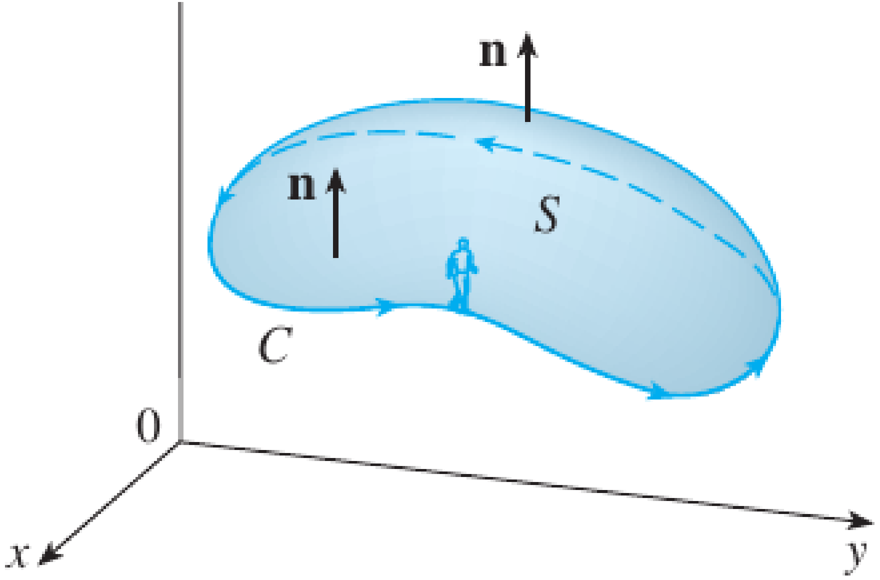
13.8 STOKES' THEOREM

13.9 THE DIVERGENCE THEOREM

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

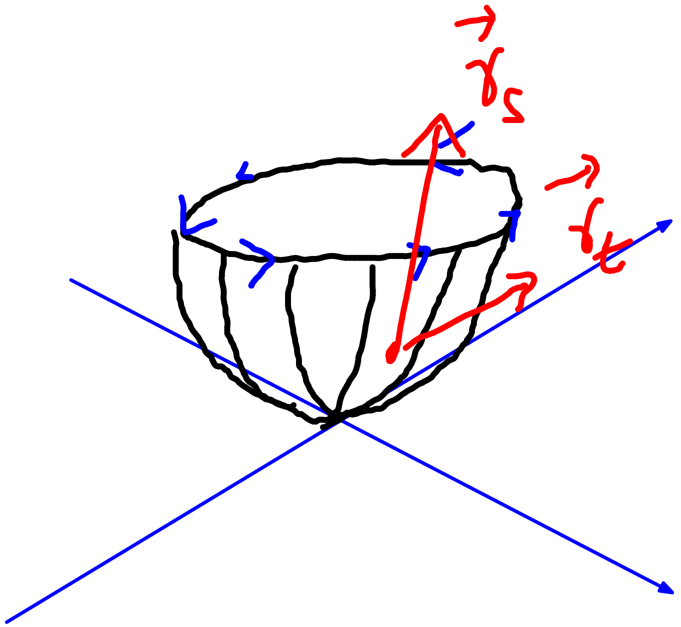
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

$\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



boundary parametric eq^n

$\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + \hat{k}$ $\quad 0 \le t \le 2\pi$

Surface parametric eq^n

$\vec{r}(s,t) = s\cos t\,\hat{i} + s\sin t\,\hat{j} + s^2\,\hat{k}$

$0 \le s \le 1$
$0 \le t \le 2\pi$

LHS

```
syms t
x = cos(t);
y = sin(t);
z = 1;
r = [x,y,z];
F = [y^2, x, z^2];
lhs = int(sum(F.*diff(r,t)),t,0,2*
```

