligned}$$

4. For $s_1 \geq s_2$, we have

$$\begin{aligned}
\mathbb{E}[s_1] &= \mathbb{E} \left[\sum_{i=1}^n a_i I_{A_i} \right] \\
&= \sum_{i=1}^n a_i P(A_i \cap \Omega) \\
&= \sum_{i=1}^n a_i P(A_i \cap (\cup_{j=1}^m B_j)) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i P(A_i \cap B_j) \quad (\text{by M2}) \\
&= \sum_{i=1}^n \sum_{\substack{j=1 \\ A_i \cap B_j \neq \emptyset}}^m a_i P(A_i \cap B_j) \\
&\geq \sum_{i=1}^n \sum_{\substack{j=1 \\ A_i \cap B_j \neq \emptyset}}^m b_j P(A_i \cap B_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m b_j P(A_i \cap B_j) \\
&= \mathbb{E}[s_2].
\end{aligned}$$

□

We denote by $\{X > Y\} := \{\omega \in \Omega : X(\omega) > Y(\omega)\}$. Similarly we define $\{X \geq Y\}$, $\{X = Y\}$, $\{X < Y\}$ and $\{X \leq Y\}$.

Proposition 2. *Given r.v's X and Y , we have $\{X > Y\} \in \mathcal{F}$, $\{X \geq Y\} \in \mathcal{F}$, $\{X = Y\} \in \mathcal{F}$, $\{X \neq Y\} \in \mathcal{F}$, $\{X < Y\} \in \mathcal{F}$ and $\{X \leq Y\} \in \mathcal{F}$.*

Proof. Note that

$$\begin{aligned}
\{X > Y\} &= \bigcup_{q \in \mathbb{Q}} \{\omega \in \Omega : X(\omega) < q < Y(\omega)\} \\
&= \bigcup_{q \in \mathbb{Q}} \underbrace{X^{-1}([-\infty, q))}_{\in \mathcal{F}} \cap \underbrace{Y^{-1}((q, +\infty])}_{\in \mathcal{F}} \in \mathcal{F}.
\end{aligned}$$

Also,

$$\{X < Y\} = \{Y > X\} \in \mathcal{F} \quad (\text{follows from the previous case}).$$

Similarly,

$$\{X \neq Y\} = \underbrace{\{X > Y\}}_{\in \mathcal{F}} \cup \underbrace{\{X < Y\}}_{\in \mathcal{F}} \in \mathcal{F}.$$

Also,

$$\{X = Y\} = \{X \neq Y\}^c \in \mathcal{F} \text{ (since } \{X \neq Y\} \in \mathcal{F}\text{)}.$$

Also,

$$\{X \geq Y\} = \underbrace{\{X > Y\}}_{\in \mathcal{F}} \cup \underbrace{\{X = Y\}}_{\in \mathcal{F}} \in \mathcal{F}.$$

Similarly, we can show $\{X \leq Y\} \in \mathcal{F}$. □

Proposition 3. *If X, Y are r.v.s (not necessarily non-negative), then*

1. *For $c \in \mathbb{R}$, both cX and $X + c$ are r.v.s.*
2. *$X + Y$ is a r.v.*
3. *XY is a r.v.*

Proof. 1. There are 3 cases to consider:

- (i) $c = 0$. Then $cX = 0$ is a r.v.
- (ii) $c > 0$. Then, for $a \in \mathbb{R}^*$, we have

$$(cX)^{-1}([-\infty, a]) = X^{-1}[-\infty, \frac{a}{c}] \in \mathcal{F}.$$

- (iii) $c < 0$. Then, for $a \in \mathbb{R}^*$, we have

$$(cX)^{-1}([-\infty, a]) = X^{-1}[\frac{a}{c}, +\infty] \in \mathcal{F}.$$

Therefore cX is a r.v.

The case of $X + c$ can be shown similarly.

2. Note that $(X+Y)(\omega) = X(\omega) + Y(\omega)$. Since X and Y can take infinity as it values, one cannot define $(X+Y)(\omega)$ in cases where $X(\omega) = +\infty, Y(\omega) = -\infty$ and $X(\omega) = -\infty, Y(\omega) = +\infty$. Let's define

$$\begin{aligned} A = & \{\omega \in \Omega : X(\omega) = -\infty \text{ and } Y(\omega) = +\infty\} \cup \\ & \{\omega \in \Omega : X(\omega) = +\infty \text{ and } Y(\omega) = -\infty\}. \end{aligned} \quad (43)$$

Therefore we define $X + Y$ as follows:

$$(X + Y)(\omega) = \begin{cases} X(\omega) + Y(\omega), & \text{if } \omega \in A^c \\ \beta, & \text{if } \omega \in A, \text{ where } \beta \in \mathbb{R}^*. \end{cases} \quad (44)$$

To show that $X+Y$ is a r.v., we have to show that $(X+Y)^{-1}([-\infty, a]) \in \mathcal{F}$, $\forall a \in \mathbb{R}^*$. To verify that, note that

$$\begin{aligned} (X + Y)^{-1}([-\infty, a]) &= \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \\ &= \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap (A \cup A^c) \\ &= \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap A \end{aligned} \quad (45)$$

$$\begin{aligned} &\cup \\ &\{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap A^c \end{aligned} \quad (46)$$

We treat Parts (45) and (46) separately. We will show that (45) $\in \mathcal{F}$ and (46) $\in \mathcal{F}$.

$$\begin{aligned}
(45) &\Rightarrow \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap A \\
&= \{\omega \in A : (X + Y)(\omega) \leq a\} \\
&= \begin{cases} \emptyset, & \text{if } a < \beta, \\ A, & \text{if } a \geq \beta. \end{cases} \\
&\in \mathcal{F}
\end{aligned} \tag{47}$$

For Part (46), there are 3 cases to consider.

