

# ADVANCED TOPICS IN MICROECONOMETRICS: MATCHING MODELS AND THEIR APPLICATIONS

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Lecture 9. Demand models

## IMPORTANT ANNOUNCEMENT REGARDING NEXT LECTURES

- ▶ Mar 5, 2018: class from 8am to 12noon
- ▶ Mar 12, 2018: regular schedule
- ▶ Mar 19, 2018: class from 8am to 12noon
- ▶ Mar 26, 2018: regular schedule
- ▶ April 2, 2018: no class (holiday)
- ▶ April 9, 2018: class cancelled and replaced
- ▶ April 16, 2018: class cancelled and replaced
- ▶ April 23, 2018: class cancelled and replaced
- ▶ April 30, 2018: regular schedule
- ▶ May 7, 2018: exam 10am-12noon

- ▶ Beyond GEV: the pure characteristics models, the random coefficient logit model, the probit model
- ▶ Simulation methods: AR, GHK, and SARS
- ▶ The inversion theorem

- ▶ [OTME], Ch. 9.2
- ▶ McFadden (1981). “Econometric Models of Probabilistic Choice,” in C.F. Manski and D. McFadden (eds.), *Structural analysis of discrete data with econometric applications*, MIT Press.
- ▶ Berry, Levinsohn, and Pakes (1995). “Automobile Prices in Market Equilibrium,” *Econometrica*.
- ▶ Berry and Pakes (2007). “The pure characteristics demand model”. *International Economic Review*
- ▶ Train (2009). *Discrete Choice Methods with Simulation*. 2nd Edition. Cambridge University Press.
- ▶ G, Salanié (2017). “Cupids invisible hand”. Working paper.
- ▶ Bonnet, G and Shum (2017). “Yogurts choose consumers? Identification of Random Utility Models via Two-Sided Matching”. Working paper.

# Section 1

## THEORY

- ▶ The GEV models are convenient analytically, but not very flexible.
  - ▶ The logit model imposes zero correlation across alternatives
  - ▶ The nested logit allows for nonzero correlation, but in a very rigid way (needs to define nests).
- ▶ A good example is the probit model, where  $\varepsilon$  is a Gaussian vector. For this model, there is no close-form solution neither for  $G$  nor for  $G^*$ .
- ▶ More recently, a number of modern models don't have closed-form either. These models require simulation methods in order to approximate them by discrete models.

- ▶ The pure characteristics model (Berry and Pakes, 2007) can be motivated as follows. Assume  $y$  stands for the number of bedrooms. The logit model would assume that the random utility associated with a 2-BR is uncorrelated with a 3-BR, which is not realistic.
- ▶ Let  $\zeta_y$  is the typical size of a bedroom of size  $y$ , one may introduce  $\epsilon$  as the valuation of size; in which case the utility shock associated with  $y$  should be  $\varepsilon_y = \epsilon \zeta_y$ . More generally, the characteristics  $\zeta_y$  is a  $d$ -dimensional (deterministic) vector, and  $\epsilon \sim \mathbf{P}_\epsilon$  is a (random) vector of the same size standing for the valuations of the respective dimensions, so that

$$\varepsilon_y = \epsilon^\top \zeta_y.$$

- ▶ For example, if each alternative  $y$  stands for a model of car, the first component of  $\zeta_y$  may be the price of car  $y$ ; the other components may be other characteristics such as number of seats, fuel efficiency, size, etc. In that case, for a given dimension  $y \in \mathcal{Y}_0$ ,  $\epsilon_y$  is the (random) valuation of this dimension by the consumer with taste vector  $\epsilon$ .

- ▶ Assume without loss of generality that  $\varepsilon_y = 0$ , that is  $\xi_0 = 0$  as we can always reduce the setting to this case by replacing  $\xi_y$  by  $\xi_y - \xi_0$ .
- ▶ Letting  $Z$  be the  $|\mathcal{Y}| \times d$  matrix of  $(y, k)$ -term  $\xi_y^k$ , this rewrites as

$$\varepsilon = Z\epsilon.$$

- ▶ Hence, we have

$$G(U) = \mathbb{E} [\max \{U + Z\epsilon, 0\}].$$

and

$$\sigma_y(U) = \Pr \left( U_y - U_z \geq (Z\epsilon)_y - (Z\epsilon)_z \quad \forall z \in \mathcal{Y}_0 \setminus \{y\} \right).$$



- When  $d = 1$  (scalar characteristics), one has  
 $\sigma_y(U) = \Pr(U_y - U_z \geq (\xi_y - \xi_z)\epsilon \ \forall z \in \mathcal{Y}_0 \setminus \{y\})$ , and thus

$$\sigma_y(U) = \Pr\left(\max_{z:\xi_y > \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\} \leq \epsilon \leq \min_{z:\xi_y < \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}\right)$$

with the understanding that  $\max_{z \in \emptyset} f_z = -\infty$  and  $\min_{z \in \emptyset} f_z = +\infty$ .

