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ADVANCED ENGINEERING MATHEMATICS

Ordinary Differential Equations (ODE)

in ODE the unknown(s) are functions

e.g. find $y(x)$ s.t. $\frac{d^2y}{dx^2} = \sin(x)$ an ODE

e.g. find $y(x)$ s.t. $(y(x))^2 + \sin(x) = 0$

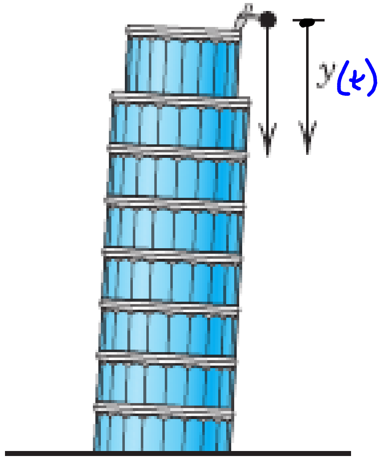
not an ODE

Equation in which we have a term which contains a derivative of the "unknown" is a differential equation

$F = ma$

$mg = my''$

$\boxed{y'' = g}$

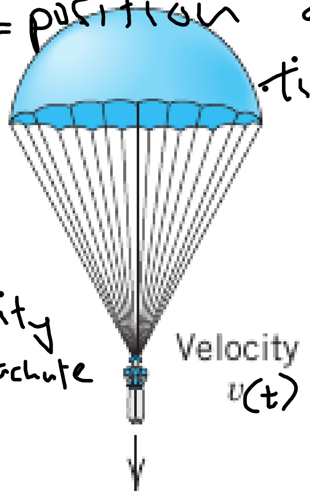


Falling stone

$y'' = g = const.$
(Sec. 1.1)

$y(t)$ = position of time t

$v(t)$ = velocity of parachute

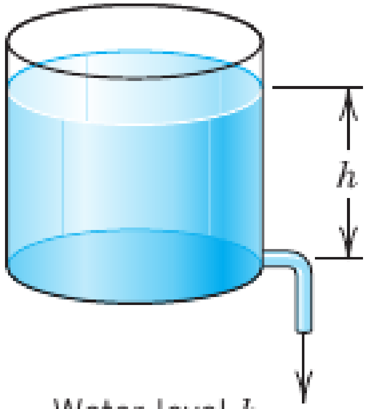


Parachutist

$$\boxed{mv' = mg - bv^2}$$

(Sec. 1.2)

$$my'' = mg - b(y(t))^2$$

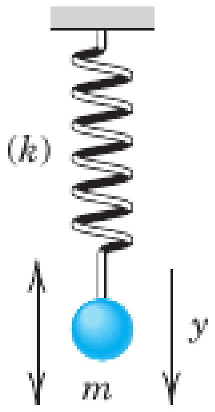


Water level h

Outflowing water

$$\boxed{h' = -k\sqrt{h}}$$

(Sec. 1.3)

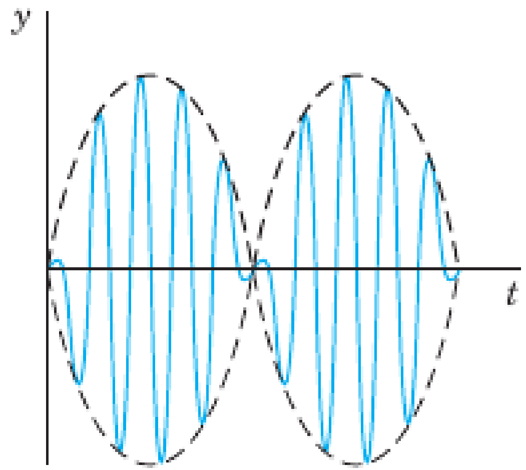


Displacement y

Vibrating mass on a spring

$my'' + ky = 0$

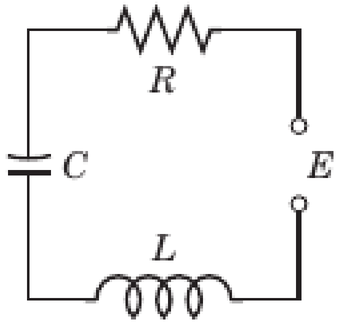
(Secs. 2.4, 2.8)