(i) $a \in \mathbb{R}$: In this case, we have

$$\begin{aligned}
(46) &\Rightarrow \{\omega \in A^c : (X + Y)(\omega) \leq a\} \\
&= \{\omega \in A^c : X(\omega) + Y(\omega) \leq a\} \\
&= \{\omega \in A^c : X(\omega) \leq a - Y(\omega)\} \\
&= \{X \leq a - Y\} \cap A^c \in \mathcal{F} \text{ follows from Proposition 2.}
\end{aligned} \tag{48}$$

(ii) $a = +\infty$: In this case, we have

$$\begin{aligned}
(46) &\Rightarrow \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap A^c \\
&= \Omega \cap A^c \\
&= A^c \in \mathcal{F}.
\end{aligned} \tag{49}$$

(iii) $a = -\infty$: In this case, we have

$$\begin{aligned}
(46) &\Rightarrow \{\omega \in \Omega : (X + Y)(\omega) \leq a\} \cap A^c \\
&= \{\omega \in \Omega : (X + Y)(\omega) \leq -\infty\} \cap A^c \\
&= \{\omega \in \Omega : (X + Y)(\omega) = -\infty\} \cap A^c \\
&= \{\omega \in A^c : (X + Y)(\omega) = -\infty\} \\
&= \{\omega \in A^c : X(\omega) + Y(\omega) = -\infty\} \\
&= (\{X = -\infty\} \cup \{Y = -\infty\}) \cap A^c \\
&\in \mathcal{F}.
\end{aligned} \tag{50}$$

3. Left as exercise. □

For $s \in \mathbb{L}_0^+$ with $s = \sum_{i=1}^n a_i I_{A_i}$, we say that *coefficients of s take non-infinity values* if $a_i \in \mathbb{R}_+$, $\forall i, 1 \leq i \leq n$. This means that none of a_i take infinity.

Proposition 4. For $s \in \mathbb{L}_0^+$, define

$$\mu(A) := \mathbb{E}[sI_A], A \in \mathcal{F}.$$

Then μ is a measure.

Proof. Let $s = \sum_{i=1}^n a_i I_{A_i}$, To show μ is a measure, we have to show two properties

1. $\mu(\emptyset) = 0$.
2. If $\{B_k\}_{k \in \mathbb{N}}$ is a disjoint collection of \mathcal{F} -sets, then $\mu(\bigcup_k B_k) = \sum_k \mu(B_k)$.

For the former case, note that

$$\mu(\emptyset) = \mathbb{E}[sI_{\emptyset}] = 0 \text{ since } (sI_{\emptyset})(\omega) = s(\omega)I_{\emptyset}(\omega) = 0, \forall \omega \in \Omega.$$

For the latter case, let $B^* = \bigcup_k B_k$. Now note that

$$\begin{aligned} \mu(B^*) &= \mathbb{E}[sI_{B^*}] = \sum_{i=1}^n a_i P(A_i \cap B^*) = \sum_{i=1}^n a_i P\left(A_i \cap \left(\bigcup_k B_k\right)\right) \\ &= \sum_{i=1}^n a_i P\left(\bigcup_k (A_i \cap B_k)\right) \\ &= \sum_{i=1}^n \sum_k a_i P(A_i \cap B_k) \\ &= \sum_k \sum_{i=1}^n a_i P(A_i \cap B_k) \\ &= \sum_k \mu(B_k). \end{aligned}$$

Therefore μ is a measure. \square

Lemma 3. Let $(s_n) \uparrow s$, where $s_n, s \in \mathbb{L}_0^+$ with the coefficients of s taking non-infinity values. Then $(\mathbb{E}[s_n]) \uparrow \mathbb{E}[s]$

Proof. Since $(s_n) \uparrow s$, we have

$$s_n \leq s \Rightarrow \mathbb{E}[s_n] \leq \mathbb{E}[s].$$

$$\text{Also, } s_{n+1} \geq s_n \Rightarrow \mathbb{E}[s_{n+1}] \geq \mathbb{E}[s_n]$$

Therefore the real sequence $(\mathbb{E}[s_n])$ is a monotonically increasing sequence bounded by $\mathbb{E}[s]$. Therefore

$$\lim_{n \rightarrow \infty} \mathbb{E}[s_n] \leq \mathbb{E}[s]. \quad (51)$$

For $0 < c < 1$, consider

$$B_n = \{\omega \in \Omega : s_n(\omega) \geq cs(\omega)\}. \quad (52)$$

Note that (B_n) is a monotonically increasing sequence of \mathcal{F} -sets. Indeed, B_n is an \mathcal{F} -set follows from Proposition 2. To see that it is monotonically increasing note that

$$\begin{aligned} \omega \in B_n &\Rightarrow s_{n+1}(\omega) \geq s_n(\omega) \geq cs(\omega) \Rightarrow \omega \in B_{n+1}. \\ &\Rightarrow B_n \subseteq B_{n+1}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} B_n = \bigcup_n B_n. \quad (53)$$

Since the coefficients of s are finite, we have $cs(\omega) < +\infty$, $\forall \omega \in \Omega$. Also, since $0 < c < 1$, we have $cs(\omega) < s(\omega)$, $\forall \omega \in \Omega$. This implies that for each $\omega \in \Omega$, there exists an $N_\omega \in \mathbb{N}$ (depending on ω) s.t. $s_n(\omega) > cs(\omega)$, $\forall n \geq N_\omega$. This implies that $\omega \in B_n$, $\forall n \geq N_\omega$. This further implies that

$$\bigcup_n B_n = \Omega. \quad (54)$$

Now we will build certain inequalities here:

1. Define $\nu(A) = \mathbb{E}[csI_A]$, $A \in \mathcal{F}$. From Proposition 4, we have $\nu : \mathcal{F} \rightarrow \mathbb{R}_+^*$ is a measure. Therefore, by Remark 3, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu(B_n) &= \nu\left(\bigcup_n B_n\right) = \nu(\Omega) \\ \parallel & \parallel \\ \lim_{n \rightarrow \infty} \mathbb{E}[csI_{B_n}] &= \mathbb{E}[csI_\Omega] = \mathbb{E}[cs] = c\mathbb{E}[s]. \end{aligned} \quad (55)$$

- 2.

$$\begin{aligned} s_n I_{B_n} \leq s_{n+1} I_{B_{n+1}} &\Rightarrow \mathbb{E}[s_n I_{B_n}] \leq \mathbb{E}[s_{n+1} I_{B_{n+1}}] \\ &\Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[s_n I_{B_n}] \text{ exists.} \end{aligned} \quad (56)$$

- 3.

$$\begin{aligned} cs I_{B_n} \leq s_n I_{B_n} &\Rightarrow \mathbb{E}[cs I_{B_n}] \leq \mathbb{E}[s_n I_{B_n}] \\ &\Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[cs I_{B_n}] \leq \lim_{n \rightarrow \infty} \mathbb{E}[s_n I_{B_n}]. \end{aligned} \quad (57)$$

