CHAPTER 6 Laplace Transforms

$\mathcal{L}(y)$ $\qquad$ $y = \mathcal{L}^{-1}(\mathcal{L}(y))$

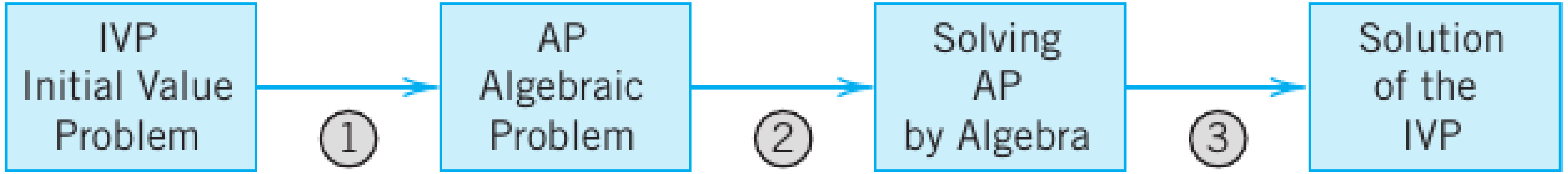


Fig. 113. Solving an IVP by Laplace transforms

solving for y

Input : $f(t)$

Output : $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$F(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\, 1\, dt = \int_0^\infty e^{-st}\,dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty = \lim_{t\to\infty} -\frac{1}{s}\left[\underbrace{e^{-st}}_{\to 0 \text{ only if } s>0} - 1 \right]$$

$$= \begin{cases} \frac{1}{s} & \text{if } s>0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathscr{L}(f)$.

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{undefined} & \text{if } s \leq a \end{cases}$$

e.g. $\mathcal{L}(e^{3t}) = ?? = \frac{1}{s-3} \quad (s > 3)$

$\mathcal{L}(e^{-t}) = \frac{1}{s+1} \quad (s > -1)$

$$\mathcal{L}(t) = ?? = \int_0^\infty e^{-st}\, t\, dt = \frac{1}{s^2} \quad (s>0)$$

$$\mathcal{L}(t^2) = \int_0^\infty e^{-st} t^2\, dt = \underbrace{\left| t^2 \frac{e^{-st}}{-s} \right|_0^\infty}_{=0 \ \text{why??}} + \underbrace{\frac{1}{s}\int_0^\infty 2t\, e^{-st}}_{= \frac{2}{s}\mathcal{L}(t) = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}} =$$

$$= \underbrace{\left. t^2 \frac{e^{-st}}{-s} \right|_0^\infty}_{=0} + \underbrace{\frac{1}{s}\int_0^\infty 2t\, e^{-st}}_{\frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3}} =$$

why ??

$t = 0 \qquad \frac{t^2 e^{-st}}{-s} = 0$

$$\lim_{t\to\infty} t^2 e^{-st} = \lim_{t\to\infty} \frac{t^2}{e^{st}} \quad \frac{\infty}{\infty} = \lim_{t\to\infty} \frac{2t}{s e^{st}} = \lim_{t\to\infty} \frac{2}{s^2 e^{st}}$$

$$= \frac{2}{\infty} = 0$$

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s}\mathcal{L}(t^n) = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

Proof ?!

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st}\, t^{n+1}\,dt$$

$$= \underbrace{\left| t^{n+1}\frac{e^{-st}}{-s}\right|_0^\infty}_{=0} + \frac{(n+1)}{s}\int_0^\infty e^{-st}\,t^n\,dt = \frac{n+1}{s}\mathcal{L}(t^n)$$

why ??
L'hopital's rule

$$= \frac{(n+1)}{s}\,\frac{n}{s}\,\mathcal{L}(t^{n-1})$$

$$= \frac{n+1}{s}\,\frac{n}{s}\,\frac{n-1}{s}\,\mathcal{L}(t^{n-2})$$

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s}\mathcal{L}(t^n) = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

