Single-shot Tomographic Shape Sensing

Ajinkya Kadu
Utrecht University, Netherlands

jointly with **Tristan van Leeuwen** and **K. Joost Batenburg**

NWO-XS project

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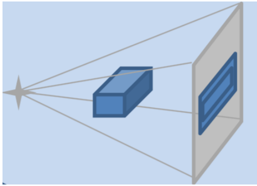
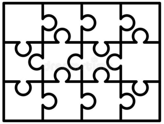
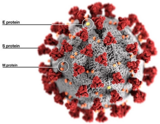
Motivation

- **Is a single cone-beam projection powerful enough?**

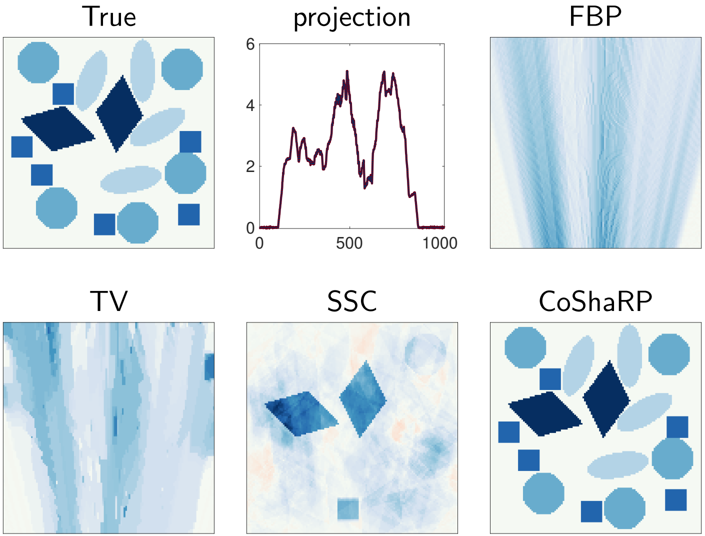


- **What if shapes are known?**





Motivation - Example



Outline

Single-shot Tomography

Shape Recovery

CoShaRP

Optimization

Numerical Experiments

Cone-beam Projection

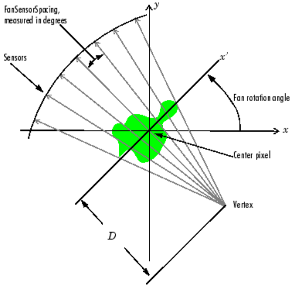
$$\left(A_C x\right)(\mathbf{r}, \boldsymbol{\vartheta}) = \int_0^\infty x\left(\mathbf{r} + t\boldsymbol{\vartheta}\right) \, \mathrm{d}t$$

- source position $\mathbf{r} \in \mathbb{R}^d$, angular vector $\boldsymbol{\varphi} \in \mathbb{S}^{d-1}$
- measurements for various $\boldsymbol{\varphi}_j \in \boldsymbol{\Phi}$ and $\mathbf{r}_i \in \mathbf{R}$

$$y_{i,j} = \left(A_C x\right)\left(\mathbf{r}_i, \boldsymbol{\vartheta}_j\right) = \sum_{k=1}^{K} a_{i,j,k} x\left(\mathbf{r}_i + t_k \boldsymbol{\vartheta}_j\right)$$

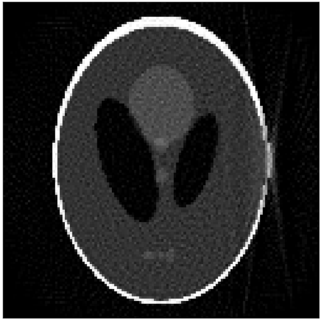
- **Cone-beam tomography**: find $\mathbf{x} \in \mathbb{R}^n$ from measurement vector $\mathbf{y} \in \mathbb{R}^m$

$$\mathbf{y} = A\mathbf{x}$$

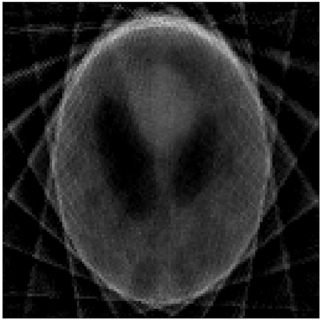


Full-sampling to single-shot

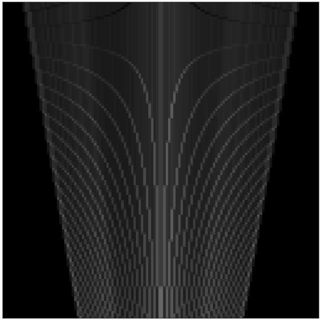
60 angles



10 angles



1 angle



Outline

Image model

$$x(\mathbf{r}) = \sum_{i=1}^{S} \sum_{j=1}^{k_i} u_i\big(R\left(\boldsymbol{\theta}_{i,j}\right)\mathbf{r} + \mathbf{s}_{i,j}\big)$$

