'MATH+ECON+CODE' MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

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Day 5, January 18 2019: "Empirical matching models"
Block 13. Separable models of matching

► Matching with unobserved heterogeneities
► Estimation of matching models

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► [CSW] Chiappori, Salanié, and Weiss (2017). "Partner Choice and the Marital College Premium," AER.

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► [DG] Dupuy and Galichon (2014). "Personality traits and the marriage market," *Journal of Political Economy*.

► [COQ2] Chiappori, Oreffice and Quintana-Domeque (2016). "Bidimensional Matching with Heterogeneous Preferences in the Marriage Market," Preprint.

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Section 1

THEORY

MOTIVATION

▶ In the footsteps of Becker, empirical studies on the marriage market had long been focused on one-dimensional models and positive assortative matching (PAM).

▶ However, it is desirable to move beyond PAM:
  ▶ PAM is always loosely true, never precisely
  ▶ there are often many observed characteristics, and it is not always the case that the sorting can be captured by a single-dimensional model
  ▶ PAM is a theoretical prediction stemming from assumptions of supermodularity of the surplus function which do not necessarily hold
  ▶ optimal transport provide tools to study multidimensional models

▶ However, any model of matching based on optimal transport will not be exploitable because it will generate far too strong predictions, namely that some matchings will never hold. This is rather counterfactual: in the data, one observes virtually any combination of type.

▶ Hence, need to regularize the matching model, and we shall do so by introducing unobserved heterogeneity. The model so obtained will be exploitable for estimation and identification purposes. The first such model (with transfers) is the model by Choo and Siow (2006). We shall see a generalization of this model by G and Salanié (2015).

- ▶ Our data are Choo and Siow's original data. Choo and Siow wanted to study the impact of the legalization of abortion by the Roe vs. Wade decision by the supreme court on the "value of marriage". Roe vs. Wade decreased the role of marriage in covering out-of-the-wedlocks pregnancies ("shotgun weddings").

- ▶ The decision did, however, not make a change uniformly in the United States as a number of states had already legalized abortion (reform states). Choo and Siow thus offer a diffs-in-diffs approach in order to compute the change in the value of marriage.

- ▶ Choo and Siow's data are thus made of the marriages between men and women in reformed states (R) vs nonreformed states (NR), in 1972 and in 1982. One should expect to see a higher drop in marriage value in NR states.

► The reference for this lecture is [CS] and [GS]. Consider a heterosexual marriage matching market. The set of types (observable characteristics) is $\mathcal{X}$ for men, and $\mathcal{Y}$ for women. There are n_x men of type x, and m_y women of type y.

► Assume that if a man $i \in \mathcal{I}$ of type x_i and a woman $j \in \mathcal{J}$ of type y_j match, they get respective utilities

$$\alpha_{x_i y_j} + w_{ij} + \varepsilon_{i y_j}$$
$$\gamma_{x_i y_j} - w_{ij} + \eta_{x_i j}$$

where w_{ij} is the transfer from i to j. If they remain single i and j get respectively ε_{i0} and η_{0j}.

► The random utility vectors (ε_y) and (η_x) are drawn from probability distributions $\mathbf{P}_x$ and $\mathbf{Q}_y$, respectively. In the sequel we shall work with a finite number of agents of each type, and then we'll investigate the limit of these results.

OPTIMAL MATCHING

▶ The matching surplus between i and j is therefore

$$\tilde{\Phi}_{ij} = \Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x_i j}$$

where $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$. The value of optimal matching is thus, under its dual form,

$$\min_{u_i, v_j} \sum_{i \in \mathcal{I}} u_i + \sum_{j \in \mathcal{J}} v_j$$
$$s.t. \ u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x_i j}$$
$$u_i \geq \varepsilon_{i0}$$
$$v_j \geq \eta_{j0}$$

▶ Written like this, the lp has $|\mathcal{I}| + |\mathcal{J}|$ variables and $|\mathcal{I}| \times |\mathcal{J}| + |\mathcal{I}| + |\mathcal{J}|$ constraints. Assuming that there are K individuals per type for each type, this is $K(|\mathcal{X}| + |\mathcal{Y}|)$ variables and $K^2(|\mathcal{X}| \times |\mathcal{Y}|) + K(|\mathcal{X}| + |\mathcal{Y}|)$ constraints.

▶ The number of constraints is **quadratic** with respect to K. Fortunately, economic reasoning will help us reduce this complexity.