- Therefore, letting  $\mathbf{F}_\epsilon$  be the c.d.f. associated with the distribution of  $\epsilon$ , one has a closed-form expression for  $\sigma_y$ :

$$\sigma_y(U) = \mathbf{F}_\epsilon\left(\left[\max_{z:\xi_y > \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}, \min_{z:\xi_y < \xi_z} \left\{ \frac{U_y - U_z}{\xi_y - \xi_z} \right\}\right]\right)$$

- ▶ When  $\mathbf{P}_\epsilon$  is the  $\mathcal{N}(0, S)$  distribution, then the pure characteristics model is called a Probit model; in this case,

$$\varepsilon \sim \mathcal{N}(0, \Sigma) \text{ where } \Sigma = ZSZ^\top.$$

- ▶ Note the distribution  $\varepsilon$  will not have full support unless  $d \geq |\mathcal{Y}|$  and  $Z$  is of full rank.
- ▶ Computing  $\sigma$  in the Probit model thus implies computing the mass assigned by the Gaussian distribution to rectangles of the type

$$[l_y, u_y].$$

When  $\Sigma$  is diagonal (random utility terms are i.i.d. across alternatives), this is numerically easy. However, this is computationally difficult in general (more on this later).

- The random coefficient logit model (Berry, Levinsohn and Pakes, 1995) may be viewed as an interpolant between the random characteristics model and the logit model. In this case,

$$\varepsilon = (1 - \lambda) Z\epsilon + \lambda\eta$$

where  $\epsilon \sim \mathbf{P}_\epsilon$ ,  $\eta$  is an EV1 distribution independent from the previous term, and  $\lambda$  is a interpolation parameter ( $\lambda = 1$  is the logit model, and  $\lambda = 0$  is the pure characteristics model).

- In this case, one may compute the Emax operator as

$$\begin{aligned} G(U) &= \mathbb{E} \left[ \max_{y \in \mathcal{Y}_0} \left\{ U_y + (1 - \lambda) (Z\epsilon)_y + \lambda\eta_y \right\} \right] \\ &= \mathbb{E} \left[ \mathbb{E} \left[ \max_{y \in \mathcal{Y}_0} \left\{ U_y + (1 - \lambda) (Z\epsilon)_y + \lambda\eta_y \right\} \mid \epsilon \right] \right] \\ &= \mathbb{E} \left[ \lambda \log \sum_{y \in \mathcal{Y}_0} \exp \left( \frac{U_y + (1 - \lambda) (Z\epsilon)_y}{\lambda} \right) \right] \end{aligned}$$

► Recall

$$G(U) = \mathbb{E} \left[ \lambda \log \sum_{y \in \mathcal{Y}_0} \exp \left( \frac{U_y + (1 - \lambda)(Z\epsilon)_y}{\lambda} \right) \right].$$

- The demand map in the random coefficients logit model is obtained by derivation of the expression of the  $E_{\max}$ , i.e.

$$\sigma_y(U) = \mathbb{E} \left[ \frac{\exp \left( \frac{U_y + (1 - \lambda)(Z\epsilon)_y}{\lambda} \right)}{\sum_{y' \in \mathcal{Y}_0} \exp \left( \frac{U_{y'} + (1 - \lambda)(Z\epsilon)_{y'}}{\lambda} \right)} \right].$$

- ▶ In a number of cases, one cannot compute the choice probabilities  $\sigma(U)$  using a closed-form expression. In this case, we need to resort to simulation to compute  $G$ ,  $G^*$ ,  $\sigma$  and  $\sigma^{-1}$ .
- ▶ The idea is that:
  - ▶ one is able to compute  $G$  and  $G^*$  for discrete distributions (more on this later)
  - ▶ the sampled versions of  $G$ ,  $G^*$ ,  $\sigma$  and  $\sigma^{-1}$  converge to the populations objects when the sample size is large.

- One simulates  $N$  points  $\varepsilon^i \sim P$ . The Emax operator associated with the empirical sample distribution  $P_N$  is

$$G_N = N^{-1} \sum_{i=1}^N \max_{y \in \mathcal{Y}} \{U_y + \varepsilon_y^i\}$$

and the demand map is given by

$$\sigma_{N,y}(U) = N^{-1} \sum_{i=1}^N 1 \left\{ U_y + \varepsilon_y^i \geq U_z + \varepsilon_z^i \quad \forall z \in \mathcal{Y}_0 \right\}$$

- In the literature,  $\sigma_N$  is called the *accept-reject simulator*.

