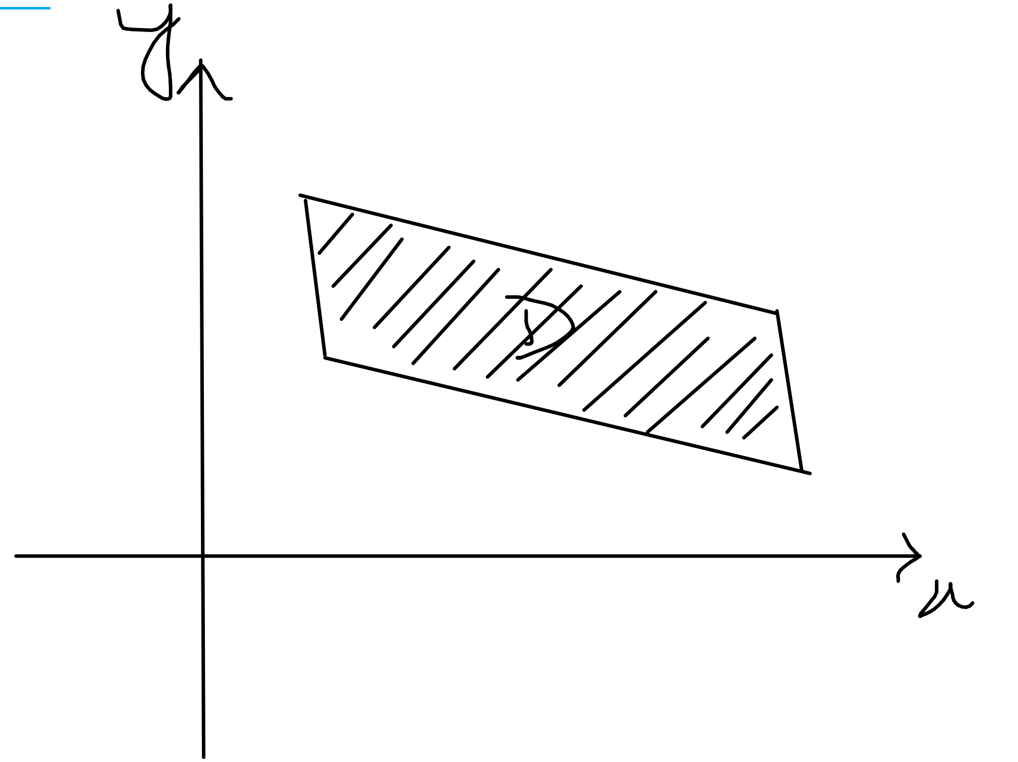
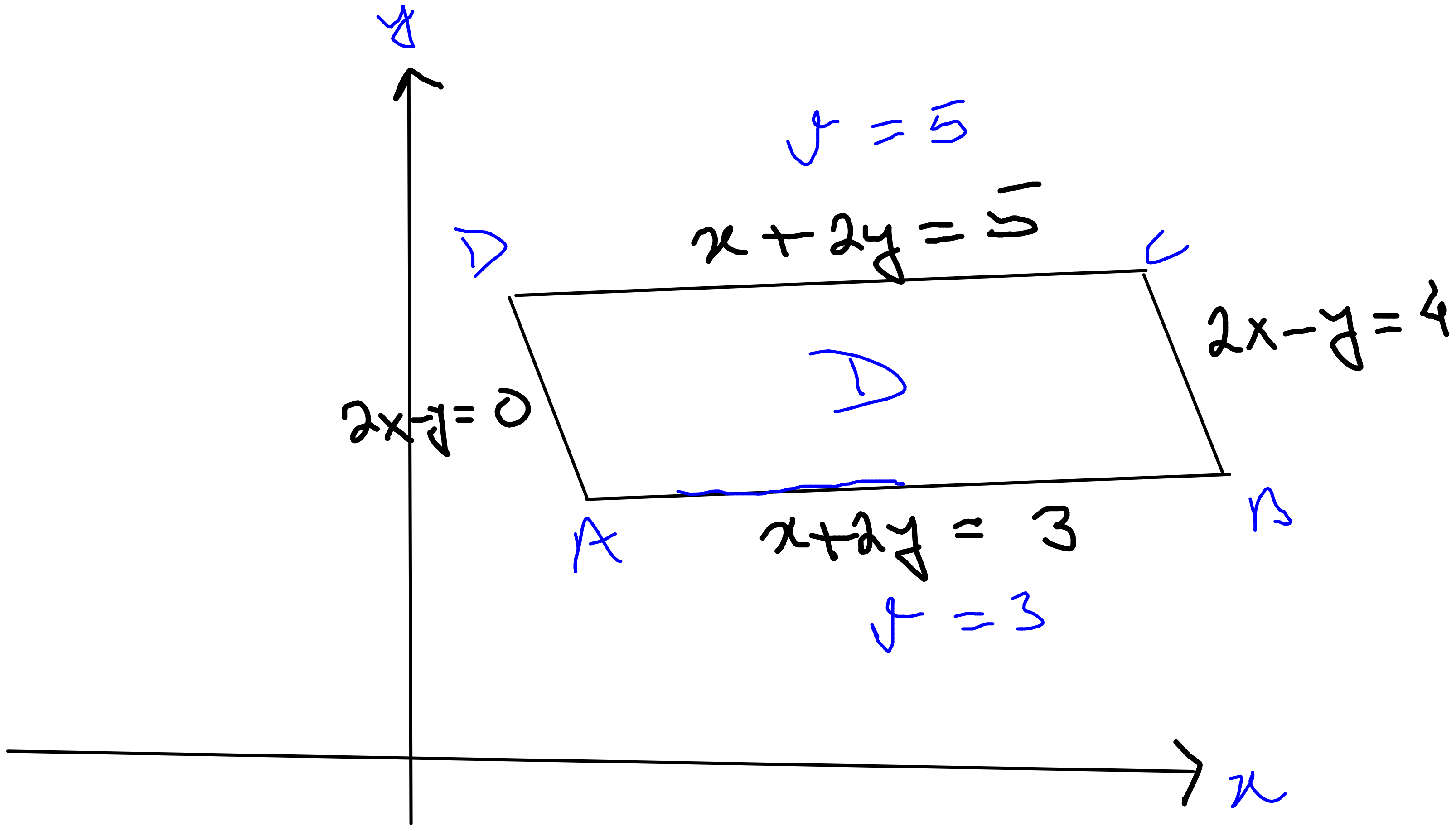
12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

→ $\displaystyle\iint_D f(x,y)\,dA$ — given



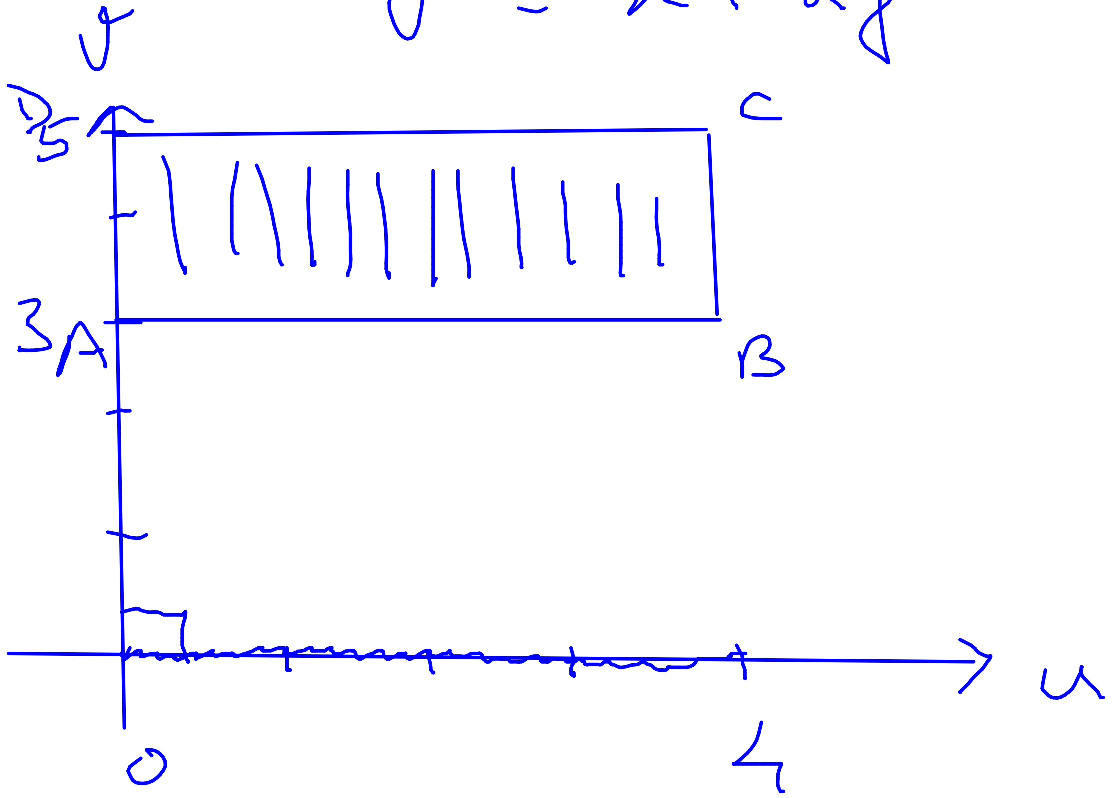
→ D : will be mildly complicated

→ D : will be simplified with change of variables



$u = 2x - y$

$v = x + 2y$



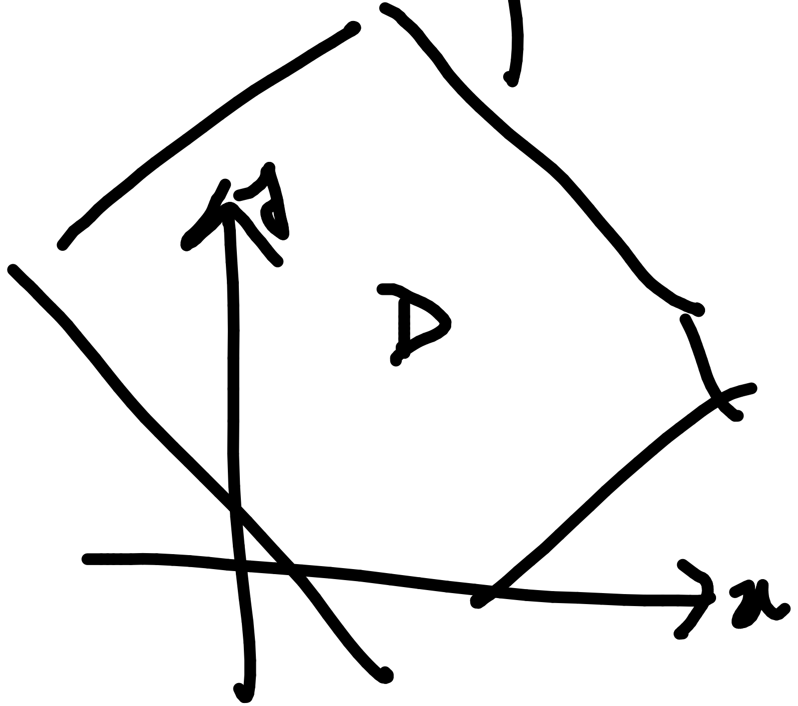
Find the Jacobian of the transformation.

$$x = u + 4v, \quad y = 3u - 2v$$

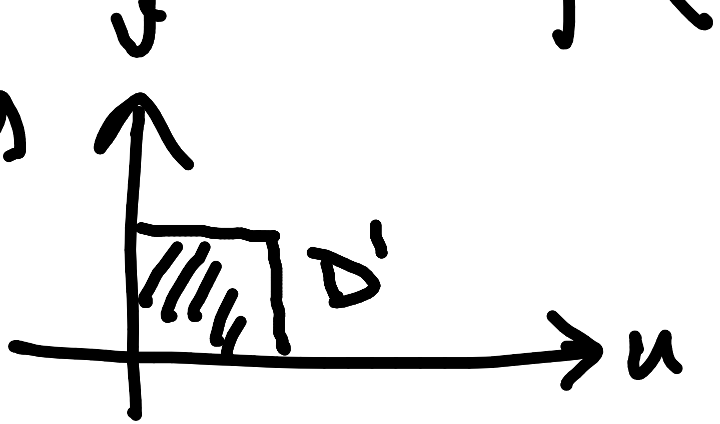
$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \text{abs}\left[\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right]$$

$$= \text{abs}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = |-14| = 14$$

$$\iint_D f(x,y)\,dx\,dy$$



$D' \approx 14$ times smaller than D.

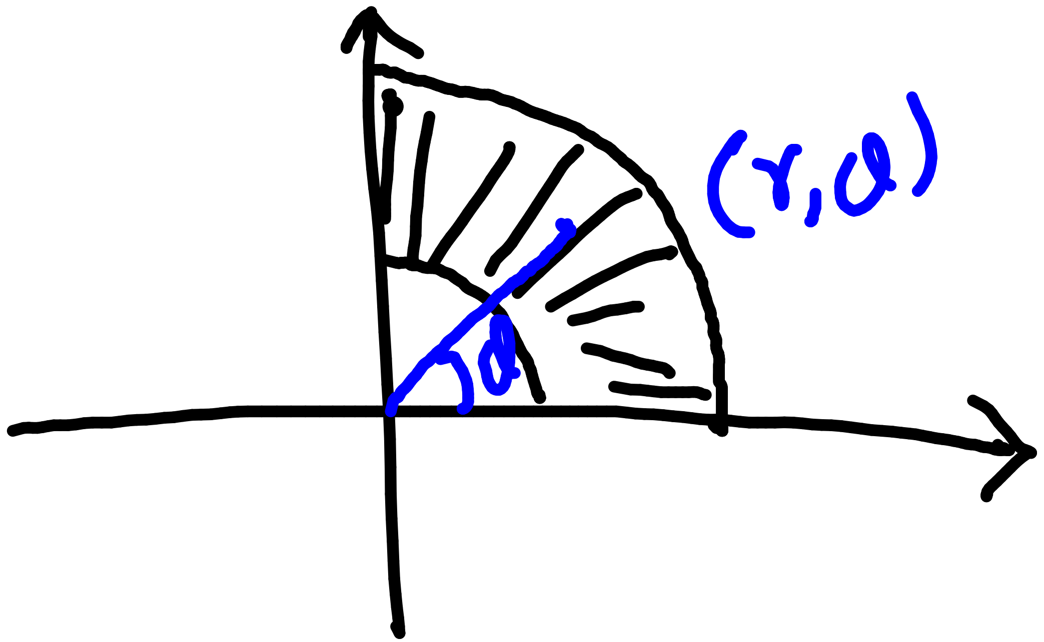


$$\iint$$

Find the Jacobian of the transformation.