Beats of a vibrating system

$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 = \omega$

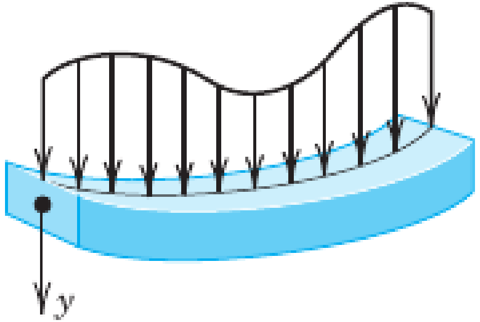
(Sec. 2.8)



Current I in an RLC circuit

$\boxed{LI'' + RI' + \frac{1}{C} I = E'}$

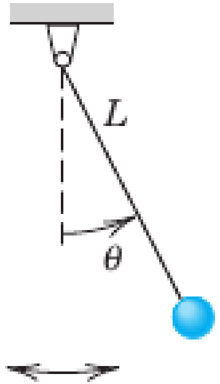
(Sec. 2.9)



Deformation of a beam

$$EIy^{iv} = f(x)$$

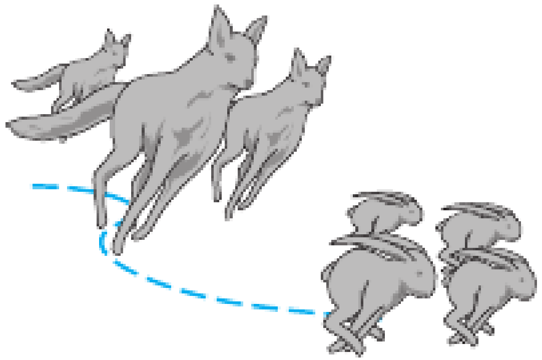
(Sec. 3.3)



Pendulum

$$L\theta'' + g \sin\theta = 0$$

(Sec. 4.5)



Lotka–Volterra
predator–prey model

$$y_1' = ay_1 - by_1y_2$$
$$y_2' = ky_1y_2 - ly_2$$

(Sec. 4.5)

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(1) order 1 $$y' = \cos x$$

(2) order 2 $$y'' + 9y = e^{-2x}$$

(3) order 3 $$\underline{y'y'''} - \tfrac{3}{2}y'^2 = 0$$

$$\cancel{y^2 + \frac{d}{dx}(\sin(x))} = 0$$

not an ODE

Q. $$y' + (y'')^3 + y''' = 5$$ order 3

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

$$F(x, y, y') = 0 \tag{4}$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

(a) Verify that y is a solution of the ODE. **(b)** Determine from y the particular solution of the IVP. **(c)** Graph the solution of the IVP.

$$y' + 4y = 1.4, \quad y = ce^{-4x} + 0.35, \quad \underline{y(0) = 2}$$

IVP: Initial Value Problem:

a) plug in $y = ce^{-4x} + 0.35$ in $y' + 4y = 1.4$

verify LHS = RHS

$$\begin{aligned} LHS &= y' + 4y \\ &= -4\cancel{ce^{-4x}} + 4(\cancel{ce^{-4x}} + 0.35) \\ &= 1.4 \end{aligned}$$

$y = ce^{-4x} + 0.35$ is a "general solution"

b) Particular Solution: $y(x)$ must satisfy ODE + extra condition given

find $y(x)$ which solves $y' + 4y = 1.4$ & $y(0) = 2$

$$y = ce^{-4x} + 0.35$$

$$y(0) = 2$$

$$ce^{-4(0)} + 0.35 = 2$$

$$c = 1.65$$

Particular Solution

$$y = 1.65e^{-4x} + 0.35$$

(a) Verify that y is a solution of the ODE. **(b)** Determine from y the particular solution of the IVP. **(c)** Graph the solution of the IVP.

