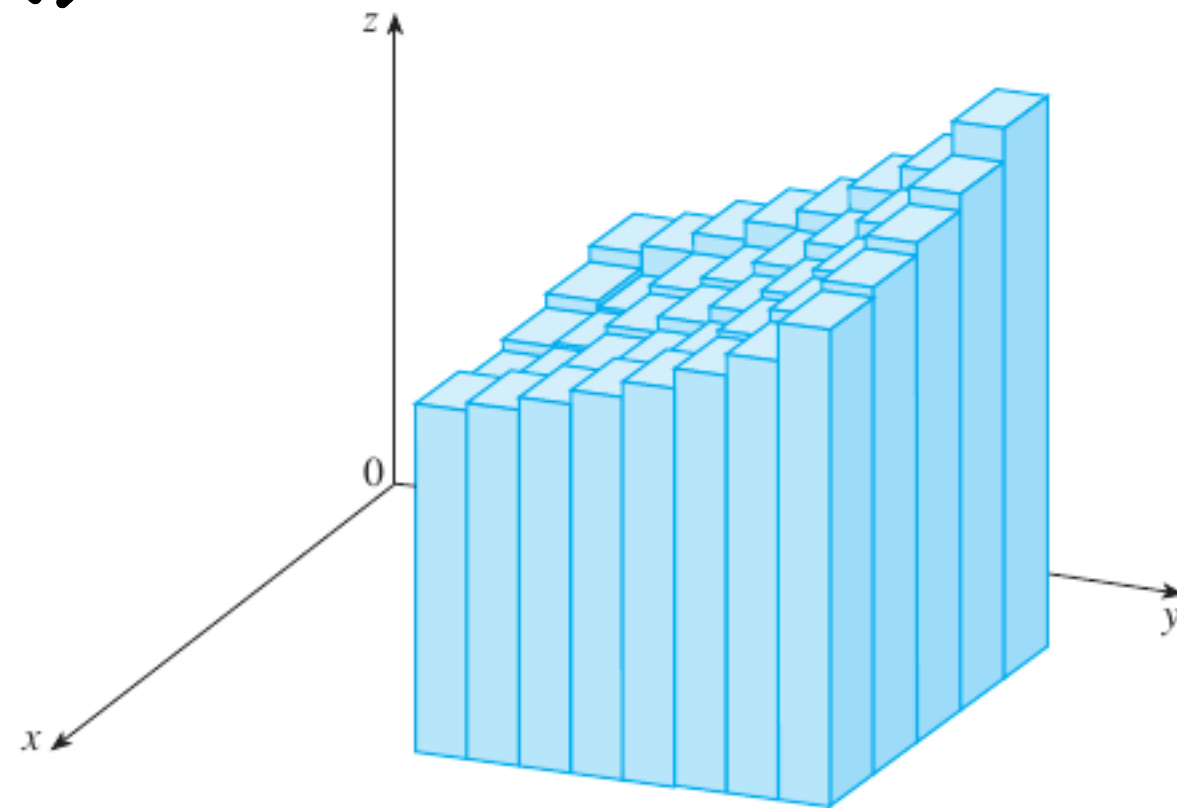
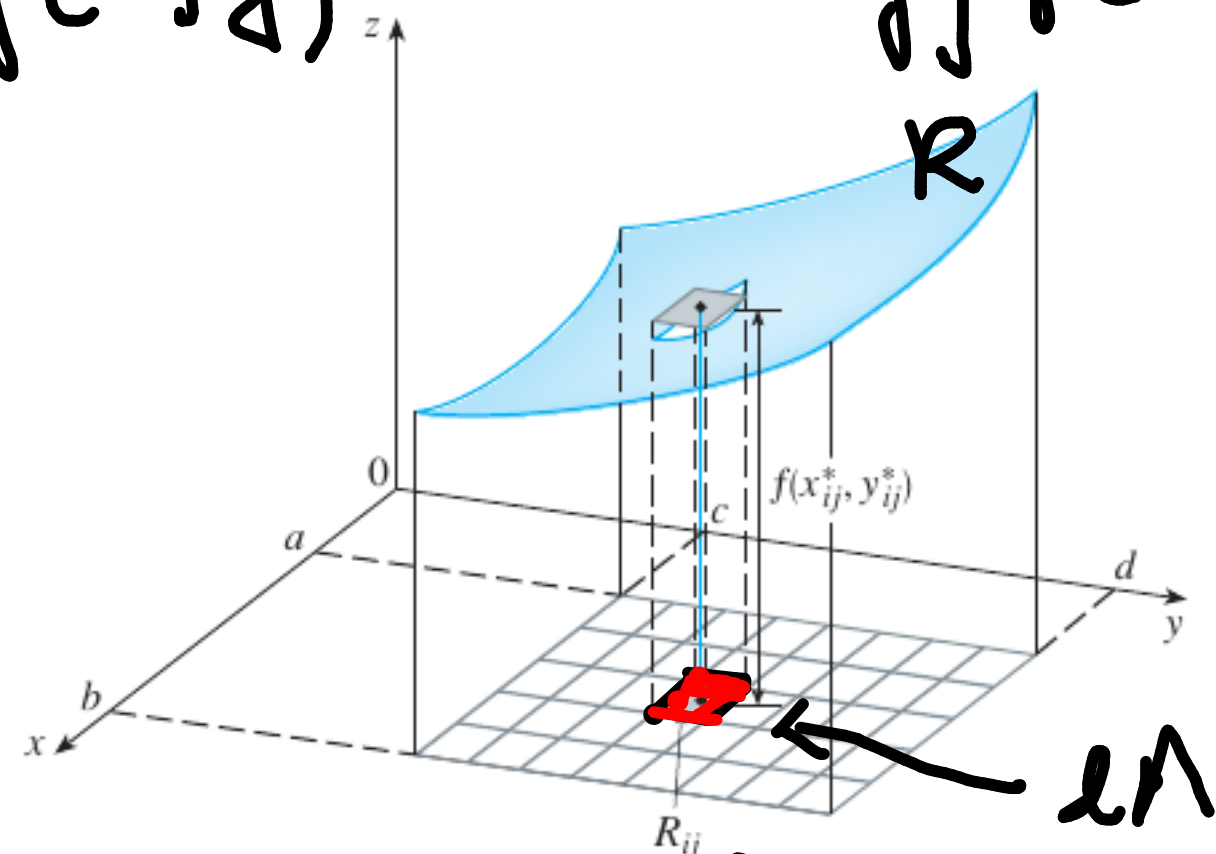
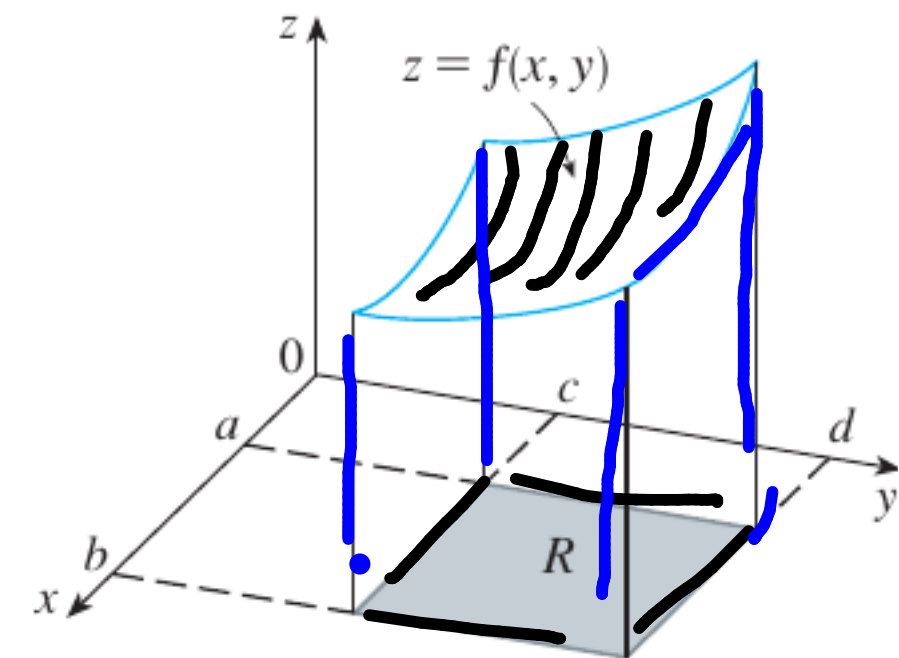