- $u_i : \Omega \mapsto \mathbb{R}$, $i = 1, \ldots, S$ shape functions
- $R \in \mathbb{R}^{d\times d}$ is a rotation matrix, *i.e.*, $R^T R = I, \det(R) = 1$
- **Trans-rotation parameters**: $\boldsymbol{\theta}$ governs rotation while s denotes shift
- **Recovery problem**

$$\text{find} \quad (\boldsymbol{\theta}, \mathrm{s}) \quad \text{subject to} \quad A\mathbf{x} \approx \mathbf{y}, \quad x(\mathbf{r}_t) = \sum_{i,j} u_i\big(R\left(\boldsymbol{\theta}_{i,j}\right)\mathbf{r}_t + \mathbf{s}_{i,j}\big) \quad t = 1, \ldots, n$$

Dictionary Approach

- **Form dictionary of shapes**

$$\Psi(\mathbf{r}) = \left[\widehat{\Psi}_1(\mathbf{r}) \quad \ldots \quad \widehat{\Psi}_S(\mathbf{r})\right],$$

$$\text{with} \quad \widehat{\Psi}_i(\mathbf{r}) = \left[u_i\left(\langle \boldsymbol{\theta}_j, \mathbf{r}\rangle + \mathbf{s}_j\right)\right]_{j=1}^{J}, \qquad\qquad i = 1, \ldots, S,$$

- **Represent image as a linear combination of shape elements**

$$x(\mathbf{r}) = \sum_{i=1}^{p} z_i \psi_i(\mathbf{r}) \qquad \text{with} \quad z_i \in \{0,1\}, \quad i = 1, \ldots, p$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \psi_1 & \ldots & \psi_p \\ | & & | \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_p \end{bmatrix}$$

Shape Recovery Problem

$$\text{find} \quad \mathbf{z} \in \{0,1\}^p \qquad \text{subject to} \quad A\Psi\mathbf{z} \approx \mathbf{y}, \quad \mathbf{z}^T\mathbf{1} = K$$

- Assumptions
  - Shapes are non-overlapping
  - Shapes (in the target image) are present in the dictionary
- Integer Program: NP-hard
- Unique Solution (no noise):
  - Each $A\Psi\mathbf{e}_i$ must be unique
  - For every $\mathbf{x} = \Psi\mathbf{z}$ with $\mathbf{z} \in \{0,1\}^p$, there must a unique $\mathbf{z}$

Outline

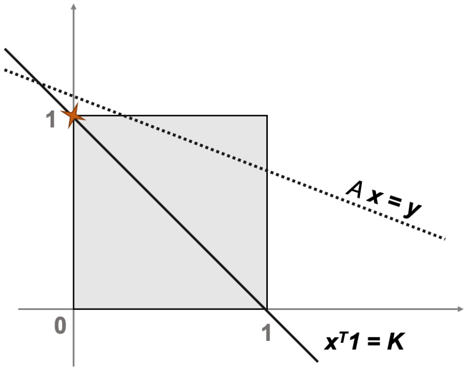
Convex Program

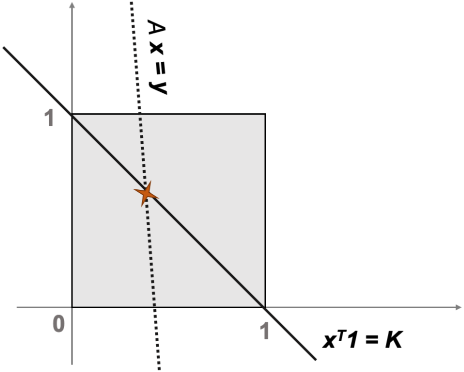
$$\underset{\mathbf{z}\in\mathbb{R}^p}{\text{minimize}} \quad \frac{1}{2}\left\|A\Psi\mathbf{z}-\mathbf{y}\right\|^2 \qquad \text{subject to} \quad \begin{aligned} &\mathbf{z}^T\mathbf{1}=K, \\ &\mathbf{0}\le\mathbf{z}\le\mathbf{1}. \end{aligned}$$

