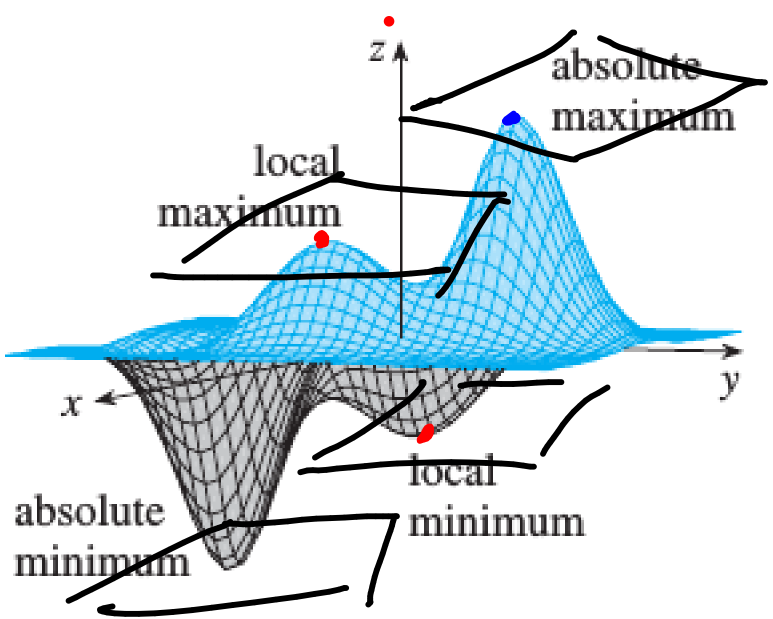
11.7 MAXIMUM AND MINIMUM VALUES

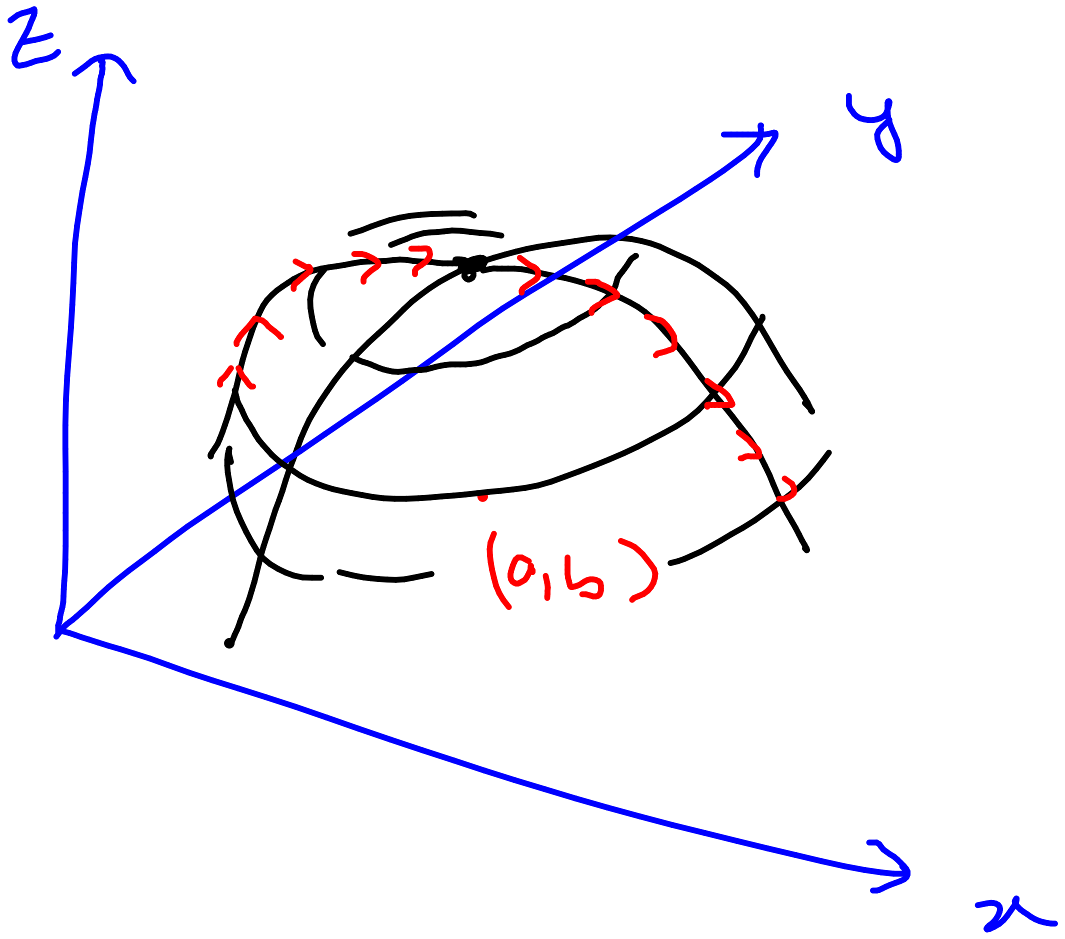
→ at local max/min points tangent plane will be horizontal



