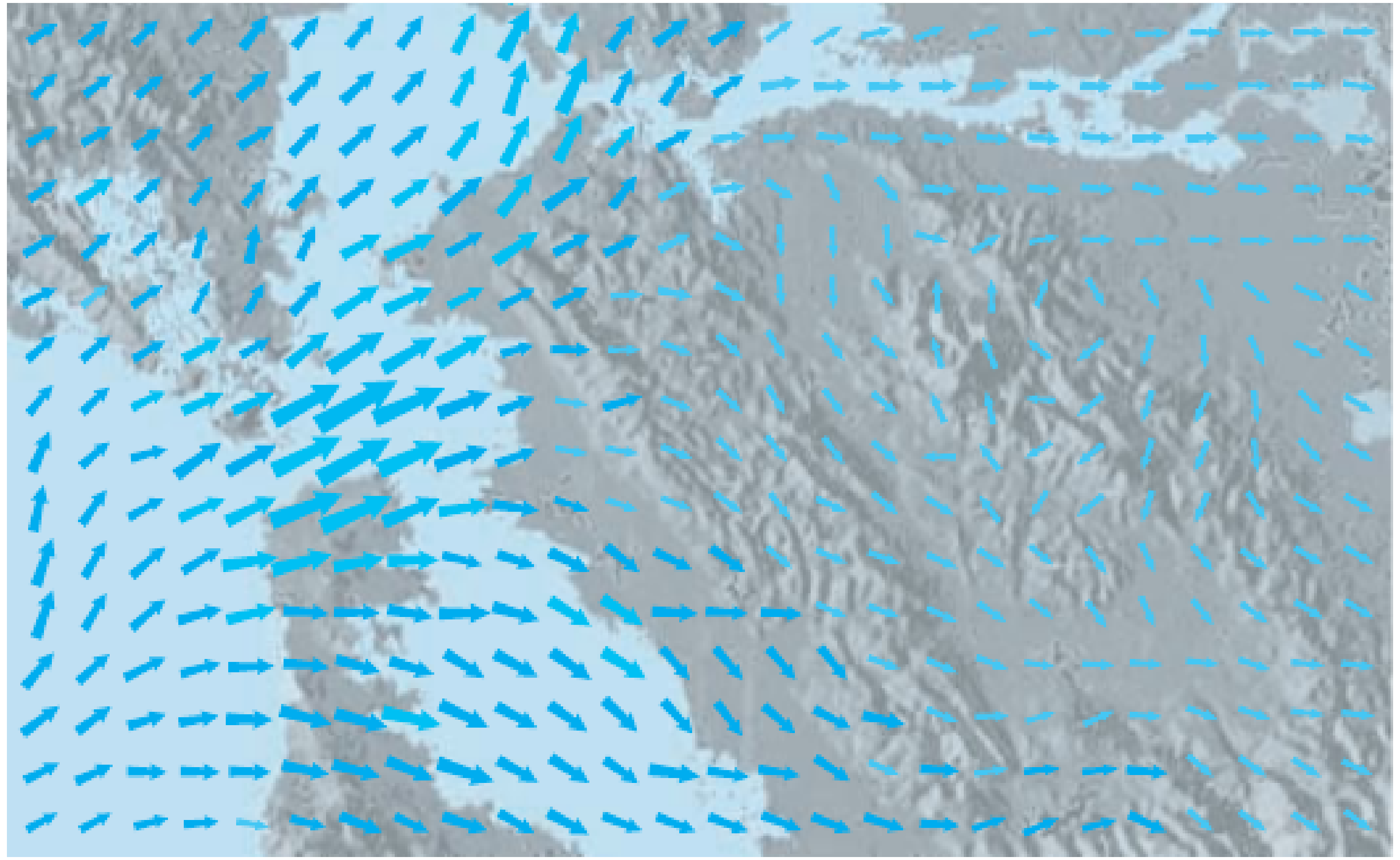
13 VECTOR CALCULUS

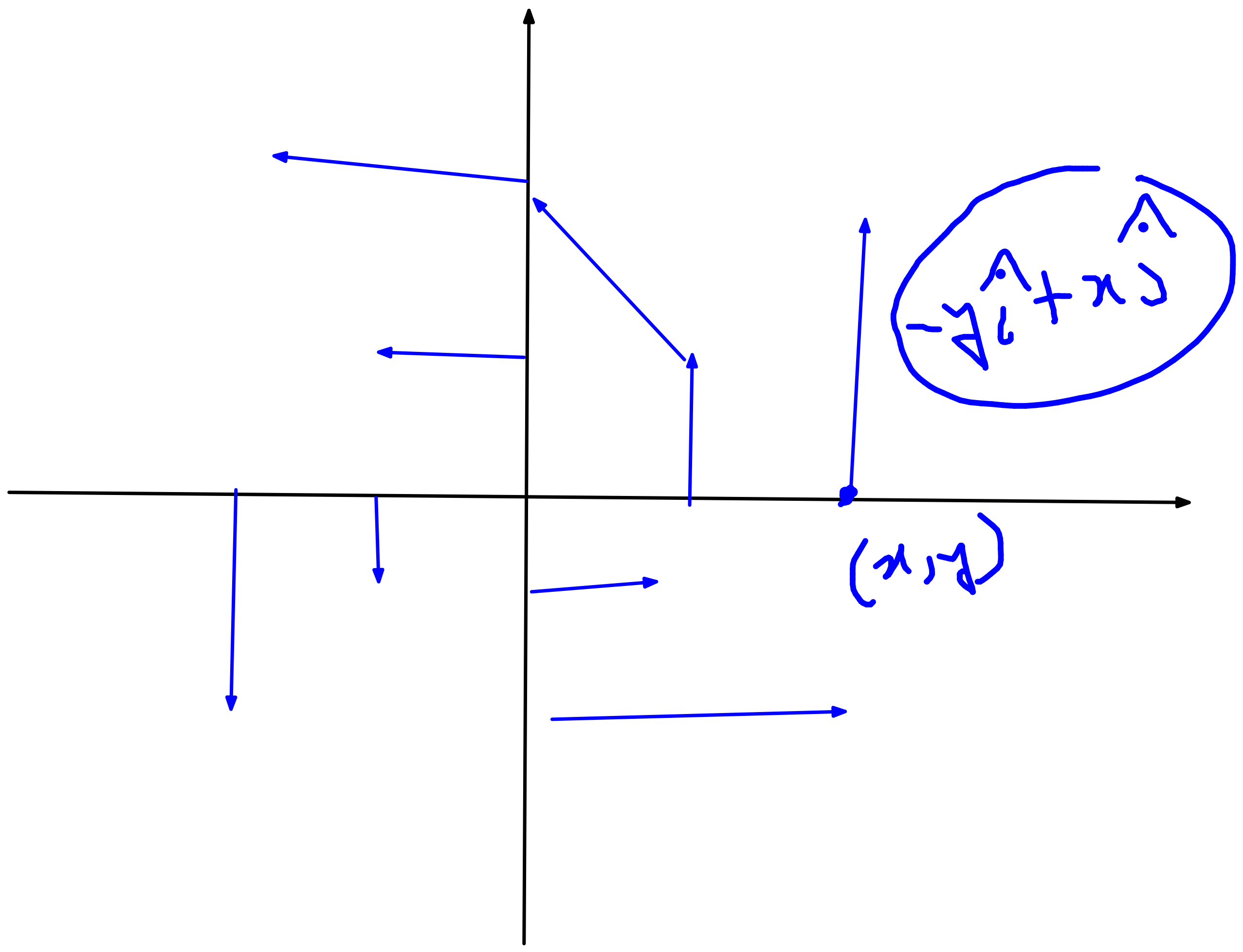
started last time

13.1 VECTOR FIELDS



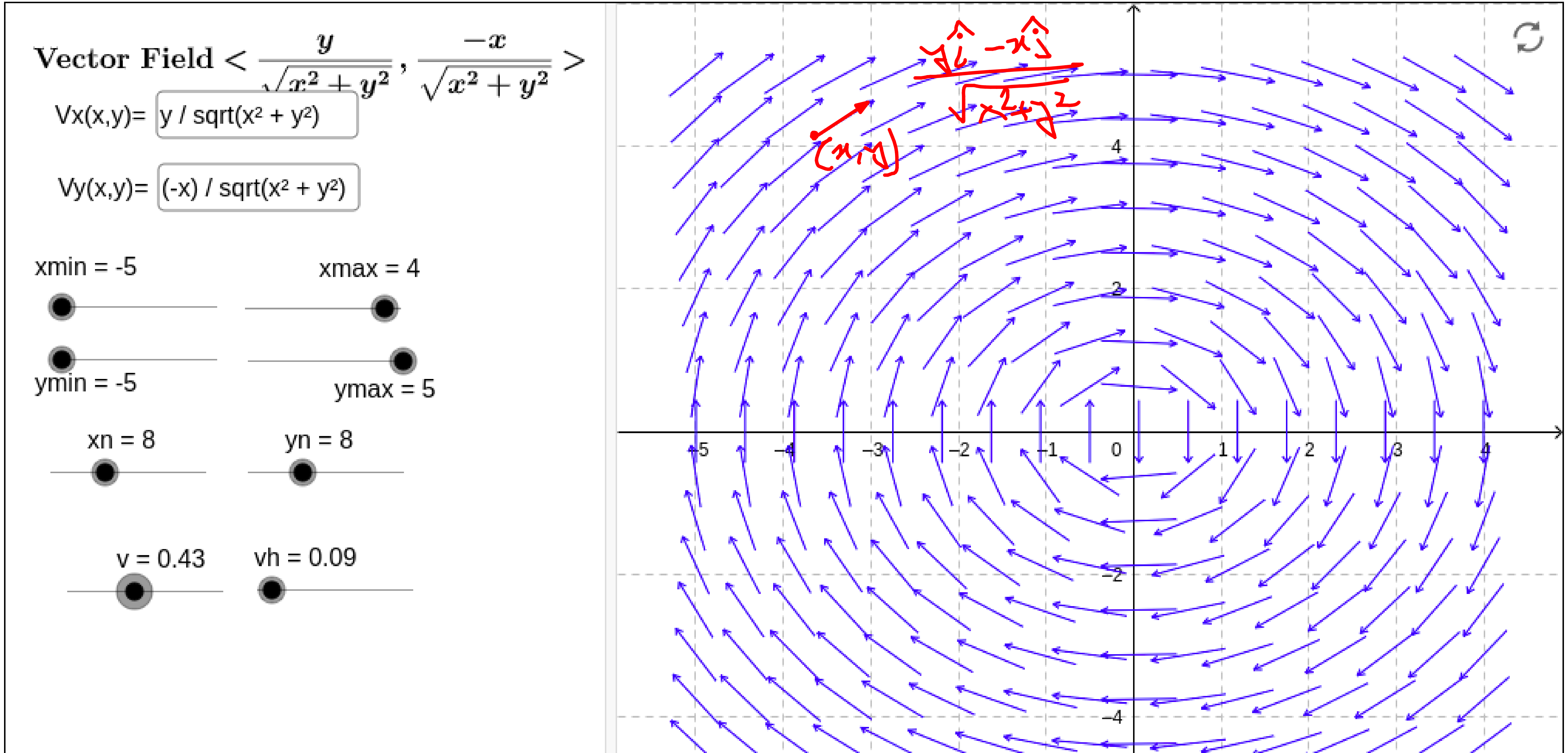
Sketch the vector field **F**

$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}.$$



Sketch the vector field $\mathbf{F}$ $\quad \mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

geogebra.org/m/QPE4PaDZ

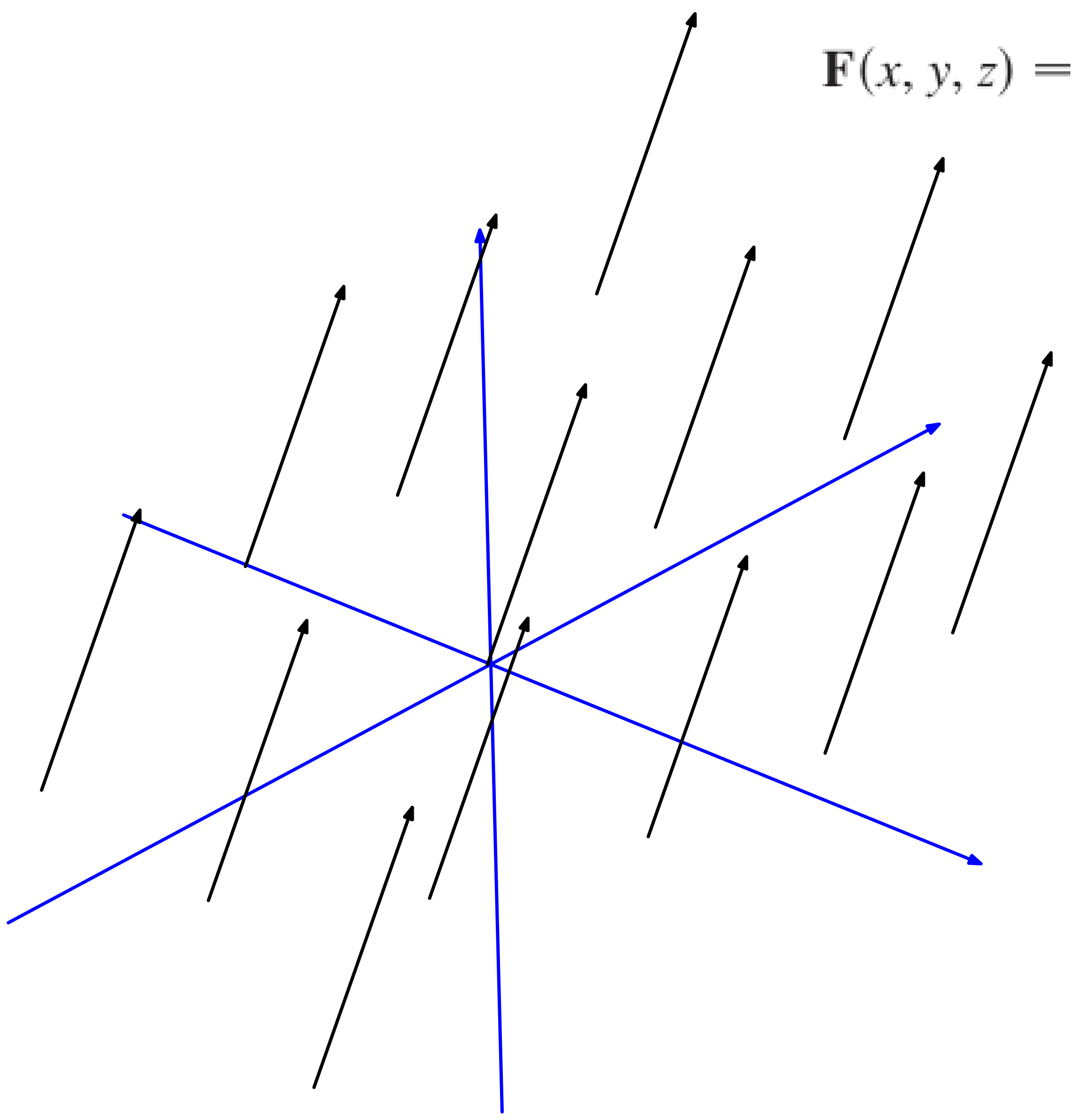


Sketch the vector field $\mathbf{F}$

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$$

Sketch the vector field **F**

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$$



Sketch the vector field $\mathbf{F}$

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$$

Vector Fields

(point in plane) $\rightarrow$ vector

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x,y) = P\,\hat{i} + Q\,\hat{j}$$

$$= F_1\,\hat{i} + F_2\,\hat{j}$$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\vec{F}(x,y,z) = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$$

$$= F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

GRADIENT FIELDS

13.1

later

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

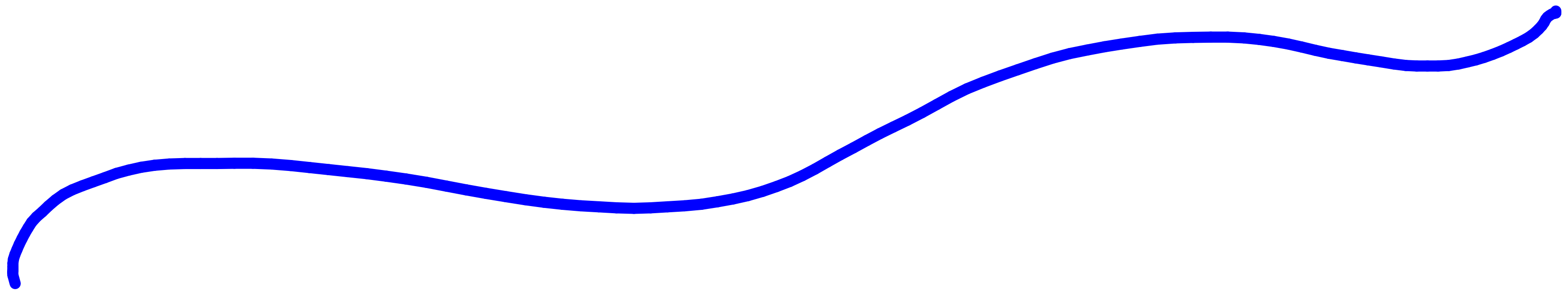
V EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