- ▶ GHK simulator (Geweke, Hajivassiliou and Keane) improves on the AR simulator.
- ▶ Without loss of generality, we shall focus on the market share of  $y = 0$ , and label the elements of  $\mathcal{Y}$  as  $\mathcal{Y} = \{1, \dots, M\}$  where  $M = |\mathcal{Y}|$ . Then if  $\mathbf{F}_\eta$  is the c.d.f. of the random vector  $\eta$  valued in  $\mathbb{R}^{\mathcal{Y}}$  defined by  $\eta_y := \varepsilon_y - \varepsilon_0$  for  $z \in \mathcal{Y}$ , then

$$\mu_0 = 1 - \mathbf{1}'_y \nabla G(U) = \mathbf{F}_\eta(z),$$

where  $z_y = -U_y$ , for all  $y \in \mathcal{Y}$ .

- ▶ Note that  $\mathbf{F}_\eta(z) = \Pr(\eta_1 \leq z_1, \dots, \eta_M \leq z_M)$  can be expressed as

$$\mu_0 = \prod_{j=1}^M \Pr(\eta_j \leq z_j | \eta_1 \leq z_1, \dots, \eta_{j-1} \leq z_{j-1}) \quad (1)$$

with the understanding that the first term in this product (associated with  $j = 1$ ) is simply the unconditional probability distribution  $\Pr(\eta_1 \leq z_1)$ .

- The fundamental assumption behind the GHK method is that the Rosenblatt quantile associated with the distribution of  $\eta$  is known.
- Recall from the exercise discussed in the previous lecture that the Rosenblatt quantile is the map  $T$  such that  $T\#\mu = P$ , and such that the Jacobian  $DT$  of  $T$  is lower triangular with nonnegative diagonal. That is,

$$\begin{cases} \eta_1 = T_1(U_1) \\ \eta_2 = T_2(U_1, U_2) \\ \dots \\ \eta_M = T_M(U_1, U_2, \dots, U_M) \end{cases},$$

where  $U \sim U([0, 1]^d)$ , and  $T_i(u)$  depends only on  $u_1, \dots, u_i$  and is a nondecreasing function of  $u_i$ . In order to evaluate quantity  $\mathbf{F}_\eta(z)$  in (1), one needs to evaluate  $\pi_j = \Pr(\eta_j \leq z_j | \eta_1 \leq z_1, \dots, \eta_{j-1} \leq z_{j-1})$ , that is

$$\pi_j = \mathbb{E} [1 \{ T_j(U_1, U_2, \dots, U_j) \leq z_j \} | T_1(U_1) \leq z_1, \dots, T_{j-1}(U_1, U_2, \dots, U_{j-1}) \leq z_{j-1} ] \quad (2)$$



- Denote  $T_i^{-1}(z; u_1, \dots, u_{i-1})$  the inverse of  $u_i \mapsto T_i^{-1}(u_1, \dots, u_{i-1}, u_i)$  for fixed values of  $u_1, \dots, u_{i-1}$ .
- Then, observe that if  $\tilde{U} \sim U([0, 1]^d)$ , and if

$$\begin{cases} \hat{U}_1 = \tilde{U}_1 T_1^{-1}(z_1) \\ \hat{U}_2 = \tilde{U}_2 T_2^{-1}(z_2; \hat{U}_1) \\ \dots \\ \hat{U}_M = \tilde{U}_M T_M^{-1}(z_M; \hat{U}_1, \dots, \hat{U}_{M-1}) \end{cases} . \quad (3)$$

- Then the conditional expectation

$$\pi_j = \mathbb{E} [1 \{ T_j(U_1, U_2, \dots, U_j) \leq z_j \} \mid T_1(U_1) \leq z_1, T_2 \leq \text{etc.}]$$

coincides with the unconditional expectation

$$\pi_j = \mathbb{E} \left[ T_j^{-1}(z_j; \hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}) \right] . \quad (4)$$

- ▶ The GHK simulator can be interpreted as an importance sampling simulation procedure.
- ▶ Indeed, expression

$$\pi_j = \mathbb{E} [1 \{ T_j (U_1, U_2, \dots, U_j) \leq z_j \} \mid T_1 (U_1) \leq z_1, T_2 \leq \text{etc.}]$$

is a conditional expectation; one may compute it by accept-reject but this is computationally suboptimal, as it leads us to discard a fraction of draws – which can be a significant fraction.

