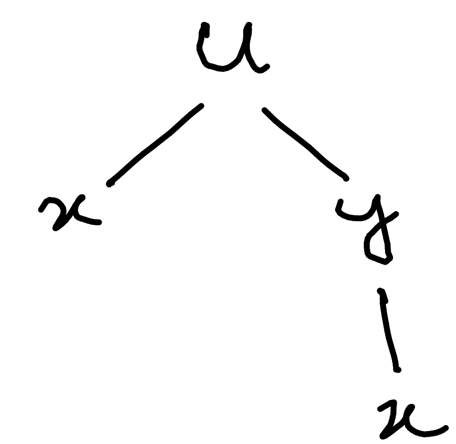
1.4 Exact ODEs. Integrating Factors

$$\underbrace{M(x,y) + N(x,y)\frac{dy}{dx}} = 0 \quad \text{———} \quad (1)$$

lets hope that there exist an eqution

$$u(x,y) = c \quad \text{——} \quad (2)$$

from which we get eq (1) by differentiation



$$\frac{du}{dx} = \underbrace{\frac{\partial u}{\partial x}}_{M} + \underbrace{\frac{\partial u}{\partial y}}_{N}\frac{dy}{dx}$$

Recall: ODE

Exact ODEs $\rightarrow$

$$M + N\frac{dy}{dx} = 0$$

$$\Downarrow$$

$$M\,dx + N\,dy = 0$$

find a formula $u(x,y)$ s.t. $\frac{du}{dx} = M + N\frac{dy}{dx}$

this helps because we will solve for y from

$$u(x,y) = C$$

Objective: Solve for y from eqn of the form

$$M + N\frac{dy}{dx} = 0$$

Plan: find a formula $u(x,y)$ s.t.

$$\frac{du}{dx} = M + N\frac{dy}{dx}$$

When can we do this??

$\rightarrow$ ODE: $$\frac{du}{dx} = 0$$

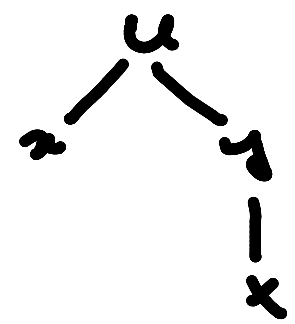
$\rightarrow \quad u(x,y) = C$

$\rightarrow$ solve for y from $u(x,y) = C$

$$M + N\frac{dy}{dx} = 0$$

1 **Aim:** find $u(x,y)$ s.t.

$$\frac{du}{dx} = M + N\frac{dy}{dx}$$



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

if $M + N\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ for some $u(x,y)$

or: if there exist $u(x,y)$ s.t.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} = M\right) \ \& \ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} = N\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} \stackrel{?}{=} \frac{\partial^2 u}{\partial y \partial x}$$

$\rightarrow$ Check for existence of such $u(x,y)$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$M + N\frac{dy}{dx} = 0$$

$$\Leftrightarrow \quad M\,dx + N\,dy = 0$$

Q: $\underbrace{\cos(x+y)}_{M}\,dx + \underbrace{\left(3y^2+2y+\cos(x+y)\right)}_{N}\,dy = 0$

→ is it exact ??

→ $M\,dx + N\,dy = 0$ is exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x}$$

exact

→ there exist $u(x,y)$ s.t. $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

→ now find this $u(x,y)$

$$\frac{\partial u}{\partial x} = \cos(x+y)$$

$$u = \underbrace{\sin(x+y) + g(y)}$$

[where $g(y)$ is to to found]

$$\rightarrow \left(\frac{\partial u}{\partial y} = M \right)$$

$$\cancel{\cos(x+y)} + \frac{dg}{dy} = 3y^2 + 2y + \cancel{\cos(x+y)}$$

$$\frac{dg}{dy} = 3y^2 + 2y$$

$$g(y) = y^3 + y^2 + c$$

→ finally: $u(x,y) = \sin(x+y) + y^3 + y^2 + c$

→ solve for y from:

$$u(x,y) = \text{constant}$$

→ $$\boxed{\sin(x+y) + y^3 + y^2 = C}$$

↳ y is given implicitly by this eqⁿ.

Q:

$$(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0, \qquad y(1) = 2.$$

$$\underbrace{(\cos y \sinh x + 1)}_{M}\,dx \underbrace{- \sin y \cosh x}_{N}\,dy = 0$$

Check for exactness.

$\cosh x = \dfrac{e^{x}+e^{-x}}{2}$ | $\dfrac{d}{dx}(\cosh x) = \sinh x$

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$ | $\dfrac{d}{dx}(\sinh x) = \cosh x$

$\dfrac{\partial M}{\partial y} = -\sin y \sinh x = \dfrac{\partial N}{\partial x}$ exact

$\rightarrow$ find u s.t. $\dfrac{\partial u}{\partial x} = M$, $\dfrac{\partial u}{\partial y} = N$

$$\frac{\partial u}{\partial x} = \cos y \sinh x + 1$$

$$u = \cos y \cosh x + x + g(y)$$

$$\frac{\partial u}{\partial y} = \cancel{-\sin y \cosh x} + \frac{dg}{dy} = \underbrace{\cancel{-\sin y \cosh x}}_{M}$$

$$\frac{dg}{dy} = 0 \Rightarrow g = c$$

$$\Rightarrow u(x,y) = \cos y \cosh x + x + c$$

$-$) y satisfies $u = c$

$$\boxed{\cos y \cosh x + x = C}$$

$$\boxed{\cos(2) \cosh(1) + 1 = C}$$

find C by $y(1) = 2$

Reduction to Exact Form. Integrating Factors

Today's agenda:

$M\,dx + N\,dy = 0$ | $M + N\frac{dy}{dx} = 0$

→ not exact $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

find a formula I s.t.

