ECON-GA 3002.015 AND MATH.GA 2840.002
'MATH & ECON + CODE' MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS
Final exam (take-home), January 27-28, 2018

Please submit at or before Jan 28 2018, 12pm noon EST. This is a strict deadline. If you don't submit an exam or if you submit after the deadline, you will be automatically considered to opt for a course paper submission.

Please submit your answers as a single pdf document (no multiple files and no separate code files).

Please make sure you submit your exam by replying to this thread – otherwise it may get lost. Also please make sure hit 'reply to all' when you reply so that the three of us get your exams.

Communication with other students or anyone else regarding this exam is prohibited. The consultation of the course material (which is available online at https://github.com/alfredgalichon/math-econ-code_masterclass) as well as personal notes taken during the course are acceptable.

Please answer FIVE of the eight independent questions below. If you answer more than five, we will only consider the five first in the order you answered.

Question 1. Assume that there are three types of men, $\mathcal{X} = \{x_1, ..., x_3\}$, four types of women, $\mathcal{Y} = \{y_1, ..., y_4\}$, one individual of each type (that is $n_{x_1} = ... = n_{x_3} = 1$ and $m_{y_1} = ... = m_{y_4} = 1$), and let

$$\Phi = \begin{pmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 4 & 1 \\ 2 & 1 & 2 & 0 \end{pmatrix} \tag{1}$$

(i) In the optimal assignment problem with singles, compute the value of the problem using a linear programming solver such as Gurobi. Please include the code you used in your answer.

(ii) Write down an optimal primal solution.

(iii) Write down an optimal dual solution

(iv) Assume Φ_{22} is increased by some small amoung $\epsilon > 0$. By how much is W increased?

(v) Same question when Φ_{34} is increased by ϵ.

Question 2. Consider a continuous assignment model of firms to CEOs, where there is a continuous distribution of firms and CEOs, with total mass one on each side. The CEOs are characterized by their talent x, the firms by their size y. The value created by a match between a CEO of talent x with a firm of size y is assumed to be of the form

$$\Phi(x, y) = xy^{\kappa}, \tag{2}$$

with $\kappa > 0$; and the cdf of CEOs' talent F_X and firms' size F_Y are assumed to be such that their inverse have expressions

$$\begin{aligned} F_X^{-1}(t) &= \frac{B}{\beta} - \frac{B}{\beta}(1-t)^{\beta} \\ F_Y^{-1}(t) &= A(1-t)^{-\alpha}. \end{aligned}$$

Let $w(x)$ be the salary of the CEO of talent x. It is assumed that if a CEO of talent x is matched with a firm of size y, this CEO will get utility $w(x)$ and the firm will get $\Phi(x,y) - w(x)$ (in other words, the CEOs are indifferent as to which firm they work with, salary being equal).

(i) Show that Φ is supermodular. What qualitative property can be deduced about the optimal assignment of firms to CEOs, which will be denoted $y = T(x)$?

(ii) Give an expression of $T(x)$.

(iii) What is the relation between its derivative w' and T? deduce an expression for $w(x)$ (up to a additive constant).

Question 3. Consider a bipartite matching market with singles. There are n_x men of type x, m_y women of type y, and the matching surplus of a pair (x,y) is Φ_{xy}. The utility of singles is normalized to zero. We shall say that a vector (u_x, v_y) is *stable* if there is $\mu_{xy} \geq 0$ such that such that $\sum_y \mu_{xy} \leq n_x$, $\sum_x \mu_{xy} \leq m_y$ and (μ, u, v) is a stable outcome.

(i) Write the primal and the dual of the Linear Programming formulation of this assignment problem. Let W be the value of this program. What are the conditions (stated in terms of W and Φ) on (u_x, v_y) so that (u,v) is feasible? what are the conditions so that (u,v) is vector of stable payoff?

Let S be the set of vectors (u_x, v_y) of stable payoffs in this problem. Define

$$\underline{u}_x = \min_{(u,v)\in S} u_x \text{ and } \bar{v}_y = \max_{(u,v)\in S} v_y.$$

(ii) Show that $(\underline{u}_x, \bar{v}_y)$ is a vector of stable payoffs.

(iii) How would you compute $(\underline{u}_x, \bar{v}_y)$ by linear programming? hint: you may want to compute the value W of the assignment problem first.

(iv) Write down the code to implement your answer in (iii).