- K-simplex: $\mathcal{C}=\{\mathbf{z}\in\mathbb{R}^n \,|\, z_1+\cdots+z_n=K, \quad 0\le z_i\le 1, \quad \forall\, i=1,\ldots,n\}$
- **Interpretation**: Find a closest point to the hyperplane $A\Psi z=y$ that lies on the intersection of the hyperplane $\mathbf{z}^T\mathbf{1}=K$ and a hypercube $\mathbf{0}\le\mathbf{z}\le\mathbf{1}$

Geometry

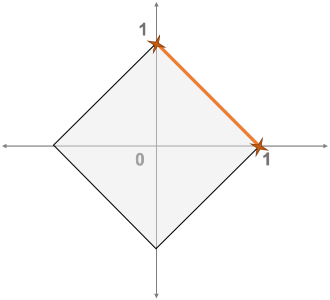
Find a closest point to the hyperplane $A\Psi z = y$ that lies on the intersection of the hyperplane $\mathbf{z}^T\mathbf{1} = K$ and a hypercube $\mathbf{0} \leq \mathbf{z} \leq \mathbf{1}$





Comparison with SSC

$$\text{minimize} \quad \|A\Psi\mathbf{z}-\mathbf{y}\|^2 \quad \text{subject to} \quad \|\mathbf{z}\|_1 \le K$$



- SSC allows far more solutions than CoShaRP.
- CoShaRP achieves sharp recovery results.

(trivial) Extensions

Outline

Primal-Dual Algorithm

- **Optimization Problem**

$$\text{minimize} \quad f(\Psi \mathbf{z}) + g(\mathbf{z}),$$

$$\text{where} \quad f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{y}\|^2$$

$$g(\mathbf{z}) = \delta_{\mathcal{C}}(\mathbf{z}) \quad \text{with} \quad \mathcal{C}(\mathbf{z}) = \left\{\mathbf{z} \in \mathbb{R}^n \,|\, \mathbf{z}^T\mathbf{1} = K, \, \mathbf{0} \leq \mathbf{z} \leq \mathbf{1}\right\},$$

- **Alternating Strategy**

$$\mathbf{z}_{t+1} = \mathbf{prox}_{\gamma g}\left(\mathbf{z}_t - \gamma \Psi^T \mathbf{u}_t\right),$$

$$\mathbf{u}_{t+1} = \mathbf{prox}_{\gamma f^\star}\left(\mathbf{u}_t - \gamma \Psi \left(\mathbf{z}_t - 2\mathbf{z}_{t+1}\right)\right),$$

Proximal Operators

- **Proximal of (conjugate of) f**

$$\mathbf{prox}_{\gamma f^\star}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{w}} \left\{ \frac{1}{2\gamma} \left\| \mathbf{w} - \mathbf{x} \right\|^2 + \frac{1}{2} \left\| \mathbf{w} - A^T \mathbf{y} \right\|^2_{(A^T A)^{-1}} \right\}$$
$$= \mathbf{x} - \gamma \left(\gamma I + A^T A \right)^{-1} \left(\mathbf{x} + A^T \mathbf{y} \right).$$

- **Proximal of g**

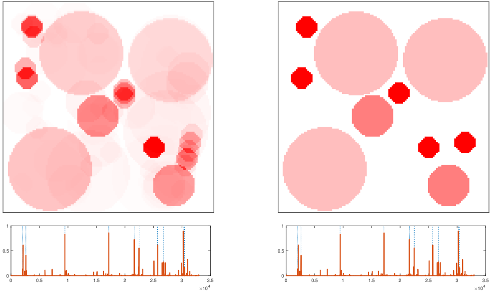
$$\mathbf{prox}_{\gamma \delta_{\mathcal{C}}}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{z}} \left\{ \frac{1}{2\gamma} \left\| \mathbf{z} - \mathbf{x} \right\|^2 \quad \text{s.t.} \quad \mathbf{z}^T \mathbf{1} = K, \quad \mathbf{0} \le \mathbf{z} \le \mathbf{1} \right\}$$
$$= \mathcal{P}_{[\mathbf{0},\mathbf{1}]} \left(\mathbf{x} - \mu^\star \mathbf{1} \right)$$

where $\mu^\star$ is a solution to the equation

$$\mathbf{1}^T \mathcal{P}_{[\mathbf{0},\mathbf{1}]} \left(\mathbf{x} - \mu \mathbf{1} \right) = K.$$

Image formation

- **Non-binary solutions**
  - CoShaRP may not always admit binary solution.
  - Terminating optimization earlier results in non-binary solution.
- **Proposed strategy**
  - Sort the coefficients
  - Form the image from sorted coefficients that are consistent with the measurements