12

MULTIPLE INTEGRALS

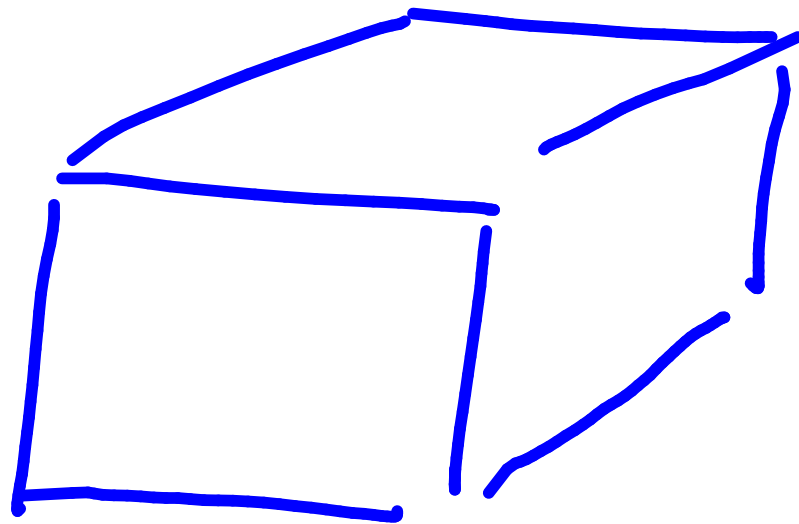
 $f(x, y)$

$$\iint_R f(x, y) \, dA$$

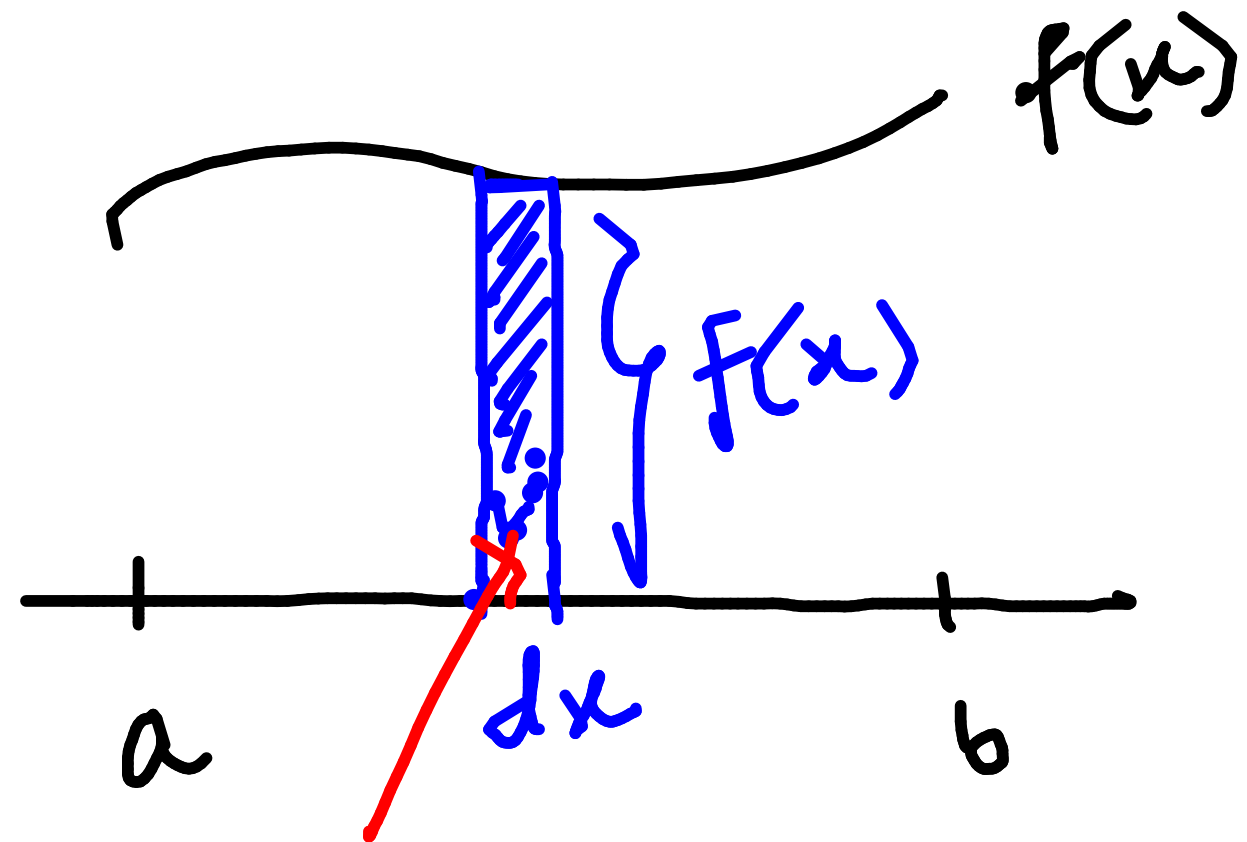


$$\iint_R f(x, y) \, dA = \text{volume with } f(x, y) \text{ on top}$$

\mathbb{Q} as base



$$\iiint_V f(x, y, z) \, dv$$



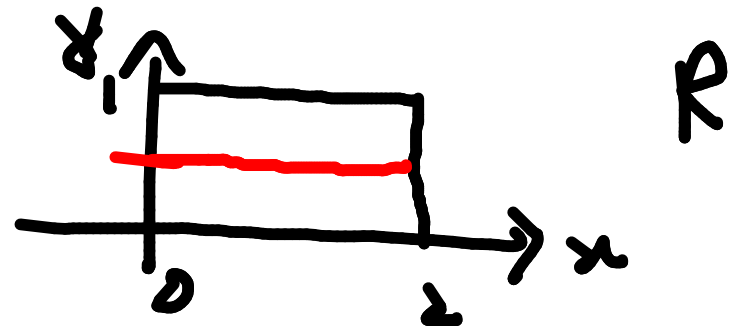
$$\text{area} = f(x) dx$$

$$\int f(x) dx = \text{infinite sum of } f(x) dx$$

$$\int_a^b f(x) dx$$

$$f(x) dx$$

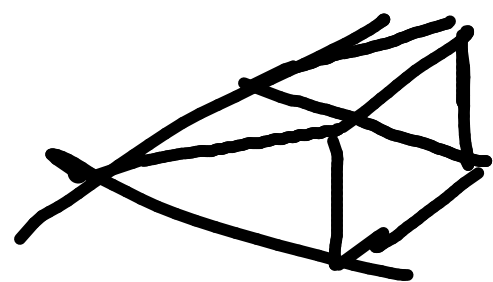
2. $f(x, y) = x$



$$\iint_R x \, dA = ?? = \int_0^2 \left(\int_0^1 x \, dy \right) dx$$

$$= \int_0^2 \left[xy \right]_{y=0}^{y=1} dx = \int_0^2 x \, dx$$

$$= 2$$



$$= \int_0^1 \left(\int_0^2 x \, dx \right) dy$$

$$= \int_0^1 2 \, dy = 2$$

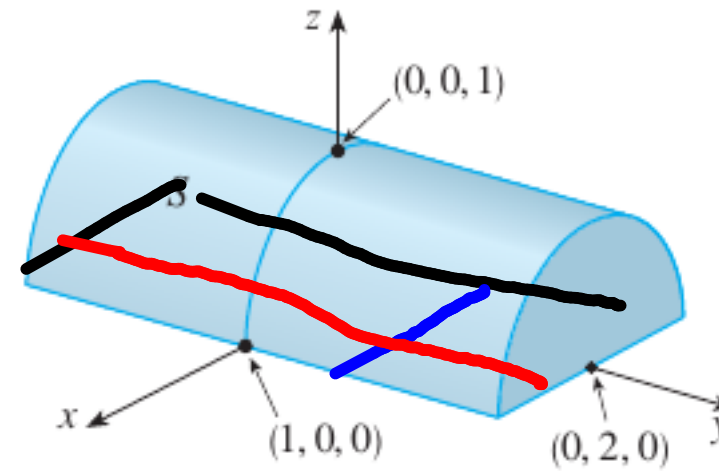
EXAMPLE 2 If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1-x^2} \, dA$$

$$f(x, y) = \sqrt{1-x^2}$$

$$z = \sqrt{1-x^2}$$

$$z^2 + x^2 = 1$$



$$= \int_{-2}^2 \int_{-1}^1 \sqrt{1-x^2} \, dx \, dy$$

$$= \int_{-1}^1 \int_{-2}^2 \sqrt{1-x^2} \, dy \, dx$$

$$= \int_{-1}^1 4\sqrt{1-x^2} \, dx = 2\pi$$

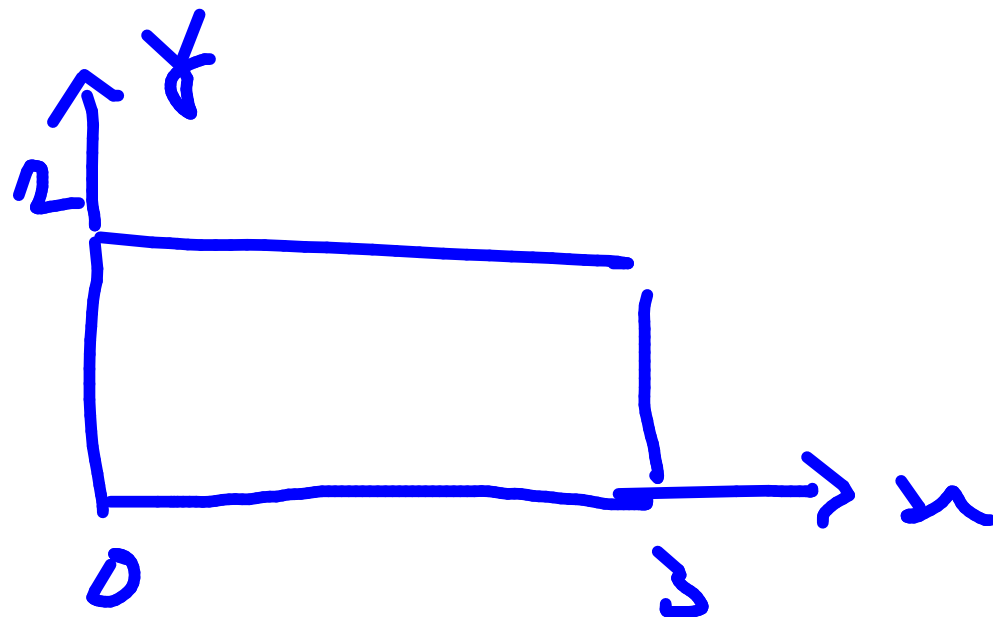
EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

$$\frac{27}{2}$$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

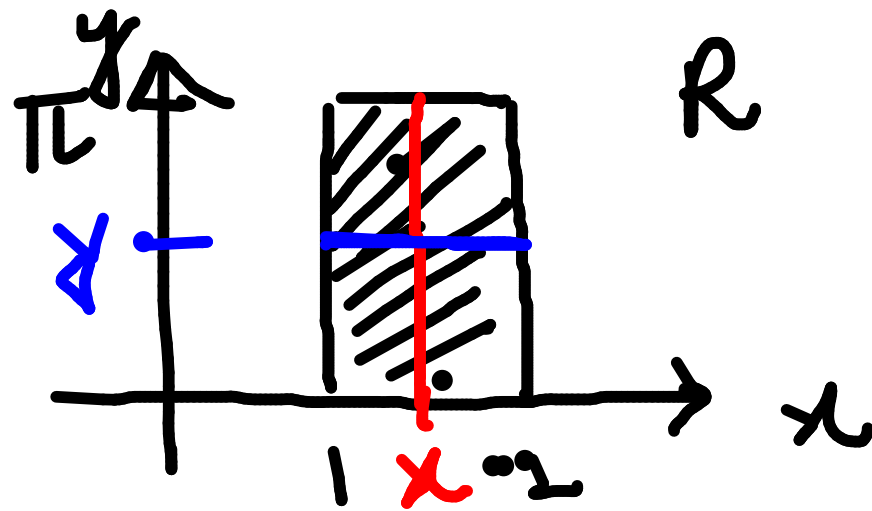
$$\frac{27}{2}$$



10 FUBINI'S THEOREM If f is continuous on the rectangle
 $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Evaluate $\iint_R y \sin(xy) dA$, where $R = [1, 2] \times [0, \pi]$.



$$\iint_R y \sin(xy) dA = \int_1^2 \left(\int_0^\pi y \sin(xy) dy \right) dx$$

$$= \int_0^\pi \left(\int_1^2 y \sin(xy) dx \right) dy$$

Then, start integrating inside out

$$\int_1^2 y \sin(xy) dx = \left[-\cancel{y} \frac{\cos(xy)}{\cancel{y}} \right]_{x=1}^{x=2} \\ = \cos(y) - \cos(2y)$$

$$\iint_R y \sin(xy) dA = \int_0^{\pi} [\cos(y) - \cos(2y)] dy \\ = 0$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

