



# ERWIN KREYSZIG

## ADVANCED ENGINEERING MATHEMATICS

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GUIDES AND MANUALS	
Maple Computer Guide Mathematica Computer Guide	
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same  
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## PART A Ordinary Differential Equations (ODEs)

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What's ODE?

# Ordinary Differential Equation

Idea: we know something about  $\frac{df}{dx}$  or  $\frac{d^2f}{dx^2}$

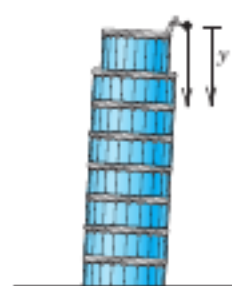
& we wish to find out about  $f$ .

$$\left. \begin{array}{l} \frac{df}{dx} = 2, \text{ find } f(x) \\ \frac{df}{dx} + f = 5 \end{array} \right\}$$

$$\left. \begin{array}{l} \frac{d^2f}{dx^2} + \frac{df}{dx} + f = e^x \\ \text{find } f(x) \end{array} \right\}$$



Why care about knowing to solve ODEs:



Falling stone

$$y'' = g = \text{const.}$$

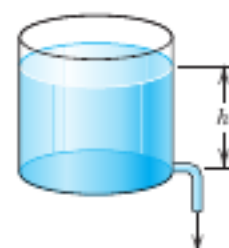
(Sec. 1.1)



Parachutist

$$mv' = mg - bv^2$$

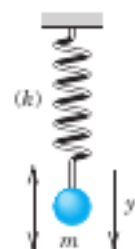
(Sec. 1.2)



Water level  $h$

$$h' = -k\sqrt{h}$$

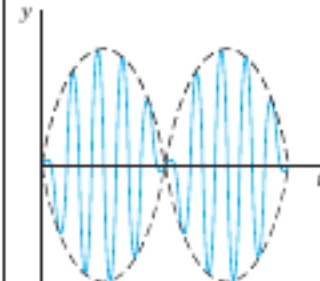
(Sec. 1.3)



Displacement  $y$

$$my'' + ky = 0$$

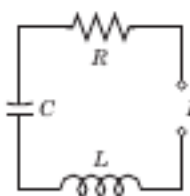
(Secs. 2.4, 2.8)



Beats of a vibrating system

$$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 \approx \omega$$

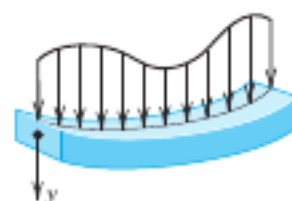
(Sec. 2.8)



Current  $I$  in an  $RLC$  circuit

$$LI'' + RI' + \frac{1}{C}I = E'$$

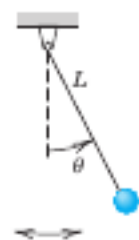
(Sec. 2.9)



Deformation of a beam

$$EIy'''' = f(x)$$

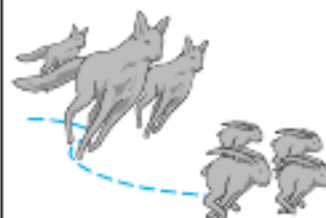
(Secs. 3.2, 3.3)



Pendulum

$$L\theta'' + g \sin \theta = 0$$

(Secs. 4.1, 4.2)



Lotka-Volterra predator-prey model

$$\begin{aligned} y_1' &= ay_1 - by_1y_2 \\ y_2' &= ky_1y_2 - ly_2 \end{aligned}$$

(Secs. 5.1, 5.2)

H.W. → Just read section 1.1 & 1.2.

→ Don't solve exercise problems.

mathematical modelling: converting real world scenarios  
into mathematical equations.  
↳ in particular into ODE eqns.

Q. What's ODE??

Solving for a function  $f(x)$

from an equation where

$f'(x)$ ,  $f''(x)$ , etc comes into picture

## 1.3 Separable ODEs

1st order  
& separable ODEs

$$\left[ \frac{dy}{dx} + y^2 = \sin(x) \right], \text{ solve for } y$$

$$\left[ \frac{d^2 y}{dx^2} + y^2 = \sin(x) \right] \text{ 2nd order ODE}$$

$$\left[ \frac{dy}{dx} + y = \frac{d^2}{dx^2}(e^x) \right] \text{ 1st order ODE}$$

Q. //

$$y' = (x+1)e^{-x}y^2$$

$$\frac{dy}{dx} = (1+x)e^{-x}y^2$$

$$\frac{1}{y^2} dy = (1+x)e^{-x} dx$$

$$\int \frac{1}{y^2} dy = \int (1+x)e^{-x} dx$$

$$\frac{1}{y} = -(x+2)e^{-x} + C$$

&

$$y = \frac{1}{(x+2)e^{-x} + C}$$

Use separation of variables  
→ try to move all  $x$  in one side  
& all  $y$  in other side  
& integrate

$C$  is an arbitrary constant



d.