→ $z - z_0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

⇒ if (a,b) is a point of max/min

then $\boxed{f_x(a,b) = f_y(a,b) = 0}$



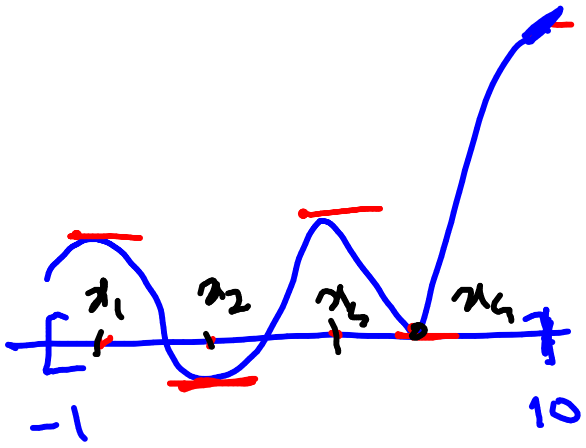
$$f_x(a,b) = 0$$

$$f_y(a,b) = 0$$

Recall questions like

$$f(x) = x^2 + \sin(x) + 2$$

find the max / min

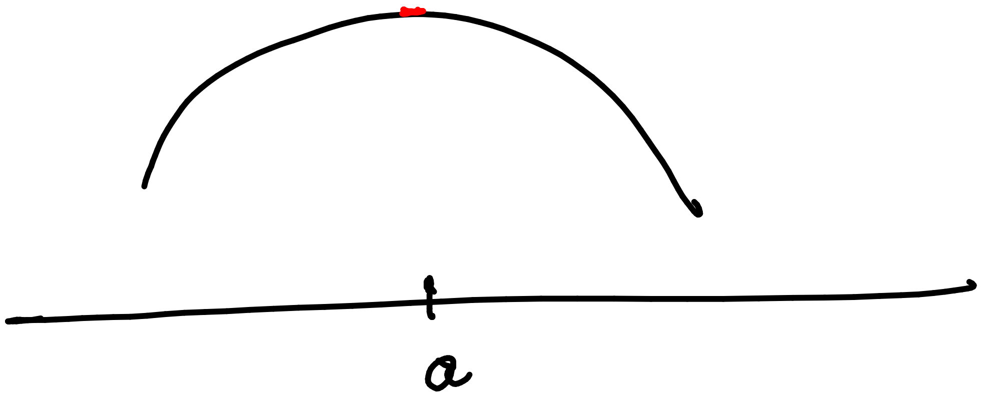


$\Rightarrow$ solve $f'(x) = 0$, $\Rightarrow$ $x = x_1, x_2, x_3, x_4$

max value $= \max\{f(x_1), f(x_2), f(x_3), f(x_4), f(a), f(b)\}$

min value $= \min\{f(x_1), f(x_2), f(x_3), f(x_4), f(a), f(b)\}$

$f(x)$



$f'(a) = 0$

identify a a
a a local max
or a local min

f''

