Exercice: Tiago de Oliveira's formula

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Exercise (Tiago de Oliveira's formula). The goal of this exercise is to show a formula due to Tiago de Oliveira: assume that $F(x,y) = \exp(-(\exp(-x)+\exp(-y))\,k(y-x))$. Then

$$cov(X,Y) = -\int_{-\infty}^{+\infty} \log k(w)\,dw. \tag{1}$$

(a) Show Hoeffding's formula:

$$cov(X,Y) = \int (F(x,y) - F(x)F(y))\,dxdy.$$

(b) Consider the change of variables $w = y-x$ and $v = -\log(\exp(-x)+\exp(-y))$. Show that

$$cov(X,Y) = \int \exp(-\exp(-v)\,k(w)) - \exp(-\exp(-v))\,dvdw.$$

(c) Deduce Tiago de Oliveira's formula (1).

(d) Application: when

$$\mathbf{F}_{\varepsilon}(x,y) = \exp\left(-\left(e^{-x/\lambda}+e^{-y/\lambda}\right)^{\lambda}\right),$$

show that

$$cor(\varepsilon_1,\varepsilon_2) = \frac{\pi^2}{6}\left(1-\lambda^2\right).$$

[Hint: one may admit without a proof that

$$\int_0^{+\infty} \log\left(1+e^{-w}\right)dw = \pi^2/12$$

which is $\eta(2)$, where η is Dirichlet's eta function.]