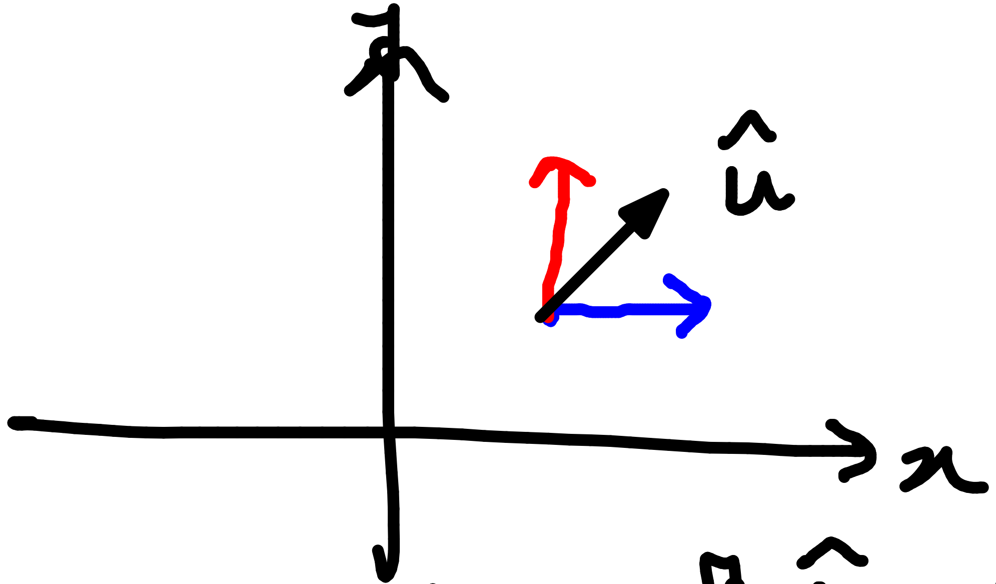
Quick Review: $f(x,y)$



- partial derivatives

$$\left[\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}\right] = \nabla f \leftarrow \text{gradient}$$

$\left[\hat{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$

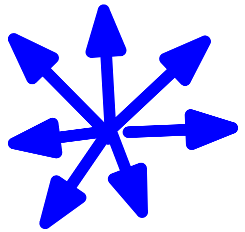
- Directional derivatives

rate of change of f in the direction of $\hat{u}$

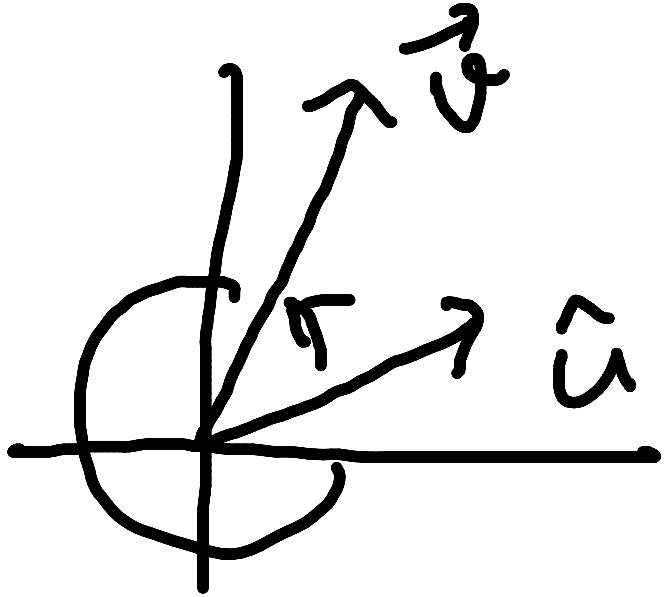
$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \nabla f \cdot \hat{u}$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$



Q: what direction $\hat{u}$ we should choose s.t. $D_{\hat{u}} f$ is maximum among all possible $\hat{u}$.



