

# Hartman Grobman and Hamiltonian System

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Consider the finite horizon optimal control problem for the control affine dynamical system.

$$J^*(\mathbf{x}) = \min_{\mathbf{u}([0,T])} \frac{1}{2} \psi(\mathbf{x}(T)) + \frac{1}{2} \int_0^T q(\mathbf{x}(t)) + \mathbf{u}^\top \mathbf{C} \mathbf{u}(t) dt \quad (1)$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (2)$$

The Hamiltonian associated with the optimal control problem can be defined using a co-state variable  $\mathbf{p} \in \mathbb{R}^n$  as follows.

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}) = \mathbf{p}^\top (\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}) + \frac{1}{2} (q(\mathbf{x}) + \mathbf{u}^\top \mathbf{C} \mathbf{u}) \quad (3)$$

The optimal input  $\mathbf{u}$  is obtained after minimizing the Hamiltonian w.r.t.  $\mathbf{u}$  as

$$\mathbf{u}^* = \frac{\partial H}{\partial \mathbf{u}} = 0 \implies \mathbf{u}^* = -\mathbf{C}^{-1} \mathbf{g}^\top(\mathbf{x}) \mathbf{p}$$

Substituting the optimal value of  $\mathbf{u}$  in (3) and after simplification, we obtain

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}^*) = \mathbf{p}^\top \mathbf{f} - \frac{1}{2} \mathbf{p}^\top \mathbf{g} \mathbf{C}^{-1} \mathbf{g}^\top \mathbf{p} + \frac{1}{2} q(\mathbf{x}) \quad (4)$$

The Hamiltonian dynamical system associated with the Hamiltonian is defined as

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial H}{\partial \mathbf{p}} = \mathbf{f} - \mathbf{g} \mathbf{C}^{-1} \mathbf{g}^\top \mathbf{p} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial \mathbf{f}^\top}{\partial \mathbf{x}} \mathbf{p} + \frac{1}{2} \frac{\partial}{\partial \mathbf{x}} (\mathbf{p}^\top \mathbf{g} \mathbf{C}^{-1} \mathbf{g}^\top \mathbf{p}) - \frac{1}{2} \frac{\partial q}{\partial \mathbf{x}} \end{aligned} \quad (5)$$

The solution to the optimal control problem is obtained from the solution of the Hamiltonian system. However, solving the Hamiltonian system is a two point boundary value problem with the initial state specified as  $\mathbf{x}(0) = \mathbf{x}_0$  and the terminal state given in terms of the co-state variable  $\mathbf{p}(T) = \frac{\partial \psi}{\partial \mathbf{x}}^\top(\mathbf{x}(T))$ . We split the Hamiltonian system into linear and nonlinear part as follows.

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & -\mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \\ -\mathbf{Q} & -\mathbf{A}^\top \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{x}} \end{pmatrix} - \begin{pmatrix} \mathbf{A} & -\mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \\ -\mathbf{Q} & -\mathbf{A}^\top \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} \quad (6)$$

where

$$\mathbf{E} := \begin{pmatrix} \mathbf{A} & -\mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \\ -\mathbf{Q} & -\mathbf{A}^\top \end{pmatrix} := \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0) & -\mathbf{g}(0) \mathbf{C}^{-1} \mathbf{g}^\top(0) \\ -\frac{1}{2} \frac{\partial^2 q}{\partial \mathbf{x}^2}(0) & -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0)^\top \end{pmatrix} \quad (7)$$

$$\mathbf{F}_n = \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{x}} \end{pmatrix} - \begin{pmatrix} \mathbf{A} & -\mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \\ -\mathbf{Q} & -\mathbf{A}^\top \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} \quad (8)$$

Using the above definitions the Hamiltonian system can be decomposed into linear and nonlinear parts as follows.