- ▶ In contrast, in expression

$$\pi_j = \mathbb{E} \left[ T_j^{-1} (z_j; \hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}) \right].$$

is an unconditional expectation; one shall compute it by drawing  $K$  i.i.d. draws of  $\tilde{U} \sim U([0, 1]^d)$ , computing the  $\hat{U}$ 's, and averaging over all the values of  $T_j^{-1} (\hat{U}_1, \hat{U}_2, \dots, \hat{U}_{j-1}, z_j)$  simulated that way. In the second method, we have not discarded any draws, which is more efficient.

**Algorithm.**

For  $k = 1, \dots, K$ : Draw  $\tilde{U}^k \sim \mathcal{U}([0, 1]^d)$ . Compute  $(\hat{U}_1^k, \dots, \hat{U}_M^k)$  from  $(\tilde{U}_1^k, \dots, \tilde{U}_M^k)$  using transformation (3). Compute  $\pi_j^K = K^{-1} \sum_{k=1}^K T_j^{-1} \left( z_j; \hat{U}_1^k, \hat{U}_2^k, \dots, \hat{U}_{j-1}^k \right)$  for  $j = 1, \dots, M$ . Return the GHK simulator

$$\mu_0 = \prod_{j=1}^M \pi_j^K.$$

**Remark:** The practical difficulty with the implementation of this algorithm is the knowledge of the Rosenblatt quantile in closed form. A leading example where this object is readily available is given by the case when  $\mathbf{P}$  is Gaussian, which is called the probit model.

- The Probit model is characterized by  $\mathbf{P} = \mathcal{N}(0, \Sigma)$ , with  $\Sigma$  is a  $M \times M$  symmetric semidefinite positive matrix, hence  $\text{cov}(\varepsilon_y, \varepsilon_{y'}) = \Sigma_{yy'}$ . In this case, the Rosenblatt quantile is known. Let  $\Sigma = LL^\top$  be the Choleski decomposition of  $\Sigma$ , where  $L$  is lower triangular with a positive diagonal. Then the Rosenblatt quantile  $T$  is such that

$$\begin{cases} T_1(u) = L_{11}\Phi^{-1}(u_1) \\ T_2(u) = L_{21}\Phi^{-1}(u_1) + L_{22}\Phi^{-1}(u_2) \\ \dots \\ T_M(u) = L_{M1}\Phi^{-1}(u_1) + \dots + L_{MM}\Phi^{-1}(u_M) \end{cases}$$

where  $\Phi$  is the c.d.f. of the standard normal (univariate) distribution.

In this case,  $\hat{U}$  is obtained from  $U \sim \mathcal{U}([0, 1]^M)$  by

$$\begin{cases} \hat{U}_1 = \tilde{U}_1\Phi\left(\frac{z_1}{L_{11}}\right) \\ \hat{U}_2 = \tilde{U}_2\Phi\left(\frac{z_2 - L_{21}\Phi^{-1}(\hat{U}_1)}{L_{22}}\right) \\ \dots \\ \hat{U}_M = \tilde{U}_M\Phi\left(\frac{z_M - L_{M1}\Phi^{-1}(\hat{U}_1) - \dots - L_{M(M-1)}\Phi^{-1}(\hat{U}_{M-1})}{L_{MM}}\right) \end{cases}$$

- ▶ McFadden's smoothed accept-reject simulator (SARS) consists in sampling  $\varepsilon \sim P$ :  $\varepsilon^1, \dots, \varepsilon^N$ , and replacing the max by the smooth-max

$$\sigma_{N,T,y}(U) = \sum_{i=1}^N \frac{1}{N} \frac{\exp((U_y + \varepsilon_y^i)/T)}{\sum_z \exp((U_z + \varepsilon_z^i)/T)}$$

- ▶ One seeks  $U$  so that the induced choice probabilities are  $s$ , that is

$$s_y = \sum_{i=1}^N \frac{1}{N} \frac{\exp((U_y + \varepsilon_y^i)/T)}{\sum_z \exp((U_z + \varepsilon_z^i)/T)}.$$

- ▶ The associated Emax operator is

$$G_{N,T}(U) = \mathbb{E}_{\mathbf{P}_N} \left[ G_{\text{logit}}(U + \varepsilon^i) \right]$$

so the underlying random utility structure is a random coefficient logit.

