Recall: we were discussing exact ODE:

$$M + N\frac{dy}{dx} = 0$$

if exact: we find $u(x, y(x))$

$$\frac{du}{dx} = M + N\frac{dy}{dx}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Downarrow$

exact

if not find I s.t.

$$IM + IN\frac{dy}{dx} = 0 \quad \text{is exact}$$

1.5 Linear ODEs.

$$y' + p(x)y = r(x)$$

$$I(x) = e^{\int p(x)\,dx}$$

$$I(x)\,y' + I(x)\,p(x)\,y \stackrel{?}{=} \frac{d}{dx}\left(I(x)\,y\right)$$

RHS: $\frac{d}{dx}\left(I(x)\,y\right) = \frac{dI}{dx}\,y + I\,\frac{dy}{dx}$

$$I(x) = e^{\int p(x)dx}$$

$$\frac{dI}{dx} = e^{\int p(x)dx} \cdot \frac{d}{dx}\left(\int p(x)\,dx\right) = e^{\int p(x)dx}\,p(x) \cdot = I\,p(x)$$

$$y' + p(x)\,y = r(x)$$

$$y' e^{\int p(x)\,dx} + p(x)\,y\,e^{\int p(x)\,dx} = r(x)\,e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\left(y\,e^{\int p(x)\,dx} \right) = r(x)\,e^{\int p(x)\,dx}$$

