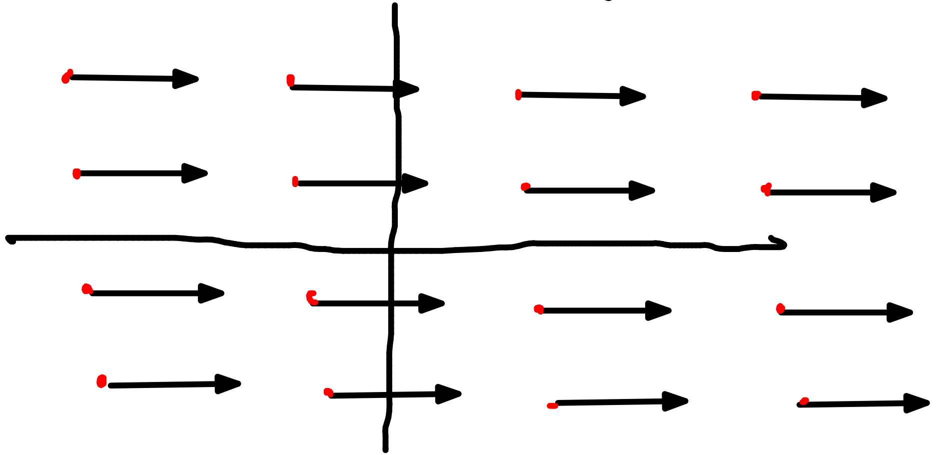
13.1 VECTOR FIELDS

functions whose range set are vector sets

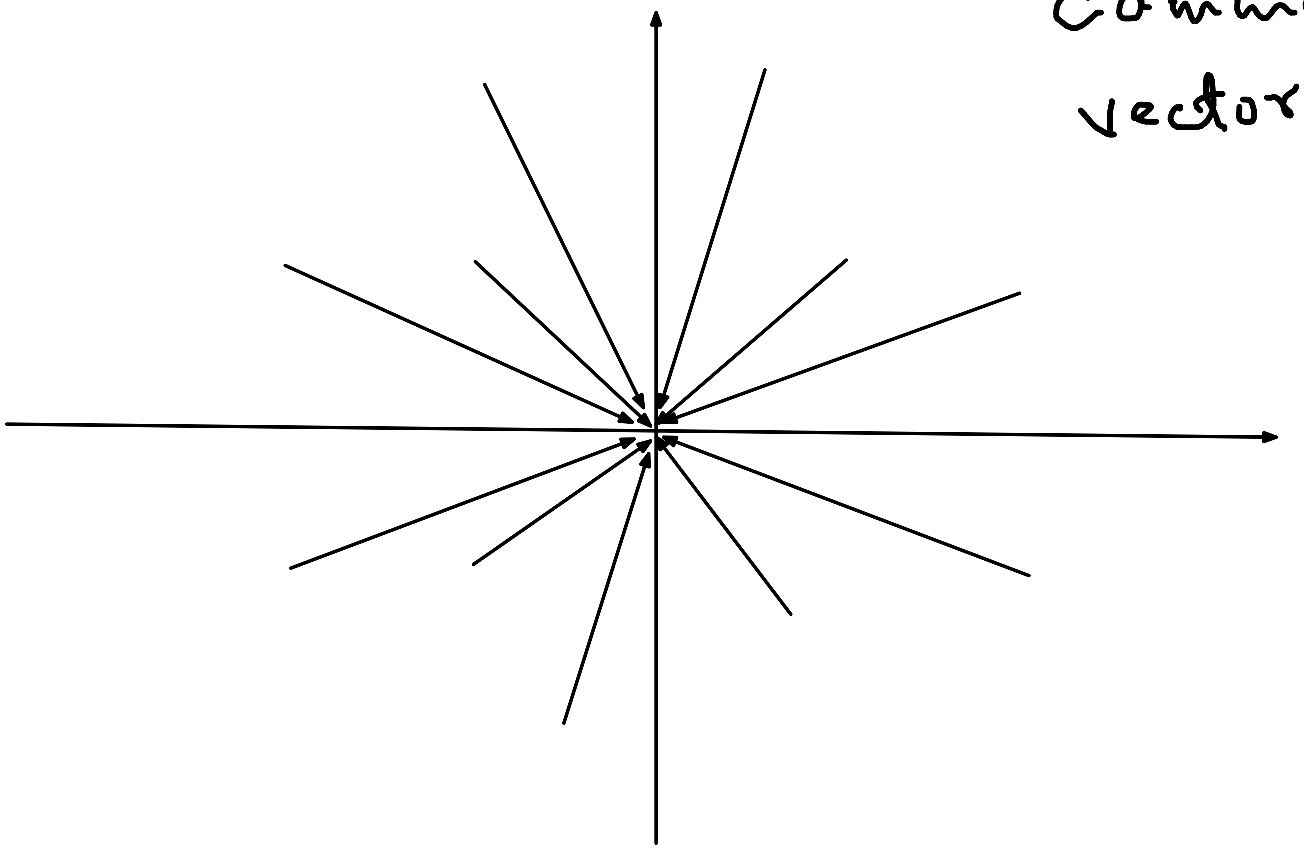
e.g. $\vec{F}(x,y) = \hat{i}$

VF $\mathbb{R}^2 \to \mathbb{R}^2$

maps a point in $\mathbb{R}^2$ to a 2 dimensional vector



$$\vec{F}(x,y) = -x\hat{i} - y\hat{j}$$