Solve  $y' = -2xy$ ,  $y(0) = 1.8$ .

initial condition

$$\frac{dy}{dx} = -2xy$$

$$\frac{1}{y} dy = -2x dx$$

$$\ln y = -x^2 + C$$

$y(0) = 1.8$

$$\ln 1.8 = -0^2 + C$$

$$C = \ln 1.8$$

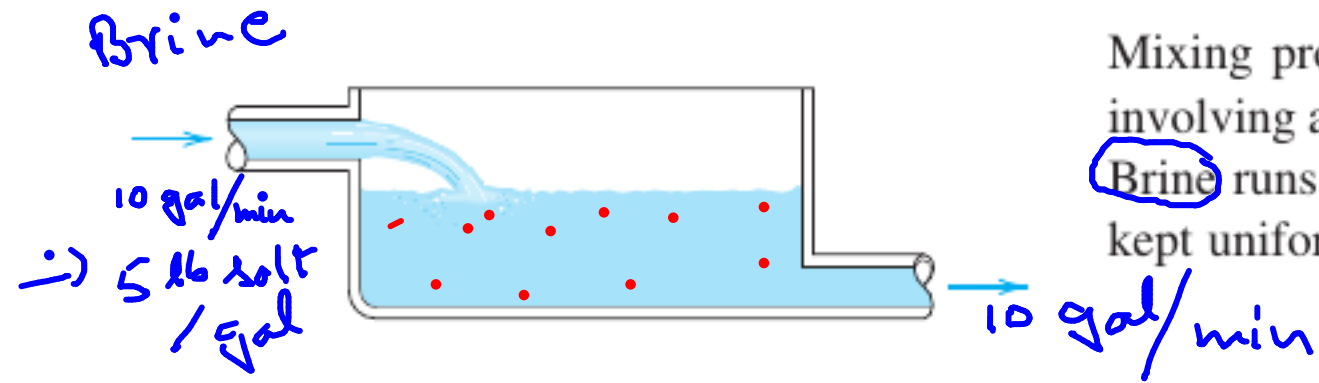
Solve using separation of variables  
use the extra condition to eliminate the constant  $C$

$$\ln y = -x^2 + \ln 1.8$$

$$\ln\left(\frac{y}{1.8}\right) = -x^2$$
$$\frac{y}{1.8} = e^{-x^2}$$

$$y = 1.8 e^{-x^2}$$

## EXAMPLE 5 Mixing Problem



Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time  $t$ .

$y(t)$  : amount of salt in the tank at time  $t$

$$y(t) = ??$$

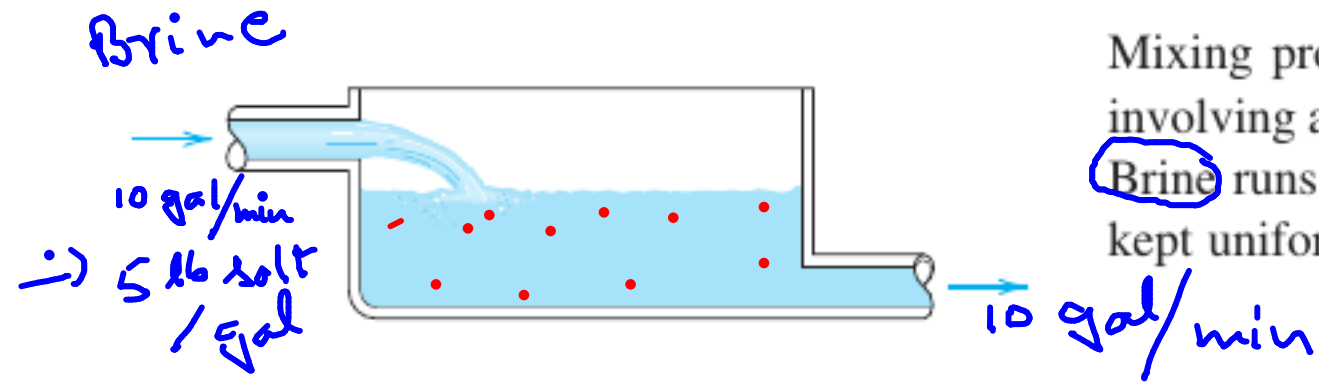
$$y(0) = 100$$

initial salt density ??  
0.1 lb / gal

$$\lim_{t \rightarrow \infty} y(t) = \text{guess} = 5000 \text{ lb}$$

Brine  $\leftrightarrow$  salt + water

## EXAMPLE 5 Mixing Problem



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$y(t)$  : amount of salt in the tank at time  $t$

