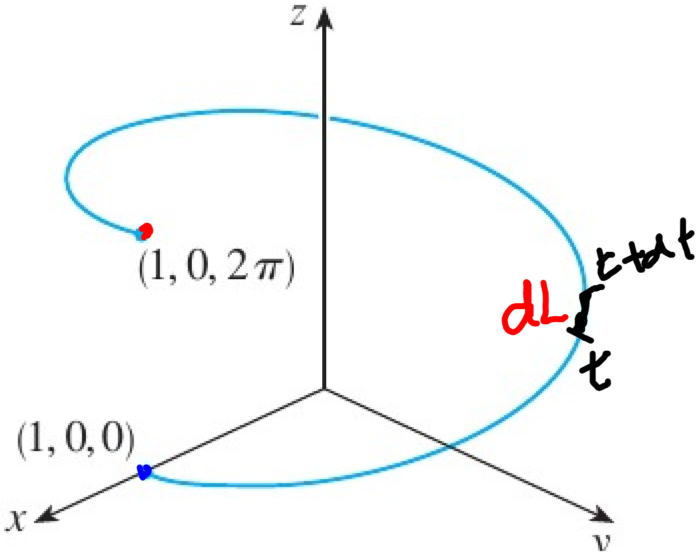
Today’s topic

- Arc Length and Curvature (Sec 10.8)

Q1. Whats the length of a curve?

Q2. How curved is the curve?

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



$0 \le t \le 2\pi$

$t \qquad\qquad t+dt$

dt: small enough to assume that speed of the particle is constant in the period $[t, t+dt]$

$$dL = (\text{speed})\,dt$$
$$= |r'(t)|\,dt$$

$$L = \int dL = \int_0^{2\pi} |r'(t)|\,dt$$

$$r'(t) = -\sin(t)\,\hat{i} + \cos(t)\,\hat{j} + \hat{k}$$

$$|r'(t)| = \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

Suppose the curve is described by the formula

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad\qquad a \le t \le b$$

Length can be evaluated by integrating the **speed**

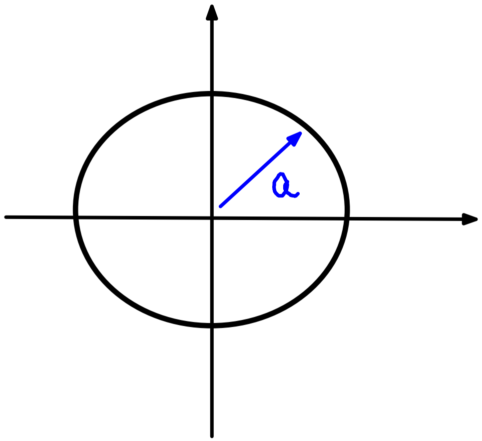
$$\begin{aligned}
\text{length} =: L &= \int_a^b \text{speed}\,\mathrm{d}t \\
&= \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,\mathrm{d}t
\end{aligned}$$

Q. Using this formula, verify that formula of circumference of a circle.

→ take a circle

→ take a parametric eqⁿ of circle

$$\vec{\gamma}(t) = a\cos t\,\hat{i} + a\sin(t)\,\hat{j}$$



→ find Length

$$L = \int_0^{2\pi} |\gamma'(t)|\,dt$$

$$= \int_0^{2\pi} a\,dt = 2\pi a$$

Find the length of the curve.

