Midterm exam

Friday, November 10, 2023 8:50 AM

1] $\dot{x}_1 = -x_1 + x_2^2$ - ①

$\dot{x}_2 = -3x_2 + 3x_1^2$ - ②

$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2$

a] Zero solⁿ is asymptotically stable.

$\therefore \dot{V}(x) = \frac{dV}{dx_1} \times \frac{dx_1}{dt} + \frac{dV}{dx_2} \times \frac{dx_2}{dt}$ .. chain rule

$= \frac{dV}{dx}\dot{x}_1 + \frac{dV}{dx}\dot{x}_2$

$= x_1\dot{x}_2 + \frac{1}{2}x_2\dot{x}_2$

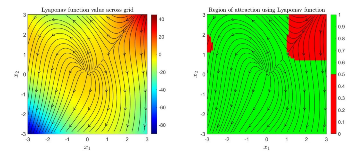
$= x_1(-x_1 + x_2^2) + \frac{1}{2}x_2(-3x_2 + 3x_1^2)$ ... from 1 & 2

$= -x_1^2 + x_1x_2^2 + \frac{3x_1^2x_2}{2} - \frac{3x_2^2}{2}$

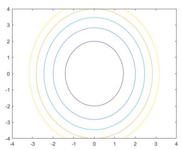
→ Evaluating $\dot{V}(x)$ at $(0,0) \rightarrow 0$; asymptotically stable

Now, the ellipse eqⁿ $V(x)$ can be used to find largest level curve in the domain of attraction.

Plots with Lyaponav function



Level curves of Lyapunov function: Solving fmincon for intersection with ROA in above figure, the max level set is 1.00325



2] $\dot{x}_1 = x_2$ - ①

$\dot{x}_2 = x_1(1 - x_1^2) - x_2$ - ②

a] Find eq pts.

$\frac{df}{dx} = 0;$

$\therefore 0 = x_2$

Substitute in 2.

$\therefore 0 = x_1 - x_1^3 - 0$

$\therefore x_1 = x_1^3$

i.e $x_1 = \pm 1$

$\therefore$ Eq pts $(0,0)$; $(1,0)$; $(-1,0)$

b] Evaluate eq pts.

Jacobian → $\left[\frac{df}{dx}\right] = \begin{bmatrix} \partial f_1/x_1 & \partial f_1/x_2 \\ \partial f_2/x_1 & \partial f_2/x_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -1 \end{bmatrix}$

$\therefore \left.\frac{df}{dx}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$; $\lambda_{1,2} = -1.61, 0.61$ ... Saddle pt.

$\left.\frac{df}{dx}\right|_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$; $\lambda_{1,2} = -0.5 \pm 1.32i$ ... Stable focus

$\left.\frac{df}{dx}\right|_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$; $\lambda_{1,2} =$ .. " .. ... Stable focus

c] Construct Lyaponav fⁿ.

→ Lyaponav's direct method.

Let sIm be $\dot{x} = Ax$. & $V(x) = x^T P x$.

$\therefore \dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

$= (Ax)^T P x + x^T P (Ax)$

$= A^T x^T P x + x^T P A x$

$= x^T \underbrace{(A^T P + PA)}_{Q} x$

For stability $\dot{V}(x)$ is -ve definite.

i.e $\dot{V}(x) = x^T(-Q)x$ s.t $-Q = A^T P + PA$

→ For stable eq pts $(1,0)$, $(-1,0)$; $A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$; $Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}$

$-Q = A^T P + PA$

$\therefore -Q = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} -2P_{21} & -2P_{22} \\ P_{11} - P_{21} & P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{12} & P_{11} - P_{12} \\ -2P_{22} & P_{21} - P_{22} \end{bmatrix}$

also $P_{12} = P_{21}$ ... So just making $P_{21} \Rightarrow P_{12}$.

$\therefore -Q = \begin{bmatrix} -2P_{12} & -2P_{22} \\ P_{11} - P_{12} & P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{12} & P_{11} - P_{12} \\ -2P_{22} & P_{12} - P_{22} \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4P_{12} & P_{11} - P_{12} - 2P_{22} \\ P_{11} - P_{12} - P_{22} & 2P_{12} - 2P_{22} \end{bmatrix}$

$\therefore -4P_{12} = -1$ .. c_1

$P_{11} - P_{12} - 2P_{22} = 0$ .. c_2

$2P_{12} - 2P_{22} = -1$ ... c_3

$\therefore P_{12} = 1/4$ .. from c_1

Sub P_{12} in c_3

$\therefore \frac{1}{2} - 2P_{22} = -1$

$-2P_{22} = -\frac{3}{2}$

$\therefore P_{22} = \frac{3}{4}$

Sub P_{12}, P_{22} in c_2

$\therefore P_{11} - \frac{1}{4} - 2 \times \frac{3}{4} = 0$

$\therefore 4P_{11} - 1 - 6 = 0$

$\therefore P_{11} = 7/4$

i.e $P = \begin{bmatrix} 7/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$; Det (A) = 1.25 ... Positive definite.