command for plotting vector fields in matlab/octave

"quiver"

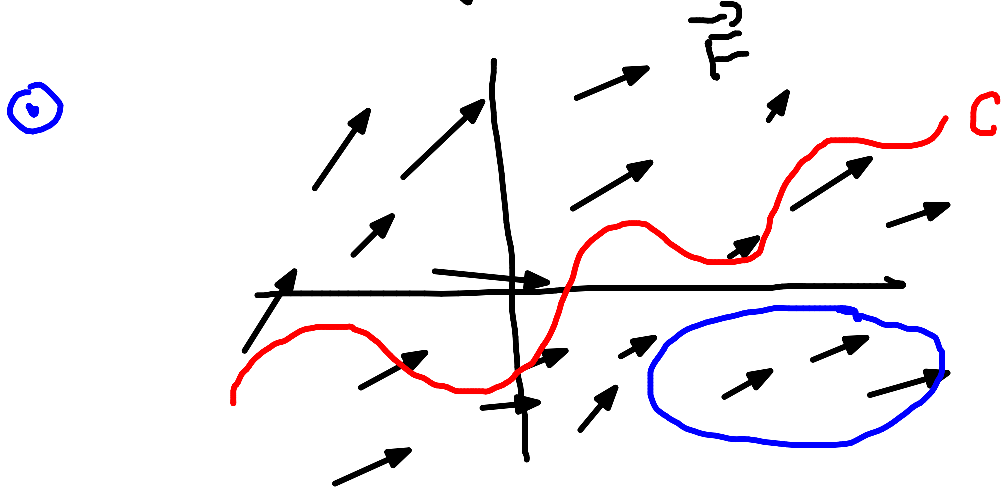
$$\vec{F}(x, y, z) = F_1(x, y, z)\,\hat{i} + F_2(x, y, z)\,\hat{j} + F_3(x, y, z)\,\hat{k}$$

Force field

$$\vec{v}(x, y, z) = v_1(x, y, z)\,\hat{i} + v_2(x, y, z)\,\hat{j} + v_3(x, y, z)\,\hat{k}$$

