Properties and algorithms for Minimum Steiner Tree problem

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INTRODUCTION

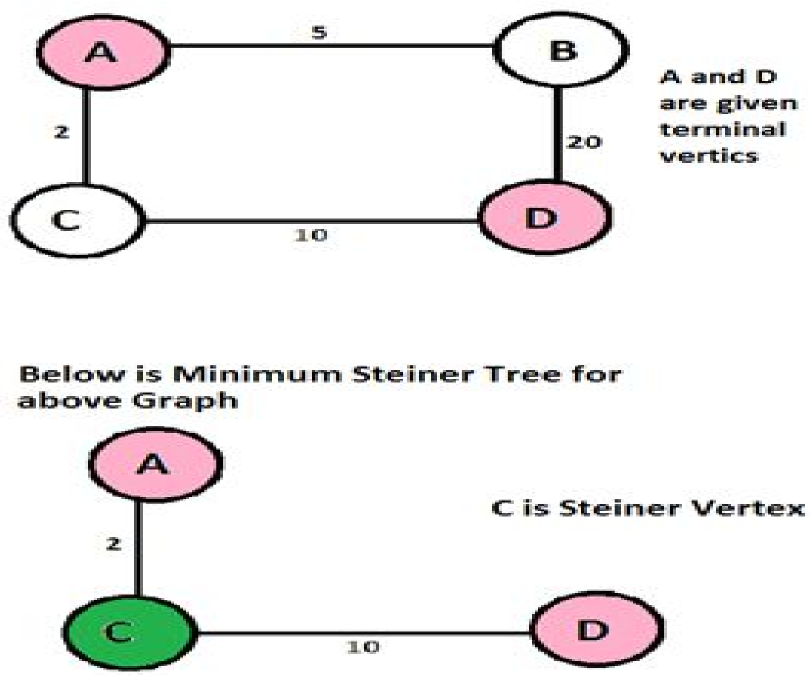
What is Steiner Tree?

Steiner Minimal Trees

While a spanning tree spans all vertices of a given graph, a Steiner tree spans a given subset of vertices. In the Steiner minimal tree problem, the vertices are divided into two parts: ***Terminal*** and ***Non-Terminal Vertices***. The terminals are the given vertices which must be included in the solution. The cost of a Steiner tree is defined as the total edge weight. A Steiner tree may contain some non terminal vertices to reduce the cost. Let V be a set of vertices. In general, we are given a set $L \subset V$ of terminals and a metric defining the distance between any two vertices in V. The objective is to find a connected subgraph spanning all the terminals of minimal total cost. Since the distances are all nonnegative in a metric, the solution is a tree structure.

Given a graph and a **subset** of vertices in the graph, a steiner tree spans though the given subset. The Steiner Tree may contain some vertices which are not in given subset but are used to connect the vertices of subset.

The Steiner Tree Problem is to find the minimum cost Steiner Tree. See below for an example.



Spanning Tree vs Steiner Tree

Minimum Spanning Tree is a minimum weight tree that spans through **all** vertices.

If given subset (or terminal) vertices is equal to set of all vertices in Steiner Tree problem, then the problem becomes Minimum Spanning Tree problem. And if the given subset contains only two vertices, then it shortest path problem between two vertices.

Finding out Minimum Spanning Tree is polynomial time solvable, but Minimum Steiner Tree problem is NP Hard and related decision problem is NP-Complete.

Types of Steiner tree:

1) Euclidean Steiner tree

2) Rectilinear minimum Steiner tree(RMST)

1)Euclidean Steiner tree : The original problem was stated in the form that has become known as the Euclidean Steiner tree problem or geometric Steiner tree problem: Given N points in the plane, the goal is to connect them by lines of minimum total length in such a way that any two points may be interconnected by line segments either directly or via other points and line segments. It may be shown that the connecting line segments do not intersect each other except at the endpoints and form a tree, hence the name of the problem.

2) Rectilinear minimum Steiner tree(RMST):
The rectilinear Steiner tree problem is a variant of the geometric Steiner tree problem in the plane, in which the Euclidean distance is replaced with the rectilinear distance. The problem arises in the physical design of electronic design automation. In VLSI circuits, wire routing is carried out by wires that are often constrained by design rules to run only in vertical and horizontal

directions, so the rectilinear Steiner tree problem can be used to model the routing of nets with more than two terminals.

Steiner ratio:

The Steiner ratio is the supremum of the ratio of the total length of the minimum spanning tree to the minimum Steiner tree for a set of points in the Euclidean plane. In the Euclidean Steiner tree problem, the Steiner ratio is conjectured to be 2/root(3) . Despite earlier claims of a proof, the conjecture is still open. In the rectilinear Steiner tree problem, the Steiner ratio is 3/2.

