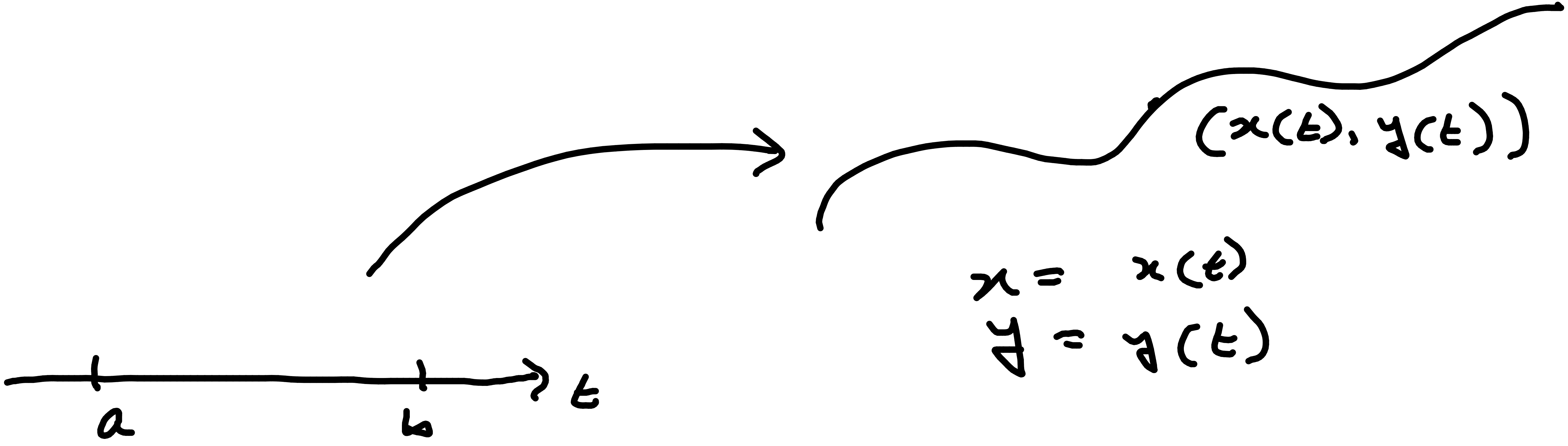
Recall parametric curves:



$$x = x(t)$$
$$y = y(t)$$

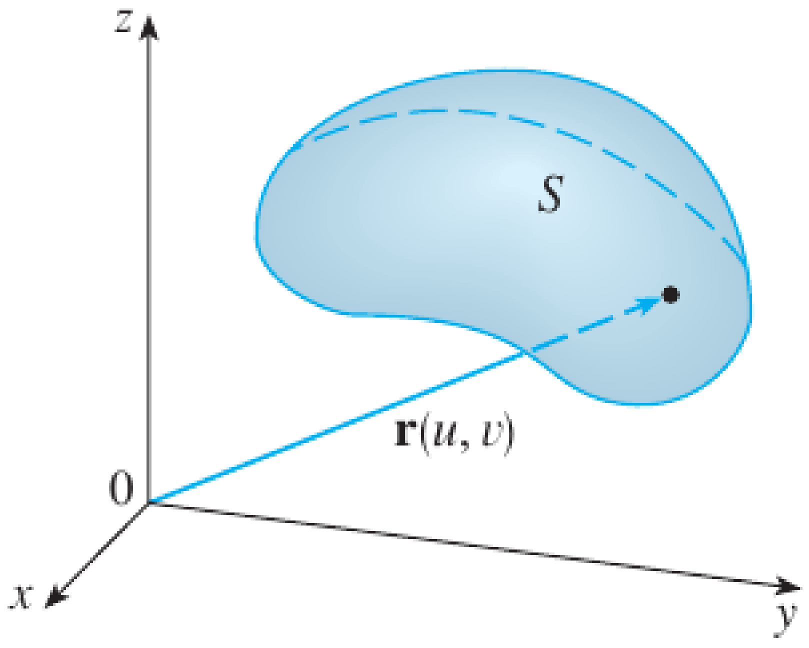
Parametric Surfaces

$$x = x(u,v)$$

$$y = y(u,v)$$

$$z = z(u,v)$$

13.6 PARAMETRIC SURFACES AND THEIR AREAS

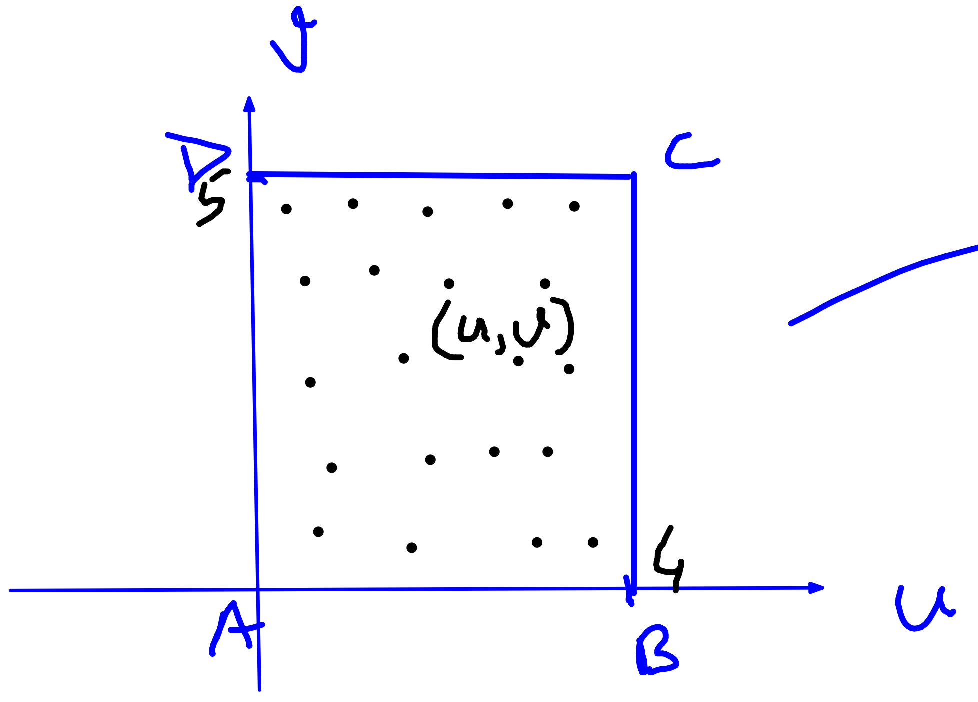


$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

Q.

$0 \le u \le 4$

$0 \le v \le 5$

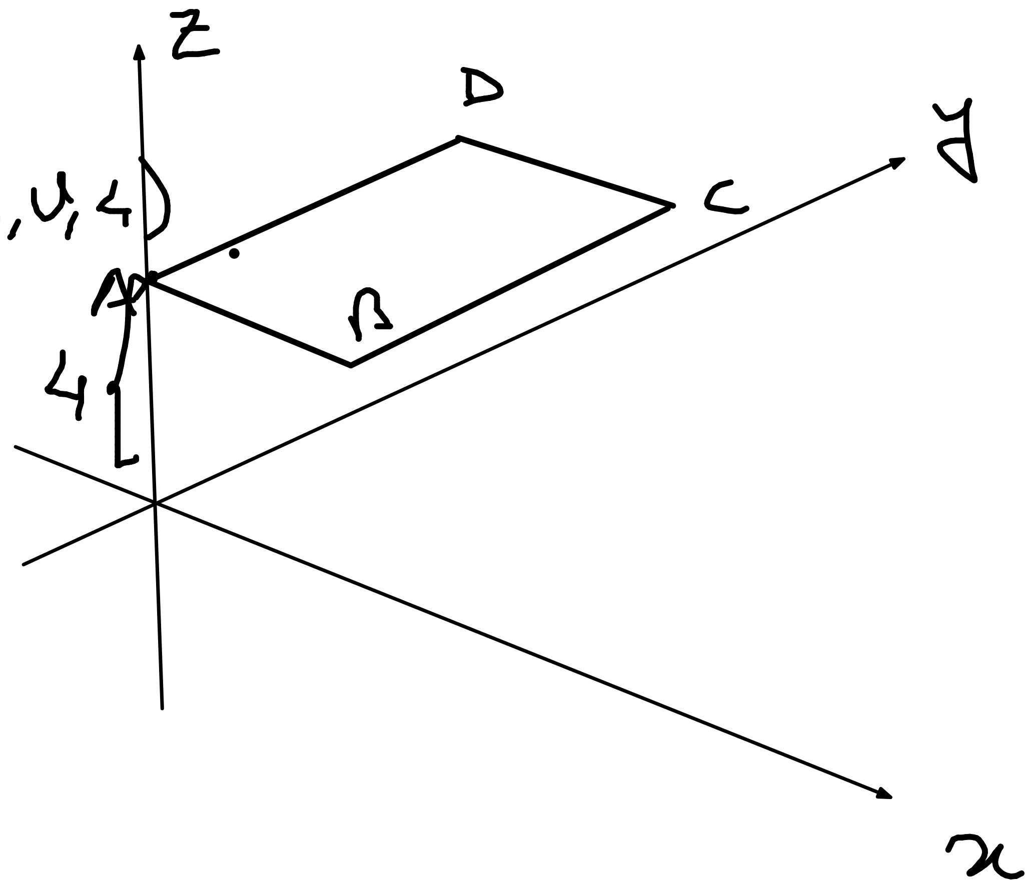


$x = u$

$y = v$

$z = 4$

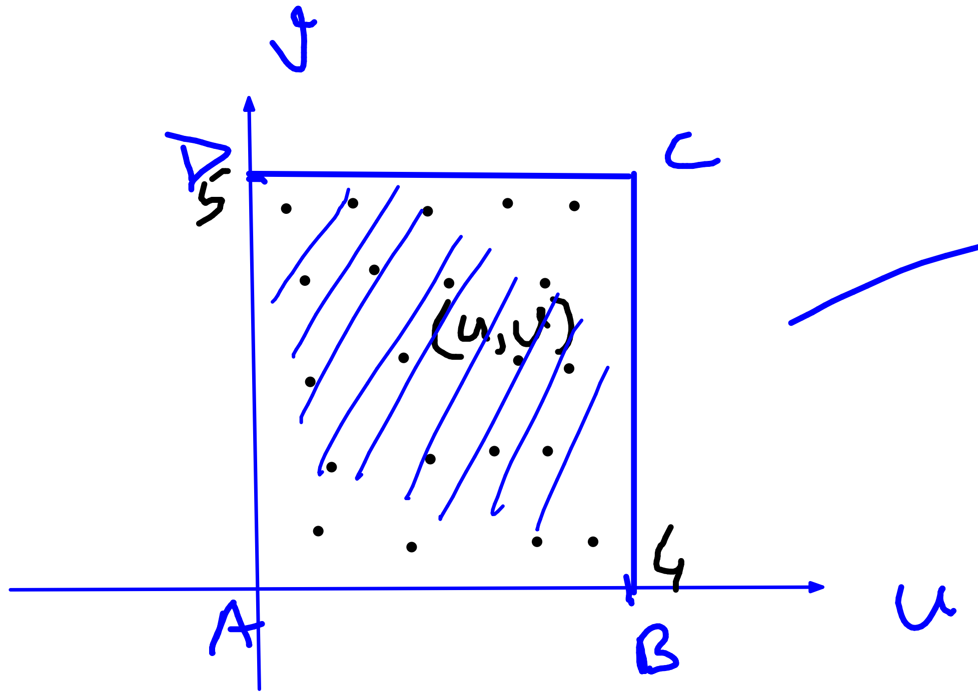
$(x,y,z) = (u,v,4)$



Q.