critical point

points in the domain of $f(x,y)$ where

$$f_x = 0 \quad \& \quad f_y = 0$$

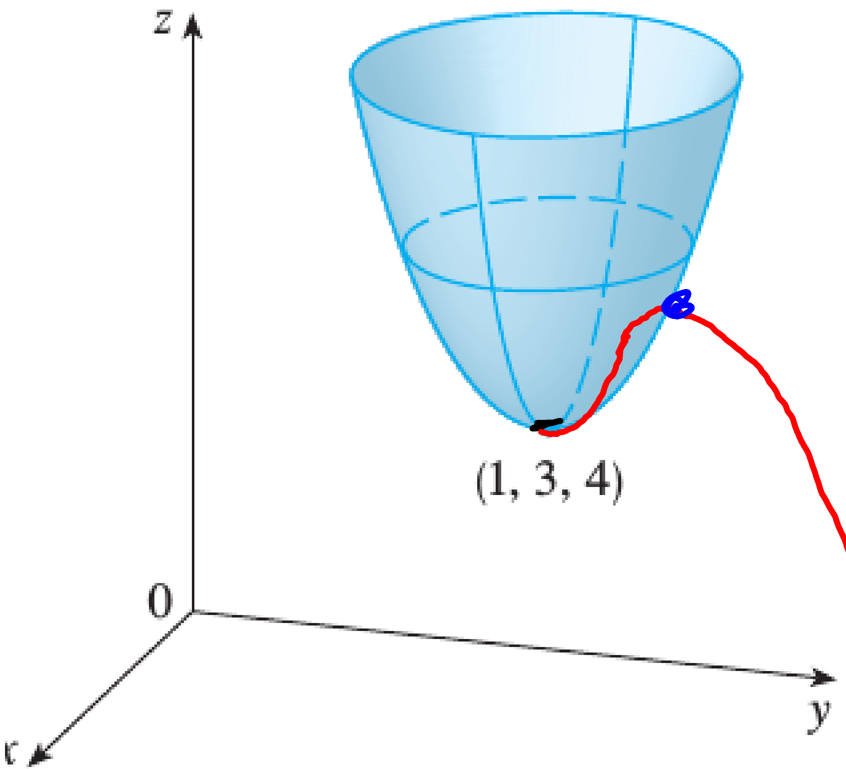
EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Q: find points of local min or max

→ find critical points

$f_x = 0$ $\qquad$ $f_y = 0$

$2x - 2 = 0$ $\qquad$ $2y - 6 = 0$

$x = 1$ $\qquad$ $y = 3$

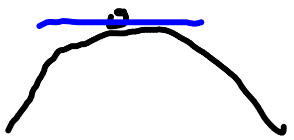


now check if it is a point of max/min/neither

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

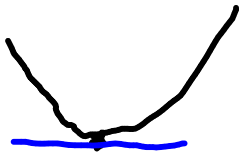
$\underbrace{\det(H) = 4 \quad \& \quad f_{xx} = 2 > 0} \Longrightarrow (1,3)$ is a point of local min

Classification of critical points



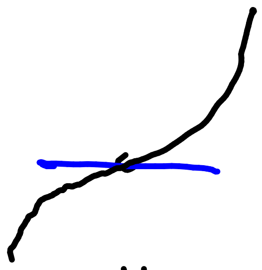
$f'' < 0$

f is concave down



$f'' > 0$

f is concave up



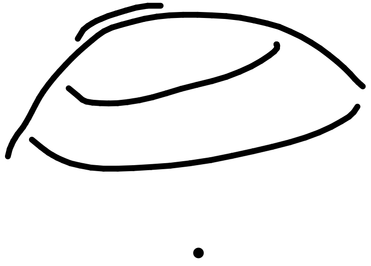
$f'' = 0$

Hessian Matrix

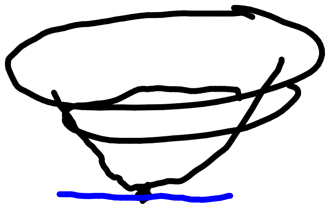
$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

local max	local min	neither
$\det(H) > 0$	$\det(H) > 0$	$\det(H) < 0$
& $f_{xx} < 0$	& $f_{xx} > 0$	

