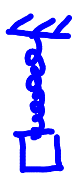
6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (*t*-Shifting)

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0).$$

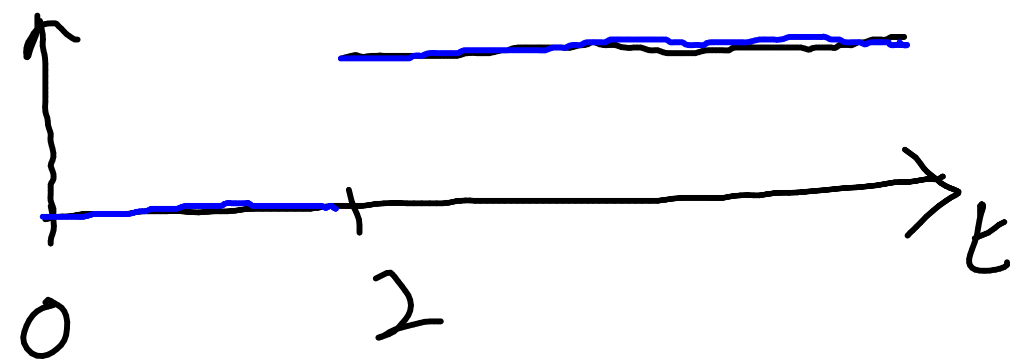


$my'' + cy' + ky = r(t)$

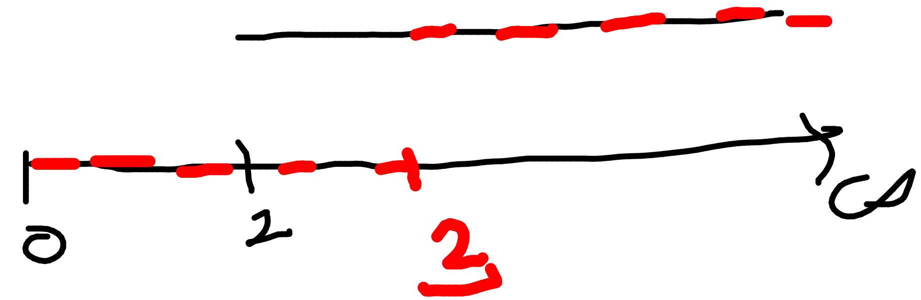
→ chapter 2
deals with only continuous $r(t)$