$$y(t) = ??$$

$$y(0) = 100$$

$$\begin{aligned} \frac{dy}{dt} &= \text{rate of change of salt mass} \\ &= \text{inflow rate} - \text{outflow rate} \\ &= 50 - \frac{10}{1000} y \end{aligned}$$

Brine  $\leftrightarrow$  salt + water

initial salt density ??

$$0.1 \text{ lb / gal}$$

Solve  $y$  from

$$\frac{dy}{dt} = 50 - \frac{y}{100}, \quad y(0) = 100$$

Solve  $\frac{dy}{dt} = 50 - \frac{y}{100}$  using separation of variables.

$$\frac{dy}{dt} = \frac{5000 - y}{100}$$

$$\frac{1}{5000 - y} dy = \frac{1}{100} dt$$

$$\int \frac{1}{5000 - y} dy = \int \frac{1}{100} dt$$

$$-\ln(5000 - y) = \frac{t}{100} + C$$

$$t = 0$$

$$y = 100$$

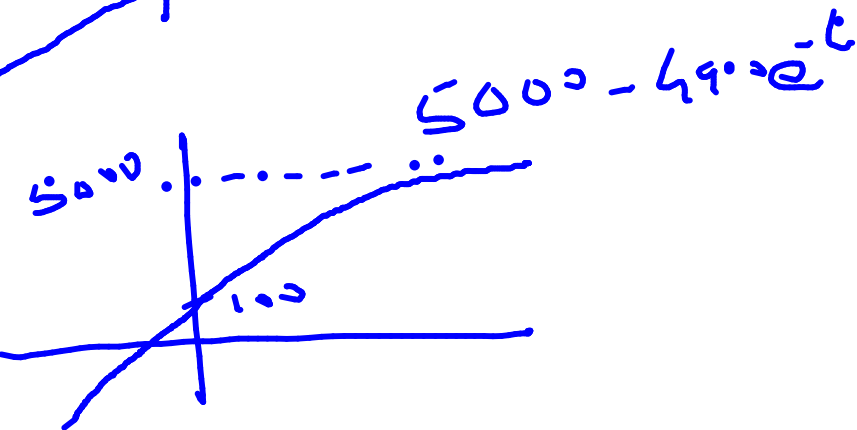
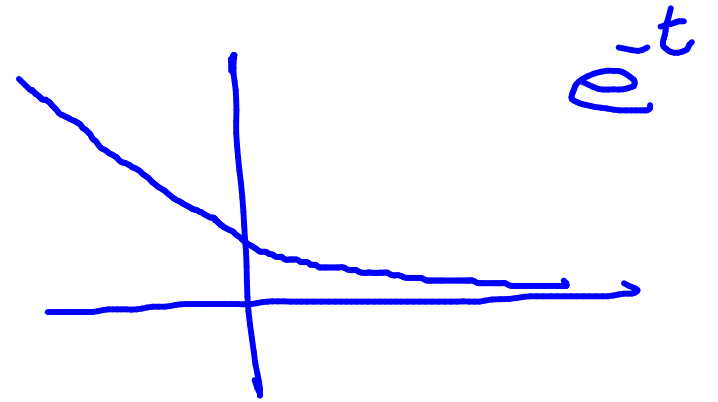
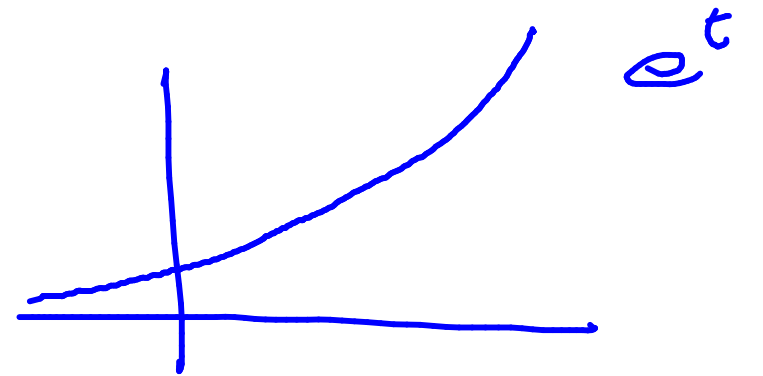
$$C = -\ln 4900$$