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$



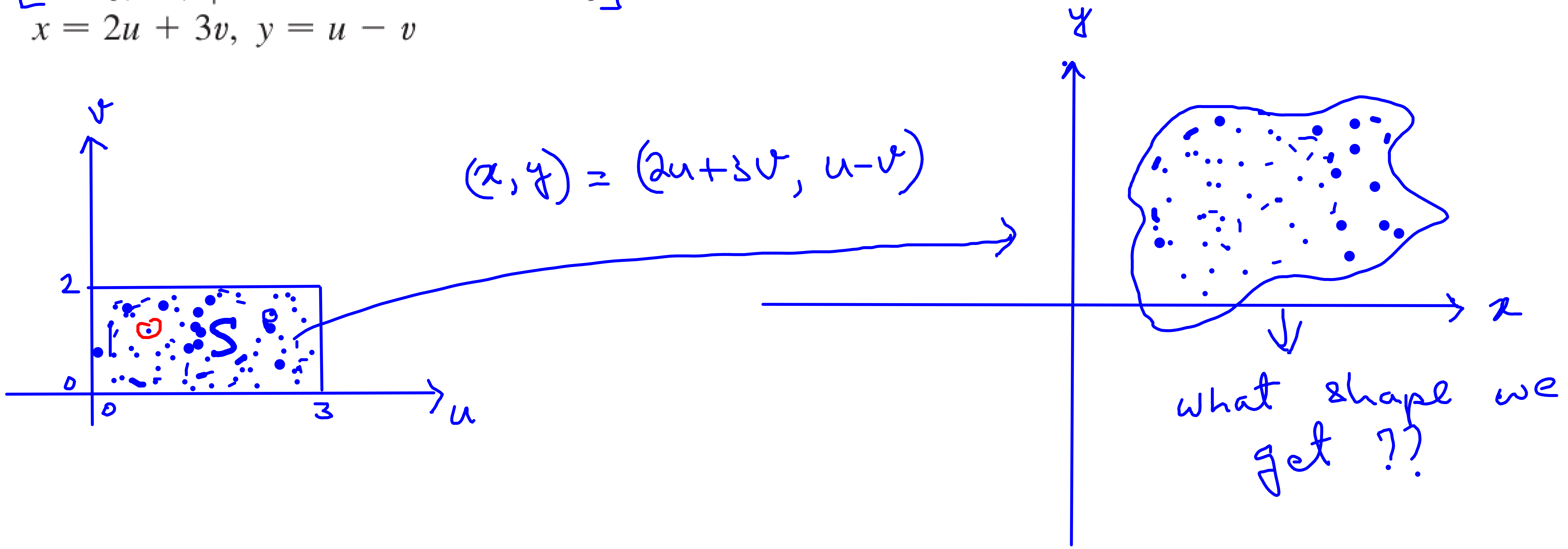
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \underline{\underline{r}}$$

$$dx\,dy = r\,dr\,d\theta$$

Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; ?? shape ??

$x = 2u + 3v,\ y = u - v$



Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; ?? shape ??

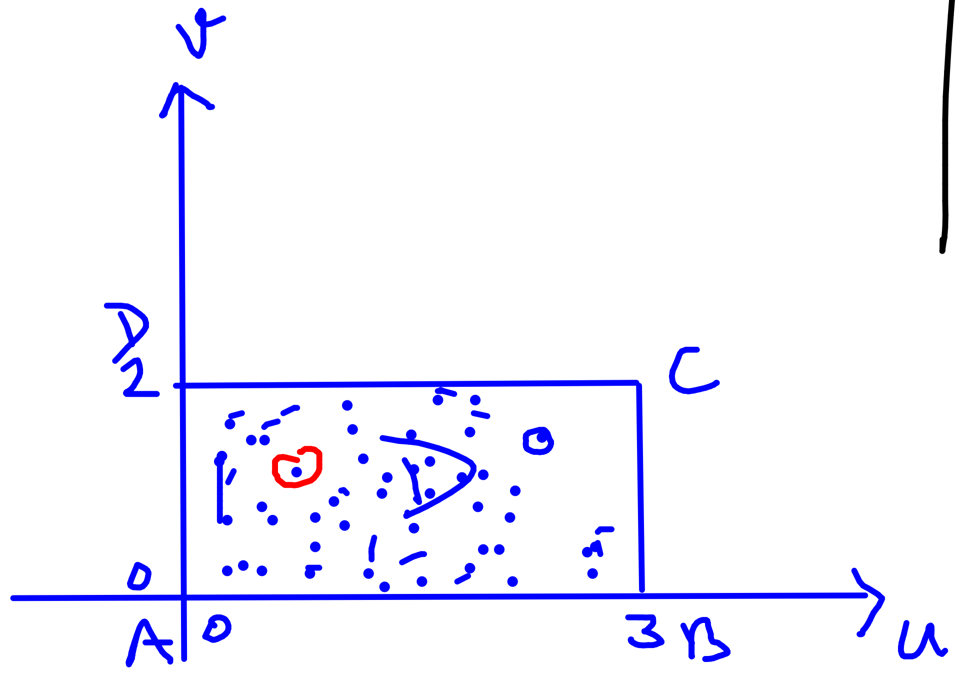
$x = 2u + 3v,\ y = u - v$

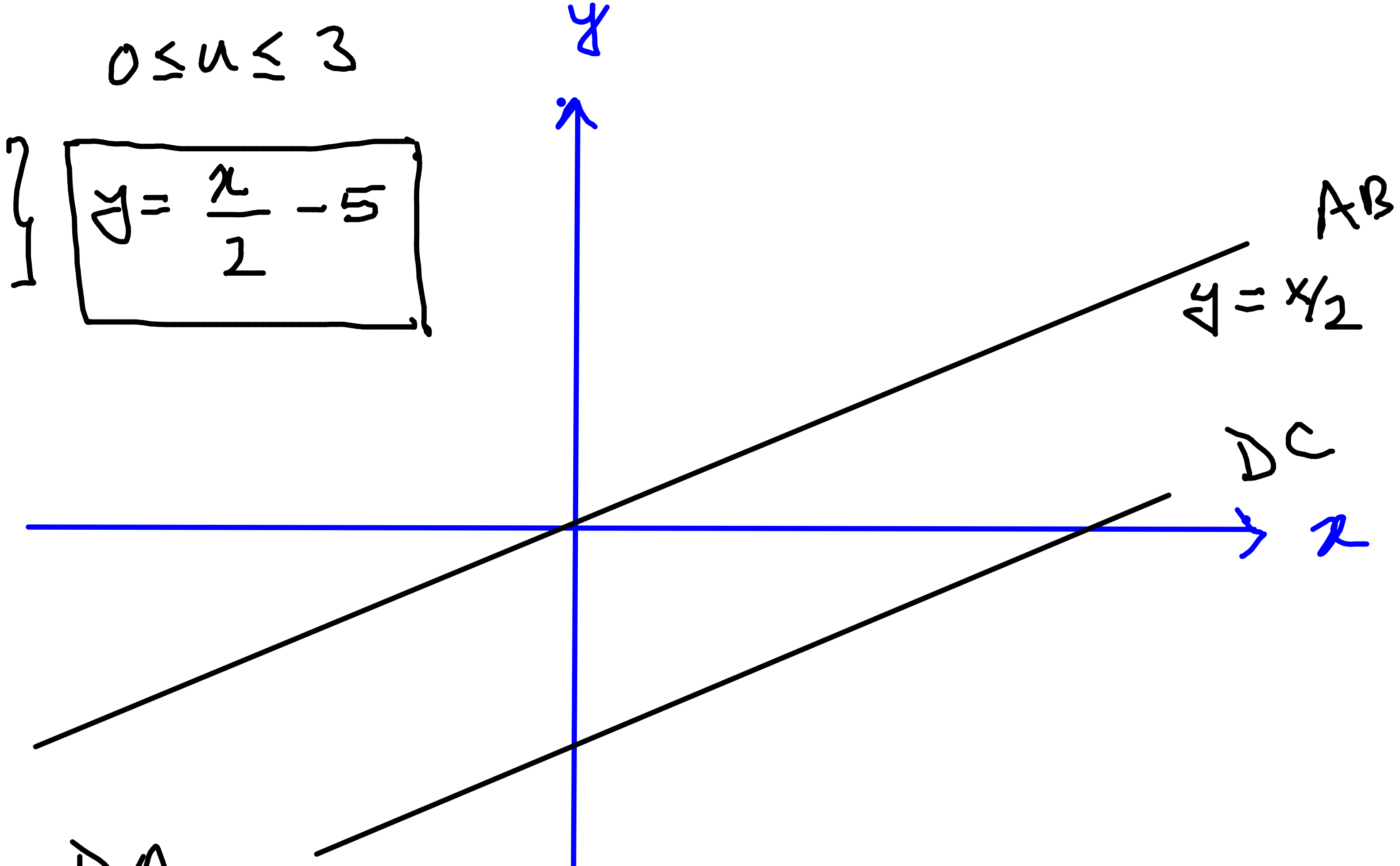
AB , $v = 0$, $0 \le u \le 3$

$\left.\begin{array}{l} x = 2u \\ y = u \end{array}\right\}$ $x = 2y$

DC , $v = 2$ $0 \le u \le 3$

$\left.\begin{array}{l} x = 2u + 6 \\ y = u - 2 \end{array}\right\}$ $\boxed{y = \frac{x}{2} - 5}$





strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

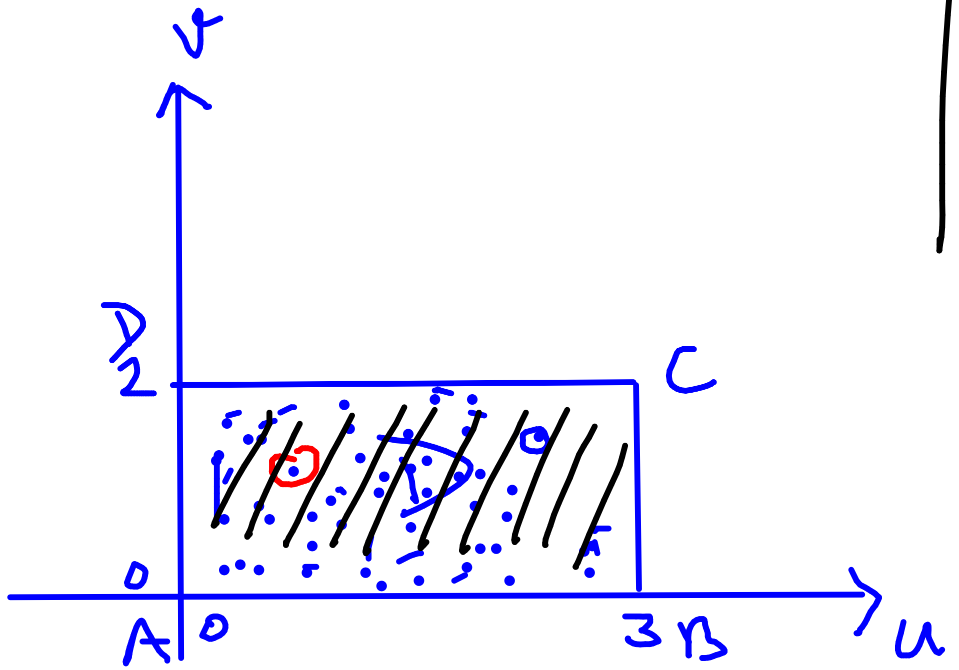
Find the image of the set S under the given transformation.

$$\boxed{S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};}$$?? shape??

$x = 2u + 3v,\ y = u - v$

AD : $u = 0$, $0 \leq v \leq 2$

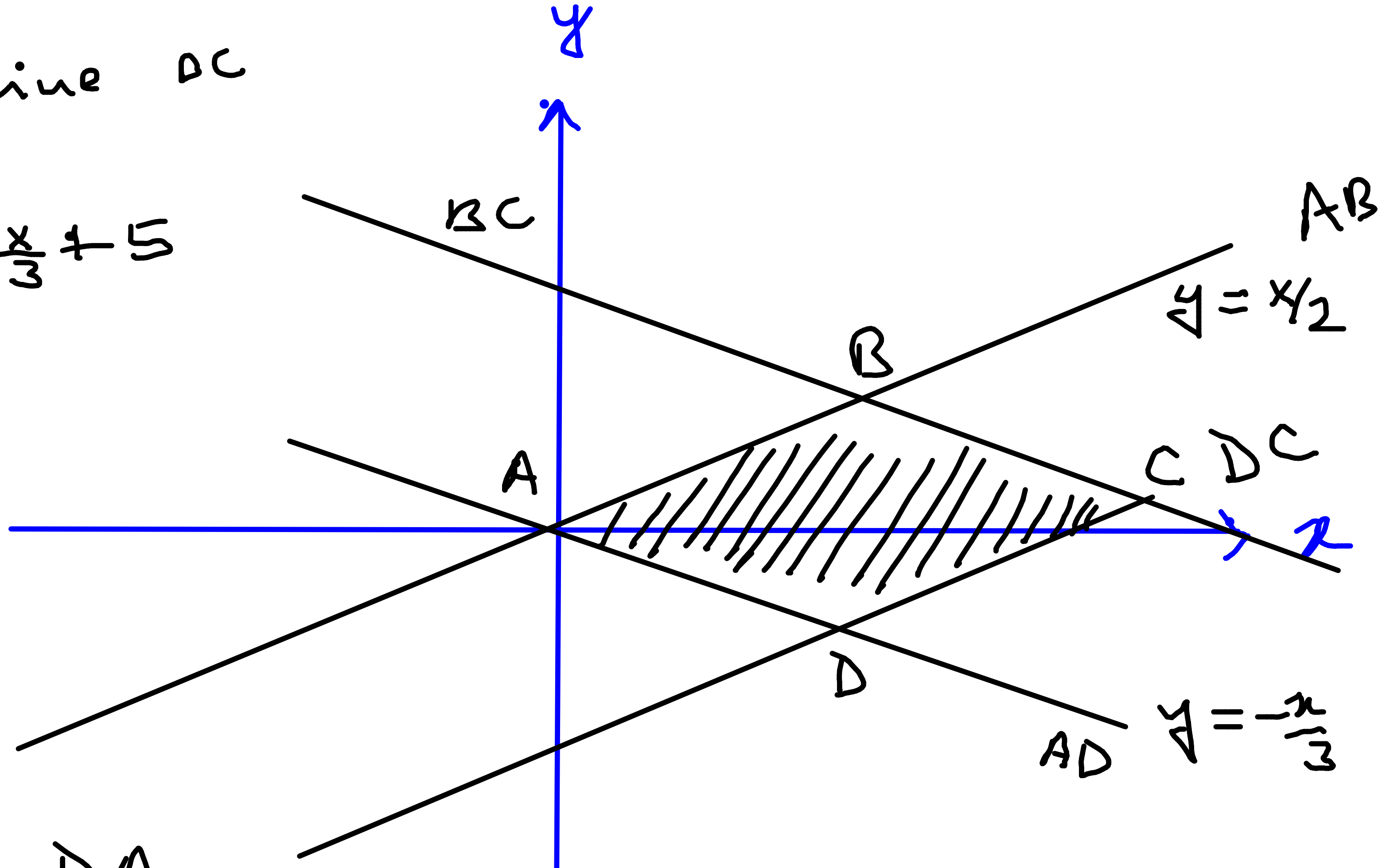
$$\left.\begin{aligned} x &= 3v \\ y &= -v \end{aligned}\right] \quad \begin{aligned} x &= -3y \\ y &= \frac{-1}{3}x \end{aligned}$$