→ $IM\,dx + IN\,dy = 0$ is exact

$\Rightarrow \quad \frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN)$

$I(x,y)$ → too complicated

Try $I = I(x)$

Try $I = I(y)$

Q-

$$-y + x\frac{dy}{dx} = 0$$

→ test for exactness

→ Find an integration factor I st.

$$-Iy + Ix\frac{dy}{dx} = 0 \quad \text{is exact}$$

→ then solve

Exact? $\frac{\partial M}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial x} \Rightarrow$ not exact

Q:

$$-y + x\frac{dy}{dx} = 0$$

Suppose there exist an IF $I = I(x)$ s.t.

$$-I(x)y + I(x)x\frac{dy}{dx} = 0 \quad \text{is exact}$$

$$\Rightarrow \quad \frac{\partial}{\partial y}\left(-I(x)y\right) = \frac{\partial}{\partial x}\left(I(x)x\right)$$

$$-I = \frac{dI}{dx}x + I$$

$$-2I = \frac{dI}{dx}x$$

$$-\frac{2}{x}dx = \frac{1}{I}dI$$

$$-2\ln x = \ln I$$

$$I = \frac{1}{x^2}$$

[no constant required $\because$ we need only one IF] new ODE:

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0$$

solve this ODE: find $u(x,y)$ s.t.

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2} \quad \& \quad \frac{\partial u}{\partial y} = \frac{1}{x}$$

$\rightarrow y$ can be solved from

$$\frac{y}{x} = c$$

$$\Rightarrow \quad \boxed{y = cx}$$

Ans

$$u = \frac{y}{x}$$

Q. $\underbrace{(e^{x+y} + y e^{y})}_{M} dx + \underbrace{(x e^{y} - 1)}_{N} dy = 0$

→ check for exactness

→ make it exact if not

→ solve

$$\frac{\partial M}{\partial y} = e^{x+y} + y e^{y} + e^{y}$$

$$\neq$$

$$\frac{\partial N}{\partial x} = e^{y}$$

Q:
$$\underbrace{(e^{x+y} + y e^{y})}_{M} dx + \underbrace{(x e^{y} - 1)}_{N} dy = 0$$

$$I = I(x)$$

$$\frac{\partial}{\partial y}(I M) = \frac{\partial}{\partial x}(I N)$$

$$I(x)\left(e^{x+y} + y e^{y} + \cancel{e^{y}}\right) = I(x)\cancel{e^{y}} + (x e^{y} - 1)\frac{dI}{dx}$$

can't eliminate y

↓

abandon

↓

Try $I = I(y)$

$$I = I(y)$$

$$\frac{\partial}{\partial y}\big(I(y)\, M\big) = \frac{\partial}{\partial x}\big(I(y)\, N\big)$$

$$\frac{\partial}{\partial y}\left(I(y)\,M\right) = \frac{\partial}{\partial x}\left(I(y)\,N\right) \quad]$$

$$I\left(e^{x+y} + ye^{y} + \cancel{e^{y}}\right) + \left(e^{x+y} + ye^{y}\right)\frac{dI}{dy} = I\cancel{e^{y}}$$

$$\underbrace{\left(I + \frac{dI}{dy}\right)}_{=0}\left(e^{x+y} + ye^{y}\right) = 0$$

$$\frac{dI}{dy} = -I$$

$$-\frac{1}{I}\,dI = dy$$

$$-\ln I = y$$

$$I = e^{-y}$$

now ODE

$$e^{-y}\left[(e^{x+y} + y e^{y})\,dx + (x e^{y} - 1)\,dy = 0\right]$$

$\rightarrow \quad (e^{x} + y)\,dx + (x - e^{-y})\,dy = 0$

$\rightarrow$ find u s.t. $\dfrac{\partial u}{\partial x} = e^{x} + y$ & $\dfrac{\partial u}{\partial y} = x - e^{-y}$

$\rightarrow \quad \dfrac{\partial u}{\partial x} = e^{x} + y$

$$u = e^{x} + yx + \underbrace{g(y)}_{??}$$

$\rightarrow \quad \dfrac{\partial u}{\partial y} = x - e^{-y}$

$$\cancel{x} + \frac{dg}{dy} = \cancel{x} - e^{-y}$$

$$\frac{dg}{dy} = -e^{-y}$$

$$g(y) = e^{-y}$$

$u = e^x + xy + e^{-y}$

finally: solution of the ODE

$\to$ y implicitly defined by

$$\boxed{e^x + xy + e^{-y} = C}$$