$$y\,e^{\int p(x)\,dx} = \int r(x)\,e^{\int p(x)\,dx} + c$$

& solve for y

Q: But how did anybody come up with the formula of $IF = e^{\int p(x)dx}$

Recall the routine

$$M + N\frac{dy}{dx} = 0$$

$$\left[y' + p(x)y = r(x) \right]$$

$$\downarrow$$

$$y' + p(x)y = 0$$

$$\underbrace{p(x)y}_{M} + \underbrace{y'}_{N} = 0$$

→ Try to find IF → $e^{\int p(x)dx}$

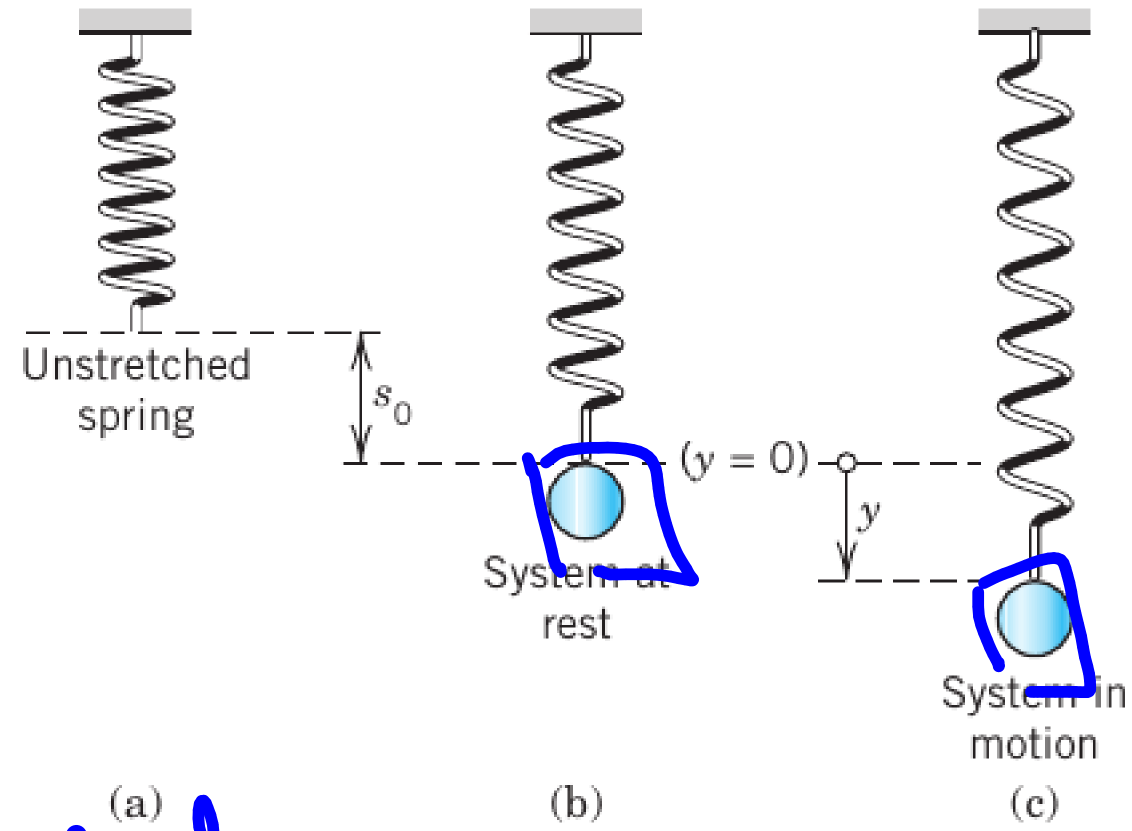
CHAPTER 2

Second-Order Linear ODEs

$=$

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c\,y = r(x)$$

2.4 Modeling of Free Oscillations of a Mass–Spring System



→ mechanical :

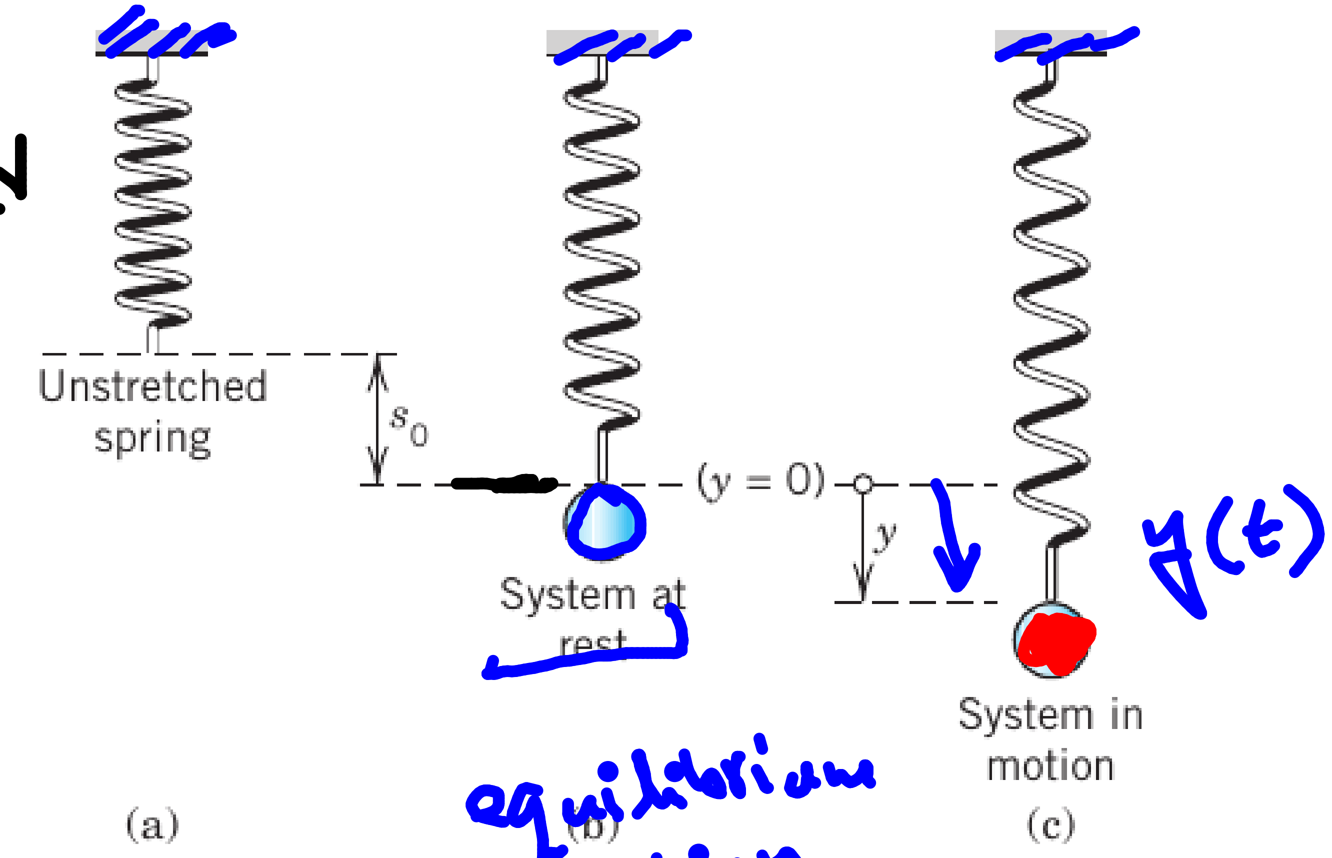
mechanical vibration

& strength of material.