similarly: line DC

$x + 3y = 15$

$y = -\frac{x}{3} + 5$



strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ y = u - v$$

Find the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

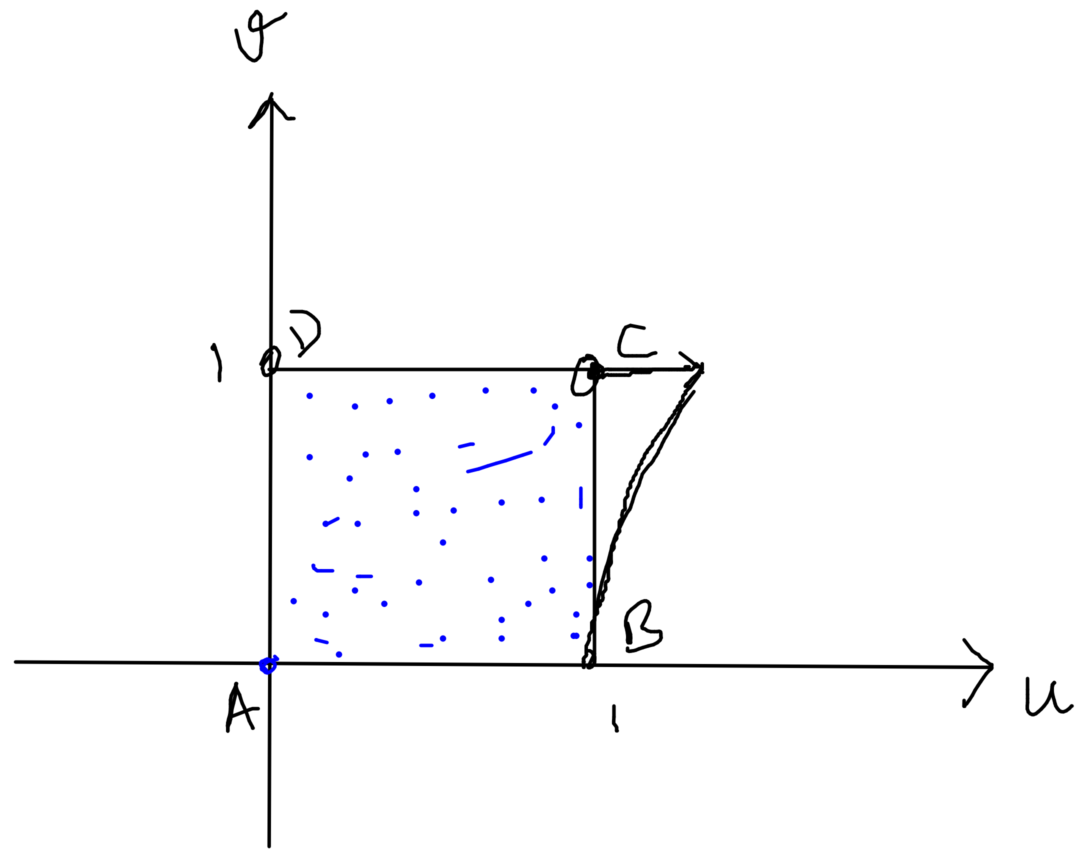
$$dx\,dy = J\,du\,dv$$

$$= \left| \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \right| = |-5|$$

$$= 5$$

Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$



AB

$v = 0$

$x = 0$, $y = u$ $\Big\}$ $x = 0$, $0 \le y \le 1$

AD

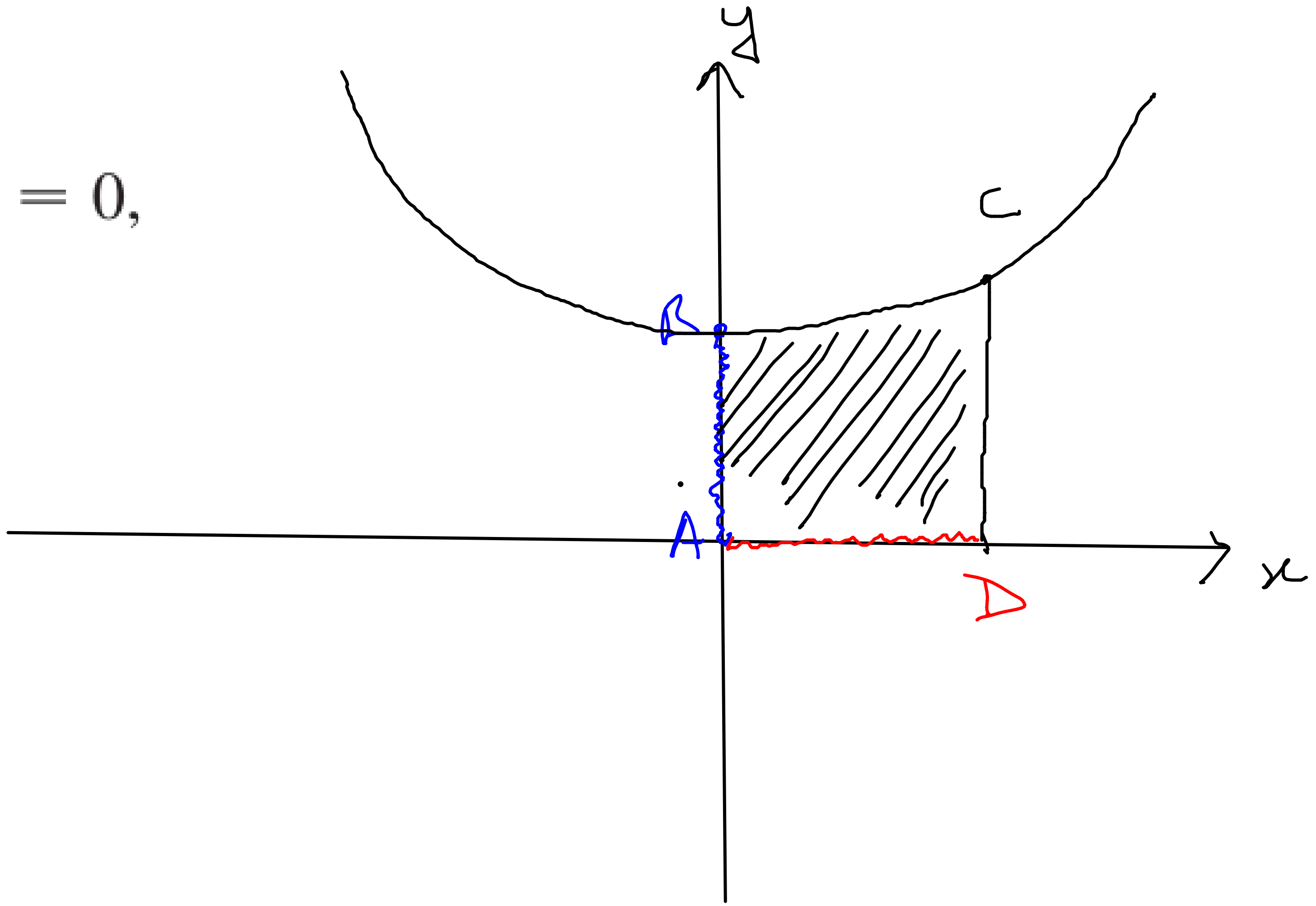
$u = 0$, $0 \le v \le 1$

$x = v$, $y = 0$ $\Big\}$ $y = 0$, $0 \le x \le 1$

BC : $u = 1$, $0 \le v \le 1$

$x = v$, $y = 1 + v^2$ $\Big\}$ $y = 1 + x^2$, $0 \le x \le 1$

DC



Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

Find the Jacobian

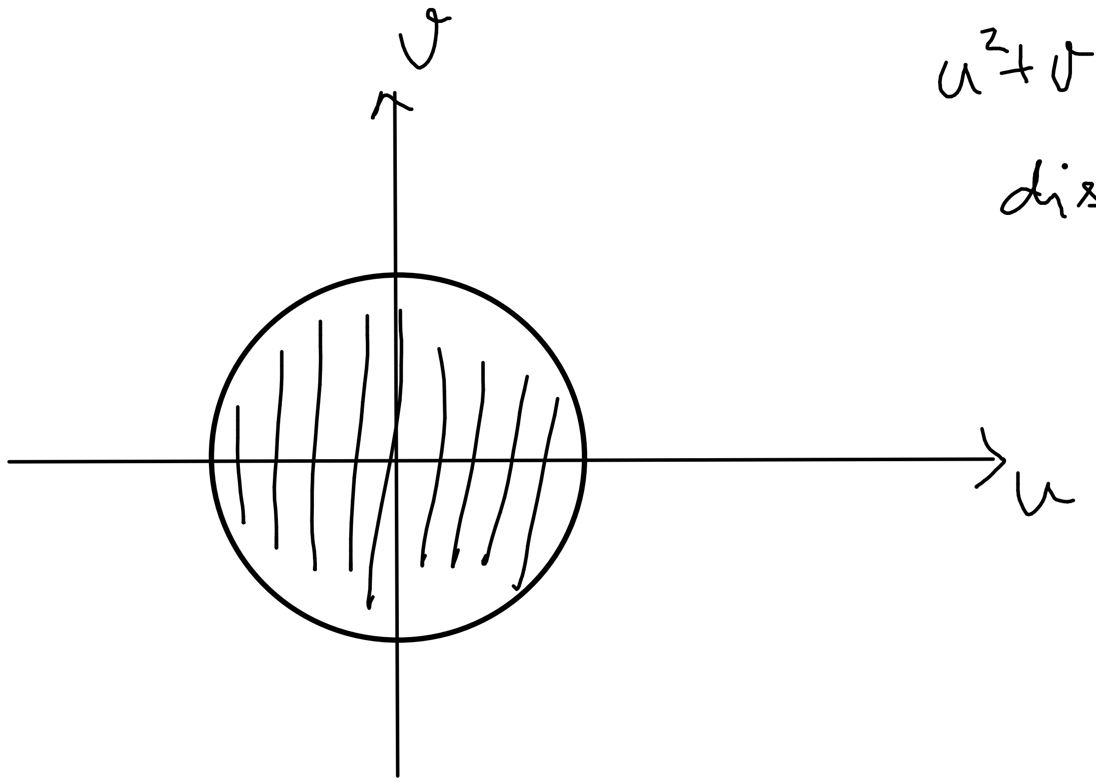
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} =$$

$$= 1+v^2$$

Find the image of the set S under the given transformation.

$$S \text{ is the disk given by } u^2 + v^2 \leqslant 1; \quad x = au, \; y = bv$$

for simplicity, assume $a = 3$, $b = 2$



$u^2+v^2 \leq 1$

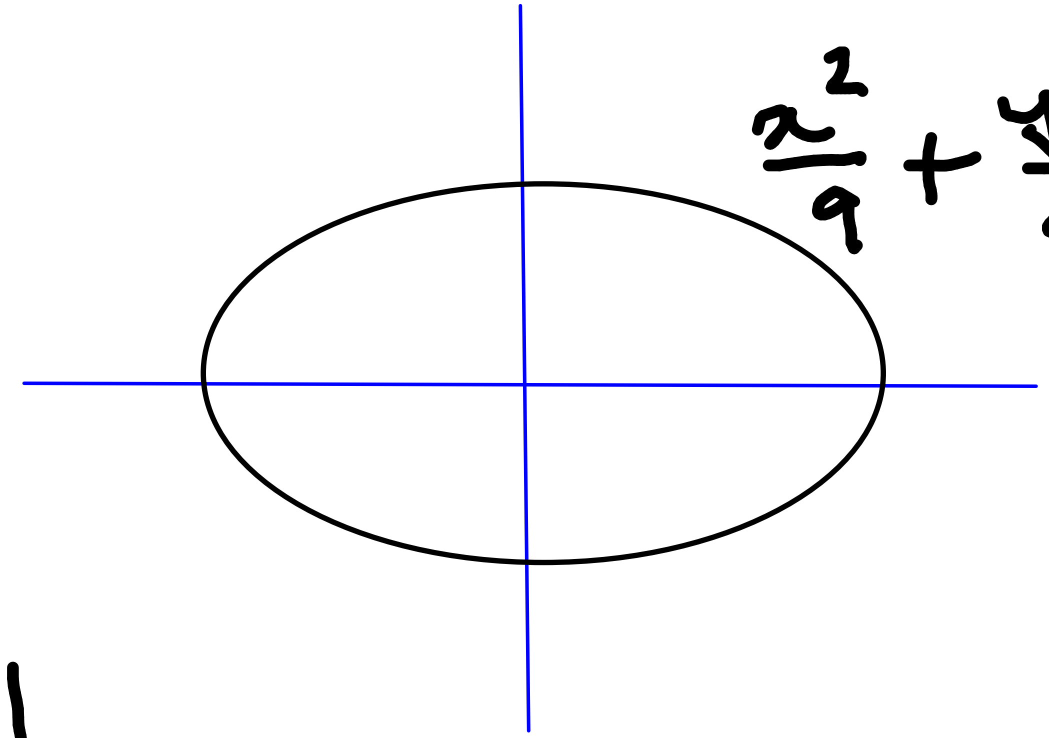
disc

$x = 3u$

$y = 2v$

$u^2+v^2=1$

$\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$

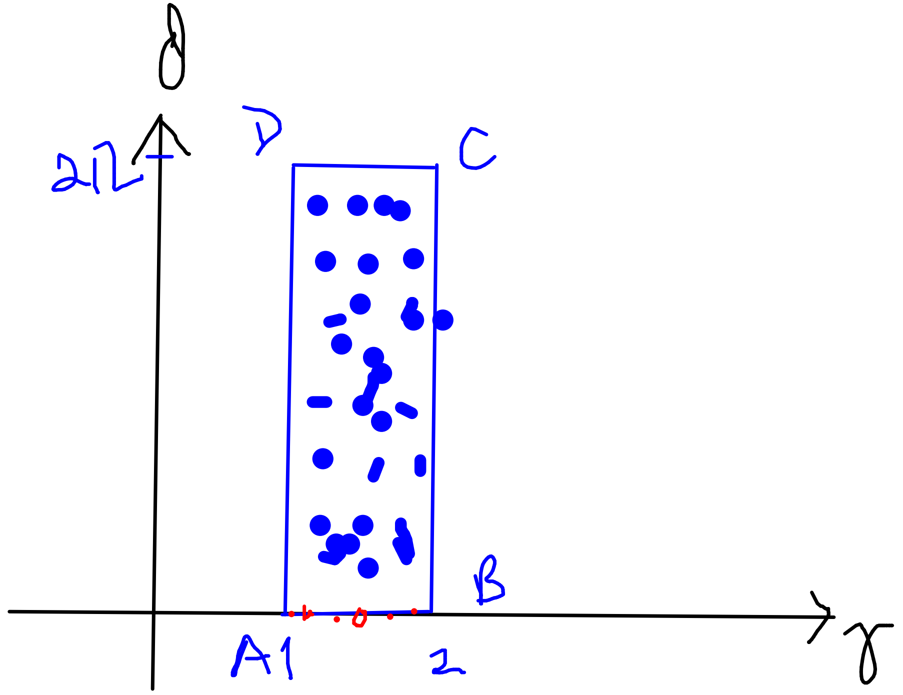


$\frac{x^2}{9}+\frac{y^2}{4}=1$

Q. Find the image of S under the given transformation:

$$S = \{ (r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 2\pi \}$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$