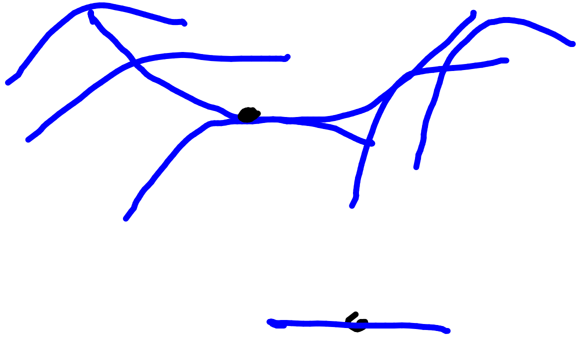
Classification of critical points



f is concave down



f is concave up



neither

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

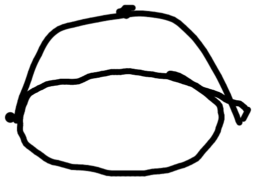
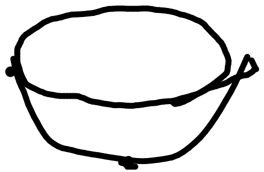
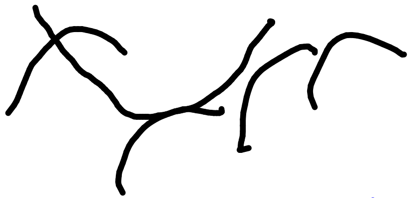
$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

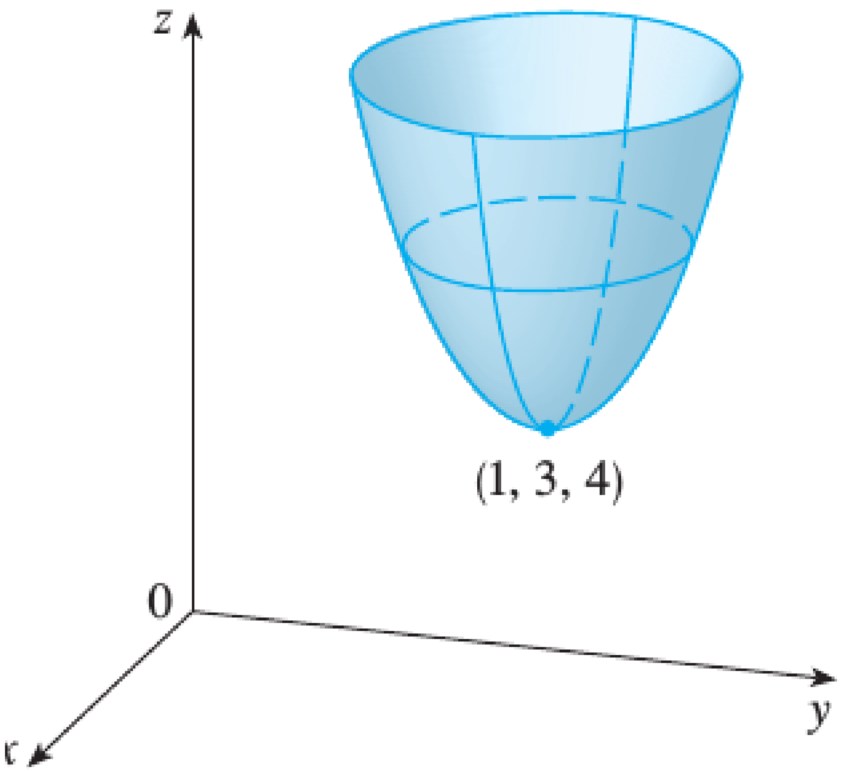
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

$$D = \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}}_{\text{Hession matrix}}$$

max	min	neither
		
concave down	concave up	neither
$D > 0$	$D > 0$	$D < 0$
$f_{xx} < 0$	$f_{xx} > 0$	

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Q: find points of local min or max