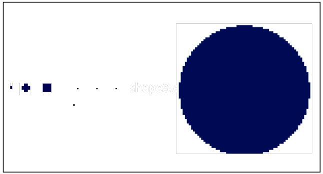


Outline

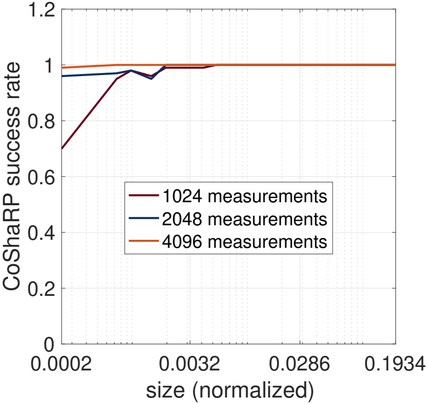
Resolution Analysis

discs with different sizes

shapes



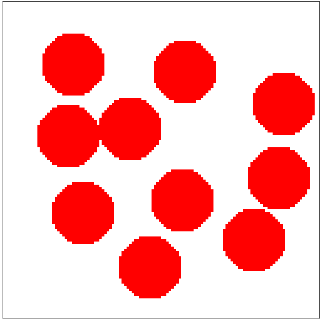
performance



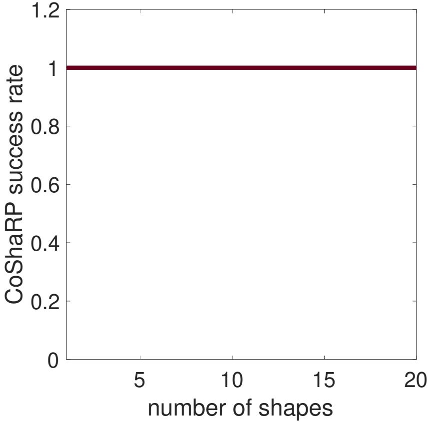
Sparsity Invariance

discs with diameter of $0.1\ m$

example (10 shapes)



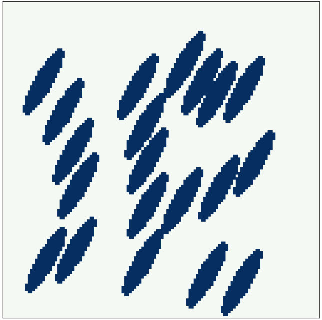
performance



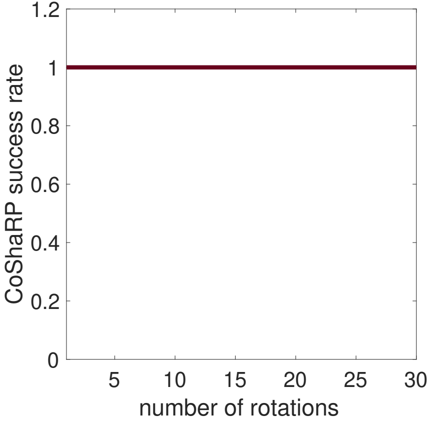
Rotational Invariance

ellipses with axial lengths $0.2\ m$ **and** $0.05\ m$

example (10 rotations)



