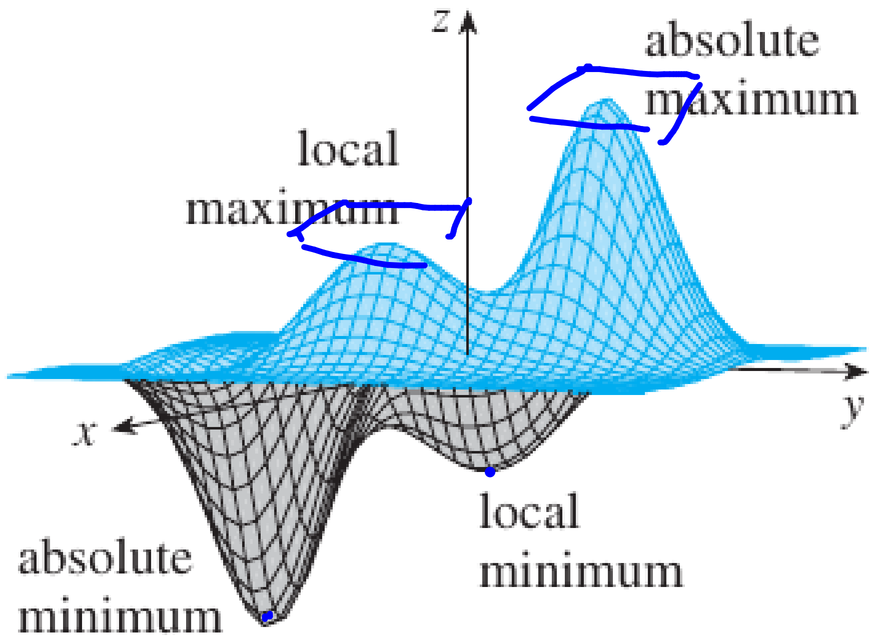
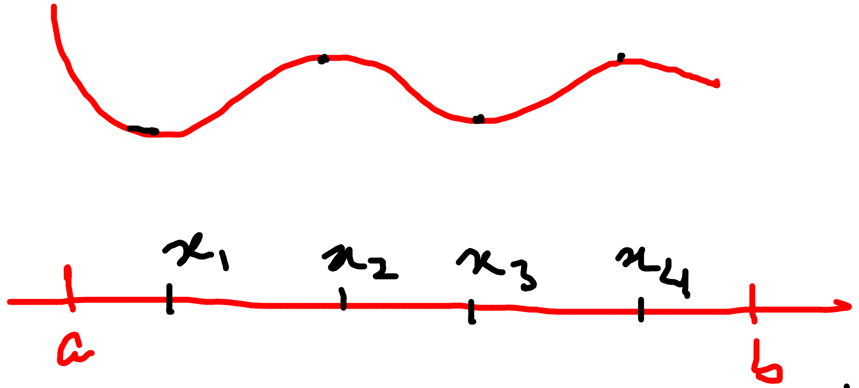
Aim: find max/min of g in $[a,b]$

$g(x)$

11.7 MAXIMUM AND MINIMUM VALUES





find critical points

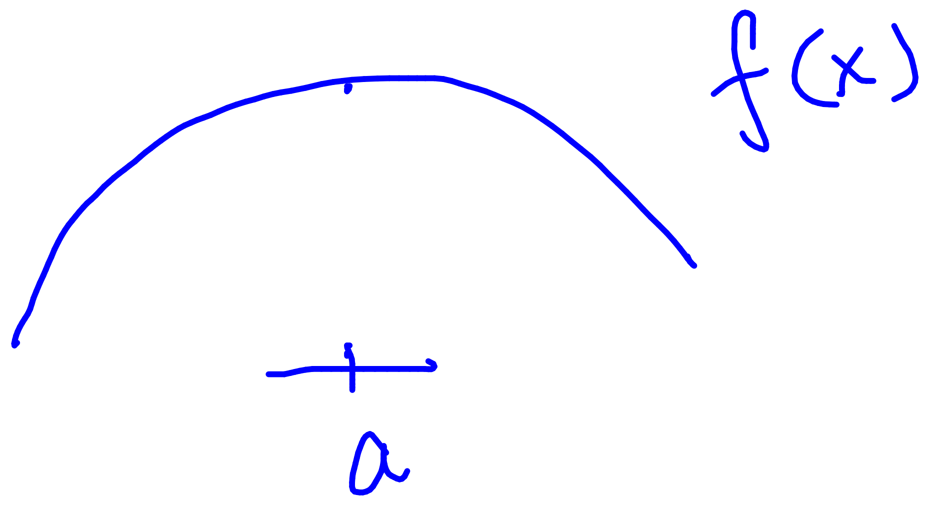
$g'(x) = 0$

Then max of g on the interval $[a, b]$

$= \max\{g(x_1), g(x_2), g(x_3), g(x_4), g(a), g(b)\}$

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

for one variable calculus



$$f'(a) = 0$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$\times \quad \cancel{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}} = 0$$