It is possible to find

$\mathcal{L}(t^a)$, where $a > 0$ & real

$\mathcal{L}(t^{1.5})$ e.t.c

But these require "Gamma functions"

Gamma Functions is not in syllabus

$$\mathscr{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathscr{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

$$\mathscr{L}(\cos \Omega t) = \frac{s}{s^2 + \Omega^2} \qquad \mathscr{L}(\sin \Omega t) = \frac{\Omega}{s^2 + \Omega^2}$$

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	$\cancel{t^a \ (a \text{ positive})}$	$\cancel{\dfrac{\Gamma(a+1)}{s^{a+1}}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

$\cosh at = \dfrac{e^{at}+e^{-at}}{2}$

Euler

$$\mathcal{L}(f(t)) = F(s)$$

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

25–32 INVERSE LAPLACE TRANSFORMS

find $f(t)$ ع.ف $\mathcal{L}(f(t)) = \dfrac{5s+1}{s^2-25}$

26. $\dfrac{5s+1}{s^2-25}$

$$\mathcal{L}^{-1}\left(\frac{5s+1}{s^2-25}\right) = \mathcal{L}^{-1}\left(\frac{A}{s-5}\right) + \mathcal{L}^{-1}\left(\frac{B}{s+5}\right),$$

$$= A e^{5t} + B e^{-5t}$$

$A = 13/5$

$B = 12/5$

25–32 INVERSE LAPLACE TRANSFORMS

27. $\dfrac{s}{L^2 s^2 + n^2\pi^2}$

L, n are some const.

Recall

$$= \frac{1}{L^2}\left[\frac{s}{\frac{n^2\pi^2}{L^2} + s^2}\right]$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{\omega^2 + s^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{L^2\left\{\frac{n^2\pi^2}{L^2} + s^2\right\}}\right] = \frac{1}{L^2}\mathcal{L}^{-1}\left(\frac{s}{\frac{n^2\pi^2}{L^2} + s}\right) = \frac{1}{L^2}\cos\left(\frac{n\pi}{L}t\right)$$

25–32 **INVERSE LAPLACE TRANSFORMS**

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

29. $\dfrac{12}{s^4} - \dfrac{228}{s^6}$

$$\mathcal{L}^{-1}\left(\frac{12}{s^4} - \frac{228}{s^6}\right)$$

$$= \frac{12}{3!} \cdot \mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) - \frac{228}{5!}\mathcal{L}^{-1}\left(\frac{5!}{s^6}\right)$$

$$= \frac{12}{3!} \cdot t^3 - \frac{228}{5!} t^5$$

First Shifting Theorem, s-Shifting

if $\mathcal{L}(f(t)) = F(s)$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

Proof: Pending

e.g. $\mathcal{L}(t^2) = ?? = \frac{2}{s^3}$

$\mathcal{L}(e^{5t}t^2) = \frac{2}{(s-5)^3}$

$\mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(e^{10t}\cdot 1) = \frac{1}{s-10}$

$$\mathcal{L}(t^2) = ?? = \frac{2}{s^3}$$

$$\mathcal{L}(e^{5t} t^2) = \frac{2}{(s-5)^3}$$

$$\mathcal{L}(e^{-10t} t^2) = \frac{2}{(s+10)^3}$$

$$\mathcal{L}(\sin \pi t) = \frac{\pi}{s^2 + \pi^2}$$

$$\mathcal{L}(e^{-2t} \sin \pi t) = \frac{\pi}{(s+2)^2 + \pi^2}$$

33–45 **APPLICATION OF s-SHIFTING** find the inverse transform.