$\hat{u}$: free to revolve

$\vec{v} \cdot \hat{u} = |\vec{v}| \cos\theta$ is max when $\theta = 0$

in conclusion:

$$D_{\hat{u}}\hat{f} = \nabla f \cdot \hat{u} \quad \text{is maximum}$$

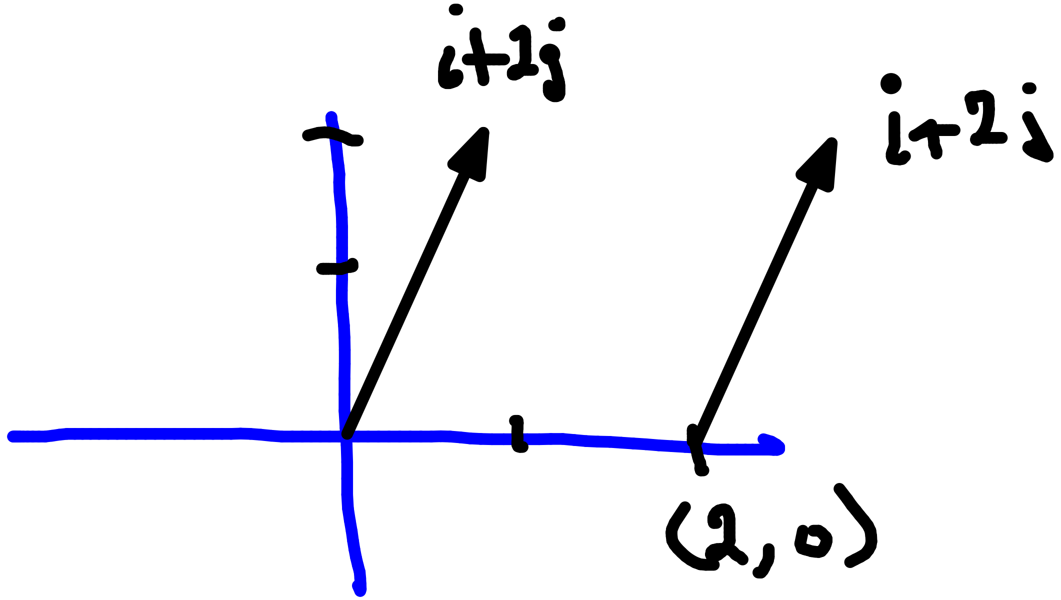
$$\text{if } \hat{u} = \frac{\nabla f}{|\nabla f|} = \text{parallel to } \nabla f$$

EXAMPLE 5

α (a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. x

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

$f(x,y) = xe^y$



$\nabla f(2,0)$

$=$ direction of max change $= \hat{i} + 2\hat{j}$

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Aim: find the direction of fastest increase in temperature at point $(1,1,-2) = P$

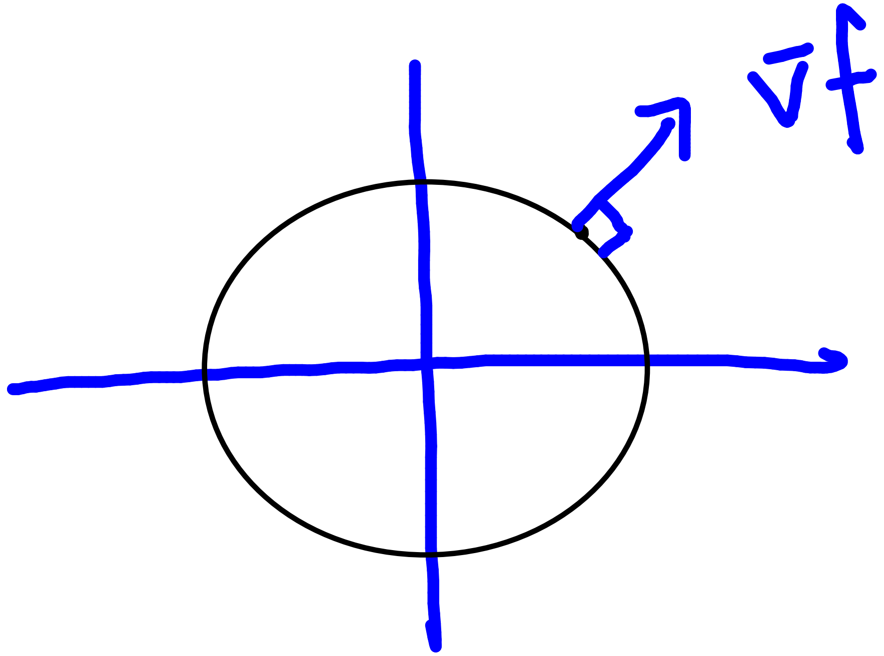
$\rightarrow$ $$\left.\begin{aligned}\nabla T(P) &= \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} \\ &= \frac{5}{8}(-\hat{i} - 2\hat{j} + 6\hat{k})\end{aligned}\right]$$ direction of fastest increase