Velocity field

Preview of the chapter



work done by $\vec{F}$ on moving a particle along the given path C

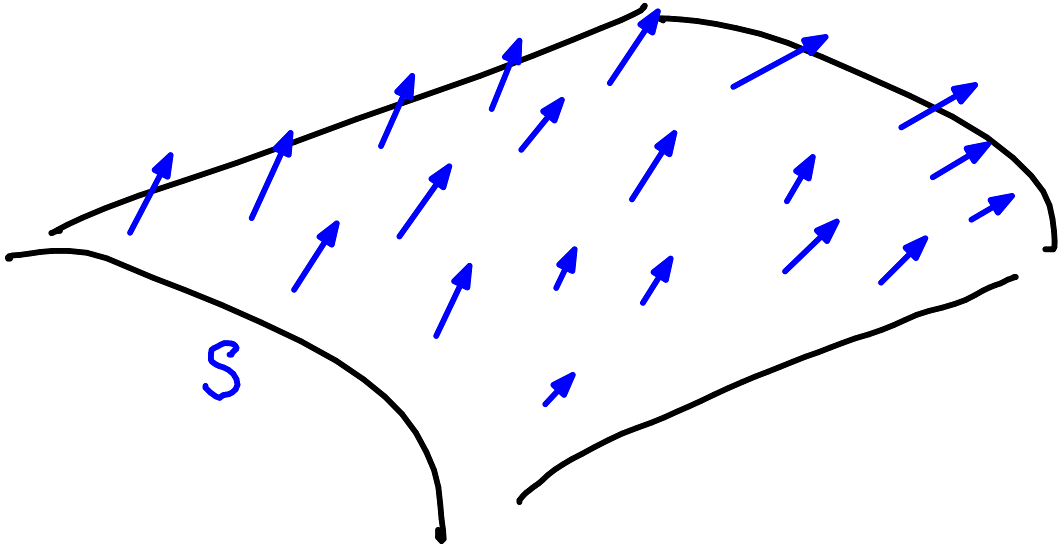
$$\int_C \vec{F}\cdot d\vec{r}$$

- Greens theorem
- Stokes theorem

simplification in $\int_C \vec{F}\cdot d\vec{r}$ if C is a closed loop

- Conservative Vector Fields

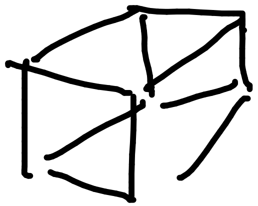
Later half of Chapter 13



$$\iint_S \vec{F} \cdot dA$$

flux of vector fields

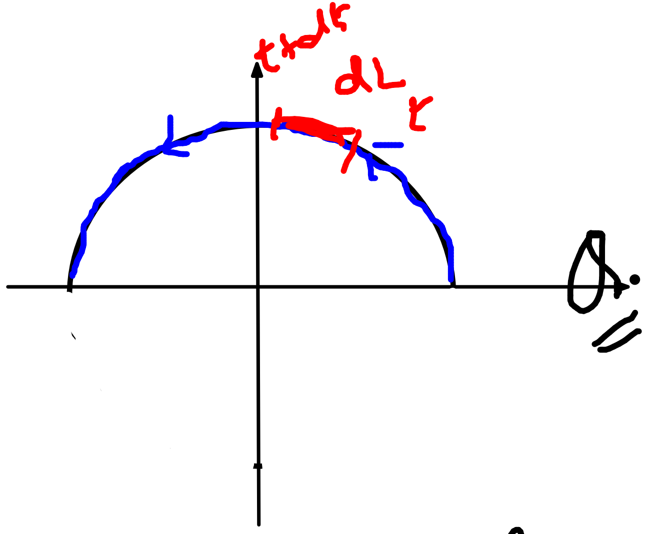
→ Divergence theorem:



13.2 LINE INTEGRALS

EXAMPLE 1 Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

$f = 2 + x^2y$: mass per unit length at point (x, y)



$\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$

$0 \le t \le \pi$

$$\int_C (2 + x^2y)\,ds = \text{mass of } C$$

Can you find length of C ??