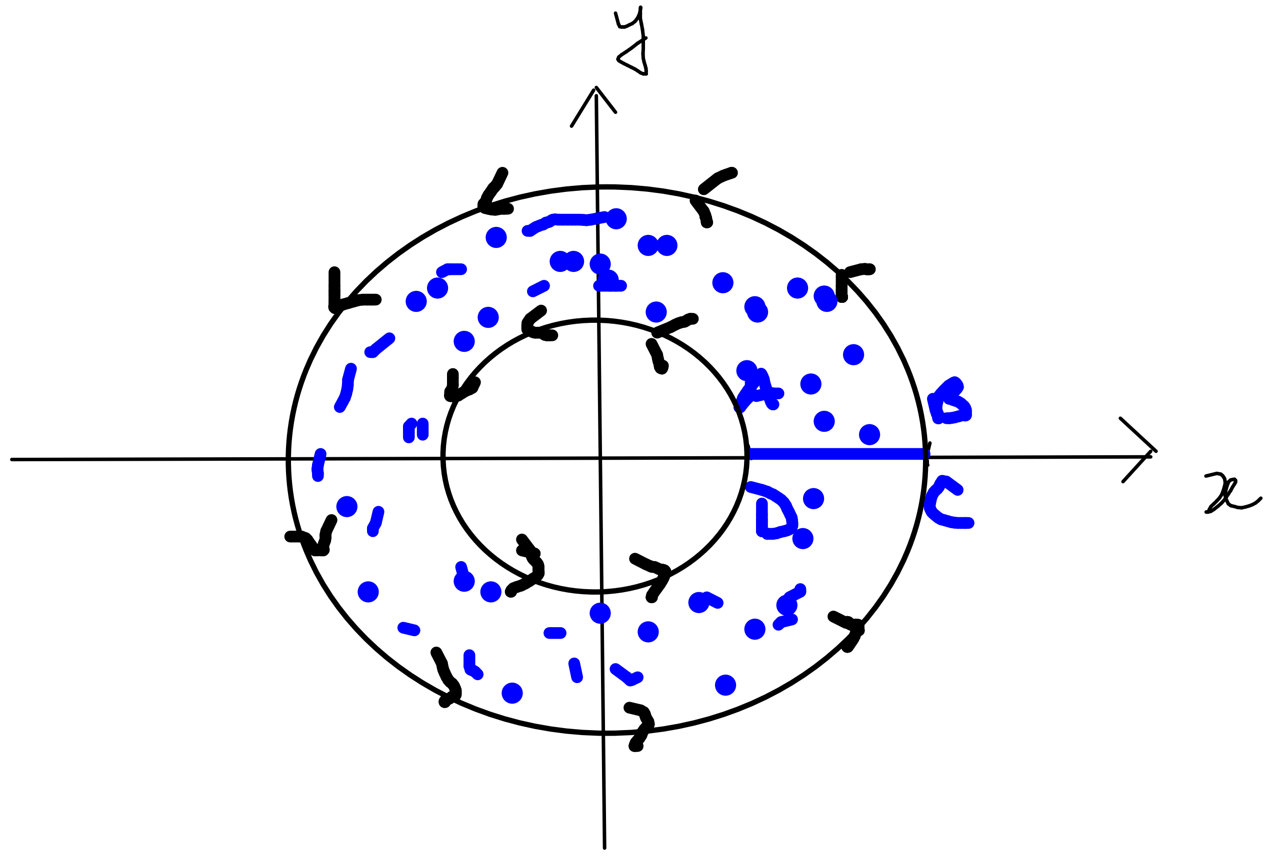
AB

$\theta = 0 \quad , 1 \le r \le 2$

BC

$r = 2,\ 0 \le \theta \le 2\pi$

$x = 2\cos\theta,\ y = 2\sin\theta$



DC, $\theta = 2\pi$, $1 \le r \le 2$

$x = r\cos 2\pi$, $y = r\sin 2\pi$

$x = r$, $y = 0$

$0 \le x \le 2$

AD:

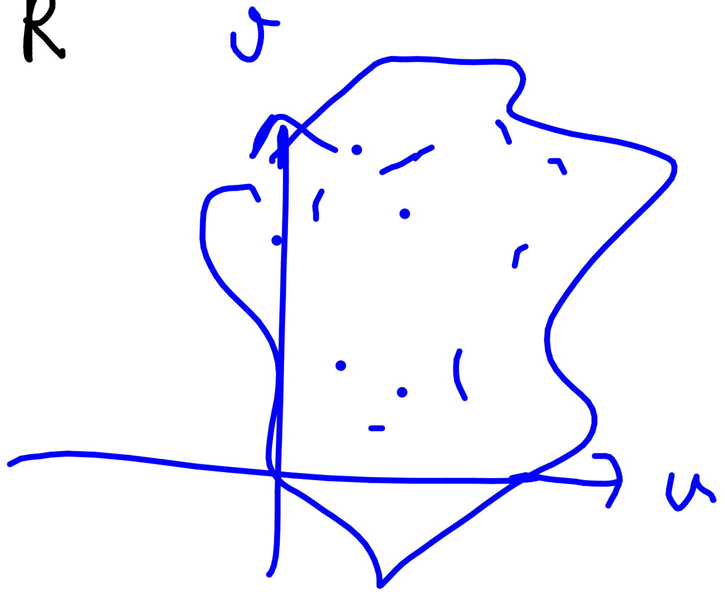
$r = 1$ $\quad 0 \le \theta \le 2\pi$

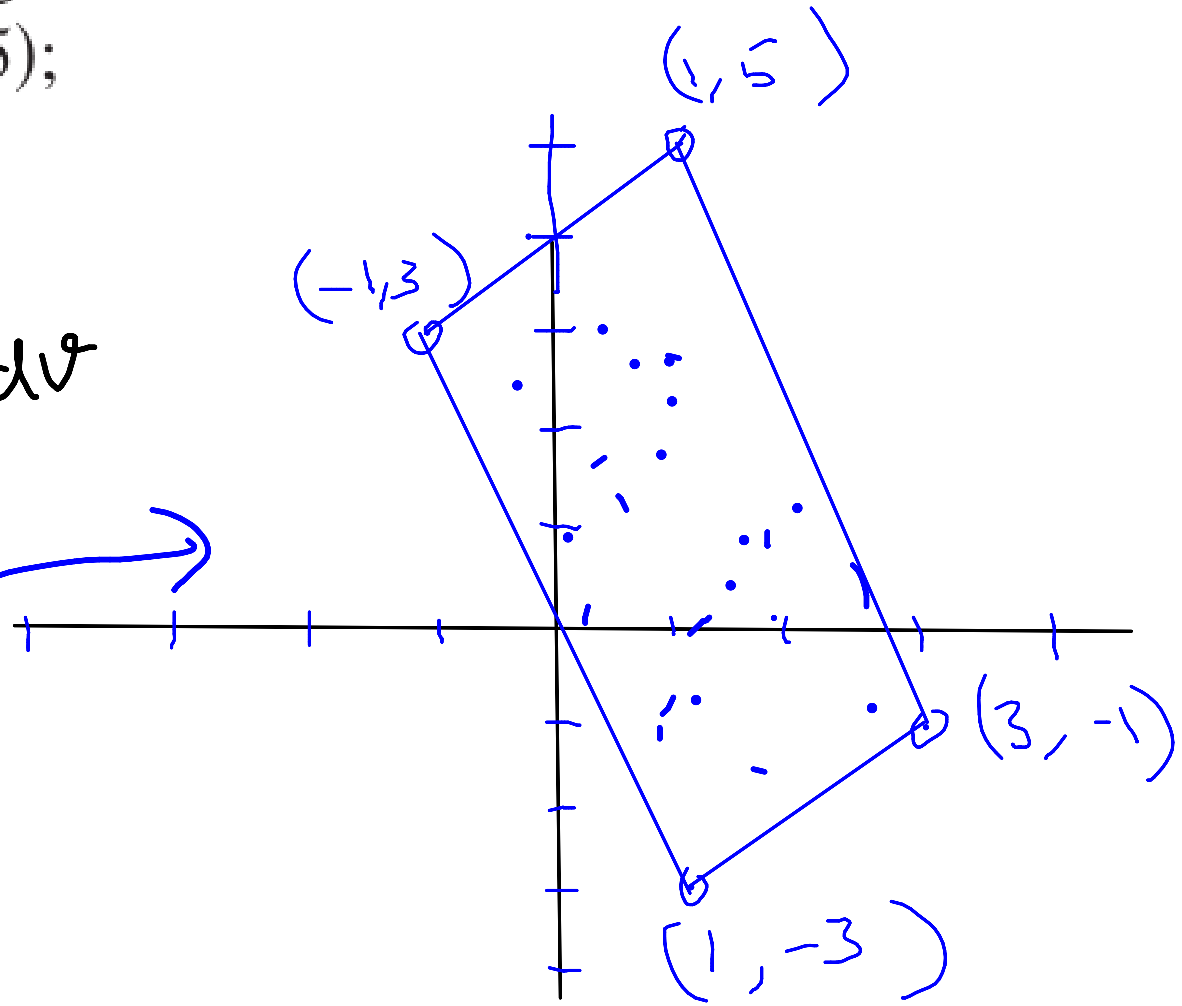
$x = \cos\theta$ $\quad y = \sin\theta$

Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\,dA = \int_{??}^{??}\!\!\int (\;??\;)\,du\,dv$$

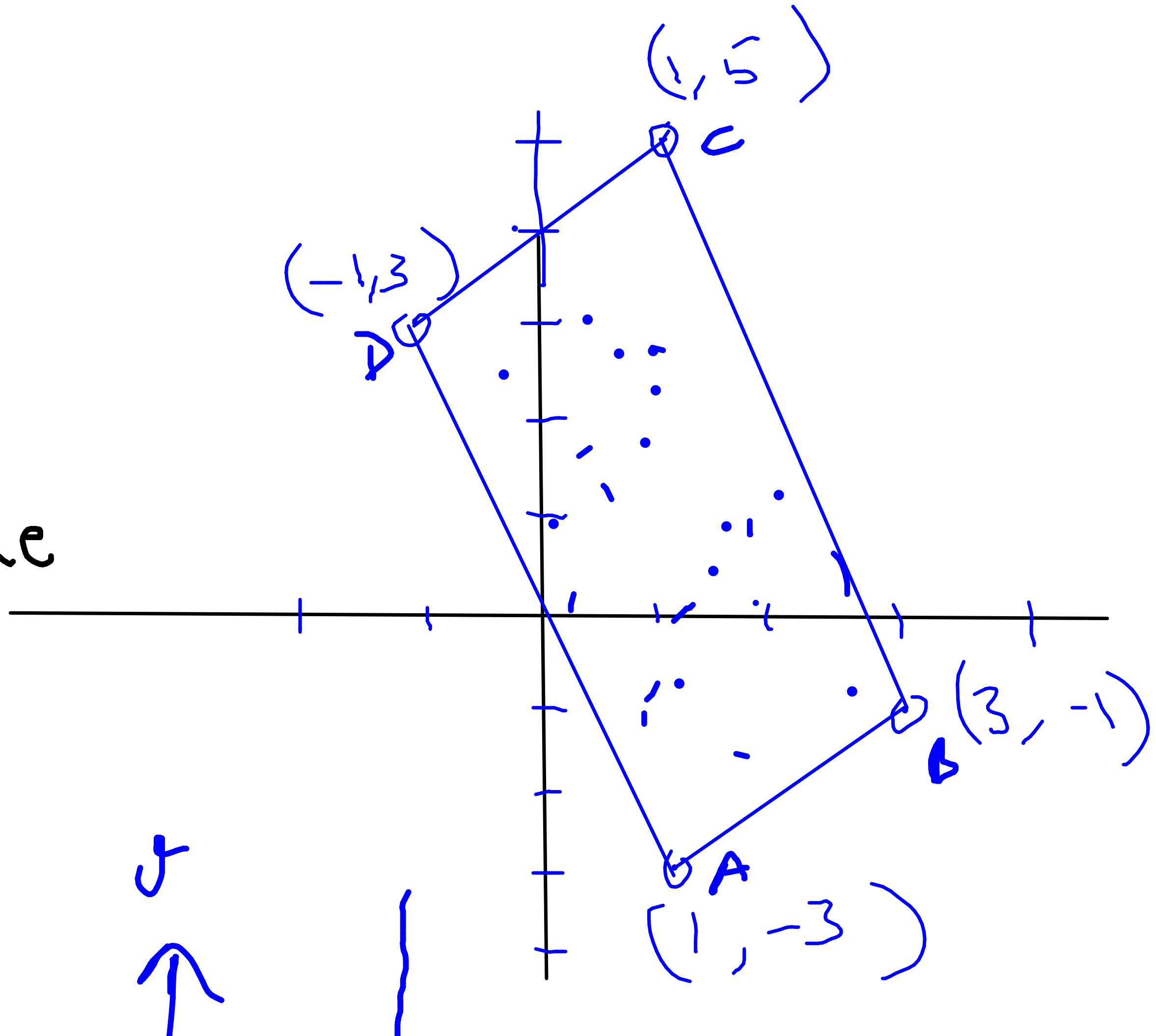




Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

find out the shape of ABCD in the uv-plane



→ AB: eqn of AB in xy variable.