- 4.

$$\begin{aligned} s_n I_{B_n} \leq s_n &\Rightarrow \mathbb{E}[s_n I_{B_n}] \leq \mathbb{E}[s_n] \\ &\Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[s_n I_{B_n}] \leq \lim_{n \rightarrow \infty} \mathbb{E}[s_n]. \end{aligned} \quad (58)$$

From Eqs: (55, 57, 58), we get

$$c\mathbb{E}[s] \leq \lim_{n \rightarrow \infty} \mathbb{E}[s_n]. \quad (59)$$

Note that the above inequality holds for any $0 < c < 1$. Therefore,

$$\mathbb{E}[s] \leq \lim_{n \rightarrow \infty} \mathbb{E}[s_n]. \quad (60)$$

Therefore, from Eqs: (51, 60), we have

$$\lim_{n \rightarrow \infty} \mathbb{E}[s_n] = \mathbb{E}[s].$$

□

4.2 Expectation of non-negative random variables

We define the expectation of a non-negative *r.v.* in the following manner:

Definition: For the non-negative *r.v.* $X : \Omega \rightarrow \mathbb{R}_+^*$, we define

$$\mathbb{E}[X] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n], \text{ where } (s_n) \uparrow X \text{ with } s_n \in \mathbb{L}_0^+ \text{ and } s_n \text{ having non-infinity coefficients.} \quad (61)$$

We have to show that the above definition is well-defined. When I say well-defined, it means that there should not be any scope for ambiguity. Of course we know from Theorem 4 that there exists $(s_n) \uparrow X$, where $s_n \in \mathbb{L}_0^+$ with non-infinity coefficients. Therefore, the existence of $\mathbb{E}[X]$ is guaranteed. But the ambiguity is in its uniqueness. Because one can ask if $(s_n) \uparrow X$ and $(s'_n) \uparrow X$ be two different sequences monotonically converging to X , then whether $\lim_n \mathbb{E}[s_n]$ and $\lim_n \mathbb{E}[s'_n]$ are the same. If they do, then the uniqueness is also guaranteed.

Theorem 6. *Let X be a non-negative *r.v.* Let (s_n) and (s'_n) be two distinct non-negative simple function sequences (with non-infinity coefficients) monotonically increasing to X . Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[s_n] = \lim_{n \rightarrow \infty} \mathbb{E}[s'_n].$$

Proof. Note that

$$s_n \geq s_n \wedge s'_m \quad (62)$$

Now consider the sequence $(s_n \wedge s'_m)_{n \in \mathbb{N}}$. Note that

$$\lim_{n \rightarrow \infty} s_n \wedge s'_m = s'_m. \quad (63)$$

To see this, consider the pointwise convergence, *i.e.*, for each $\omega \in \Omega$, observe the real (extended) sequence

$$((s_n \wedge s'_m)(\omega))_{n \in \mathbb{N}} = (s_n(\omega) \wedge s'_m(\omega))_{n \in \mathbb{N}} = (\min\{s_n(\omega), s'_m(\omega)\})_{n \in \mathbb{N}}. \quad (64)$$

Here the sequence is running over n keeping m fixed. Since $s'_m(\omega) \leq X(\omega)$, there are two possibilities to consider:

1. $s'_m(\omega) < X(\omega)$: In this case, observe that since $(s_n(\omega))$ is monotonically increasing to $X(\omega)$, there exists an $N \in \mathbb{N}$ s.t. $s_n(\omega) > s'_m(\omega)$, $\forall n \geq N$. From Eq. (64), we have

$$\begin{aligned} ((s_n \wedge s'_m)(\omega))_{n \in \mathbb{N}} &= (s'_m(\omega)), \forall n \geq N \\ \lim_{n \rightarrow \infty} (s_n \wedge s'_m)(\omega) &= s'_m(\omega). \end{aligned}$$

2. $s'_m(\omega) = X(\omega)$: Since $(s_n(\omega)) \uparrow X(\omega)$, we have $s_n(\omega) \leq X(\omega) = s'_m(\omega)$, $\forall n$. Now from Eq. (64), we have

$$\begin{aligned} ((s_n \wedge s'_m)(\omega))_{n \in \mathbb{N}} &= (s_n(\omega) \wedge s'_m(\omega))_{n \in \mathbb{N}} = (s_n(\omega) \wedge X(\omega))_{n \in \mathbb{N}} = (s_n(\omega))_{n \in \mathbb{N}} \\ \Rightarrow \lim_{n \rightarrow \infty} (s_n \wedge s'_m)(\omega) &= \lim_{n \rightarrow \infty} s_n(\omega) = X(\omega) = s'_m(\omega). \end{aligned}$$

This proves Eq. (63).

Now from Eqs: (62-63) and Lemma 3, we get

$$\begin{aligned} \mathbb{E}[s_n] &\geq \mathbb{E}[s_n \wedge s'_m] \\ \Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[s_n] &\geq \lim_{n \rightarrow \infty} \mathbb{E}[s_n \wedge s'_m] = \mathbb{E}[s'_m] \\ \Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[s_n] &\geq \lim_{m \rightarrow \infty} \mathbb{E}[s'_m]. \end{aligned} \quad (65)$$

The above inequality is obtained starting from Eq. (62). Now instead of Eq. (62), if we start with the following inequality $s'_m \geq s_n \wedge s'_m$, then we get

$$\lim_{m \rightarrow \infty} \mathbb{E}[s'_m] \geq \lim_{n \rightarrow \infty} \mathbb{E}[s_n]. \quad (66)$$

Therefore, from Eqs:(65, 66), we get

$$\lim_{m \rightarrow \infty} \mathbb{E}[s'_m] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n].$$

□

Remark 5. Another definition of $\mathbb{E}[X]$ (where X is a non-negative r.v.) commonly found in textbook is the following:

$$\mathbb{E}[X] = \sup \{ \mathbb{E}[s] : s \leq X, s \in \mathbb{L}_0^+ \}. \quad (67)$$

Properties of expectation of non-negative r.v.