general sol^n

$$y' = y - y^2, \quad \boxed{y = \frac{1}{1 + ce^{-x}},} \quad y(0) = 0.25$$

a) $y' = \dfrac{ce^{-x}}{(1+ce^{-x})^2} = \text{LHS}$

$y - y^2 = \dfrac{\cancel{1}+ce^{-x} - 1}{(1+ce^{-x})^2} = \dfrac{ce^{-x}}{(1+ce^{-x})^2} = \text{RHS}$

Particular Sol^n

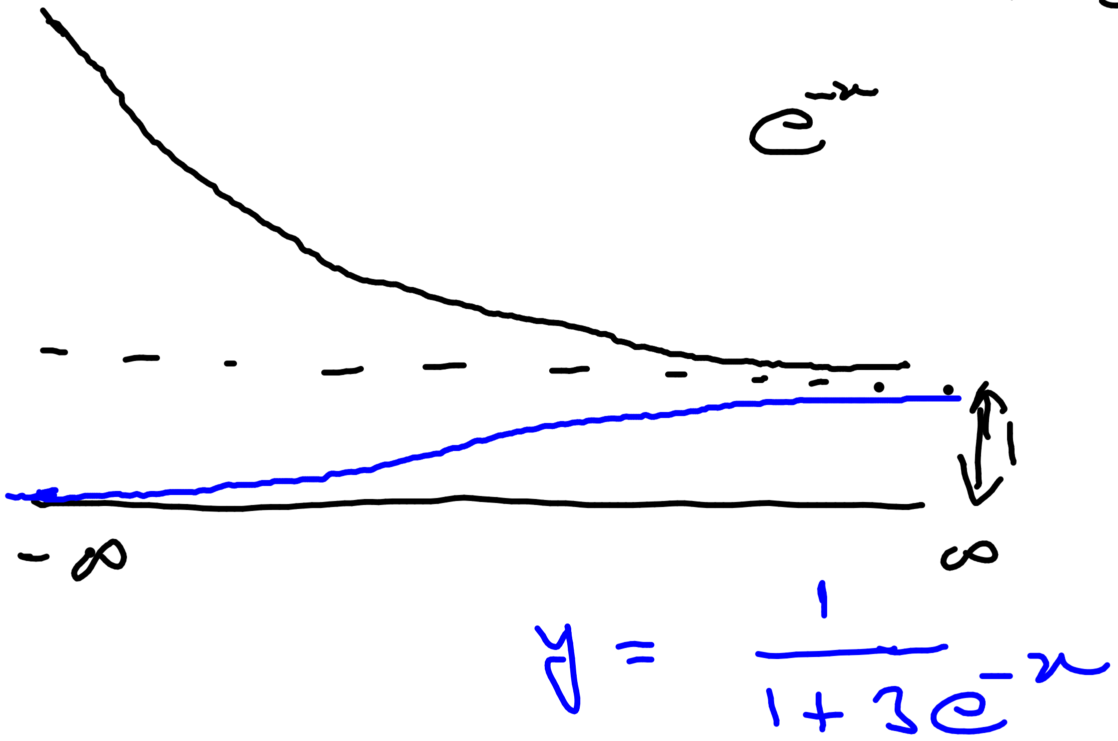
$y(0) = 0.25$

$\dfrac{1}{1+c} = 0.25$

$c = 3$

$y(x) = \dfrac{1}{1+3e^{-x}}$

c) Graph of $y(x) = \dfrac{1}{1+3e^{-x}}$



Solve the ODE by integration or by remembering a differentiation formula.

$$\boxed{y' + 2 \sin 2\pi x = 0}$$

$$y' = -2 \sin 2\pi x$$

$$\boxed{y(x) = \frac{\cos(2\pi x)}{\pi} + C}$$

general solution

Solve the ODE by integration or by remembering a differentiation formula.

$$y' = -1.5y$$

try $y(x) = e^{-1.5x}$

general solution: $y(x) = ce^{-1.5x}$

Solve the ODE by integration or by remembering a differentiation formula.