PROPERTIES OF MINIMUM STEINER TREES

- Given a graph and a subset of vertices in the graph, a Steiner tree spans through the given subset.

- The Steiner Tree may contain some vertices, which are not in given subset but connects the vertices of subset.

- The Steiner Tree Problem is to find the minimum cost Steiner Tree.

- If given subset (or terminal) vertices is equal to set of all vertices in Steiner Tree problem, then the problem becomes Minimum Spanning Tree problem.

- If the given subset contains only two vertices, then it is shortest path problem between two vertices.

- Minimum Steiner Tree problem is NP Hard.

- There are exactly three edges at every Steiner vertex.

- The angles between the edges meeting at a Steiner vertex is 120 degrees.

- If there are N vertices then we can use a maximum of N-2 Steiner vertices.

HISTORY AND LITERATURE SURVEY

The “Steiner problem” is named after the Swiss mathematician Jacob Steiner (1796-1863), who solved and popularized the problem of joining three villages by a system of roads having minimum total length (he also addressed the general case of this problem, and made many fundamental contributions to projective geometry).

However, while Jacob Steiner’s work on this problem was independent of its predecessors, about two centuries earlier Pierre de Fermat (1601-1665) proposed this problem to Evangelista Torricelli (1608-1647), who solved it and passed it along to his student Vincenzo Viviani (1622-1703), who in turn published his own solution as well as Torricelli’s in 1659 . An even earlier (and presumably independent) published discussion of this problem,is found in a 1647 book by the Italian mathematician Bonaventura Francesco Cavalieri (1598-1647).

Luckily, today we refer to this problem simply as the Steiner problem, instead of the more accurate but considerably less wieldy title “the Fermat-Torricelli-Viviani-Cavalieri-Steiner problem”.

More recent research progress on the Steiner minimal tree (SMT) problem has been historically driven by several main results.

- In 1966 Hanan showed that for a pointset P there exists an SMT whose Steiner points S are all chosen from the Hannan grid, namely the intersections of all the horizontal and vertical lines passing through every point of P (see Figure Below). Snyder generalized Hanan’s theorem to all higher-dimensional Manhattan geometries; on the other hand, extensions of Hanan’s theorem to λ-geometries are less straightforward.

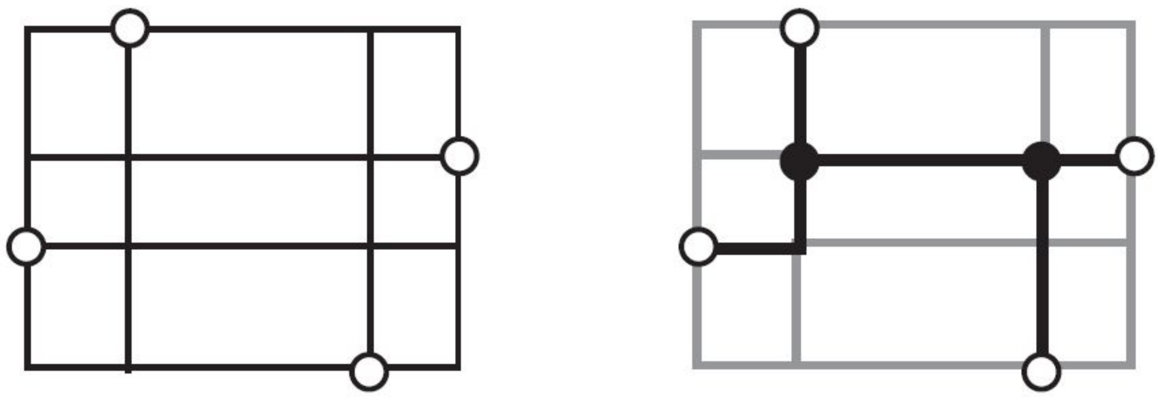


Figure : Hanan’s theorem: there exists an SMT with Steiner points chosen from the Hanan grid, i.e., intersection points of all horizontal and vertical lines drawn through the points.

- . In 1977 Garey and Johnson showed that despite restricting the Steiner points to lie on the Hanan grid, the rectilinear SMT problem is NP-complete . Only a very few special cases have been solved optimally (e.g., a linear-time solution exists when all points of P lie on the boundary of a rectangle). Many heuristics have been proposed for the general problem.