$$x - y = 4$$

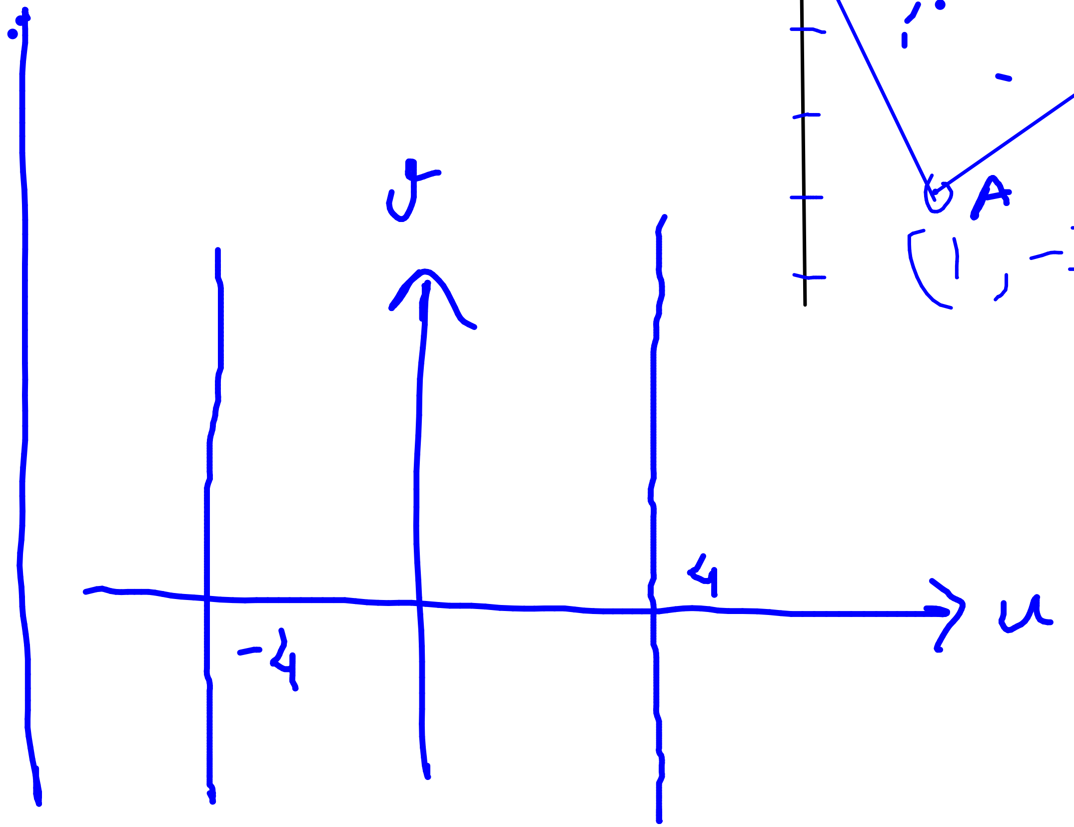
$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = 4$$

$$u = 4$$

→ DC: $x - y = -4$

$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = -4$$

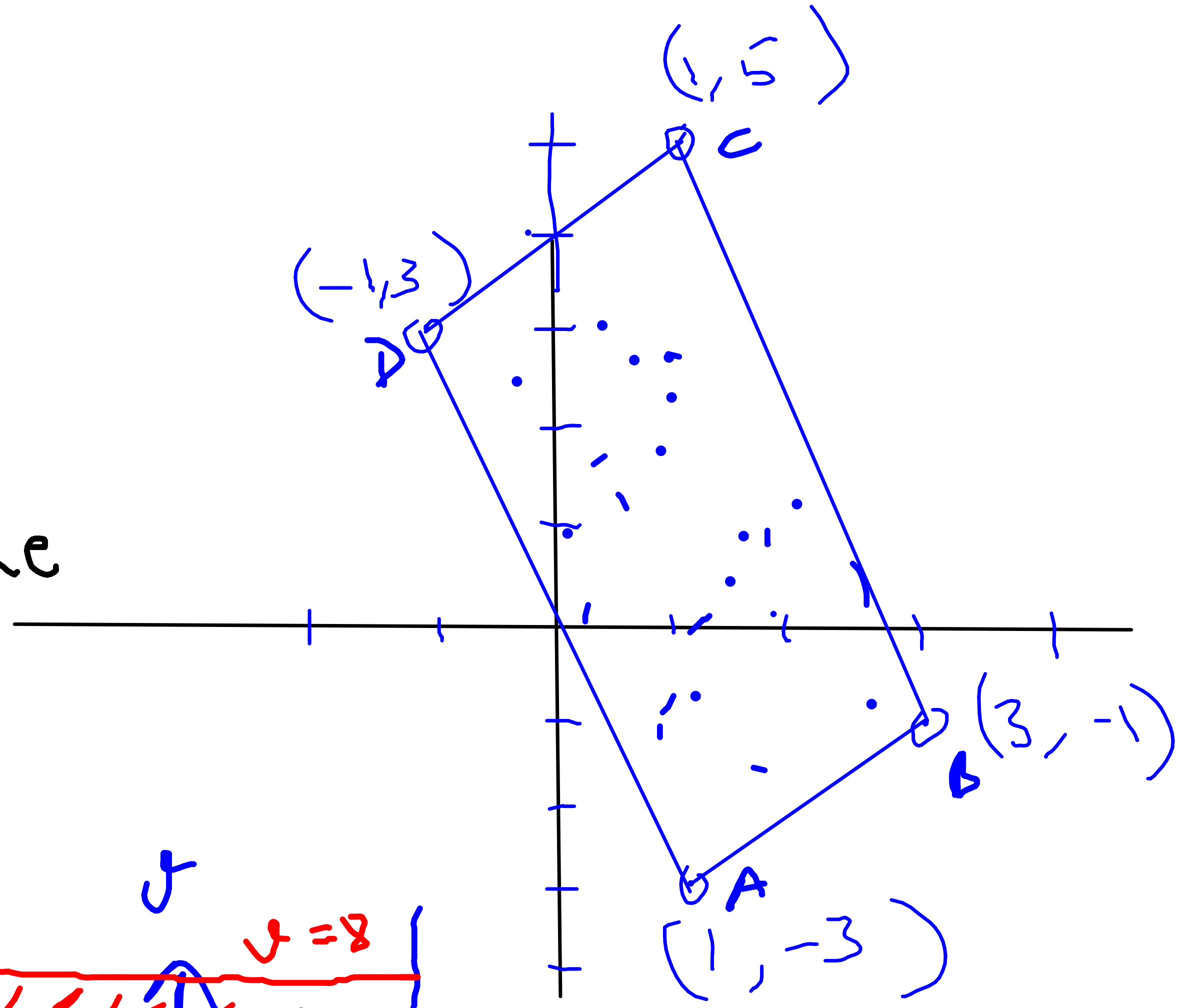
$$u = -4$$



Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

find out the shape of ABCD in the uv-plane



→ AD , $y = -3x$

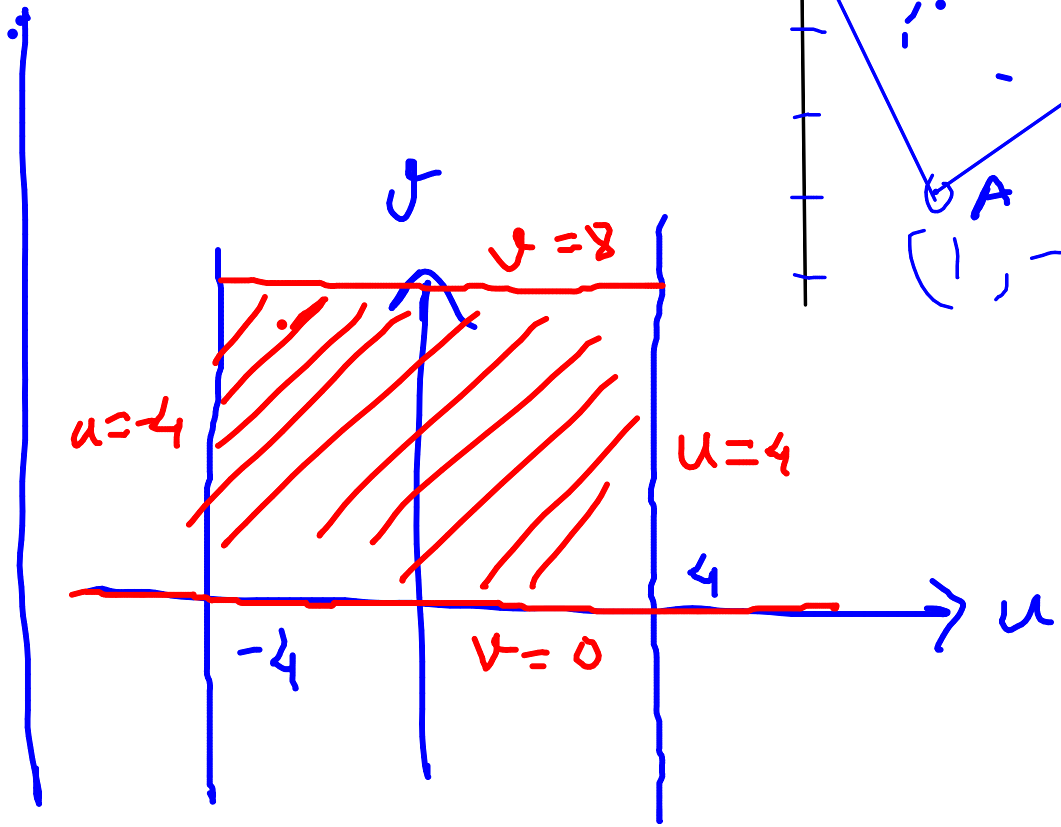
$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)$$

$$v = 0$$

→ BC , $y = -3x+8$

$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v) + 8$$

$$v = 8$$



Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

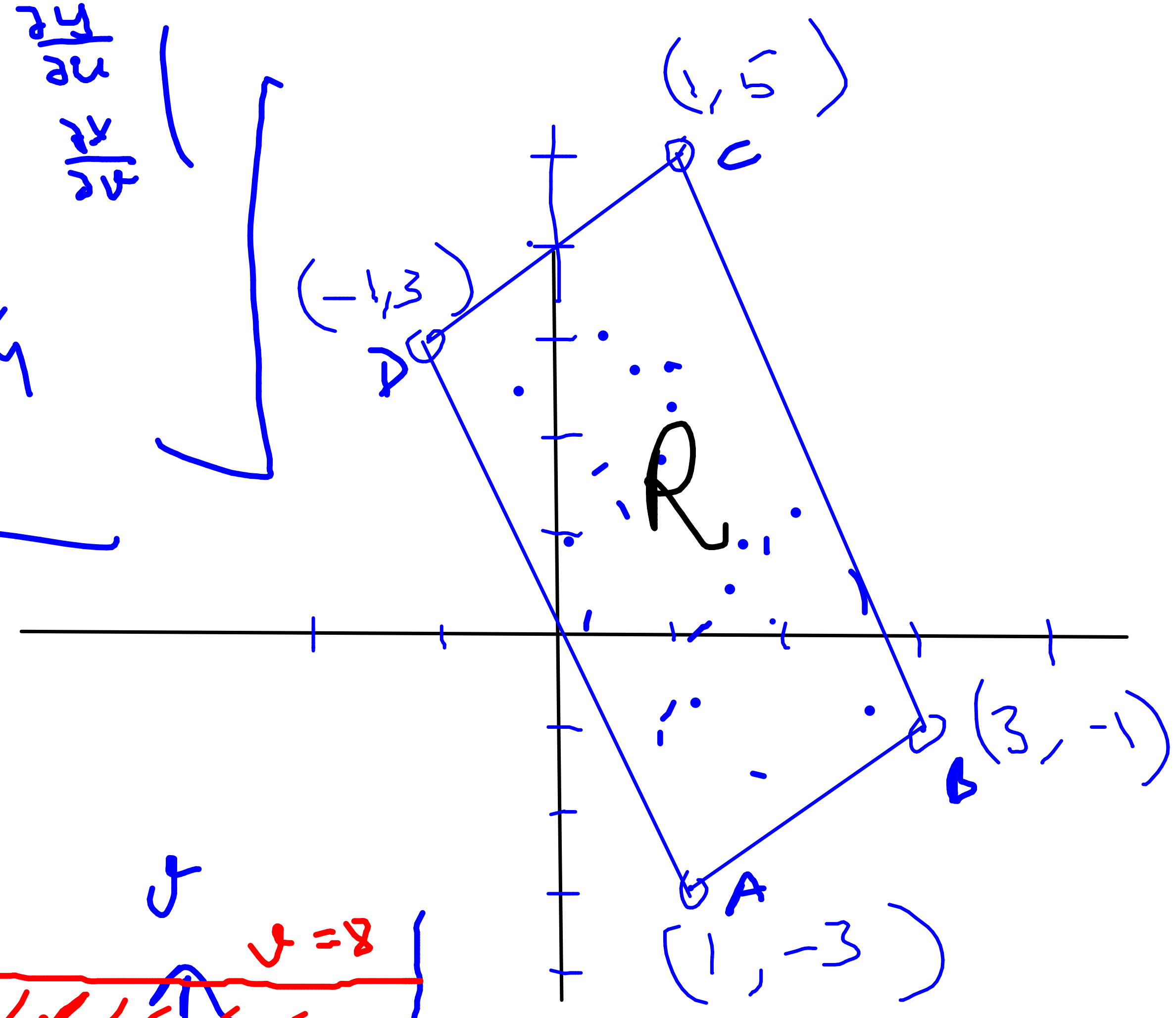
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = 1/4$$