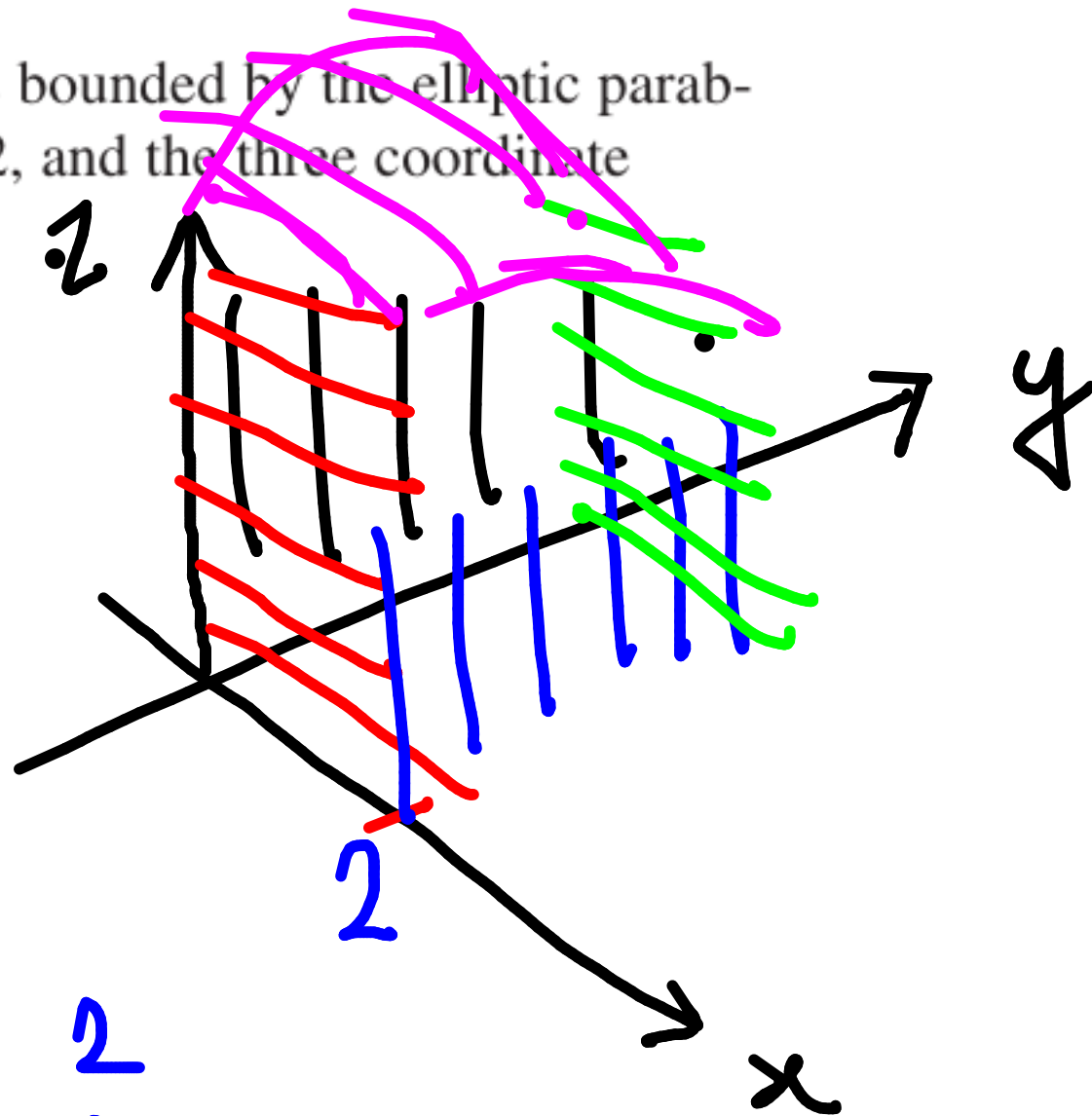
$$x^2 + 2y^2 + z = 16$$

$$z = 16 - x^2 - 2y^2$$

$$V = \iint_{[0,2] \times [0,2]} (16 - x^2 - 2y^2) \, dA$$

$$[0,2] \times [0,2]$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dy \, dx$$



$$\int_0^2 (16 - x^2 - 2y^2) dy = \left[16y - x^2y - \frac{2y^3}{3} \right]_{y=0}^{y=2}$$

$$= 32 - \frac{16}{3} - 2x^2$$

$$\int_0^2 \left(32 - \frac{16}{3} - 2x^2 \right) dx = 48$$

PROPERTIES OF DOUBLE INTEGRALS

$$\text{12} \quad \iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$$

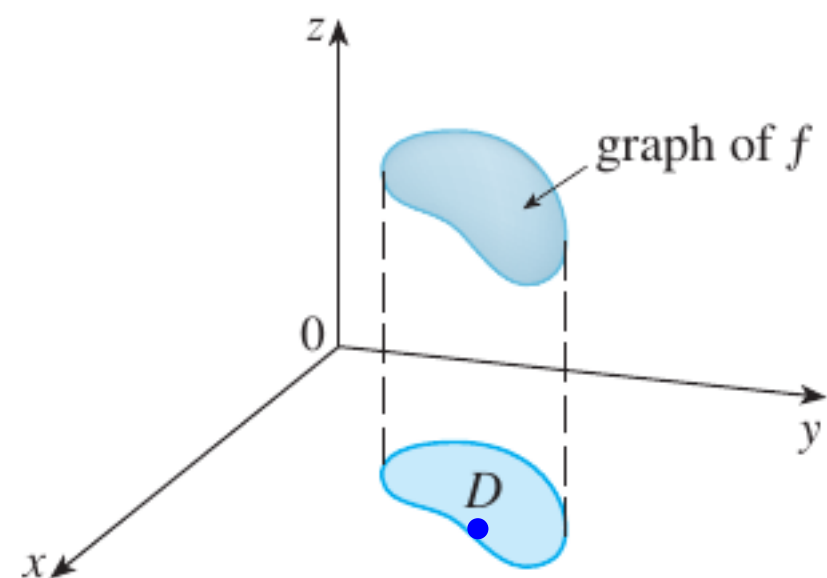
$$\text{13} \quad \iint_R cf(x, y) dA = c \iint_R f(x, y) dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in R , then

$$\text{14} \quad \iint_R f(x, y) dA \geq \iint_R g(x, y) dA$$

12.2

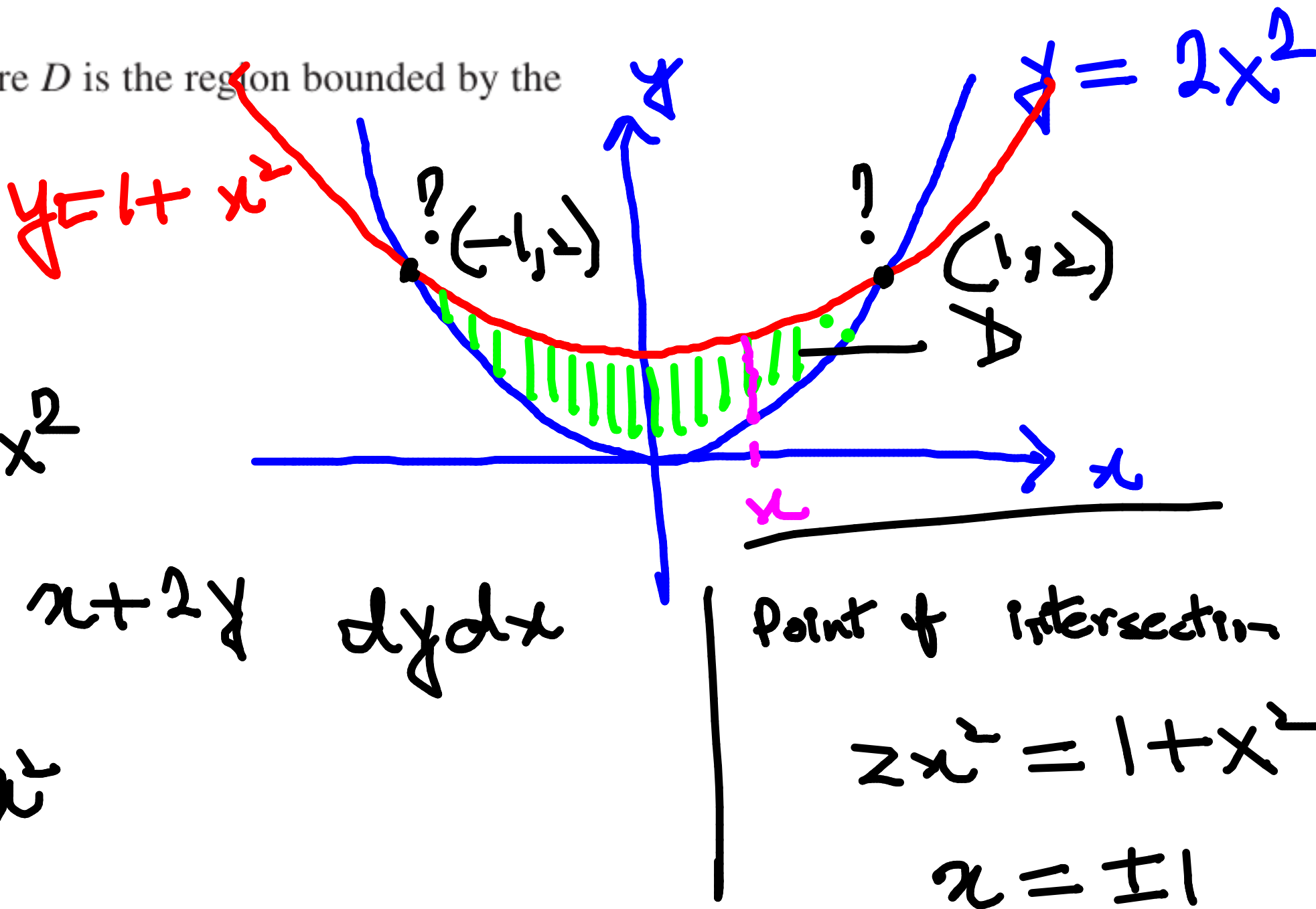
DOUBLE INTEGRALS OVER GENERAL REGIONS



EXAMPLE I Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$z = x + 2y$$

$$\iint_D (x + 2y) dA = \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx$$