$$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$dm = (\text{density})\,dL$$

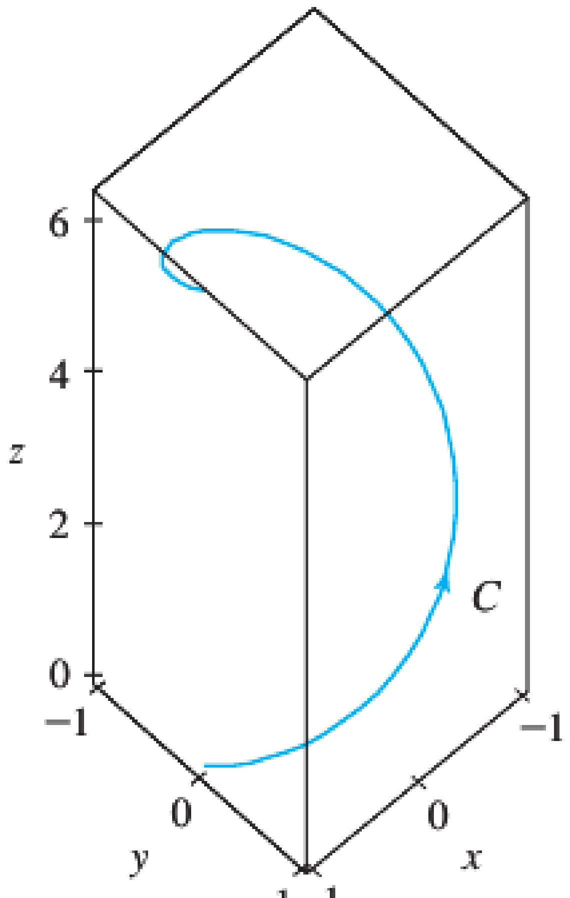
$$= (2+x^2 y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Total mass

$$m = \int_0^{\pi} dm = \int_0^{\pi} (2+x^2 y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi} \left(2+\cos^2 t \sin t\right) dt = \text{whatever}\checkmark$$

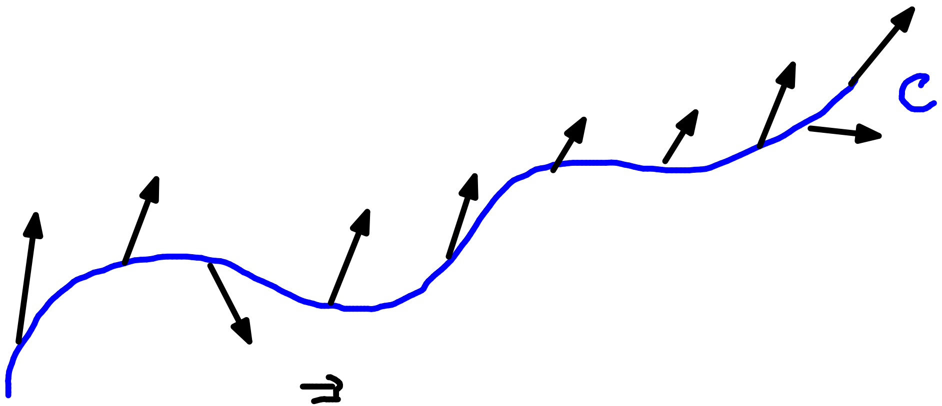
EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)



$$\int_C y \sin z \, ds = \int_0^{2\pi} y \sin z \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