$0 \le u \le 4$

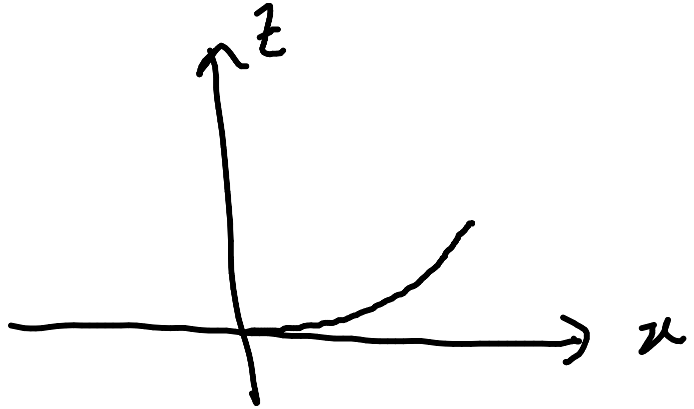
$0 \le v \le 5$

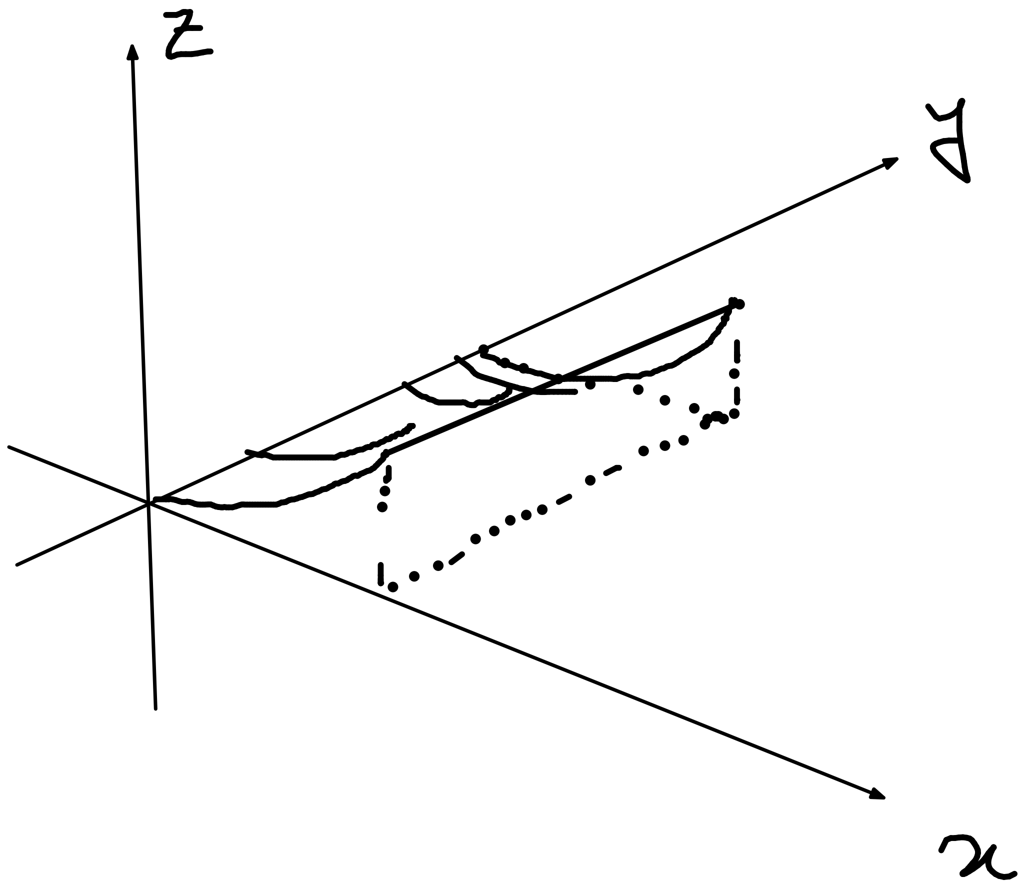


$x = u$

$y = v$

$z = u^2$

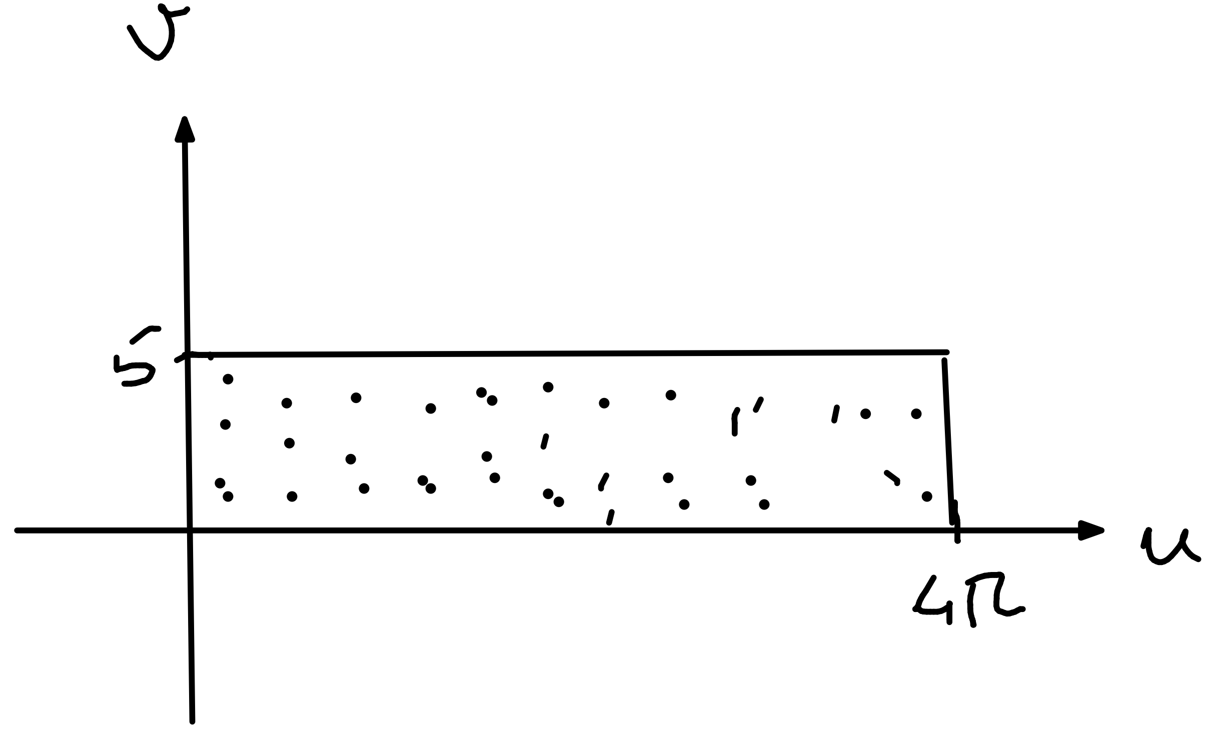




Q.