Recall position function from physics

→ a particle is moving in a path



→ $\vec{r}(t)$ = position of particle at time t

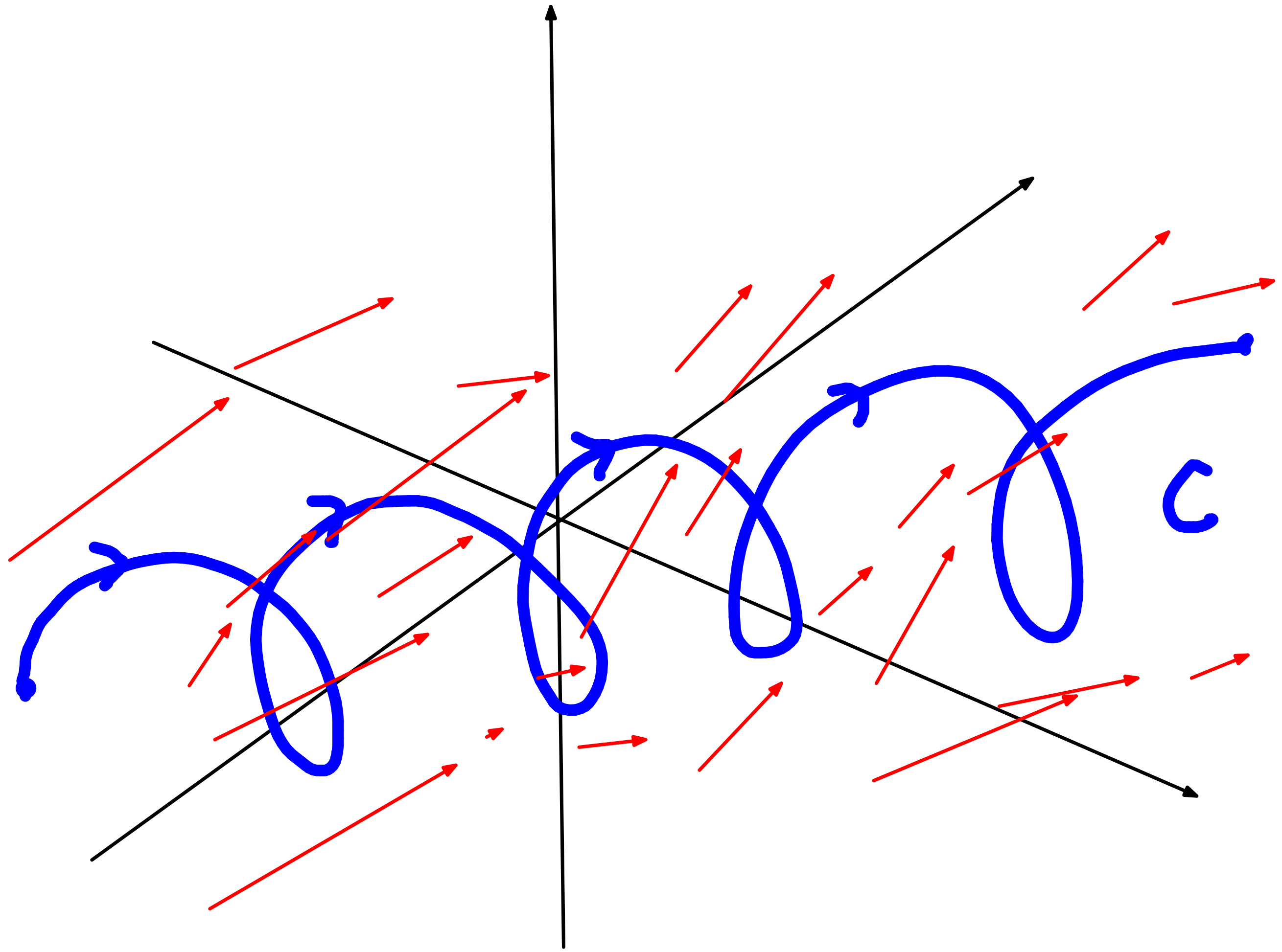
$= (x(t), y(t), z(t))$

→ Here we refer to these $\vec{r}(t)$ as curves.

& t is a parameter for the curve

We will focus for now:

Calculus on curves or paths.



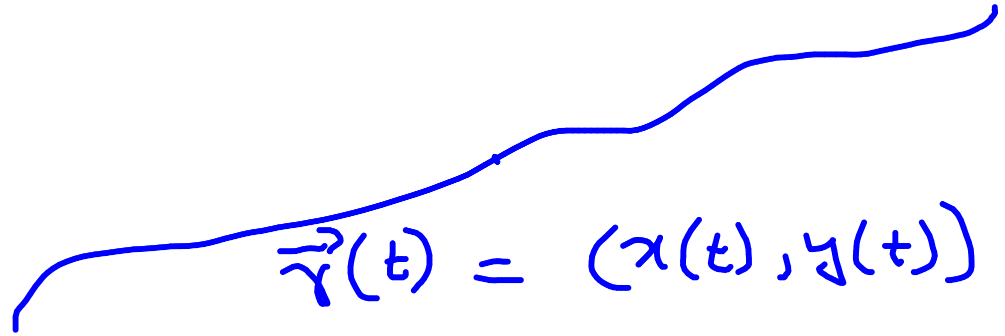
→ Length of the curve?

→ $\int_C f\,d\vec{r}$: integration of scalar function

→ $\int_C \vec{F}\cdot d\vec{r}$: integration of vector functions on curves.

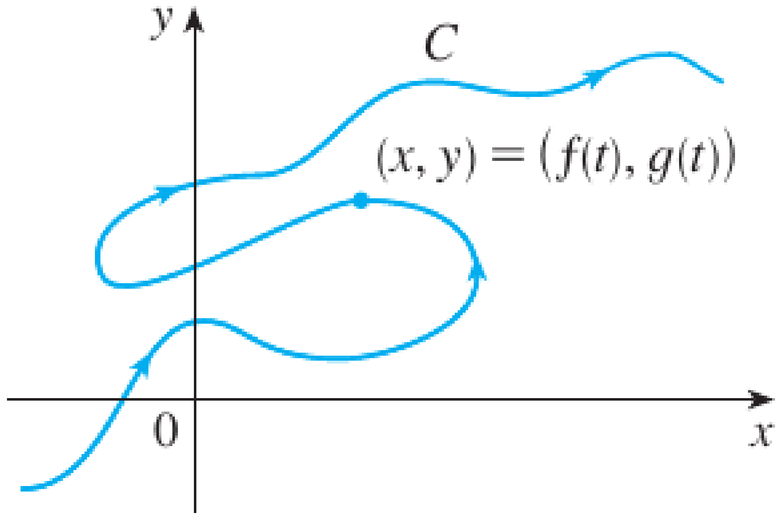
13.2 LINE INTEGRALS

next time



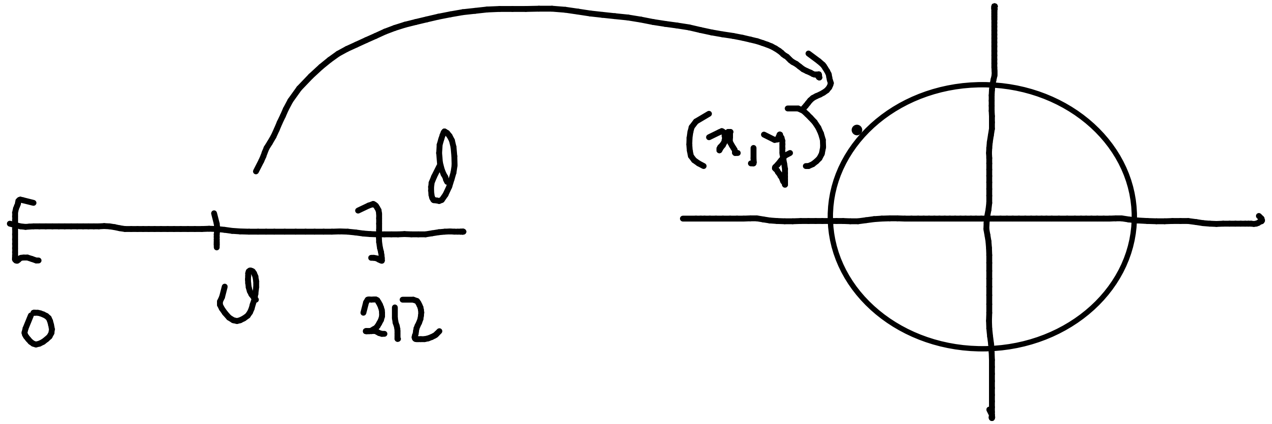
$\vec{r}(t) = (x(t), y(t))$

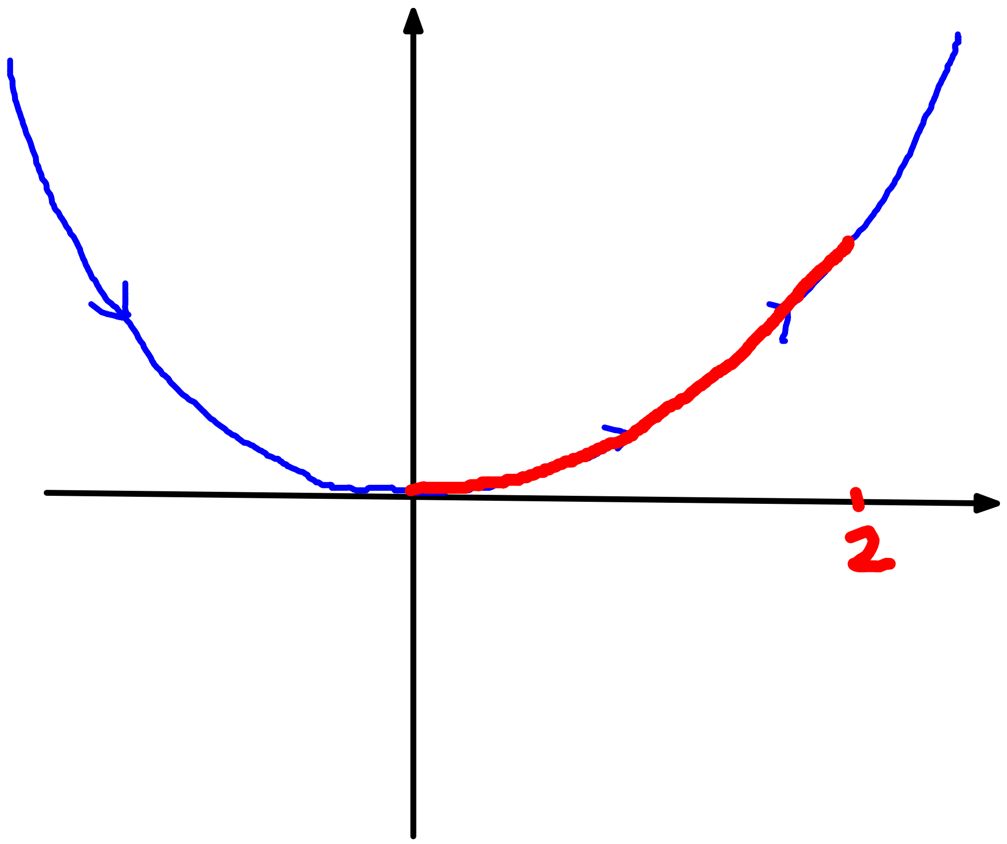
9.1 PARAMETRIC CURVES



:

$$\boxed{\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned} \qquad 0 \le \theta \le 2\pi}$$

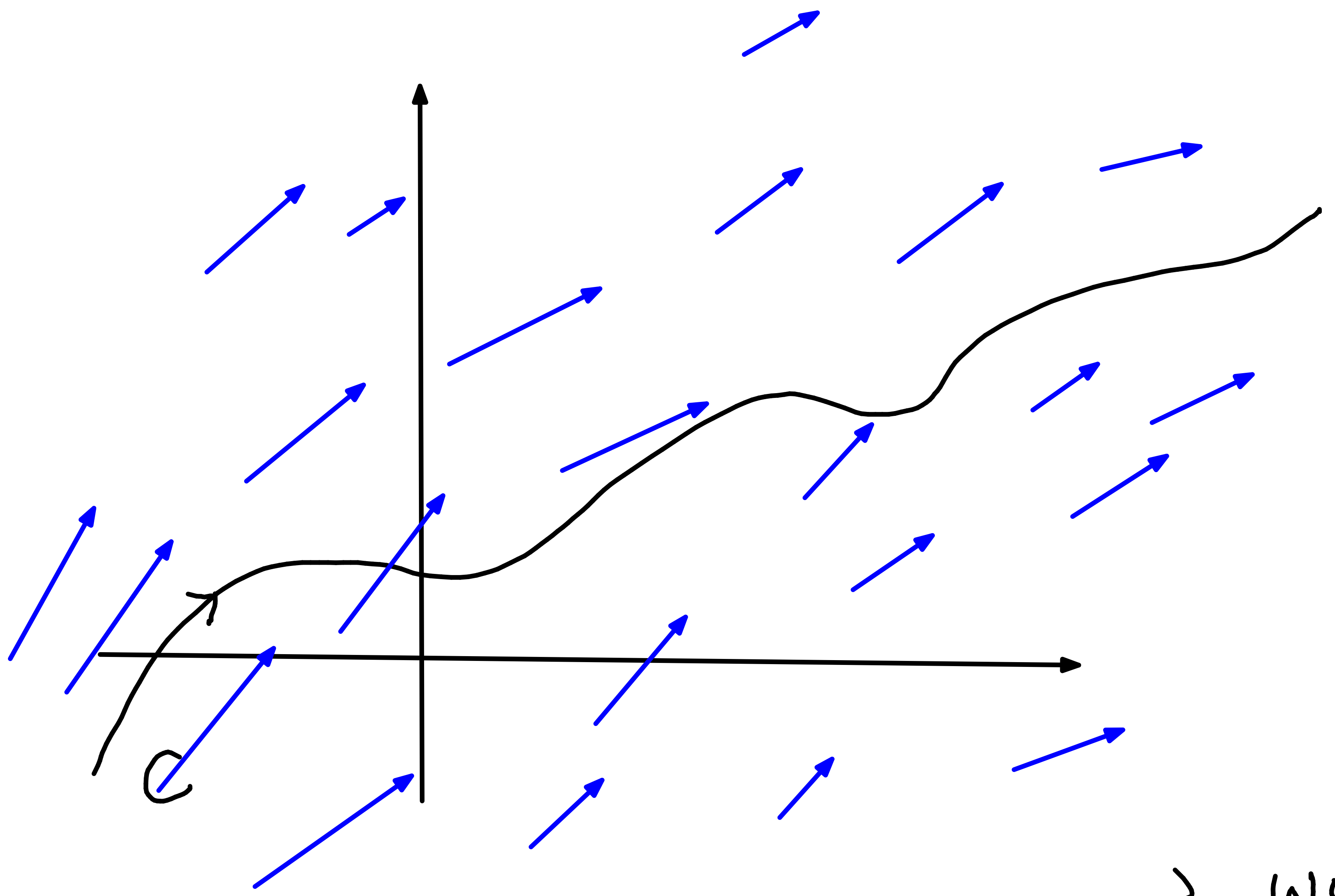




$y = x^2$

$$\begin{bmatrix} -\infty \leq t \leq \infty \\ x = t \\ y = t^2 \end{bmatrix}$$

$$\boxed{\begin{array}{l} 0 \leq t \leq 2 \\ x = t \\ y = t^2 \end{array}} \longrightarrow \text{what curve ??}$$