$r(t) = \sin(t)$

$$u(t-2) = \begin{cases} 0 & , t < 2 \\ 1 & , t > 2 \end{cases}$$

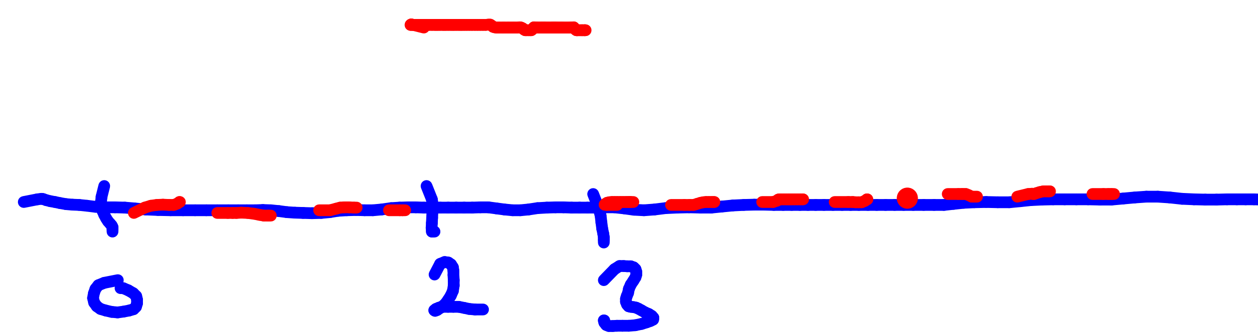


$u(t-2)$



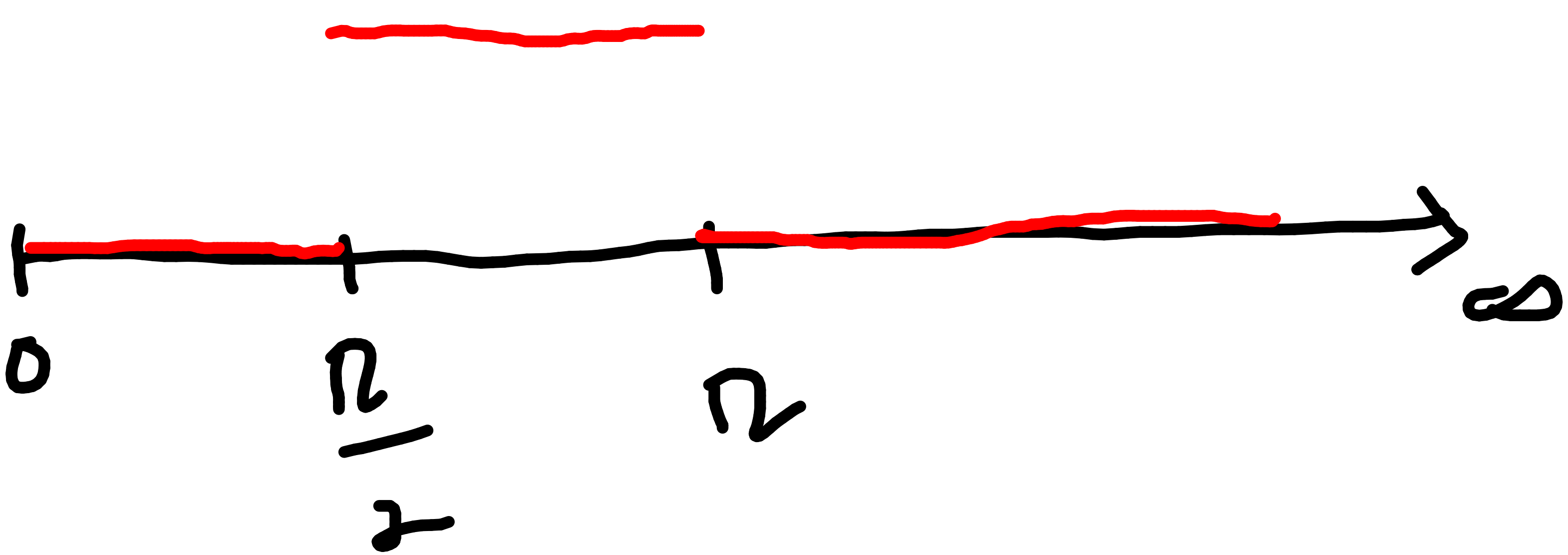
$u(t-3)$

$u(t-2) - u(t-3)$



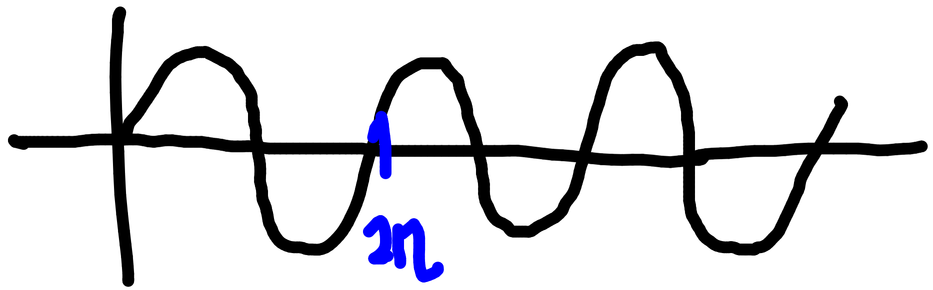
d. Create a switch which is on only in the interval $[\pi/2, \pi]$

$$u(t-\pi/2) - u(t-\pi)$$

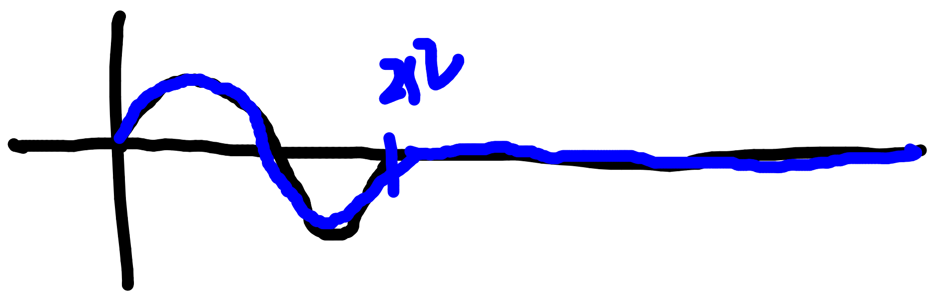


Q.