Question 4. Consider a labor market where firms are endowed with a job of complexity $y > 0$ drawn from a continuous distribution whose cdf is F_Y. Workers are endowed with skills (ability) $x > 0$ drawn from a continuous distribution whose cdf is F_X. To produce one unit of output at job y, a worker x must work for $c(x,y) > 0$ units of time, where $c(x,y)$ is a continuous and twice differentiable function. Let $w(x)$ be the wage function indicating the per unit time wage for worker x (endogenously determined). The total costs for a firm y of employing worker x is then given as: $C(x,y) = c(x,y)\,w(x)$. It is assumed that the output is not differentiated within workers. Firm y's cost minimization problem, which determines the assignment of worker to firm $y = T(x)$, is hence

$$\min_x c(x,y)\,w(x).$$

(i) Show $T(x)$ is determined given by a matching problem with surplus function

$$\Phi(x,y) = -\log c(x,y).$$

(ii) Show that, when c is log-submodular, i.e. $\partial^2_{xy} \log c(x,y) \leq 0$, there is positive assortative matching in this economy. Give the expression of T as a function of F_X and F_Y, and provide the equation determining the equilibrium wage.

Question 5. Consider a Dagsvik-Menzel model, where there are three types of men, $\mathcal{X} = \{x_1, ..., x_3\}$, four types of women, $\mathcal{Y} = \{y_1, ..., y_4\}$, and assume $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$ has expression (1). Let n_x be the number of men of each type $x \in \mathcal{X}$, and m_y be the number of women of each type $y \in \mathcal{Y}$. Recall that the equilibrium matching is given by $\mu_{xy} = \mu_{x0}\mu_{0y} \exp \Phi_{xy}$, where the number of single individuals μ_{x0} and μ_{0y} are obtained by

$$\begin{cases} n_x = \mu_{x0} + \sum_y \mu_{x0}\mu_{0y} \exp \Phi_{xy} \\ m_y = \mu_{0y} + \sum_y \mu_{x0}\mu_{0y} \exp \Phi_{xy} \end{cases}$$

(i) Assume that there is one individual of each type (that is $n_{x_1} = ... = n_{x_3} = 1$ and $m_{y_1} = ... = m_{y_4} = 1$). Compute the number of matched couples, i.e.

$$\frac{2\sum_{xy} \mu_{xy}}{\sum_x n_x + \sum_y m_y}.$$

Include the code in your answer.

(ii) Same question if there are ten individuals of each type (that is $n_{x_1} = ... = n_{x_3} = 10$ and $m_{y_1} = ... = m_{y_4} = 10$). Comment. Would you get this in a Choo-Siow model? why?

Question 6. Consider a labour market in which workers decide on their careers. There is a finite number of types of worker $x \in \mathcal{X}$. There is a large total number of workers, and the fraction of workers of type x is denoted p_x, so that

$$\sum_{x \in \mathcal{X}} p_x = 1.$$

There is a finite number of job types $y \in \mathcal{Y}$. The wage offered to an individual of type x working job y is denoted w_{xy}. It is assumed that an individual worker i of type $x_i = x$ has utility on job y equal to

$$w_{xy} + \varepsilon_{iy}$$

where (ε_{iy}) is a random utility term, which is a random vector on $\mathbb{R}^{|\mathcal{Y}|}$ with distribution $\mathbf{P}_x$. Every worker is forced to choose a job.

In questions (a) and (b), w_{xy} is assumed to be fixed, and the number of each job chosen is determined by the fraction of workers of each type choosing this job.

(a) Write down the expression of the social welfare $G(w)$ (sum of the worker's utilities).

(b) Let $\pi_{y|x}$ be the fraction of workers of type x choosing job y. What is the relation between $\pi_{y|x}$ and $G(.)$? Deduce an expression for q_y, the overall fraction of workers choosing job y.

Now assume that there is a fixed supply of jobs, so that there is the same number of jobs as workers, and that the fraction of jobs that are of type y is q_y, which is a fixed, exogenous quantity. The equilibrium wage is now endogenous: the wages adjust to that the fraction of workers choosing y is q_y.

(c) How is the equilibrium wage w_{xy} determined? (Hint: show that at equilibrium, w_{xy} does not depend on x).

Question 7. We consider a random utility model with six alternatives, the utility of the last one being normalized to zero, i.e $U_6 = 0$. Assume that the model is Probit with

$$\varepsilon \sim N\left(0, \begin{pmatrix} 1 & 0.5 & \cdots & 0.5 \\ 0.5 & \ddots & & \vdots \\ \vdots & & \ddots & 0.5 \\ 0.5 & \cdots & 0.5 & 1 \end{pmatrix}\right).$$

One observes that the vector of market shares is given by

$$s = c(0.216191, 0.263915, 0.141353, 0.176406.0.113032.0.089103)$$

Give an estimator of the U. Include the code in your answer.

Question 8. Consider a variant of the min-cost flow model described in lecture 2, where we add an entropic regularization. With the same notations as in lecture 2, the problem is

$$\begin{aligned} \min_{\pi \geq 0} & \sum_{(x,y)\in\mathcal{A}} \pi_{xy} c_{xy} + \sigma \sum_{xy} \pi_{xy} \ln \pi_{xy} \\ s.t.\ & \nabla^{\intercal} \pi = n \end{aligned} \tag{3}$$

(a) Write down the dual problem and the optimality conditions.

(b) Suggest several (at least two) algorithms to solve this problem.

(c) Write down the code for these algorithms.