- Force field $\vec{F}$
- a particle is moving along a curve C

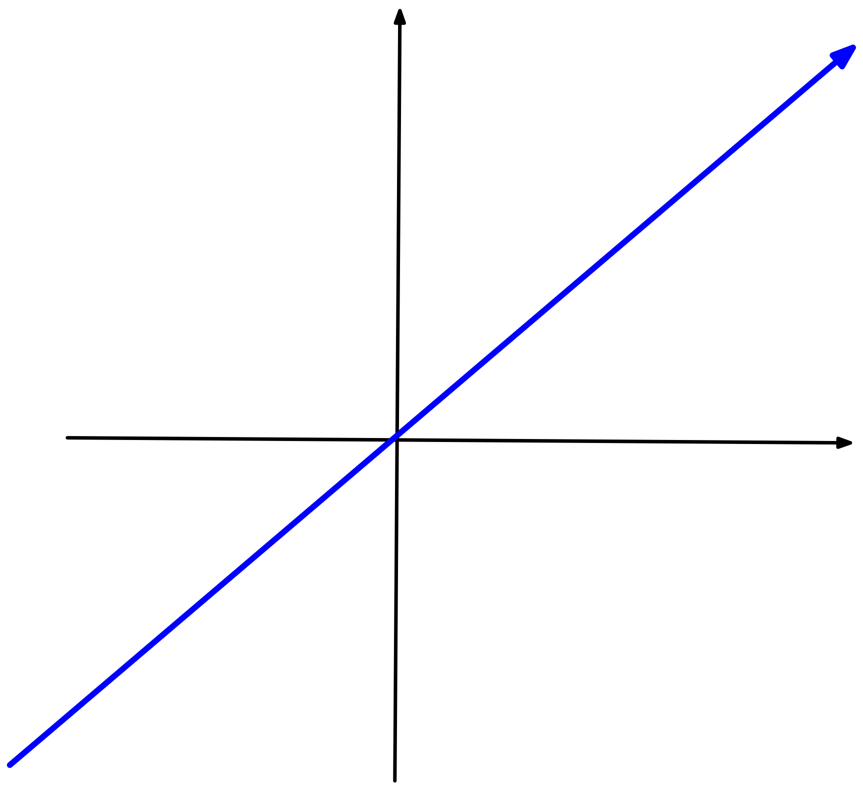
Q: find work done by $\vec{F}$ in moving the particle along the curve ??

→ we need a better precise description of whats a curve or path.

$-6\pi \leq t \leq 6\pi$

$x = t$

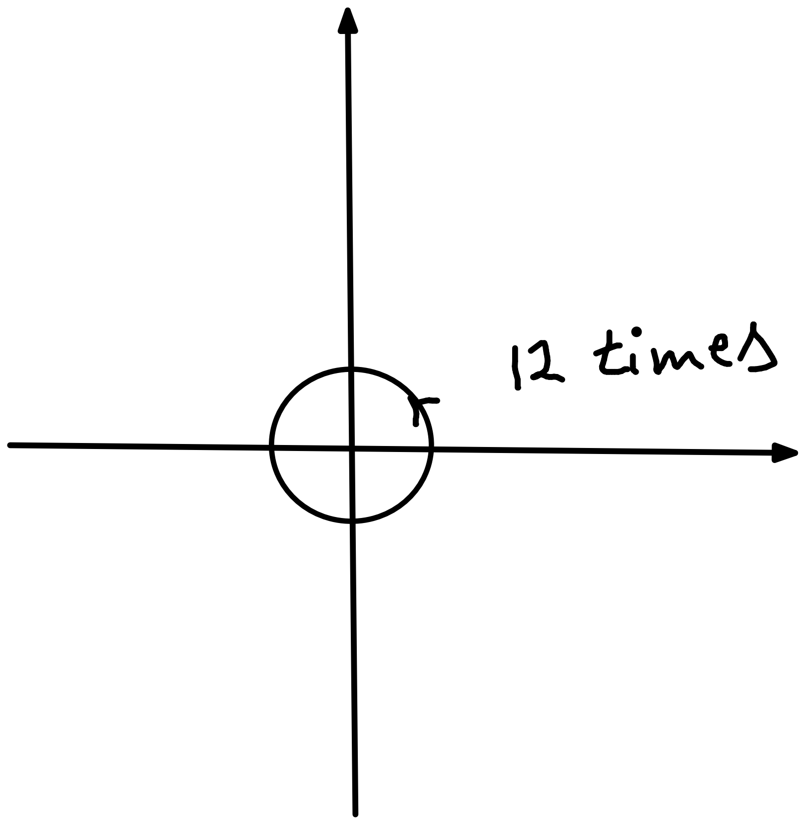
$y = t$



$-6\pi \leq t \leq 6\pi$

$x = 0.1\cos(2t)$

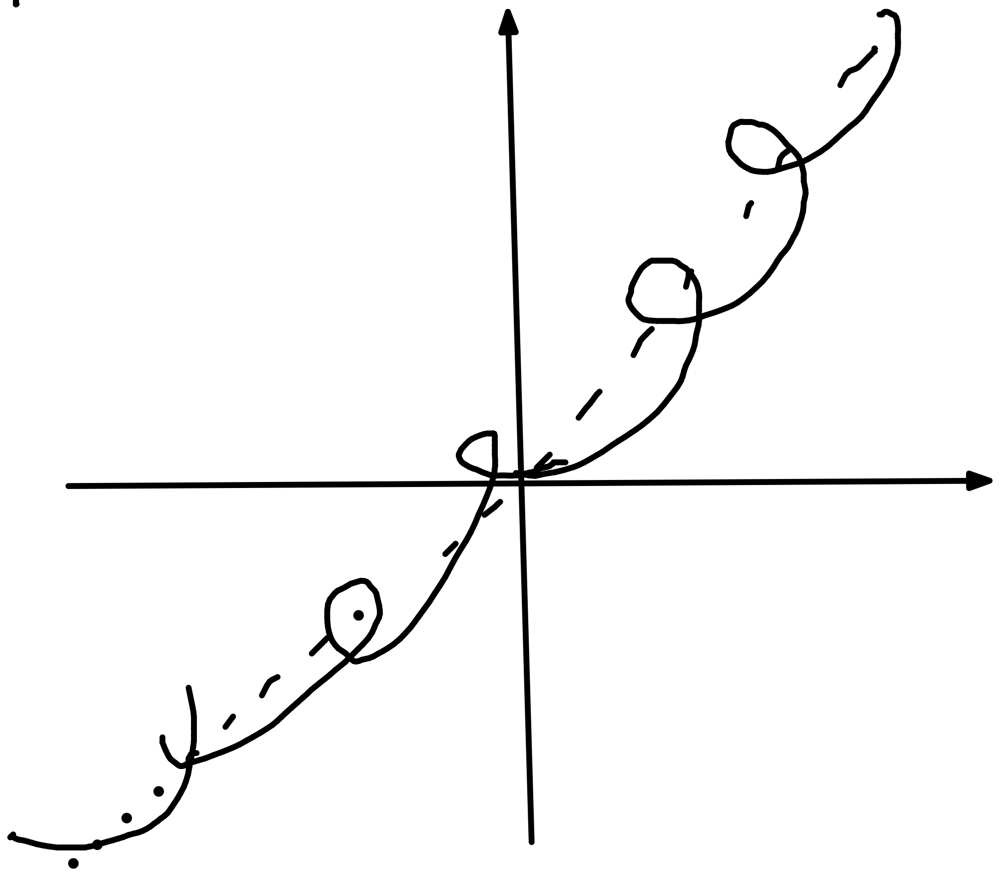
$y = 0.1\sin(2t)$

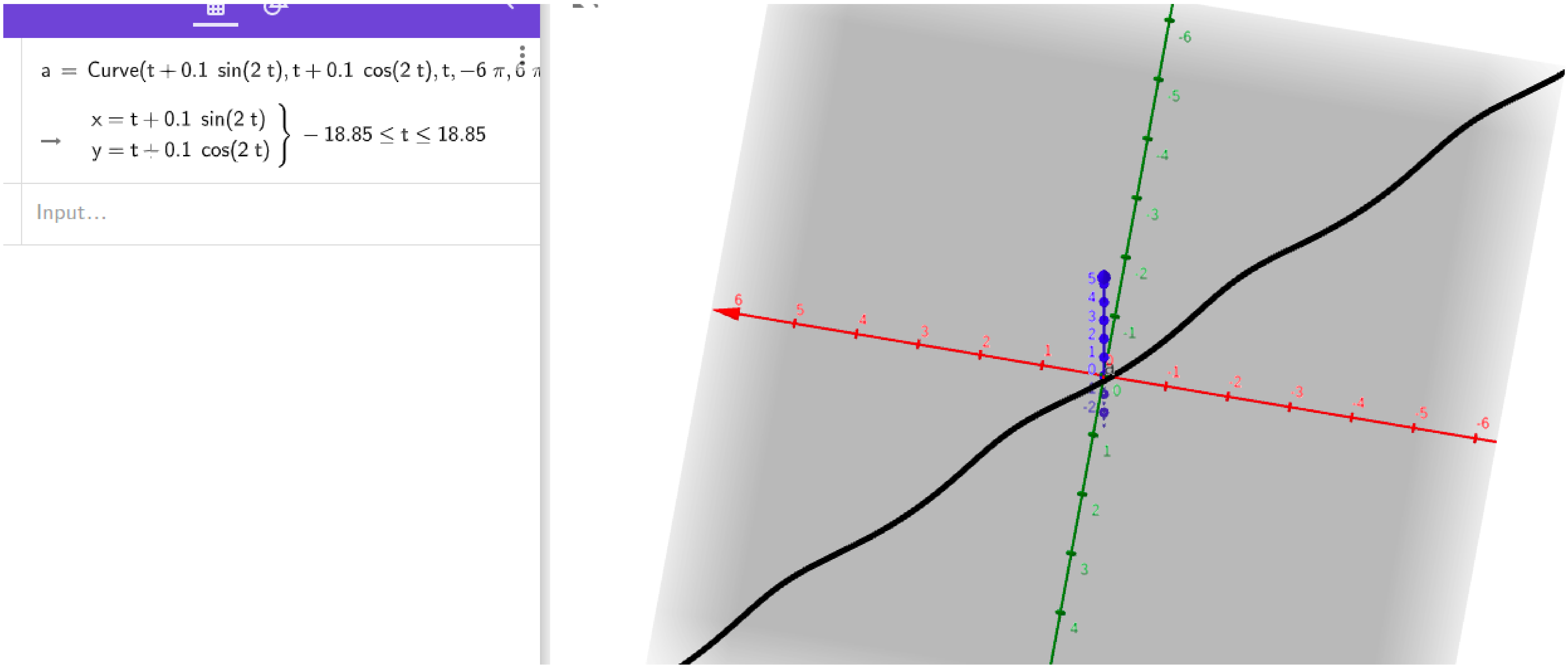


$-6\pi \leq t \leq 6\pi$

$x = t + 0.1\cos(2t)$

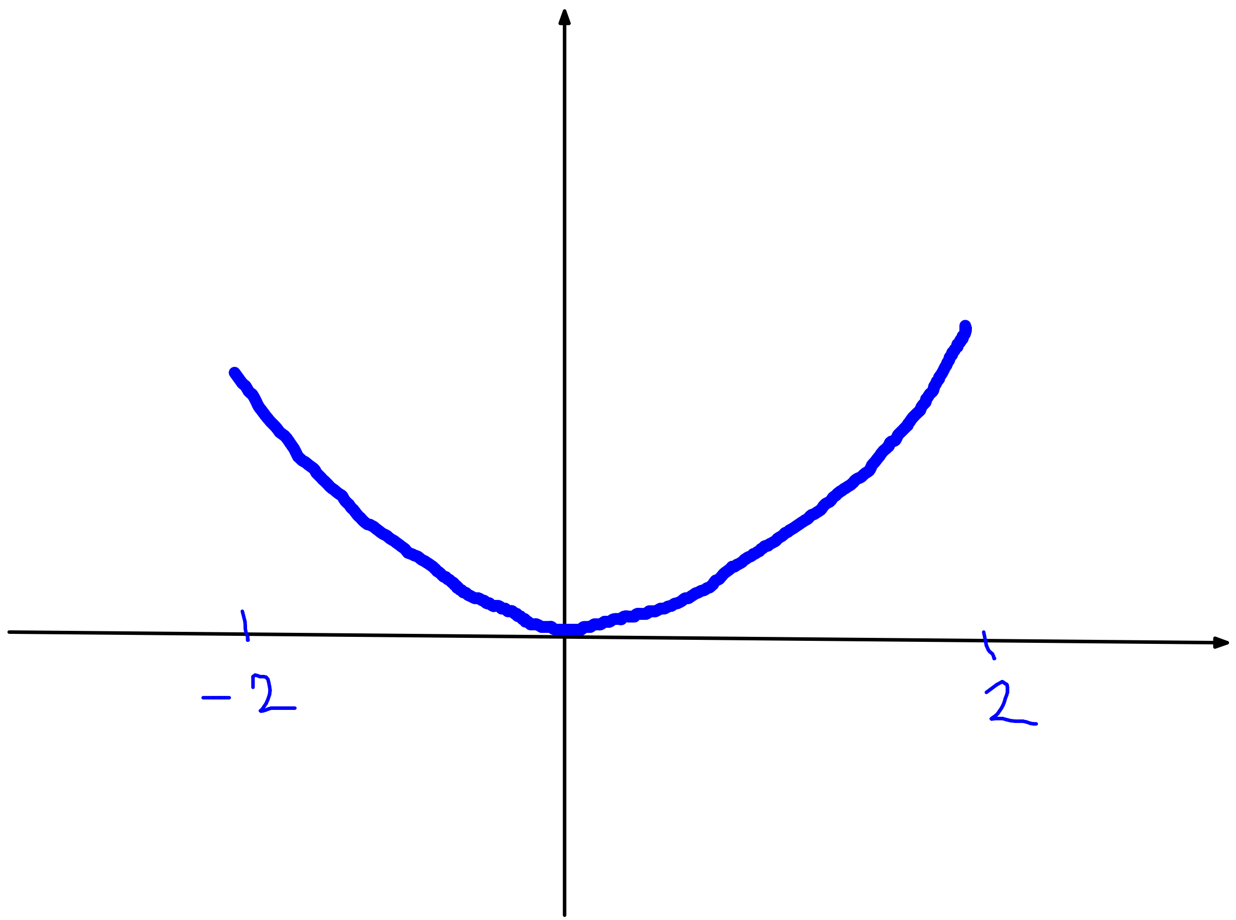
$y = t + 0.1\sin(2t)$




a = Curve(t + 0.1 sin(2 t), t + 0.1 cos(2 t), t, −6 π, 6 π
x = t + 0.1 sin(2 t)
y = t + 0.1 cos(2 t)
− 18.85 ≤ t ≤ 18.85
Input…