Ans = π

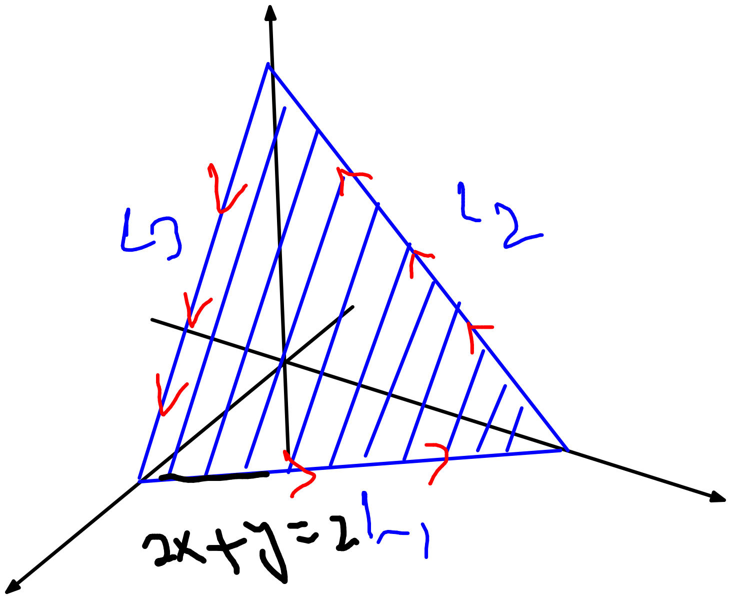
```
F = [y^2, x, z^2]
cF = curl(F)
```

RHS

```
% problem 1
% rhs

syms s t
x = s*cos(t);
y = s*sin(t);
z = s^2;
r = [x,y,z];
cF = [0, 0, 1 - 2*y];
c = simplify(cross(diff(r,s),diff(r,t))

rhs = int(int(sum(c.*cF),t,0,2*pi),s,0,1
```

Verify that Stokes' Theorem is true for the given vector field **F** and surface S.

$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward



boundary integration

$$= \int_{L_1} \vec{F}\cdot d\vec{r} + \int_{L_2} \vec{F}\cdot d\vec{r} + \int_{L_3} \vec{F}\cdot d\vec{r}$$

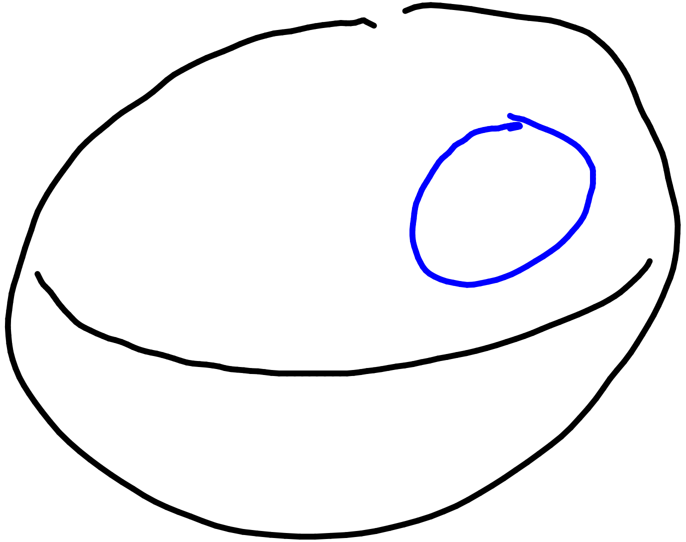
parametric eqn for surface

$x = x$ $\qquad 0 \le x \le 1$

$y = y$

$z = 2 - 2x - y$ $\qquad 0 \le y \le 2 - 2x$

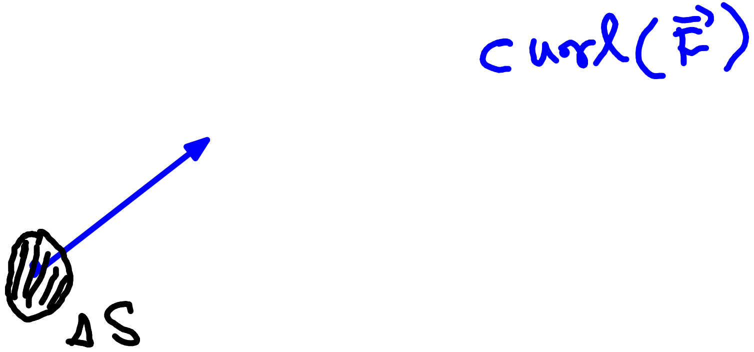
15. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$.