$$\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$$

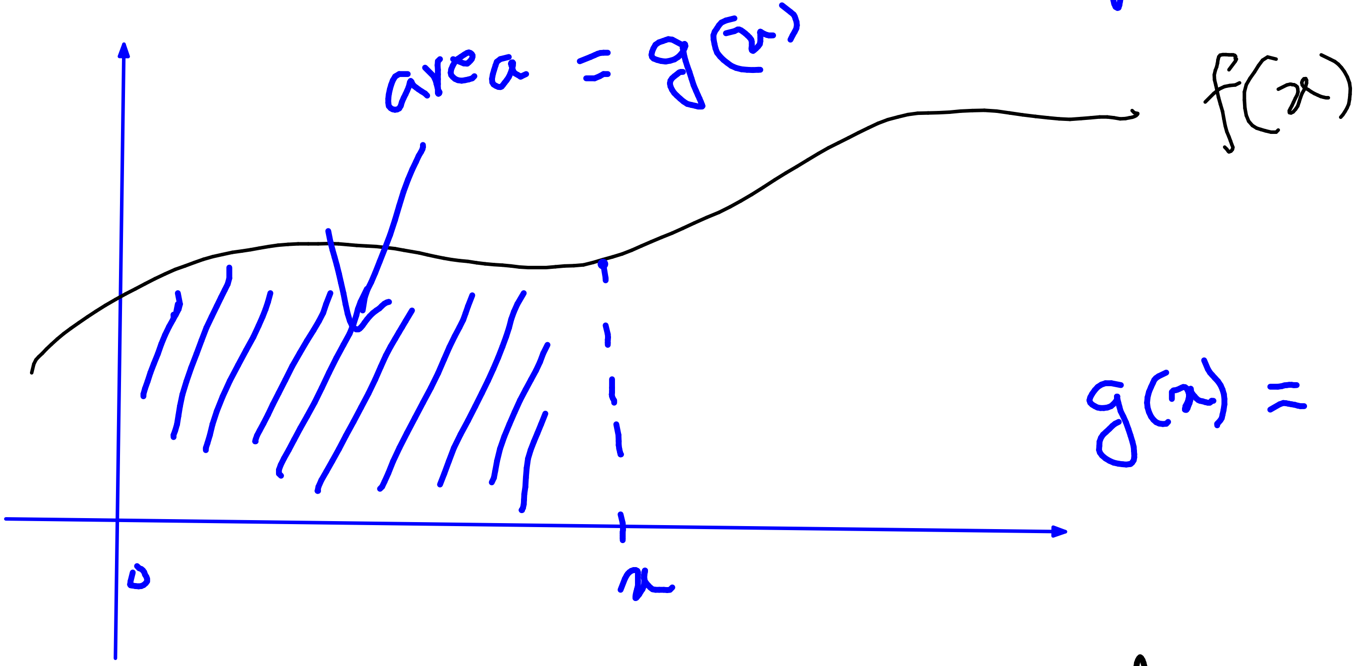
$$|r'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4+9t^2}$$

u - substitution

$$L = \int_0^1 t\sqrt{4+9t^2}\,dt \qquad t^2 = u \qquad du = 2t\,dt$$

$$= \frac{1}{2}\int_0^1 \sqrt{4+9u}\,du = \frac{1}{\cancel{2}}\left|\frac{(4+9u)^{3/2}}{\frac{3}{\cancel{2}}}\cdot\frac{1}{9}\right|_0^1 =$$

Fundamental Theorem of Calculus



$$g(x) = \int_0^x f(u)\,du$$

$$\frac{dg}{dx} = f(x)$$

arc length function s

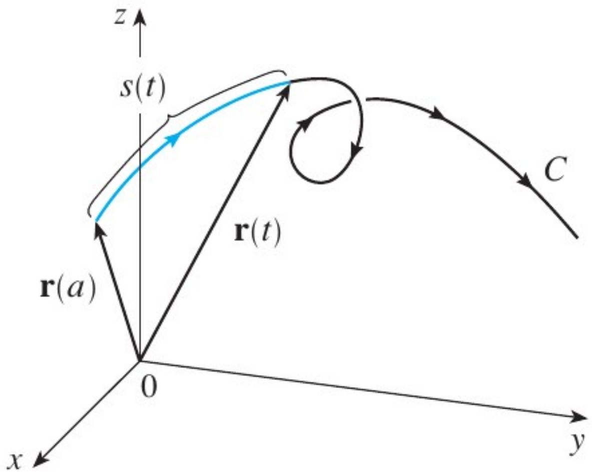
$a \le t \le b$

distance travelled till time t

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

why??