$$\ln\left(\frac{5000 - y}{4900}\right) = -\frac{t}{100}$$

$$y = ??$$

$$5000 - y = 4900 e^{-t/100}$$

$$y = 5000 - 4900 e^{-t/100}$$



Recall: what did we start ?? ODE

In differential equations, we solve for a function:

↳ we have some information about derivatives

e.g.  $\frac{dx}{dt} = 2$ , find  $x(t)$ .

$2 \frac{d^2x}{dt^2} = 7$ , find  $x(t)$

→ We learnt last time: the first thing we should try when solving an ODE is variable separable

$$\boxed{x \frac{dy}{dx} = 5x^2 y}$$

is this an ODE ??

$$\frac{1}{y} dy = \frac{5x^2}{x} dx$$

$$\int \frac{1}{y} dy = \int 5x dx$$

$$\sin\left(\frac{dy}{dx}\right) = x$$
$$\frac{dy}{dx} = \sin^{-1}(x)$$



# Extended Method: Reduction to Separable Form

$$y' = f\left(\frac{y}{x}\right)$$

$$y_x = v$$

$$2xyy' = y^2 - x^2.$$

$$2xy \frac{dy}{dx} = y^2 - x^2 \quad \text{try to move } x \text{ \& } y \text{ in two sides}$$

$$\downarrow ??$$
$$(\text{only } y \text{ terms}) dy = (\text{only } x \text{ terms}) dx$$

$$2 \frac{dy}{dx} = \frac{y^2 - x^2}{xy}$$

$$\frac{dy}{dx} = \frac{1}{2} \left[ \left( \frac{y}{x} \right) - \left( \frac{x}{y} \right) \right]$$

& eliminate  $y$

$$v = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{1}{2} \left[ v - \frac{1}{v} \right]$$

↙

$$y = vx$$

$$\frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$= v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = \frac{1}{2} \left[ v - \frac{1}{v} \right]$$

↳ is this separable??

$$x \frac{dv}{dx} = -\frac{1}{2} \left[ \frac{v^2 + 1}{v} \right]$$

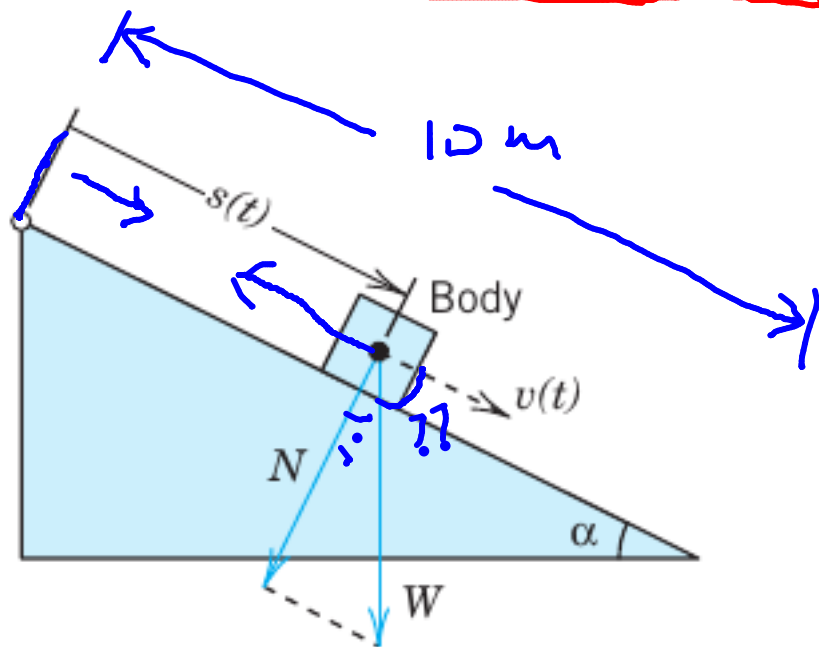
$$\frac{2v}{v^2 + 1} dv = -\frac{dx}{x} \quad \rightarrow \quad \ln(v^2 + 1) = -\ln x + \ln C$$