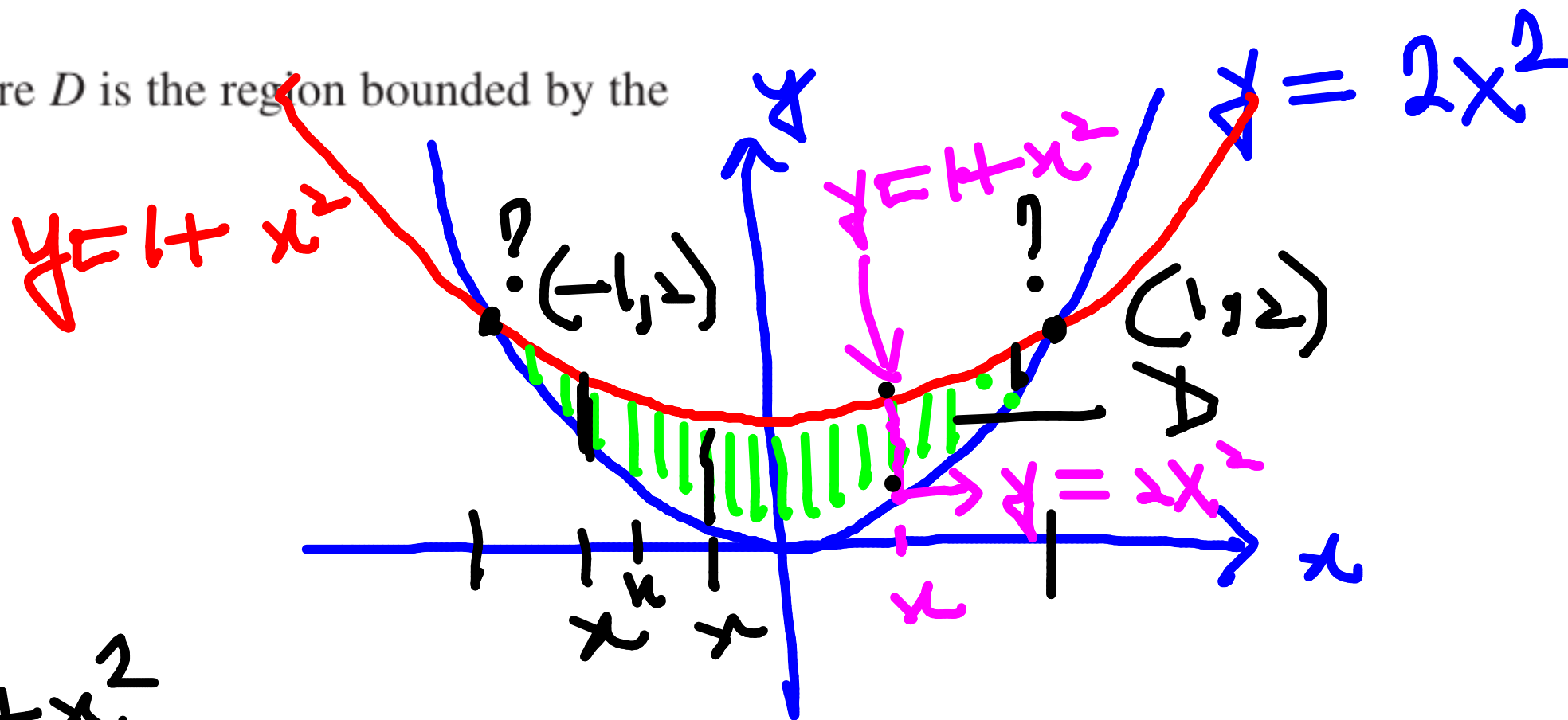
EXAMPLE I Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$z = x + 2y$$

