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Hartman Grobman and Hamiltonian System

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Consider the finite horizon optimal control problem for the control affine dynamical system.

$$J^\star(\mathbf{x}) = \min_{\mathbf{u}([0,T])} \frac{1}{2}\psi(\mathbf{x}(T)) + \frac{1}{2}\int_0^T q(\mathbf{x}(t)) + \mathbf{u}^\top \mathbf{C}\mathbf{u}(t)dt \tag{1}$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

The Hamiltonian associated with the optimal control problem can be defined using a co-state variable $\mathbf{p} \in \mathbb{R}^n$ as follows.

$$H(\mathbf{x},\mathbf{p},\mathbf{u}) = \mathbf{p}^\top(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}) + \frac{1}{2}(q(\mathbf{x}) + \mathbf{u}^\top\mathbf{C}\mathbf{u}) \tag{3}$$

The optimal input $\mathbf{u}$ is obtained after minimizing the Hamiltonian w.r.t. $\mathbf{u}$ as

$$\mathbf{u}^\star = \frac{\partial H}{\partial \mathbf{u}} = 0 \implies \mathbf{u}^\star = -\mathbf{C}^{-1}\mathbf{g}^\top(\mathbf{x})\mathbf{p}$$

Substuiting the optimal value of $\mathbf{u}$ in (3) and after simplification, we obtain

$$H(\mathbf{x},\mathbf{p},\mathbf{u}^\star) = \mathbf{p}^\top\mathbf{f} - \frac{1}{2}\mathbf{p}^\top\mathbf{g}\mathbf{C}^{-1}\mathbf{g}^\top\mathbf{p} + \frac{1}{2}q(\mathbf{x}) \tag{4}$$

The Hamiltonian dynamical system associated with the Hamiltonian is defined as

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial H}{\partial \mathbf{p}} = \mathbf{f} - \mathbf{g}\mathbf{C}^{-1}\mathbf{g}^\top\mathbf{p} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^\top\mathbf{p} + \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{p}^\top\mathbf{g}\mathbf{C}^{-1}\mathbf{g}^\top\mathbf{p}\right) - \frac{1}{2}\frac{\partial q}{\partial \mathbf{x}}^\top \\ &\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{\partial V}{\partial \mathbf{x}}\mathbf{g}\mathbf{C}^{-1}\mathbf{g}^\top\frac{\partial V}{\partial \mathbf{x}}^\top + q(\mathbf{x}) = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} V &= \lambda \log J \\ \frac{\partial V}{\partial \mathbf{x}} &= \lambda\frac{1}{J}\frac{\partial J}{\partial \mathbf{x}} \end{aligned}$$

The solution to the optimal control problem is obtained from the solution of the Hamiltonian system. However, solving the Hamiltonian system is a two point boundary value problem with the initial state specified as $\mathbf{x}(0) = \mathbf{x}_0$ and the terminal state given in terms of the co-state variable $\mathbf{p}(T) = \frac{\partial \psi}{\partial \mathbf{x}}^\top(\mathbf{x}(T))$. We split the Hamiltonian system into linear and nonlinear part as follows.

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{p}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & -\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\\ -\mathbf{Q} & -\mathbf{A}^\top\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} + \begin{pmatrix}\frac{\partial H}{\partial \mathbf{p}}\\ -\frac{\partial H}{\partial \mathbf{x}}\end{pmatrix} - \begin{pmatrix}\mathbf{A} & -\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\\ -\mathbf{Q} & -\mathbf{A}^\top\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} \tag{6}$$

where

$$\mathbf{E} := \begin{pmatrix}\mathbf{A} & -\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\\ -\mathbf{Q} & -\mathbf{A}^\top\end{pmatrix} := \begin{pmatrix}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0) & -\mathbf{g}(0)\mathbf{C}^{-1}\mathbf{g}^\top(0)\\ -\frac{1}{2}\frac{\partial^2 q}{\partial \mathbf{x}^2}(0) & -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0)^\top\end{pmatrix} \tag{7}$$

$$\mathbf{F}_n = \begin{pmatrix}\frac{\partial H}{\partial \mathbf{p}}\\ -\frac{\partial H}{\partial \mathbf{x}}\end{pmatrix} - \begin{pmatrix}\mathbf{A} & -\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\\ -\mathbf{Q} & -\mathbf{A}^\top\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} \tag{8}$$

Using the above definitions the Hamiltonian system can be decomposed into linear and nonlinear parts as follows.

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}) = \mathbf{E}\mathbf{z} + \mathbf{F}_n(\mathbf{z}) \tag{9}$$

where we have used the notation $\mathbf{z} = (\mathbf{x}^\top, \mathbf{p}^\top)^\top$. Consider the following change of coordinates

$$(\mathbf{x},\mathbf{p}) \to (\mathbf{X},\mathbf{P})$$

such that in the new coordinates the nonlinear Hamiltonian system gets transformed to linear system of the form

$$\dot{\mathbf{Z}} = \mathbf{E}\mathbf{Z}. \tag{10}$$

Lemma 1. *The coordinates* $\mathbf{X}$ *in the coordinates transformation* $\mathbf{Z}$ *transforming the nonlinear Hamiltonian system (5) to linear Hamiltonian system (10) is independent of* $\mathbf{p}$, *i.e.,* $\frac{\partial \mathbf{X}}{\partial \mathbf{p}} = 0$.