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}) = \mathbf{E} \mathbf{z} + \mathbf{F}_n(\mathbf{z}) \quad (9)$$

where we have used the notation  $\mathbf{z} = (\mathbf{x}^\top, \mathbf{p}^\top)^\top$ . Consider the following change of coordinates

$$(\mathbf{x}, \mathbf{p}) \rightarrow (\mathbf{X}, \mathbf{P})$$

such that in the new coordinates the nonlinear Hamiltonian system gets transformed to linear system of the form

$$\dot{\mathbf{Z}} = \mathbf{E} \mathbf{Z}. \quad (10)$$

**Lemma 1.** *The coordinates  $\mathbf{X}$  in the coordinates transformation  $\mathbf{Z}$  transforming the nonlinear Hamiltonian system (5) to linear Hamiltonian system (10) is independent of  $\mathbf{p}$ , i.e.,  $\frac{\partial \mathbf{X}}{\partial \mathbf{p}} = 0$ .*

The results of the above Lemma can be used for the computation of the coordinates. In particular, we assume that the transformation  $\mathbf{Z}$  is of the form

$$\mathbf{Z} = \mathbf{z} + \mathbf{H}(\mathbf{z}) \quad (11)$$

Using the result of Lemma 1, we obtain following equation to be satisfied by  $\mathbf{H}$ .

$$\mathbf{H}(\mathbf{z}) = (\mathbf{H}_1(\mathbf{x})^\top, \mathbf{H}_2(\mathbf{z}))^\top \quad (12)$$

$$\left( \mathbf{I} + \frac{\partial \mathbf{H}}{\partial \mathbf{z}} \right) (\mathbf{E}\mathbf{z} + \mathbf{F}_n) = \mathbf{E}(\mathbf{I}\mathbf{z} + \mathbf{H}(\mathbf{z})) \quad (13)$$

$$\frac{\partial \mathbf{H}}{\partial \mathbf{z}} \mathbf{F}(\mathbf{z}) - \mathbf{E}\mathbf{H}(\mathbf{z}) + \mathbf{F}_n = 0 \quad (14)$$

$$\left( \frac{\partial \mathbf{H}_1}{\partial \mathbf{x}} \mathbf{F}_1(\mathbf{z}) \right) - \left( \mathbf{A}\mathbf{H}_1(\mathbf{x}) - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top \mathbf{H}_2(\mathbf{z}) \right) + \begin{pmatrix} \mathbf{F}_{n1}(\mathbf{z}) \\ \mathbf{F}_{n2}(\mathbf{z}) \end{pmatrix} \quad (15)$$

Next we construct the finite dimensional approximation of the  $\mathbf{H}(\mathbf{z})$ . For this purpose we make following choice of basis function using the results of Lemma 1.

Finite-dimensional approximation of the coordinates  $\mathbf{Z}$ . Let

$$\Gamma_1(\mathbf{x}) = (\gamma_{11}(\mathbf{x}), \dots, \gamma_{1N}(\mathbf{x})) \in \mathbb{R}^N, \quad \Gamma_2(\mathbf{z}) = (\gamma_{21}(\mathbf{z}), \dots, \gamma_{2M}(\mathbf{z})) \in \mathbb{R}^M$$

be the basis functions. The choice of  $\Gamma_2(\mathbf{z})$  is assumed to be of the form

$$\Gamma_2(\mathbf{z}) = (\Xi(\mathbf{p})^\top, \Xi(\mathbf{p}) \otimes \Gamma_1(\mathbf{x})^\top)^\top = (\Xi(\mathbf{p})^\top, \Xi(\mathbf{p})^\top \gamma_{11}(\mathbf{x}), \dots, \Xi(\mathbf{p})^\top \gamma_{1N}(\mathbf{x}))$$

We then approximate

$$\mathbf{H}_1(\mathbf{x}) \approx \mathbf{D}_1^\top \Gamma_1(\mathbf{x}), \quad \mathbf{H}_2(\mathbf{z}) \approx \mathbf{D}_2^\top \Gamma_2(\mathbf{z})$$

with  $\mathbf{D}_1 \in \mathbb{R}^{N \times n}$ ,  $\mathbf{D}_2 \in \mathbb{R}^{M \times n}$ .

## II. EXAMPLE 1

$$\dot{x} = x - x^3 + u \quad (16)$$

with cost function

$$J = \int_0^T x^2 + u^2$$

## III. EXAMPLE 2

$$\dot{x}_1 = x_2 \quad (17)$$

$$\dot{x}_2 = -x_1 - x_2(1 - (1 + 2 \sin x_1)^2) + (1 + 2 \sin x_1)u.$$

with cost function

$$J = \int_0^T x^2 + u^2$$