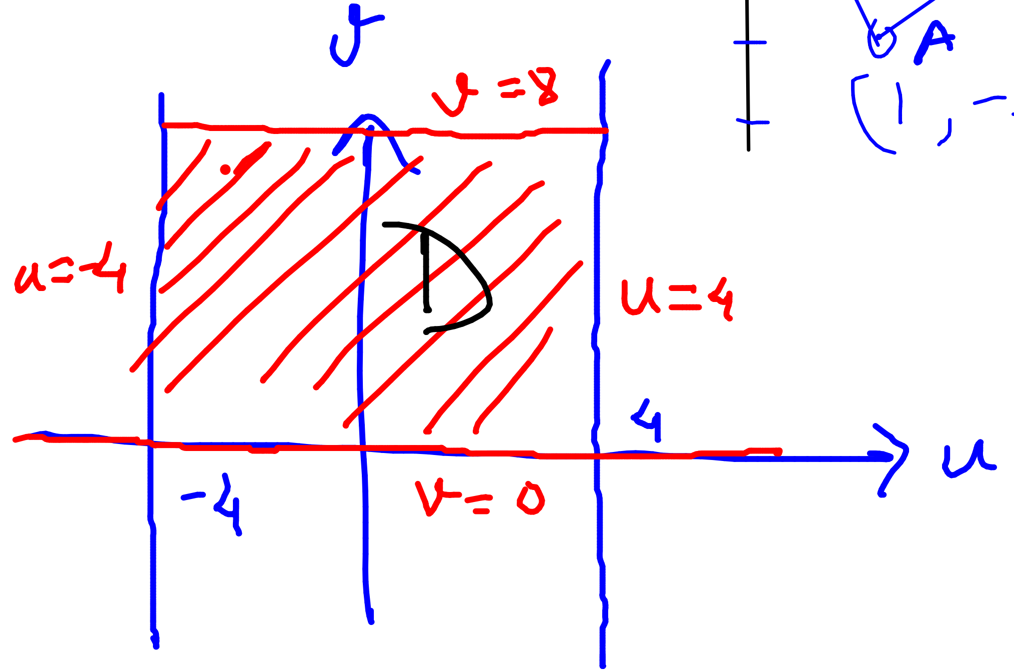
$$4x + 8y = 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u)$$
$$= 3v - 5u$$

$$\iint_R (4x+8y)\, dR = \iint_D (3v-5u)\,(\text{Jacobian})\, dD$$

$$= \int_0^8 \int_{-4}^{4} (3v - 5u)\left(\frac{1}{4}\right) du\, dv$$

$$= 192$$





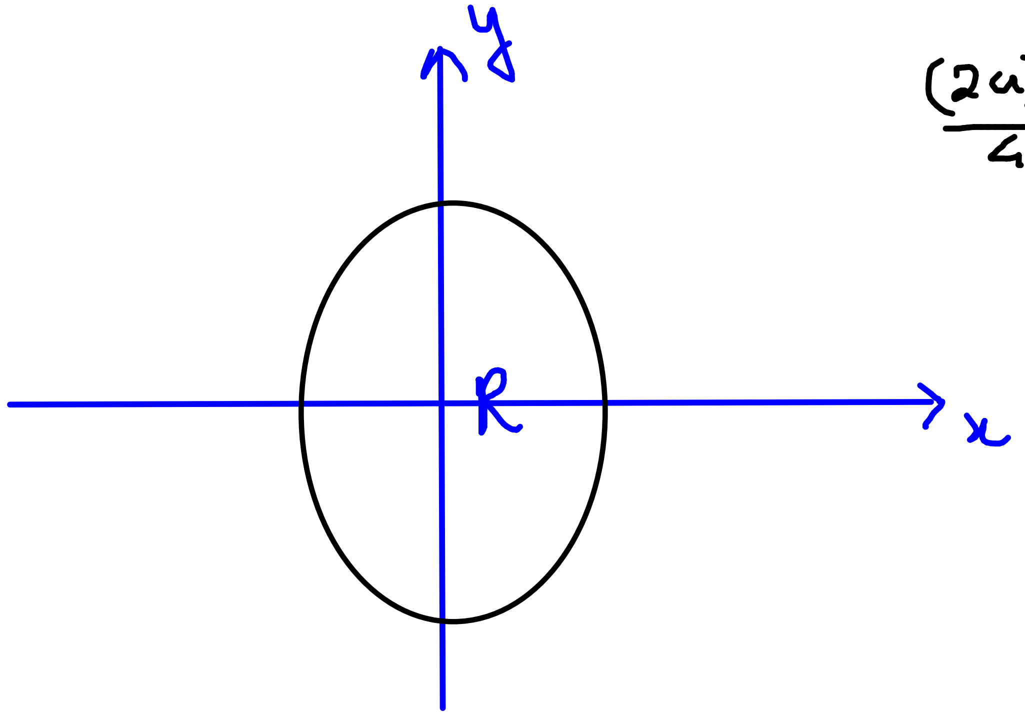
Use the given transformation to evaluate the integral.

$\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $\quad x = 2u,\ y = 3v$

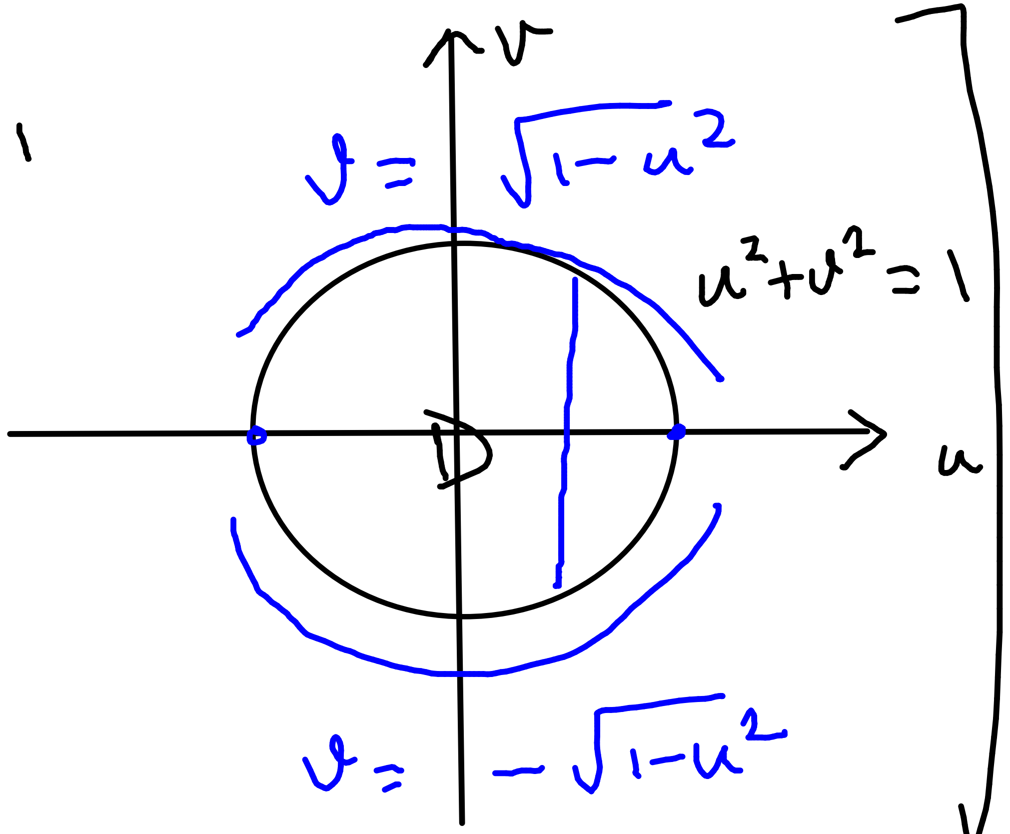
$$\iint_R x^2\, dA = \iint_D (2u)^2 \cdot (\text{Jacobian})\, dO$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{(2u)^2}{4} + \frac{(3v)^2}{9} = 1$$



Jacobian = 6



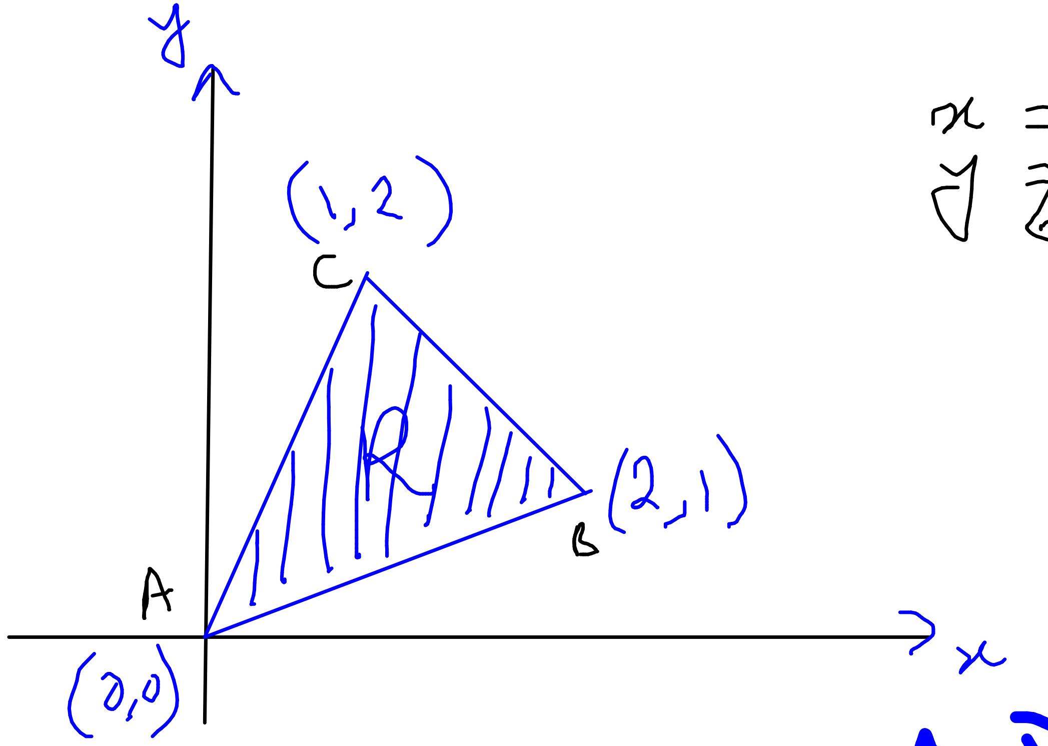
$$= \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 4u^2 \cdot 6 \, dv\, du$$

$$= \underline{\qquad\qquad}$$

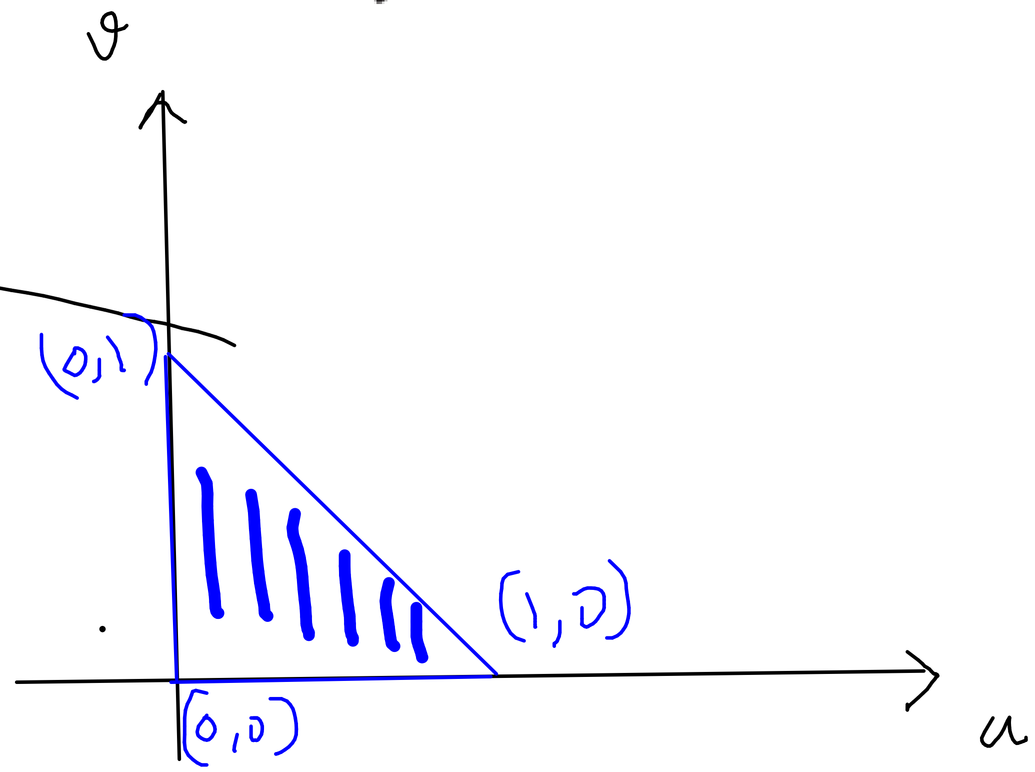
complete it.

Use the given transformation to evaluate the integral.

$\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $x = 2u + v$, $y = u + 2v$



$x = 2u + v$
$y = u + 2v$



AB

$x = 2y$

$2u + v = 2(u + 2v)$

$v = 0$

AC

$y = 2x$

$u + 2v = 2(2u + v)$

$u = 0$

BC

$x + y = 3$

$(2u + v) + (u + 2v) = 3$

$u + v = 1$

$$\iint_R (x - 3y)\, dA = \int_B^A \int_C^D (\ E\ )(\ F\ )\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} (-u-5v)(3)\, dv\, du$$

Jacobian

Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$x - 3y = (2u+v) - 3(u+2v)$$

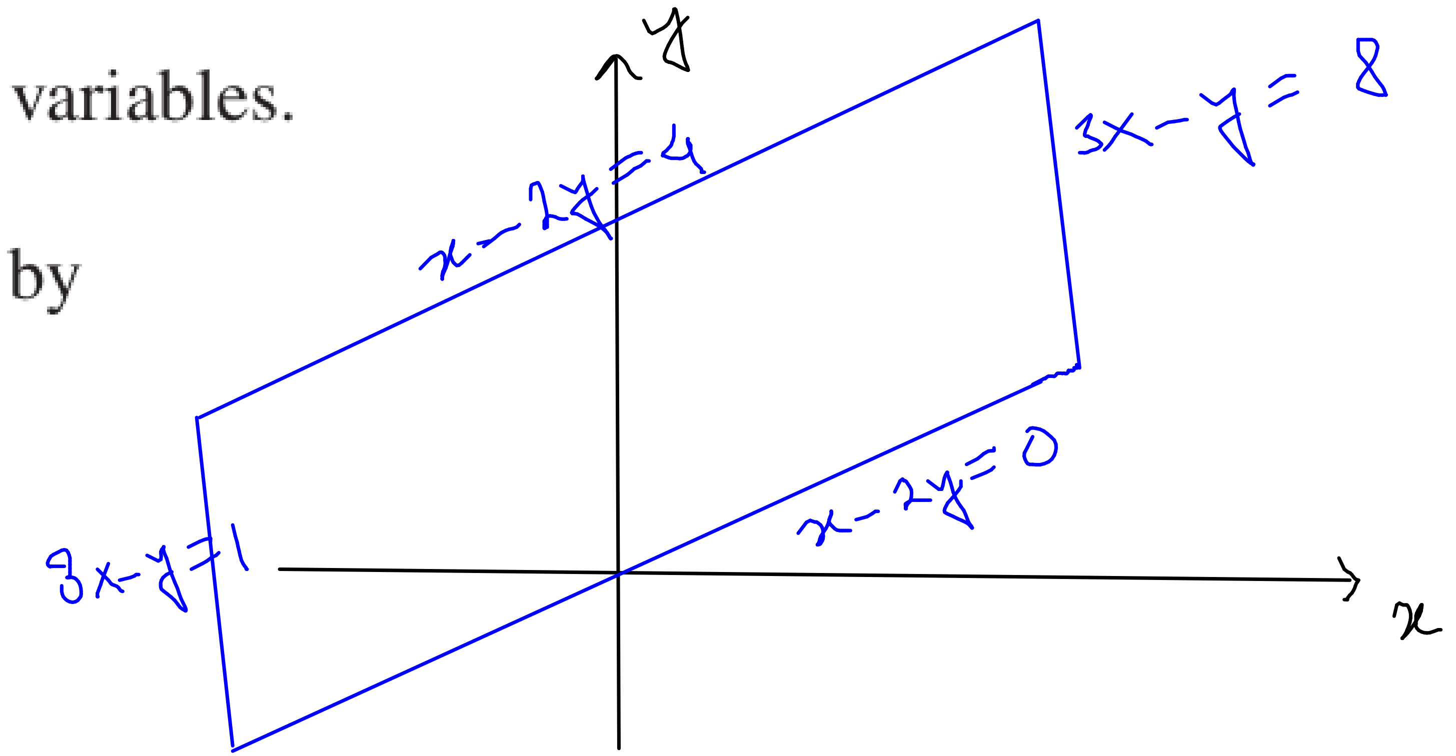
$$= -3$$

Evaluate the integral by making an appropriate change of variables.

