Advanced topics in microeconometrics: Matching Models and their Applications

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Lecture 11. Matching models 2: models with imperfectly transferable utility

- Matching with imperfectly transferable utility
- Collective models, private and public consumption, endogenous sharing rule
- Galois connections, distance-to-frontier function
- nonlinear complementary slackness
- equilibrium transport
- collective models, sharing rule, Pareto weights

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Section 1

MATCHING FUNCTION EQUILIBRIA

- A matching function is a function $M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$ which is isotone and such that
$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right).$$
- A matching function equilibrium (MFE) is the set of equations
$$\left\{ \begin{array}{c} n_x = \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) \\ m_y = \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) \end{array} \right.$$
- Today, we will first provide examples of models that reformulate as MFEs; then we will discuss existence, computation, and uniqueness of a MFE, and then we'll discuss comparative statics.

- This model was seen last time. Consider a labor market where $\mathcal{X}$ are the types of the workers and $\mathcal{Y}$ are the types of the firms. There are n_x workers of type x, and m_y firms of type y.
- Let w_{xy} be the equilibrium salary of a worker of type x working for a firm y. Assume that worker $x \in \mathcal{X}$ has utility for matching with firm of type y equal to

$$\alpha_{xy} + w_{xy} + \varepsilon_y$$

  and ε_0 if remains unemployed, where the random utility vector ε is a vector of i.i.d. Gumbel distributions drawn by each worker, and α is a term that captures job amenity.
- Similarly, the profit of the firm is

$$\gamma_{xy} - w_{xy} + \eta_y$$

  and η_0 if it does not hire, where the random utility vector η is a vector of i.i.d. Gumbel distributions drawn by each worker, and γ is a term that captures job productivity.

- The quantities (α, γ, n, m) as well as the distributions of ε and η are exogenous: they are primitives of the model. The equilibrium quantities are the matching patterns (μ_{xy}), as well as the equilibrium wages w_{xy}.
- The conditional choice probabilities on the side of workers is $\mu_{y|x} = \mu_{xy} / n_x$, and the CCPs on the side of firms is $\mu_{x|y} = \mu_{xy} / m_y$. Because we are in a logit model, we have by the log-odds ratio formula that

$$\alpha_{xy} + w_{xy} = \ln \frac{\mu_{xy}}{\mu_{x0}} \text{ and } \gamma_{xy} - w_{xy} = \ln \frac{\mu_{xy}}{\mu_{0y}},$$

so by summation, $\alpha_{xy} + \gamma_{xy} = 2 \ln \mu_{xy} - \ln \mu_{x0} - \ln \mu_{0y}$, thus $\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}$, where $K_{xy} = \exp\left(\frac{\alpha_{xy}+\gamma_{xy}}{2}\right)$, that is

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}. \tag{1}$$

- Note that TU models with general heterogeneities do not have a MMF formulation.

- A *matching function equilibrium* is a solution of the following system with unknowns μ_{x0} and μ_{0y}:

$$\left\{ \begin{array}{c} \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = m_y \end{array} \right. . \tag{2}$$

- In the sequel we will consider the following questions:
  - Existence of an equilibrium
  - Algorithms for the determination of an equilibrium
  - Uniqueness of an equilibrium

We will assume the following about the matching functions:

ASSUMPTION

M is such that for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$:
(i) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is continuous.
(ii) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is weakly isotone, i.e. if $a \leq a'$ and $b \leq b'$, then $M_{xy}(a, b) \leq M_{xy}(a', b')$.
(iii) For each $a > 0$, $\lim_{b \to 0^+} M_{xy}(a, b) = 0$, and for each $b > 0$, $\lim_{a \to 0^+} M_{xy}(a, b) = 0$.

ALGORITHM

Step 0. Fix the initial value of μ_{0y} at $\mu_{0y}^{0} = m_y$.
Step $2t+1$. Keep the values μ_{0y}^{2t} fixed. For each $x \in \mathcal{X}$, solve for the value, μ_{x0}^{2t+1}, of μ_{x0} so that

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x.$$

Step $2t+2$. Keep the values μ_{x0}^{2t+1} fixed. For each $y \in \mathcal{Y}$, solve for which is the value, μ_{0y}^{2t+2}, of μ_{0y} so that

$$\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y.$$

The following theorem from [GKW] ensures that the algorithm converges to a matching function equilibrium.