$$\frac{6}{(s+1)^3}$$

$$\mathcal{L}^{-1}\left(\frac{6}{s^3}\right) = 3t^2$$

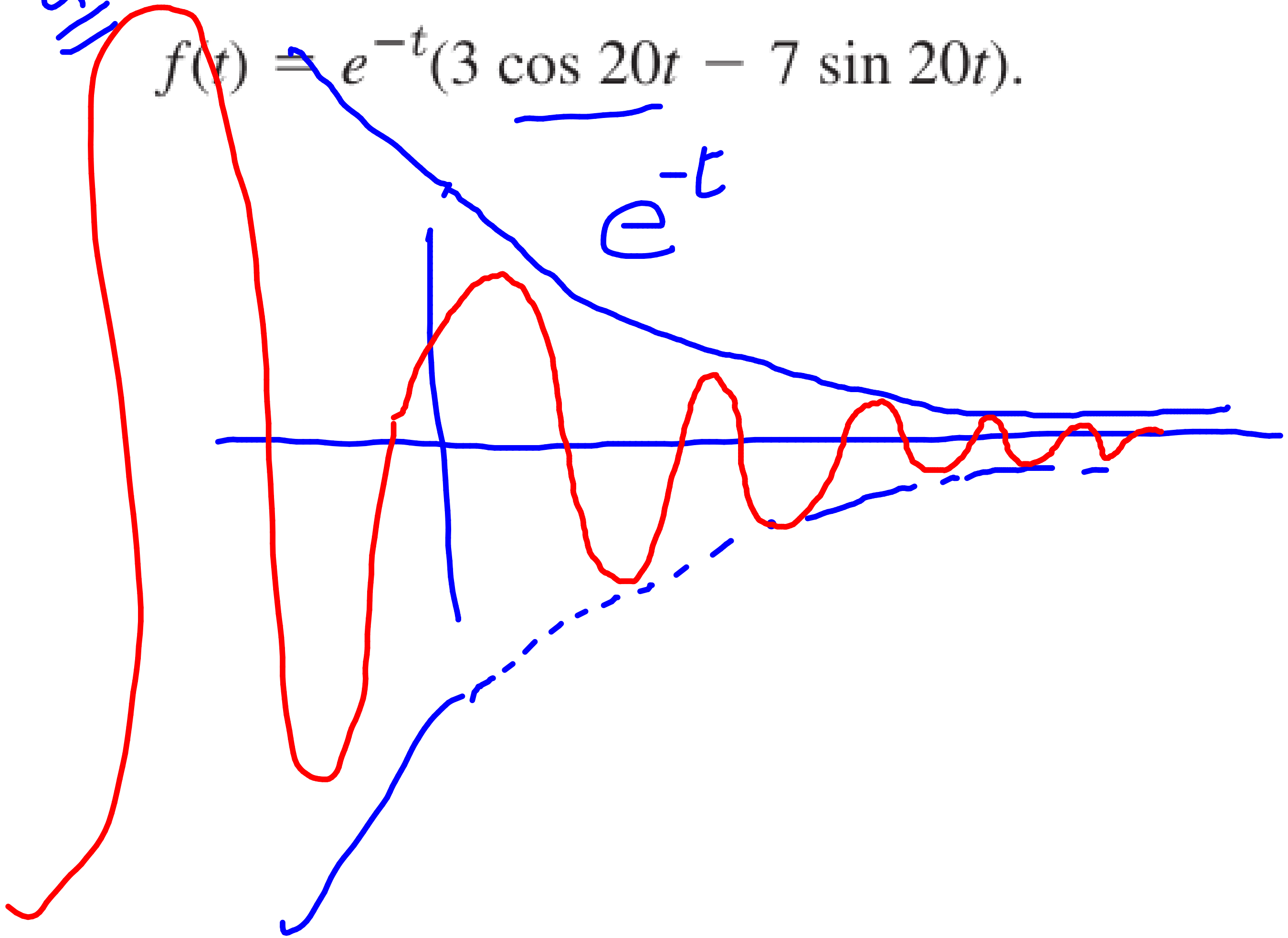
$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(3t^2) = \frac{6}{s^3}$$

$$\mathcal{L}(e^{-t}3t^2) = \frac{6}{(s+1)^3}$$

8.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$



33–45 APPLICATION OF s-SHIFTING

find the inverse transform.

$$\frac{4}{s^2 - 2s - 3}$$

$\rightarrow$ two approaches

partial fractions

complete the square

$$s^2 - 2s - 3 = (s-3)(s+1)$$

$$\mathcal{L}^{-1}\left(\frac{4}{s^2-2s-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= e^{3t} - e^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$$

