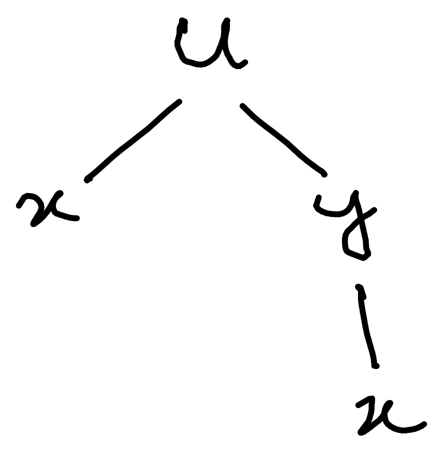
1.4 Exact ODEs. Integrating Factors

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad \text{—— } (1)$$

lets hope that there exist an eqution

$$u(x,y) = c \qquad \text{—— } (2)$$

from which we get eq (1) by differentiation



$$\frac{du}{dx} = \underbrace{\frac{\partial u}{\partial x}}_{M} + \underbrace{\frac{\partial u}{\partial y}}_{N}\frac{dy}{dx}$$

find a formula $u(x,y)$ s.t. $\frac{du}{dx} = M + N\frac{dy}{dx}$

this helps because we will solve for y from

$$u(x,y) = C$$

Objective: Solve for y from eqn of the form

$$M + N\frac{dy}{dx} = 0$$

Plan: find a formula $u(x,y)$ s.t.

$$\frac{du}{dx} = M + N\frac{dy}{dx}$$

When can we do this??

$\rightarrow$ ODE: $\frac{du}{dx} = 0$

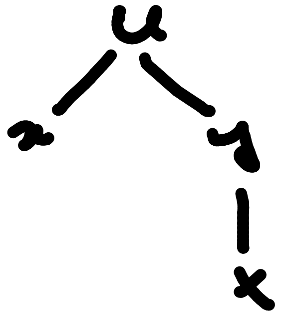
$\rightarrow \qquad u(x,y) = C$

$\rightarrow$ solve for y from $u(x,y) = C$

$$M + N\frac{dy}{dx} = 0$$

1 Aim: find $u(x,y)$ s.t.

$$\frac{du}{dx} = M + N\frac{dy}{dx}$$



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

if $M + N\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ for some $u(x,y)$

or: if there exist $u(x,y)$ s.t.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} = M\right) \;\&\; \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} = N\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} \stackrel{?}{=} \frac{\partial^2 u}{\partial y \partial x}$$

$\rightarrow$ Check for existence of such $u(x,y)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M + N\frac{dy}{dx} = 0$$

$$\Leftrightarrow \quad M\,dx + N\,dy = 0$$

d. $\underbrace{\cos(x+y)}_{M}\,dx + \underbrace{(3y^2+2y+\cos(x+y))}_{N}\,dy = 0$

→ is it exact ??

→ $M\,dx + N\,dy = 0$ is exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x}$$ exact

→ there exist $u(x,y)$ s.t. $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

→ now find this $u(x,y)$

$$\frac{\partial u}{\partial x} = \cos(x+y)$$

$$u = \underbrace{\sin(x+y) + g(y)}$$

[where $g(y)$ is to be found]

$$\rightarrow \quad \frac{\partial u}{\partial y} = N$$

$$\cancel{\cos(x+y)} + \frac{dg}{dy} = 3y^2 + 2y + \cancel{\cos(x+y)}$$

$$\frac{dg}{dy} = 3y^2 + 2y$$

$$g(y) = y^3 + y^2 + c$$

→ finally: $u(x,y) = \sin(x+y) + y^3 + y^2 + c$

→ solve for y from:

$$u(x,y) = \text{constant}$$

→ $\boxed{\sin(x+y) + y^3 + y^2 = C}$

↳ y is given implicitly by this eqn.

a//

$$(\cos y \sinh x + 1)\, dx - \sin y \cosh x\, dy = 0, \qquad y(1) = 2.$$

$$\underbrace{(\cos y \sinh x + 1)}_{M} dx \underbrace{- \sin y \cosh x}_{N} dy = 0$$

check for exactness.

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{\partial M}{\partial y} = -\sin y \sinh x = \frac{\partial N}{\partial x} \quad \text{exact}$$

$\rightarrow$ find u s.t. $\frac{\partial u}{\partial x} = M, \ \frac{\partial u}{\partial y} = N$

$$\frac{\partial u}{\partial x} = \cos y \sinh x + 1$$

$$u = \cos y \cosh x + x + g(y)$$

$$\frac{\partial u}{\partial y} = -\cancel{\sin y \cosh x} + \frac{dg}{dy} = \underbrace{-\cancel{\sin y \cosh x}}_{M}$$

$$\frac{dg}{dy} = 0 \Rightarrow g = c$$

$$\Rightarrow u(x,y) = \cos y \cosh x + x + c$$

—) y satisfies $u = c$

$$\boxed{\cos y \cosh x + x = C}$$

$$\boxed{\cos(2) \cosh(1) + 1 = C}$$

find C by $y(1) = 2$

Reduction to Exact Form. Integrating Factors

Next time

$$-y\,dx + x\,dy = 0.$$

$$(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0$$