$$F = m \frac{d^2y}{dx^2}$$

2.4 Modeling of Free Oscillations of a Mass–Spring System

+ve dir



m : attached

y(t) : position of the body at time t

Aim: find Eqn for y(t) :

$$m \frac{d^2y}{dt^2} = \text{net force on the body at time } t.$$

$$m \frac{d^2y}{dt^2} = \text{gravity} + \text{spring force} + \text{medium resistance}$$

$$m\frac{d^2y}{dt^2} = mg - k(y+s_0) - \underbrace{c\frac{dy}{dt}}_{\text{experimental}}$$

$\rightarrow$ this eq^n completes the eq^n for y

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = mg - ks_0 \quad , \quad \left.\begin{array}{l} y(0) = y_0 \\ y'(0) = v_0 \end{array}\right\} \text{2 extra conditions required}$$

$$\frac{d^2y}{dt^2} = 5$$

2.9 Modeling: Electric Circuits

$I(t) = ??$

$E_0 \sin\omega(t)$ = sum of voltage drop accross each component

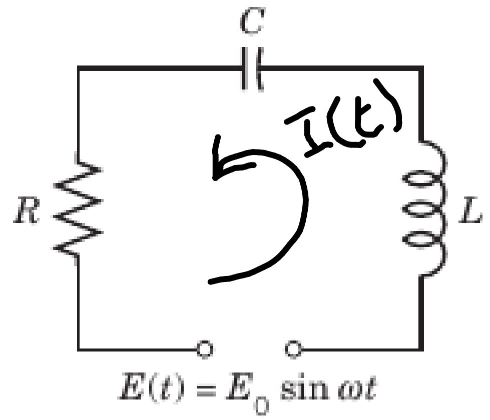


Fig. 61. *RLC*-circuit

$$E_0 \sin\omega(t) = L\frac{dI}{dt} + \frac{1}{C}q + RI$$

Name	Symbol	Notation	Unit	Voltage Drop
Ohm's Resistor		R Ohm's Resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

Fig. 62. Elements in an *RLC*-circuit

$$I = \frac{dq}{dt}$$

$$q = \int I(t)\,dt$$

$$E_0 \sin\omega(t) = L\frac{dI}{dt} + \frac{1}{C}\int I(\tau)\,d\tau + RI$$

$$\boxed{\omega E_0 \cos(\omega t) = L\frac{d^2I}{dt^2} + \frac{1}{C}I + R\frac{dI}{dt}}$$

Recall: 2nd order ODEs

→ linear

→ constant coefficient

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = r(x)}$$

↳ first learn to solve it with $r(x) = 0$

↳ then we learn some methods with $r(x) \neq 0$

2.2 Homogeneous Linear ODEs with Constant Coefficients

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Q: find a formula for $y(x)$ which satisfies

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \qquad \Big| \qquad \frac{d}{dx} = D$$

$$(D^2 - 5D + 6)y = 0$$

$$(D-3)\underbrace{(D-2)y} = 0$$

find a y s.t.

$$(D-2)y = 0$$

$$\frac{dy}{dx} = 2y$$

$$\rightarrow y = e^{2x}$$

$$(D-3)y = 0$$

$$\frac{dy}{dx} = 3y$$

$$y = e^{3x}$$

Q: find a formula for $y(x)$ which satisfies

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$y(0) = 1$

$y'(0) = 5$

$$(D^2 - 5D + 6)y = 0$$

$$(D-3)\underbrace{(D-2)y}_{} = 0$$

$y -$

$y_1 = c_1 e^{2x}$ ✓

$y_2 = c_2 e^{3x}$ ✓

$$y = c_1 e^{2x} + c_2 e^{3x}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \qquad D = \frac{d}{dx}$$

$$(D^2 - 5D + 6)\, y = 0$$

→ find roots of

$$D^2 - 5D + 6 = 0 \quad \Big] \text{ auxillicary eq}^n$$

$$\alpha = 3, 2$$

→ solution = $y = c_1 e^{3x} + c_2 e^{2x}$

General steps for solving

$$\left[ay'' + by' + cy = 0 \right]$$

steps:

↳ auxilliary eqn

$$a\alpha^2 + b\alpha + c = 0$$

↳ solve for roots

$$\alpha_1 \quad \alpha_2$$

↳ general solution

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$

Q: Solve

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$$

↳ auxilliary eqⁿ $\alpha^2 + 7\alpha + 10 = 0$

↳ find roots α_1, α_2

roots $= -2, -5$

↳ general solution =

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$

$$y = c_1 e^{-2x} + c_2 e^{-5x}$$

Problem: what if roots of

$$a\alpha^2 + b\alpha + c = 0$$

are complex?!