LEMMA

Consider the set $\mathcal{I}_{xy}$ of men of type x matched to women of type y at equilibrium. If $\mathcal{I}_{xy}$ is nonempty, then $u_i - \varepsilon_{iy}$ is a constant across $\mathcal{I}_{xy}$.

PROOF.

For $i \in \mathcal{I}$ such that $x_i = x$,

$$u_i = \max_{j \in \mathcal{J}} \left\{ \tilde{\Phi}_{ij} - v_j, \varepsilon_{i0} \right\}$$
$$= \max_{y \in \mathcal{Y}} \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$$

where $U_{xy} = \max_{j:y_j=y} \left\{ \Phi_{xy} + \eta_{x_i j} - v_j \right\}$, thus $u_i \geq U_{xy} + \varepsilon_{iy}$ with equality on $\mathcal{I}_{xy}$. With similar notations, $v_j \geq V_{xy} + \eta_{xj}$ with equality on $\mathcal{J}_{xy}$. As a result, if $\mathcal{I}_{xy}$ is nonempty, then $U_{xy} + V_{xy} = \Phi_{xy}$ and $\forall i \in \mathcal{I}_{xy}$, $u_i = U_{xy} + \varepsilon_{iy}$. $\qquad\square$

▶ In the sequel, we shall see that *adding* an auxiliary variable to the previous lp will lead to *decreasing* the computational complexity of the problem.

▶ Observe that the first set of constraints is reexpressed by saying that, for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$,

$$\min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} + \min_{j:y_j=y} \{v_j - \eta_{xj}\} \geq \Phi_{xy}.$$

▶ Hence, letting $U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\}$ and $V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{xj}\}$, a solution of the previous lp should satisfy

$$u_i = \max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \text{ and } v_j = \max_{x \in \mathcal{X}} \{V_{xy} + \varepsilon_{xj}, \varepsilon_{0j}\}.$$

▶ The problem rewrites as

$$\min_{u_i, v_j, U_{xy}, V_{xy}} \sum_{i \in \mathcal{I}} u_i + \sum_{j \in \mathcal{J}} v_j \tag{1}$$

$$s.t. \ U_{xy} + V_{xy} \geq \Phi_{xy} \ [\mu_{xy} \geq 0]$$
$$u_i \geq U_{x_i y} + \varepsilon_{i y_j} \ [\mu_{iy}]$$
$$v_j \geq V_{x y_j} + \eta_{x_i j} \ [\mu_{xi}]$$
$$u_i \geq \varepsilon_{i0} \ [\mu_{i0}]$$
$$v_j \geq \eta_{j0} \ [\mu_{0x}]$$

▶ This problem has $K \left(|\mathcal{X}| + |\mathcal{Y}| \right) + |\mathcal{X}| \times |\mathcal{Y}|$ variables and $\left(|\mathcal{X}| \times |\mathcal{Y}| \right) + K \left(2 |\mathcal{X}| \times |\mathcal{Y}| + |\mathcal{X}| + |\mathcal{Y}| \right)$ constraints.

▶ The number of constraint is now **linear** with respect to K.

- ▶ Lagrange multipliers:
    - ▶ The Lagrange multiplier μ_{xy} is interpreted as the number of matchings between types x and y.
    - ▶ The Lagrange multiplier μ_{iy} ($y \in \mathcal{Y}_0$) is interpreted as a 0-1 indicator that man i chooses a type y
    - ▶ The Lagrange multiplier μ_{xj} ($x \in \mathcal{X}_0$) is interpreted as a 0-1 indicator that woman j chooses a type x
- ▶ Utilities:
    - ▶ i solves a discrete choice problem $u_i = \max_{y \in \mathcal{Y}} \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$
    - ▶ j solve a discrete choice problem $v_j = \max_{x \in \mathcal{X}} \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\}$.
- ▶ U_{xy} and V_{xy} are related by $U_{xy} + V_{xy} \geq \Phi_{xy}$ with equality if $\mu_{xy} > 0$.

▶ [GS] look at the limit of previous markets when the number of market participants gets large, holding fixed the frequency of each types.

▶ In the large population limit n_x and m_y are now interpreted as the mass distribution of respective types x and y.

▶ We shall from now on assume that $\mathbf{P}_x$ and $\mathbf{Q}_y$, the distributions of random utility vectors (ε_y) and (η_x), have a density with full support. This will ensure that G, G^*, H and H^* are continuously differentiable.