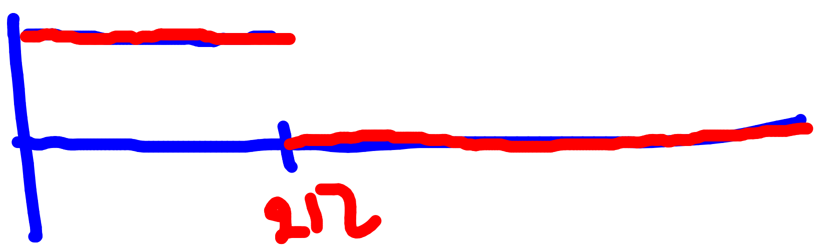
$\sin(t)$



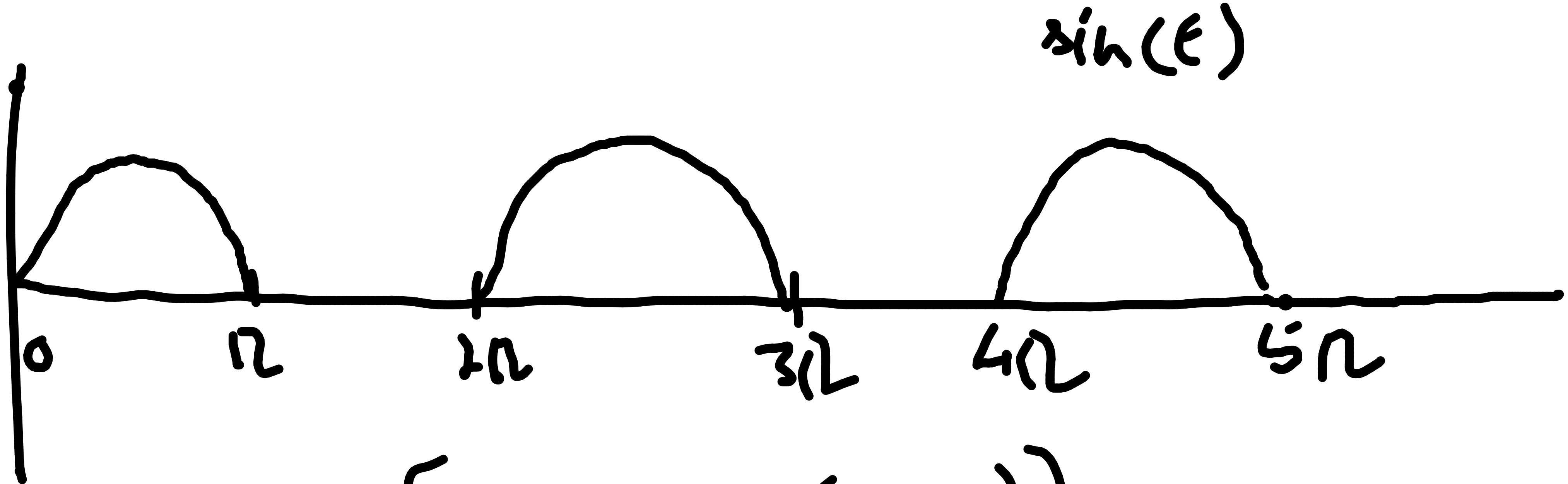
$\sin(t)\left[u(t) - u(t-2\pi)\right]$



$u(t) - u(t-2\pi)$



Q:



$$\sin(t)\left[u(t) - u(t-\pi)\right]$$
$$+ \sin(t)\left[u(t-2\pi) - u(t-3\pi)\right]$$
$$+ \left[\underline{\sin}(t)\left[u(t-4\pi) - u(t-5\pi)\right]\right]$$