Proposition 5. For X_1, X_2 are non-negative r.v.s, we have

1. $\mathbb{E}[X_1] \geq 0$.
2. $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$.
3. $\mathbb{E}[cX_1] = c\mathbb{E}[X_1]$, $c \geq 0$.
4. If $X_1 \leq X_2$, then $\mathbb{E}[X_1] \leq \mathbb{E}[X_2]$.

Proof. 1. Since X_1 is a non-negative r.v., there exists a $(s_n^1) \uparrow X_1$ (follows from Theorem 4) and

$$\mathbb{E}[X_1] = \lim_{n \rightarrow \infty} \underbrace{\mathbb{E}[s_n^1]}_{\geq 0 \text{ since } s_n \in \mathbb{L}_0^+} \geq 0.$$

2. Since X_1, X_2 are non-negative r.v.s, there exist $(s_n^1) \uparrow X_1$ and $(s_n^2) \uparrow X_2$ and

$$\mathbb{E}[X_1] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n^1] \text{ and } \mathbb{E}[X_2] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n^2].$$

Therefore, $(s_n^1 + s_n^2) \uparrow X_1 + X_2$, with $s_n^1 + s_n^2 \in \mathbb{L}_0^+$ (follows from Proposition 1) and

$$\begin{aligned} \mathbb{E}[X_1 + X_2] &= \lim_{n \rightarrow \infty} \mathbb{E}[s_n^1 + s_n^2] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n^1] + \mathbb{E}[s_n^2] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[s_n^1] + \lim_{n \rightarrow \infty} \mathbb{E}[s_n^2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]. \end{aligned}$$

3. Since $(s_n^1) \uparrow X_1$, we have $(cs_n^1) \uparrow cX_1$ with $cs_n^1 \in \mathbb{L}_0^+$ and

$$\mathbb{E}[cX_1] = \lim_{n \rightarrow \infty} \mathbb{E}[cs_n^1] = \lim_{n \rightarrow \infty} c\mathbb{E}[s_n^1] = c\mathbb{E}[X_1].$$

4. We use the characterization of $\mathbb{E}[\cdot]$ provided in Remark 5. Note that since $X_1 \leq X_2$, we have,

For $s \in \mathbb{L}_0^+$, if $s \leq X_1$ then $s \leq X_2$.

Therefore, from Eq. (67), we have $\mathbb{E}[X_1] \leq \mathbb{E}[X_2]$.

□

4.3 Expectation of random variable (which takes both non-negative and negative values)

To define the expectation of a *r.v.* which takes both non-negative and negative values, we represent the *r.v.* as the difference of two non-negative *r.v.s*. Since we have already defined the expectation of non-negative *r.v.s* in the previous section, we can thus define the expectation of a *r.v.* which takes both negative and non-negative values as the difference of expectations of the two non-negative components.

For a random variable $X : \omega \rightarrow \mathbb{R}^*$, we define

$$X^+(\omega) = \begin{cases} X(\omega), & \text{if } X(\omega) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (68)$$

Similarly, we define

$$X^-(\omega) = \begin{cases} -X(\omega), & \text{if } X(\omega) < 0, \\ 0, & \text{otherwise.} \end{cases} \quad (69)$$

Observe that for $\omega \in \Omega$, $X(\omega) = X^+(\omega) - X^-(\omega)$ and $|X(\omega)| = X^+(\omega) + X^-(\omega)$. Therefore,

$$X = X^+ - X^- \text{ and } |X| = X^+ + X^-. \quad (70)$$

Note that,

$$(X^+)^{-1}([-\infty, a]) = \begin{cases} \underbrace{\emptyset}_{\in \mathcal{F}}, & \text{if } a < 0, \\ \underbrace{X^{-1}([-\infty, a])}_{\in \mathcal{F}} & \text{if } a \geq 0. \end{cases}$$

Therefore, X^+ is a non-negative *r.v.*. Similarly, X^- is a non-negative *r.v.*. (This further implies $|X|$ is also a non-negative *r.v.*) Hence, we can talk about $\mathbb{E}[X^+]$

and $\mathbb{E}[X^-]$. We want to define $\mathbb{E}[X]$ as $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$. But the problem with this definition is that if $\mathbb{E}[X^+] = +\infty$ and $\mathbb{E}[X^-] = +\infty$, then $\mathbb{E}[X] = +\infty - +\infty$ which is not defined. So we separate out these two situations.

Definition: For a *r.v.* X , we say $\mathbb{E}[X]$ exists, if $\mathbb{E}[X^+] < +\infty$ and $\mathbb{E}[X^-] < +\infty$. In this case, we define

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]. \quad (71)$$

It follows directly from the above definition that

$$\mathbb{E}[|X|] = \mathbb{E}[X^+ + X^-] = \mathbb{E}[X^+] + \mathbb{E}[X^-] < +\infty.$$

Therefore, another way to say the same thing is

Definition: For a *r.v.* X , we say $\mathbb{E}[X]$ exists, if $\mathbb{E}[|X|] < +\infty$.

Proposition 6. For *r.v.s* X and Y which are integrable, we have

1. $X + Y$ is integrable and $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
2. cX is integrable for $c \in \mathbb{R}$ and $\mathbb{E}[cX] = c\mathbb{E}[X]$.

Proof. 1. Note that

$$\begin{aligned} |X + Y| \leq |X| + |Y| &\Rightarrow \mathbb{E}[|X + Y|] \leq \mathbb{E}[|X| + |Y|] \\ &= \underbrace{\mathbb{E}[|X|]}_{< +\infty} + \underbrace{\mathbb{E}[|Y|]}_{< +\infty} < +\infty. \end{aligned}$$

Therefore $X + Y$ is integrable.