$$\iint_D (x + 2y) dA$$

=

$$\int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx$$



$$\int_{2x^2}^{1+x^2} (x+2y) dy = \left[xy + y^2 \right]_{y=2x^2}^{y=1+x^2}$$

$$= \underline{x(1-x^2)} + (1+x^2)^2 - (2x^2)^2$$

$$\int \int (x+2y) dy = \int_{-1}^1 x(1-x^2) + (1+x^2)^2 - (2x^2)^2 dx$$

$$= \text{whatever} = \frac{32}{15}$$

Today's agenda

— 12.2 continued

Q. $\iint_R (x+2y) dA$,

where

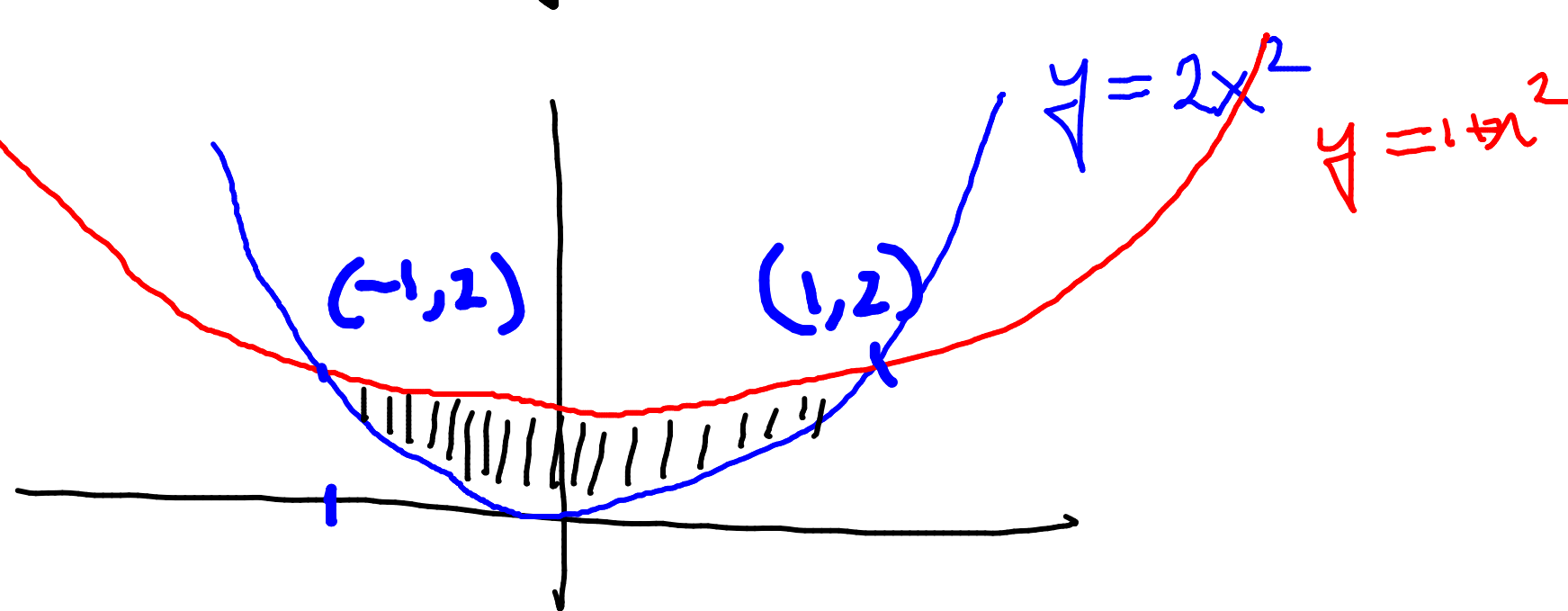
R is the region bounded
by the graph of

$y = 2x^2$ & $y = 1+x^2$

$= \int_A^B \left(\int_C^D (x+2y) dy \right) dx$

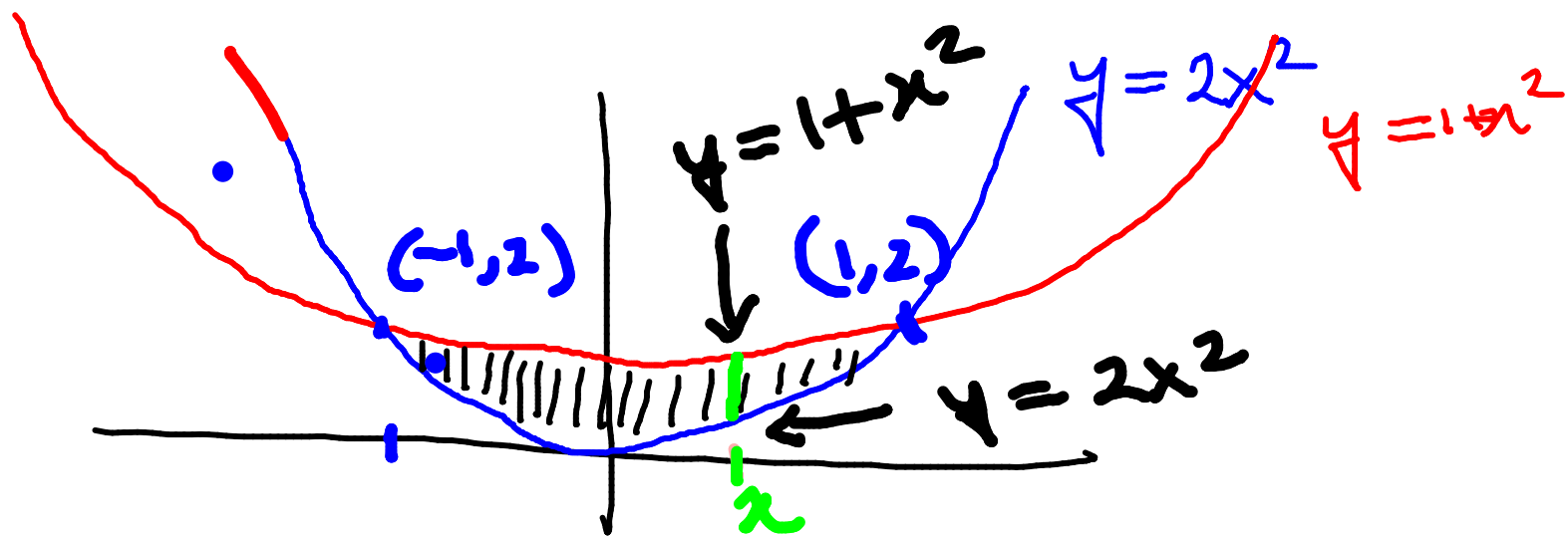
Q. $A, B, C, D = ??$

$A = -1, B = 1, C = 2x^2$ & $D = 1+x^2$



Q. $\iint_R (x+2y) dA$

where
 R is the region bounded
 by the graph of
 $y = 2x^2$ & $y = 1+x^2$



$$2x^2 = 1 + x^2$$

$$x = \pm 1, y = 2$$

$$\int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx$$

Q. $\iint_R (x+2y) dA$

where
R is the region bounded
by the graph of

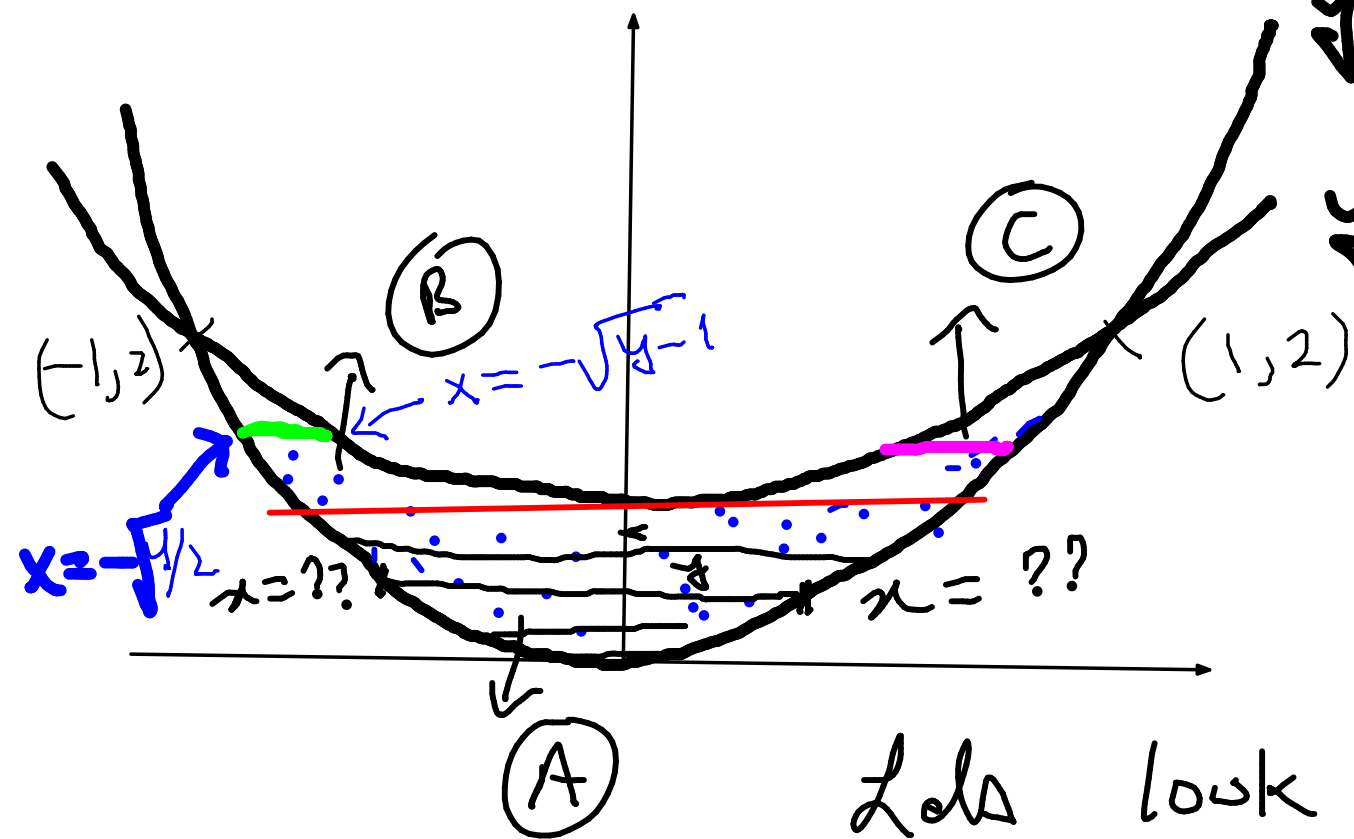
$y = 2x^2$ and $y = 1+x^2$

$y = 2x^2$ & $y = 1+x^2$

$\int_0^2 \int_{-1}^1 (x+2y) dx dy$

$= \iint_A + \iint_B + \iint_C$

look for limits of x in (A), (B), (C)



$$\iint (x+2y) dx dy = \int_1^2 \int_{\sqrt{y-1}}^{\sqrt{y}} (x+2y) dx dy$$

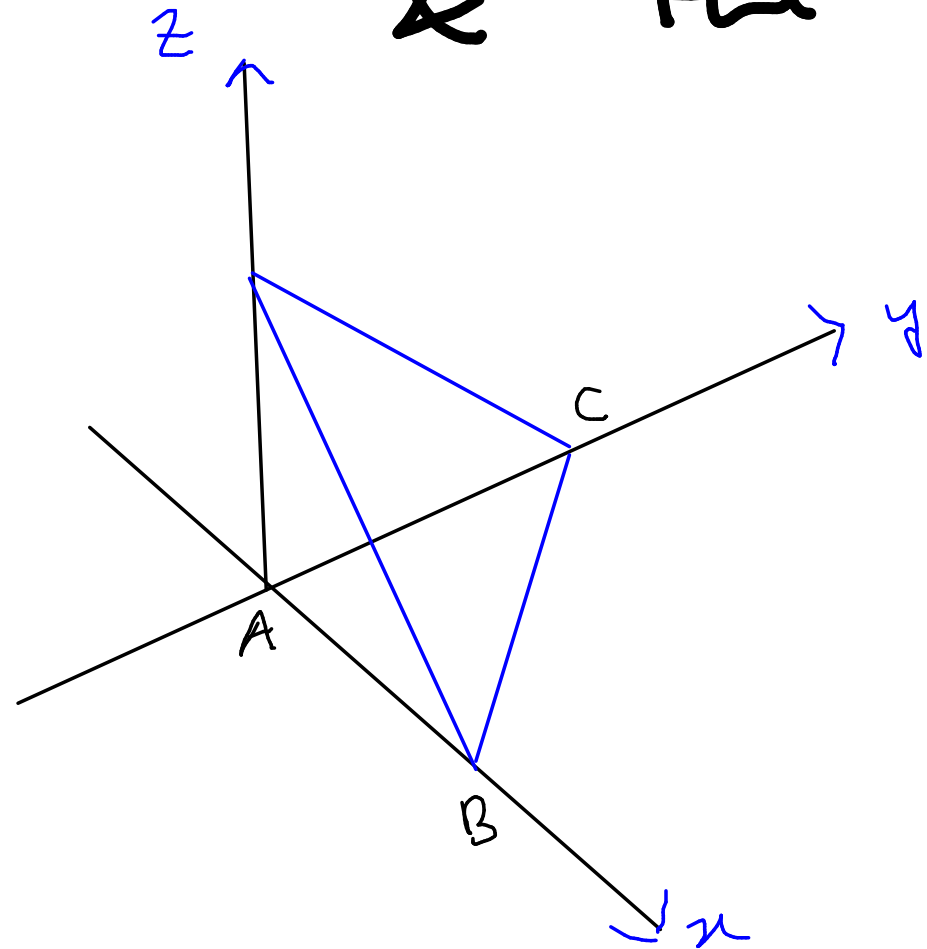
②

$$\textcircled{B} \quad \iint (x+2y) \, dx \, dy = \int_1^2 \int_{-\sqrt{y}/2}^{\sqrt{y}-1} (x+2y) \, dx \, dy$$

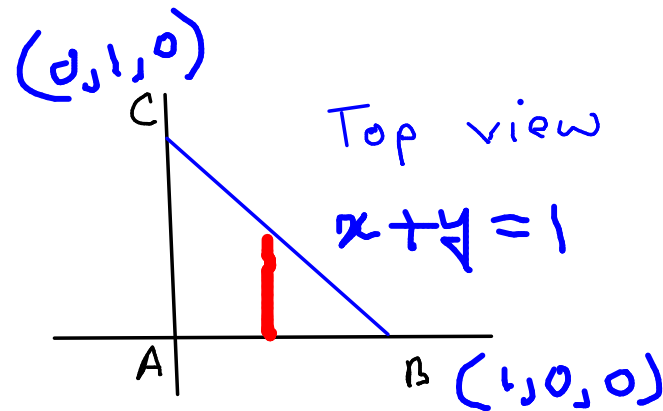
$$\textcircled{A} \quad y = 2x^2, \quad x = \pm \sqrt{\frac{y}{2}}, \quad \begin{array}{ll} x = -\sqrt{y}/2 & \text{left half} \\ x = \sqrt{y}/2 & \text{right half} \end{array}$$

$$\textcircled{A} \quad \iint (x+2y) \, dx \, dy = \int_{-\sqrt{y}/2}^{\sqrt{y}/2} \int (x+2y) \, dx \, dy$$

Q. Find the volume of the tetrahedron bounded by the plane $x+y+z=1$ & the other standard planes, xy, yz, xz



$$V = \iint_{\Delta ABC} (1-x-y) dA$$



$$= \int_0^1 \int_0^{1-x} (1-x-y) dy dx$$

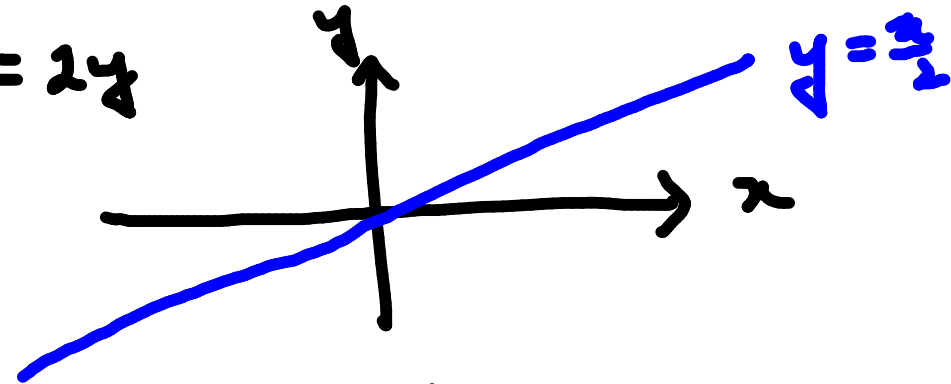
$$= \int_0^1 \int_0^{1-y} (1-x-y) dx dy$$