Q. Sketch the path

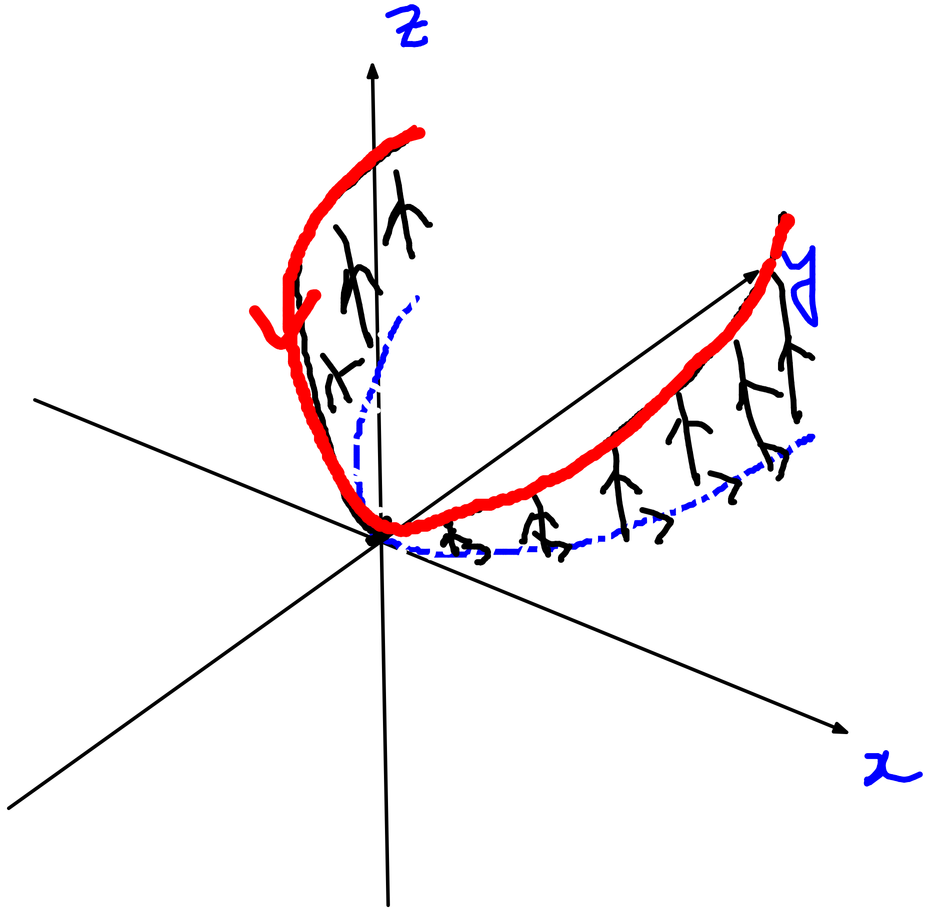
$$x = t \quad , \quad y = t^2 \quad , \qquad -2 \leq t \leq 2$$



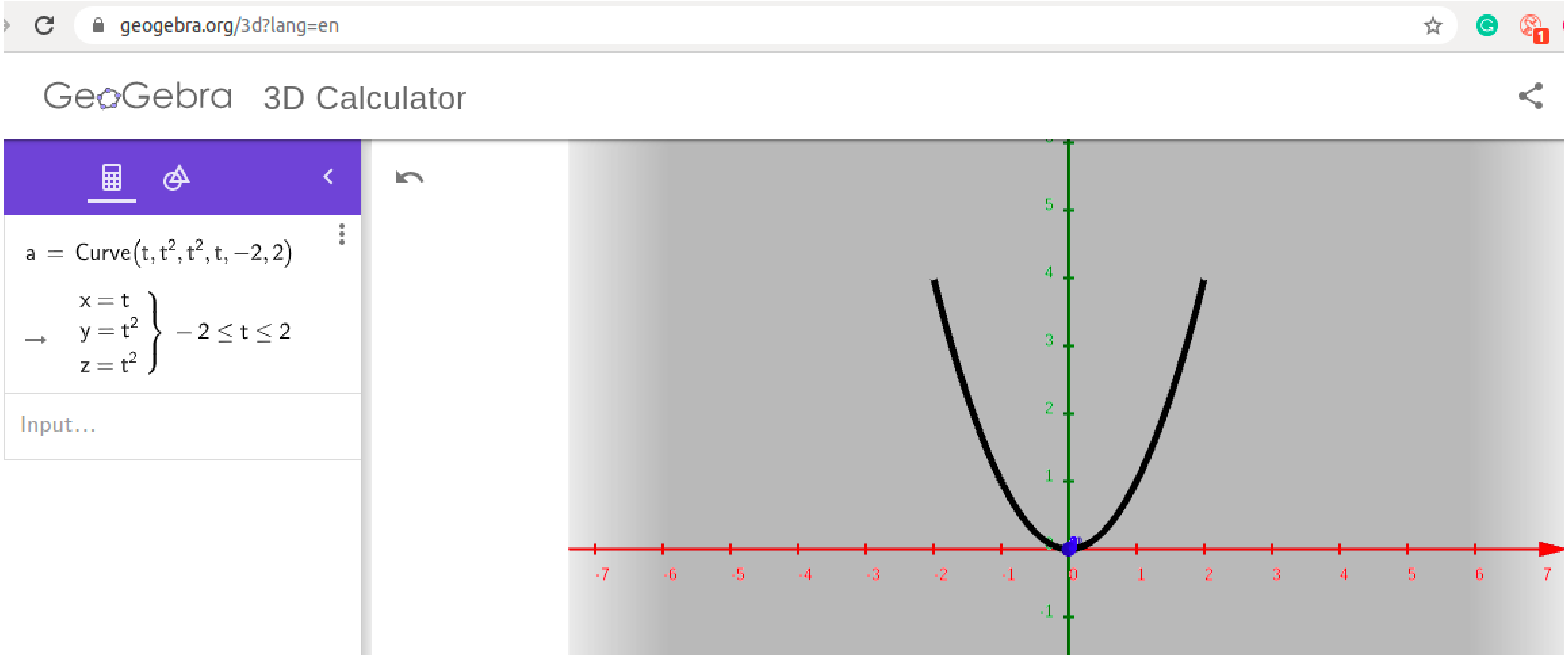
Q. Sketch the path

$$\underbrace{x = t \quad , \quad y = t^2}_{y = x^2} \quad , \quad z = t^2$$

$$-2 \le t \le 2$$



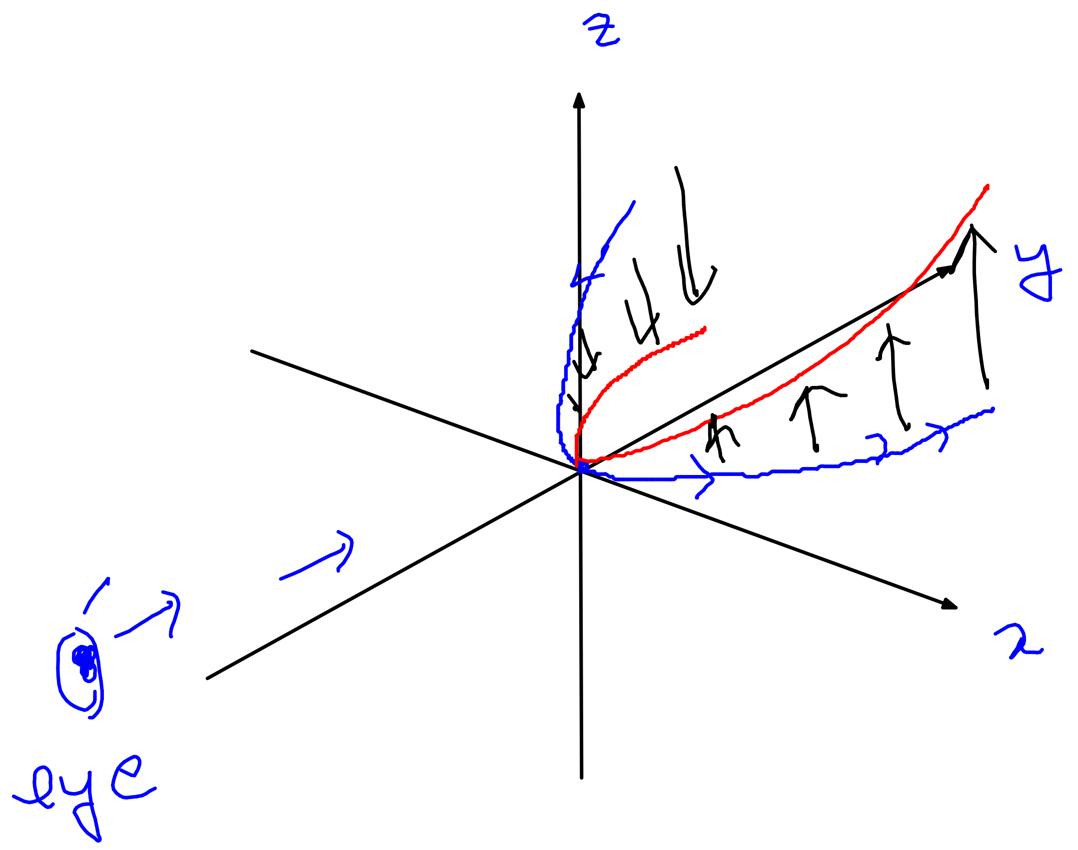
$t = 0 \qquad (x, y, z) = (0, 0, 0)$

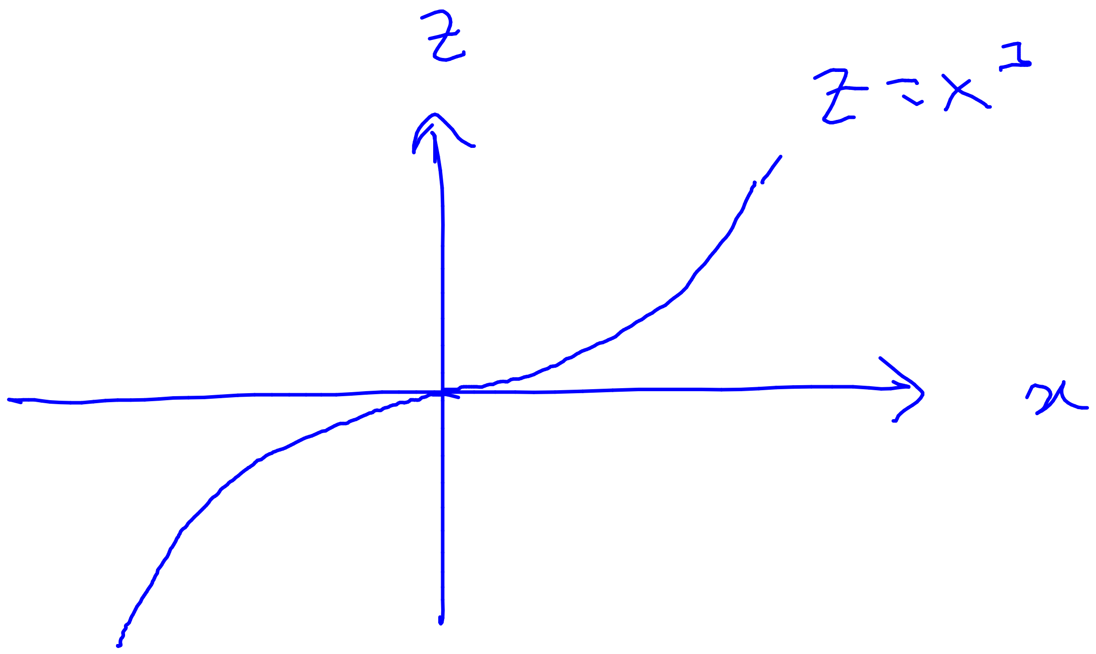

geogebra.org/3d?lang=en
GeoGebra 3D Calculator
a = Curve(t, t², t², t, −2, 2)
x = t
y = t²
z = t²
−2 ≤ t ≤ 2
Input…