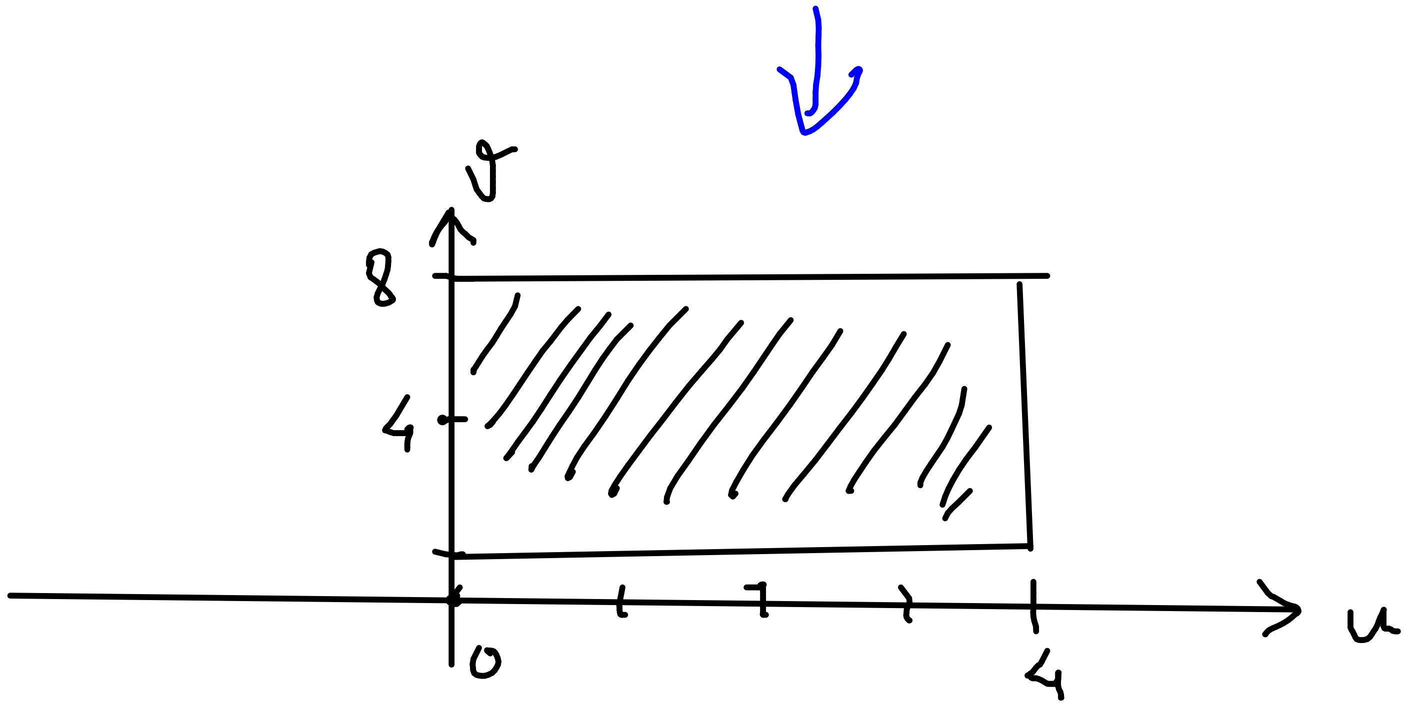
$\displaystyle\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$

$u = x-2y$ | $x=(2v-u)/5$

$v = 3x-y$ | $y=(v-3u)/5$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{-3}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}$$

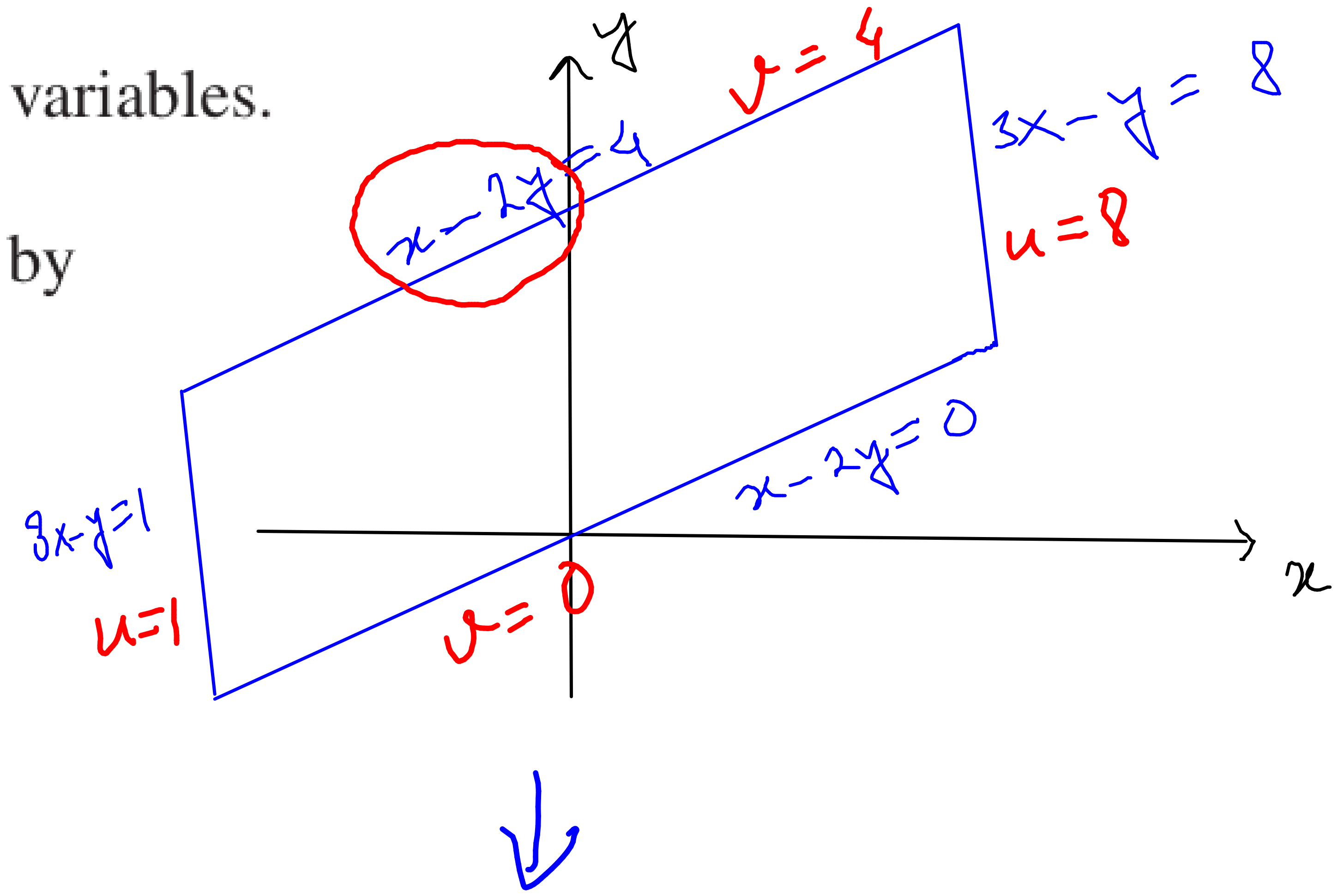


$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4\int_1^8 \frac{u}{v}\left(\frac{1}{5}\right) dv\,du$$

$$= (8\log 8)/5$$

Evaluate the integral by making an appropriate change of variables.

$$\iint_R \frac{x - 2y}{3x - y}\,dA,$$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$



$$u = 3x - y$$

$$v = x - 2y$$

Evaluate the integral by making an appropriate change of variables.

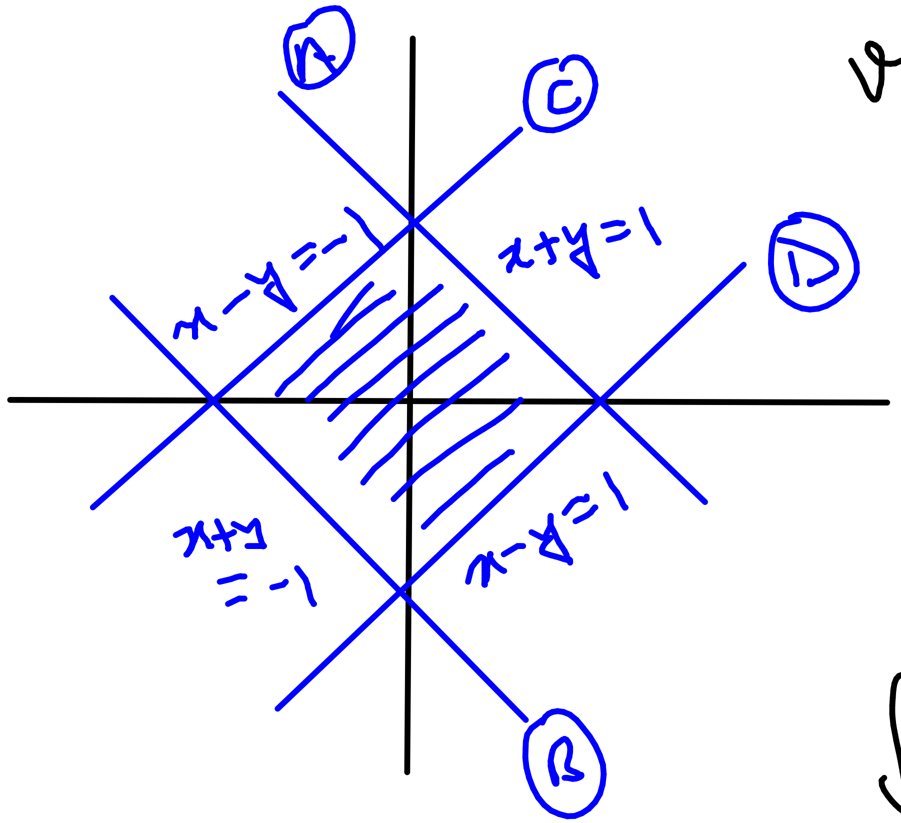
$\iint_R e^{x+y}\, dA$, where R is given by the inequality
$|x| + |y| \leqslant 1$

$$|x| + |y| \le 1$$

$$u = x+y \qquad x = (u+v)/2$$
$$v = x-y \qquad y = (u-v)/2$$

$$(\pm x) + (\pm y) \le 1$$

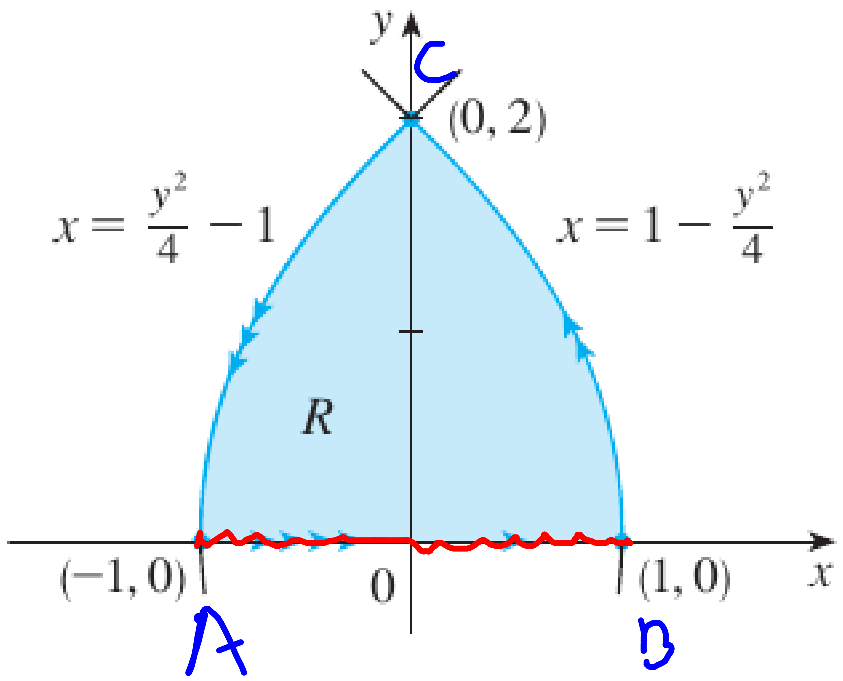
$x+y \le 1$	$x-y \le 1$	$-x+y \le 1$	$-x-y \le 1$
A	B	C	D



$$\iint_R e^{x+y}\, dA = \int_A^B \int_C^D (\ E\ )(\ F\ )\, du\, dv = \int_{-1}^{1}\int_{-1}^{1} e^u \left(\tfrac{1}{2}\right) du\, dv$$

$$= e - \frac{1}{e}$$

V EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.