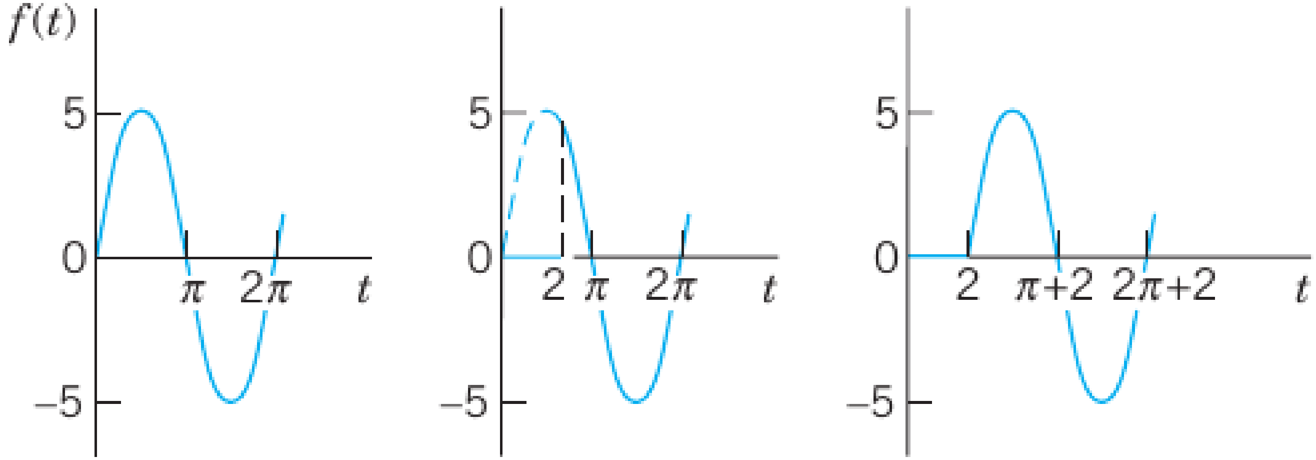
$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= -\frac{1}{s}\Big[\underbrace{0}_{\text{for } s>0} - e^{-sa} \Big] = \frac{e^{-as}}{s}$$

$$[s>0]$$



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin (\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

EXAMPLE 1

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \qquad t > \frac{1}{2}\pi. \end{cases}$$

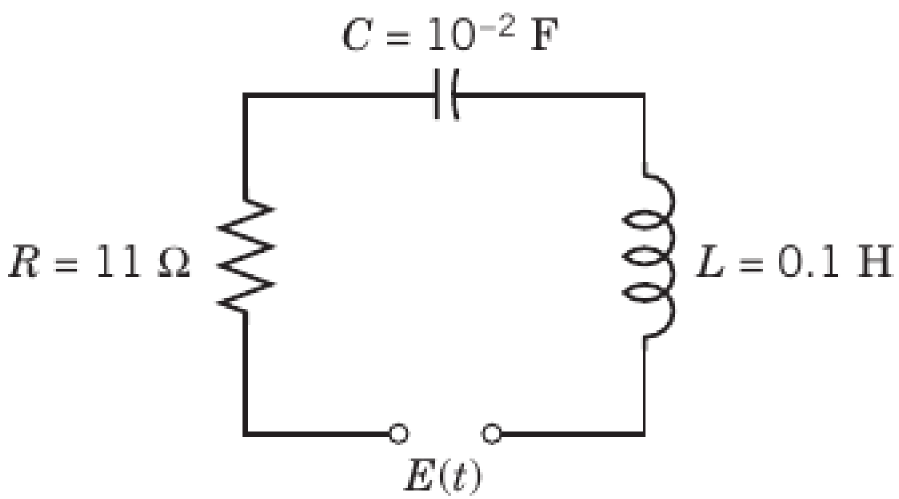
$$f(t) = 2\left(u(t) - u(t-1)\right)$$
$$+ \frac{1}{2}t^2\left[u(t-1) - u\left(t - \frac{\pi}{2}\right)\right]$$
$$+ \cos(t)\left[u(t-\pi/2)\right]$$

EXAMPLE 4

Find the response (the current) of the *RLC*-circuit in Fig. 125, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t) = 100 \sin 400t \quad \text{if } 0 < t < 2\pi \qquad \text{and} \qquad E(t) = 0 \text{ if } t > 2\pi$$

and current and charge are initially zero.



yourself

Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the "**shifted function**"

(3) $$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. *That is, if* $\mathcal{L}\{f(t)\} = F(s)$, *then*

