2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$

undetermined coefficients

- → easier
- → range of $r(x)$ is small

variation of parameters

- → slightly tedious
- → more general

2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x) \qquad (*)$$

step ① solve a corresponding homogenous eqⁿ

$$y'' + py' + qy = 0$$

→ let solution y_h

step ② find a particular solution y_p

$$y'' + py' + qy = r$$

either by
→ undetermined coefficients
→ variation of parameter

↳ final step : $y = y_h + y_p$ [general solution of Eq ⊛]

Recall the ODE :

→ $y'' + py' + qy = r$

→ $y_h'' + p y_h' + q y_h = 0$

→ $y_p'' + p y_p' + q y_p = r$

$$(y_h + y_p)'' + p\,(y_h + y_p)' + q\,(y_h + y_p) = r$$

End Sem :

May 7

→ mode : similar to mid term

→ syllabus : everything covered this semester

So far we have been solving homogeneous 2nd order ODE

$$y'' + ay' + by = 0$$

now we will solve non-homogeneous 2nd order ODE

$$y'' + ay' + by = r(x) \quad \text{———} \quad (*)$$

Step ① find a general solution of the corresponding homogeneous solⁿ:

$$y_h : \qquad y_h'' + a y_h' + b y_h = 0$$

Step ② find a particular solⁿ: y_p which solves Eq $(*)$

↳ we will see how

Q. Does $y = y_h + y_p$ solve Eq (*)

Yes

$y = y_h + y_p$ is called general solution of Eq (*)

how to find y_p

method of undetermined coefficients

Today

Variations of parameters

Sec 2.10

Method of Undetermined Coefficients

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Q. Solve: $y'' + y = 1$

→ Solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = c_1 \cos x + c_2 \sin x$$

→ find y_p using undetermined coefficients:

$$r(x) = 1$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
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$ke^{\alpha x} \sin \omega x$	

∵ our $r(x)$ is a constant

The table is suggesting that y_p will also be a constant

$$y_p = C \qquad [\, C \text{ needs to be found} \,]$$

↳ we substitute $y_p = c$ in the given ODE & solve for C

$$(c)'' + (c) = 1$$

$$c = 1$$

general sol^n

$$y = c_1 \cos x + c_2 \sin x + 1$$

Q. Solve: $y'' + y = 5$

→ Solve the corresponding homogeneous part.

$y'' + y = 0$

$y_h = C_1 \cos x + C_2 \sin x$

→ find y_p:

$y_p = C$

$y_p = 5$ will work

→

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Q. Solve: $y'' + y = e^{-2x}$

→ Solve the corresponding homogenous part.

$$y'' + y = 0$$

$$y_h = c_1 \cos x + c_2 \sin x$$

→ find y_p using undetermined coefficients

$r(x) = e^{-2x}$

Try $y_p = C e^{-2x}$, need to find C

→ plug in $y_p = Ce^{-2x}$ in $y'' + y = e^{-2x}$

$$4Ce^{-2x} + Ce^{-2x} = e^{-2x}$$

$$5C\cancel{e^{-2x}} = \cancel{e^{-2x}}$$

$$5C = 1$$

$$C = 1/5$$

$\Rightarrow y_p = \frac{1}{5} e^{-2x}$

general solution

$$y = c_1 \cos x + c_2 \sin x + \frac{e^{-2x}}{5}$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$\boxed{ke^{\gamma x}}$	$\boxed{Ce^{\gamma x}}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

EXAMPLE 1

Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5. \tag{5}$$

$y_h = c_1 \cos x + c_2 \sin x$

what should be y_p ?

$y_p = k_0 + k_1 x + k_2 x^2$

we need to find k_0, k_1, k_2 by substituting y_p in $y'' + y = 0.001x^2$

$k_2 x^2 + k_1 x + (k_0 + 2k_2) = 0.001 x^2$

→ matching the x, x^2, & constants in LHS & RHS

$k_2 = 0.001$

$k_1 = 0$

$k_0 = -0.002$

general solⁿ:

$y = c_1 \cos x + c_2 \sin x - 0.002 + 0.001 x^2$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$\boxed{kx^n \ (n = 0, 1, \cdots)}$	$\boxed{K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0}$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

r	y_p
1	k_0
x	$k_0 + k_1 x$
x^2	$k_0 + k_1 x + k_2 x^2$
x^3	$k_0 + k_1 x + k_2 x^2 + k_3 x^3$

remaining work:

find c_1 & c_2 using $y(0) = 0$

$y'(0) = 1.5$

$\downarrow$ $\boxed{\text{work}}$

$\rightarrow$ $c_1 = 0.002$, $c_2 = 1.5$

EXAMPLE 2

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{6}$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

$\rightarrow y_h = ??$

$\lambda^2 + 3\lambda + 2.25 = 0$

$\lambda = \dfrac{-3 \pm \sqrt{9 - 9}}{2} = -3/2$

$y_h = c_1 e^{-3/2 x} + c_2 x e^{-3/2 x}$

$\rightarrow y_p = C e^{-1.5x}$ should have worked but it's not.

trick: try $y_p = C x e^{-1.5x}$ X

try $y_p = C x^2 e^{-1.5x}$

$$2.25\left(y_p = c\cancel{x^2 e^{-1.5x}} \right)$$

$$+\quad 3\left(y_p' = c\,2\cancel{x e^{-1.5x}} - \frac{3c}{\cancel{2}}\cancel{x^2 e^{-1.5x}} \right)$$

$$+\quad \left(y_p'' = c\,2 e^{-1.5x} + 2(2x)\cancel{\left(-\frac{3}{2}\right)e^{-1.5x}} + c x^2 \cancel{\frac{9}{4} e^{-1.5x}} \right)$$

$$-10 e^{-1.5x} = 2c\, e^{-1.5x}$$

$$c = -5$$

$\Rightarrow$ general sol^n : $y = c_1 e^{-1.5x} + c_2 x e^{-1.5x} - 5x^2 e^{-1.5x}$

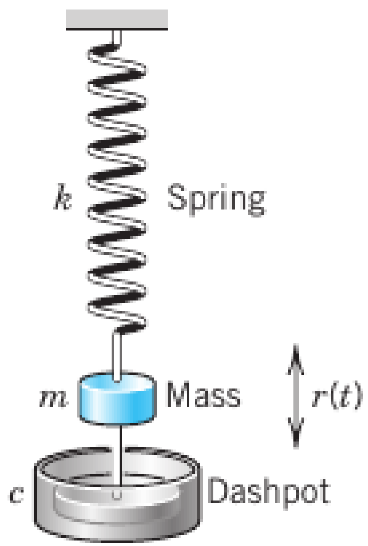
then find c_1 & c_2 using $y(0) = 1$, $y'(0) = 0$

EXAMPLE 3

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) = -0.43. \tag{7}$$

2.8 Modeling: Forced Oscillations. Resonance



$$my'' + cy' + ky = F_0 \cos \omega t.$$

https://www.youtube.com/watch?v=XwIZBJIp1AA