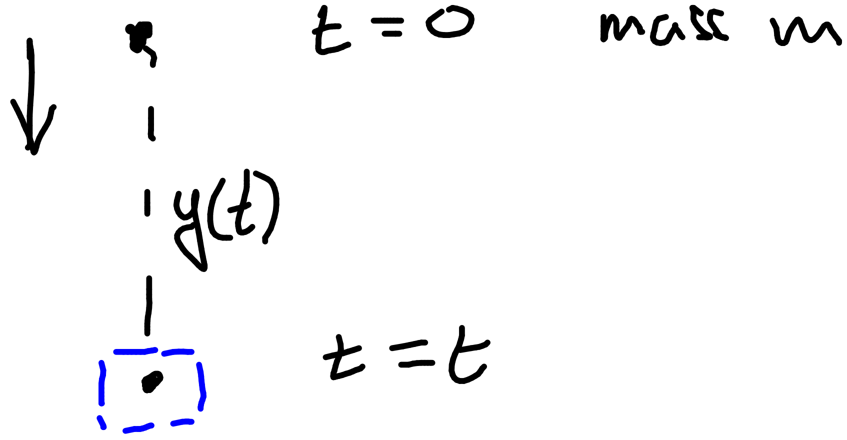
$$y'' = -y$$

$y(x) = c_1 \sin(x) + c_2 \cos(x)$

general sol^n :

19. Free fall. In dropping a stone or an iron ball, air resistance is practically negligible. Experiments show that the acceleration of the motion is constant (equal to $g = 9.80\ \mathrm{m/sec^2} = 32\ \mathrm{ft/sec^2}$, called the **acceleration of gravity**). Model this as an ODE for $y(t)$, the distance fallen as a function of time t. If the motion starts at time $t = 0$ from rest (i.e., with velocity $v = y' = 0$), show that you obtain the familiar law of free fall

$$y = \tfrac{1}{2} g t^2.$$



$m\,a = F$

ODE which solves $y(t)$

$$\boxed{m y'' = mg, \quad y(0) = 0, \quad y'(0) = 0}$$

$\rightarrow \quad \frac{d^2y}{dt^2} = g$

$$\left[\frac{dy}{dt} = gt + c_1 \right.$$

$$y(t) = \frac{1}{2}gt^2 + c_1 t + c_2 \quad \Big] \text{ general solution}$$

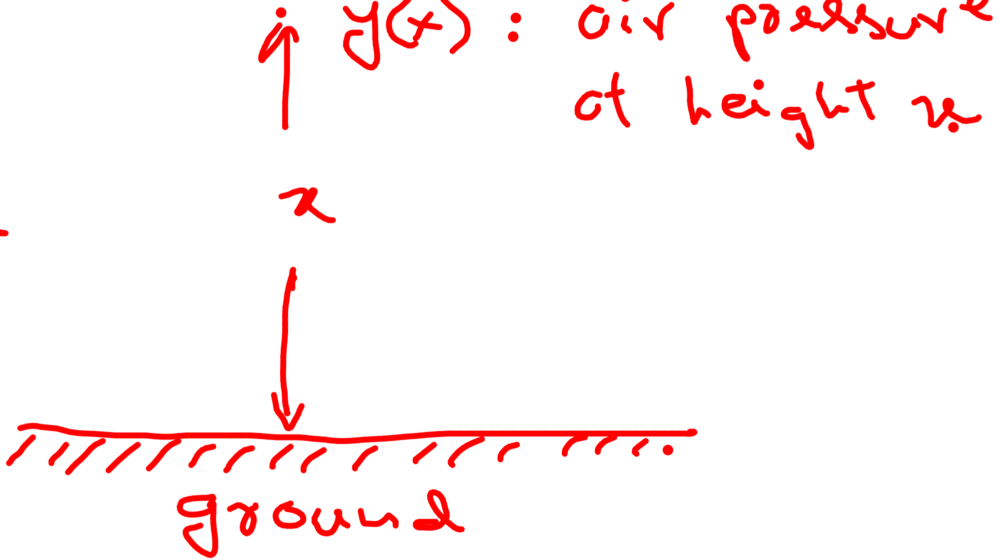
find c_1 & c_2 using $y(0) = 0$ & $y'(0) = 0$