$\therefore$ The system is stable at (1,0), (-1,0)

Constructing Lyaponav f^n

$$V(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} P_{11}x_1 + P_{12}x_2 \\ P_{21}x_1 + P_{22}x_2 \end{bmatrix}$$

$$= x_1 P_{11} x_1 + x_1 P_{12} x_2 + x_2 P_{21} x_1 + x_2 P_{22} x_2$$

$$= x_1^2 P_{11} + 2(x_1 P_{12} x_2) + x_2^2 P_{22}$$

$$V(x) = \frac{7}{4}x_1^2 + \frac{2}{4}x_1x_2 + \frac{3}{4}x_2^2$$

$$\dot{V}(x) = \left(\frac{7}{2}x_1 + \frac{1}{2}x_2\right)\dot{x}_1 + \left(\frac{3}{2}x_2 + \frac{1}{2}x_1\right)\dot{x}_2$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

i.e $\dot{x}_1 = x_2$

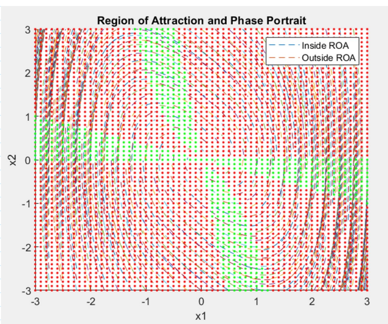
$\dot{x}_2 = -2x_1 - x_2$.

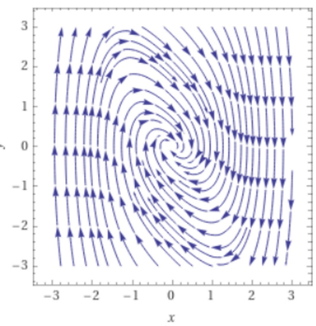
$$\dot{V}(x) = \left(\frac{7}{2}x_1 + \frac{1}{2}x_2\right)x_2 + \left(\frac{3}{2}x_2 + \frac{1}{2}x_1\right)(-2x_1 - x_2)$$

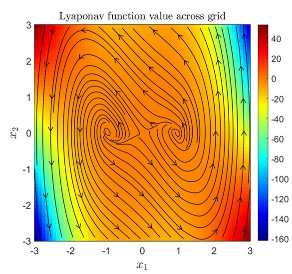
$$= \cancel{\frac{7}{2}x_1x_2} + \frac{1}{2}x_2^2 + \left(\cancel{-\frac{6}{2}x_1x_2} - \frac{3}{2}x_2^2 - \frac{1}{2}x_1^2 - \cancel{\frac{1}{2}x_1x_2}\right)$$

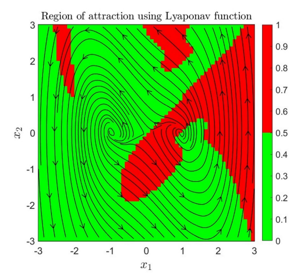
$$= -\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 \Big\} \rightarrow \text{Asymptotically Stable at } (1,0); (-1,0)$$

C,D] Regions of attraction from Lyapunov function we derived along with the phase portrait of the system









Question 3

Part 1:

A] Considering system from Question 1

$\dot{x}_1 = -x_1 + x_2^2$ - ①

$\dot{x}_2 = -3x_2 + 3x_1^2$ - ②

$A \rightarrow \lambda$.

λ.

Linearize the s/m.

$\frac{df}{dx} = 0$ $\therefore$ $0 = -x_1 + x_2^2$ $\therefore x_1 = x_2^2$

Sub in ②

$\therefore$ $0 = -3x_2 + 3x_2^4$

$\therefore$ $1 = x_2^3$

$\therefore$ $x_2 = 1$

Sub x_2 in ①.

$\therefore$ $0 = -x_1 + 1$ $\therefore x_1 = 1$

Eq pt $\rightarrow$ (1, 1).