$0 \le u \le 4\pi$

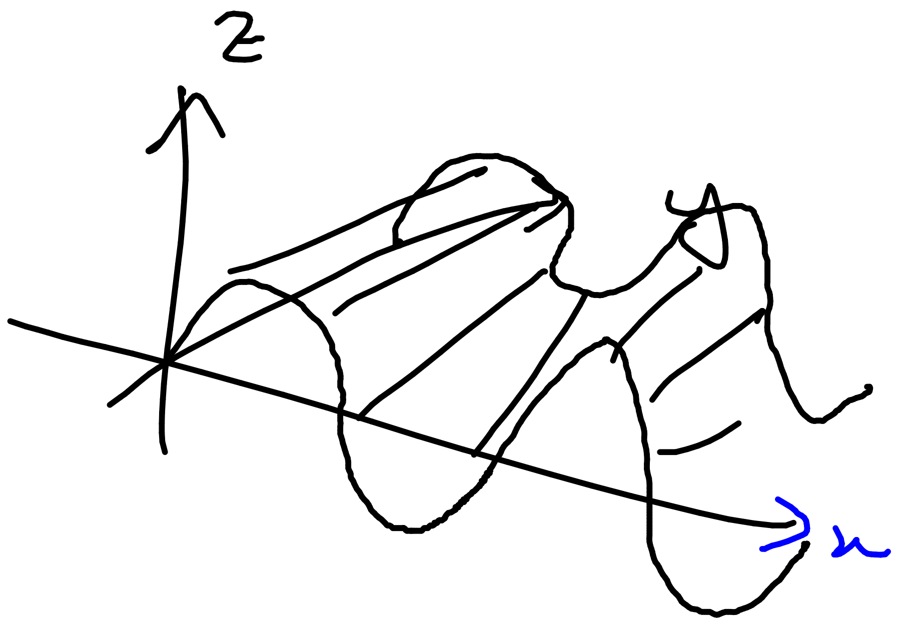
$0 \le v \le 5$

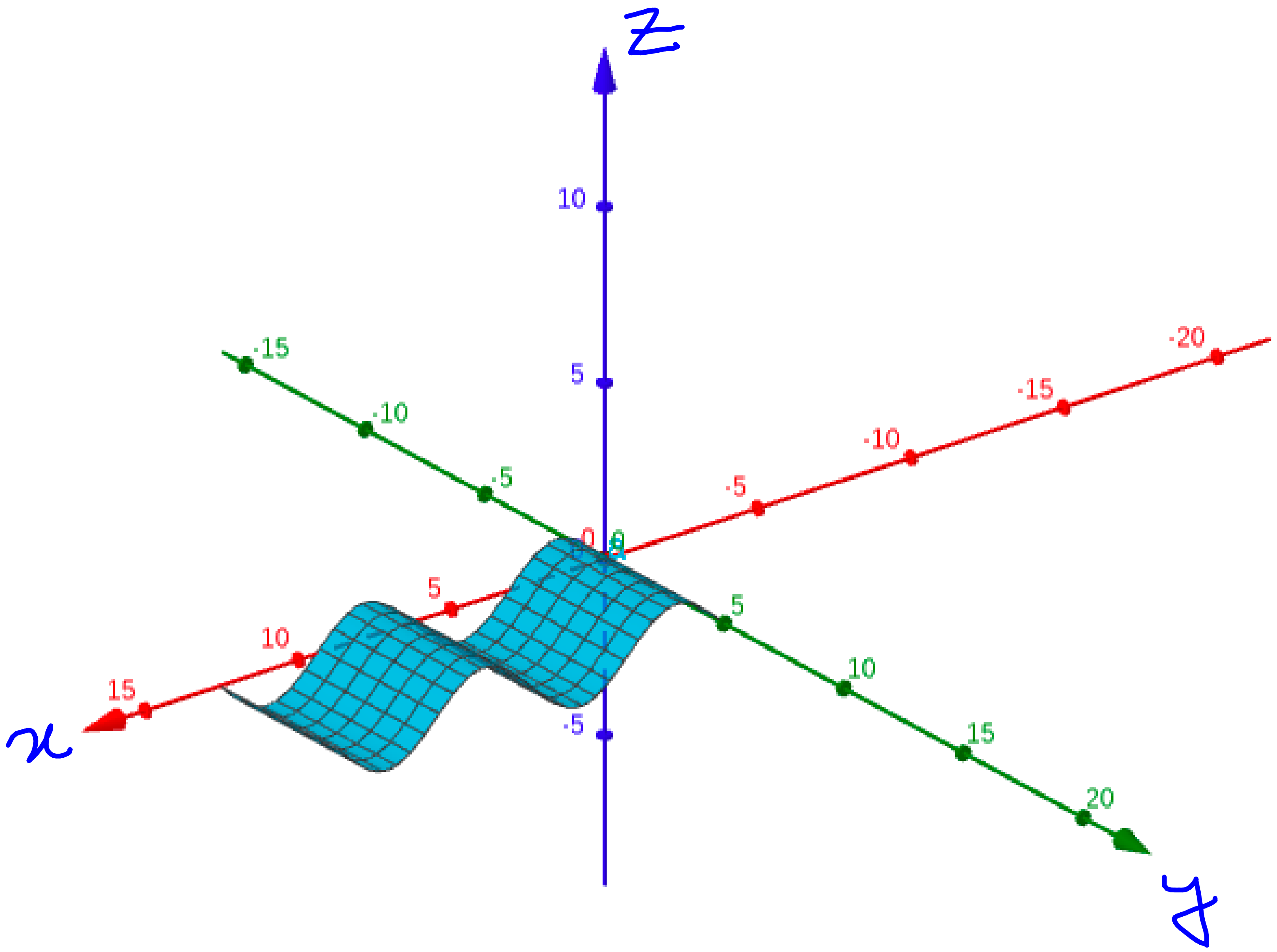


$x = u$

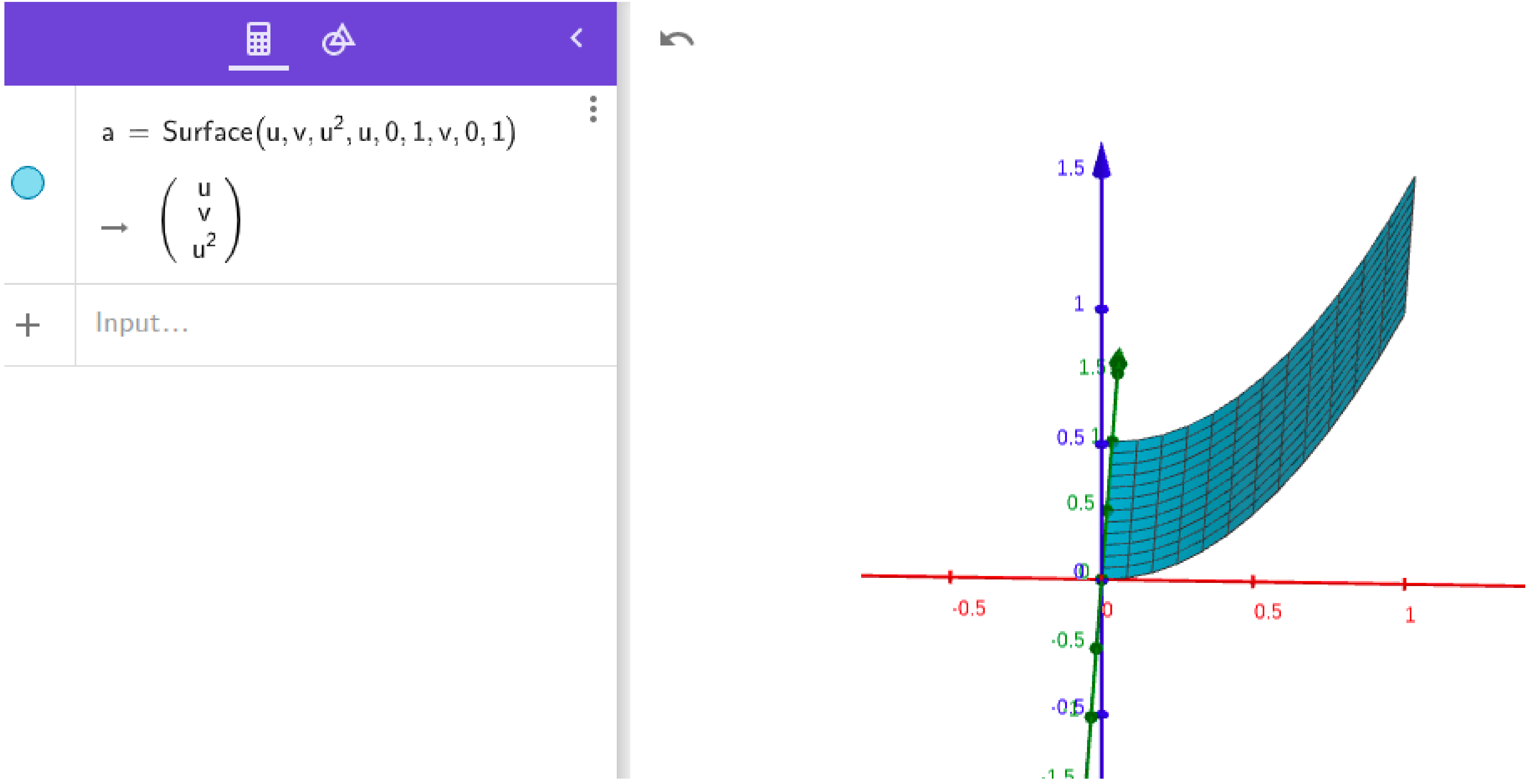
$y = v$

$z = \sin(u)$



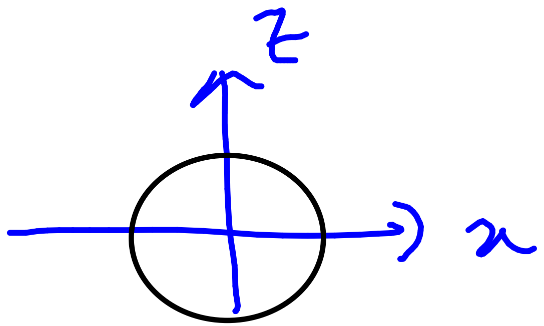


z
10
5
-5
-15
-10
-5
5
10
15
20
y
-20
-15
-10
-5
5
10
15
x
0


a = Surface(u, v, u^2, u, 0, 1, v, 0, 1)
→ (u, v, u^2)
Input…
1.5
1
0.5
0
-0.5
0.5
1

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = \underbrace{2\cos u}_{x}\,\mathbf{i} + \underbrace{v}_{y}\,\mathbf{j} + \underbrace{2\sin u}_{z}\,\mathbf{k}$$



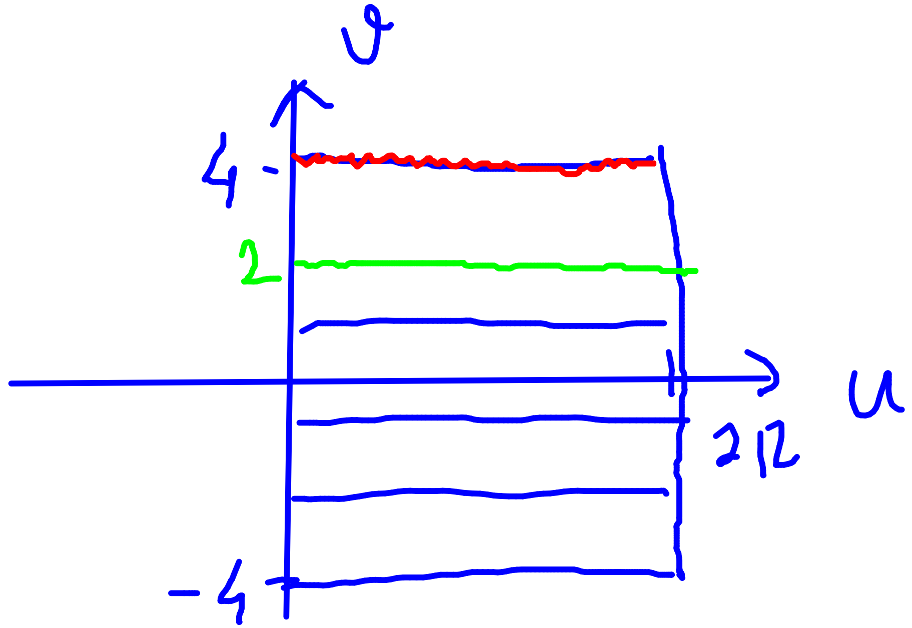
$x = 2\cos u$

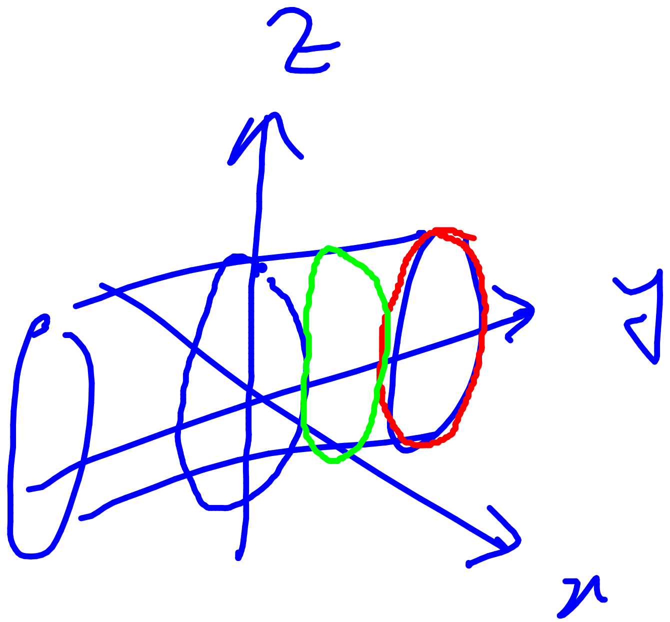
$y = v$

$z = 2\sin u$

$0 \le u \le 2\pi$

$-4 \le v \le 4$

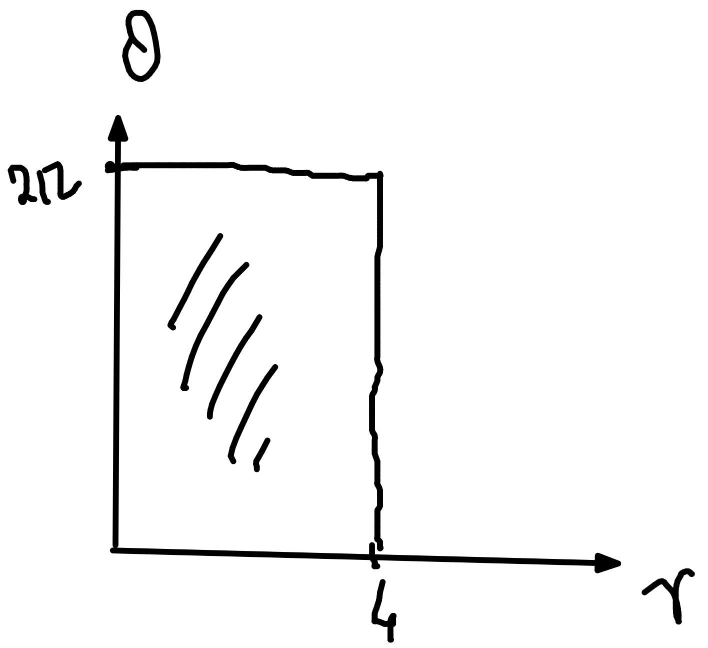




EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

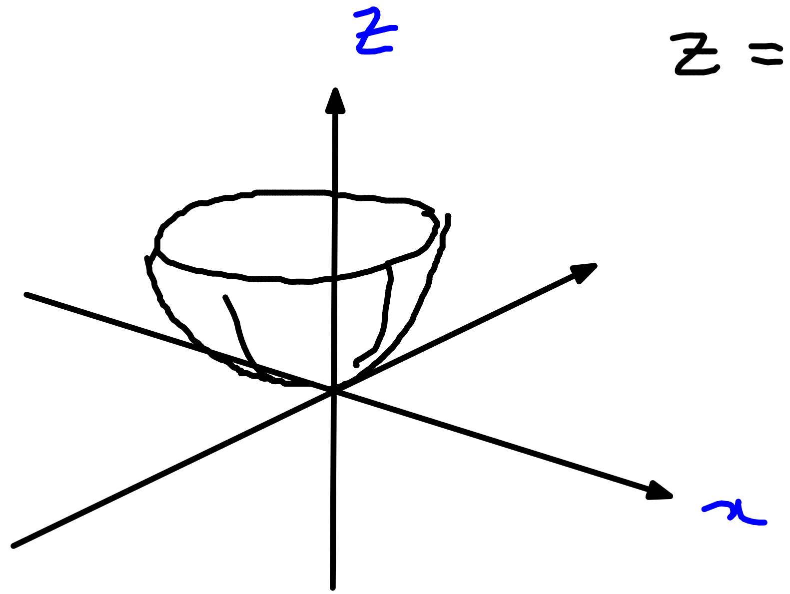
d.



