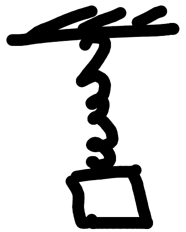
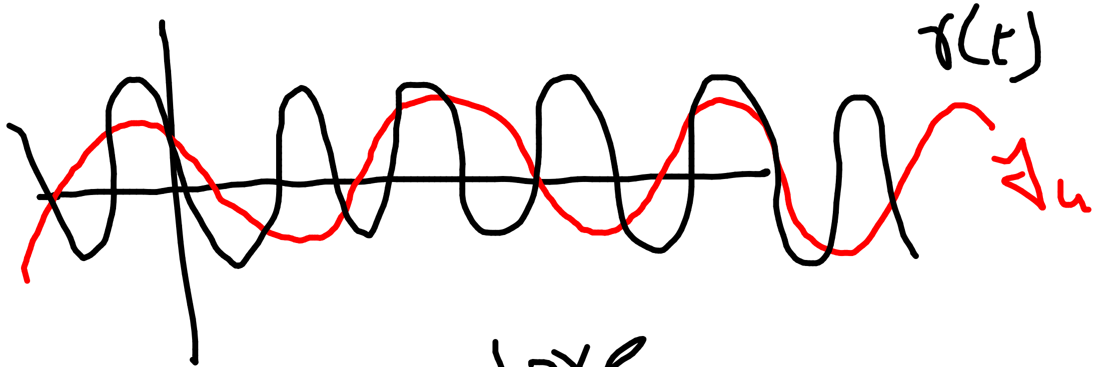
End Sem
on 8th
May

Syllabus: Everything after midsem
+ selected section from
midsem
(email on this)

Last time:



$$y'' + ky = \sin \omega t$$



we have so far:

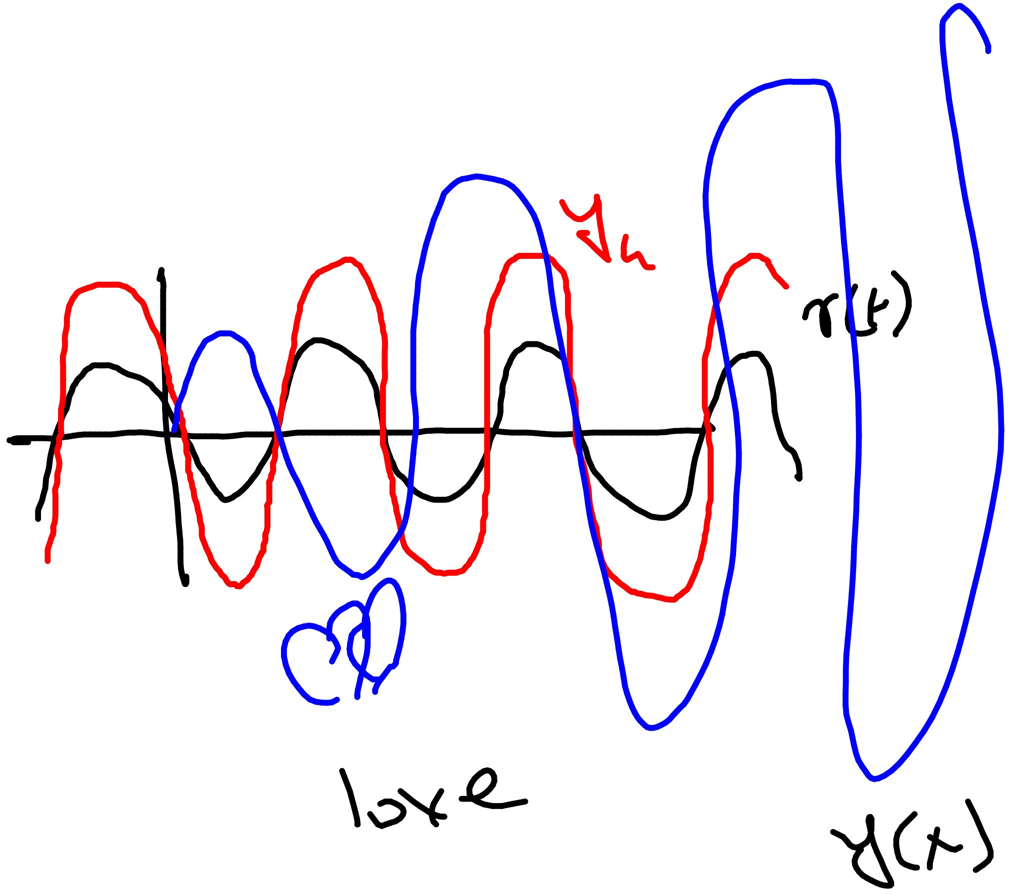
→ solve homogenous sol^n y_h

→ solve for y_p:

e.g. $y_h = 5 \sin 2t$

$r(t) = 3 \sin 5t$

resonance ??