(4) $$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).}$$

Or, if we take the inverse on both sides, we can write

(4*) $$\boxed{f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.}$$

say $f(t) = t$

$F(s) = \frac{1}{s^2}$

$a = 5$

$\tilde{f}(t) = (t-5)u(t-5)$

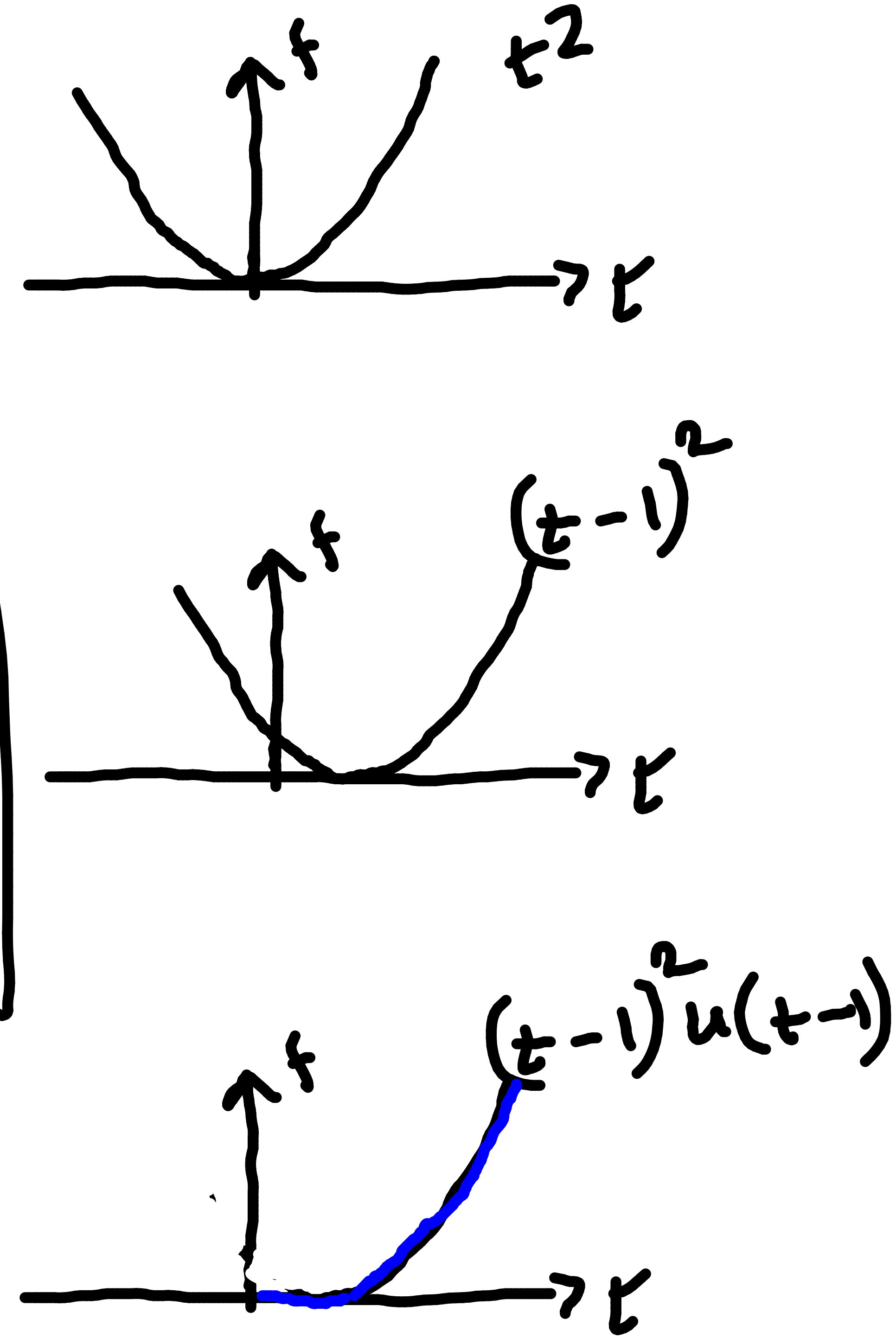
$\mathcal{L}(\tilde{f}) = \frac{e^{-5s}}{s^2}$