Jacobian:

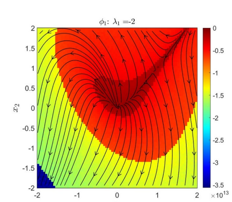
$$\frac{df}{dx} = \begin{bmatrix} -1 & 2x_2 \\ 6x_1 & -3 \end{bmatrix}$$

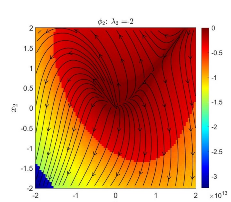
$$\left.\frac{df}{dx}\right|_{(1,1)} = \begin{bmatrix} -1 & 2 \\ 6 & -3 \end{bmatrix}; \quad \lambda_{1,2} = 1.6, -5.6056$$

Additionally, (0,0) is an equilibrium point.

Finding the coordinate transformation, Lyapunov function and ROA using the path-integral method.
The following results can be observed.

Eigen function Plot





Lyapunov function and ROA for the eigenfunction coordinates; The ROA is approximated in a binary class based on the gradient of Lyapunov.
In the plot on the left, a simple gradient was calculated along the Lyapunov function value grid. On the right, the gradient for Lyapunov function was calculated by the finite difference method between points in Phi dynamics and the LP function value. The results on the right highlight the difference.

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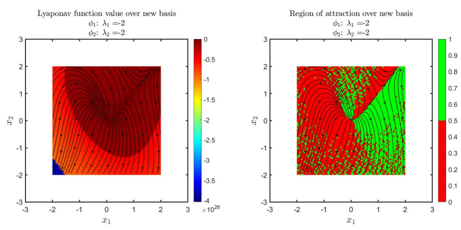
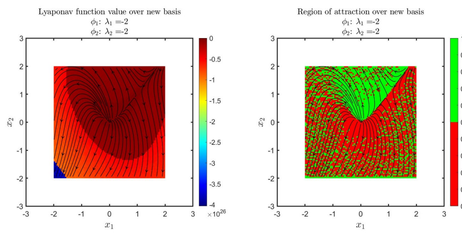
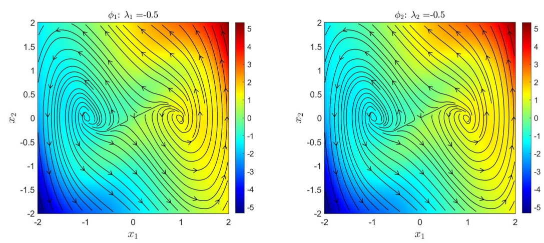


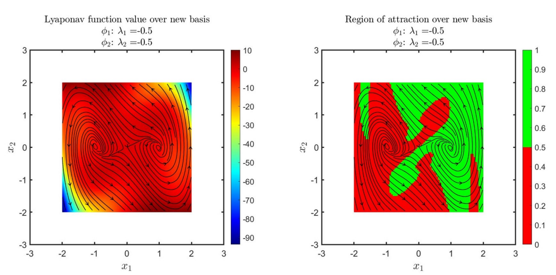


Part2: Eigenfunction coordinates and Lyapunov function for Q2.
In Q2, we already established that the equlibrium points of the duffing oscillator are at (0,0) (1,0) and (-1,0).
The later 2 are stable focus.

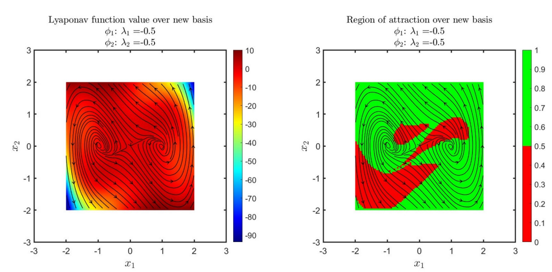
Generating eigenfunction coordinate transform along these equilibrium points, we can construct a data-driven Lyapunov function, get the derivatives and generate the ROA.

The following plots highlight the obtained eigenfunction coordinates and Lyapunov function with ROA for the duffing oscillator.





Lyaponav function plot and taking derivative with finite-difference method for calculation of ROA. This method is much more accurate in representing region of attraction towards eq points (1,0) and (-1,0).
You can clearly see the saddle point near (0.0)



C] For the final problem, I just made extensions to the above code to run the path integral formula to obtain eigenfunction coordinates for the saddle point equilibrium at (0,0). The coordinates corresponding to the positive eigen value were then filtered into a binary class.
1st set being the values close to zero within a certain threshold and other being the set excluding the first set. These sets were then color coded, and the one's corresponding to zero (blue region) are the stable manifolds corresponding to the unstable eigenvalues, which satisfy:

$$\frac{\partial \phi_+}{\partial x} f(x) = \lambda_+ \phi_+(x)$$