$0 = c_2$ | $y'(0) = 0$, $g(0) + c_1 = 0$ | $c_1 = 0$

Particular solution

$$\boxed{y(t) = \frac{1}{2}gt^2}$$

20. Exponential decay. Subsonic flight. The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near 35,000 ft. Find the air pressure $y(x)$ at this height. *Physical information*. The rate of change $y'(x)$ is proportional to the pressure. At 18,000 ft it is half its value $y_0 = y(0)$ at sea level. *Hint*. Remember from calculus that if $y = e^{kx}$, then $y' = ke^{kx} = ky$. Can you see without calculation that the answer should be close to $y_0/4$?



$$\frac{dy}{dx} = ky \qquad y(0) = y_0 \qquad y(18000) = \frac{1}{2} y_0$$

$$y(x) = C e^{kx}$$

use $y(0) = y_0$ to find C

$$y(x) = y_0 e^{kx}$$

We can find k, using $y(18000) = \frac{1}{2} y_0$

$$\cancel{y_0} e^{k(18000)} = \frac{1}{2} \cancel{y_0}$$

$$k = \frac{-\ln(2)}{18000} = -3.85 \times 10^{-5}$$

Air pressure at height x

$$y(x) = y_0 e^{kx}, \text{ where } k = -3.85 \times 10^{-5}$$