Q: Sketch

$$x = t, \quad y = t^2, \quad z = t^3$$

$$-2 \le t \le 2$$





$$z = y^{3/2}$$

between

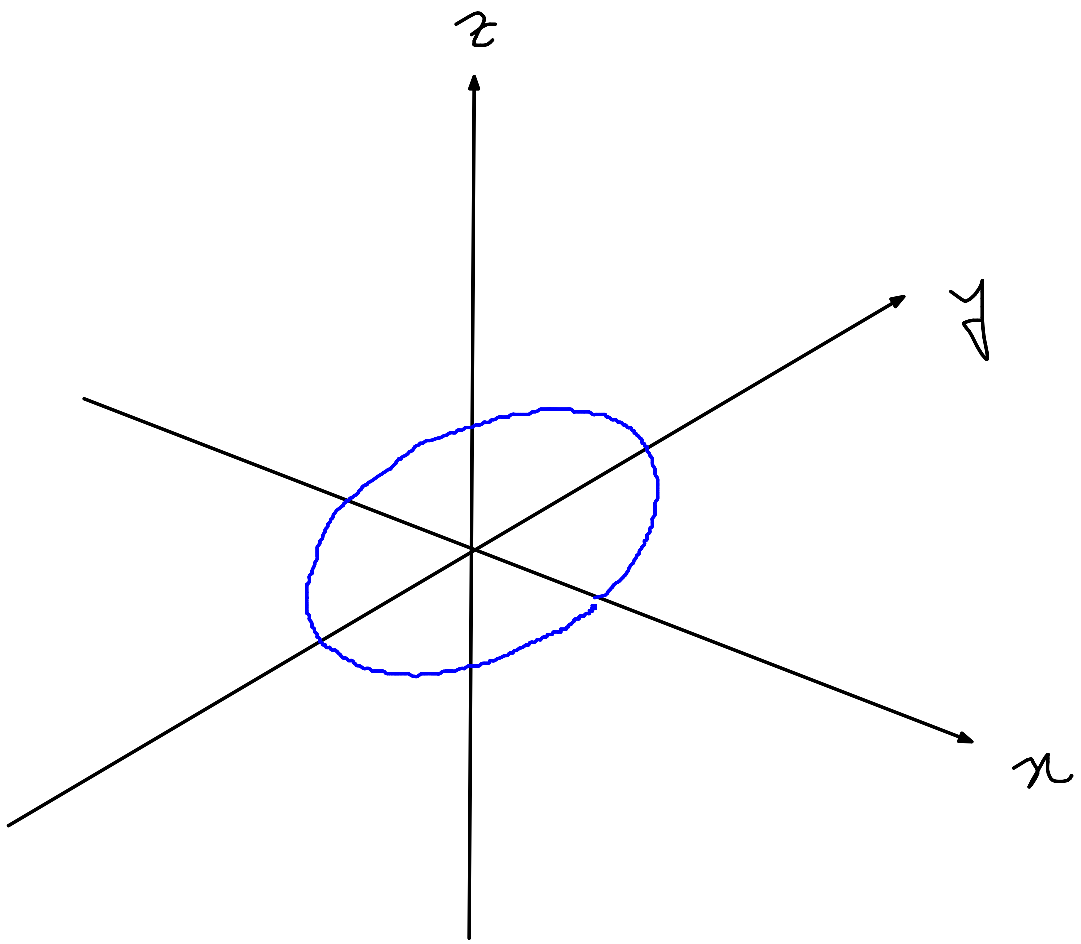
$z = y$

$z = y^2$

Q. Sketch

$x = \cos(t) \qquad y = \sin(t) \qquad z = 0$

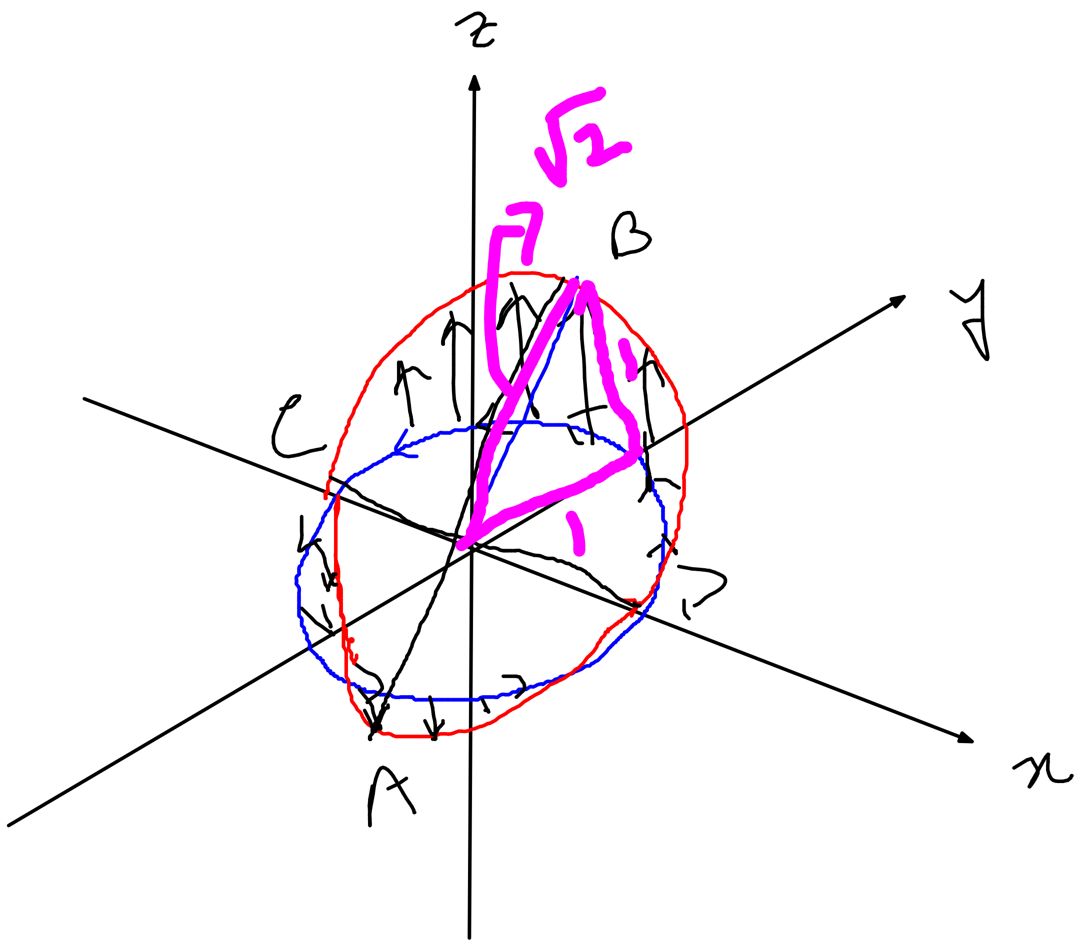
$0 \le t \le 6\pi$



Q. Sketch

$x = \cos(t)$ $y = \sin(t)$ $z = \sin(t)$

$0 \leq t \leq 6\pi$



$AB \overset{??}{=} CD$

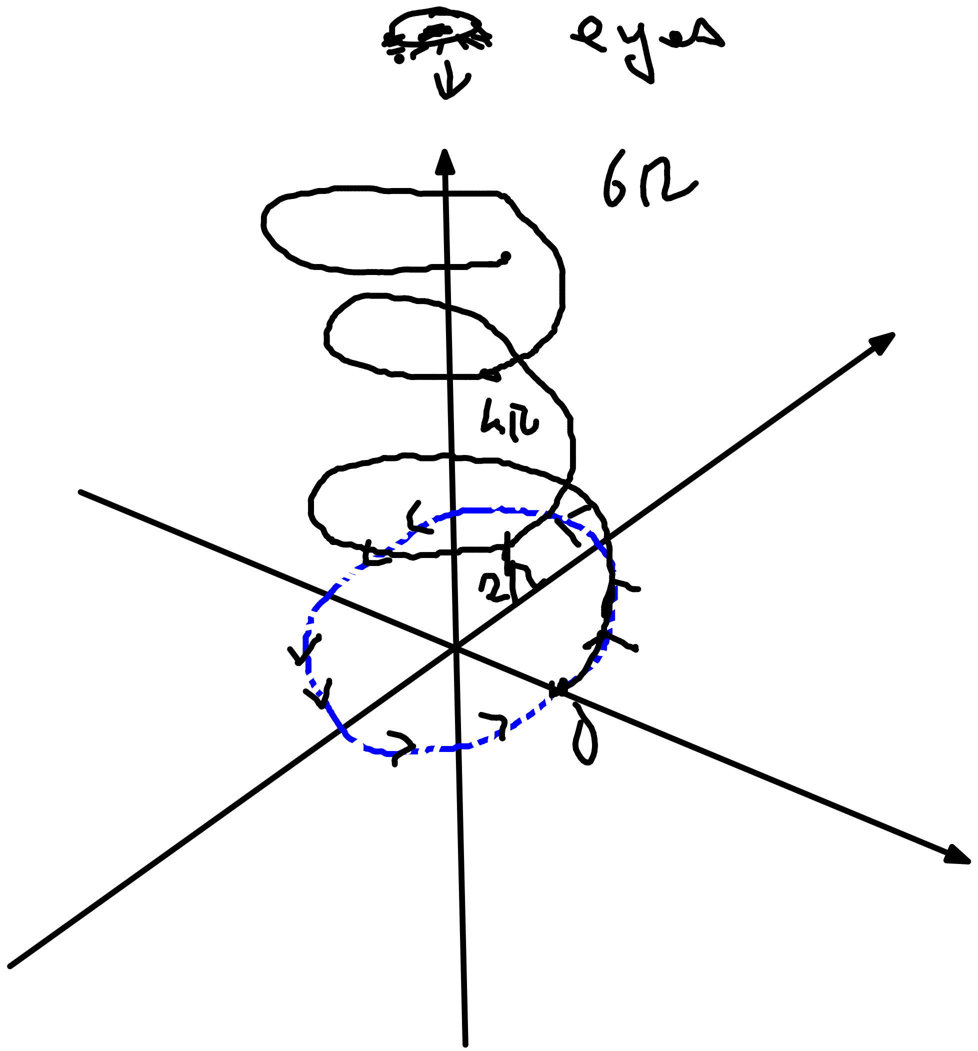
$2\sqrt{2}$ 2

ellipse

Q. Sketch

$x = \cos(t)$ $\qquad y = \sin(t)$, $\qquad z = t$

$0 \le t \le 6\pi$



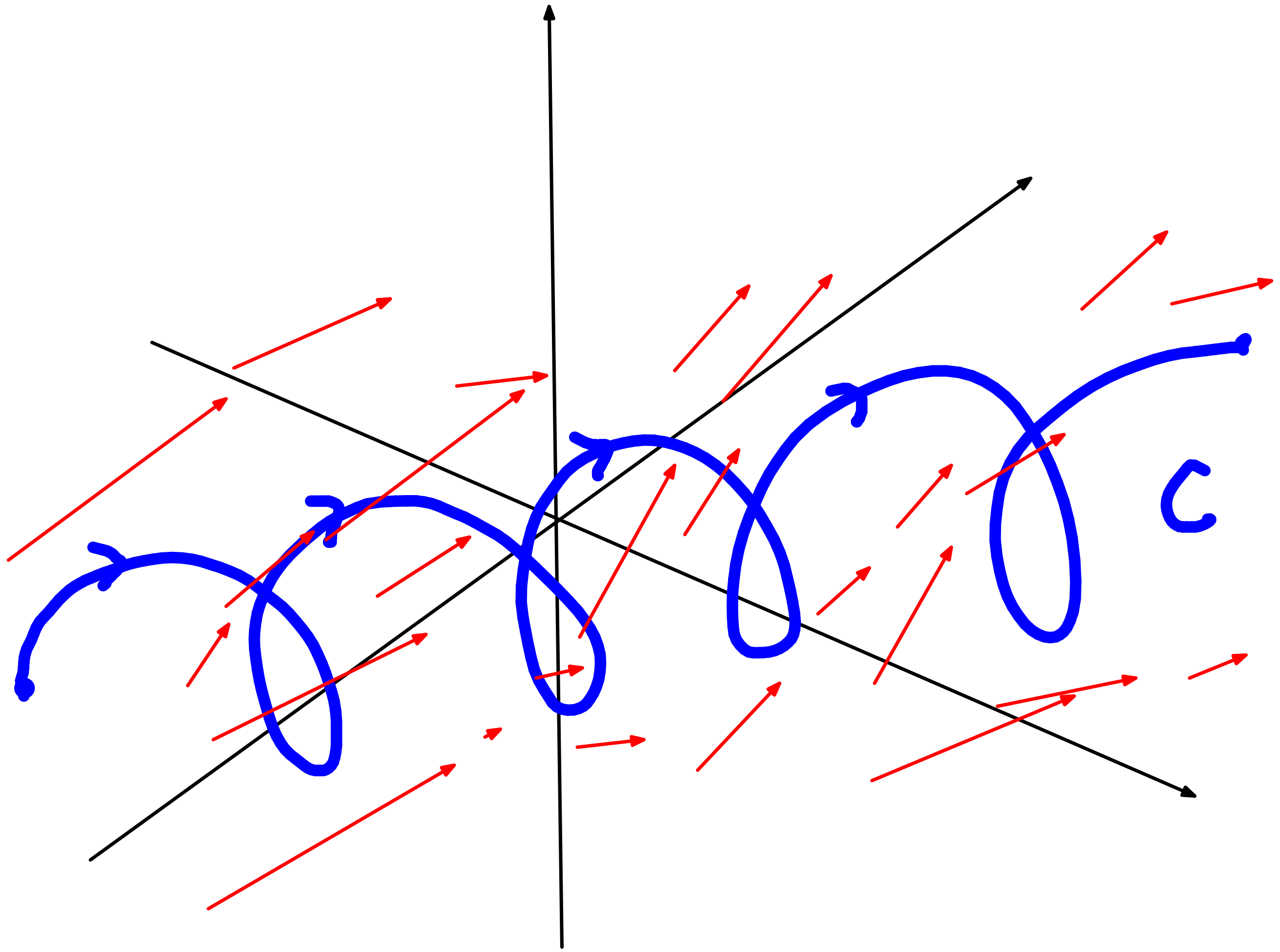
Calculus on curves

— next time

We will focus for now:

next time

Calculus on curves or paths.



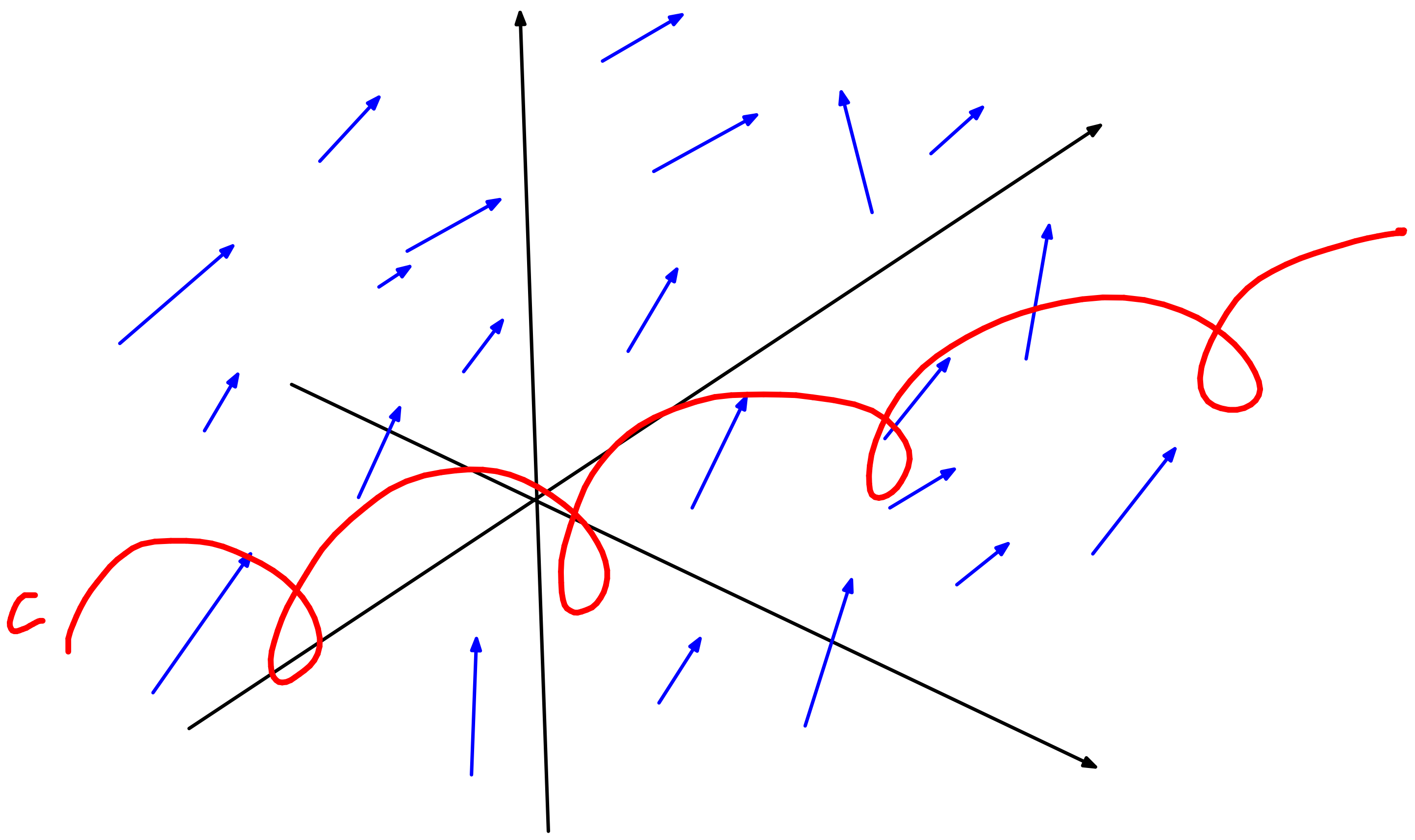
→ Length of the curve?

→ $\int_C f \, d\vec{r}$: integration of scalar function

→ $\int_C \vec{F} \cdot d\vec{r}$: integration of vector functions on curves.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$