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

$z = 0 \rightarrow xy$ plane

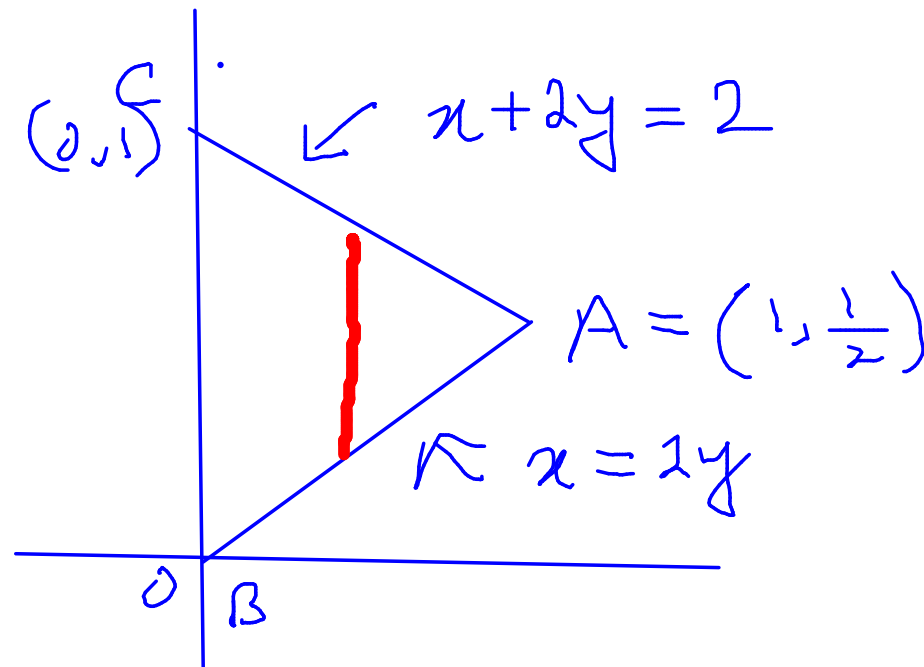
$x = 0 \rightarrow yz$ plane

$x = 2y$

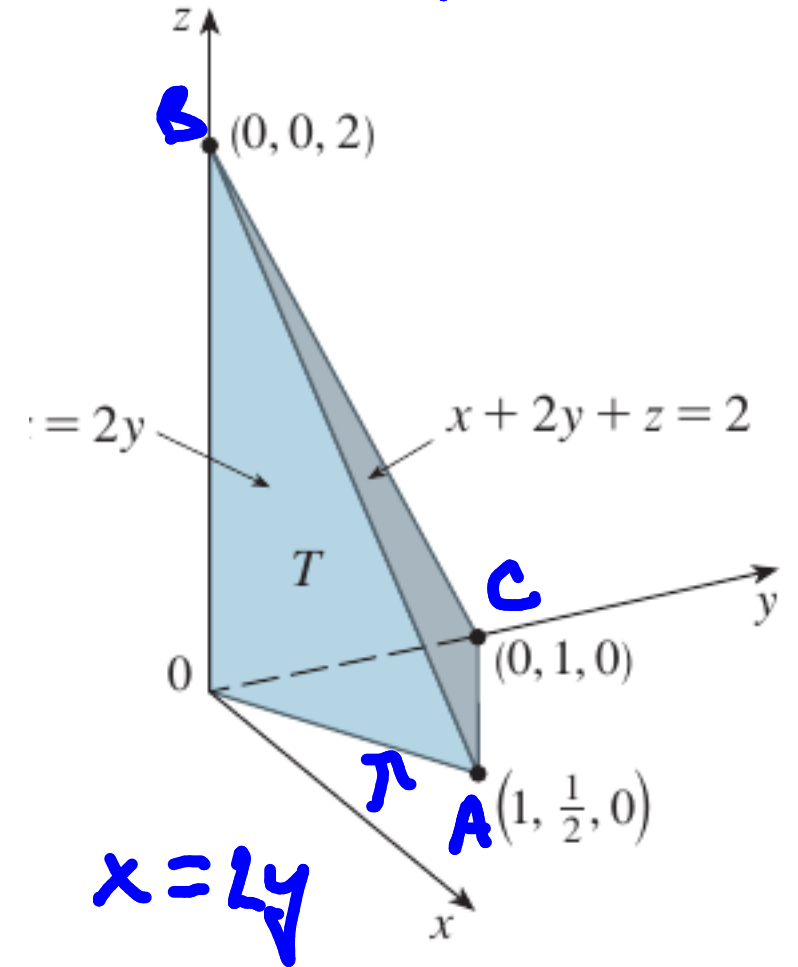


$$\text{volume} = \iint (2 - x - 2y) dA$$

ΔOAC



$$\Delta ABC: x + 2y + z = 2$$



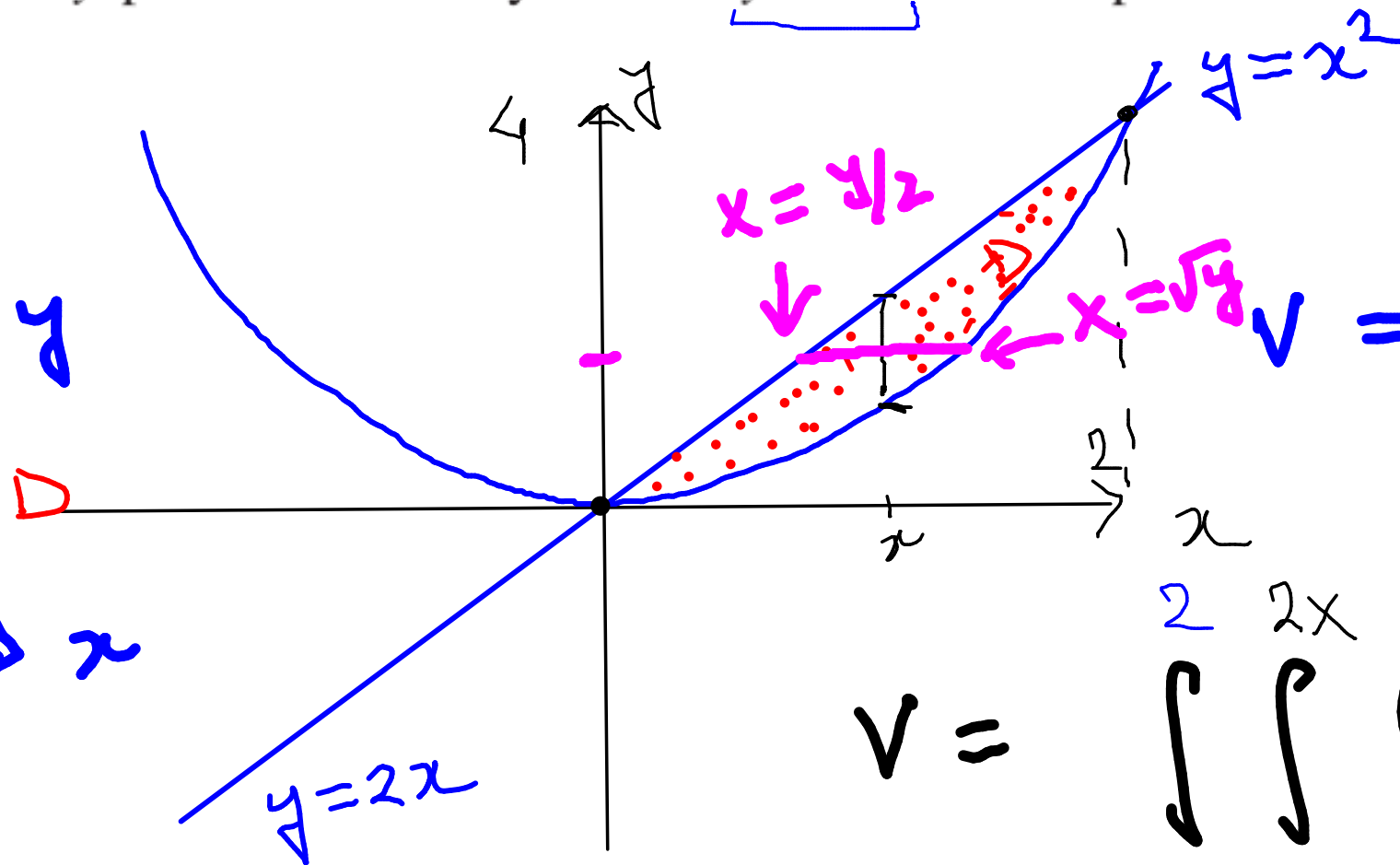
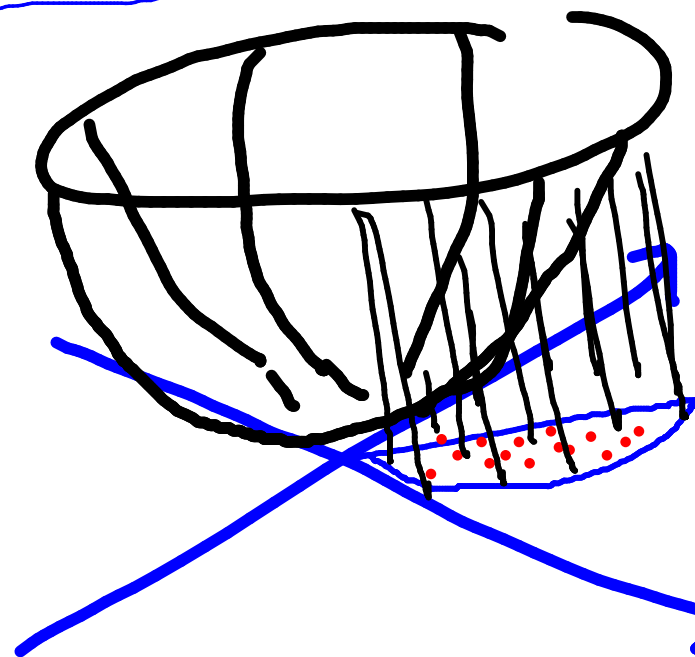
ΔOAB : vertical plane
above the line $x = 2y$

$$= \int_0^1 \int_{x/2}^{(2-x)/2} (2-x-2y) \, dy \, dx = \text{Ans} \quad ?? = \frac{7}{12}$$

H.W.

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

graph??