Pressure at height 35000 ft

$y(35000)$

1.3 Separable ODEs. Modeling

Skipping 1.2

d. Solve by separating variables

$$y' = 1 + y^2$$

$$\frac{dy}{dx} = 1+y^2$$

$\frac{1}{1+y^2} dy = dx$ & integrate both sides

$$\int \frac{1}{1+y^2} dy = \int dx$$

$$\tan^{-1}(y) = x + c$$

$$y = \tan(x+c)$$

Verify - easy

Q. Solve using separation of variables

$$y' = (x+1)e^{-x}y^2$$

$$\frac{dy}{dx} = (x+1)e^{-x}y^2$$

$$\frac{1}{y^2}\,dy = (x+1)e^{-x}\,dx$$

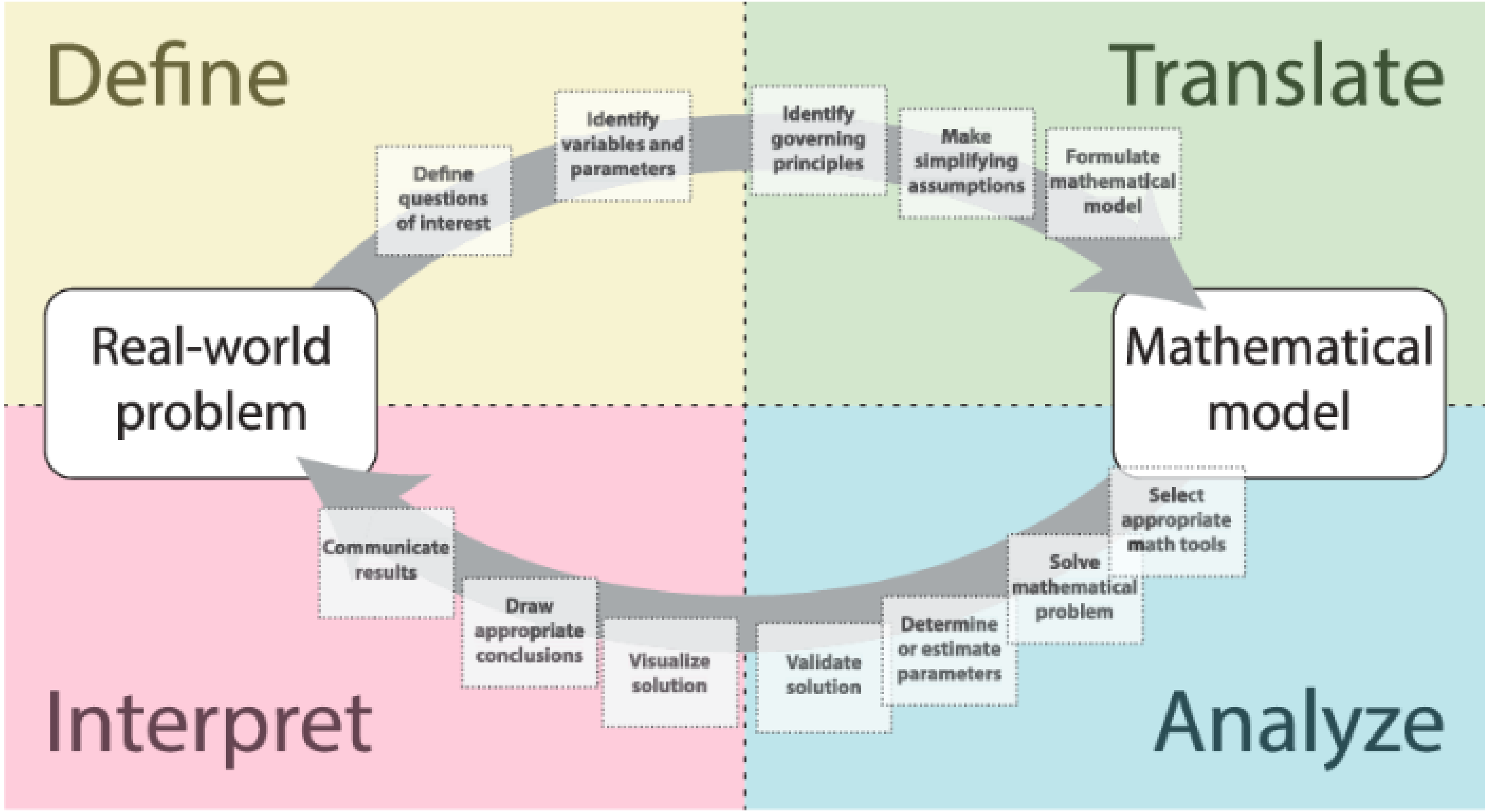
$$\int \frac{1}{y^2}\,dy = \int (x+1)e^{-x}\,dx$$

$$-\frac{1}{y} = -e^{-x} - (1+x)e^{-x} + c$$

$$y = \frac{1}{2e^{-x} + xe^{-x} + c}$$

verify this later

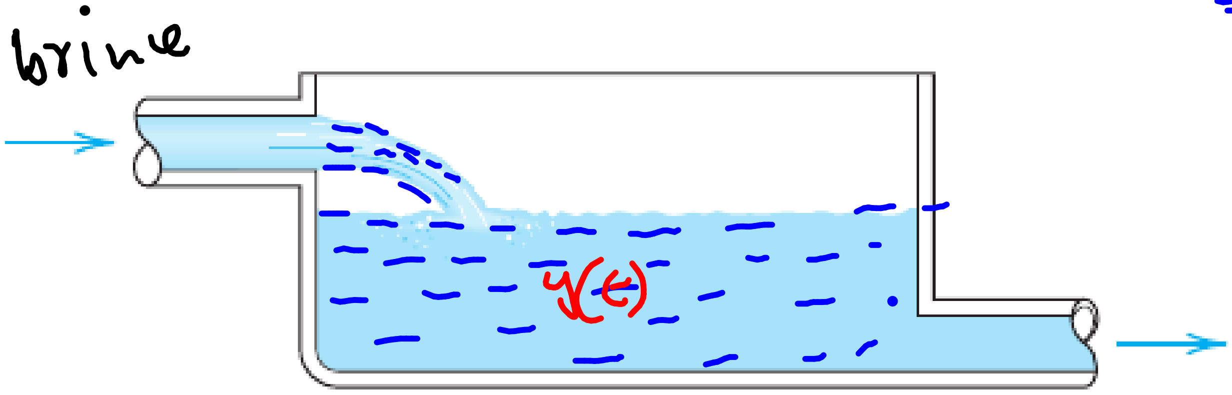
Modeling



researchgate.net/figure/Schematic-representation-of-the-mathematical-modeling-process-which-was-a-significant_fig1_283805990

EXAMPLE 5 **Mixing Problem**

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.