$\int_C ds$: find length of curve

$\int_C f.ds$: integration of scalar f^n function

$\left[\int_C \vec{F}.d\vec{r} \text{ : work done by } \vec{F}\right]$

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

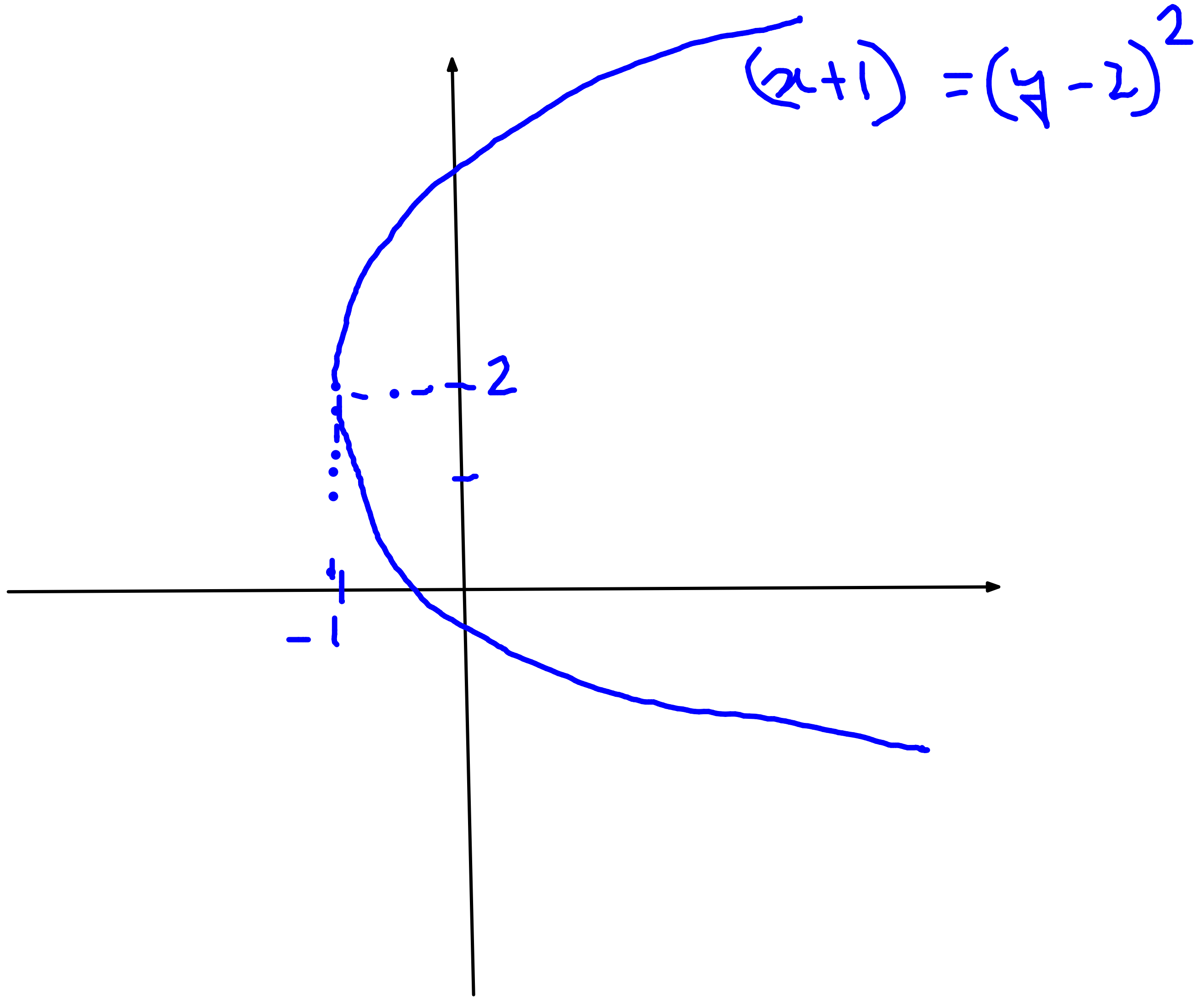
$x = t^2 - 2t$

$y = t + 1$

?? Sketch the shape

$$x = (y-1)^2 - 2(y-1)$$

$$(x+1) = (y-2)^2$$



$x = t + 2\sin 2t$
$y = t + 2\cos 5t$

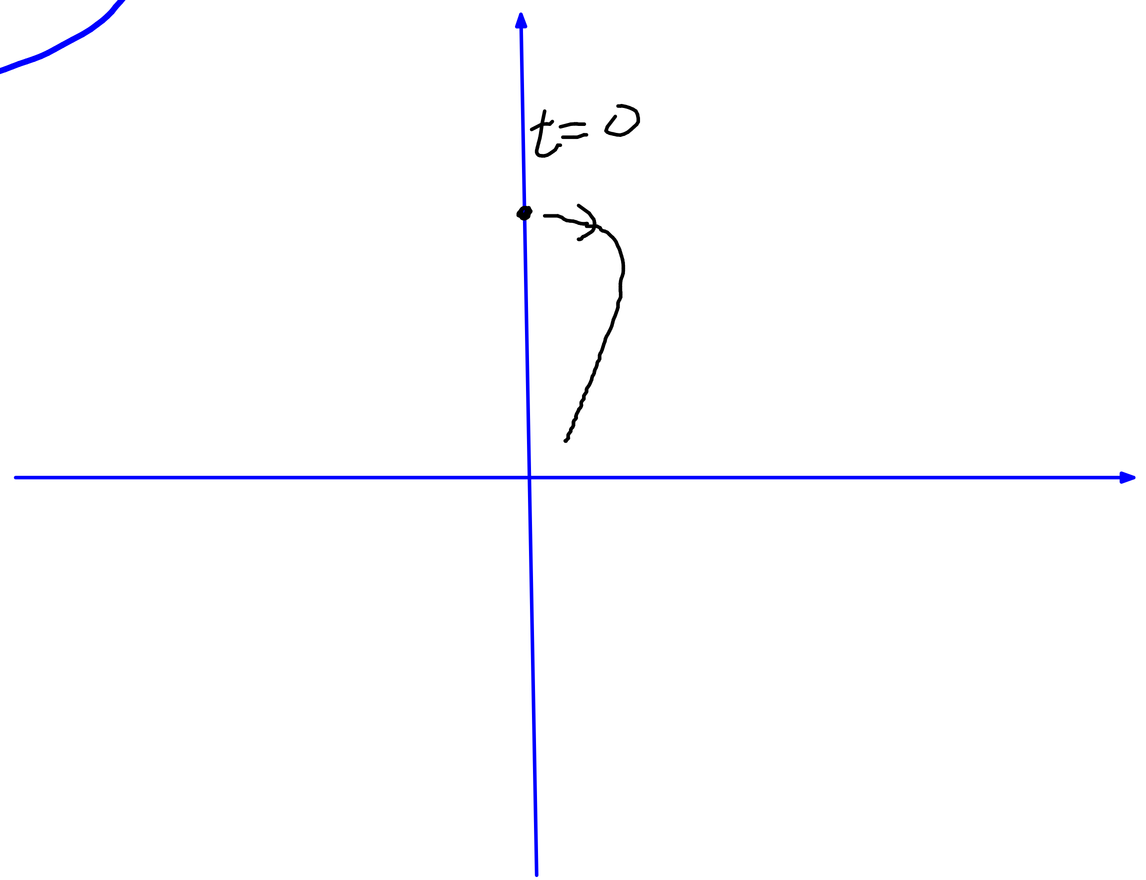
Sketch by intuition first

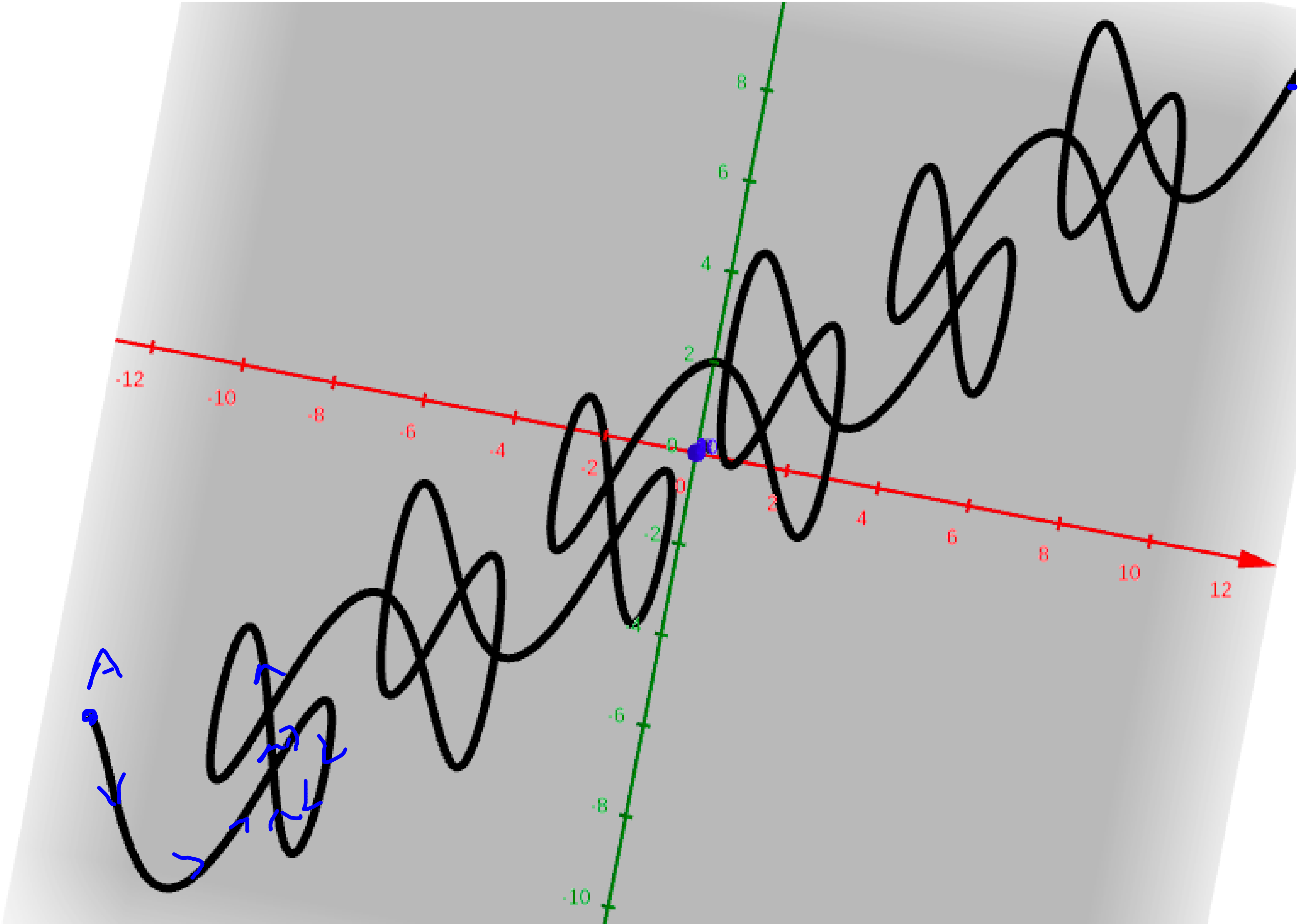
$x = t$
$y = t$

$+$

$x = 2\sin 2t$
$y = 2\cos 5t$

$t = 0$
$\pi/6$
$\pi/4$
$\pi/3$
$\pi/2$





-12
-10
-8
-6
-4
-2
2
4
6
8
10
12
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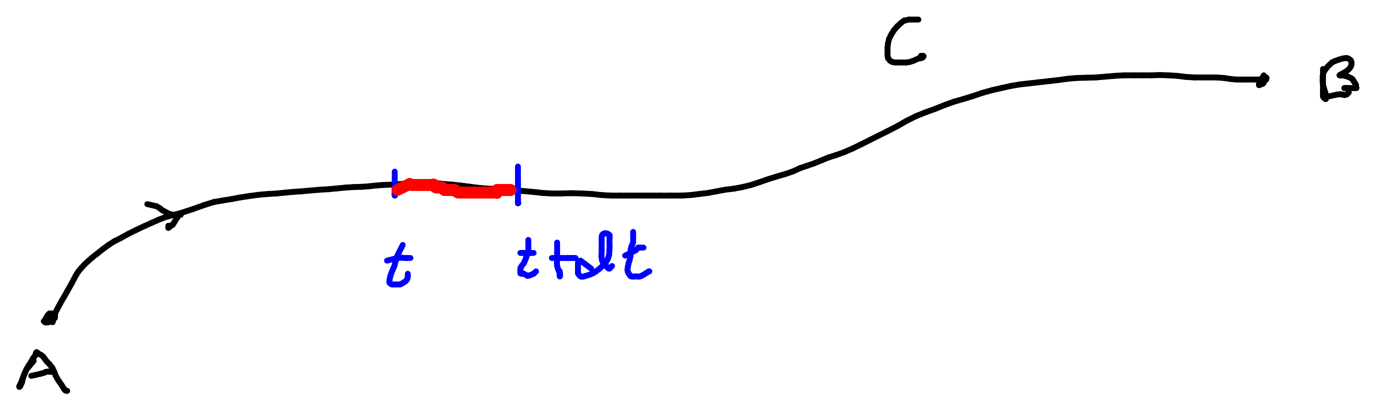
-10
A

Q.



parametric eq^n

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$a \le t \le b$$

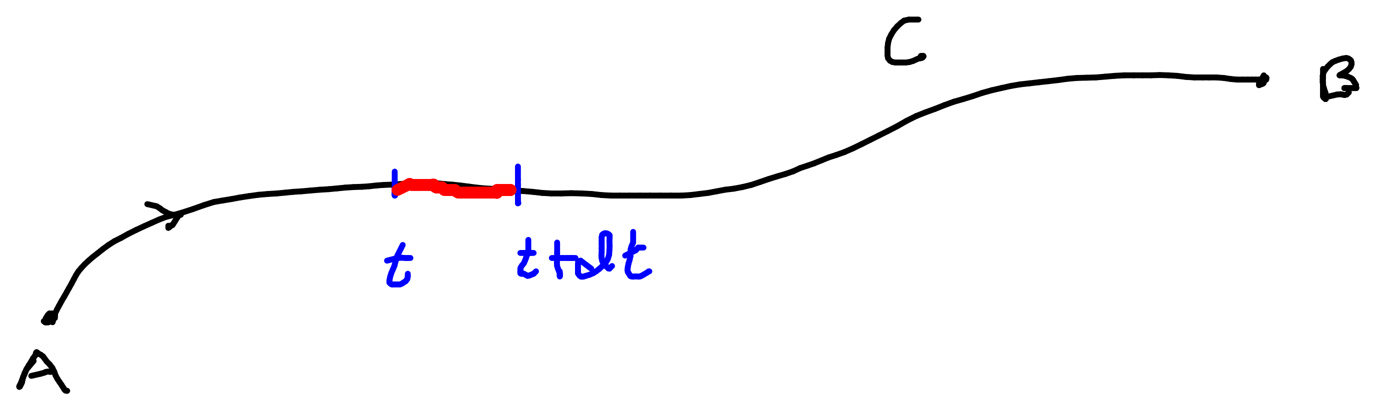
is there a distance formula to measure the length of the curve ??

L : total length

dL : distance travelled in t to $t+dt$; & dt is small enough to assume that speed was constant in the interval $(t, t+dt)$

$$dL = (\text{speed at time } t)\, dt$$

Q.



parametric eq^n

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$a \leq t \leq b$$

is there a distance formula to measure the length of the curve ??

L : total length

dL : distance travelled in t to $t+dt$; & dt is small enough to assume that speed was constant in the interval $(t, t+dt)$

$$dL = \underbrace{(\text{speed at time } t)}_{??}\, dt$$

$$= \left| \vec{r}\,'(t) \right| \, dt$$

position $\quad \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

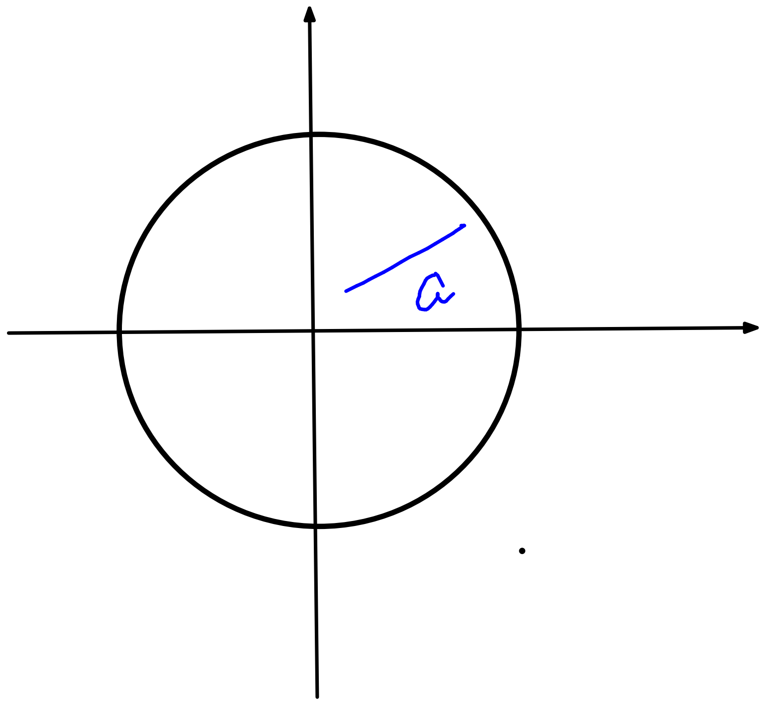
velocity $\quad \vec{r}\,'(t) = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j}$

speed $\quad \left| \vec{r}\,'(t) \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$L = \int dL = \int_a^b (\text{speed})\, dt = \int_a^b \left| \vec{r}\,'(t) \right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Q.