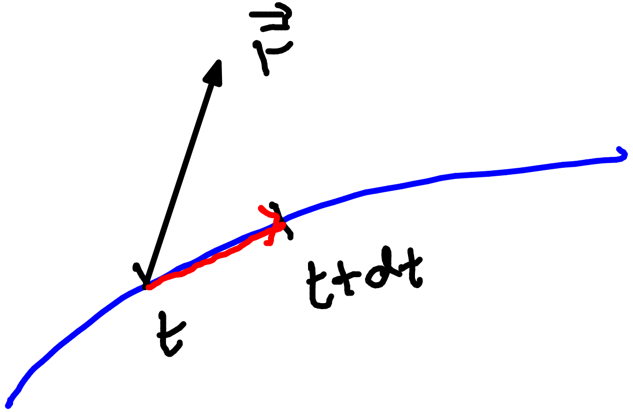
$$= \int_0^{2\pi} \sin^2(t) \sqrt{2} \, dt = \text{whatever}$$

LINE INTEGRALS OF VECTOR FIELDS



$\vec{F}$ integrate $\vec{F}$ along C

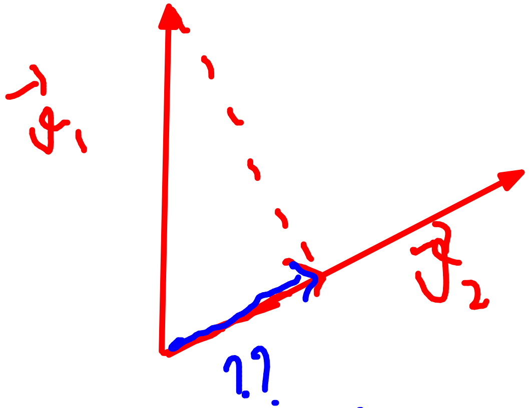
$$\int_C \vec{F}\cdot d\vec{r} = \text{work done by } \vec{F} \text{ in moving a particle along the curve}$$



$W = \vec{F}\cdot\vec{d} = \left(\text{component of } \vec{F} \text{ in the direction of displacement}\right) \times \left(\text{distance travelled}\right)$

$$dW = \left(\text{tangential component of } \vec{F}\right) dL$$

Tangential Component of $\vec{F} = \vec{F} \cdot \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



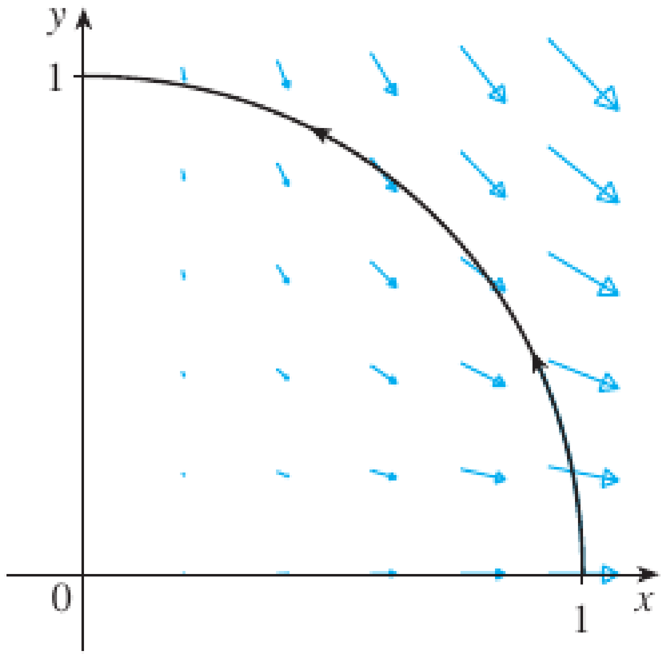
$= \vec{v}_1 \cdot \dfrac{\vec{v}_2}{|\vec{v}_2|}$

$$dW = \vec{F} \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \cdot \cancel{|r'(t)|}\; dt$$

$$= \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$W = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.



$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F} \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\pi/2} \left(\cos^2 t,\ -\cos t \sin t\right) \cdot \left(-\sin t,\ \cos t\right) dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \, dt = -\frac{2}{3}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt = \frac{27}{28}$$

13.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

11. $\mathbf{F}(x, y) = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

13. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$