$x = r\cos\theta$

$y = r\sin\theta$

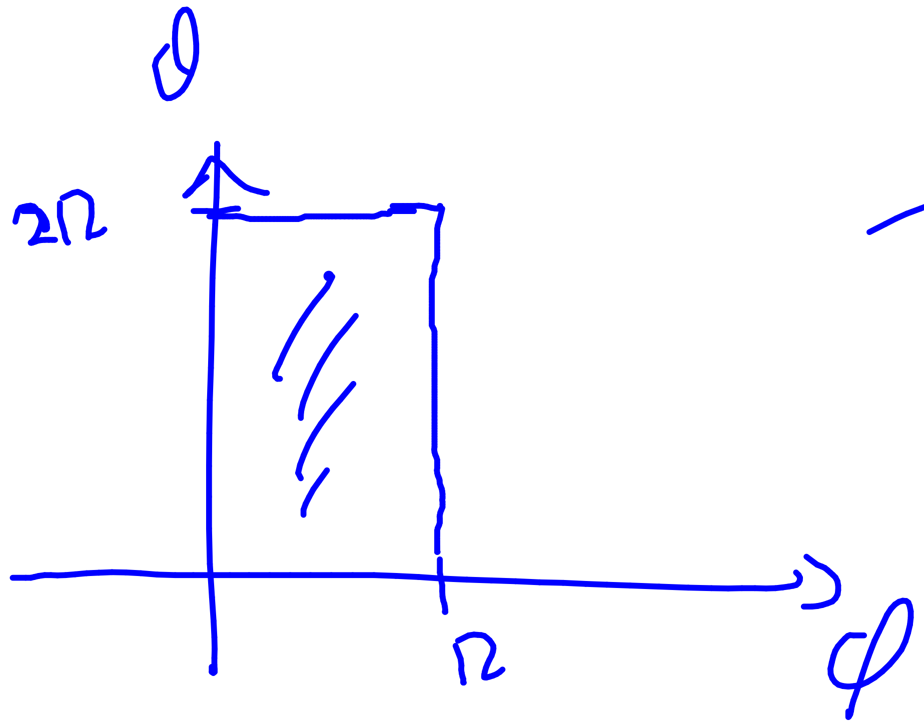
$z = r^2$



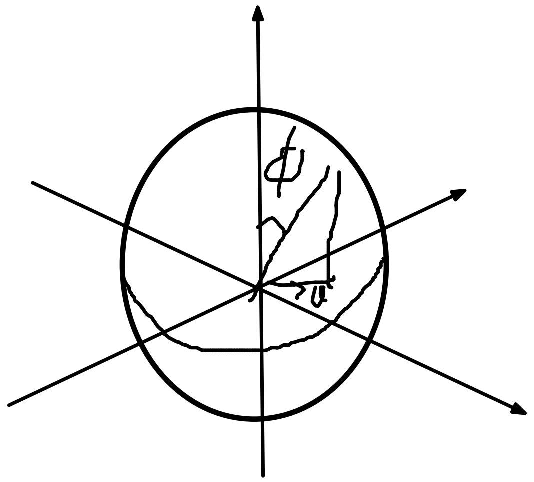
$z = x^2 + y^2$

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 4 Find a parametric representation of the sphere $x^2 + y^2 + z^2 = 3^2$



$$\boxed{\begin{aligned} x &= 3\sin\phi\cos\theta \\ y &= 3\sin\phi\sin\theta \\ z &= 3\cos\phi \end{aligned}}$$



EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$

$x = 2\cos\theta$

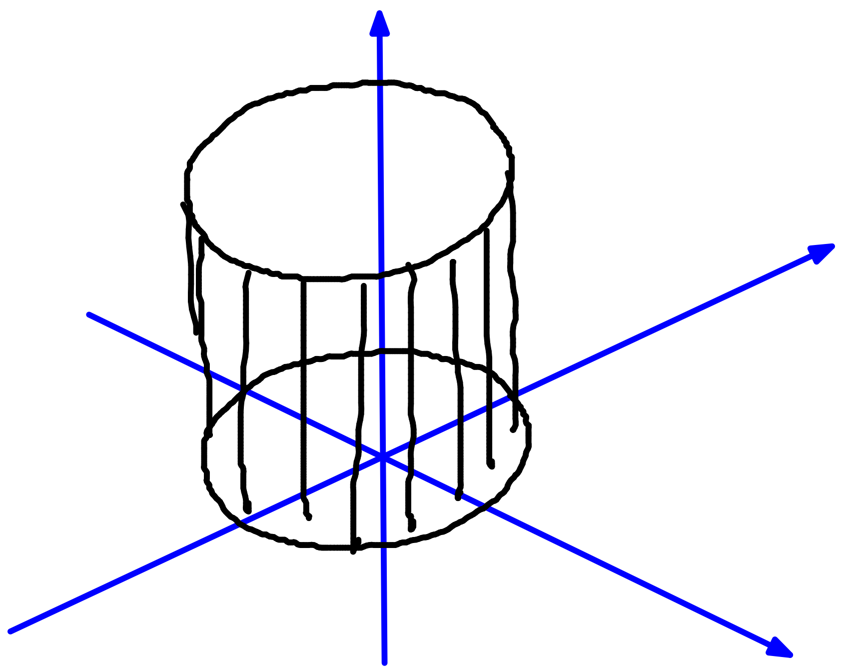
$y = 2\sin\theta$

$z = z$

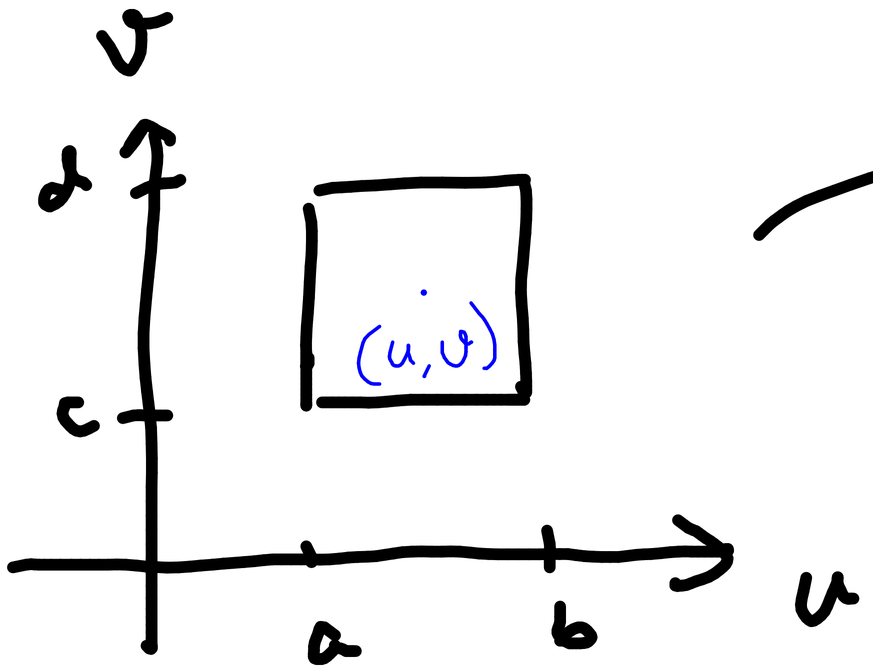
range for θ & z

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 1$



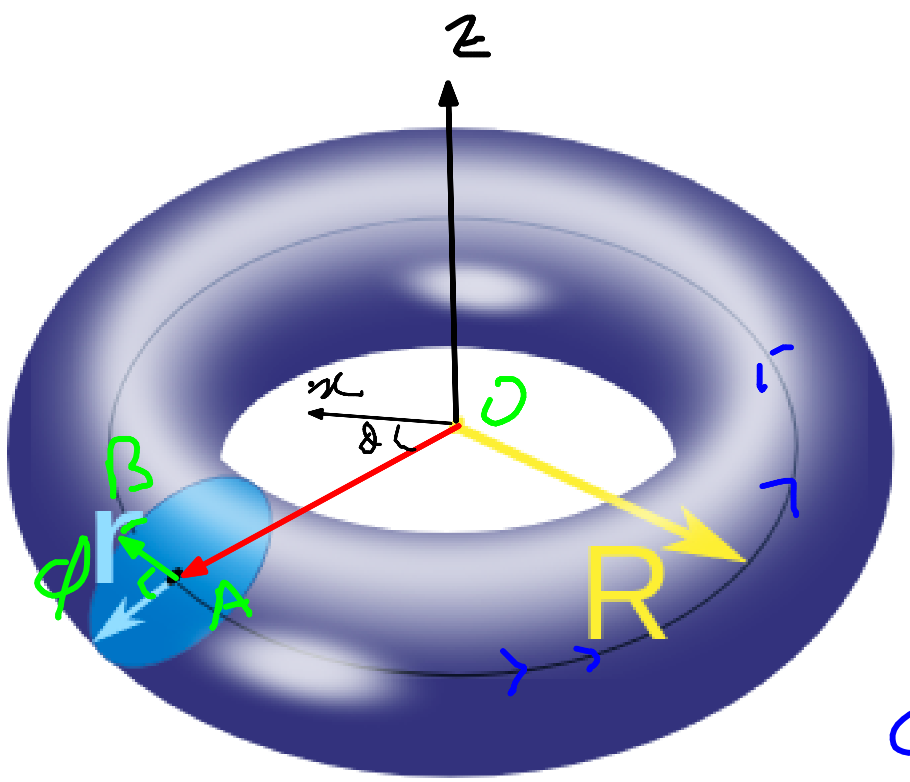
Parametric Surfaces:



$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

Q. find a parametric representation for a torus



ϕ : angle between the vector $\vec{AB}$ & $\vec{OA}$

$0 \le \phi \le 2\pi$

$0 \le \theta \le 2\pi$

Q: find coordinates of the $\boxed{\text{point } B:}$ in terms of r, R, θ, ϕ

$x = ?$

$y = ?$

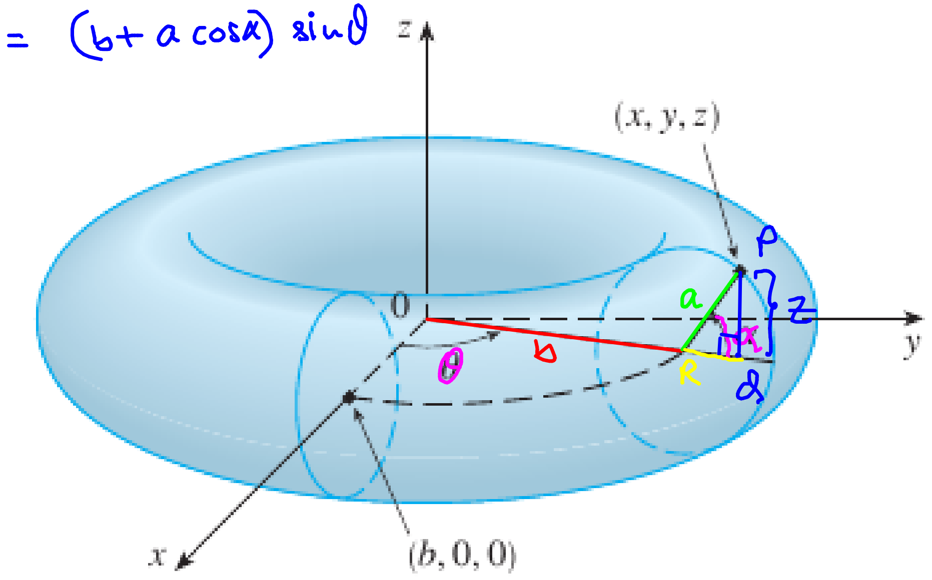
$z = ?$

H. W.

$x = OQ\cos\theta = (b + a\cos\alpha)\cos\theta$

$y = OQ\sin\theta = (b + a\cos\alpha)\sin\theta$

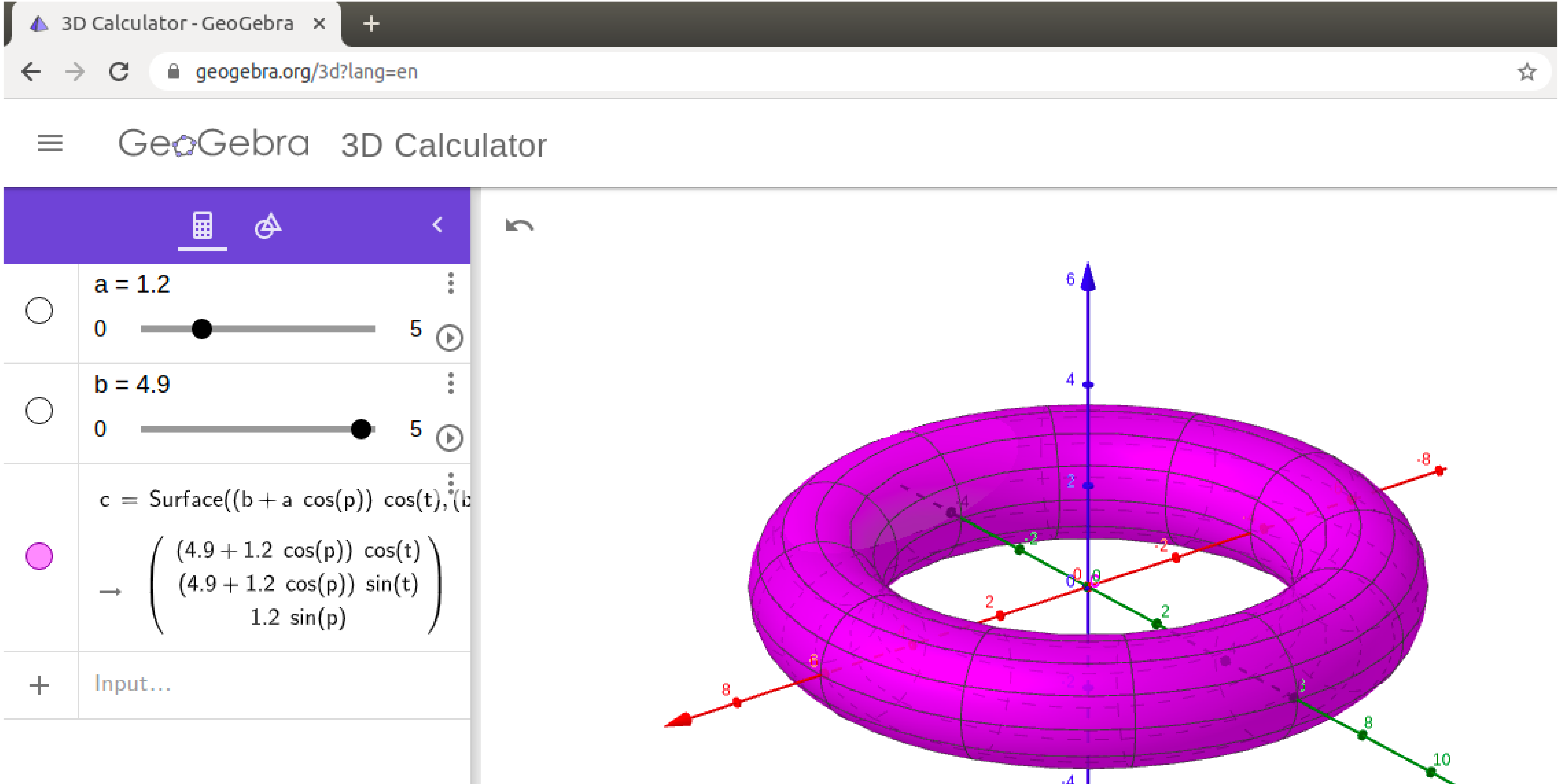
$z = a\sin\alpha$



Torus: radius : b

radius of cross sectional circle $\to a$

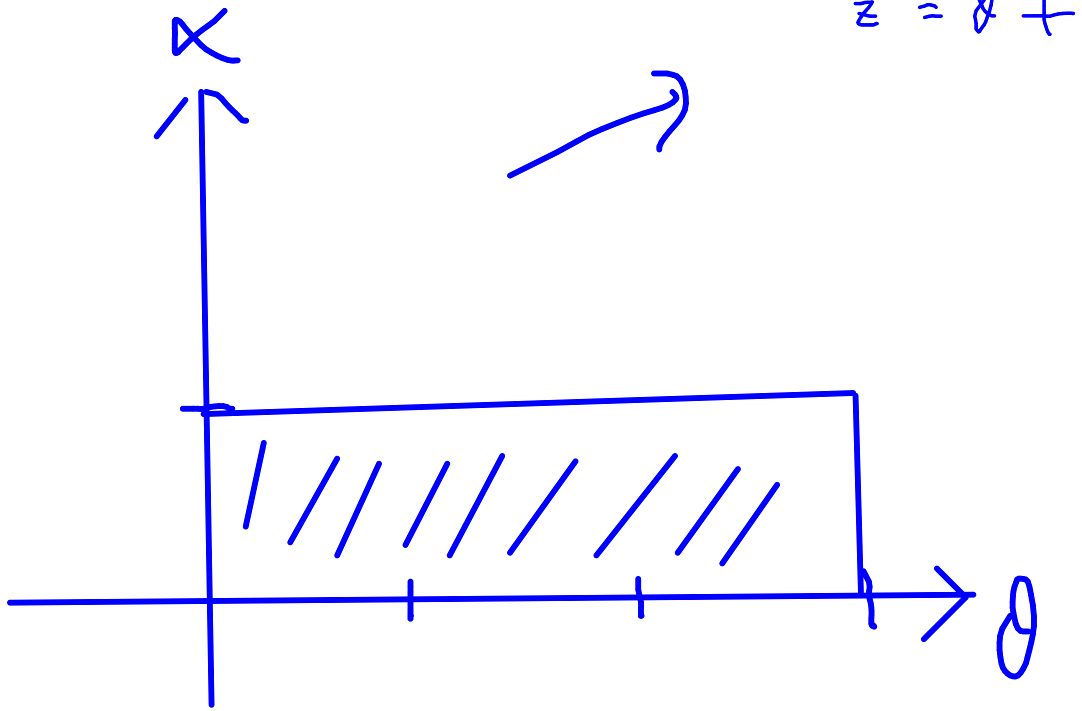
(x, y, z) : will be uniquely identified if we know θ & α.

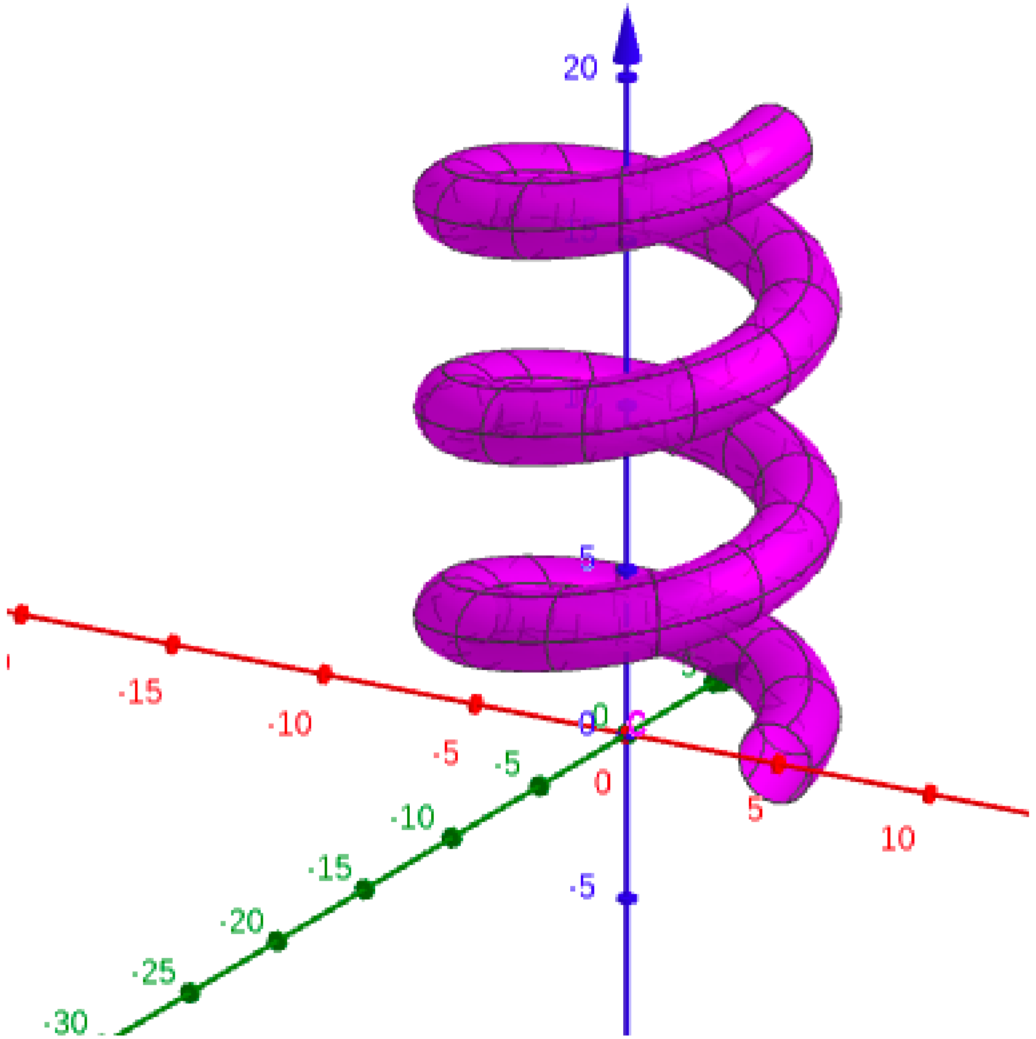

3D Calculator - GeoGebra
geogebra.org/3d?lang=en
GeoGebra 3D Calculator
a = 1.2
0 5
b = 4.9
0 5
c = Surface((b + a cos(p)) cos(t), (b
→ ((4.9 + 1.2 cos(p)) cos(t), (4.9 + 1.2 cos(p)) sin(t), 1.2 sin(p))
Input…