33–45 APPLICATION OF s-SHIFTING find the inverse transform.

$$\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}$$

$$\mathcal{L}^{-1}\left(F(s)+G(s)\right)=\mathcal{L}^{-1}(F(s))+\mathcal{L}^{-1}(G(s))$$

$$\mathcal{L}^{-1}\left(\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}\right)=a\mathcal{L}^{-1}\left(\frac{s+k}{(s+k)^2+\pi^2}\right)+b\mathcal{L}^{-1}\left(\frac{\pi}{(s+k)^2+\pi^2}\right)$$

$$=a e^{-kt}\cos(\pi t)+b e^{-kt}\sin(\pi t)$$

Proof of Shifting theorem

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty \underline{e^{-st}\, e^{at}} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$\therefore$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

6.2 Transforms of Derivatives and Integrals. ODEs

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

→ lets apply this to solve few ODEs

[→ prove this]

↳ today

& 6-3

EXAMPLE 4 **Initial Value Problem: The Basic Laplace Steps**

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$\mathcal{L}(y'' - y) = \mathcal{L}(t)$

$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t)$

plan → → create an eq^n where the unknown is $\mathcal{L}(y)$

→ solve for $\mathcal{L}(y) =$

→ get $y = \mathcal{L}^{-1}(\mathcal{L}(y))$

$\underbrace{s^2\mathcal{L}(y) - sy(0) - y'(0)}_{\text{clear ??}} - \mathcal{L}(y) = \dfrac{1}{s^2}$

$$s^2\mathcal{L}(y) - s - 1 - \mathcal{L}(y) = \frac{1}{s^2}$$

$$(s^2-1)\,\mathcal{L}(y) = \frac{1}{s^2} + s + 1$$

$$\mathcal{L}(y) = \frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$$

Finally, get y

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}\right]$$

H.W.

can you do this??

Partial fraction will do the job

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

EXAMPLE 5 Solve the initial value problem

$$y'' + y' + 9y = 0. \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

$$\mathcal{L}(y'') + \mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}(0)$$

$$\underbrace{s^2\mathcal{L}(y) - s\,y(0) - y'(0)}_{\mathcal{L}(y'')} + \underbrace{s\mathcal{L}(y) - y(0)}_{\mathcal{L}(y')} + 9\mathcal{L}(y) = 0$$

using given conditions

$$(s^2 + s + 9)\mathcal{L}(y) = 0.16(s+1)$$

$$\mathcal{L}(y) = 0.16\,\frac{s+1}{s^2+s+9}$$

use $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+\omega^2}\right) = \cos(\omega t)$

$$\frac{s+1}{s^2+s+9} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \left(9-\frac{1}{4}\right)} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{35}}{2}\right)^2}$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{35}}{2}\right)^2} + \frac{1/2}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{35}}{2}\right)^2}$$

$$\frac{1/2}{\sqrt{35}/2} \, \frac{\sqrt{35}/2}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{35}}{2}\right)^2}$$

$\mathcal{L}^{-1}$

$$e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{35}}{2} t\right)$$

$\mathcal{L}^{-1}$

$$+ \frac{1}{\sqrt{35}} e^{-t/2} \sin\left(\frac{\sqrt{35}}{2} t\right)$$

$$\mathcal{L}\left(\sin(\omega t)\right) = \frac{\omega}{s^2+\omega^2}$$

$$\omega = \sqrt{35}/2$$

Proof of

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left| e^{-st} f(t) \right|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \left| e^{-st} f(t) \right|_0^\infty + s\mathcal{L}(f)$$

$$= s\mathcal{L}(f) + \underbrace{\lim_{t\to\infty} e^{-st} f(t)}_{=0\ ??} - f(0)$$

? all the time

? for all $f(t)$

[see discussion next slide]

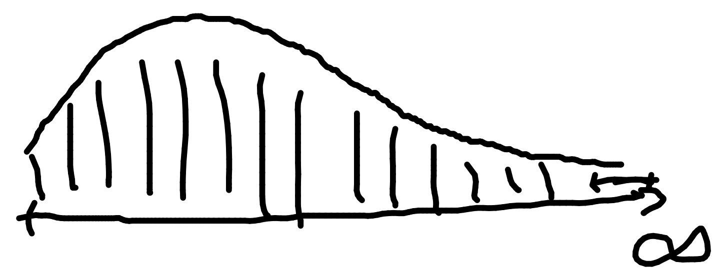
Lets study, should Laplace transform exist for all f.