the boundary of a sphere is empty set.

$$\int_{\emptyset} \vec{F} \cdot d\vec{r} = 0$$

Q. Physical interpretation of curl$(\vec{F})$.



$$\boxed{\iint_{\Delta S} \operatorname{curl}(\vec{F}) \cdot d\vec{s} = \oint_{\partial S} \vec{F} \cdot d\vec{r}}$$

$\Delta S \perp \operatorname{curl}(\vec{F})$

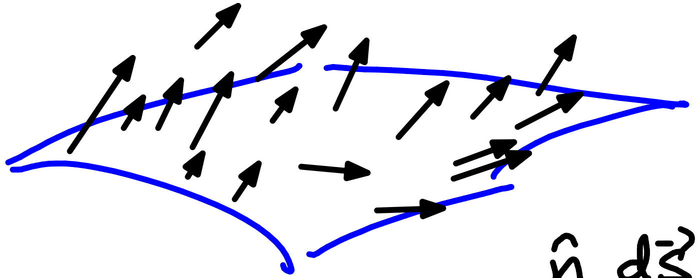
& ΔS is so small s.t. curl$(\vec{F})$ is constant.

$$\operatorname{curl}(\vec{F}) \, |\Delta S| = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\boxed{\operatorname{curl}(\vec{F}) = \lim_{|\Delta S| \to 0} \frac{\oint_{\partial S} \vec{F} \cdot d\vec{r}}{|\Delta S|}}$$

kind of rotational energy density ⟸

Recall Surface Integration of Vector Fields



$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S (\vec{F}\cdot\hat{n})\, ds =$$

$$\hat{n}\cdot d\vec{s} = \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}_{\hat{n}} \underbrace{|\vec{r}_u \times \vec{r}_v|\, du\, dv}_{ds}$$

Divergence theorem:

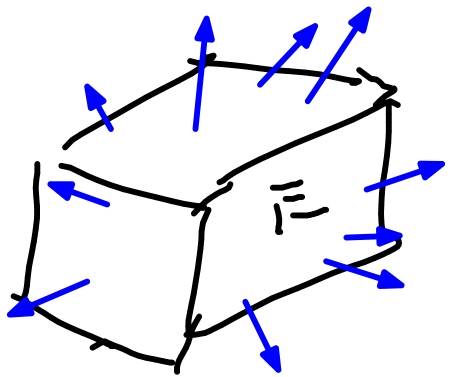
if Surface is closed, then flux can be calculated by a volume integration.

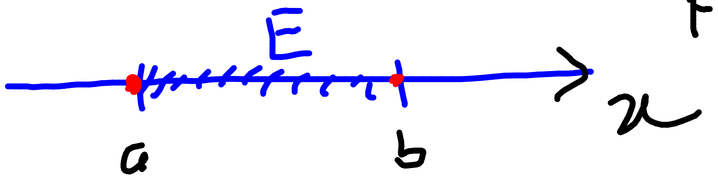
THE DIVERGENCE THEOREM Let E be a "simple" solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$





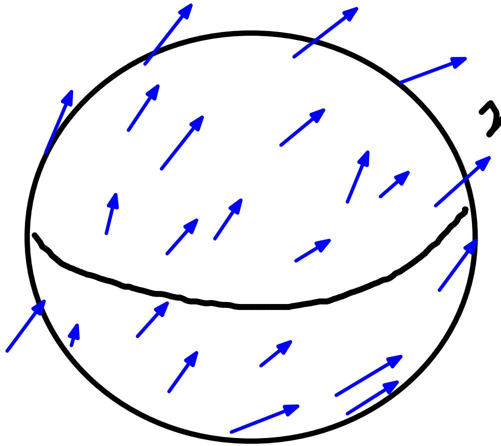
$$F(x) = f(x)$$

$$\operatorname{div}(\vec{F}) = f'(x)$$

$$f(b) - f(a) = \int_E f'(x)\,dx$$

V EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\text{flux} = \iint_S \vec{F} \cdot d\vec{s} = \iiint_V 1\, dV = \frac{4}{3}\pi$$