$$x = \rho \cos\theta = (b + a\cos\alpha)\cos\theta$$

$$y = \rho \sin\theta = (b + a\cos\alpha)\sin\theta$$

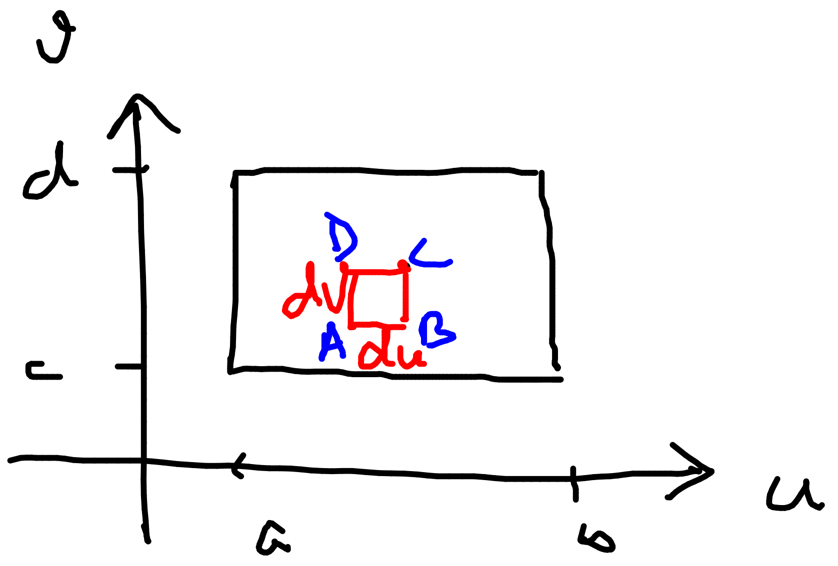
$$z = \theta + a\sin\alpha$$

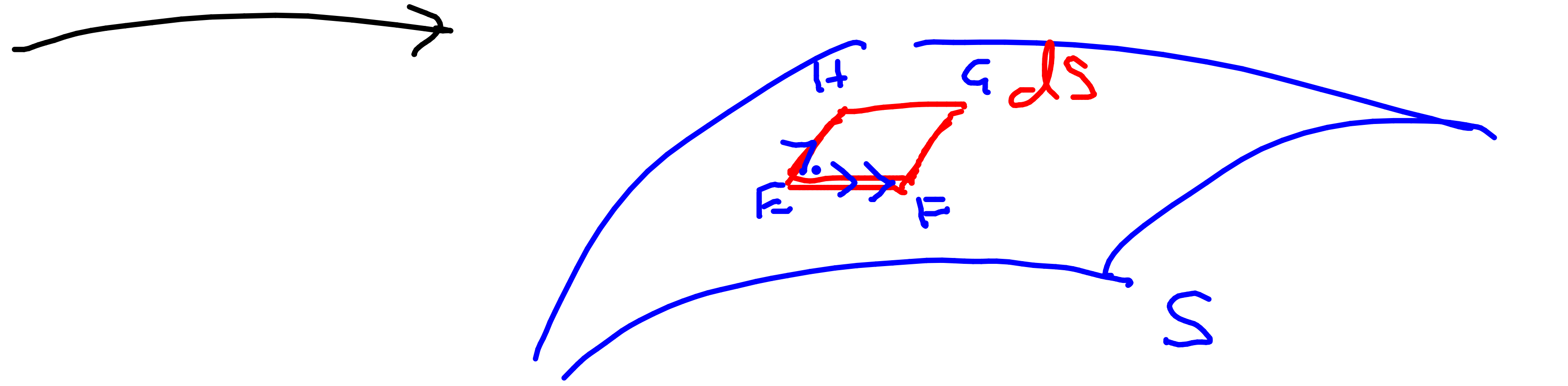




Next task: find area of parametric surfaces:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$





Q: Keeping v constant & changing u to $u+du$ how much distance we are covering in surface

$EF \sim \frac{\partial \vec{r}}{\partial u} du$

$EH \sim \frac{\partial \vec{r}}{\partial v} dv$

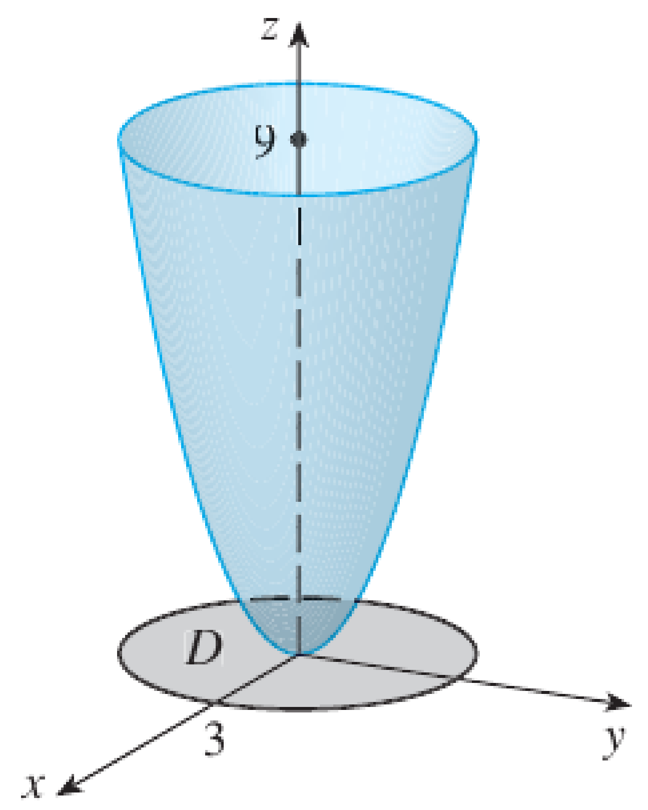
$$\text{area}(S) = \iint_S dS$$

$$dS = |\vec{EF} \times \vec{EH}|$$

$$dS = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right| = \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}_{\text{think this like Jacobian}} du\, dv$$

$$\text{Total area} = \int_c^d \int_a^b \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

V EXAMPLE 10 Find the area of the part of the paraboloid $\underbrace{z = x^2 + y^2}$ that lies under the plane $z = 9$.



$$\vec{r}(s,\theta) = s\cos\theta\,\hat{i} + s\sin\theta\,\hat{j} + s^2\,\hat{k}$$

$$0 \le s \le 3$$

$$0 \le \theta \le 2\pi$$

$$\text{area} = \int_0^{2\pi}\int_0^{3} \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| ds\, d\theta$$

$$\frac{\partial \vec{r}}{\partial s} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 2s\,\hat{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -s\sin\theta\,\hat{i} + s\cos\theta\,\hat{j} + 0\,\hat{k}$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2s \\ -s\sin\theta & s\cos\theta & 0 \end{vmatrix} = -2s^2\cos\theta\,\hat{i} - 2s^2\sin\theta\,\hat{j} + s\,\hat{k}$$

$$\left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{4s^4 + s^2}$$

$$\text{area} = \int_0^{2\pi}\int_0^{3} \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| ds\, d\theta = \int_0^{2\pi}\int_0^{3} s\sqrt{4s^2+1}\; ds\, d\theta = \text{whatever}$$

$$= \frac{\pi}{6}\left(37^{3/2} - 1\right)$$

MAT 104 :