It is often useful to **parametrize a curve with respect to arc length**

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Basically, write $\vec{r}$ as a function of s

either $t = f(s)$

or $s = f(t)$

s = distance travelled in $0 \le u \le t$

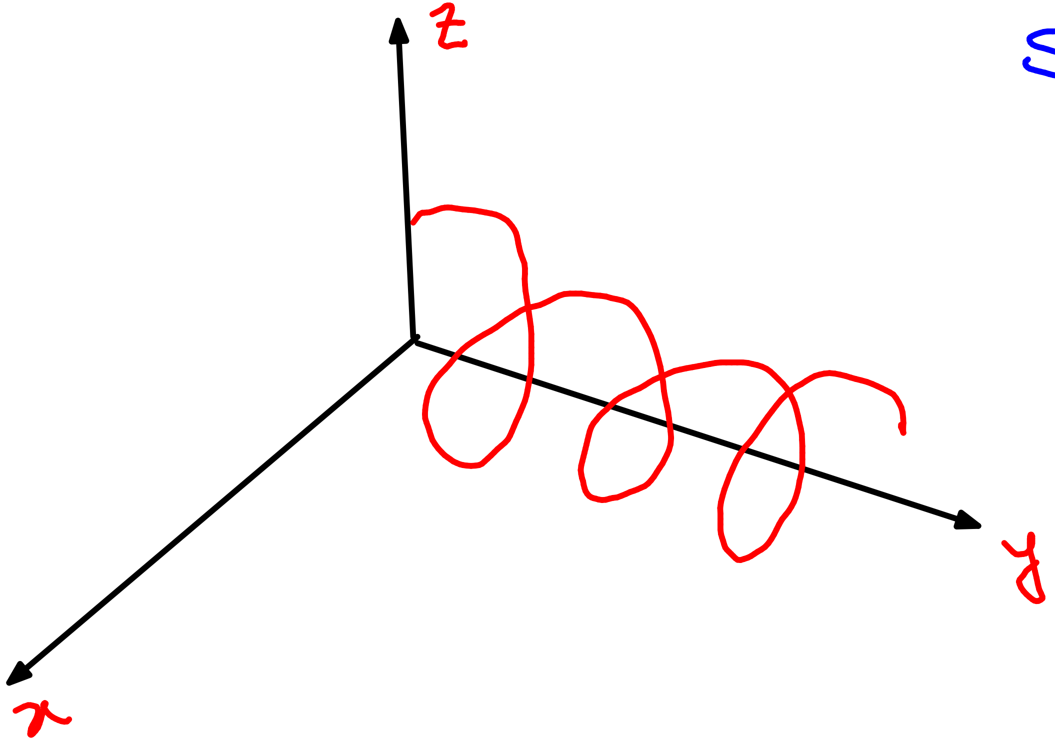
$$= \int_0^t (\text{speed})\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

$$\vec{r}(s) = \cos\left(s/\sqrt{2}\right)\hat{i} + \sin\left(s/\sqrt{2}\right)\hat{j} + s/\sqrt{2}\,\hat{k}$$

$$0 \le s < \infty$$

$\rightarrow t = 0$

9. Suppose you start at the point (0, 0, 3) and move 5 units along the curve $x = 3 \sin t$, $y = 4t$, $z = 3 \cos t$ in the positive direction. Where are you now?



$s = 5$, find t

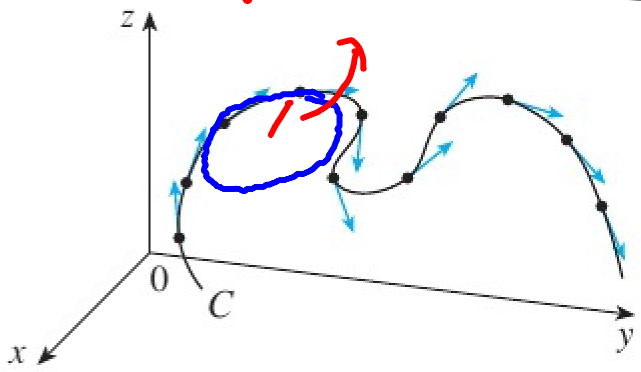
$$5 = \int_0^t (5)\, du$$

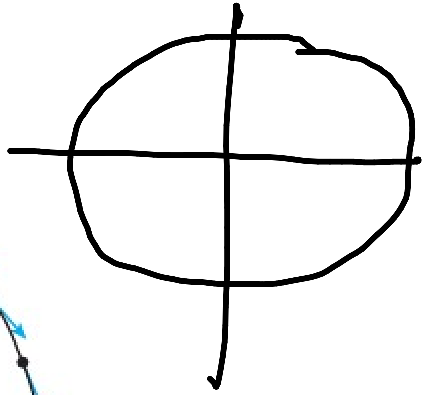
$$\Rightarrow \boxed{t = 1}$$

Ans.

