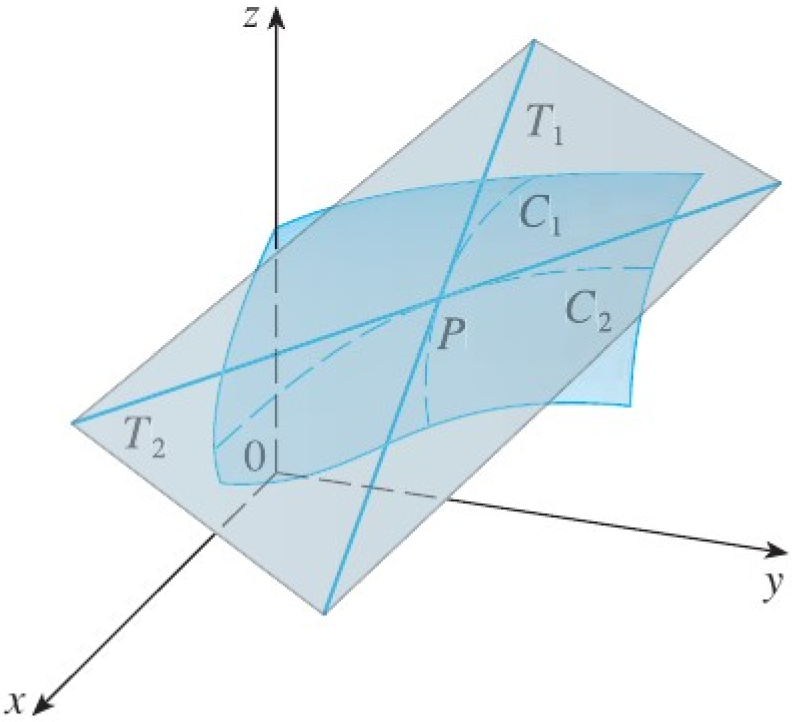
11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS



$f(x,y)$

$P: \ (x_0, y_0, f(x_0,y_0))$

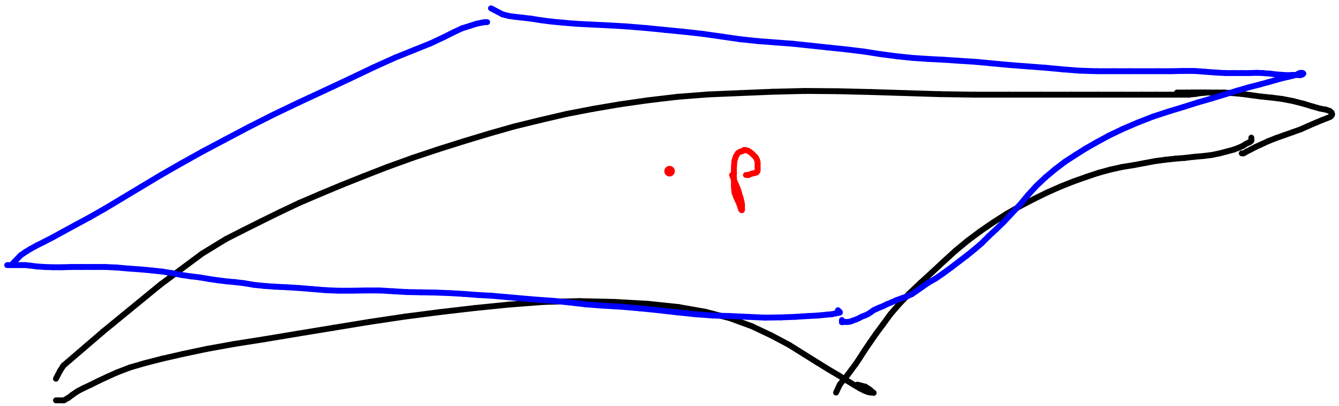
$z_0 = f(x_0, y_0)$

$$\left[\ z - z_0 = \underbrace{f_x(x_0, y_0)}(x - x_0) + f_y(x_0, y_0)\underbrace{(y - y_0)}\ \right]$$

why tangent plane has this formula?

H.W.