say: roots are $\alpha = \alpha \pm i\beta$

→ general solution

$$y = c_1 \underbrace{e^{(\alpha+i\beta)x}}_{} + c_2 e^{(\alpha-i\beta)x}$$

→ fine but imaginary formulas are not welcome in most places

$$e^{(5+2i)} = e^5 \cdot e^{2i}$$

$$e^{2i} := \cos 2 + i \sin 2$$

$$e^{5+2i} = e^5 \left(\cos 2 + i \sin 2\right)$$

$$e^{(\alpha+i\beta)x} = e^{\alpha x + i\beta x}$$

$$= e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right]$$

Solution of ODE :

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Q: Solve: $y'' + y' + y = 0$ ——— (1)

aux eqn $\quad d^2 + d + 1 = 0$

$$d = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

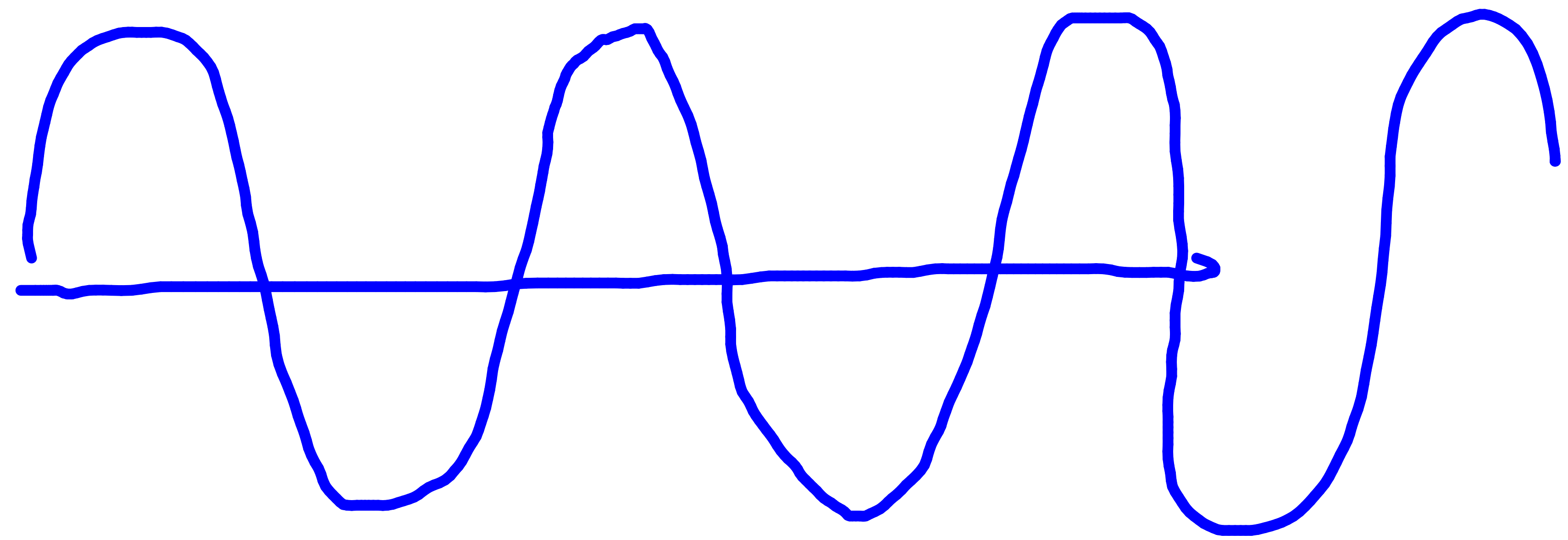
$$e^{\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)x} = e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) + i \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

general solution of (1)