CURVATURE

radius $= \frac{1}{\kappa}$





$(\cos(t), \sin(t))$ $\quad 0 \le t \le 2\pi$

$(\cos(2\pi t), \sin(2\pi t))$ $\quad 0 \le t \le 1$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

unit tangent vectors

8 DEFINITION The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where **T** is the unit tangent vector.

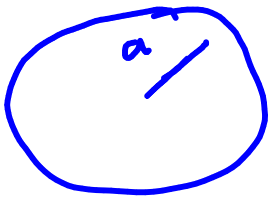
10 THEOREM The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

working formula

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$



$$\vec{r}(t) = a\cos(t)\,\hat{i} + a\sin(t)\,\hat{j}$$

$$\left.\begin{aligned} r'(t) &= -a\sin(t)\,\hat{i} + a\cos(t)\,\hat{j} \end{aligned}\right\} \; |r'(t)| = a$$

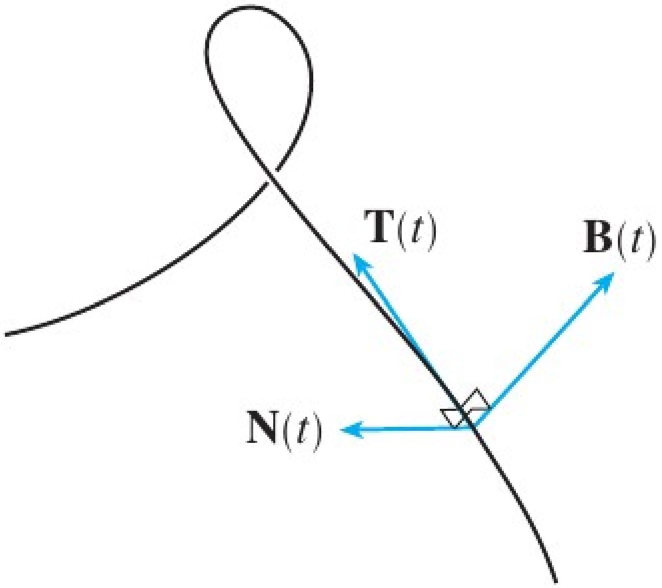
$$r''(t) = -a\cos(t)\,\hat{i} - a\sin(t)\,\hat{j}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2\hat{k}$$

$$\Rightarrow \quad \kappa = a^2/a^3 = \frac{1}{a}$$

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

THE NORMAL AND BINORMAL VECTORS



$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

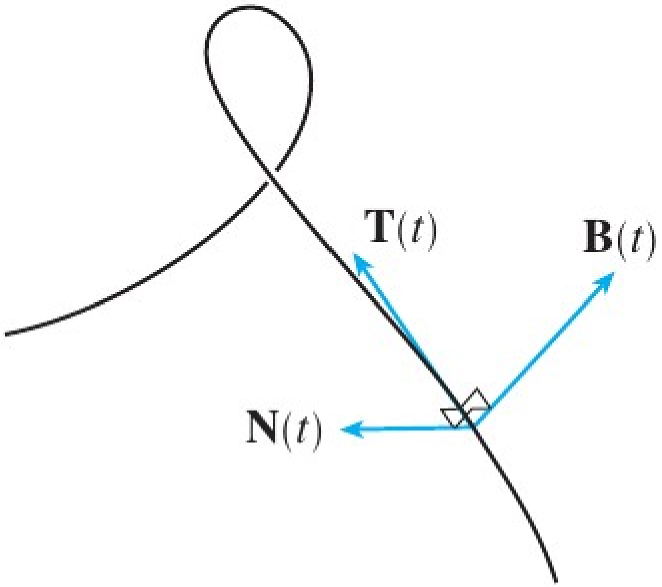
principal unit normal vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

binormal vector

The plane determined by the normal and binormal vectors **N** and **B** at a point *P* on a curve *C* is called the **normal plane** of *C* at *P*. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the **osculating plane** of *C* at *P*. The name comes from the Latin *osculum*, meaning "kiss."

THE NORMAL AND BINORMAL VECTORS



$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

principal unit normal vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

binormal vector

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**)

40. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$. Graph both osculating circles and the parabola on the same screen.