$$v^2 + 1 = \frac{C}{x^2}$$

$$\boxed{\frac{y^2}{x^2} + 1 = \frac{c}{x}}$$

$y$  is given implicitly by this eq<sup>n</sup>

**32. Friction.** If a body slides on a surface, it experiences friction  $F$  (a force against the direction of motion). Experiments show that  $|F| = \mu|N|$  (Coulomb's<sup>6</sup> law of kinetic friction without lubrication), where  $N$  is the normal force (force that holds the two surfaces together; see Fig. 15) and the constant of proportionality  $\mu$  is called the *coefficient of kinetic friction*. In Fig. 15 assume that the body weighs 45 nt (about 10 lb; see front cover for conversion).  $\mu = 0.20$  (corresponding to steel on steel),  $\alpha = 30^\circ$ , the slide is 10 m long, the initial velocity is zero and air resistance is negligible. Find the velocity of the body at the end of the slide.



$$W = 45 \text{ N} \quad \mu = 0.2$$

$$\alpha = 30^\circ$$

$s(t)$ : distance from the peak of the slide

→ we use  $F = ma$  to get an eq<sup>n</sup> for  $s(t)$

$$m \frac{d^2 s}{dt^2} = W \cos(60^\circ) - \mu W \cos(30^\circ)$$

$$\cancel{\frac{W}{g}} \frac{d^2 s}{dt^2} = \cancel{W} \cos(60^\circ) - \mu \cancel{W} \cos(30^\circ)$$

$$\frac{d^2 s}{dt^2} = g \left[ \frac{1}{2} - 0.2 \frac{\sqrt{3}}{2} \right]$$

$$= \underbrace{\quad}_A \approx 3.202$$

$$\frac{d^2 s}{dt^2} = A,$$

$$s(0) = 0$$

$$s'(0) = 0$$

Q: what  $\frac{ds}{dt}$  when  $s = 10$  ??

Can you complete this ??

$$\frac{d^2 s}{dt^2} = A$$

$$\frac{ds}{dt} = At + C$$

$$v(t) = \boxed{\frac{ds}{dt} = At}$$

★

$$s(t) = \frac{At^2}{2} + D$$

$$\boxed{s(t) = \frac{At^2}{2}}$$

$$\begin{cases} C = ?? \\ \frac{ds}{dt}(0) = 0 \\ C = 0 \end{cases}$$

$$\begin{cases} D = ?? \\ s(0) = 0 \\ D = 0 \end{cases}$$

$$\begin{aligned} \frac{At^2}{2} &= 10 \\ t &= \sqrt{\frac{20}{A}} \\ v\left(\sqrt{\frac{20}{A}}\right) &= A \sqrt{\frac{20}{A}} = \sqrt{20A} \end{aligned}$$

## 1.4 Exact ODEs. Integrating Factors

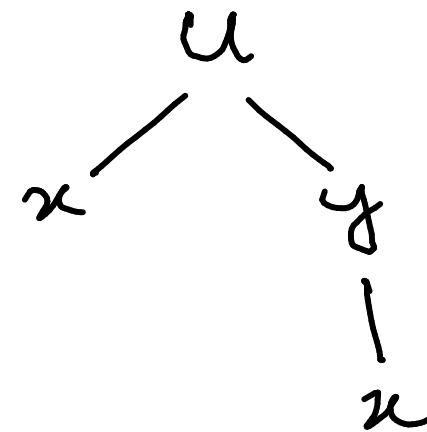
$$M(x,y) + N(x,y) \frac{dy}{dx} = 0$$

lets hope that there exist an equation

$$u(x,y) = c \quad \text{--- (2)}$$

from which we get eq (1) by  
differentiation

--- (1)



$$\frac{du}{dx} = \underbrace{\frac{\partial u}{\partial x}}_M + \underbrace{\frac{\partial u}{\partial y}}_N \frac{dy}{dx}$$



find a formula

$u(x, y)$

s.t.  $\frac{du}{dn} = M + N \frac{dy}{dx}$

this helps

because

we will solve

for

$y$

from

$$u(x, y) = C$$

Objective: Solve for  $y$  from eqn of  
the form

$$M + N \frac{dy}{dx} = 0$$

Plan: find a formula  $u(x, y)$  s.t.

$$\frac{du}{dx} = M + N \frac{dy}{dx}$$

When  
can we  
do this??