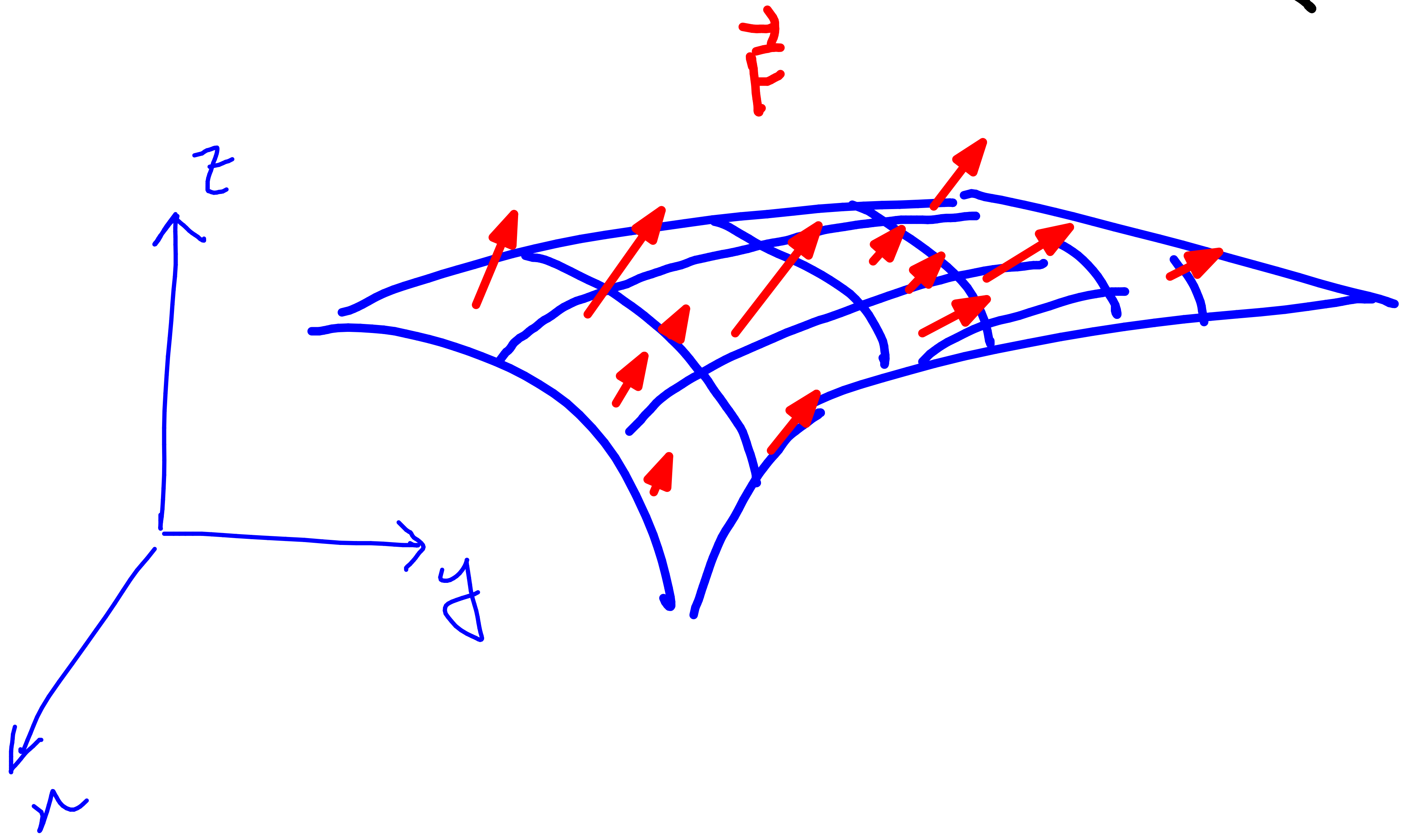
mid term exam

Tue Mar 23

→ syllabus : whatever we have covered this sem till today

→ blackboard / Perfectice | MCQ + subjective + oral + video

Recall: we were discussing surfaces & calculus on them



Q. find area

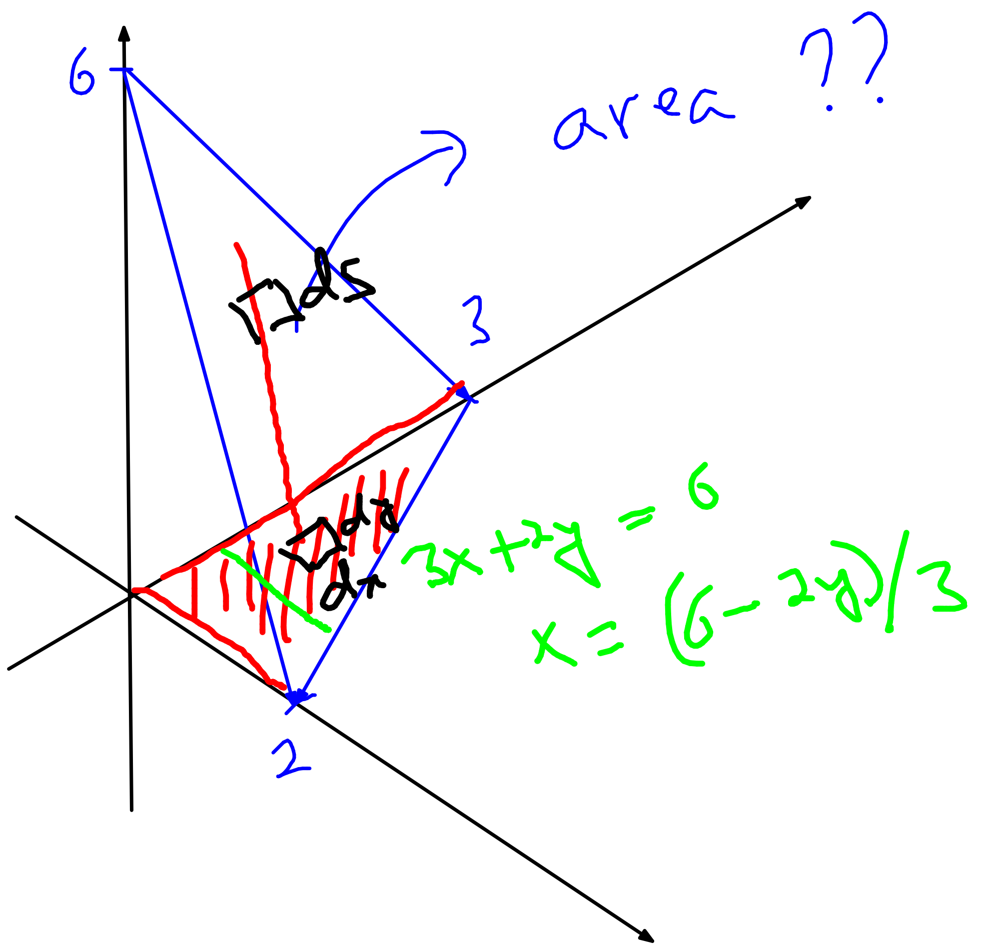
Q. find mass

Q. flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

Q. Find the area of the surface.

The part of the plane $3x + 2y + z = 6$ that lies in the first octant



sketch this plane first

we need to represent the surface with a parametric eqn

$x = x$
$y = y$
$z = 6-3x-2y$

where $(x,y) \in$ red triangle

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + (6-3x-2y)\hat{k}$$

$\vec{r}$: △ ⟶ △

$$dS = \left|\frac{\partial\vec{r}}{\partial x} \times \frac{\partial\vec{r}}{\partial y}\right| dx\,dy$$

$$\text{Total area} = \iint dS = \int_0^3 \int_0^{(6-2y)/3} \underbrace{\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right|}_{??} \, dx\, dy$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \left| 3\hat{i} + 2\hat{j} + \hat{k} \right| = \sqrt{14}$$

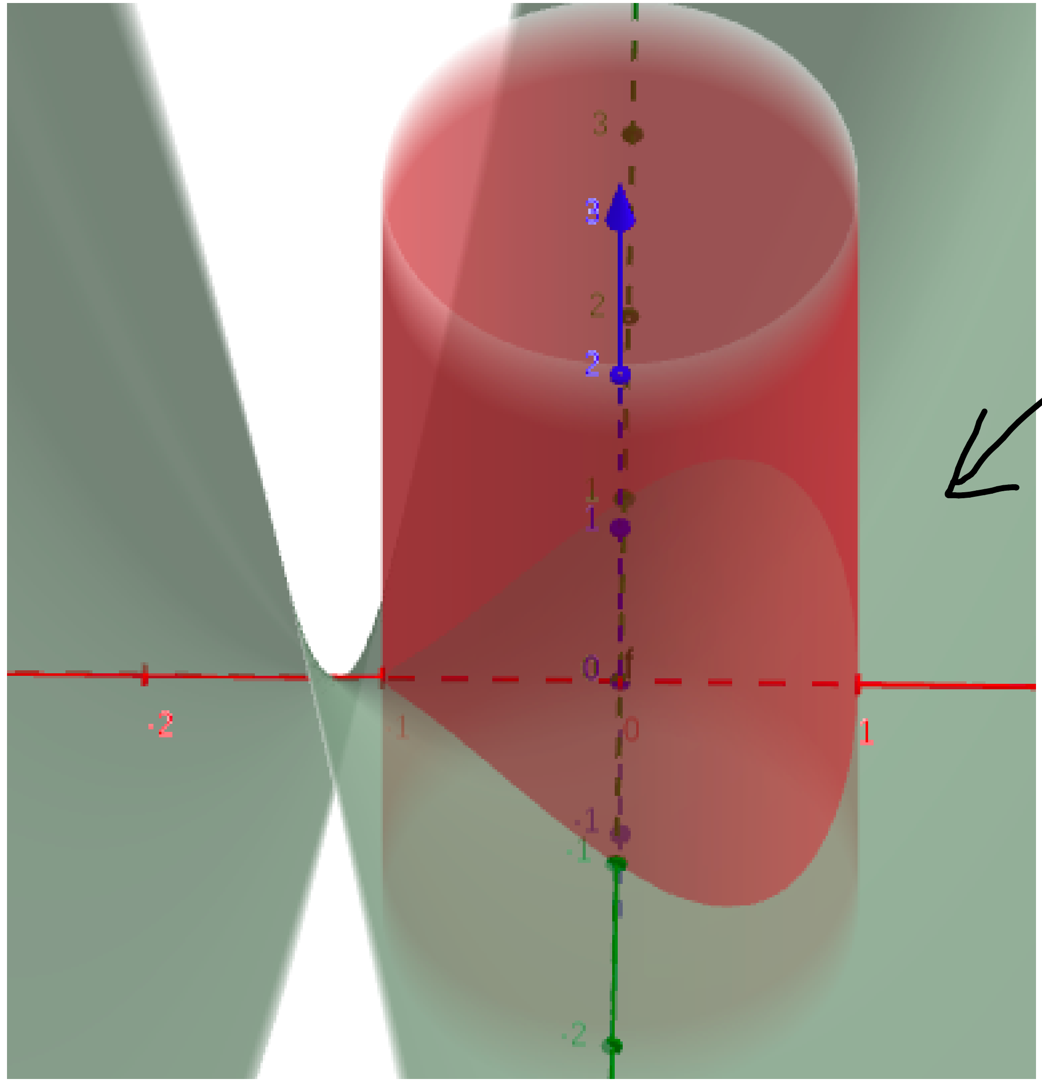
$$\text{area} = \int_0^3 \int_0^{(6-2y)/3} \sqrt{14} \, dx\, dy = 3\sqrt{14}$$

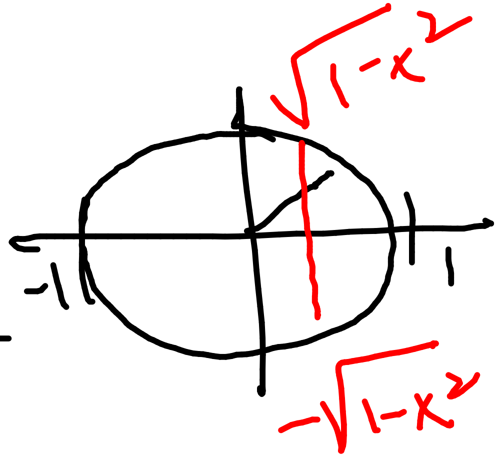
Find the area of the surface.

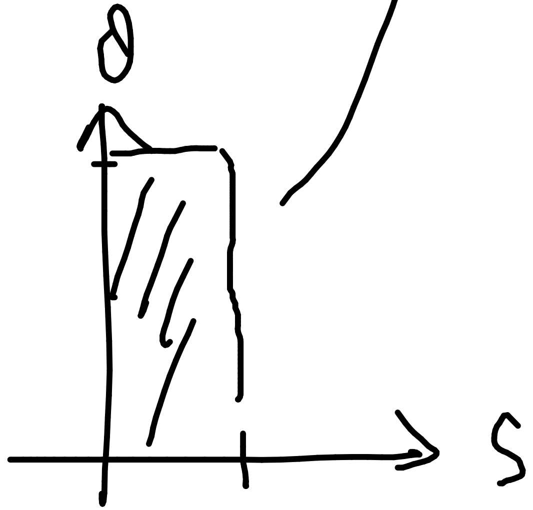
→ sketch the relevant surface

→ find parametrization ✓??