$$V = \iint_D (x^2 + y^2) dA$$

$$V = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx = \frac{216}{35}$$

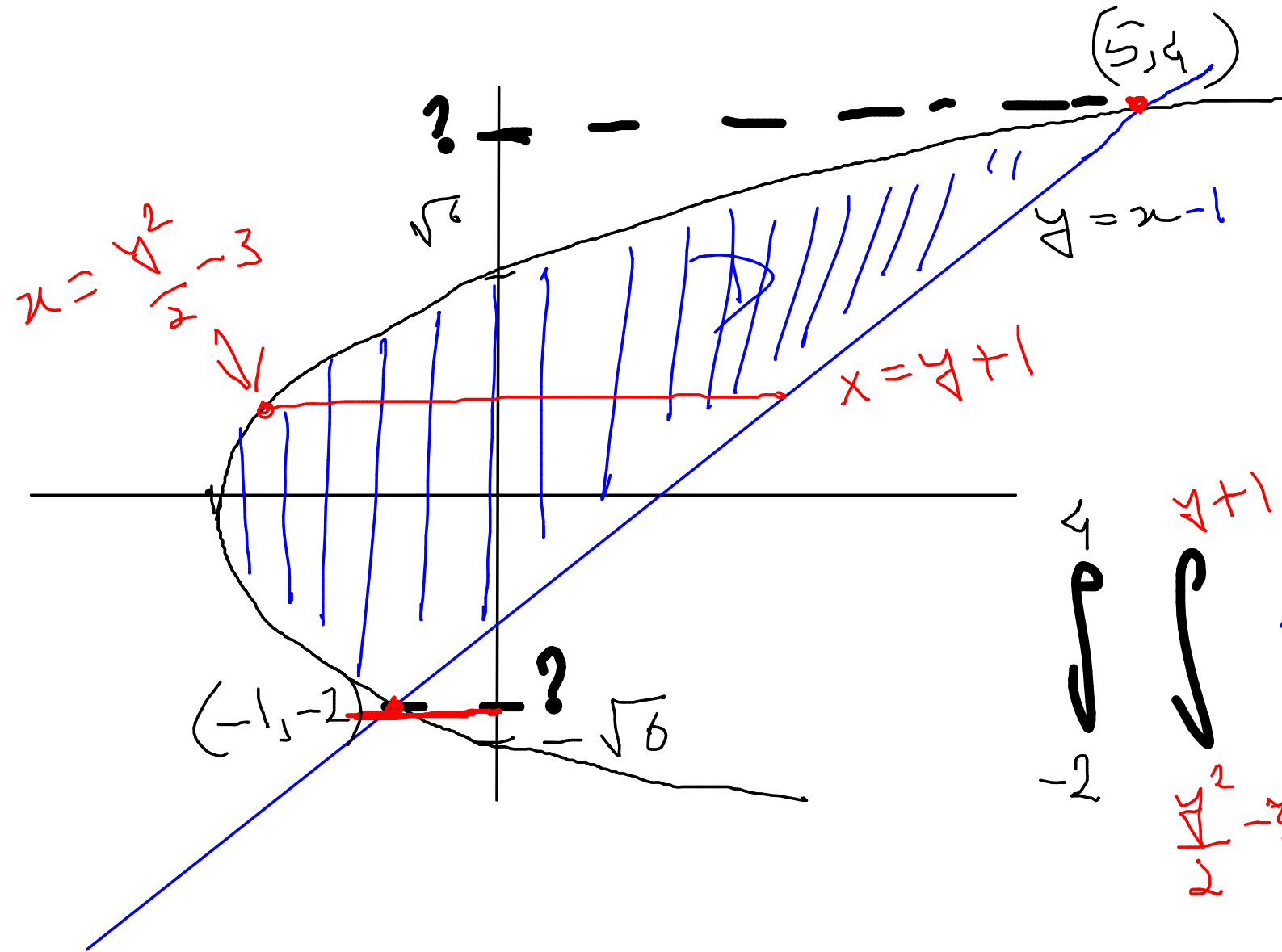
$$2x = x^2$$

$$x = 0, 2$$

$$V = \int_{?}^{?} \int_{?}^{?} (x^2 + y^2) \, dx \, dy = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2) \, dx \, dy$$

$$= \frac{216}{35}$$

EXAMPLE 3 Evaluate $\iint_D xy \, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$$\int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx \, dy = 36$$

$$y^2 = 2x + 6$$

$$\frac{y^2 - 6}{2} = x$$

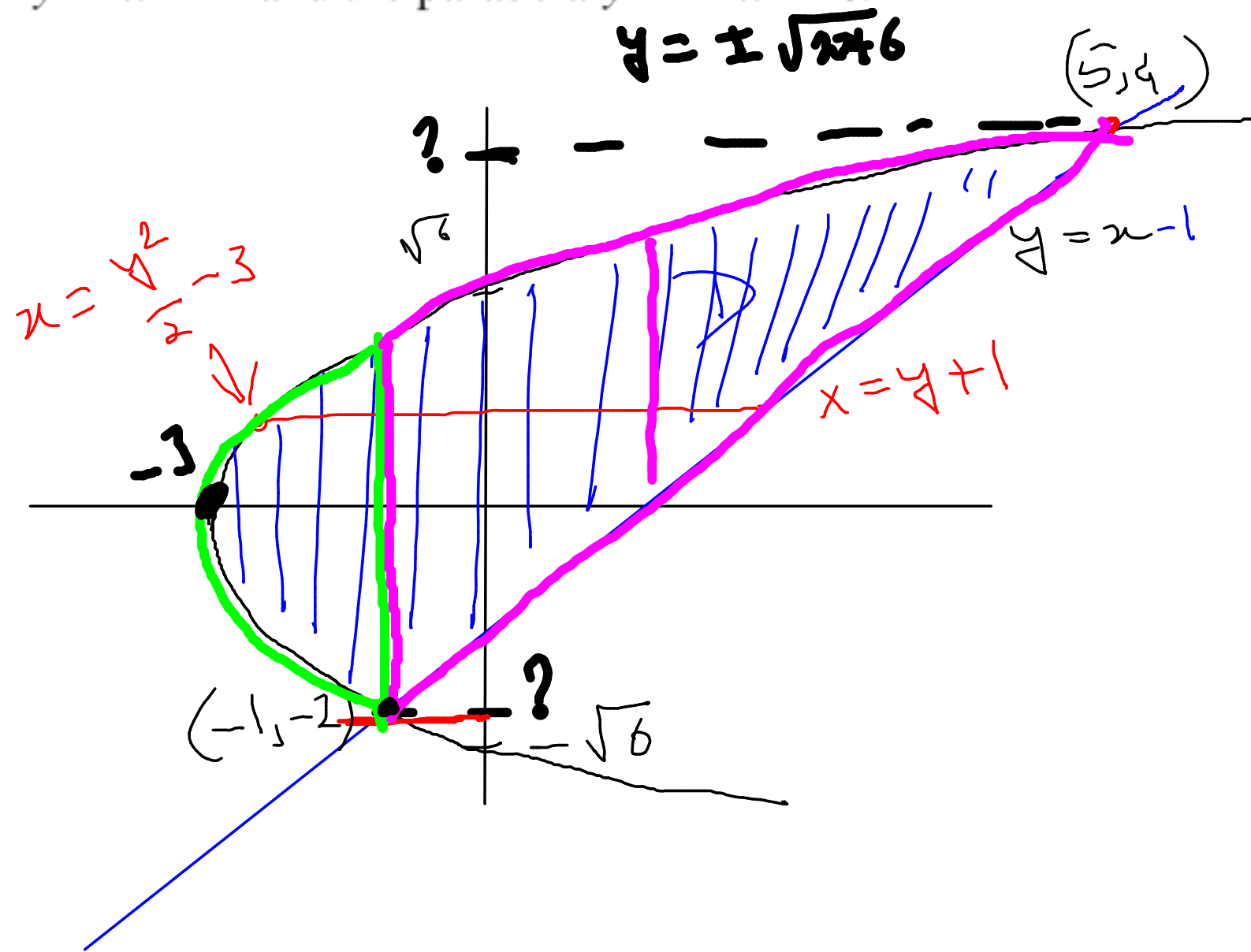
$$\frac{y^2}{2} - 3 = x$$

parabola open
rightward

x intercepts ?? $(-3, 0)$

y intercepts ?? $y^2 - 6 = 0$
 $y = \pm\sqrt{6}$

EXAMPLE 3 Evaluate $\iint_D xy \, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



Integration in the order reverse

$$\int_{-3}^4 \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \, dy \, dx$$