Aim: find a formula $y(t)$ which gives amount of salt in the tank at time t

given: $y(0) = 100$

rate of change of y $= \dfrac{dy}{dt}$ = inflow rate of salt − outflow rate of salt

$$\frac{dy}{dt} = 50 - \frac{10}{1000} y$$

mathematical model.

$$\boxed{\begin{array}{l} \dfrac{dy}{dt} = 50 - \dfrac{y}{100} \\ y(0) = 100 \end{array}}$$

Solve :

$$\frac{dy}{dt} = 50 - \frac{y}{100}$$

$$\int \frac{100}{5000 - y} dy = \int dt$$

$$-100 \ln(5000 - y) = t + C$$

$t = 0, \quad y = 100$

$$-100 \ln(4900) = C$$

$$-100 \ln(5000-y) = t - 100 \ln(4900)$$

$$\ln(5000-y) = -\frac{t}{100} + \ln(4900)$$

$$5000 - y = 4900 e^{-t/100}$$

$$y = 5000 - 4900 e^{-t/100}$$

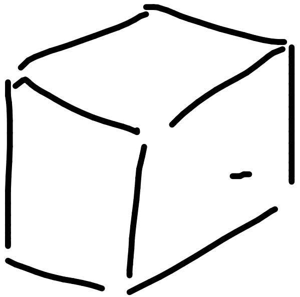
graph of $y(t)$??

EXAMPLE 6 Heating an Office Building (Newton's Law of Cooling[3])

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, for example, as a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium **(Newton's law of cooling).**

$T(t)$: Temperature at time t



T_0

$$\boxed{\frac{dT}{dt} = k(T - T_0)}$$

given

T_0 : outside temperature

Extended Method: Reduction to Separable Form

$$y' = f\left(\frac{y}{x}\right).$$

try changing variables

$v = \frac{y}{x}$

Eliminate y from the ODE

=

$y = v(x)\, x$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

Eg.

$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)$

$v + x\frac{dv}{dx} = v^2 - v$

– separable?

Yes

$$x\frac{dv}{dx} = v^2 - 2v$$

$$\frac{1}{v^2-2v}\,dv = \frac{1}{x}\,dx$$

$$\frac{1}{v(v-2)} = \frac{1}{2}\left[\frac{1}{v-2} - \frac{1}{v}\right]$$

$$\frac{1}{2}\left[\frac{1}{v-2} - \frac{1}{v}\right]dv = \ln x$$

$$\ln\frac{v-2}{v} = 2\ln x + \ln(c)$$

$$\ln\left(\frac{v-2}{v}\right) = \ln\left(cx^2\right)$$

$$\frac{v-2}{v} = cx^2$$

$$\frac{\frac{y}{x}-2}{\frac{y}{x}} = cx^2$$

$$\frac{y-2x}{y} = cx^2$$

$$\boxed{y = \frac{2x}{1-cx^2}}$$

EXAMPLE 8 **Reduction to Separable Form**

Ans $y^2 - x^2 = \frac{x}{c}$

$$2xyy' = y^2 - x^2$$

$v = \frac{y}{x}$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$y' = \frac{y^2 - x^2}{2xy}$

$y' = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)}$

$x\frac{dv}{dx} + v = \frac{v^2 - 1}{2v}$

$x\frac{dv}{dx} = -\frac{(1+v^2)}{2v}$

$\int \frac{2v}{1+v^2}dv = \frac{1}{x}dx$

$-\ln(1+v^2) = \ln(xc)$

$1 + v^2 = \frac{1}{xc}$

$$1+\frac{y^2}{x^2}=\frac{1}{xc}$$

$$x^2+y^2=\frac{x}{c}$$

$$\boxed{x^2+y^2=cx}$$

y is defined implicitly in this eqⁿ!