2.9 Modeling: Electric Circuits

sum of voltage drop = voltage supplied
accross each component

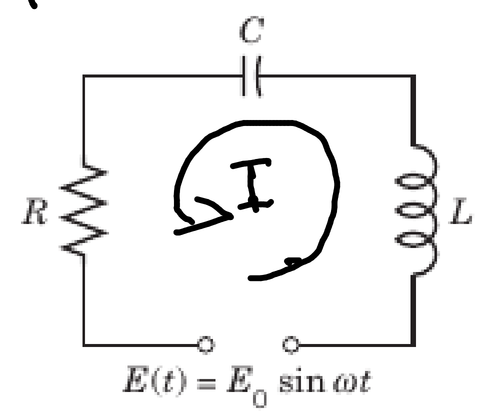
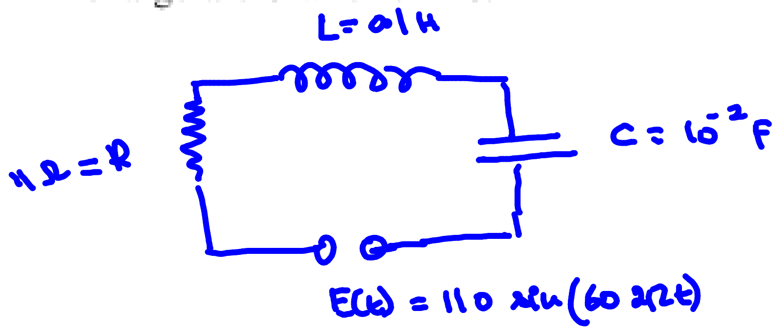


Fig. 61. *RLC*-circuit

Name	Symbol	Notation	Unit	Voltage Drop
Ohm's Resistor		R Ohm's Resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L \frac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

RLC-Circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$ F (farad), which is connected to a source of EMF $E(t) = 110 \sin(60 \cdot 2\pi t) = 110 \sin 377\, t$ (hence 60 Hz = 60 cycles/sec, the usual in the U.S. and Canada; in Europe it would be 220 V and 50 Hz). Assume that current and capacitor charge are 0 when $t = 0$.



$I(0) = 0, \quad q(0) = 0$

$$\begin{pmatrix}\text{voltage drop}\\ \text{over}\\ R\end{pmatrix} + \begin{pmatrix}\text{voltage drop}\\ \text{over}\\ L\end{pmatrix} + \begin{pmatrix}\text{voltage drop}\\ \text{over}\\ C\end{pmatrix} = 110 \sin 377 t$$

$$\left[11 I + 0.1 \frac{dI}{dt} + 100 q = 110 \sin 377 t \right]$$

problem! we don't know q.

$\left(I = \frac{dq}{dt} \right)$ so what??

differentiate the eqn

$$0.1 \frac{d^2 I}{dt^2} + 11 \frac{dI}{dt} + 100 I = (110)(377) \cos(377t)$$

easily solvable

8–14 Find the **steady-state current** in the *RLC*-circuit in Fig. 61 for the given data. Show the details of your work.

$$R = 4\ \Omega,\ L = 0.5\ \mathrm{H},\ C = 0.1\ \mathrm{F},\ E = 500 \sin 2t\ \mathrm{V}$$

8–14 Find the **steady-state current** in the *RLC*-circuit in Fig. 61 for the given data. Show the details of your work.

$$R = 4\ \Omega,\ L = 0.1\ \mathrm{H},\ C = 0.05\ \mathrm{F},\ E = 110\ \mathrm{V}$$

2.10 Solution by Variation of Parameters

$y'' + ay' + by = r(x)$

Recall the process:

→ solve homogeneous eq^n

$y_h = c_1 y_1 + c_2 y_2$

→ $y = y_h + y_p$

how to find y_p

variation of parameters

undetermined coeffs

(Wronskian) $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$u = -\int \frac{y_2 r}{W}\,dx, \qquad v = \int \frac{y_1 r}{W}\,dx.$$

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

Solve the nonhomogeneous ODE

$$y'' + y = \sec x$$

homogenous part

$$y'' + y = 0$$

$$y_h = C_1 \underbrace{\cos x}_{y_1} + C_2 \underbrace{\sin(x)}_{y_2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u = -\int \frac{y_2 r}{W} dx = -\int \frac{\sin x \sec x}{1} dx = \log|\cos x|$$

$$v = \int \frac{y_1 r}{W} dx = \int \frac{\cos x \sec x}{1} dx = x$$

$$y_p = u y_1 + v y_2 = (\log|\cos x|)\cos x + x \sin x$$

→ solve homogenous part get $y_h = c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$r = \sec x$

$$u = -\int \frac{y_2 r}{W} dx, \qquad v = \int \frac{y_1 r}{W} dx.$$

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

Q:

$$y'' + 9y = \csc 3x$$

$$y_h = A \underbrace{\cos 3x}_{y_1} + B \underbrace{\sin 3x}_{y_2}$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$u = -\int \frac{y_2 r}{W} dx = -\int \frac{\cancel{\sin 3x}\,\cancel{\csc 3x}}{3} dx = \frac{-x}{3}$$

$$v = \int \frac{y_1 r}{W} dx = \frac{1}{3} \int \frac{\cos 3x \cdot \csc 3x}{} dx$$

$$= \frac{1}{9} \log|\sin 3x|$$

$$y_p = \frac{-x}{3} \cos 3x + \frac{1}{9} \log|\sin 3x| \sin 3x$$

→ solve homogenous part get $y_h = c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$r = \sec x$

$$u = -\int \frac{y_2 r}{W} dx, \qquad v = \int \frac{y_1 r}{W} dx.$$

Q: $y'' + 5y' + 6y = e^{-x}$.