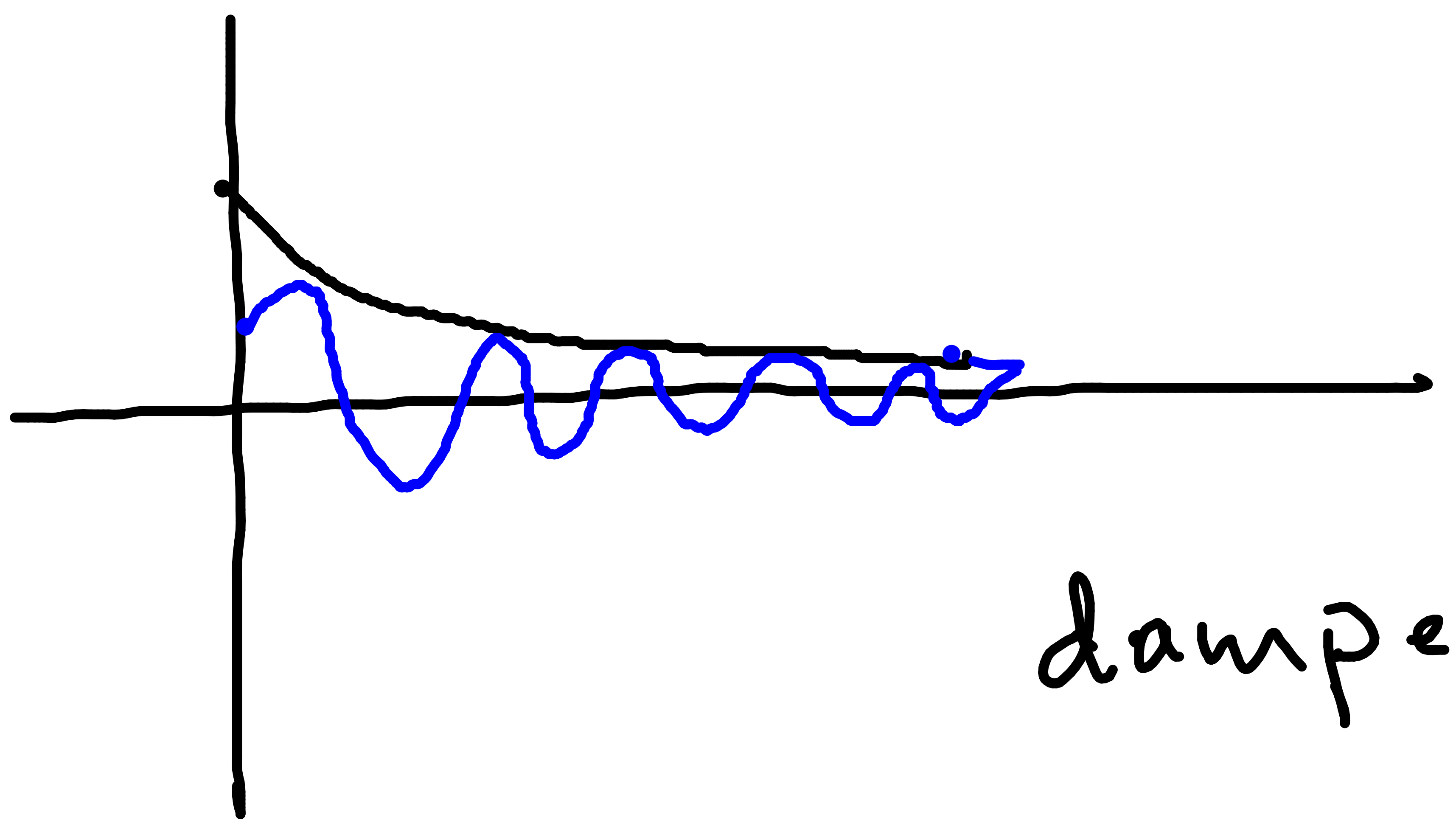
$$y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

graph of $e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$??

$e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right)$



$e^{-x/2}$



damped oscillation

Q: Solve:

$$y'' + 2y' + 2y = 0$$

aux eqⁿ: $\alpha^2 + 2\alpha + 2 = 0$

$$\alpha = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

gen solⁿ:

$$y = e^{-x}\left[c_1 \cos(x) + c_2 \sin(x)\right]$$

Q:

$$y'' - 2y' + y = 0 \qquad \Big| \qquad y(0) = 2, \quad y'(0) = 0$$

$$\alpha^2 - 2\alpha + 1 = 0 \qquad \alpha = 1$$

$$y(x) = c_1 e^{x} + c_2 (x e^{x})$$

→ idea: if roots are repeated α

→ check $e^{\alpha x}$ $\quad$ $x e^{\alpha x}$

→ general sol^n: $y = c_1 e^{\alpha x} + c_2 \underbrace{x e^{\alpha x}}$

a.

$$y'' - 2y' + y = 0$$

$$\alpha^2 - 2\alpha + 1 = 0 \qquad \alpha = 1$$

$y(0) = 2$

$y'(0) = 0$

→ check: xe^x solves

$$y = xe^x$$

$$-2\left(y' = xe^x + e^x\right)$$

$$y'' = xe^x + 2e^x$$

$$0 = 0$$