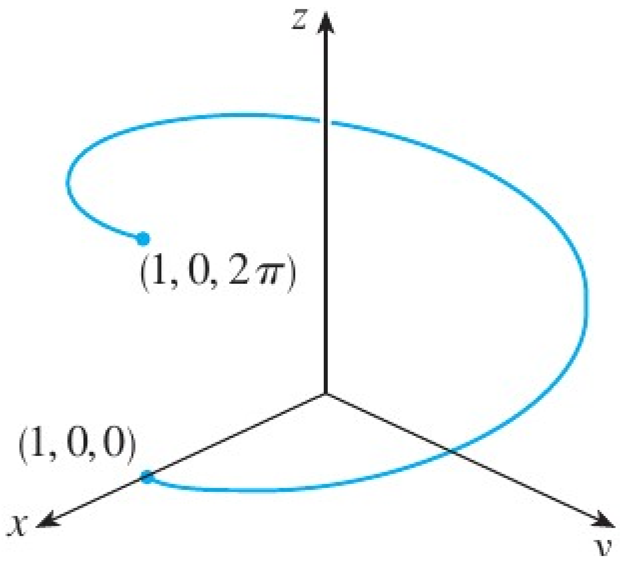
Today’s topic

- Arc Length and Curvature (Sec 10.8)

Q1. Whats the length of a curve?

Q2. How curved is the curve?

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



Suppose the curve is described by the formula

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad\qquad a \le t \le b$$

Length can be evaluated by integrating the **speed**

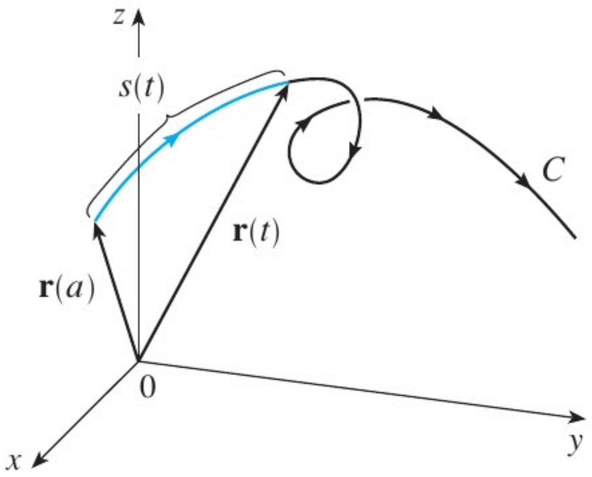
$$\begin{aligned}
\text{length} =: L &= \int_a^b \text{speed}\,\mathrm{d}t \\
&= \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,\mathrm{d}t
\end{aligned}$$

Q. Using this formula, verify that formula of circumference of a circle.

Find the length of the curve.

$$\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$$

arc length function s



$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

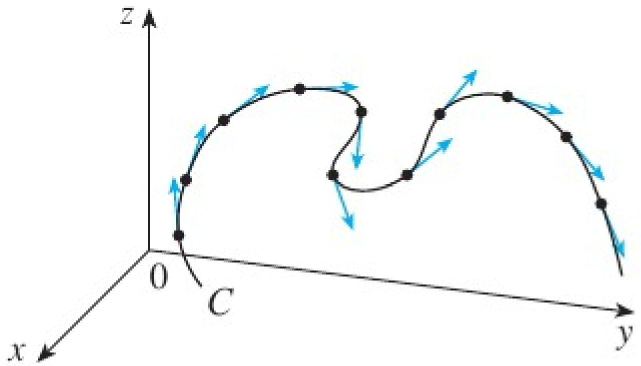
$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length**

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

9. Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3 \sin t$, $y = 4t$, $z = 3 \cos t$ in the positive direction. Where are you now?

CURVATURE



$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

8 DEFINITION The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

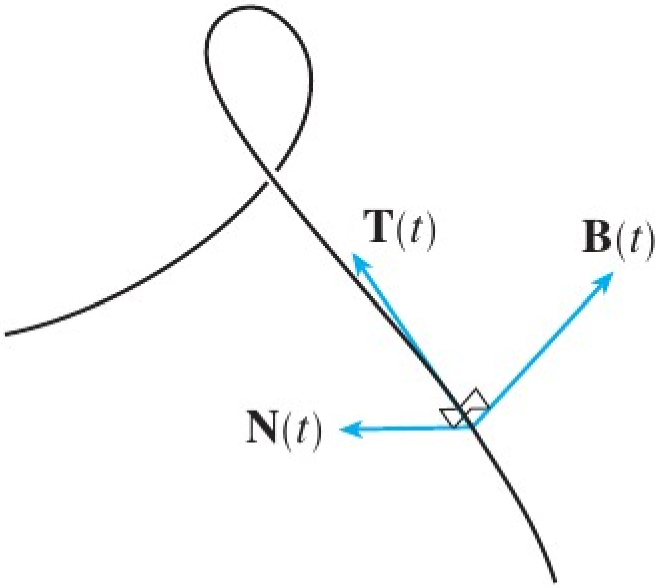
10 THEOREM The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

THE NORMAL AND BINORMAL VECTORS



$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

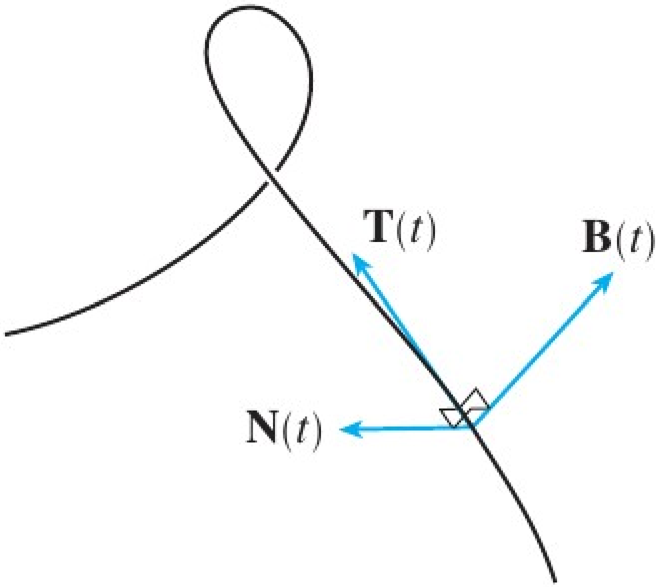
principal unit normal vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

binormal vector

The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P. The name comes from the Latin *osculum*, meaning "kiss."

THE NORMAL AND BINORMAL VECTORS



$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

principal unit normal vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

binormal vector

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**)

40. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$. Graph both osculating circles and the parabola on the same screen.