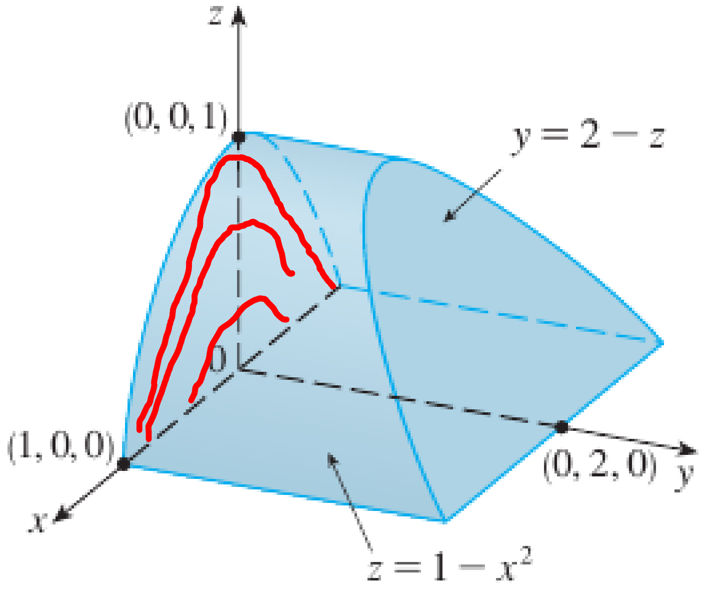


$x^2 + y^2 + z^2 = 1$

EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)



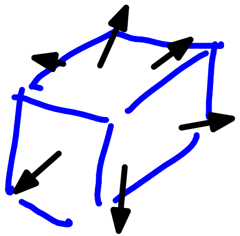
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{F})\, dV = \iiint_V 3y\, dV$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y \; dy\, dz\, dx$$

```
syms x y z

i1 = int(3*y,y,0,2-z)

i2 = int(i1,z,0,1-x^2)

i3 = int(i2,x,-1,1)
```

$$= \frac{184}{35}$$

Recall divergence theorem to understand $\operatorname{div}(\vec{F})$



outward flux from S $= \iiint_V \operatorname{div}(\vec{F})$

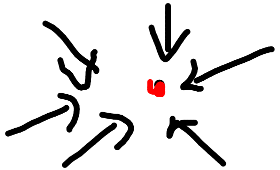
ssume volume is shrinking.

$$\iint_S \vec{F} \cdot d\vec{S} = \operatorname{div}(\vec{F}) \, |\text{volume of } V|$$

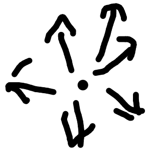
$$\operatorname{div}(\vec{F}) = \lim_{|V| \to 0} \frac{1}{|V|} \iint_S \vec{F} \cdot ds$$

divergence of a vector field gives us outward flux density of $\vec{F}$

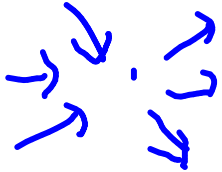
$\vec{F}$ is a vector field



$\operatorname{div}(\vec{F}) < 0$ sink



$\operatorname{div}(\vec{F}) > 0$ source



$\operatorname{div}(\vec{F}) = 0$ incompressible

13 REVIEW

TRUE-FALSE QUIZ

[H.W. Find example of $P\hat{i} + Q\hat{j}$ s.t. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ but $\oint_C P\,dx + Q\,dy \neq 0$]

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

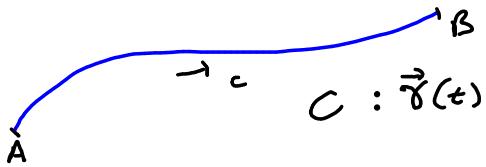
1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field. F
2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field. T
3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$. T
4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$. T
5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative. Not always
6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$ F [But $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$]
7. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$. T
8. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Chaitanya's theorem

$\operatorname{div}(\operatorname{curl} F) = 0$

prove this thm



$C : \vec{r}(t) \qquad a \le t \le b$

$$\int_{-C} \vec{F}\cdot d\vec{r} = \int_b^a \vec{F}\cdot \vec{r}\,'(t)\,dt = -\int_a^b \vec{F}\cdot r'(t)\,dt$$

$$= -\int_C \vec{F}\cdot dr$$