OB $y = 0$, $x \geq 0$

$u = 0$ or $v = 0$ $[u^2 \geq v^2]$ $\Rightarrow$

BC :

$u = \pm 1$

AC

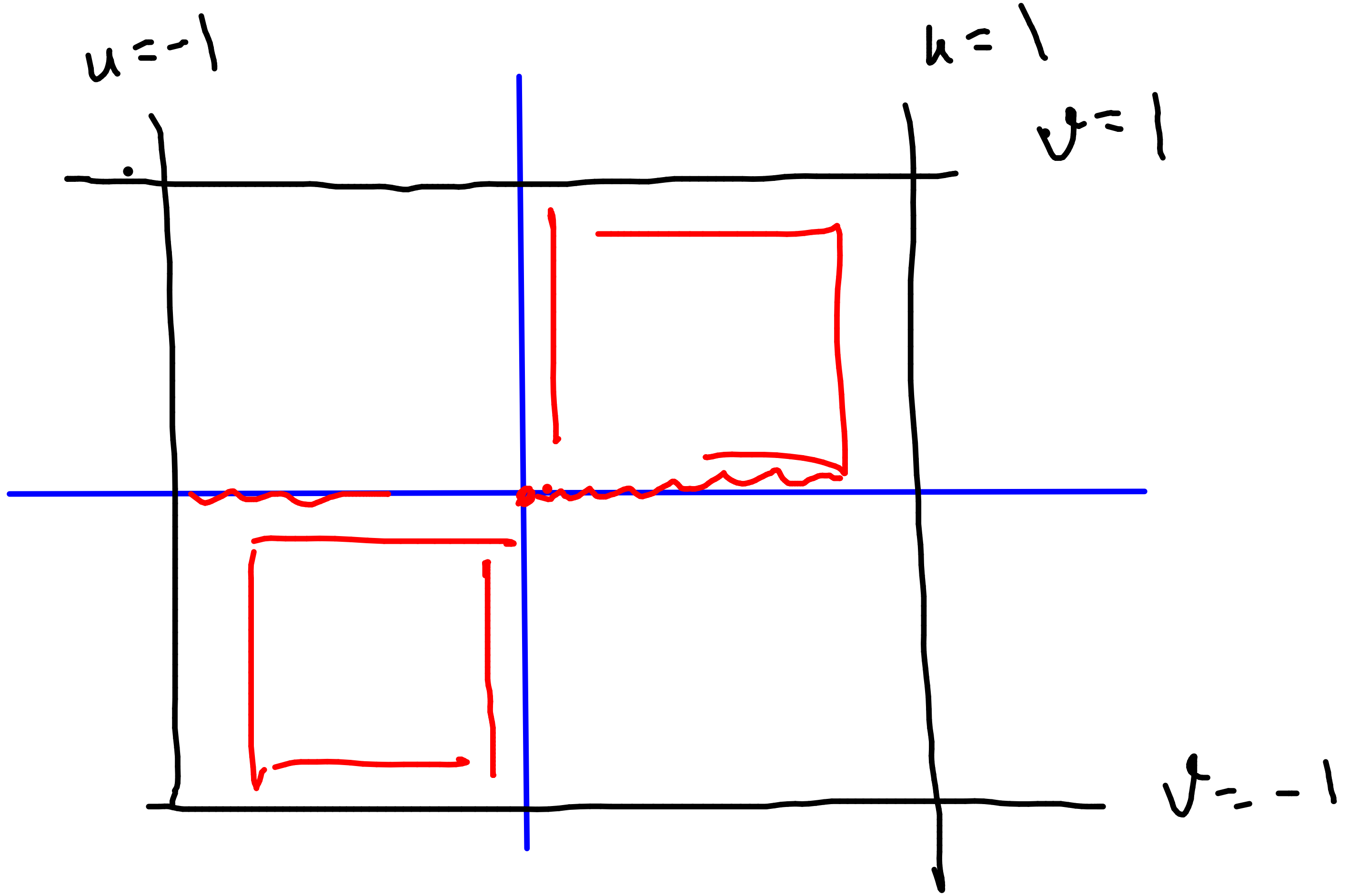
$y^2 = 4 + 4x$

$u^2 v^2 = 1 + u^2 - v^2$

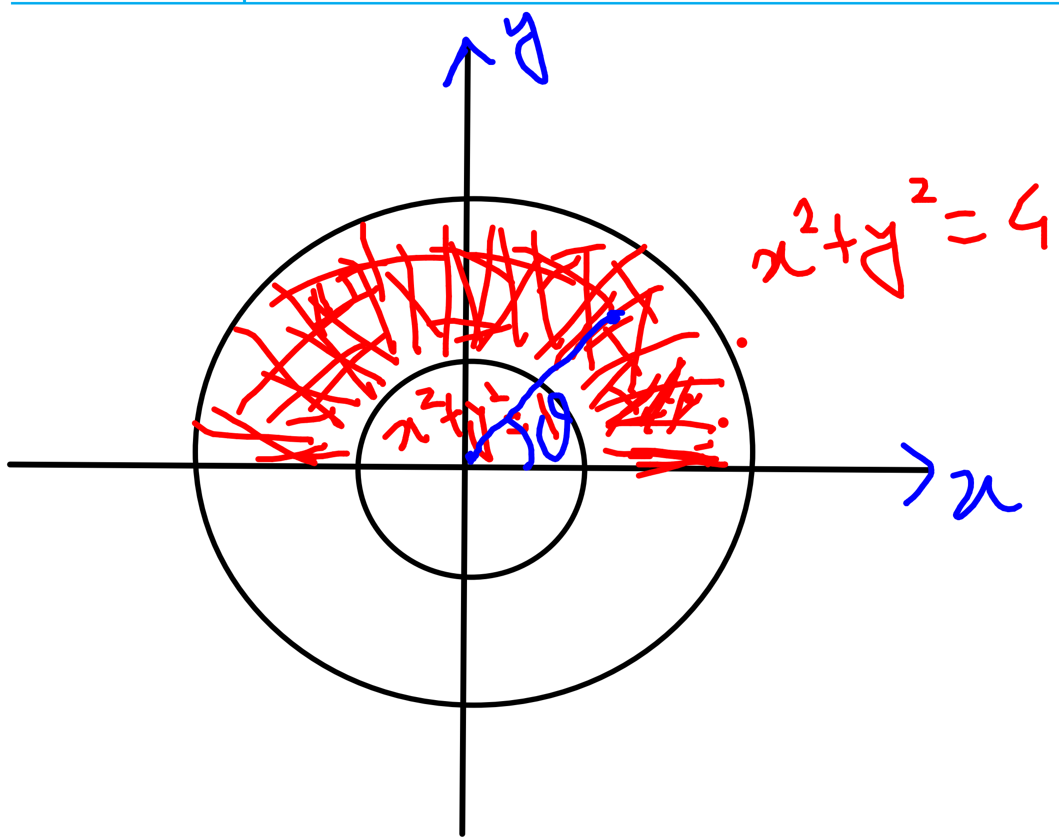
$1 + u^2 - v^2 - u^2 v^2 = 0$

$\cancel{(1+u^2)}(1 - v^2) = 0$

$v = \pm 1$



12.3 DOUBLE INTEGRALS IN POLAR COORDINATES



range of r & θ for the shaded region

$0 \le \theta \le \pi$

$1 \le r \le 2$

$x = r\cos\theta$

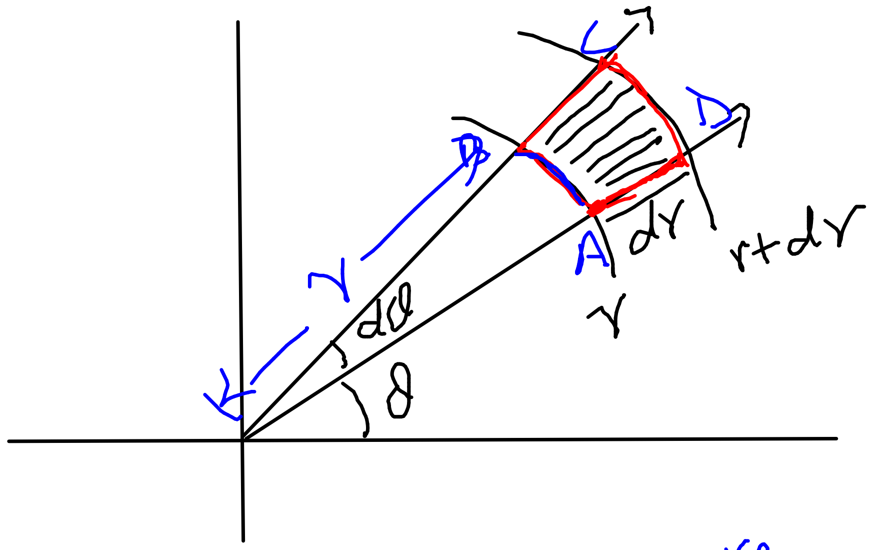
$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$dx\,dy \approx r\,dr\,d\theta$$

$\square$ dy

dx

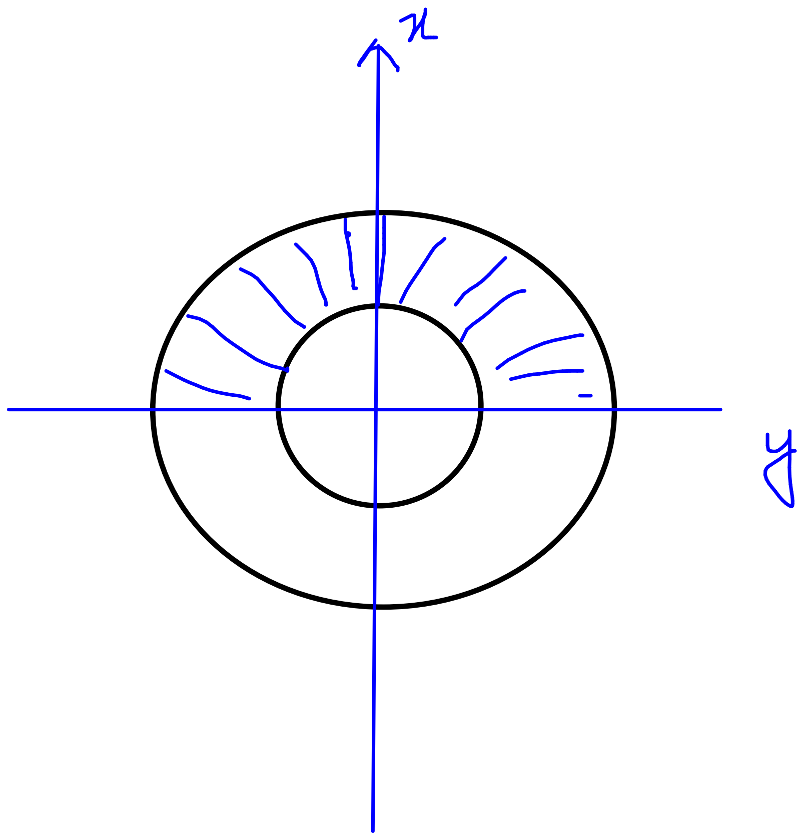


$AB \approx r\,d\theta$

$AD = dr$

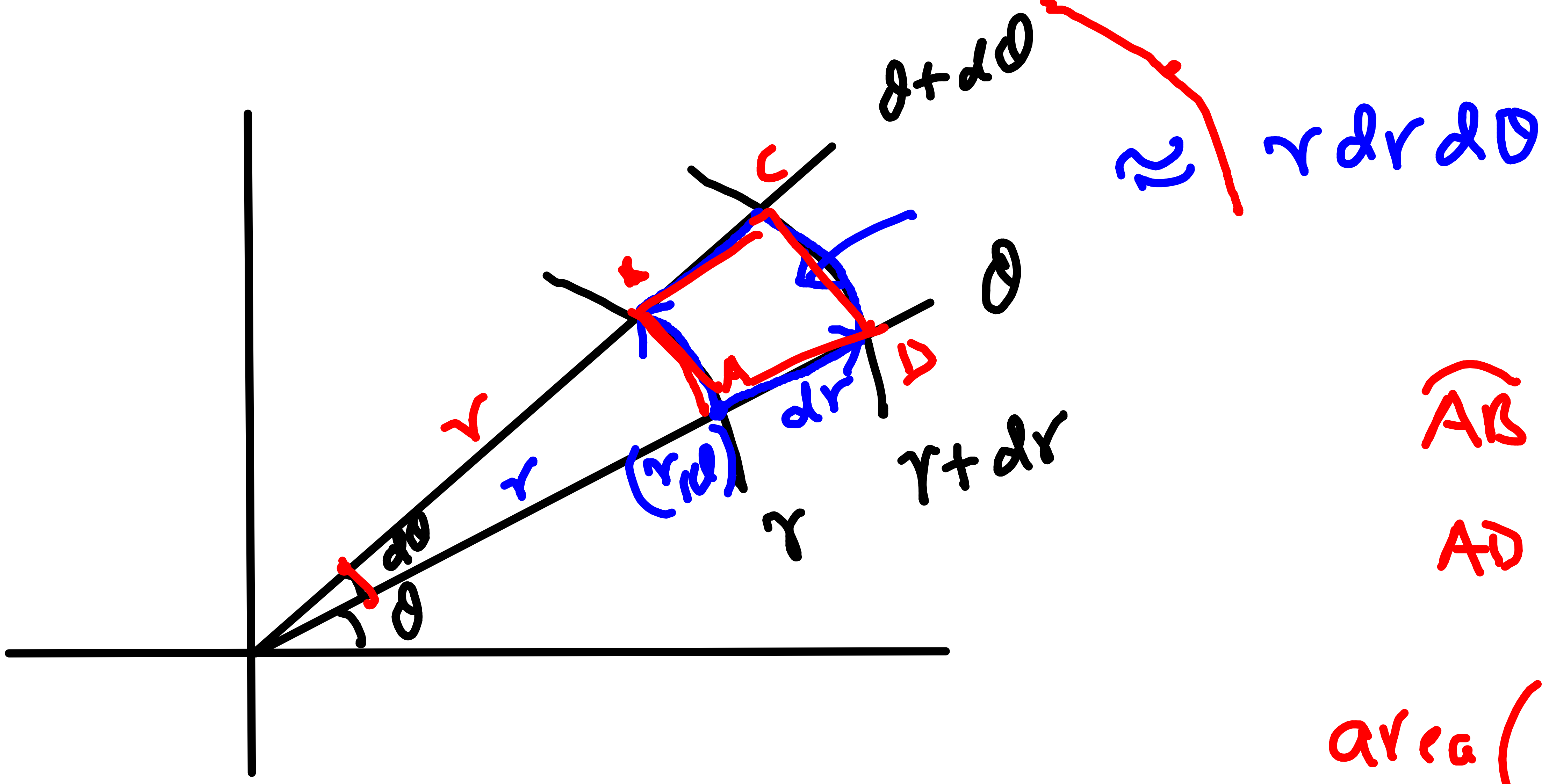
area $(ABCD) = (r\,d\theta)\,dr = r\,dr\,d\theta$

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$\int_1^2 \int_0^{\pi} \left(3r\cos\theta + 4r^2\sin^2\theta\right) r \, d\theta \, dr$$

$$= \frac{15\pi}{2}$$



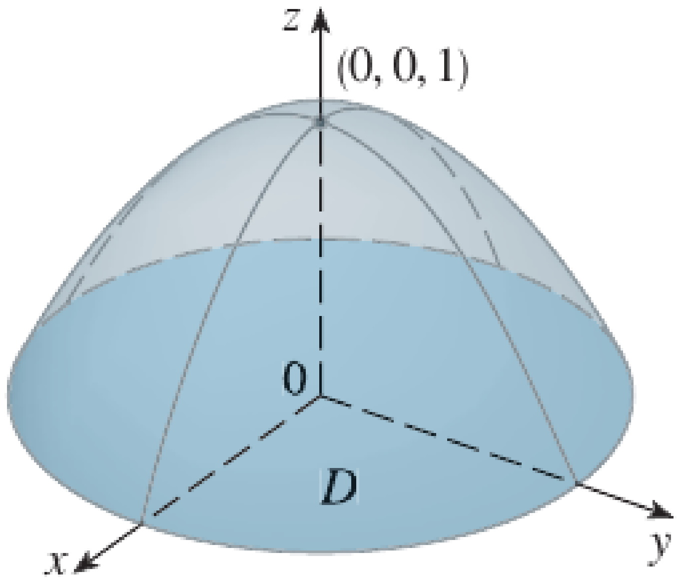
$$\overset{\frown}{AB} = r\,d\theta$$

$$AD = dr$$

$$\text{area}(ABCD) \approx (r\,d\theta)(dr)$$

$$= r\,dr\,d\theta$$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



V EXAMPLE 3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.