Now note that

$$X + Y = (X + Y)^+ - (X + Y)^- \text{ and} \quad (72)$$

$$X = X^+ - X^-; \quad Y = Y^+ - Y^-. \quad (73)$$

Combining the above two equations we get

$$\begin{aligned} X^+ - X^- + Y^+ - Y^- &= (X + Y)^+ - (X + Y)^- \\ \Rightarrow X^+ + Y^+ + (X + Y)^- &= (X + Y)^+ + X^- + Y^-. \end{aligned}$$

Now taking expectation on both sides (this is possible since all the terms in the LHS are non-negative *r.v.s* and similarly on the RHS).

$$\begin{aligned} \mathbb{E}[X^+ + Y^+ + (X + Y)^-] &= \mathbb{E}[(X + Y)^+ + X^- + Y^-] \\ \Rightarrow \mathbb{E}[X^+] + \mathbb{E}[Y^+] + \mathbb{E}[(X + Y)^-] &= \mathbb{E}[(X + Y)^+] + \mathbb{E}[X^-] + \mathbb{E}[Y^-] \\ &\quad \text{(follows from Proposition 5(2))} \\ \Rightarrow \mathbb{E}[X^+] - \mathbb{E}[X^-] + \mathbb{E}[Y^+] - \mathbb{E}[Y^-] &= \mathbb{E}[(X + Y)^+] - \mathbb{E}[(X + Y)^-] \\ \Rightarrow \mathbb{E}[X] + \mathbb{E}[Y] &= \mathbb{E}[X + Y]. \end{aligned}$$

2. Left as exercise. □

4.4 Convergence theorems

Proposition 7. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of r.v.s (not necessarily non-negative), then $\inf_n X_n$, $\sup_n X_n$, $\liminf_n X_n$ and $\limsup_n X_n$ are r.v.s. Additionally, if $\lim_n X_n$ exists, then it is also a r.v.*

Proof. (i) We will first show that $\inf_n X_n$ is a r.v.. For that, consider for $a \in \mathbb{R}^*$,

$$\begin{aligned} (\inf_n X_n)^{-1}([a, +\infty]) &= \{\omega \in \Omega : (\inf_n X_n)(\omega) \geq a\} \\ &= \{\omega \in \Omega : \inf_n X_n(\omega) \geq a\} \\ &= \{\omega \in \Omega : X_n(\omega) \geq a, \forall n\} \\ &= \bigcap_n X_n^{-1}([a, +\infty]) \in \mathcal{F} \text{ (follows since each } X_n \text{ is a r.v.)} \end{aligned}$$

Therefore $\inf_n X_n$ is a r.v..

(ii) Now consider for $a \in \mathbb{R}^*$,

$$\begin{aligned} (\sup_n X_n)^{-1}([-\infty, a]) &= \{\omega \in \Omega : (\sup_n X_n)(\omega) \leq a\} \\ &= \{\omega \in \Omega : \sup_n X_n(\omega) \leq a\} \\ &= \{\omega \in \Omega : X_n(\omega) \leq a, \forall n\} \\ &= \bigcap_n X_n^{-1}([-\infty, a]) \in \mathcal{F} \text{ (follows since each } X_n \text{ is a r.v.)} \end{aligned}$$

Therefore $\sup_n X_n$ is a r.v..

(iii) Now note that

$$\liminf_n X_n = \sup_k \inf_{n \geq k} X_n \quad (74)$$

Let $Y_k = \inf_{n \geq k} X_n$. Then, $\liminf_n X_n = \sup_k Y_k$. We know that Y_k is a r.v. from part (i) of the proof. Therefore, it follows from part (ii) of the proof that $\liminf_n X_n$ is a r.v.

Also,

$$\limsup_n X_n = \inf_k \sup_{n \geq k} X_n \quad (75)$$

Again by the same argument as above, we can show that $\limsup_n X_n$ is a r.v.

(iv) Now if $\lim_n X_n$ exists, then

$$\lim_n X_n = \liminf_n X_n = \limsup_n X_n \quad (76)$$

Therefore, since $\liminf_n X_n$ is a r.v., we have $\lim_n X_n$ is also a r.v. \square

Theorem 7 (Montone Convergence Theorem (MCT)). *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of monotonically increasing non-negative r.v.s. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E} \left[\lim_{n \rightarrow \infty} X_n \right].$$

**Note that we can talk about $\mathbb{E}[\lim_n X_n]$ since $\lim_n X_n$ is a r.v. by Proposition 7.*

Proof. Since $(X_n)_{n \in \mathbb{N}}$ is monotonically increasing, we have $\lim_{n \rightarrow \infty} X_n$ exists and is a r.v.. Let

$$\lim_{n \rightarrow \infty} X_n = X. \quad (77)$$

Further, note that

$$X_n \leq X \Rightarrow \mathbb{E}[X_n] \leq \mathbb{E}[X], \forall n$$

Also, $X_{n+1} \geq X_n \Rightarrow \mathbb{E}[X_{n+1}] \geq \mathbb{E}[X_n]$ which implies that $(\mathbb{E}[X_n])_{n \in \mathbb{N}}$ is monotonically increasing. Therefore, $\lim_{n \rightarrow \infty} \mathbb{E}[X_n]$ exists and

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \mathbb{E}[X]. \quad (78)$$

Also, since X_n is a non-negative r.v. for each $n \in \mathbb{N}$, we have $(s_m^n) \uparrow X_n$, where $s_m^n \in \mathbb{L}_0^+$ (with non-infinity coefficients) and

$$\lim_{m \rightarrow \infty} \mathbb{E}[s_m^n] = \mathbb{E}[X_n]. \quad (79)$$

Now define

$$Y_m = s_m^1 \vee s_m^2 \vee s_m^3 \cdots \vee s_m^m. \quad (80)$$

(See notation section for the definition of \vee)

Note that, for $\omega \in \Omega$, we have

$$\begin{aligned} Y_{m+1}(\omega) &= s_{m+1}^1(\omega) \vee s_{m+1}^2(\omega) \cdots \vee s_{m+1}^m(\omega) \vee s_{m+1}^{m+1}(\omega) \\ &\geq s_m^1(\omega) \vee s_m^2(\omega) \cdots \vee s_m^m(\omega) = Y_m(\omega). \end{aligned}$$

Also, $Y_m \in \mathbb{L}_0^+$ (follows from Proposition 1). Thus $(Y_m)_{m \in \mathbb{N}}$ is a monotonically increasing sequence of non-negative simple functions (with non-infinity coefficients). Therefore $\lim_{m \rightarrow \infty} Y_m$ exists and let

$$Y = \lim_{m \rightarrow \infty} Y_m \text{ and } \mathbb{E}[Y] = \lim_{m \rightarrow \infty} \mathbb{E}[Y_m]. \quad (81)$$

$$\begin{array}{ccccccc}
s_1^1 & s_2^1 & s_3^1 & \dots & s_m^1 & \dots & \rightarrow X_1 \\
s_1^2 & s_2^2 & s_3^2 & \dots & s_m^2 & \dots & \rightarrow X_2 \\
\cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\
\cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\
s_1^m & s_2^m & s_3^m & \dots & s_m^m & \dots & \rightarrow X_m \\
\cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\
\cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot
\end{array}$$

$$\begin{array}{ccccccc}
& & & & & & \downarrow \\
& & & & & & X \\
Y_1 & Y_2 & Y_3 & \dots & Y_m & \dots & \rightarrow Y
\end{array}$$