LARGE POPULATION LIMIT

▶ Under these assumptions, Problem (1) becomes

$$\min_{U,V} G(U) + H(V)$$
$$s.t.\ U_{xy} + V_{xy} \geq \Phi_{xy}\ [\mu_{xy}]$$

where

$$G(U) = \sum_{x \in \mathcal{X}} n_x \mathbb{E}_{\mathbf{P}} \left[\max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \right]$$

$$H(V) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E}_{\mathbf{Q}} \left[\max_{x \in \mathcal{X}} \{V_{xy} + \eta_{xj}, \eta_{0j}\} \right]$$

▶ By first order conditions,

$$\frac{\partial G(U)}{\partial U_{xy}} = \mu_{xy} = \frac{\partial H(V)}{\partial V_{xy}}.$$

and $\mu_{xy} > 0$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

SOCIAL PLANNER'S PROBLEM

▶ The primal problem corresponding to (1) is

$$\max_{\mu_{xy} \geq 0} \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy} \Phi_{xy} - \mathcal{E}(\mu)$$

where

$$\mathcal{E}(\mu) = G^*(\mu) + H^*(\mu)$$

▶ Recall $G^*(\mu) = \max \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\}$ is the Legendre transform of G, and similarly for H^*.

▶ By first order conditions, we get the identifcation formula of Φ

$$\Phi_{xy} = \frac{\partial G^*(\mu)}{\partial \mu_{xy}} + \frac{\partial H^*(\mu)}{\partial \mu_{xy}}.$$

CHOO AND SIOW'S TU-LOGIT MODEL

▶ In Choo and Siow's model [CS], the heterogeneities in tastes are Gubmel, this

$$\mathcal{E}\left(\mu\right) = 2 \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy} \log \mu_{xy} + \sum_{x \in \mathcal{X}} \mu_{x0} \log \mu_{x0} + \sum_{y \in \mathcal{Y}} \mu_{0y} \log \mu_{0y}.$$

Note that $\mathcal{E}\left(\mu\right) < \infty$ if and only if $\mu \in \mathcal{M}\left(n, m\right)$.

▶ We call this model the *TU-logit* model. Tomorrow, we shall see how to generalize this to *ITU-logit* models, including *NTU-logit*.

▶ By first order conditions above, Choo-Siow's TU-logit model implies the following matching function:

$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right) := \sqrt{\mu_{x0}}\sqrt{\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right).$$

IDENTIFICATION IN THE TU-LOGIT MODEL

▶ As a result, $\partial \mathcal{E}(\mu) / \partial \mu_{xy} = 2 \log \mu_{xy} - \log \mu_{x0} - \log \mu_{0y}$, which implies that Φ_{xy} is estimated by *Choo and Siow's identification formula*

$$\hat{\Phi}_{xy} = \log \frac{\hat{\mu}_{xy}^2}{\hat{\mu}_{x0} \hat{\mu}_{0y}}.$$

▶ Consistent with our previous remarks, only the joint surplus $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$ is identified. However, if the transfers $\hat{w}_{xy}$ are observed too (e.g. wages in labour market), then $U_{xy} = \alpha_{xy} + w_{xy}$ and $V_{xy} = \gamma_{xy} - w_{xy}$, so that α and γ are separately identified by

$$\begin{cases} \hat{\alpha}_{xy} = \log \frac{\hat{\mu}_{xy}}{\hat{\mu}_{x0}} - \hat{w}_{xy} \\ \hat{\gamma}_{xy} = \log \frac{\hat{\mu}_{xy}}{\hat{\mu}_{0y}} + \hat{w}_{xy} \end{cases}$$

▶ Write down the equilibrium equations in the TU-logit model:

$$\begin{cases} \sum_{y \in \mathcal{Y}} \sqrt{\mu_{x0}}\sqrt{\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{x0} = n_x \\ \sum_{x \in \mathcal{X}} \sqrt{\mu_{x0}}\sqrt{\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{0y} = m_y \end{cases}$$

▶ Setting $a_x = \sqrt{\mu_{x0}}$, $b_y = \sqrt{\mu_{0y}}$, and $K_{xy} = \exp\left(\Phi_{xy}/2\right)$, this rewrites as

$$\begin{cases} \sum_{y \in \mathcal{Y}} K_{xy} a_x b_y + a_x^2 = n_x \\ \sum_{x \in \mathcal{X}} K_{xy} a_x b_y + b_y^2 = m_y \end{cases}$$

▶ Hence, the IPFP consists in iteratively solving quadratic equations, i.e.