THEOREM

Under Assumptions (i)–(iii) above, there exists a matching function equilibrium which is the limit of $\left(\mu_{x0}^{2t+1}, \mu_{0y}^{2t+2}\right)$ *defined in the algorithm.*

- We show that the construction of μ_{x0}^{2t+1} and μ_{0y}^{2t+2} at each step is well defined. Consider step $2t+1$. For each $x \in \mathcal{X}$, the equation to solve is

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$

but the right-hand side is a continuous and increasing function of μ_{x0}, tends to 0 when $\mu_{x0} \to 0$ and tends to $+\infty$ when $\mu_{x0} \to +\infty$. Hence μ_{x0}^{2t+1} is well defined and is in $(0, +\infty)$. The map$\left(\mu_{0y}^{2t}\right) \to \left(\mu_{x0}^{2t+1}\right)$ is antitone, meaning that $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$ for all $y \in \mathcal{Y}$ implies $\tilde{\mu}_{x0}^{2t+1} \leq \mu_{x0}^{2t+1}$ for all $x \in \mathcal{X}$.
- By the same token, the map$\left(\mu_{x0}^{2t+1}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is well defined and antitone. Thus, the map $\left(\mu_{0y}^{2t}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is isotone. But $\mu_{0y}^{2} \leq m_y = \mu_{0y}^{0}$ implies that $\mu_{0.}^{2t+2} \leq \mu_{0.}^{2t}$. Hence $\left(\mu_{0y}^{2t}\right)_{t \in \mathbb{N}}$ is a decreasing sequence, bounded from below by 0. As a result $\left(\mu_{0y}^{2t}\right)$ converges. Letting $(\bar{\mu}_{0y})$ be its limit, and letting $(\bar{\mu}_{x0})$ be the limit of $\left(\mu_{x0}^{2t+1}\right)$, it is not hard to see that $(\bar{\mu}_{0x}, \bar{\mu}_{0y})$ is a solution to (2).

THEOREM

Under Assumptions (i)–(iii) above, and under the additional assumption that the domain of M_{xy} is $\mathbb{R}_+^2$ for each x and y, the matching function equilibrium is unique.

The proof of this theorem (proven more generally in [GKW]) is based on reformulating the equilibrium as a demand system and applying the result of Berry, Gandhi and Haile (2013).

Section 2

Matching with imperfectly transferable utility

- The two main tools of Family Economics are **Matching Models** and **Collective Models**.
- Matching models are interested in sorting: "who matches with whom". They generally regard the question of bargaining within the household as a black box, assuming either no transfer (**Nontransferable Utility – NTU**) or additive transfers (**Transferable Utility – TU**).
- On the contrary, collective models seek to understand how utility is distributed within the household: agents bargain and reach the efficient frontier according to a "sharing rule". However, in most of these models, the **sharing rule is exogenous**.
- The goal of this talk is to bring together those two models, in order to open the black box of utility transfers in matching models, and endogenize the sharing rule. For this we'll need to go beyond TU and NTU models and work with more general models with **Imperfectly Transferable Utility (ITU)**.

- Transfers of utility (under the form of money or other exchanges):
  - Are sometimes clearly forbidden (e.g. **school choice problems**): NTU matching.
  - Are sometimes clearly allowed (e.g. **wages in the market for CEOs**): TU matching.
  - Sometimes are allowed but imperfect: **labor market** with nonquasilinear utilities (Kelso and Crawford, Hatfield and Milgrom); of taxes (Jaffe and Kominers); of risk aversion (Legros and Newman, Chade and Eckehout); of investments (Samuelson and Noeldeke): ITU matching**.**
- For the **marriage market**, the literature is split.
  - Indeed, there is a tradition to model the marriage market with transfers (Shapley and Shubik; Becker; Choo and Siow) and without transfers (Gale and Shapley; Dagsvik; Hitsch, Hortacsu and Ariely; Menzel).
  - It makes sense to assume that transfers whithin the couple are possible, but not efficient in the sense that the utility transfered by i to j maybe more costly to i that it is beneficial to j: ITU matching**.**

- Assume that there are groups, or clusters of men and women who share similar observable characteristics, called *types*. There are n_x men of type x, and m_y women of type y.
- Let $\mu_{xy} \geq 0$ be the number of men of type x matched to women of type y. This quantity satisfies

$$\sum_y \mu_{xy} \leq n_x$$

$$\sum_x \mu_{xy} \leq m_y$$

- We shall denote μ_{x0} and μ_{0y} the number of single men of type x and single women of type y.