Also from Eq. (80), for $\omega \in \Omega$, we have

$$\begin{aligned}
Y_m(\omega) &= s_m^1(\omega) \vee s_m^2(\omega) \cdots \vee s_m^m(\omega) \\
&\leq X_1(\omega) \vee X_2(\omega) \cdots \vee X_m(\omega) \\
&= X_m(\omega).
\end{aligned} \tag{82}$$

Taking limits on both sides, we get

$$\begin{aligned}
\lim_{m \rightarrow \infty} Y_m(\omega) &\leq \lim_{m \rightarrow \infty} X_m(\omega) \\
\Rightarrow Y(\omega) &\leq X(\omega).
\end{aligned} \tag{83}$$

Therefore,

$$Y \leq X. \tag{84}$$

Also, note that, for $\omega \in \Omega$,

$$\begin{aligned}
s_m^k(\omega) &\leq Y_m(\omega), \text{ for } 1 \leq k \leq m \\
\Rightarrow \lim_{m \rightarrow \infty} s_m^k(\omega) &\leq \lim_{m \rightarrow \infty} Y_m(\omega), \text{ for } 1 \leq k \leq m \\
\Rightarrow X_k(\omega) &\leq Y(\omega), \text{ for } 1 \leq k \leq m \\
\Rightarrow X_m(\omega) &\leq Y(\omega) \\
\Rightarrow \lim_{m \rightarrow \infty} X_m(\omega) &\leq Y(\omega) \\
\Rightarrow X(\omega) &\leq Y(\omega).
\end{aligned}$$

Therefore

$$X \leq Y. \tag{85}$$

Hence, from Eqs. (84, 85), we have

$$X = Y. \tag{86}$$

Thus

$$\begin{aligned} \mathbb{E}[X] &\stackrel{(86)}{=} \underbrace{\mathbb{E}[Y]}_{(81)} = \lim_{n \rightarrow \infty} \mathbb{E}[Y_n] \stackrel{(82)}{\leq} \lim_{n \rightarrow \infty} \underbrace{\mathbb{E}[X_n]}_{(78)} \leq \mathbb{E}[X]. \\ \Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[X_n] &= \mathbb{E}[X]. \end{aligned}$$

□

Lemma 4 (Fatou's Lemma). *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of non-negative r.v.s. Then*

$$\liminf_n \mathbb{E}[X_n] \geq \mathbb{E}\left[\liminf_n X_n\right].$$

Proof. Let

$$Y_n = \inf_{k \geq n} X_k \quad (87)$$

Note that Y_n is a r.v. (follows from Proposition 7). Also, it is easy to verify that $Y_{n+1} \geq Y_n$, $\forall n \in \mathbb{N}$. Therefore, $(Y_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence of r.v.s. Further Y_n is non-negative for all $n \in \mathbb{N}$, since X_n is non-negative for all n . Also

$$\lim_{n \rightarrow \infty} Y_n = \liminf_n X_n. \quad (88)$$

Also it is easy to verify that $Y_n \leq X_n$, $\forall n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \mathbb{E}[Y_n] &\leq \mathbb{E}[X_n], \forall n \in \mathbb{N} \\ \Rightarrow \liminf_n \mathbb{E}[Y_n] &\leq \liminf_n \mathbb{E}[X_n] \\ \Rightarrow \lim_n \mathbb{E}[Y_n] &\leq \liminf_n \mathbb{E}[X_n] \end{aligned} \quad (89)$$

The last implication follows since $\lim_n \mathbb{E}[Y_n]$ exists since $(\mathbb{E}[Y_n])_{n \in \mathbb{N}}$ is a monotonically increasing real (extended) sequence.

Now by applying MCT to the sequence $(Y_n)_{n \in \mathbb{N}}$ we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[Y_n] &= \mathbb{E}\left[\lim_{n \rightarrow \infty} Y_n\right] \\ \Rightarrow \liminf_n \mathbb{E}[X_n] &\geq \mathbb{E}\left[\liminf_n X_n\right] \text{ (follow from Eqs. (88, 89)).} \end{aligned}$$

□

Theorem 8 (Bounded Convergence Theorem (BCT)). *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of r.v.s (not necessarily non-negative). Assume that there exists an integrable r.v. Y such that $|X_n| \leq Y$, $\forall n \in \mathbb{N}$. Also let $\lim_{n \rightarrow \infty} X_n = X$. Then*

1. X is integrable.

$$2. \lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Proof. Since $\lim_n X_n = X$, we have

$$\liminf_n X_n = \limsup_n X_n = X. \quad (90)$$

Further, $\lim_{n \rightarrow \infty} |X_n| = |X|$ and since $|X_n| \leq Y, \forall n$, we have $|X| \leq Y$. Therefore, $\mathbb{E}[|X|] \leq \mathbb{E}[Y] < \infty$ (since Y is integrable). Thus X is integrable, i.e., $\mathbb{E}[X] < \infty$.

Also, note that

$$|X_n| \leq Y \Rightarrow -Y \leq X_n \leq Y \Rightarrow Y + X_n \geq 0 \text{ and } Y - X_n \geq 0. \quad (91)$$

Now consider the sequence $(Y + X_n)_{n \in \mathbb{N}}$. This is a sequence of non-negative *r.v.s* (follows from Proposition 3, Eq.(91)). By applying Fatou's lemma on this sequence, we get

$$\begin{aligned} \liminf_n \mathbb{E}[Y + X_n] &\geq \mathbb{E}\left[\liminf_n (Y + X_n)\right] \\ &\Rightarrow \liminf_n (\mathbb{E}[Y] + \mathbb{E}[X_n]) \geq \mathbb{E}\left[Y + \liminf_n X_n\right] \\ &\Rightarrow \mathbb{E}[Y] + \liminf_n \mathbb{E}[X_n] \geq \mathbb{E}[Y] + \mathbb{E}\left[\liminf_n X_n\right] \\ &\Rightarrow \liminf_n \mathbb{E}[X_n] \geq \mathbb{E}\left[\liminf_n X_n\right] = \mathbb{E}[X]. \end{aligned} \quad (92)$$