$\rightarrow$ max rate of change in the direction

$$= |\nabla T| = 5\sqrt{41}/8$$

TANGENT PLANES TO LEVEL SURFACES

Q. note the curve $x^2+y^2=1$



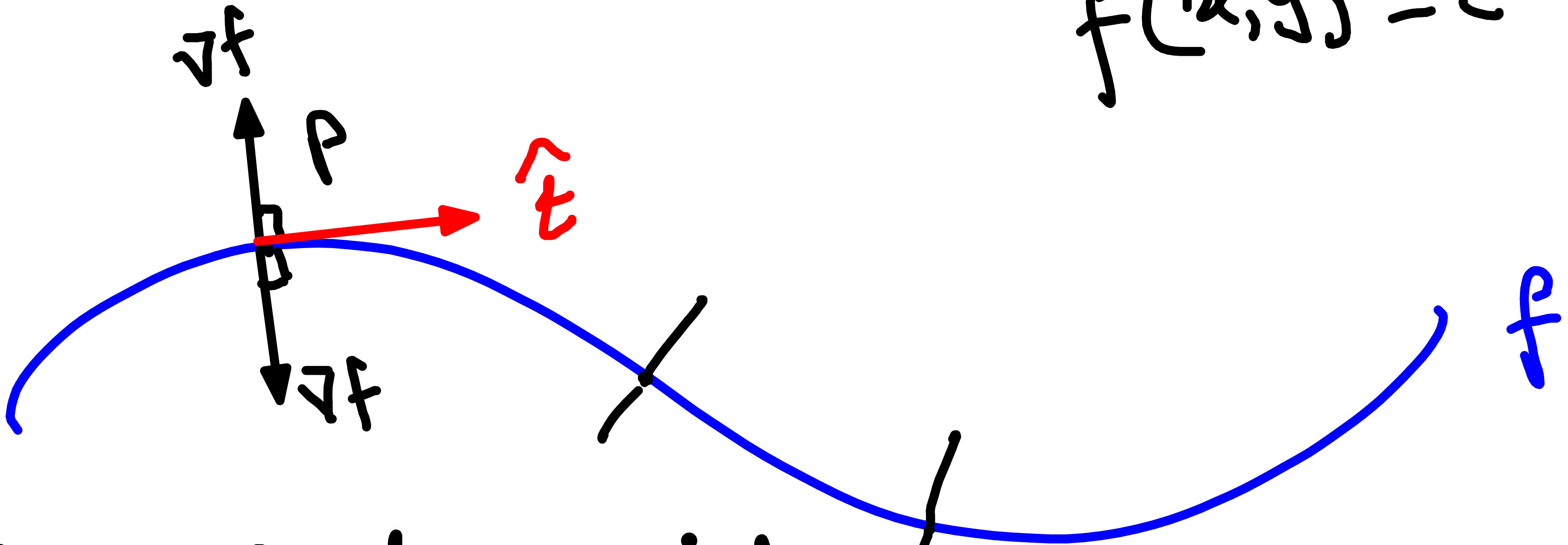
$$f(x,y) = x^2+y^2$$

P: any random point on the curve

Q: what will be the direction of $\nabla f(P)$??