- **Assumption 1**: Assume that if man i of type x and woman j of type y match, then they respectively get

$$u_i = U_i + \varepsilon_{iy}$$
$$v_j = V_j + \eta_{jx}$$

where the systematic part of the utilities U_i and V_j satisfy the feasibility equation

$$(U_i, V_j) \in \mathcal{F}_{xy}$$

where $\mathcal{F}_{xy}$ is a nonempty closed subset of $\mathbb{R}^2$ which is such that if $(u, v) \in \mathcal{F}_{xy}$, then $(u - c, v - c) \in \mathcal{F}_{xy}$ for $c \geq 0$.
- **Assumption 2**: there are a large number of invididuals per group and the ε and the η's random vectors whose components are iid type-I extreme value distributions.

- We can introduce the distance-to-frontier function associated with set $\mathcal{F}_{xy}$ as

$$D_{xy}(u,v) = \inf \{c \geq 0 : (u-c, v-c) \in \mathcal{F}_{xy}\}.$$

- One has $D_{xy}(u,v) \leq 0$ if and only if $(u,v) \in \mathcal{F}_{xy}$, and $D_{xy}(u,v) = 0$ if and only if (u,v) is on the frontier of $\mathcal{F}_{xy}$.
- Further, $D_{xy}(u+c, v+c) = D_{xy}(u,v) + c$ for any $c \in \mathbb{R}$.

- Consider a model where partners need to jointly decide on a public good $g \in G$ which is assumed to be discrete (for instance, the number of children they will have together) and on their private consumption, and assume that the utilities are given by

$$\begin{aligned} u_i &= \alpha_{xy}(g) + \ln c_i \\ v_j &= \gamma_{xy}(g) + \ln c_j \end{aligned}$$

  where $c_i + c_j = I_x + I_y - C(g)$ where $C(g)$ is the cost of the public good, and I_x and I_y are the income of the partners.
- One has

$$\exp\left(u_i - \alpha_{xy}(g)\right) + \exp\left(v_j - \gamma_{xy}(g)\right) = I_x + I_y - C(g),$$

  which defines the feasible set $\mathcal{F}_{xy}^{g}$ conditional on choosing public good g, while the feasible set $\mathcal{F}_{xy}$ is the union of $\mathcal{F}_{xy}^{g}$ over $g \in G$.

A model with private consumption and a public good: distance function

- It is easy to see that conditional on choosing public good g, the distance function associated with $\mathcal{F}_{xy}^{g}$ is

$$D_{xy}^{g}\left(u,v\right)=\log\left(\frac{\exp\left(u-A_{xy}\left(g\right)\right)+\exp\left(v-G_{xy}\left(g\right)\right)}{2}\right)$$

where

$$A_{xy}\left(g\right)=\alpha_{xy}\left(g\right)+\log\left(\frac{I_{x}+I_{y}-C\left(g\right)}{2}\right)$$
$$G_{xy}\left(g\right)=\gamma_{xy}\left(g\right)+\log\left(\frac{I_{x}+I_{y}-C\left(g\right)}{2}\right)$$

- It is also easy to see that the distance function associated with $\mathcal{F}_{xy}=\bigcup_{g}\mathcal{F}_{xy}^{g}$ is

$$D_{xy}\left(u,v\right)=\min_{g\in G}D_{xy}\left(u,v\right).$$

- We are now ready to define an equilibrium in this market.
- An *outcome* (μ, u, v) is an equilibrium if
  - (i) The matching is feasible:

$$\begin{cases} \mu_{ij} \in \{0, 1\} \\ \sum_j \mu_{ij} \leq 1, \text{ and } \sum_i \mu_{ij} \leq 1. \end{cases}$$