1. Suppose (1, 1) is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

(b) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$

a) $|H| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7 > 0$

$f_{xx} = 4 > 0$

f concave up $\Rightarrow$
(1,1) is a local min

b) $|H| = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} = -1 < 0$

(1,1) is a saddle point

Q. $f(x,y) = x^2 - y^2$

find & classify critical points

$f_x = 0$ $\quad$ $f_y = 0$

$2x = 0$ $\quad$ $2y = 0$

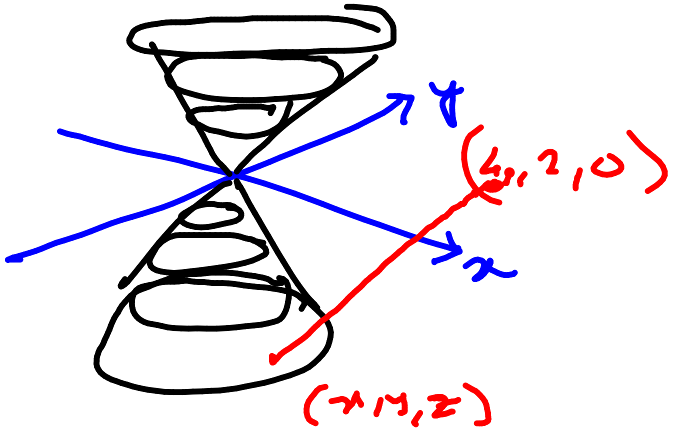
$x = 0$ $\quad$ $y = 0$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$|H| = -4 < 0$

Saddle point

33. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.



minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

s.t. (x,y,z) belongs to the surface $z^2 = x^2 + y^2$

solve it

use $z^2 = x^2 + y^2$

minimize $f = (x-4)^2 + (y-2)^2 + x^2 + y^2$

find critical points

$f_x = 2(x-4) + 2x = 0$ | $f_y = 2(y-2) + 2y = 0$

$\Rightarrow \quad x = 2$ | $y = 1$

$(2,1)$: critical point

check for max/min $\quad |H| = \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 > 0$

$\& \; f_{xx} = 4 > 0$

$(2,1)$ is a point of local min

but also absolute min (why??)

Ans: The point on the cone $z^2 = x^2 + y^2$ closest to $(4,2,0)$ is $(2,1,\sqrt{5})$ & $(2,1,-\sqrt{5})$

Q.

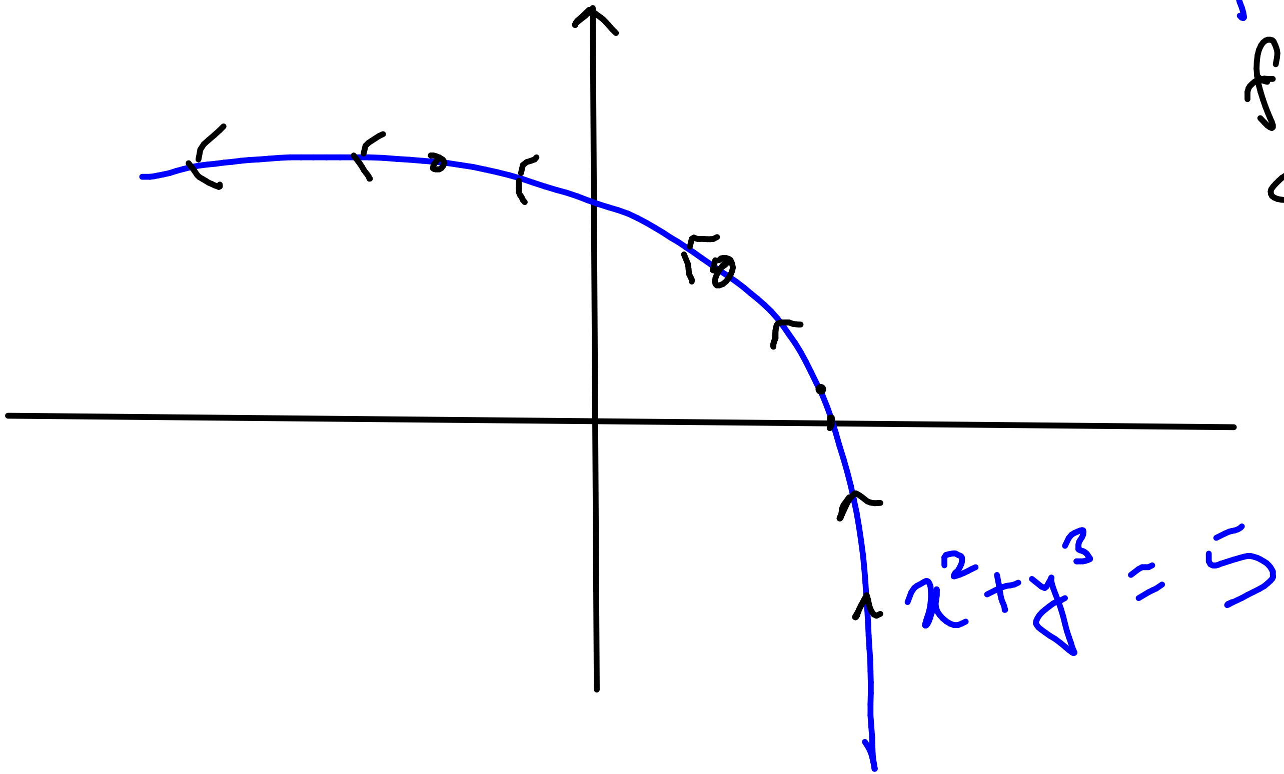
Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$

d. Find the absolute maximum and minimum values of f on the set D.