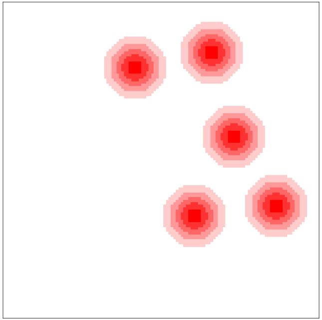
performance



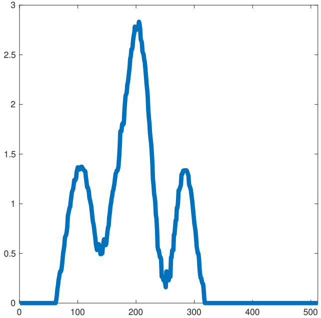
Non-homogeneous shapes

discs with 5 radial contrast levels

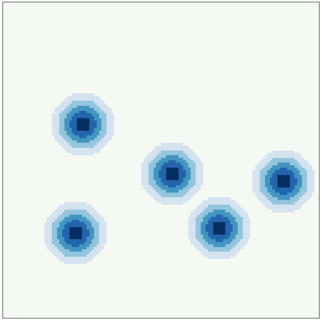
True



Measurement



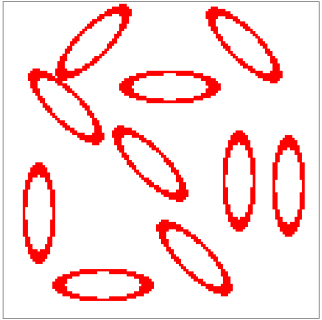
CoShaRP



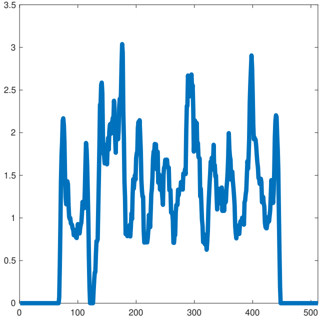
Non-convex shapes

hollow ellipses

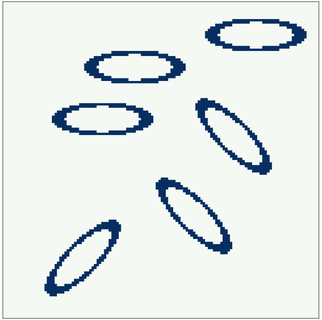
True



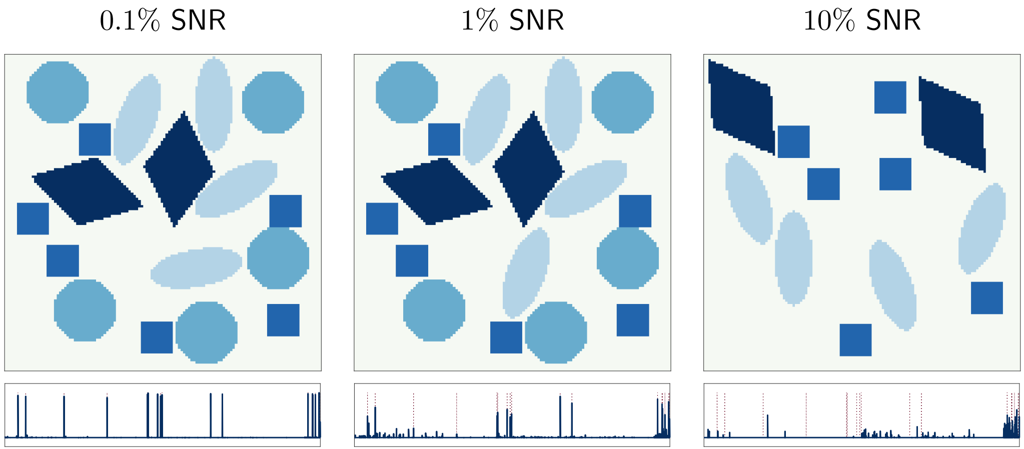
Measurement



CoShaRP

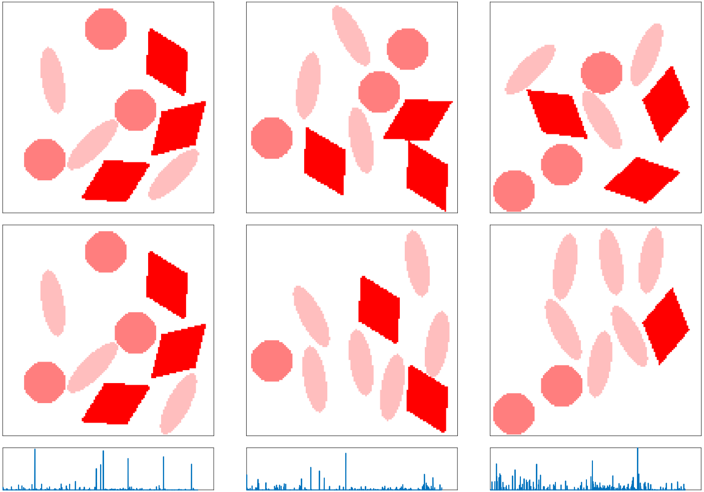


Measurement Noise



Roto-translation Noise

1% SNR | 10% SNR | 50% SNR



Conclusions

- A single cone-beam contains enough information about the images composed of (non-overlapping) shapes.

- We proposed a **Convex Shape Recovery Program** (CoShaRP) to recover shapes from a single cone-beam projection.

- Through numerical experiments:
  - CoShaRP can recover objects up to even **single pixel**
  - CoShaRP is **invariant** with respect to **sparsity** and **rotation**
  - CoShaRP can recover **non-homogeneous** as well as **non-convex shapes**.
  - CoShaRP **fails** for **moderate noise** in the measurements or roto-translation of shapes.