The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$







$$x = s\cos\theta$$

$$y = s\sin\theta$$

$$z = s^2\cos\theta\sin\theta$$

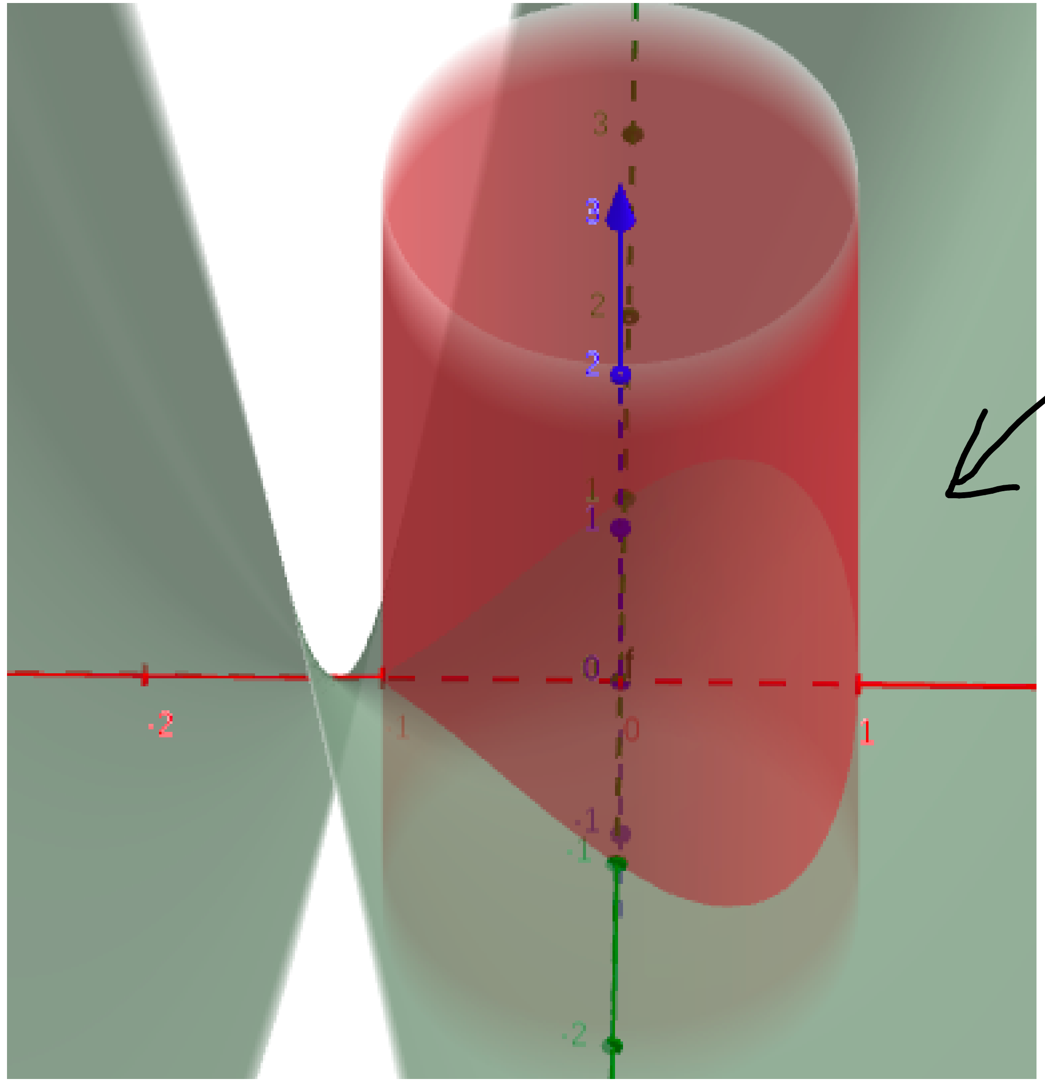
$$\vec{r}(s,\theta) = s\cos\theta\,\hat{i} + s\sin\theta\,\hat{j} + s^2\cos\theta\sin\theta\,\hat{k}$$

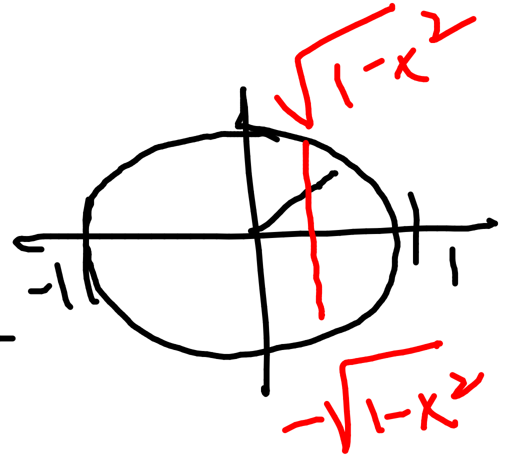
Find the area of the surface.

The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

→ sketch the relevant surface

→ find parametrization ✓ ??





$$x = x$$

$$y = y$$

$$z = xy$$

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + xy\hat{k}$$

$$\text{area} = \int_{?}^{?}\int_{?}^{?} \left| \frac{d\vec{r}}{dx} \times \frac{d\vec{r}}{dy} \right| \, dy \, dx$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{matrix} \right|$$

$$= \left| -y\hat{i} - x\hat{j} + \hat{k} \right|$$

$$= \sqrt{1+x^2+y^2}$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2} \; dy \; dx$$

= whatever

0 marks

(-1 to 1i) ntegr ×

symbolab.com/solver/step-by-step/%5Cint_%7B-1%7D%5E%7B1%7D%5Cint_%7B-%5Csqrt%7B1-x%5E%7B2%7D%7D%7

Functions

Matrices & Vectors

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1+x^2+y^2}\,dy\,dx$$

Go

$$=2\pi\frac{2\sqrt{2}-1}{3}$$

Got a different answer? Check if it's correct

Verify

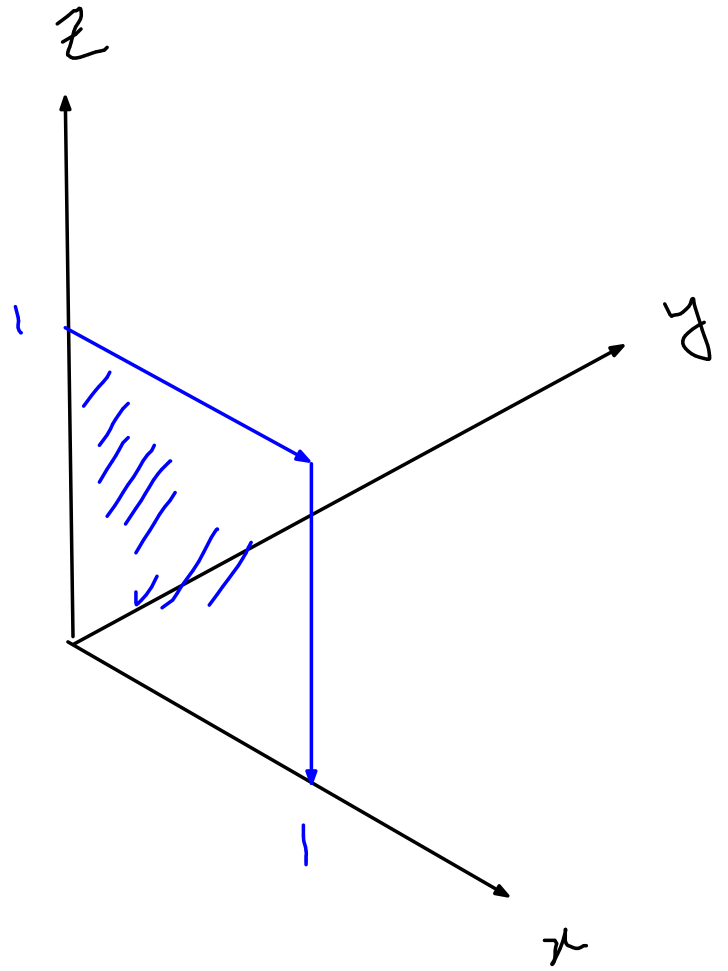
Find the area of the surface.

The part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, and $z = 1$

→ sketch the surface

→ find a parametrization

→ area = ??



$x = x$ $\quad 0 \le x \le 1$

$y = 4x + z^2$ $\quad 0 \le z \le 1$

$z = z$

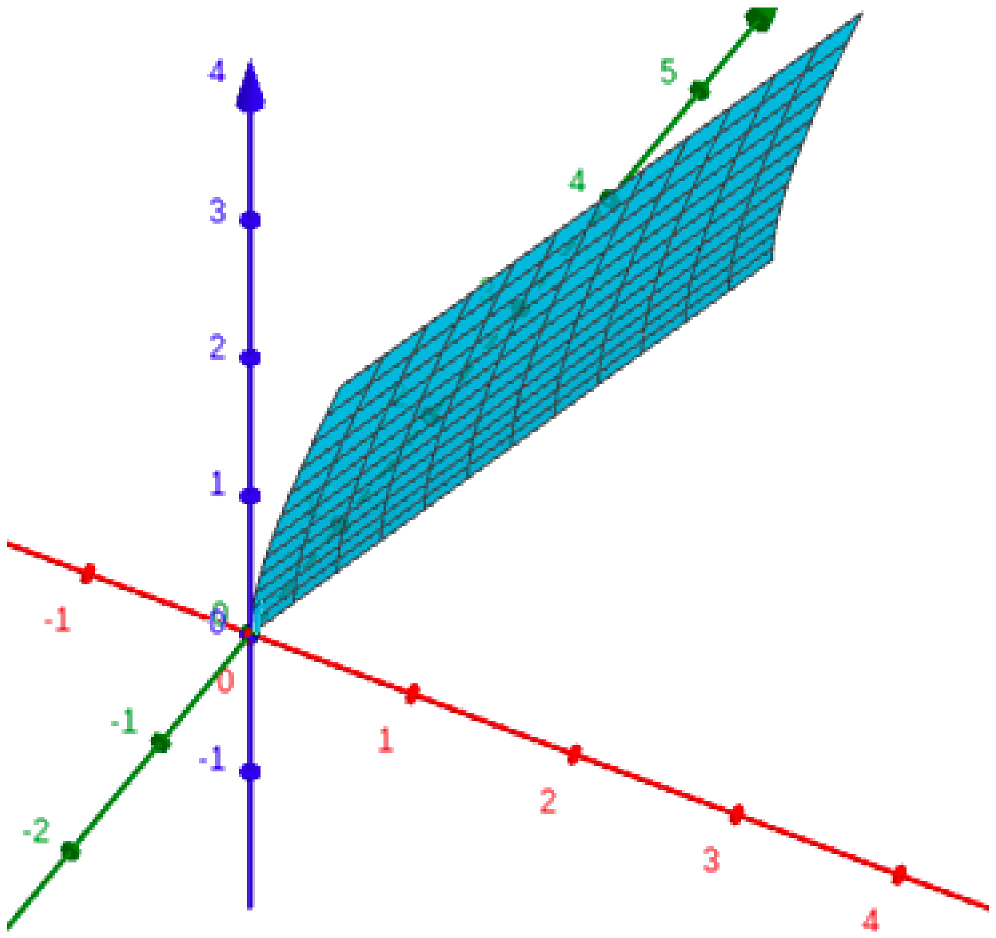
$\vec{r}(x,z) = x\hat{i} + (4x+z^2)\hat{j} + z\hat{k}$

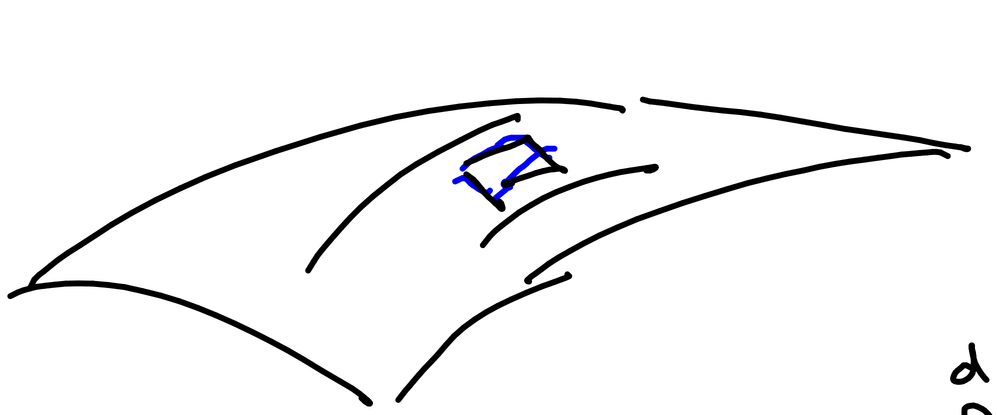
$$\left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial z}\right| = \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ 0 & 2z & 1 \end{matrix}\right| = |4\hat{i} - \hat{j} + 2z\hat{k}|$$

$$= \sqrt{17 + 4z^2}$$

$$\text{area} = \int_0^1 \int_0^1 \sqrt{17 + 4z^2} \; dx\, dz$$

$$= \text{whatever}$$





$\vec{r}(u,v)$

$a \le u \le b$

$c \le v \le d$

$$\text{area} = \int_c^d \int_a^b \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv}_{dS}$$

Q: surface density is $f(x,y,z)$.
find mass of the surface

$$dm = f \, dS$$

$$\text{total mass} = \int_c^d \int_a^b f \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

$$\iint_S f(x,y,z)\, dS$$

$$= \iint f(\ ) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

= usual calculation