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt < \infty$$

$$\boxed{\lim_{t\to\infty} e^{-st} f(t) = 0}$$

Q. Suppose $\int_0^\infty g(t)\,dt$ is finite.

$$\lim_{t\to\infty} g(t) = 0$$



$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0)$$

$$= s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}(f) - sf(0) - f'(0)$$

EXAMPLE 6 Shifted Data Problems

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Later

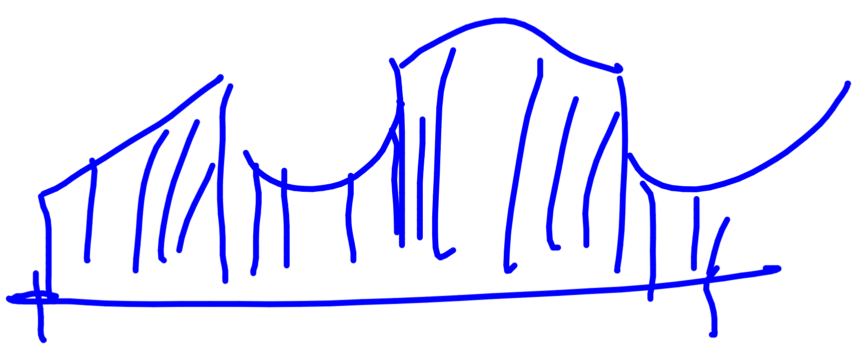
Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), *Sec.* 6.1. *Then, for $s > 0$, $s > k$, and $t > 0$,*

$$\textbf{(4)} \qquad \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Thus: if $\mathcal{L}^{-1}(F(s)) = f(t)$

$$\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \underbrace{\int_0^t f(\tau)\,d\tau}_{\text{exist because of piecewise continuity}}$$



INVERSE TRANSFORMS

$$\frac{3}{s^2 + s/4} = \frac{1}{s}\ \frac{3}{s+\frac{1}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{3}{s+\frac{1}{4}}\right) = 3e^{-t/4}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\ \frac{3}{s+\frac{1}{4}}\right) = 3\int_0^t e^{-\tau/4}\,d\tau$$

$$= 3\left[\frac{e^{-t/4}}{-1/4}\right]_0^t$$

$$= -12\left[e^{-t/4} - 1\right]$$

$$= 12\left[1 - e^{-t/4}\right]$$

$$\frac{1}{s^3 + as^2} = \frac{1}{s^2}\ \frac{1}{s+a}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\ \frac{1}{s+a}\right) = \int_0^t e^{-a\tau}\,d\tau$$

$$= \left|\frac{e^{-a\tau}}{-a}\right|_0^t$$

$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\,\frac{1}{s+a}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{s(s+a)}\right)$$

$$= \int_0^t \frac{1}{a}\left(1 - e^{-a\tau}\right) d\tau$$

$$= \frac{1}{a^2}\left(at - 1 + e^{-at}\right)$$

Q. if $\mathcal{L}^{-1}(F(s)) = f(t)$

why $\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau$??

Hint: it uses $\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$

$g(t) = \int_0^t f(\tau)\,d\tau$, $g'(t) = f(t)$

$$\mathcal{L}(f) = s\,\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) - 0$$

$$F(s) = s\,\mathcal{L}\left(\int_0^t f(\tau)\,dt\right)$$

$$\frac{1}{s} F(s) = \mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot F(s)\right) = \int_0^t f(\tau)\, d\tau$$