circumference = $2\pi a$

find the length of path

$x = a\cos t \qquad 0 \le t \le 2\pi$

$y = a\sin t$

$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

$|\vec{r}\,'(t)| = a$ = speed

$$L = \int_0^{2\pi} |\vec{r}(t)|\,dt = \int_0^{2\pi} a\,dt = 2\pi a$$

Q: Recall this curve

$x = \cos(t)$

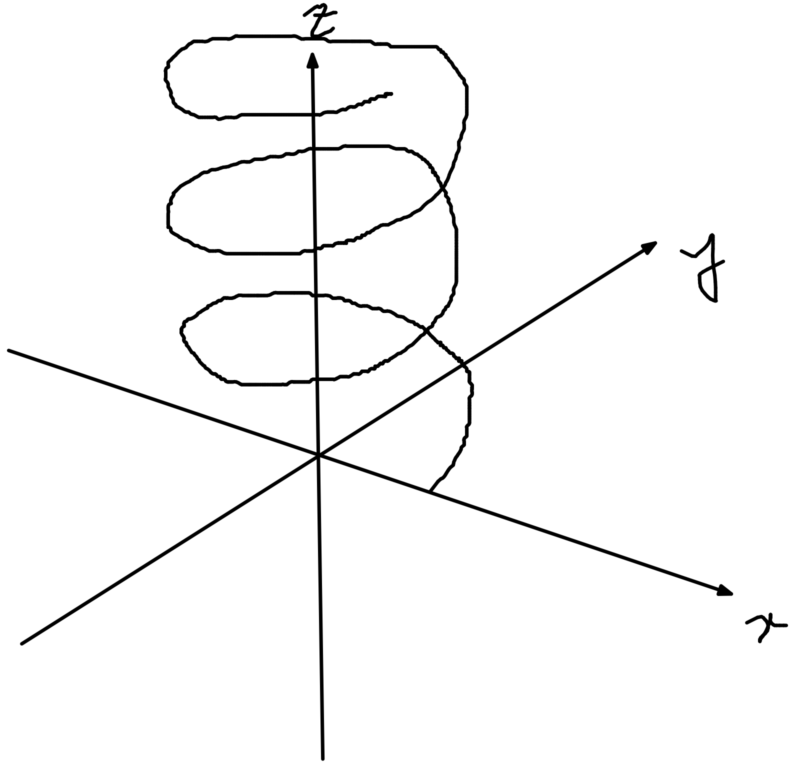
$y = \sin(t)$

$z = t$

$0 \leq t \leq 6\pi$

→ sketch

→ find length



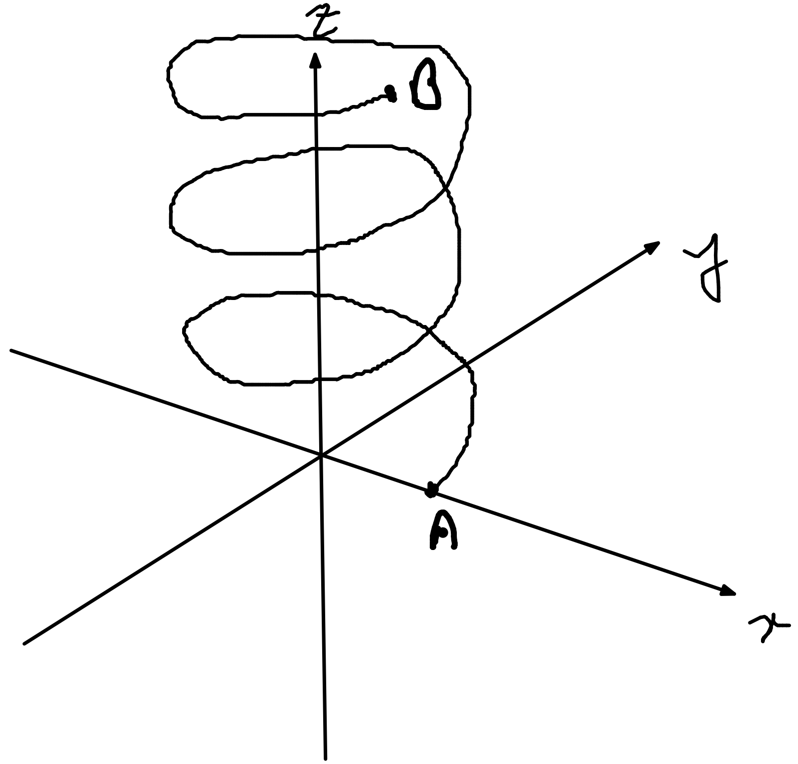
$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{2}$$

$$L = \int_0^{6\pi} \sqrt{2}\,dt = 6\pi\sqrt{2}$$

Q: Recall this curve

$$x = \cos(t) \qquad 0 \le t \le 6\pi$$
$$y = \sin(t)$$
$$z = t$$



→ think of AB as a wire

→ material used to make this wire in non-uniform

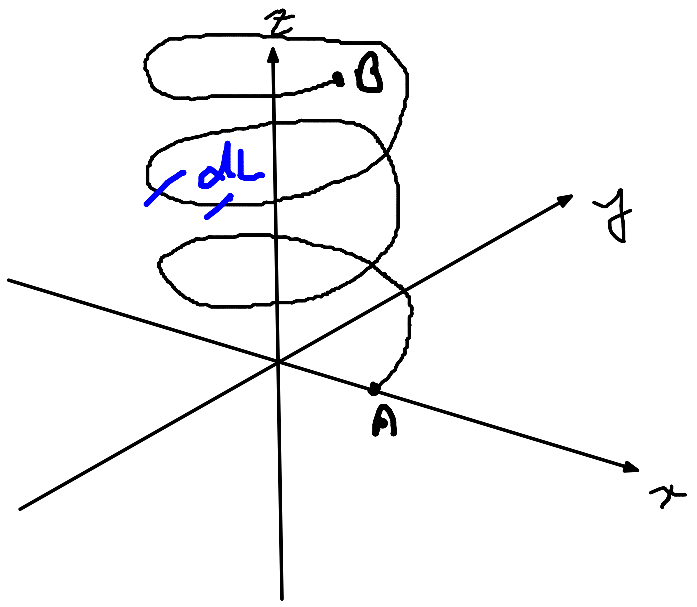
→ $f(x,y,z)$ represents linear density (mass per unit length)

→ let $f(x,y,z) = z$

→ Q: find the mass of wire AB

Q: Recall this curve

$$x = \cos(t) \qquad 0 \le t \le 6\pi$$
$$y = \sin(t)$$
$$z = t$$



→ think of AB as a wire

→ material used to make this wire in non-uniform

→ $f(x,y,z)$ represents linear density (mass per unit length)

→ let $f(x,y,z) = z$

→ Q: find the mass of wire AB

dL ~ so small that we can assume that density is constant

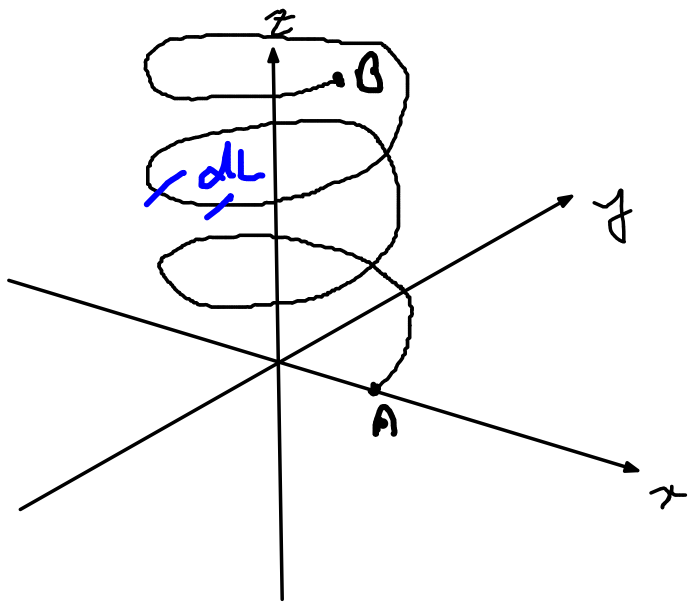
dm = mass for dL

$$dm = f\, dL$$

$$\text{Total mass} = \int dm = \int f\, dL = \int_a^b f \underbrace{|\vec{r}\,'(t)|\, dt}_{dL}$$

Q: Recall this curve

$$x = \cos(t) \qquad 0 \le t \le 6\pi$$
$$y = \sin(t)$$
$$z = t$$



→ think of AB as a wire

→ material used to make this wire in non-uniform

→ $f(x,y,z)$ represents linear density (mass per unit length)

→ let $f(x,y,z) = z$

→ Q: find the mass of wire AB

$$dm = z\, dL$$

$$m = \int dm = \int z\, dL = \int_0^{6\pi} z\ (\text{speed})\, dt$$

$$= \int_0^{6\pi} \underbrace{t}_{z} \underbrace{\sqrt{2}}_{\text{speed}}\, dt = 18\sqrt{2}\,\pi^2$$

a: Evaluate the line integral, where C is the given curve.

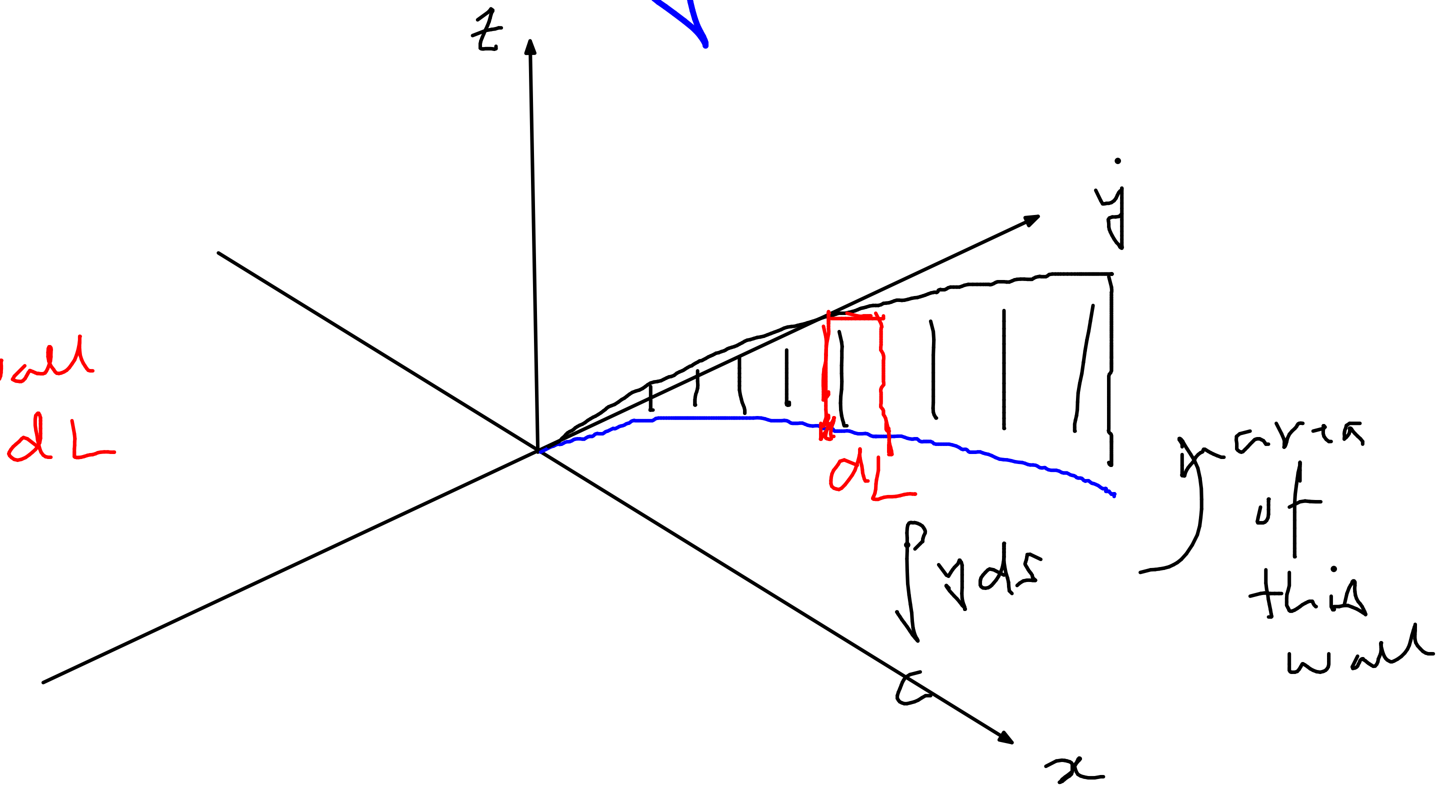
$\int_C y\,ds, \quad C: x = t^2,\ y = t,\ 0 \leqslant t \leqslant 2$

→ sketch the curve

→

$x = t^2 \qquad y = t \qquad 0 \le t \le 2$

$f = y$: linear density



$f\,dL$ = area of wall above dL

$$= \int_0^2 t\sqrt{(2t)^2 + 1^2}\,dt$$

$$= \frac{17\sqrt{17} - 1}{12}$$

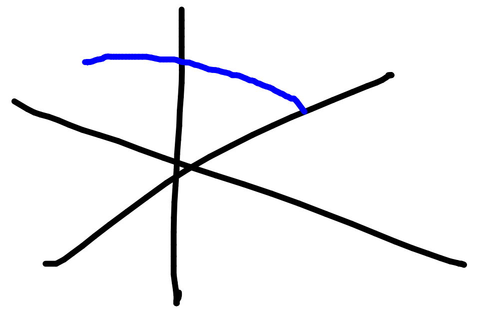
V EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

Evaluate the line integral, where C is the given curve.

Sketch ??

$$\int_C xy^3\, ds,$$

C: $x = 4\sin t,\ y = 4\cos t,\ z = 3t,\ 0 \leqslant t \leqslant \pi/2$



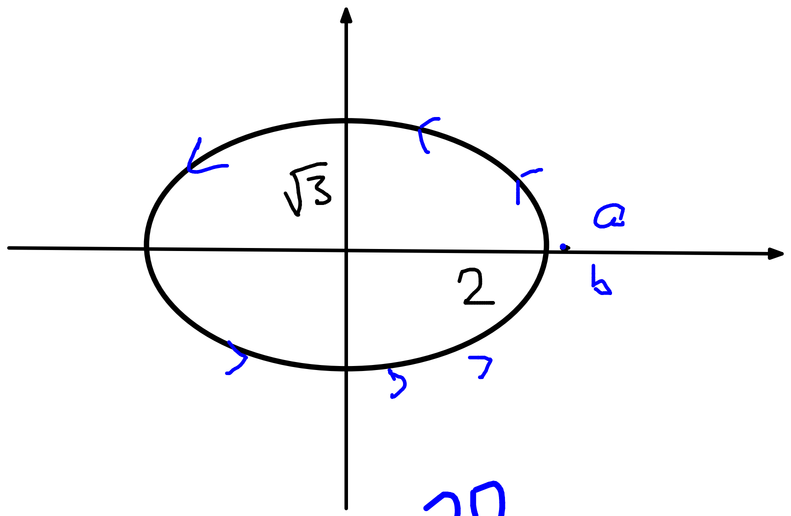
$$\vec{r}(t) = 4\sin(t)\,\hat{i} + 4\cos(t)\,\hat{j} + 3t\,\hat{k}$$

$$\text{speed} = \sqrt{16+9} = 5$$

$$\int_C xy^3\, ds = \int_0^{\pi/2} 4\sin t \cdot (4\cos t)^3\, 5\, dt = ?? = 320$$

Q. $\frac{x^2}{4}+\frac{y^2}{3}=1$

find the length of this ellipse



$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{3}}\right)^2=1$$

$x = 2\cos\theta$ $\quad 0 \le \theta \le 1$

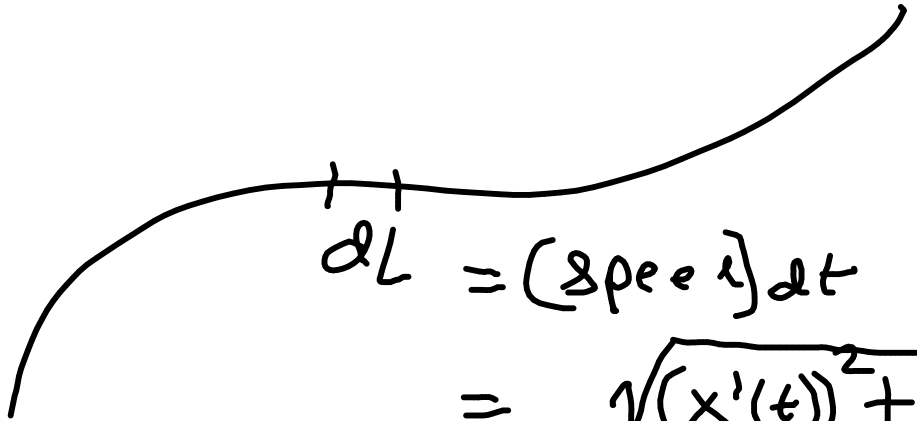
$y = \sqrt{3}\sin\theta$

$$L = \int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi}\sqrt{\left(4\sin^2\theta + 3\cos^2\theta\right)}\,d\theta$$