$$f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant 3\}$$

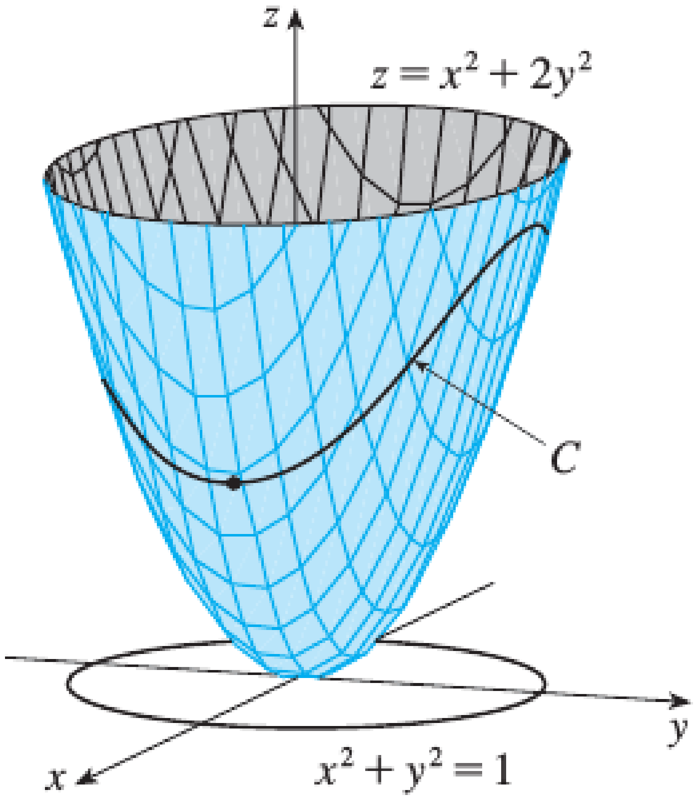
LAGRANGE MULTIPLIERS



$f(x,y) = x+y$

find point on the curve $x^2+y^3=5$ where $f(x,y)=x+y$ is lowest.

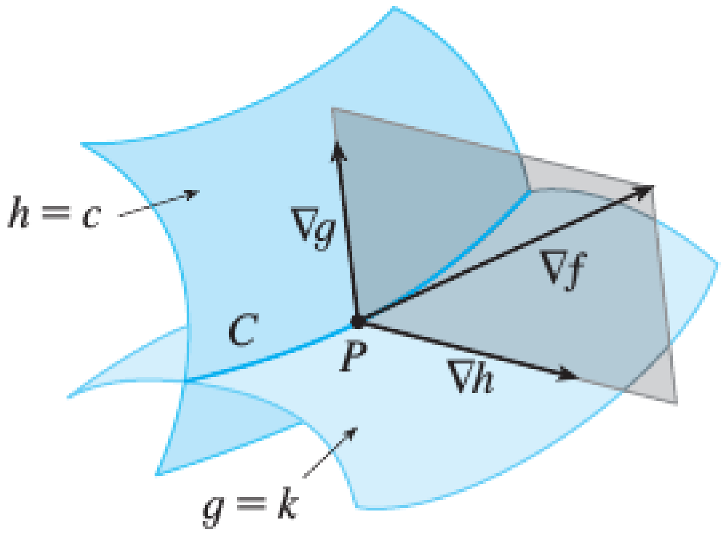
EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.



EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

EXAMPLE 4 Find the points on the circle ~~sphere~~ $x^2 + y^2 \quad = 4$ that are closest to and farthest from the point $(3, 1 \quad)$.

TWO CONSTRAINTS



$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35$$

1–15 ■ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4$$