“shifted function”

$$\tilde{f}(t) = \underline{f(t-a)}u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$f(t) = t^2$, $a = 1$

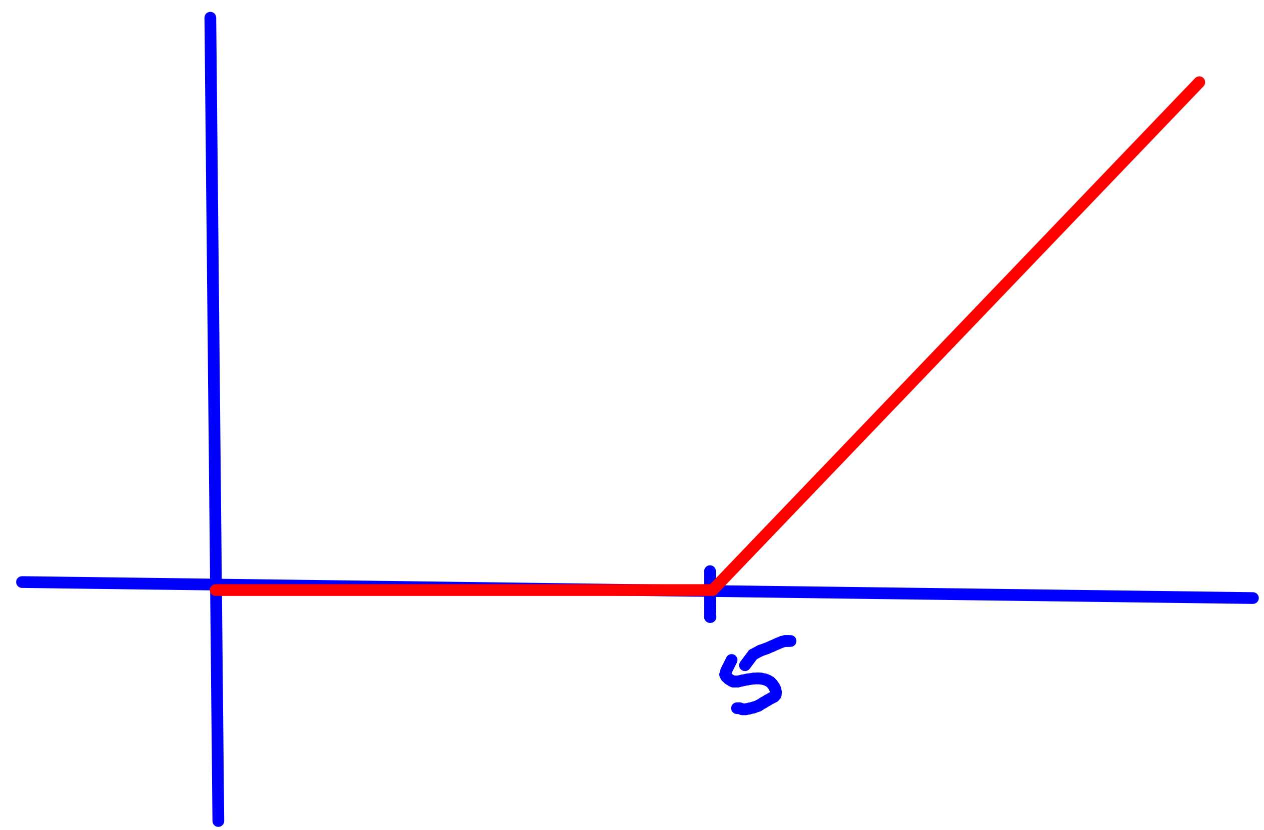
$\tilde{f}(t) = f(t-1)\,u(t-1)$



$$\tilde{f}(t) = f(t-a)\,u(t-a)$$

$f(t) = t$ $\qquad a = 5$

sketch $\tilde{f}(t)$



$$f(t) = \sin(t)$$

$$\boxed{\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}}$$

$$\mathcal{L}\left(\sin(t)\, u(t-2\pi)\right) = ??$$

$$= \mathcal{L}\left(\sin(t-2\pi)\, u(t-2\pi)\right)$$

$$= \frac{e^{-2\pi s}}{s^2+1}$$

INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

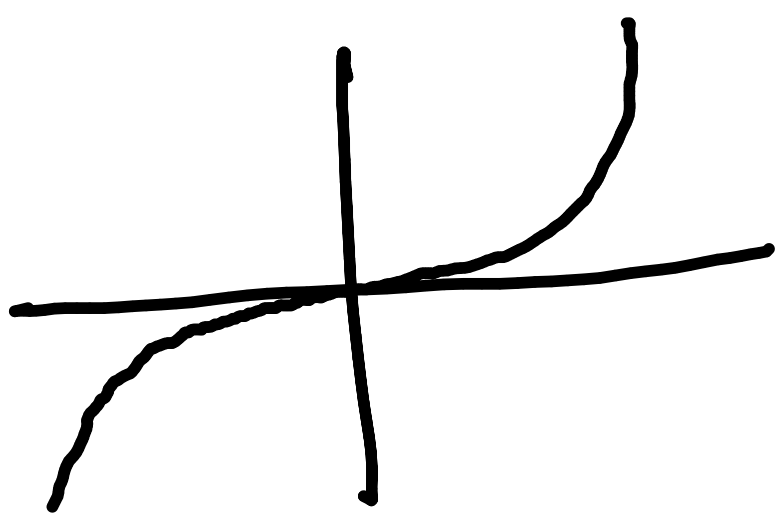
e^{-3s}/s^4

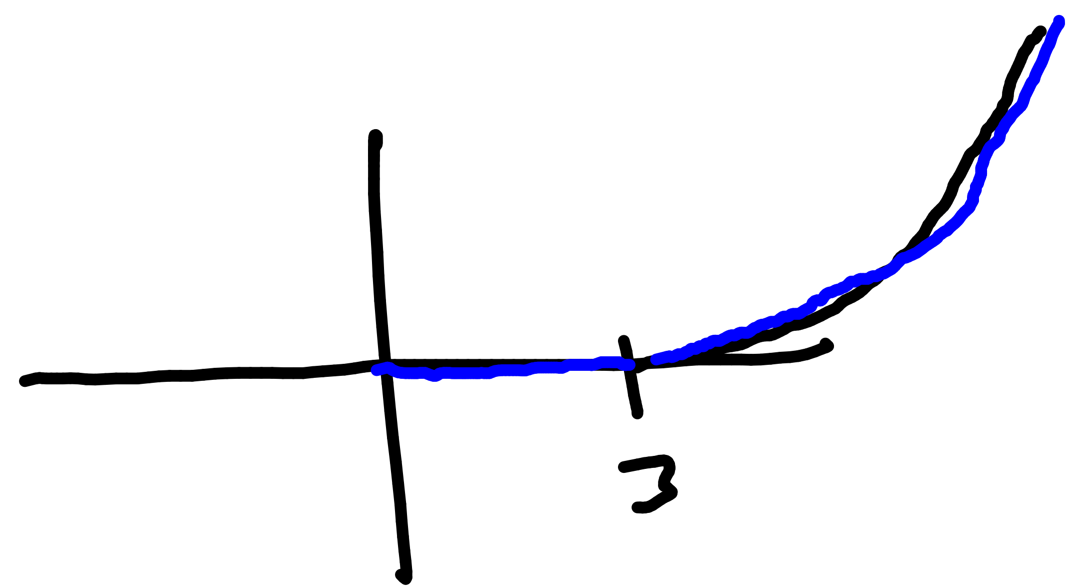
$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right)$$

$$= f(t-a)\,u(t-a)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = ?? = \frac{t^3}{6}$$

$$\mathcal{L}^{-1}\left(e^{-3s}\frac{1}{s^4}\right) = \frac{(t-3)^3}{6}u(t-3)$$





INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

$e^{-3s}/(s-1)^3$

INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

$6(1 - e^{-\pi s})/(s^2 + 9)$

Proof

Theorem:

if $\mathcal{L}^{-1}(F(s)) = f(t)$

then $\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau$

$\Leftrightarrow$ $\boxed{\frac{1}{s}F(s) = \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)}$

$$\boxed{g(t) = \int_0^t f(\tau)\,d\tau}$$

$$g'(t) = f(t)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$$

$$\mathcal{L}(f) = s\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) - 0$$

Recall

$$\mathcal{L}(f) = F(s)$$

$$F(s) = s\,\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

$$\frac{1}{s}F(s) = \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau$$