  - (ii) There is no blocking pair:

$$\begin{cases} D_{x_i y_j}\left(u_i - \varepsilon_{iy_j}, v_j - \eta_{x_i j}\right) \geq 0, \\ u_i - \varepsilon_{iy_j} \geq 0, \text{ and} \\ v_j - \eta_{x_i j} \geq 0. \end{cases}$$

  - (iii) (Nonlinear) complementary slackness holds:

$$\begin{cases} \mu_{ij} = 1 \text{ implies } D_{x_i y_j}\left(u_i - \varepsilon_{iy_j}, v_j - \eta_{x_i j}\right) = 0, \\ \sum_j \mu_{ij} < 1 \text{ implies } u_i - \varepsilon_{iy_j = 0}, \text{ and} \\ \sum_i \mu_{ij} < 1 \text{ implies } v_j - \eta_{x_i j} = 0. \end{cases}$$

- For i and j such that $x_i = x$ and $y_j = y$, one has

$$D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{jx}\right) \geq 0.$$
$$\mu_{ij} = 1 \Longrightarrow D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{jx}\right) \leq 0$$

- This allows to define

$$U_{xy} = \min_{i:x_i=x} \left\{u_i - \varepsilon_{iy}\right\} \text{ and } V_{xy} = \min_{j:y_j=y} \left\{v_j - \eta_{xj}\right\}$$

so that

$$D_{xy}\left(U_{xy}, V_{xy}\right) = 0$$

and that μ_{xy} is related to U and V by

$$\mu_{xy} = \sum_{i:x_i=x} \sum_{j:y_j=y} 1\left\{u_i = U_{xy} + \varepsilon_{iy}\right\} = \sum_{i:x_i=x} \sum_{j:y_j=y} 1\left\{v_j = V_{xy} + \eta_{xj}\right\}.$$

- As a result, at equlibrium

$$u_i = \max_{y \in \mathcal{Y}} \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\} \text{ and } v_j = \max_{x \in \mathcal{X}} \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\}$$

where

$$D_{xy} \left(U_{xy}, V_{xy} \right) = 0,$$

and

$$n_x \Pr \left(u_i = U_{xy} + \varepsilon_{iy} | x_i = x \right) = m_y \Pr \left(v_j = V_{xy} + \eta_{xj} | y_j = y \right),$$

where this common number is denoted μ_{xy}.

- This yields a set of nonlinear equations that are quite simple to express in the logit case.

- Man i and woman j (of types x and y) solve respectively

$$\max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$$
$$\max_x \left\{ V_{xy} + \eta_{jx}, \eta_{j0} \right\}$$

which are standard discrete choice problems; thus in the logit case, the log-odds ratio formula applies, and

$$\ln \frac{\mu_{xy}}{\mu_{x0}} = U_{xy}$$
$$\ln \frac{\mu_{xy}}{\mu_{0y}} = V_{xy}$$

But remember that $D_{xy}\left(U_{xy}, V_{xy}\right) = 0$, thus

$$D_{xy}\left(\ln \frac{\mu_{xy}}{\mu_{x0}}, \ln \frac{\mu_{xy}}{\mu_{0y}}\right) = 0, \text{ hence}$$
$$\mu_{xy} = \exp\left(-D_{xy}\left(-\log \mu_{x0}, -\log \mu_{0y}\right)\right).$$

- Equilibrium in the ITU problem with logit heterogeneities is therefore characterized by a matching function equilibrium with matching function

$$M_{xy}\left(\mu_{x0},\mu_{0y}\right) := \exp\left(-D_{xy}\left(-\log\mu_{x0},-\log\mu_{0y}\right)\right)$$

- Equilibrium is characterized by the set of nonlinear equations in μ_{xy}, μ_{x0} and μ_{0y}

$$\begin{aligned}\mu_{xy} &= M_{xy}\left(\mu_{x0},\mu_{0y}\right)\\ \sum_y \mu_{xy}+\mu_{x0} &= n_x\\ \sum_x \mu_{xy}+\mu_{0y} &= m_y\end{aligned}$$