$$\begin{cases} a_x^{2t+1} = \sqrt{n_x + \left(\sum_{y\in\mathcal{Y}} b_y^{2t} K_{xy}/2\right)^2} - \sum_{y\in\mathcal{Y}} b_y^{2t} K_{xy}/2 \\ b_y^{2t+2} = \sqrt{m_y + \left(\sum_{x\in\mathcal{X}} a_x^{2t+1} K_{xy}/2\right)^2} - \sum_{x\in\mathcal{X}} a_x^{2t+1} K_{xy}/2 \end{cases}$$

▶ This is implemented by:
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Kb_x = c(K_xy %*% b_y)
a_x = sqrt(n_x + Kb_x * Kb_x /4 ) - Kb_x / 2
Ka_y = c(t(a_x) %*% K_xy )
b_y = sqrt(m_y + Ka_y * Ka_y /4 ) - Ka_y / 2
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▶ Cf [GS]. The dual problem is given by

$$\min_{u,v} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y \\ +2\sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) \\ + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right) \end{array} \right\}$$

▶ Remarks:
  ▶ This problem is an unconstrained convex optimization problem, so this formulation will be quite useful.
  ▶ If (u,v) is solution, $u_x = -\log \mu_{0|x} = -\log \left(\mu_{x0}/n_x\right)$ and $v_y = -\log \left(\mu_{0|y}\right)$.
  ▶ Note that the IPFP algorithm just seen interprets as (blockwise) **coordinate descent** in the dual problem.

▶ Now assume that the scaling factor (=standard deviation) of the Gumbel is σ. Instead of ε_y and η_x, these heterogeneities are $\sigma\varepsilon_y$ and $\sigma\eta_x$. As a result, the entropy is now $\sigma\mathcal{E}(\mu)$ instead of $\mathcal{E}(\mu)$.

▶ The expression of the social welfare is now

$$\max_{\mu \in \mathcal{M}(n,m)} \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy}\Phi_{xy} - \sigma\mathcal{E}(\mu)$$

which tends to the usual optimal assignment problem when $\sigma \to 0$ (no heterogeneity). Tends to random matching solution when $\sigma \to +\infty$.

Parametric estimation of the TU-logit model

▶ Assume Φ_{xy} is parameterized as

$$\Phi_{xy}^{\lambda} = \sum_k \lambda_k \phi_{xy}^k$$

so that the welfare function can be written

$$W(\lambda) = \max_{\mu \in \mathcal{M}(n,m)} \left\{ \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy} \Phi_{xy}^{\lambda} - \sigma \mathcal{E}(\mu) \right\}$$

$$= \min_{u,v} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y + 2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^{\lambda} - u_x - v_y}{2} \right) \\ + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right) \end{array} \right\}$$

▶ As a result,

$$\frac{\partial W}{\partial \lambda_k} = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy}^{\lambda} \phi_{xy}^k$$

where μ^{λ} is the optimal μ given λ.

▶ We will pick the parameter λ such that $\sum_{xy} \mu_{xy}^\lambda \phi_{xy}^k = \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^k$ for each k.

▶ For this, do

$$\min_\lambda \left\{ W(\lambda) - \sum_{xy} \hat{\mu}_{xy} \Phi_{xy}^\lambda \right\}$$

which reexpresses using the dual expression as

$$\min_{u,v,\lambda} \left\{ \begin{array}{l} \sum_x n_x u_x + \sum_y m_y v_y + 2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) \\ + \sum_x n_x \exp(-u_x) + \sum_y m_y \exp(-v_y) - \sum_{xy} \hat{\mu}_{xy} \Phi_{xy}^\lambda \end{array} \right\}.$$

▶ The set of vectors C^λ where $C_k^\lambda = \sum_{xy} \mu_{xy}^\lambda \phi_{xy}^k$ is a convex and compact set called *covariogram*. Its boundaries correspond to solutions of the Monge-Kantorovich problem without regularization; the point 0 corresponds to the independent coupling ($\|\Phi\| / \sigma \to 0$).

▶ Note that increasing λ^k will increase the weight of ϕ^k in the optimization problem, and thus will increase the corresponding moment C_k^λ.

▶ We look for λ such that the predicted covariances C_k^λ equates the observed covariances $\hat{C}_k = \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^k$, which is computed from our sample by

$$\hat{C}_k = \frac{1}{n} \sum_{i=1}^{n} \phi_{x_i y_i}^k.$$