Solve using variation of parameters

$$y_h = c_1 \underbrace{e^{-2x}}_{y_1} + c_2 \underbrace{e^{-3x}}_{y_2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = e^{-5x}\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} = -e^{-5x}$$

$$u = -\int \frac{e^{-3x} e^{-x}}{-e^{-5x}}\, dx = \int e^{x}\, dx = e^{x}$$

$$v = \int \frac{e^{-2x} e^{-x}}{-e^{-5x}}\, dx = -\frac{e^{2x}}{2}$$

$$y_p = e^{x} e^{-2x} + \left(-\frac{e^{2x}}{2}\right) e^{-3x} = \frac{e^{-x}}{2}$$

$$\boxed{\text{general sol}^n: \quad y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{-x}}{2}}$$

→ solve homogenous part get $y_h = c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$r = \sec x$

$$\boxed{u = -\int \frac{y_2 r}{W}\, dx, \qquad v = \int \frac{y_1 r}{W}\, dx.}$$

$$y_p = u y_1 + v y_2$$

$$y = y_h + y_p$$

$$y'' + ay' + by = r(x)$$

→ solve homogenous part
get $y_h = c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$r = \sec x$

$$\boxed{u = -\int \frac{y_2 r}{W}\,dx, \qquad v = \int \frac{y_1 r}{W}\,dx.}$$

$$y_p = u y_1 + v y_2$$

$$y = y_h + y_p$$

Idea behind variation of parameters:

→ what do we want ?? y_p

→ lets hope y_p looks like

$$y_p = u y_1 + v y_2$$

where, u & v are some unknown formula, need to be solved for.

Recall:

$y_h = c_1 y_1 + c_2 y_2$

solves the homogenous part

$y'' + ay' + by = 0$

↳ how to find u, v ??

$$y_p'' + a y_p' + b y_p = r(x)$$

$$(uy_1 + vy_2)'' + a(uy_1 + vy_2)' + b(uy_1 + vy_2) = r(x) \quad \text{— (A)}$$

$$u'y_1 + v'y_2 = 0 \quad \text{— (B)}$$

$$y_p = uy_1 + vy_2$$

$$y_p' = uy_1' + vy_2' + \underbrace{u'y_1 + v'y_2}_{\text{this is assumed to be zero}}$$

$$y_p' = uy_1' + vy_2'$$

$$y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$$

$$y_p'' + ay_p' + by_p = r(x)$$

$$u(\underbrace{y_1'' + ay_1' + by_1}_{=0 \text{ why?!}}) + v(\underbrace{y_2'' + ay_2' + by_2}_{=0}) + u'y_1' + v'y_2' = r(x)$$

$$\boxed{u'y_1' + v'y_2' = r(x)} \quad \text{— } \boxed{A_2}$$

now we have two eqns for two unknowns u, v

$$u'y_1 + v'y_2 = 0 \quad \text{—— (B)}$$

$$u'y_1' + v'y_2' = r(x) \quad \text{—— } (A_2)$$

→ H.W: verify that if you solve eqn (B) & A_2

for u' , v'

& then u , v

→ you get the method of variation of parameters.

Idea of the Method.

Today:

→ discussion on variation of parameters

$$y'' + ay' + by = r(x)$$

$$u = -\int \frac{r y_2}{W}\,dx \qquad v = \int \frac{r y_1}{W}\,dx$$

$$y_p = u y_1 + u y_2$$

derivation of this formula for y_p

→ A bit of review

Quick review of ODE sections

Chapter ① 1st order ODE

→ Separation of variables

↪ may be it will become separable after change of variables

$y' = f(\frac{y}{x})$ | $v = \frac{y}{x}$

→ Exact ODE : $M\,dx + N\,dy = 0 \iff M + Ny' = 0$

hoping to find u s.t. $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

→ solution: $u(x,y) = C$

such u exist : if the eq^n is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

find this u by integrating $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

↪ if the eq^n is not exact, then we try to make it exact through Integration factors . I

$(IM)\,dx + (IN)\,dy = 0$ is exact

→ Word problems

→ Chapter 2 only covered eqns of one form

$$y'' + ay' + by = r(x)$$

a, b : constants
r(x) : not constant

→ solution of homogenous eqn :

→ $y = y_h + y_p$

→ Two methods for find y_p

↳ undetermined coeffts

↳ variation of parameters

→ applications : mass-spring system, LCR circuits.