Now consider the sequence $(Y - X_n)_{n \in \mathbb{N}}$. This is a sequence of non-negative *r.v.s* (follows from Proposition 3, Eq.(91)). Again, by applying Fatou's lemma on this sequence, we get

$$\begin{aligned} \liminf_n \mathbb{E}[Y - X_n] &\geq \mathbb{E}\left[\liminf_n (Y - X_n)\right] \\ &\Rightarrow \liminf_n (\mathbb{E}[Y] - \mathbb{E}[X_n]) \geq \mathbb{E}\left[Y - \limsup_n X_n\right] \\ &\Rightarrow \mathbb{E}[Y] - \limsup_n \mathbb{E}[X_n] \geq \mathbb{E}[Y] - \mathbb{E}\left[\limsup_n X_n\right] \\ &\Rightarrow \limsup_n \mathbb{E}[X_n] \leq \mathbb{E}\left[\limsup_n X_n\right] = \mathbb{E}[X]. \end{aligned} \quad (93)$$

Now for the real (extended) sequence $(\mathbb{E}[X_n])_{n \in \mathbb{N}}$, we have

$$\limsup_n \mathbb{E}[X_n] \geq \liminf_n \mathbb{E}[X_n]. \quad (94)$$

Therefore, from Eqs (92, 93, 94), we have

$$\begin{aligned} \limsup_n \mathbb{E}[X_n] &= \limsup_n \mathbb{E}[X_n] = \mathbb{E}[X] < \infty \text{ (since } X \text{ is integrable)} \\ &\Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]. \end{aligned}$$

□

5 Auxiliary Results

Notion of almost surely (denoted as a.s.) We say that a r.v. X satisfies the property Γ *almost surely* if there exists a set $E \in \mathcal{F}$ s.t. $P(E) = 0$ ¹ and $X(\omega)$ satisfies property Γ , $\forall \omega \in E^c$.

Examples:

1. For $c \in \mathbb{R}^*$, we say $X = c$ a.s. if there exists a set $E \in \mathcal{F}$ s.t. $P(E) = 0$ and $X(\omega) = c$, $\forall \omega \in E^c$.
2. For r.v.s X and Y , we say $X = Y$ a.s. if there exists a set $E \in \mathcal{F}$ s.t. $P(E) = 0$ and $X(\omega) = Y(\omega)$, $\forall \omega \in E^c$.
3. For the sequence $(X_n)_{n \in \mathbb{N}}$, we say $\lim_n X_n = X^*$ a.s., if there exists a set $E \in \mathcal{F}$ s.t. $P(E) = 0$ and $\lim_n X(\omega) = X^*(\omega)$, $\forall \omega \in E^c$.

Theorem 9. *Let X be a r.v.. If $X \geq 0$ and $\mathbb{E}[X] < \infty$, then*

$$P(\{X = +\infty\}) = P(\{\omega \in \Omega : X(\omega) = +\infty\}) = 0.$$

(or Equivalently, we can say $X < +\infty$ a.s.)

Proof. Define for $n \in \mathbb{N} \cup \{0\}$,

$$B_n = \{\omega \in \Omega : X(\omega) \geq n\} \quad (95)$$

Note that $B_n = X^{-1}([n, +\infty])$ and hence $B_n \in \mathcal{F}$. Also, $B_{n+1} \subseteq B_n$. Therefore, $(B_n)_{n \in \mathbb{N}}$ is a monotonically decreasing sequence of \mathcal{F} -sets. Hence, by Eq. (14), we have

$$\lim_{n \rightarrow \infty} B_n = \bigcap_{n \geq 0} B_n = \{\omega \in \Omega : X(\omega) = +\infty\}. \quad (96)$$

The last equality is easy to verify.

Also, since $X \geq 0$, we have

$$\begin{aligned} nI_{B_n} &\leq XI_{B_n} \leq X \Rightarrow \mathbb{E}[nI_{B_n}] \leq \mathbb{E}[X] \\ &\Rightarrow nP(B_n) \leq \mathbb{E}[X] \\ &\Rightarrow P(B_n) \leq \frac{\mathbb{E}[X]}{n}. \end{aligned} \quad (97)$$

Now by Theorem 2 and Eq. (96), we have

$$\begin{aligned} 0 &\leq P(\{\omega \in \Omega : X(\omega) = +\infty\}) = \lim_{n \rightarrow \infty} P(B_n) \\ &\Rightarrow 0 \leq P(\{\omega \in \Omega : X(\omega) = +\infty\}) \leq \lim_{n \rightarrow \infty} \frac{\mathbb{E}[X]}{n} \quad (\text{follows from} \\ &\quad \text{Eq. (97) and the hypothesis that } \mathbb{E}[X] < \infty). \\ &\Rightarrow P(\{\omega \in \Omega : X(\omega) = +\infty\}) = 0. \end{aligned}$$

□

¹We call \mathcal{F} -set with probability zero as null-events

Now we extend this result to general integrable *r.v.s*.

Corollary 1. *Let X be a *r.v.* If X is integrable, then*

$$P(\{X = +\infty\} \cup \{X = -\infty\}) = P(\{\omega \in \Omega : X(\omega) = +\infty\} \cup \{\omega \in \Omega : X(\omega) = -\infty\}) = 0.$$

(or Equivalently, we can say $|X| < +\infty$ a.s.)

Proof. Note that since X is integrable, we have $\mathbb{E}[|X|] < \infty$. Now since $|X| \geq 0$, apply Theorem 9 on $|X|$. We get

$$\begin{aligned} P(\{\omega \in \Omega : |X(\omega)| = +\infty\}) &= 0 \\ \Rightarrow P(\{\omega \in \Omega : X(\omega) = +\infty\} \cup \{\omega \in \Omega : X(\omega) = -\infty\}) &= 0. \end{aligned}$$

□

Theorem 10. *Let X be a *r.v.* which is integrable and $A \in \mathcal{F}$. If $P(A) = 0$, then $\mathbb{E}[XI_A] = 0$.*

Proof. Let X be a simple function with non-infinity coefficients. Let $s = \sum_{i=1}^k a_i I_{A_i}$, where $A_i \in \mathcal{F}$ and $a_i \in \mathbb{R}$. Then

$$sI_A = \sum_{i=1}^k a_i I_{A_i \cap A} \Rightarrow \mathbb{E}[sI_A] = \sum_{i=1}^k a_i P(A_i \cap A) = 0. \quad (98)$$

The last equality follows since $P(A_i \cap A) \leq P(A) = 0$.