17. $xy' = y + 3x^4\cos^2(y/x), \quad y(1) = 0$

$$x\frac{dy}{dx} = y + 3x^4\cos^2\left(\frac{y}{x}\right)$$

Try $v = \dfrac{y}{x}$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\cancel{x}\frac{dv}{dx} + \cancel{v} = \cancel{v} + 3x^{\cancel{3}\,2}\cos^2(v)$$

$$\sec^2(v)\,dv = 3x^2\,dx$$

$$\tan(v) = x^3 + c$$

General Solution

$$\tan\left(\frac{y}{x}\right) = x^3 + c$$

Particular solution

$x = 1, \quad y = 0$

$0 = 1 + c \quad | \quad c = -1$

Solution: $y(x)$ defined implicitly in the eq^n

$$\tan\left(\frac{y}{x}\right) = x^3 - 1$$

1.4 Exact ODEs. Integrating Factors

$$M(x, y) + N(x, y)y' = 0.$$

Hope: find a formula $u(x,y)$ s.t.

$$\frac{du}{dx} = M + N y'$$

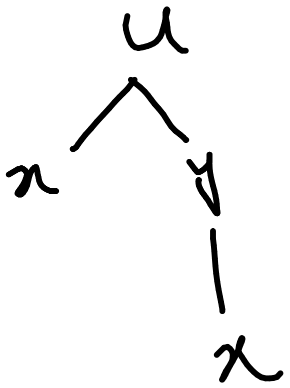
if such u is found, then y can be solved from the eqn

$$u(x,y) = C$$

1.4 Exact ODEs. Integrating Factors

$$M(x, y) + N(x, y)y' = 0.$$

$u(x, y)$



$y = y(x)$

$$\frac{du}{dx} = \underbrace{\frac{\partial u}{\partial x}}_{M} + \underbrace{\frac{\partial u}{\partial y}}_{N} y'$$

$M \qquad\qquad N$

Q: if $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \quad \Bigg| \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$$

Note. if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, There exist $u(x,y)$

s.t. $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

which make the $M + N y' = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y' = 0$$

i.e $\frac{d}{dx}(u) = 0$

$$\boxed{u = C}$$

solve for y from this eq^n

$$\underbrace{2xy}_{M}\,dx + \underbrace{x^2}_{N}\,dy = 0$$

$$M + N\frac{dy}{dx} = 0$$

$$\Leftrightarrow M\,dx + N\,dy = 0$$

Q: is it exact? $\quad \dfrac{\partial M}{\partial y} \overset{??}{=} \dfrac{\partial N}{\partial x}$

$$2x = 2x \quad \checkmark \quad \text{exact}$$

Q: find $u(x,y)$ s.t. $\quad \dfrac{\partial u}{\partial x} = M = 2xy \;\&\; \dfrac{\partial u}{\partial y} = N = x^2$

$$u = x^2 y$$

the ODE $2xy + x^2 y' = 0$

$$\Leftrightarrow \quad \frac{d}{dx}\left(x^2 y\right) = 0$$

$$\Leftrightarrow \quad x^2 y = c$$

$$\Rightarrow \quad \boxed{y = \frac{c}{x^2}}$$

Q.

$$x^3dx + y^3dy = 0$$

check for exactness.
Solve if it is.

$u = ?? = (x^4+y^4)/4$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$u = c$

$x^4+y^4 = C$

Q. Is it exact?

$$\underbrace{\sin x \cos y}_{M}\, dx + \underbrace{\cos x \sin y}_{N}\, dy = 0$$

check for exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$-\sin x \sin y \qquad -\sin x \sin y$] exact

$\rightarrow u = -\cos(x)\cos(y)$ works:

$\rightarrow$ ODE solⁿ: $\boxed{\cos(x)\cos(y) = C}$

Q: $M\,dx + N\,dy = 0$

suppose it is not exact. $\left[\text{i.e. } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right]$

→ multiply with an integration factor F

s.t. $FM\,dx + FN\,dy = 0$ becomes exact

→ How to find F ?? $$\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN)$$

to be continued next time.

$$2x \tan y \, dx + \sec^2 y \, dy = 0$$

$$e^x(\cos y \, dx - \sin y \, dy) = 0$$

1.5 Linear ODEs.

$$y' + p(x)y = r(x)$$

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

EXAMPLE 2 Electric Circuit

$$LI' + RI = E(t)$$

$$I(0) = 0$$