→ ODE:  $\frac{du}{dx} = 0$

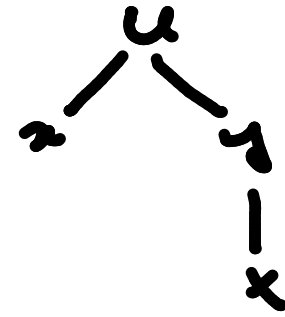
→  $u(x, y) = C$

→ solve for  $y$  from  $u(x, y) = C$

$$M + N \frac{dy}{dx} = 0$$

1 Aim: find  $u(x,y)$  s.t.

$$\frac{du}{dx} = M + N \frac{dy}{dx}$$



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

$$M = \frac{\partial u}{\partial x}$$

$$N = \frac{\partial u}{\partial y}$$

if  $M + N \frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$  for some  $u(x, y)$

or: if there exist  $u(x, y)$  s.t.

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} = M \right) \quad \& \quad \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} = N \right)$$

$$\left| \frac{\partial^2 u}{\partial x \partial y} \stackrel{?}{=} \frac{\partial^2 u}{\partial y \partial x} \right.$$

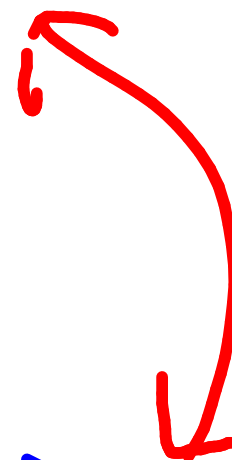
→ check for existence of such  $u(x, y)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$





$$M + N \frac{dy}{dx} = 0$$



$\Leftrightarrow$

$$M dx + N dy = 0$$

d.  $\underbrace{\cos(x+y)}_{M} dx + \underbrace{(3y^2 + 2y + \cos(x+y))}_{N} dy = 0$

→ is it exact??

→  $M dx + N dy = 0$  is exact  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x}$  exact

→ there exist  $u(x, y)$  s.t.  $\frac{\partial u}{\partial x} = M$  &  $\frac{\partial u}{\partial y} = N$

→ now find this  $u(x, y)$

$$\frac{\partial u}{\partial x} = \cos(x+y)$$

$$u = \sin(x+y) + g(y)$$

[where  $g(y)$  is to be found]

$$\rightarrow \frac{\partial u}{\partial y} = 1$$

$$\cancel{\cos(x+y)} + \frac{dg}{dy} = 3y^2 + 2y + \cancel{\cos(x+y)}$$

$$\frac{dy}{dx} = 3y^2 + 2y$$

$$g(y) = y^3 + y^2 + C$$

→ finally:  $u(x,y) = \sin(x+y) + y^3 + y^2 + C$

→ solve for  $y$  from:

$$u(x,y) = \text{constant}$$

→  $\boxed{\sin(x+y) + y^3 + y^2 = C}$

↳  $y$  is given

implicitly by  
this eq<sup>n</sup>.

Q.11

$$(\cos y \sinh x + 1) dx - \sin y \cosh x dy = 0,$$

$$y(1) = 2.$$

$$(\cos y \sinh x + 1) dx - \sin y \cosh x dy = 0$$

check for exactness.

$$\frac{\partial M}{\partial y} = -\sin y \sinh x = \frac{\partial N}{\partial x} \quad \text{Exact}$$

$$\rightarrow \text{find } u \text{ s.t. } \frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \left| \quad \frac{d}{dx}(\cosh x) = \sinh x \right.$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \left| \quad \frac{d}{dx}(\sinh x) = \cosh x \right.$$

$$\frac{\partial u}{\partial x} = \cos y \sinh x + 1$$

$$u = \cos y \cosh x + x + g(y)$$

$$\frac{\partial u}{\partial y} = -\sin y \cosh x + \frac{dg}{dy} = -\sin y \cosh x$$

$$\frac{dg}{dy} = 0 \Rightarrow g = C$$

$$\Rightarrow u(x, y) = \cos y \cosh x + x + C$$

✓ satisfies

$$u = C$$

$$\cos y \cosh x + x = C$$

$$\cos(2) \cosh(1) + 1 = C$$

find C by  
 $y(1) = 2$



# Reduction to Exact Form. Integrating Factors

Next time

$$-y \, dx + x \, dy = 0.$$

$$(e^{x+y} + ye^y) dx + (xe^y - 1) dy = 0$$