The results of the above Lemma can be used for the computation of the coordinates. In particular, we assume that the the transformation $\mathbf{Z}$ is of the form

$$\mathbf{Z} = \mathbf{z} + \mathbf{H}(\mathbf{z}) \tag{11}$$

$$\frac{\partial H}{\partial z}(0) = 0$$

$$\dot{Z} = EZ$$

$$ET = T\Lambda \implies \Lambda = T^{-1}ET$$

$$\bar{Z} = T^{-1}Z \implies Z = T\bar{Z}$$

$$\dot{\bar{Z}} = T^{-1}ET\bar{Z} = \Lambda\bar{Z}$$

$\bar{Z}$ are the eigenfunction.

Using the result of Lemma 1, we obtain following equation to be satisfied by $\mathbf{H}$.

$$\mathbf{H}(\mathbf{z}) = (\mathbf{H}_1(\mathbf{x})^\top, \mathbf{H}_2(\mathbf{z}))^\top \tag{12}$$

$$\left(\mathbf{I} + \frac{\partial \mathbf{H}}{\partial \mathbf{z}}\right)(\mathbf{E}\mathbf{z} + \mathbf{F}_n) = \mathbf{E}(\mathbf{I}\mathbf{z} + \mathbf{H}(\mathbf{z})) \tag{13}$$

$$\frac{\partial \mathbf{H}}{\partial \mathbf{z}}\mathbf{F}(\mathbf{z}) - \mathbf{E}\mathbf{H}(\mathbf{z}) + \mathbf{F}_n = 0 \tag{14}$$

$$\begin{pmatrix} \frac{\partial \mathbf{H}_1}{\partial \mathbf{x}}\mathbf{F}_1(\mathbf{z}) \\ \frac{\partial \mathbf{H}_2}{\partial \mathbf{z}}\mathbf{F} \end{pmatrix} - \begin{pmatrix} \mathbf{A}\mathbf{H}_1(\mathbf{x}) - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\mathbf{H}_2(\mathbf{z}) \\ -\mathbf{Q}\mathbf{H}_1(\mathbf{x}) - \mathbf{A}^\top\mathbf{H}_2(\mathbf{z}) \end{pmatrix} + \begin{pmatrix} \mathbf{F}_{n1}(\mathbf{z}) \\ \mathbf{F}_{n2}(\mathbf{z}) \end{pmatrix} = 0 \tag{15}$$

Next we construct the finite dimensional approximation of the $\mathbf{H}(\mathbf{z})$. For this purpose we make following choice of basis function using the results of Lemma 1.

Finite-dimensional approximation of the coordinates $\mathbf{Z}$. Let

$$\Gamma_1(\mathbf{x}) = (\gamma_{11}(\mathbf{x}), \dots, \gamma_{1N}(\mathbf{x})) \in \mathbb{R}^N, \quad \Gamma_2(\mathbf{z}) = (\gamma_{21}(\mathbf{z}), \dots, \gamma_{2M}(\mathbf{z})) \in \mathbb{R}^M$$

be the basis functions. The choice of $\Gamma_2(\mathbf{z})$ is assumed to be of the form

$$\Gamma_2(\mathbf{z}) = (\Xi(\mathbf{p})^\top, \Xi(\mathbf{p}) \otimes \Gamma_1(\mathbf{x})^\top)^\top = (\Xi(\mathbf{p})^\top, \Xi(\mathbf{p})^\top\gamma_{11}(\mathbf{x}), \dots, \Xi(\mathbf{p})^\top\gamma_{1N}(\mathbf{x}))$$

We then approximate

$$\mathbf{H}_1(\mathbf{x}) \approx \mathbf{D}_1^\top\Gamma_1(\mathbf{x}), \quad \mathbf{H}_2(\mathbf{z}) \approx \mathbf{D}_2^\top\Gamma_2(\mathbf{z}) \tag{16}$$

with $\mathbf{D}_1 \in \mathbb{R}^{N\times n}, \mathbf{D}_2 \in \mathbb{R}^{M\times n}$.

II. Example 1

$$\dot{x} = x - x^3 + u \tag{17}$$

with cost function

$$J = \frac{1}{2}\int_0^T x^2 + u^2$$

$$\dot{x} = x - x^3 - p$$

$$\dot{p} = -(1 - 3x^2)p - x$$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} -x^3 \\ 3x^2 p \end{pmatrix} \tag{18}$$

III. Example 2

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2(1-(1+2\sin x_1)^2) + (1+2\sin x_1)u.\end{aligned} \tag{19}$$

with cost function

$$J = \int_0^T x^2 + u^2$$