$\perp$ to the level curve $f(x,y) = c$

Consider a level curve $f(x,y) = c$ $\quad \nabla f(P) =$

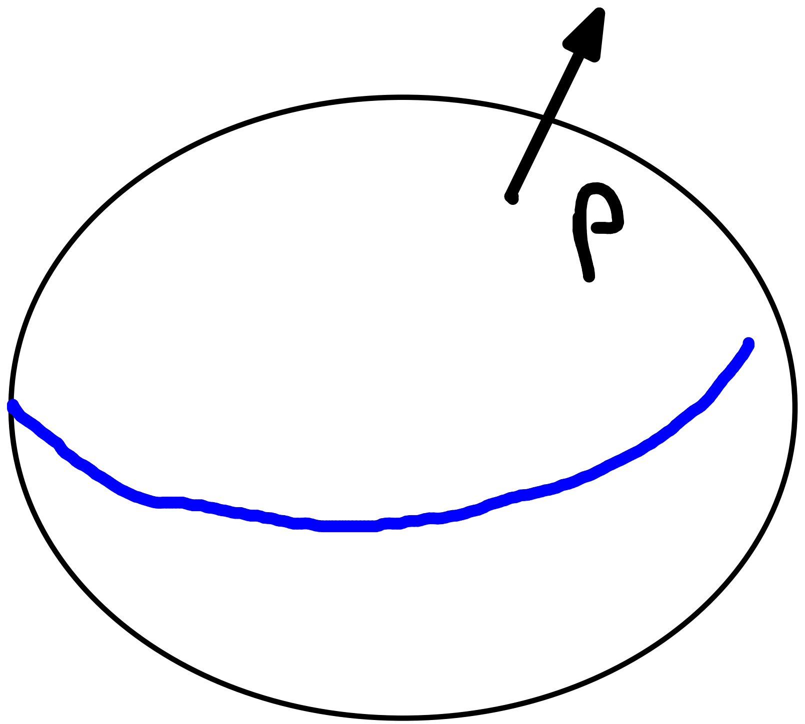


P: a random point on the curve $f(x,y) = c$

$$\underbrace{D_{\hat{t}} f(P)}_{=??} = \nabla f \cdot \hat{t} = 0 \quad \left[\because f \text{ is const along the curve} \right]$$