$$+ \int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} xy \, dy \, dx$$

1-6 ■ Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y) \, dy \, dx$

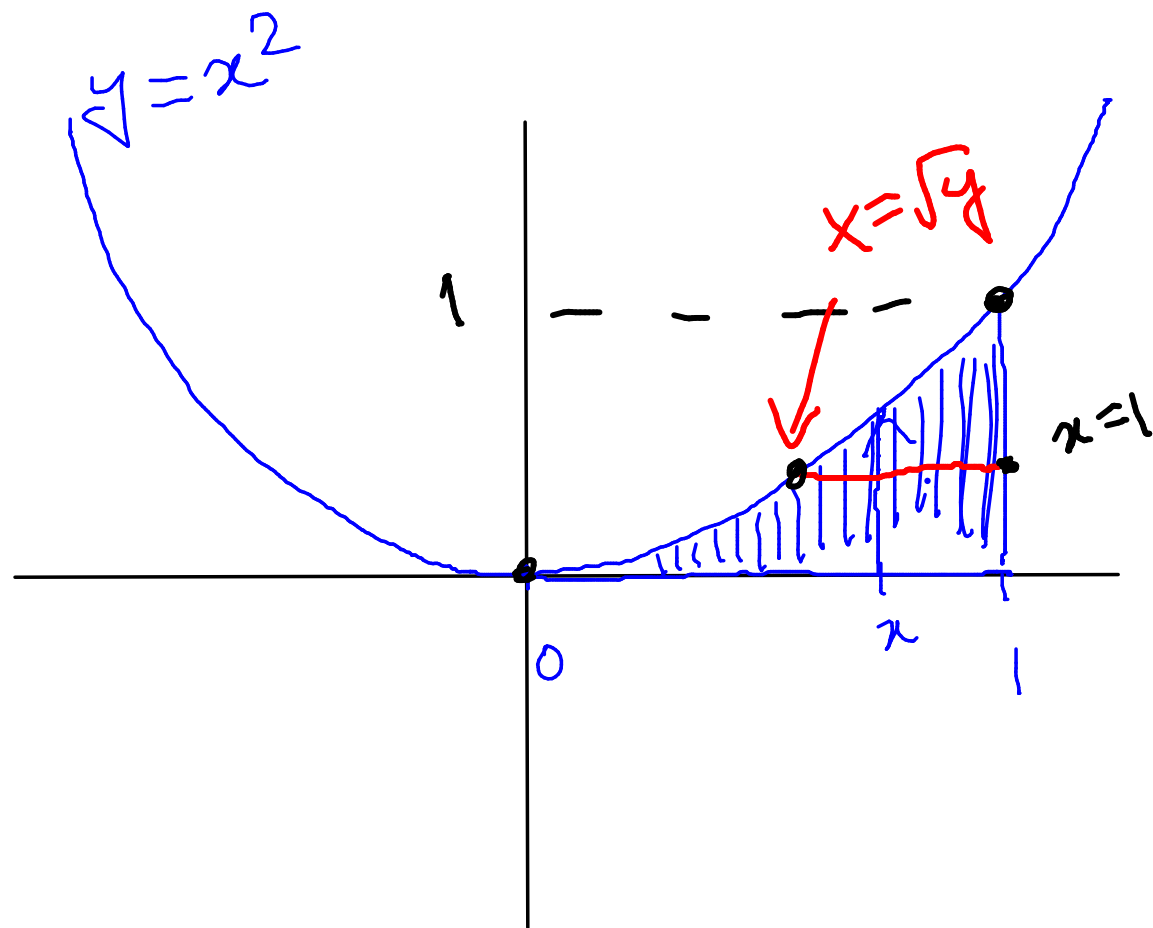
$\frac{9}{20}$

2. $\int_1^2 \int_y^2 xy \, dx \, dy$

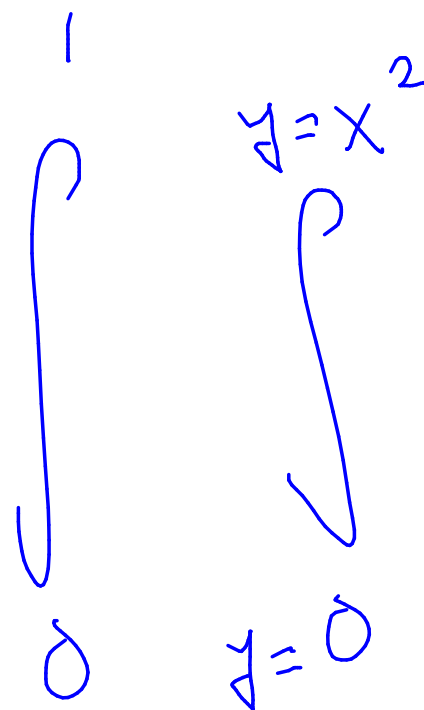
$\frac{9}{8}$

1-6 ■ Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y) dy dx$



Sketch the region of integration
& reverse the order of integration



*** $dy dx$

=

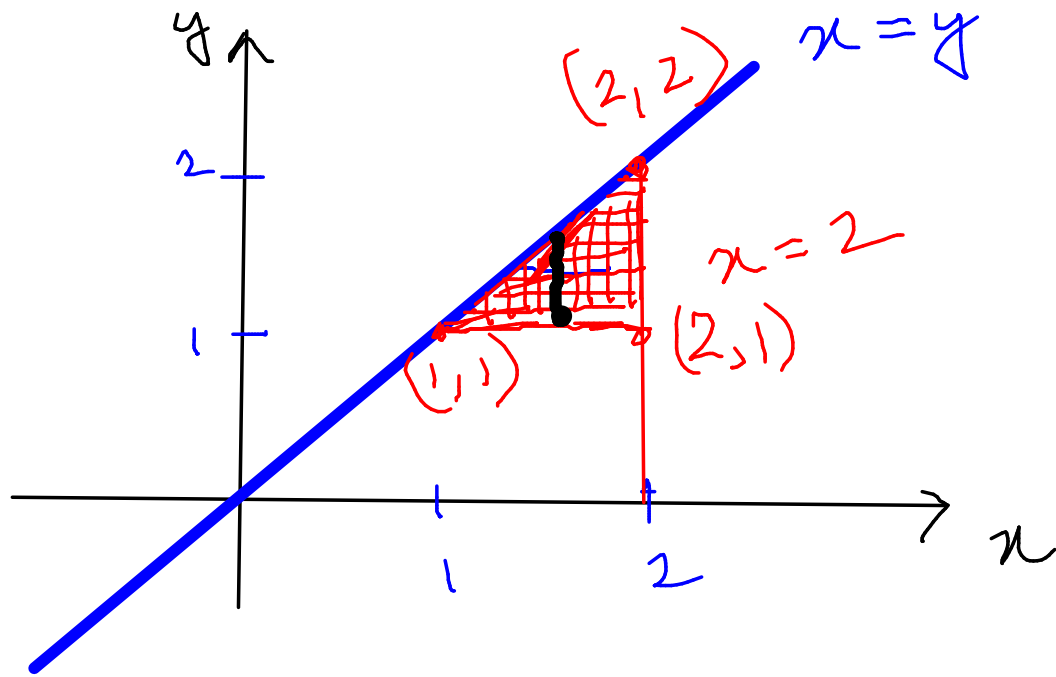
$\int_0^1 \int_{\sqrt{y}}^1 dx dy$

$dx dy$

1-6 ■ Evaluate the iterated integral.

$$2. \int_1^2 \int_y^2 xy \, dx \, dy$$

Sketch the region of integration & revert the order of integration.



reverse

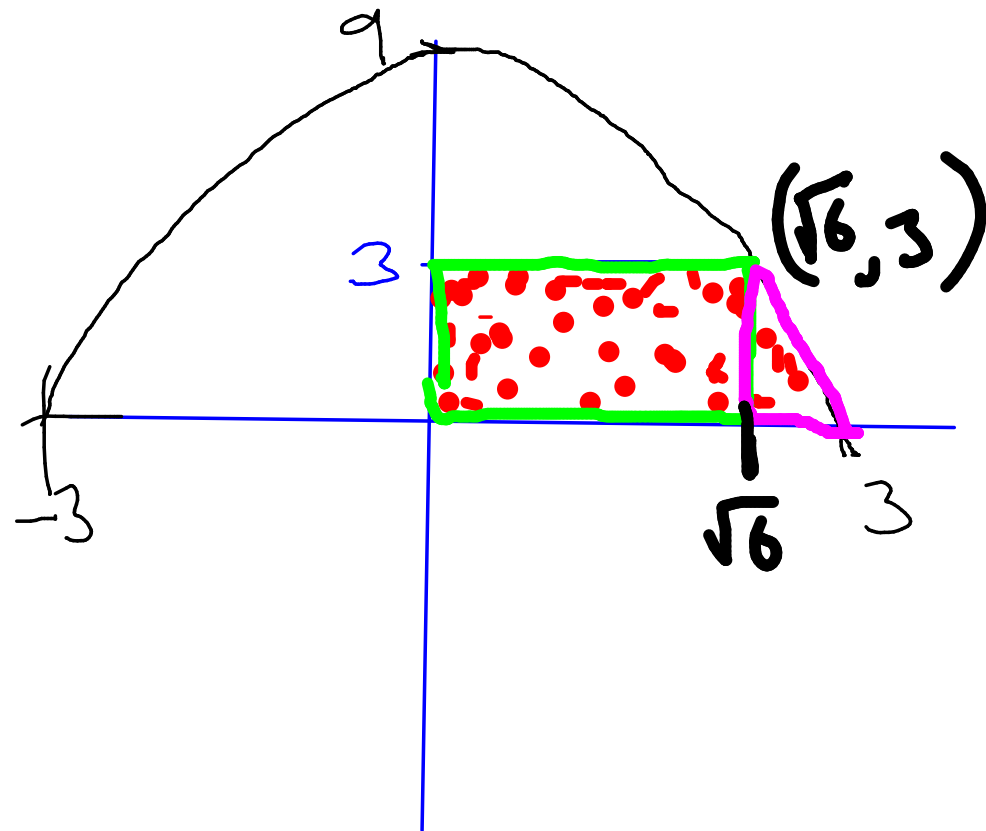
=

$$\int_1^2 \int_1^x xy \, dy \, dx$$

$$\int_1^2 \int_y^2 xy \, dx \, dy$$

31–36 ■ Sketch the region of integration and change the order of integration.

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y) \, dx \, dy$$



$$x = \sqrt{9-y}$$

$$x^2 = 9-y$$

$$y = 9-x^2$$

$$\int_0^{\sqrt{6}} \int_0^3 f(x, y) \, dy \, dx$$

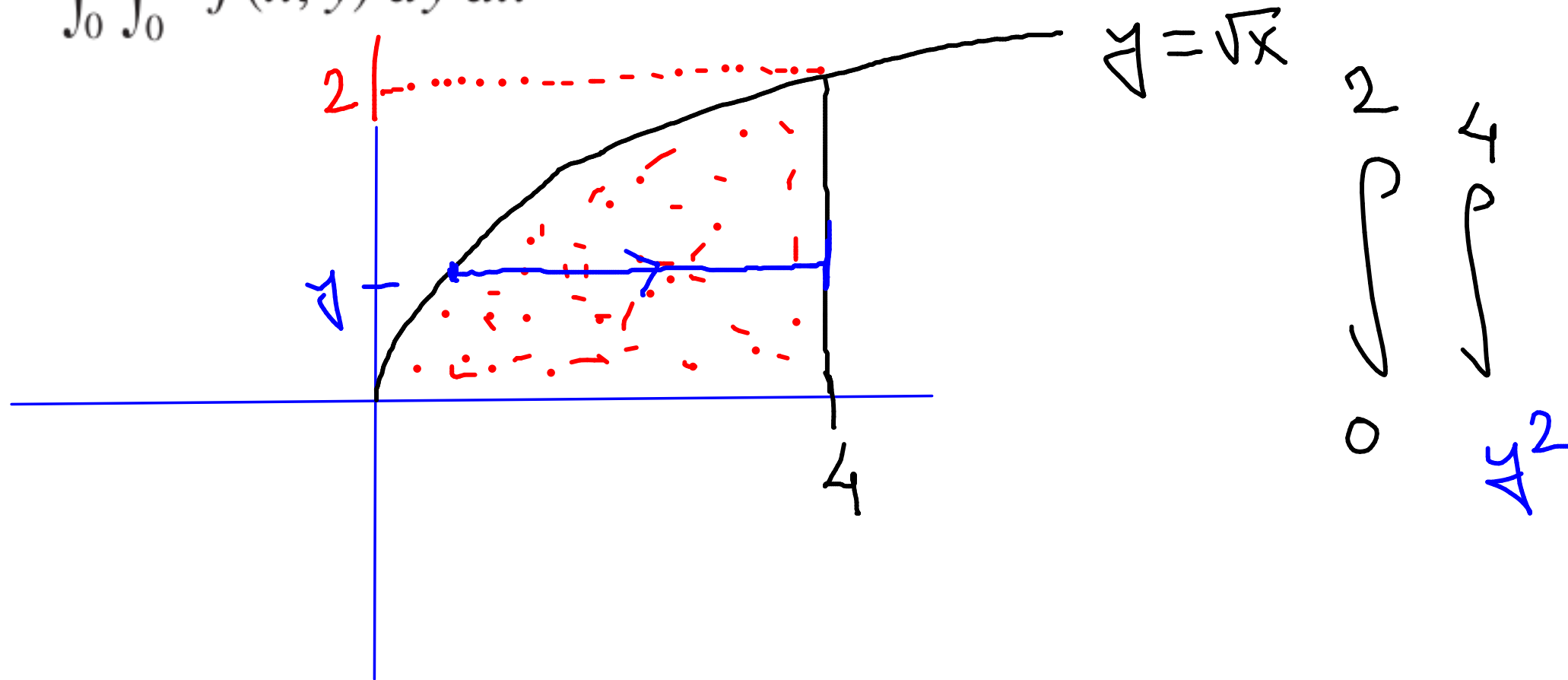
$$+ \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x, y) \, dy \, dx$$

31–36 ■ Sketch the region of integration and change the order of integration.

$$y = \sqrt{x}$$

$$x = y^2$$

$$\int_0^4 \int_0^{\sqrt{x}} f(x, y) \, dy \, dx$$



$$\int_0^2 \int_{y^2}^4 f(x, y) \, dx \, dy$$

$$f(x, y) \, dx \, dy$$

37–42 ■ Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2} dx dy$$

17. w.