Let $X \geq 0$. Then by Theorem 4, we have $(s_n)_{n \in \mathbb{N}} \uparrow X$. Therefore, $(s_n I_A)_{n \in \mathbb{N}} \uparrow XI_A$. By the Definition (61), we have

$$\mathbb{E}[XI_A] = \lim_{n \rightarrow \infty} \mathbb{E}[s_n I_A] = 0. \quad (99)$$

The last equality follows from Eq. (98).

Now let X be a integrable *r.v.* Therefore,

$$\begin{aligned} \mathbb{E}[XI_A] &= \mathbb{E}[(X^+ - X^-)I_A] \\ &= \mathbb{E}[X^+ I_A] - \mathbb{E}[X^- I_A] \\ &= 0 - 0 = 0 \text{ (follows from Eq. (99))}. \end{aligned}$$

□

Theorem 11. *Let X be a *r.v.* and $A \in \mathcal{F}$ and $P(A) > 0$. If $X(\omega) > 0, \forall \omega \in A$. Then $\mathbb{E}[XI_A] > 0$.*

Proof. Before we prove this claim, we make an important remark. Note that we never assumed that X is integrable. This is not required since we are concerned with the expectation of XI_A . Note that $XI_A > 0$ (from the hypothesis) and hence $\mathbb{E}[XI_A]$ exists (even though it may be $+\infty$) and

$$\mathbb{E}[XI_A] \geq 0 \text{ (follows from Proposition 5(1))}. \quad (100)$$

Now we will prove that $\mathbb{E}[XI_A] \neq 0$. We prove this by proof by contradiction. Therefore, assume the contrary, *i.e.*,

$$\mathbb{E}[XI_A] = 0. \quad (101)$$

Now define, for $n \in \mathbb{N}$,

$$B_n := \{\omega \in A : X(\omega) \geq \frac{1}{n}\} \quad (102)$$

Note that $B_n = X^{-1}[\frac{1}{n}, +\infty] \cap A$. Hence $B_n \in \mathcal{F}$. Also, $B_{n+1} \supseteq B_n$. Therefore, $(B_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence of \mathcal{F} -sets. By Eq. (8), we have

$$\lim_n B_n = \bigcup_{n \in \mathbb{N}} B_n = A. \quad (103)$$

The last equality is easy to verify. And from Theorem 1, we have

$$P(A) = P(\lim_n B_n) = \lim_n P(B_n). \quad (104)$$

Further

$$\begin{aligned} \frac{1}{n} I_{B_n} &\leq XI_{B_n} \leq XI_A \\ \Rightarrow \mathbb{E}\left[\frac{1}{n} I_{B_n}\right] &\leq \mathbb{E}[XI_A] = 0 \text{ (follows from the assumption (101))} \\ \Rightarrow \frac{P(B_n)}{n} &= 0 \\ \Rightarrow P(B_n) &= 0. \end{aligned} \quad (105)$$

This along with Eq. (104) further imply that $P(A) = 0$. This is a contradiction since in the hypothesis of the theorem we have $P(A) > 0$. Therefore the initial assumption (101) is wrong and therefore $\mathbb{E}[XI_A] \neq 0$. Hence by Eq. (100), we have $\mathbb{E}[XI_A] > 0$. \square

Now from Theorems 10 and 11, we obtain the following:

Theorem 12. *Let X be a r.v. which is integrable and $A \in \mathcal{F}$. Let $X(\omega) \geq 0$, $\forall \omega \in A$. Then,*

If $\mathbb{E}[XI_A] = 0$, then either

$$P(A) = 0 \text{ or } X = 0 \text{ almost surely in } A.$$

When we say “ $X = 0$ almost surely in A ”, we mean that there exists an \mathcal{F} -set E s.t. $E \subseteq A$ with $P(E) = 0$ and $X(\omega) = 0, \forall \omega \in A \setminus E$. In other words, it means that there exists a null event E (*i.e.*, $P(E) = 0$) inside A , where X is non-zero; but outside E , yet inside A , we have $X = 0$.

This result helps to derive a characterization to establish the equivalence between two *r.v.s.*

Theorem 13. Let X, Y be two r.v.s which are integrable. If $\mathbb{E}[XI_A] = \mathbb{E}[YI_A]$, $\forall A \in \mathcal{F}$, then $X = Y$ a.s.

Proof. Define

$$B = \{\omega \in \Omega : X(\omega) > Y(\omega)\}. \quad (106)$$

By Proposition 2, we have $B \in \mathcal{F}$. Now by the hypothesis of the theorem, we have

$$\mathbb{E}[XI_B] = \mathbb{E}[YI_B] \quad \Rightarrow \quad \mathbb{E}[(X - Y)I_B] = 0.$$

Then by Theorem 12, we have either $P(B) = 0$ or $X - Y = 0$ a.s. in B (this is not possible since $X(\omega) > Y(\omega)$, $\forall \omega \in B$). Therefore, $P(B) = 0$, i.e., $P(\{X > Y\}) = 0$. Similarly, one can show that $P(\{Y > X\}) = 0$. Now, note that

$$\{X \neq Y\} = \{X > Y\} \cup \{Y > X\}.$$

This further implies that

$$P(\{X \neq Y\}) = P(\{X > Y\}) + P(\{Y > X\}) = 0 + 0 = 0.$$

Therefore $X = Y$ a.s. □

Similar results exist for \geq and \leq , i.e.

Theorem 14. Let X, Y be two r.v.s which are integrable.

1. If $\mathbb{E}[XI_A] \geq \mathbb{E}[YI_A]$, $\forall A \in \mathcal{F}$, then $X \geq Y$ a.s.
2. If $\mathbb{E}[XI_A] \leq \mathbb{E}[YI_A]$, $\forall A \in \mathcal{F}$, then $X \leq Y$ a.s.

The above two results are very powerful and are used in conditional expectation to compare r.v.s.