2.5 Euler–Cauchy Equations

$$x^2y'' + axy' + by = 0$$

a, b are constants.

$$\underbrace{y = x^m}$$ trial solution

$$m(m-1)x^m + am\,x^m + b\,x^m = 0$$

$$(m^2 + (a-1)m + b)\,x^m = 0$$

Characteristic eq^n / Auxilliary eq^n

$$m^2 + (a-1)m + b = 0$$

Definition: of real arbitrary power

$$\boxed{x^a := e^{a\ln(x)}}$$

$$m^2 + (a-1)m + b = 0$$

Case 1 roots are real & distinct : say m_1, m_2

two independent real solutions: x^{m_1}, x^{m_2}

general solution of the ODE :

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

$$m^2 + (a-1)m + b = 0$$

Case (2) say the roots are in complex pairs

$$m = \alpha \pm i$$

$$y_1 = x^{(\alpha + i\beta)} \qquad y_2 = x^{(\alpha - i\beta)}$$

Job: find real & imaginary parts of $x^{(\alpha+i\beta)}$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$x^{(2+3i)} = x^2 \cdot x^{3i} = x^2 e^{3i\log x}$$

$$= x^2\left[\cos(3\log x) + i\sin(3\log x)\right]$$

$$\rightarrow \text{real}(x^{2+3i}) = x^2\cos(3\log x)$$

$$\rightarrow \text{im}(x^{2+3i}) = x^2\sin(3\log x)$$

$$x^{\alpha+i\beta} = x^\alpha\left[\cos(\beta\log(x)) + i\sin(\beta\log(x))\right]$$

$$y_1 = \text{Re}(x^{\alpha+i\beta})$$

$$y_2 = \text{Im}(x^{\alpha+i\beta})$$

H.W. if $\alpha+i\beta$ solves the char eq^n then $\text{Re}(x^{\alpha+i\beta})$ & $\text{Im}(x^{\alpha+i\beta})$ also solve the EC eq^n

$$x^2y'' - 20y = 0$$

char eqⁿ

$$m^2 + (a-1)m + b = 0$$

$$m^2 - m - 20 = 0$$

$$(m-5)(m+4) = 0$$

$$m = 5, -4$$

general solution

$$y = c_1 x^5 + \frac{c_2}{x^4}$$

$xy'' + 2y' = 0$

$x^2 y'' + 2x y' = 0$

char eq^n

$m^2 + (a-1)m + b = 0$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

solution: $y = c_1 x^0 + c_2 x^{-1}$

$= c_1 + \frac{c_2}{x}$

$$(x^2D^2 - 3xD + 10I)y = 0$$

$D = \frac{d}{dx}$

$x^2y'' - 3xy' + 10y = 0$

$Dy = \frac{dy}{dx}$

$Iy = y$

char. eqⁿ : $m^2 - 4m + 10 = 0$

$$m = \frac{4 \pm \sqrt{16 - 40}}{2} = \frac{4 \pm \sqrt{-24}}{2}$$

$$= 2 \pm i\sqrt{6}$$

$$y = x^2\left[c_1 \cos(\sqrt{6} \log x) + c_2 \sin(\sqrt{6} \log(x))\right]$$

Case ③ $x^2y'' + axy' + by = 0$

trial solⁿ : $y = x^m$

→ suppose $m^2 + (a-1)m + b = 0$

has repeated real root, say m

→ now we have one solution $y = x^m$

→ find $y_2 = u(x)\, x^m$, where $u(x)$ needs to be determined.

→ plug ux^m in the above ODE, get an eqⁿ for u

& then solve that eqⁿ to get u

$$x^2y'' + axy' + by = 0$$

Recall

$$[m^2 + (a-1)m + b]x^m = 0$$

$$b(y = ux^m)$$

$$ax(y' = mux^{m-1} + u'x^m)$$

$$x^2(y'' = m(m-1)ux^{m-2} + 2mu'x^{m-1} + u''x^m)$$

$$0 = 0 + u'ax^{m+1} + 2mu'x^{m+1} + u''x^{m+2}$$

$$0 = u'\underbrace{(2m+a)} + u''x$$

$m^2 + (a-1)\,m + b = 0$ & roots are repeated

then $m = -\dfrac{(a-1)}{2} \;\Big|\Rightarrow 2m + a = 1$

the governing eq^u for u is

$$0 = u' + u''x$$

$$0 = v + v'x$$

say $v = u'$

$$v'x = -v$$

$$\frac{1}{v} v' = \frac{-1}{x}$$

$$\log v = -\ln x$$

$$v = \frac{1}{x}$$

$$u' = \frac{1}{x}$$

$$\boxed{u = \log x}$$

Case 3 if $m^2 + (a-1)m + b = 0$ has repeated roots m,

then the general solution of the EC eqn

is $y = C_1 x^m + C_2 x^m \log x$

12. $x^2y'' - 4xy' + 6y = 0, \quad y(1) = 0.4, \quad y'(1) = 0$

$y = c_1 y_1 + c_2 y_2$