#### THEOREM (G AND SALANIÉ)

Consider a solution  $(u(\varepsilon), v_y)$  to the dual Monge-Kantorovich problem with cost  $\Phi(\varepsilon, y) = \varepsilon_y$ , that is:

$$\begin{aligned} \min_{u, v} \int u(\varepsilon) d\mathbf{P}(\varepsilon) + \sum_{y \in \mathcal{Y}_0} v_y s_y \\ \text{s.t. } u(\varepsilon) + v_y \geq \Phi(\varepsilon, y) \end{aligned} \tag{5}$$

Then:

- (i)  $U = \sigma^{-1}(s)$  is given by  $U_y = v_0 - v_y$ .
- (ii) The value of Problem (5) is  $-G^*(s)$ .

PROOF.

$\sigma^{-1}(s) = \arg \max_{U: U_0=0} \{ \sum_{y \in \mathcal{Y}} s_y U_y - G(U) \}$ , thus, letting  $v = -U$ ,  $v$  is the solution of

$$\min_{v: v_0=0} \left\{ \sum_{y \in \mathcal{Y}_0} s_y v_y + G(-v) \right\}$$

which is exactly problem (5). □

- ▶ It follows from the inversion theorem that the problem of demand inversion in the pure characteristics model is a semi-discrete transport problem, a point made in Bonnet, G and Shum (2017).
- ▶ Indeed, the correspondence is:
  - ▶ an alternative  $y$  is a fountain
  - ▶ the characteristics of an alternative is a fountain location
  - ▶ the systematic utility associated with alternative  $y$  is minus the price of fountain  $y$
  - ▶ the market share of alternative  $y$  coincides with the capacity of fountain  $y$
  - ▶ the random vector  $\epsilon$  is the location of an inhabitant



- Cf. Bonnet, G. and Shum (2017). Let  $u_i = T \log \sum_z \exp((U_z + \varepsilon_z^i)/T)$ . One has

$$\begin{cases} s_y = \sum_{i=1}^N \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T) \\ \frac{1}{N} = \sum_y \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T) \end{cases}.$$

- As a result,  $(u_i, U_y)$  are the solution of the regularized OT problem

$$\min_{u, U} \sum_{i=1}^N \frac{1}{N} u_i - \sum s_y U_y + \sum_{i,y} \frac{1}{N} \exp((U_y - u_i + \varepsilon_y^i)/T).$$

- Consider the IPFP algorithm for solving the latter problem:

$$\begin{cases} \exp(u_i^{k+1}/T) = \sum_z \exp((U_z^k + \varepsilon_z^i)/T) \\ \exp(U_y^{k+1}/T) = \frac{Ns_y}{\sum_{i=1}^N \exp((-u_i^{k+1} + \varepsilon_y^i)/T)} \end{cases}$$

- This rewrites as

$$\exp U_y^{k+1}/T = \frac{Ns_y}{\sum_{i=1}^N \frac{\exp(\varepsilon_y^i/T)}{\sum_z \exp((U_z^k + \varepsilon_z^i)/T)}}, \text{ i.e.}$$

$$U_y^{k+1} = T \log s_y - T \log \sum_{i=1}^N \frac{1}{N} \frac{\exp(\varepsilon_y^i/T)}{\sum_z \exp((U_z^k + \varepsilon_z^i)/T)}$$

which is exactly the contraction mapping algorithm of Berry, Levinsohn and Pakes (1995, appendix 1).

## Section 2

## CODING

- ▶ We shall code the AR simulator for the probit model and then invert it using the inversion theorem.

- ▶ Take a vector of systematic utilities:

```
U_y = c(1.6, 3.2, 1.1,0)
```

- ▶ Simulate the market shares using the AR simulator:

```
epsilon_iy = matrix(rnorm(nbDraws*nbY),ncol=nbY) %*%  
SqrtCovar
```

```
u_iy = t(t(epsilon_iy)+U_y)
```

```
ui = apply(X = u_iy, MARGIN = 1, FUN = max)
```

```
s_y = apply(X = u_iy - ui, MARGIN = 2,FUN = function(v)  
(length(which(v==0)))) / nbDraws
```

- To invert the market share, simply run the optimal assignment problem:  
A1 =  
kronecker(matrix(1,1,nbY),sparseMatrix(1:nbDraws,1:nbDraws))  
A2 =  
kronecker(sparseMatrix(1:nbY,1:nbY),matrix(1,1,nbDraws))  
A = rbind2(A1,A2)  
result = gurobi (  
list(A=A,obj=c(epsilon\_iy),modelsense="max",  
rhs=c(rep(1/nbDraws,nbDraws),s\_y) ,sense="="),  
params=list(OutputFlag=0) )  
Uhat\_y = - result\$pi[(1+nbDraws):(nbY+nbDraws)] +  
result\$pi[(nbY+nbDraws)]