= whatever

we need a parametrization

$$x = x(t)$$
$$y = y(t)$$



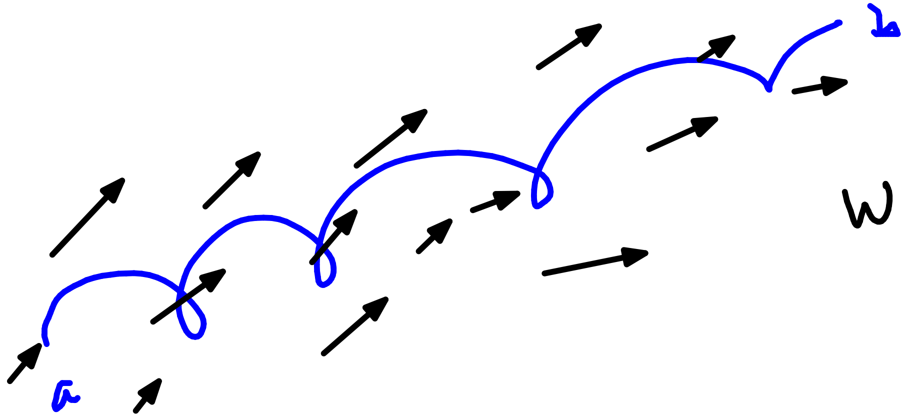
C

$$dL = (\text{speed})\,dt$$
$$= \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$L = \int_C ds = \int (\text{speed})\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

13 DEFINITION Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$



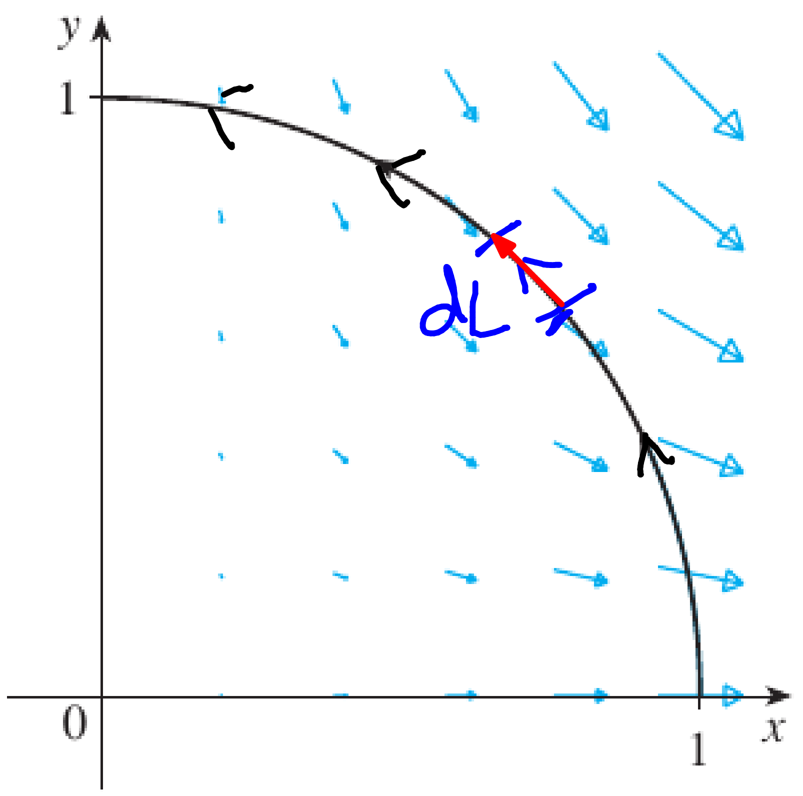
$\vec{r}(t)$: position f^n

W = work done by a force field $\vec{F}$ on a particle moving along a curve C $= \int_C \vec{F} \cdot d\vec{r}$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

$dL \sim$ straight line
& $\vec{F}$ is constant

$dW =$ work done in dL



$$d\vec{r} = \text{displacement vector}$$

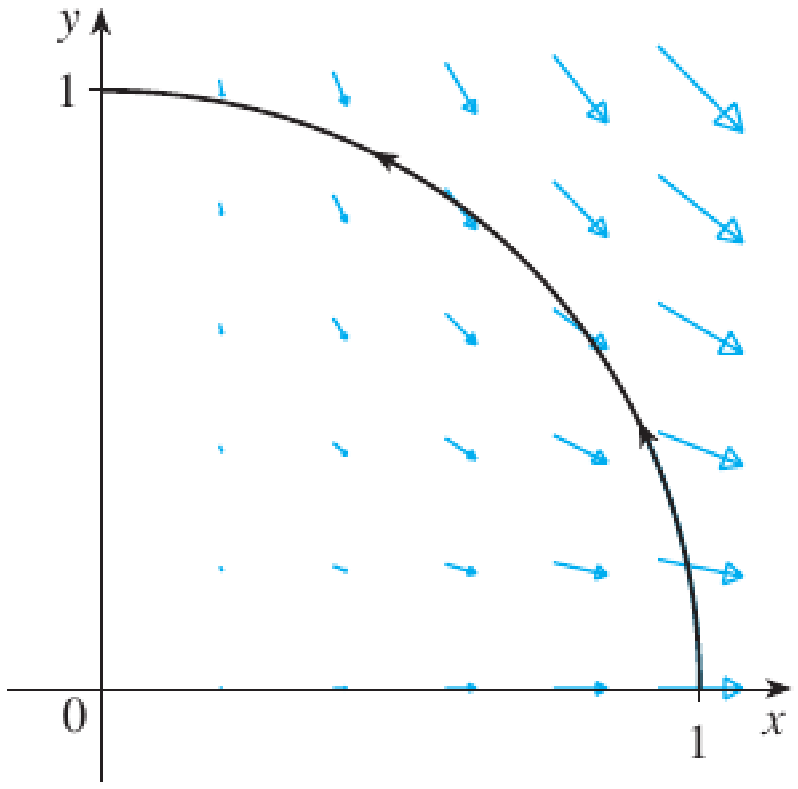
$$= \left(\frac{d\vec{r}}{dt}\right) dt$$

$$dW = \vec{F} \cdot d\vec{r}$$

$$= \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right) dt$$

$$W = \int dW = \int_a^b \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right) dt$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.



$$W = \int_a^b \vec{F}\cdot\underbrace{\left(\frac{d\vec{r}}{dt}\right) dt}_{\text{local displacement}}$$

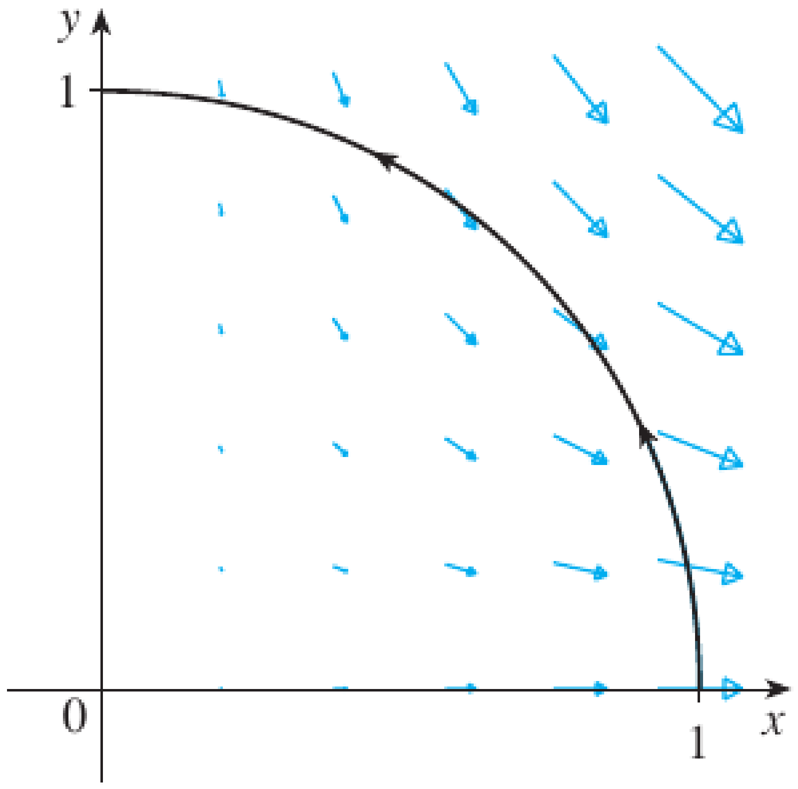
$\dfrac{d\vec{r}}{dt}$: velocity

$\left(\dfrac{d\vec{r}}{dt}\right) dt$: displacement

$$\vec{F}\cdot\left(\frac{d\vec{r}}{dr}\right) dt = dW$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\sin t}_{y}\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

$$W = \int_a^b \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right) dt$$

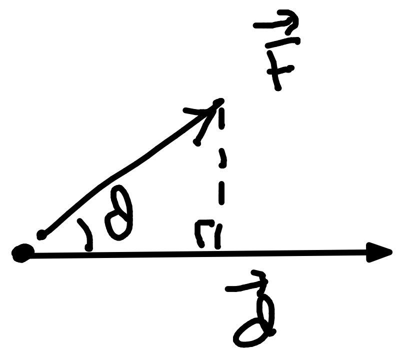


$$\frac{d\vec{r}}{dt} = -\sin(t)\,\hat{i} + \cos(t)\,\hat{j}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = x^2(-\sin(t)) + (-xy)\cos t$$

$$= \cos^2 t\,(-\sin t) + (-\cos t \sin t)\cos t$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \cos^2 t\,(-\sin t) + (-\cos t \sin t)\cos t \quad dt$$



$$W = |\vec{F}||\vec{d}| \cos\theta$$

$$= (|\vec{F}| \cos\theta)\, |\vec{d}|$$

$$= \begin{pmatrix} \text{component of} \\ \vec{F} \text{ in the direction} \\ \text{of motion} \end{pmatrix} |\vec{d}|$$

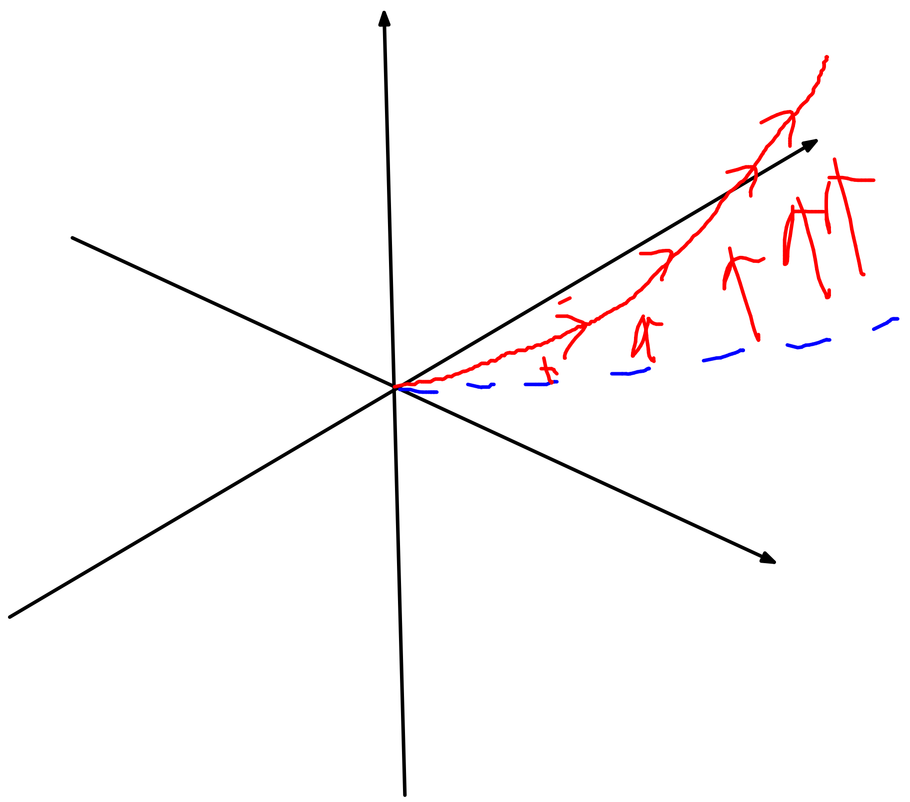
Q. can we work out $(|\vec{F}| \cos\theta)$ without explicitly calculating θ

$$|\vec{F}| \cos\theta = \vec{F} \cdot \begin{pmatrix} \text{unit vector} \\ \text{in the direction} \\ \text{of motion} \end{pmatrix} = \vec{F} \cdot \frac{\vec{d}}{|\vec{d}|}$$

$$W = \left(\vec{F} \cdot \frac{\vec{d}}{|\vec{d}|}\right) |\vec{d}| = \vec{F} \cdot \vec{d}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$



$$\vec{F} \cdot \vec{\gamma}'(t) = \text{—}$$

$$\int_0^1 \left(\vec{F} \cdot \vec{\gamma}'(t)\right) dt = ??$$

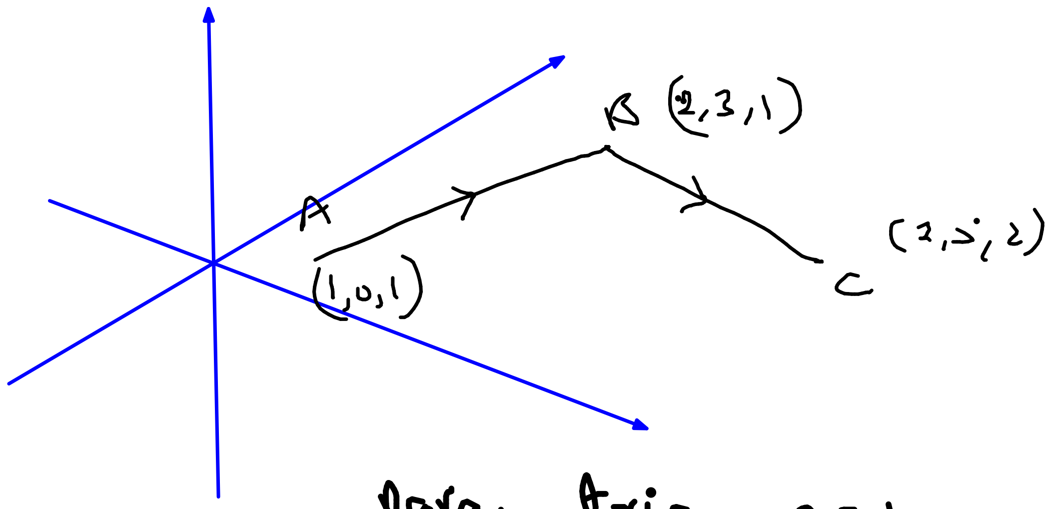
$$= \frac{27}{28}$$

13. $\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$, C consists of line segments from (1, 0, 1) to (2, 3, 1) and from (2, 3, 1) to (2, 5, 2)

$$\int_C \vec{F}\cdot d\vec{r}$$

$$\vec{F} = (x+yz)\hat{i} + 2x\hat{j} + (xyz)\hat{k}$$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz \quad , \text{where} \quad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$



$$= \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}$$

Parametric eq's

AB

$$\vec{r}(t) = \vec{r}_A + t\,\vec{r}_{AB}$$

$$= \vec{r}_A + t(\vec{r}_B - \vec{r}_A) = (\hat{i} + \hat{k}) + t\,(\hat{i} + 3\hat{j})$$

BC

$$\vec{r}(t) = \vec{r}_B + t\left(\vec{r}_C - \vec{r}_B\right)$$

$$=$$

$\rightarrow$ rest work it out yourself.