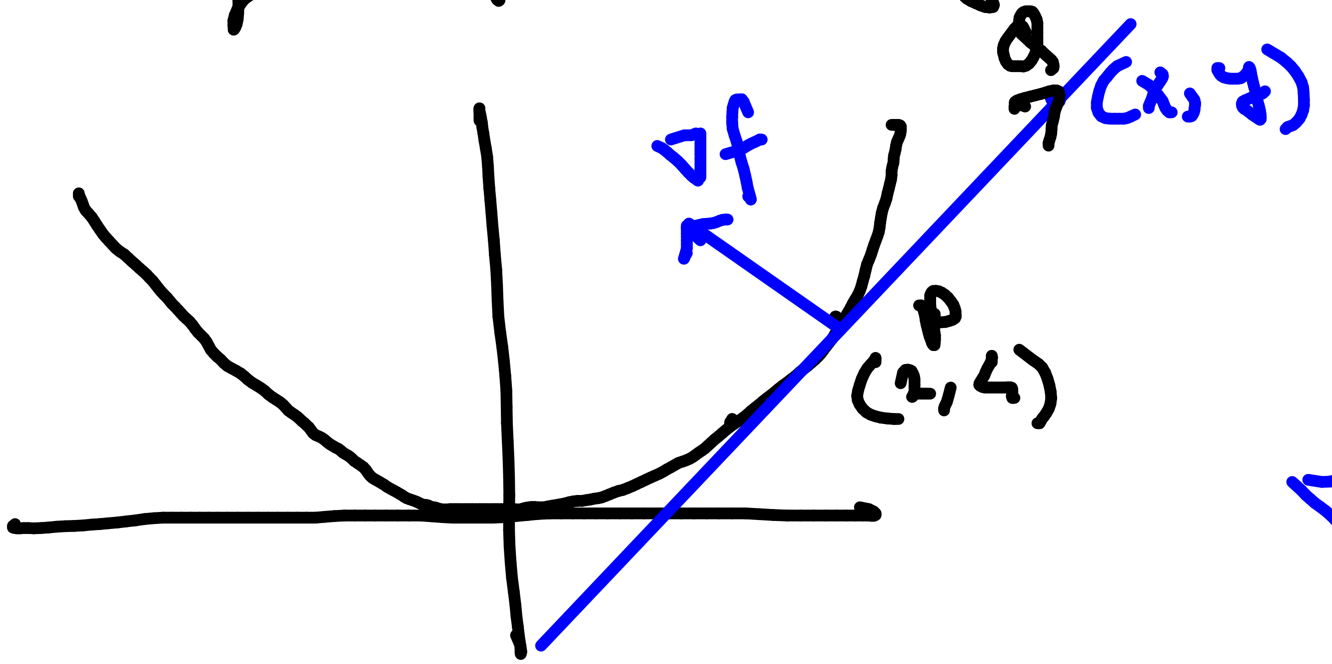
Similarly: a level surface $f(x,y,z)=c$

e.g. $x^2+y^2+z^2=1$



$\nabla f \perp$ surface

Q: find eqⁿ of tangent line



$y - x^2 = 0$

$f(x, y) = y - x^2$

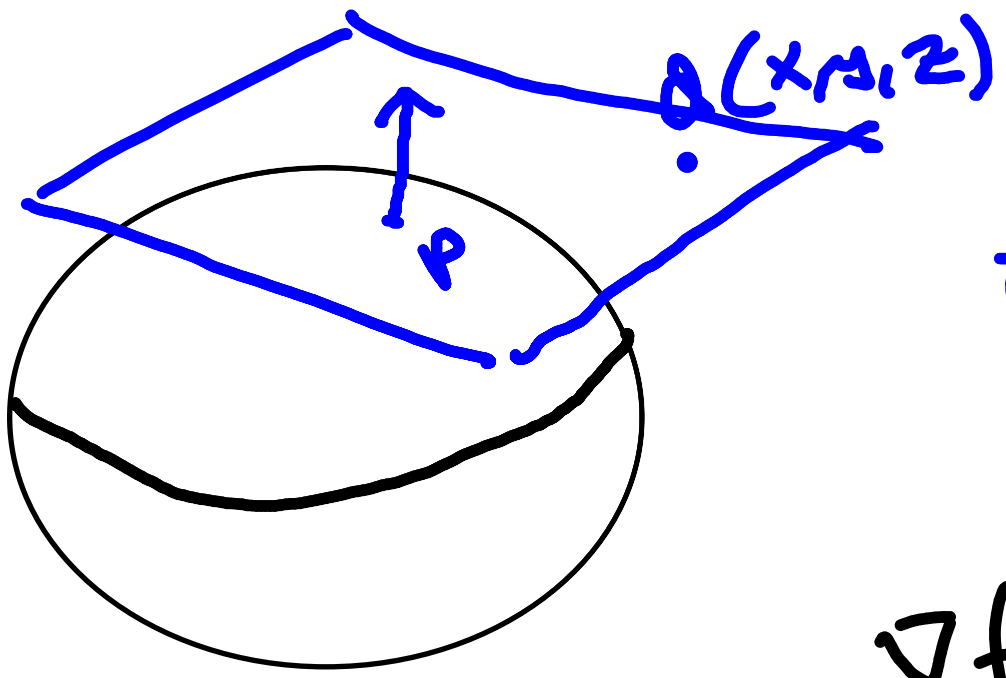
$\nabla f = -2x\hat{i} + \hat{j}$
$= -4\hat{i} + \hat{j}$

$\nabla f \perp \overrightarrow{PQ}$

$(-4\hat{i} + \hat{j}) \cdot ((x-2)\hat{i} + (y-4)\hat{j}) = 0$

$\boxed{-4(x-2) + (y-4) = 0}$ tangent line

Q: find eqⁿ of tangent plane for the surface $x^2+y^2+z^2=1$ at point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$



∇f : normal to the plane

$$\boxed{\nabla f \cdot \vec{PQ} = 0}$$ eqⁿ of plane

$$\nabla f(P) = (\sqrt{2}, \sqrt{2}, 0)$$

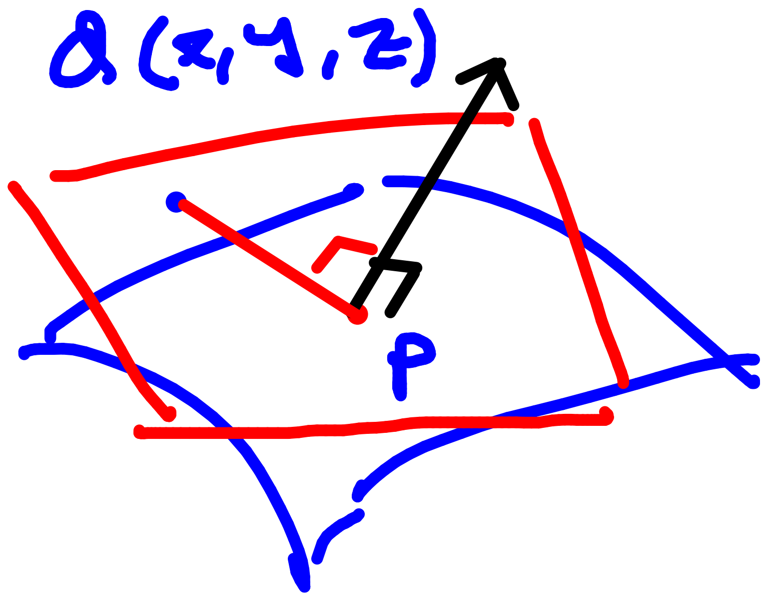
Tangent plane:

$$\sqrt{2}\left(x-\frac{1}{\sqrt{2}}\right)+\sqrt{2}\left(y-\frac{1}{\sqrt{2}}\right)+0(z-0)=0$$

Find equations of (a) the tangent plane and (b) the nor mal line to the given surface at the specified point.

work these out

$$x^2 - 2y^2 + z^2 + yz = 2, \quad (2, 1, -1)$$



Ans:

$$4x - 5y - z = 4$$

$$\nabla f \cdot \overrightarrow{PQ} = 0$$

$$yz = \ln(x + z), \quad (0, 0, 1)$$

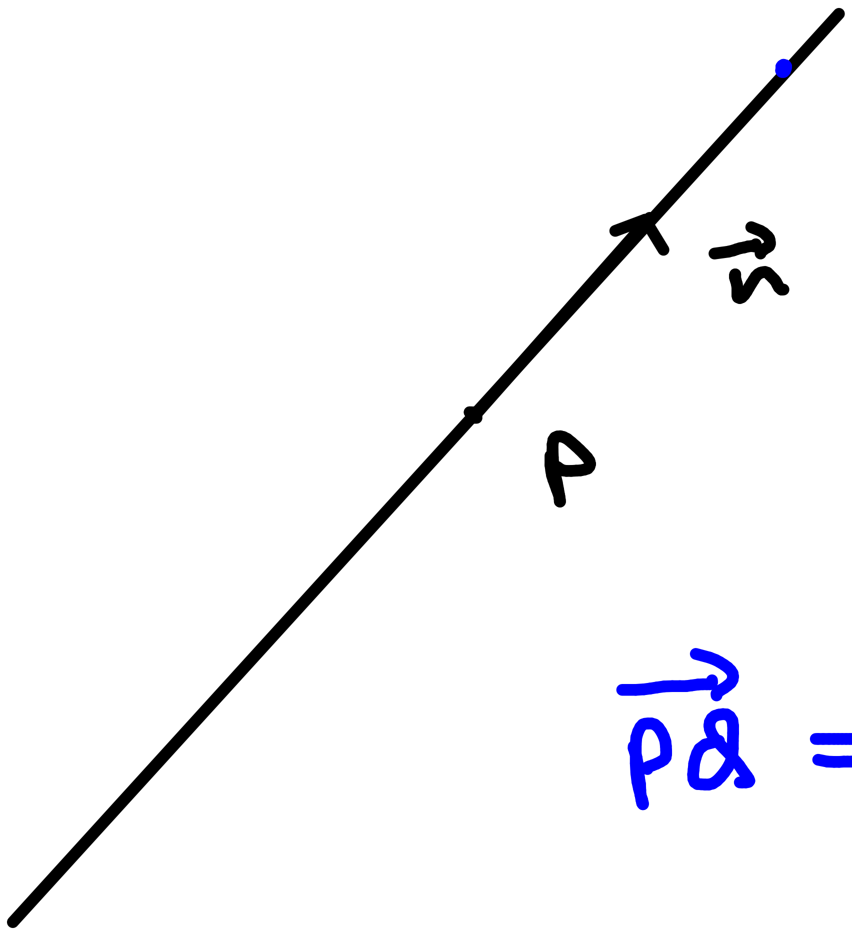
$$f(x,y,z) = yz - \ln(x+z) = 0$$

Ans:

Find equations of (a) the tangent plane and (b) the nor mal line to the given surface at the specified point.

$$yz = \ln(x + z), \quad (0, 0, 1)$$

Q.



whats the eq^n of the line passing through P & parallel to $\vec{n}$.

$$\vec{PQ} = \alpha \vec{n}$$

$$\vec{PQ} = \alpha \nabla f(\vec{P})$$