$f(x,y)$



EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

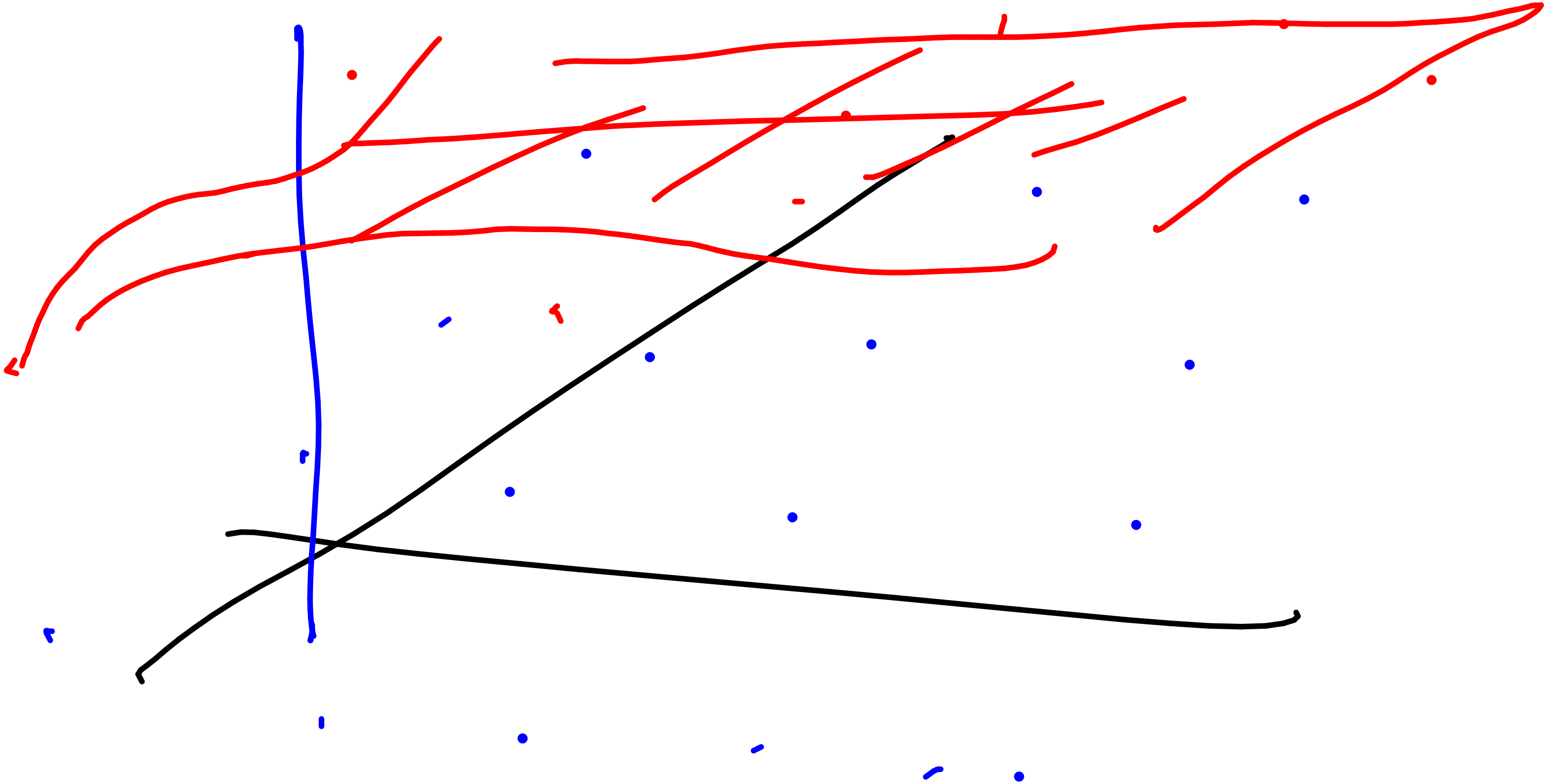
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Ans:

$$\left[z - 3 = 4(x-1) + 2(y-1) \right]$$

Lets plot f & tangent plane in matlab/octave

$$z = \underbrace{3 + 4(x-1) + 2(y-1)}_{\text{Linear approximation of } f}$$



LINEAR APPROXIMATIONS

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

7 DEFINITION If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

find linear approximation at point $(1,0)$

$$f(x,y) = xe^{xy}$$

$$f_x = e^{xy} + xye^{xy}$$

$$f_y = x^2e^{xy}$$

$$f_x(1,0) = 1$$

$$f_y(1,0) = 1$$

$$f(1,0) = 1$$

Tangent plane: $z - 1 = 1(x-1) + 1(y-0)$

Linearization $L(x,y) = 1 + (x-1) + y$

$$= x + y$$

Eq^n for tangent plane

$$\underbrace{z - z_0}_{dz} = f_x(x_0, y_0)\underbrace{(x - x_0)}_{\substack{dx \\ \text{small change} \\ \text{in } x}} + f_y(x_0, y_0)\underbrace{(y - y_0)}_{\substack{dy \\ \text{small change} \\ \text{in } y}}$$

DIFFERENTIALS

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

$$= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

EXAMPLE 3

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

$$z = x^2 + 3xy - y^2$$

$$dz = f_x\,dx + f_y\,dy$$

$$dz = (2x+3y)\,dx + (3x-2y)\,dy$$

a

a) compare dz, $\Delta z = f(2.05, 2.96) - f(2, 3)$

$$dz = 13 \cdot (0.05) + 0 \cdot (-0.04)$$

$$= 0.65$$

$$\Delta z = 0.6449$$

$$\boxed{\Delta z \approx dz}$$

EXAMPLE 5 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

$V = xyz$

Q: find dV if dx, dy, dz are all 0.2

$$dV = V_x dx + V_y dy + V_z dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

$$= 1980$$

$$\frac{dV}{V} = \left(\frac{1980}{75 \times 60 \times 40}\right)$$

$$= 1.1$$