- In 1976 Hwang [47] showed that the MST over P is a good approximation to the SMT, having performance ratio cost(MST(P)) / cost(SMT(P)) ≤ 3/2 for any pointset P in the rectilinear plane. In attacking intractable problems, a standard goal is to achieve a “provably good” heuristic having a constant-factor performance ratio (i.e., asymptotic worst-case error bounded with respect to the optimal solution). In light of the intractability of the rectilinear SMT problem, Hwang’s result implies that any Steiner approximation approach which improves upon an initial MST solution will have performance ratio at most 3/2 . Thus, many SMT heuristics in the literature are MST-improvement strategies, i.e., they resemble classic minimum spanning tree constructions. For over 15 years after the publication of, the fundamental open problem was to find a heuristic with (worst-case) performance ratio strictly less than 3/2 . A complementary research goal has been to find new practical heuristics with improved average-case solution quality. In practice, most SMT heuristics, including MST-based strategies, exhibited very similar average performance. On uniformly-distributed random instances (the typical benchmark), heuristic Steiner tree costs averaged between 7% and 9% improvement over the corresponding MST costs.

- In 1990 Kahng and Robins have shown that any Steiner tree heuristic in a general class of greedy MST-based methods has worst-case performance ratio arbitrarily close to 3/2 , i.e., the MST for certain classes of pointsets is unimprovable. Thus, the 3/2 bound is tight for a wide range of MST-based strategies in the rectilinear plane , which resolved the performance ratios for a number of heuristics in the literature with previously unknown worst-case behavior. Moreover, this established that in general, MST-based Steiner heuristics (e.g., where MST edges are “flipped” within their bounding boxes) are unlikely to achieve performance ratio better than 3/2 . Analogous constructions in higher d-dimensional

Manhattan geometry showed that all of these heuristics have performance ratio of at least 2d−1/d, which is bounded from above by 2 as the dimension grows.

- In 1992 Zelikovsky developed a rectilinear Steiner tree algorithm with a performance ratio of 11/8 times optimal [97], the first heuristic provably better than the MST. His techniques yield a general graph Steiner tree algorithm with a 11/6 performance ratio, the first graph Steiner approximation proven to beat the MST-based graph Steiner heuristic of Kou, Markowsky, and Berman [62]. This settled in the affirmative the longstanding open question of whether there exists a polynomial-time rectilinear Steiner tree heuristic with performance ratio < 3/2 , and whether there exists a polynomial-time graph Steiner tree heuristic with performance ratio < 2. In light of this sequence of developments, research on Steiner tree approximation has turned away from MST-improvement heuristics. One of the earliest and most effective Steiner tree approximation schemes to break away from the herd of MST-improvement shemes is the Iterated 1-Steiner (I1S) approach of Kahng and Robins.The I1S heuristic is simple, easy to implement, generalizes naturally to any dimension and metric (including arbitrary weighted graphs), and significantly outperforms previous approaches, as detailed below. The I1S algorithm was subsequently proven to be the earliest published Steiner approximation method to have a non-trivial performance ratio (of 1.5 times optimal) in quasi-bipartite graphs.

EXACT ALGORITHMS

As Steiner Minimum Tree is NP-hard problem.It can’t be solved in polynomial time.So, Many scientists have worked on it and proved exact algorithms having different complexities.

Theorem [Dreyfus & Wagner, Networks 1971] and [Levin 1971]

“Edge weighted Steiner Tree can be solved in time

O(3t n + 2t n^2 + n(n log n + m)).”

Improvements of the D-W Algorithm

- The 1971 D-W algorithm achieves time O(3t n + 2t n2 + n(n log n + m)) This can be improved to O(3t n + 2t (n log n + m)) by computing the distances more cleverly on demand[Erickson, Monma, Veinott Mathematics of Operations Research 1987]

- In 2007 Fuchs, Kern, and Wang [Math. Meth. Oper. Res.] improved this to O(2.684t n^O(1)) and

- Molle, Richter, and Rossmanith [STACS 2006] to O((2 + ")tnf (e1))

- later the above two groups together [Theory Comput. Syst. 2007] improved the exponent to O(("ln ")) for any 1=2 < which gives,e.g., O(2.5^t n^14.2) or O(2.1^t n^57.6)

- By using subset convolution and Mobius inversion, one can get to a running time of ~O (2tn2 + nm) for the node weighted case with bounded weights [Bjorklund, Husfeldt, Kaski, Koivisto STOC 2007

Exponential Time Algorithms

- the fast exponential algorithms are obtained by combining branching

for large values of t and FPT

algorithms for small t

- the fastest for weighted case is based on the algorithm of Molle etal.achieving O(1.42n) in exponential space

- The only paper devoted to such algorithms is by Fomin, Grandoni,

Kratsch, Lokshtanov, Saurabh

[Algorithmica 2009 / ESA 2008]

- It uses more involved branching, quasi FPT algorithm, and analyses the running time by Measure & Conquer.

- It is polynomial space and originally achieved running time O(1.59n)

for the weighted case and O(1.55n) for

the cardinality case.

- Plugging in the O(2t) algorithm of Nederlof, the running time can
  be improved to O(1.36n) for the
  cardinality case.

The Iterated 1-Steiner (I1S) Approach

This section outlines the Iterated 1-Steiner heuristic , which repeatedly finds optimum single Steiner points for inclusion into the pointset. Given two pointsets A and B, we define the MST savings of B with respect to A as: ΔMST (A, B) = cost(MST (A)) − cost(MST (A $\cup$ B)). Let H(P) denote the Steiner candidate set, i.e., the intersection points of all horizontal and vertical lines passing through points of P (as defined by Hanan's theorem). For any pointset P , a 1-Steiner point with respect to P is a point x $\in$ H(P) that maximizes ΔMST (P, {x}) > 0. Starting with a pointset P and a set S = $\varnothing$ of Steiner points, the Iterated 1-Steiner (I1S) method repeatedly finds a 1-Steiner point x for P $\cup$ S and sets S $\leftarrow$ S $\cup$ {x}. The cost of MST (P $\cup$ S) will decrease with each added point, and the construction terminates when there no longer exists any point x with ΔMST (P $\cup$ S, {x}) > 0. An optimal Steiner tree over n points has at most n−2 Steiner points of degree at least 3 (this follows from simple degree arguments). However, the I1S method can (or rare occasions) add more than n − 2 Steiner points. Therefore, at each iteration we eliminate any extraneous Steiner points which have degree ≤ 2 in the MST over P ∪ S (since such points can not contribute to the tree cost savings). To find a 1-Steiner point in the Manhattan plane, it suffices to construct an MST over |P ∪S|+1 points for each of the O(n2) members of the Steiner candidate set (i.e., Hanan grid points), and then pick a candidate which minimizes the overall MST cost. Each MST computation can be performed in O(n log n) time, yielding an O(n3 log n) time method to find a single 1- Steiner point. A more efficient algorithm can find a new 1-Steiner point within O(n2) time. A linear number of Steiner points can therefore be found in O(n3) time, and trees with a bounded number of k Steiner points require O(kn2) time. Since the MSTs between trying one candidate Steiner point and the next change very little (by only a constant number of tree edges), incremental/dynamic MST updating schemes can be employed, resulting in further asymptotic time complexity improvements. In practice, the number of iterations performed by I1S averages less than n 2 for uniformly distributed random pointsets. Furthermore, the I1S heuristic is provably optimal for 4 or less points; this is not a trivial observation, since many earlier heuristics were not optimal even for 4 points. On the other hand, the worst-case performance ratio of I1S over small pointsets is at least 7 6 and 13 11 for 5 and 9 points, and is at least 1.3 in general. The next subsection discusses a batched variant of the I1S approach, which offers runtime improvements in practice.

The Batched 1-Steiner Variant

Although a single 1-Steiner point may be found in O(n2) time, the required computational geometry techniques are complicated and not easy to implement. To address these issues, a batched variant of I1S was developed, which amortizes the computational expense of finding 1-Steiner points by adding as many “independent” 1-Steiner points as possible in every round. The Batched 1-Steiner (B1S) variant computes _MST(P, {x}) for each candidate Steiner point x ∈ H(P) (i.e., the Hanan grid candidate points). Two candidate Steiner points x and y are independent if: _MST(P, {x}) + _MST(P, {y}) ≤ _MST(P, {x, y}), introducing each of the two 1-Steiner points does not reduce the potential gain in MST cost relative of the other 1-Steiner point. Given pointset P and a set of Steiner points S, each round of B1S greedily adds into S a maximal set of independent

1-Steiner points. Termination occurs when a round fails to add any new Steiner points. The total time required for each round is O(n2 log n). In three dimensions, I1S exploits a generalization of Hanan’s theorem to higher dimensions , namely that there always exists an optimal Steiner tree whose Steiner points are selected from the O(n3) intersections of all axis-orthogonal planes passing through points of P. The three dimensional analog of Hwang’s result suggests that the Steiner ratio, i.e. the maximum cost(MST) cost(SMT) ratio for three dimensions is at most 5/3 ; however, this is only a conjecture and generalizing Hwang’s theorem to dimensions three and higher is still an open problem. An example consisting of six points located in the middle of the faces of a rectilinear cube establishes that 5/3 is a lower bound for the Steiner ratio in three dimensions. The I1S and B1S algorithms are highly parallelizable since each processor can independently compute the MST savings of different candidate Steiner points. The Iterated Steiner approach is therefore very amenable to parallel implementation on grid computers. As with I1S, the time complexity and practical runtime of B1S can be further improved using incremental / dynamic MST update techniques . Moreover, by exploiting tighter bounds on the maximum MST degree in the rectilinear metric4, further runtime improvements can be obtained.

Contributions:

Anirudh Sharma - Introduction
Abhijit Singh - Types of steiner tree and Steiner Ratio
Ajinkya - Properties

Ajay Sharma - History and literature Survey
Srinath - Exact and exponential time algorithms
Aman Garg - Iterated 1s approach and batch 1s variant

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