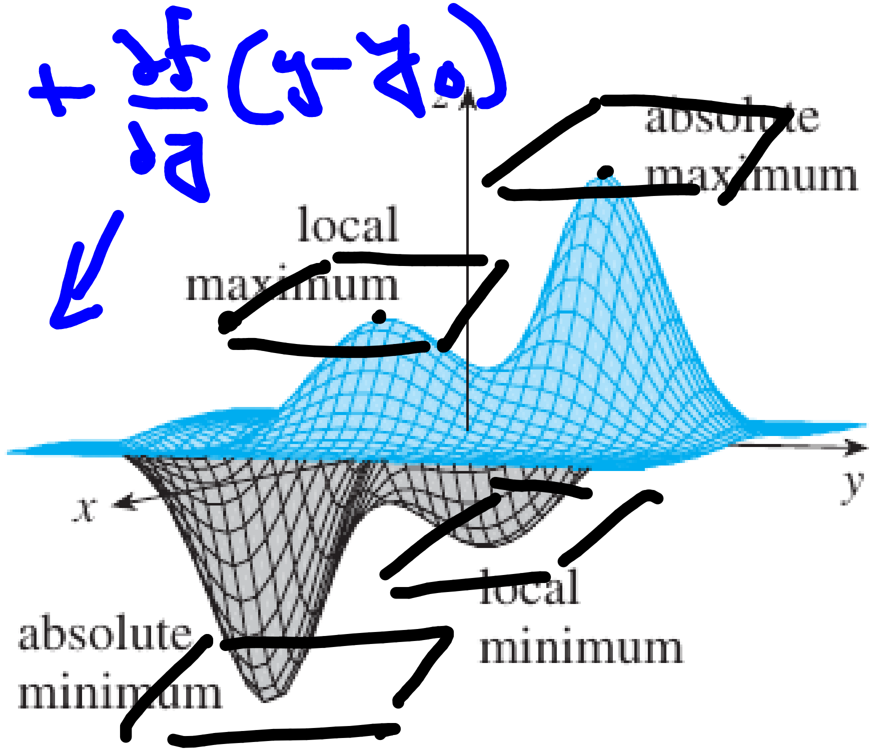
11.7 MAXIMUM AND MINIMUM VALUES

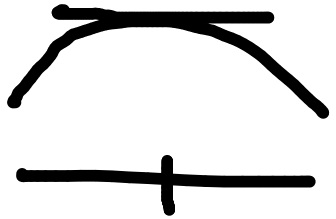
$f(x,y)$: temperature at point (x,y)

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$





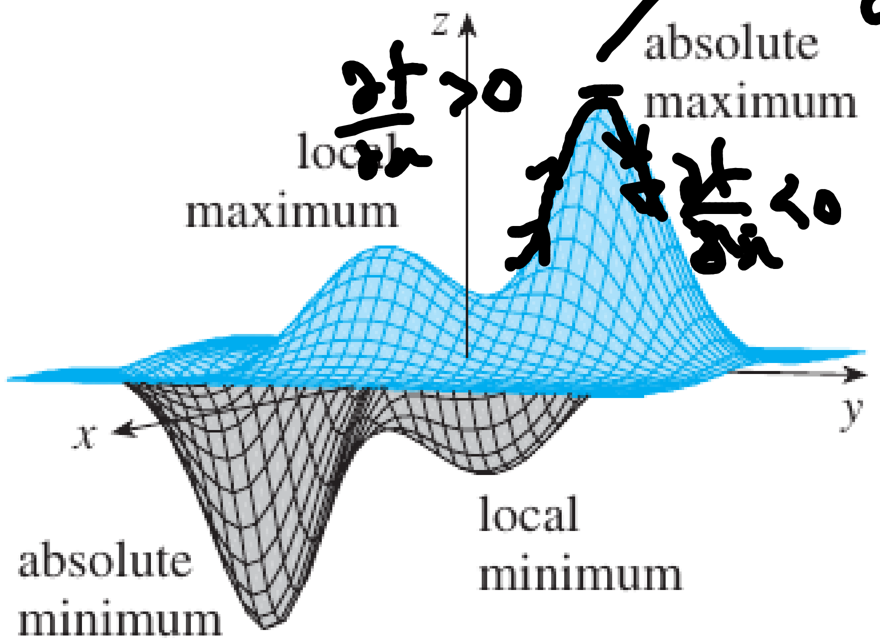
$g'(a) = 0$

horizontal tangent line

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

11.7 MAXIMUM AND MINIMUM VALUES

$\frac{\partial f}{\partial x} = 0$



black curve: only x is changing

y : constant

local max / local min

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

11.7 MAXIMUM AND MINIMUM VALUES

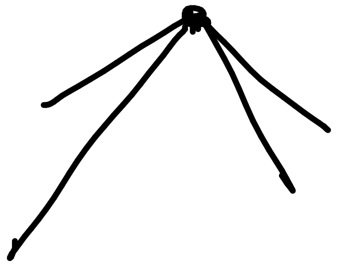
$P\ (a,b)$

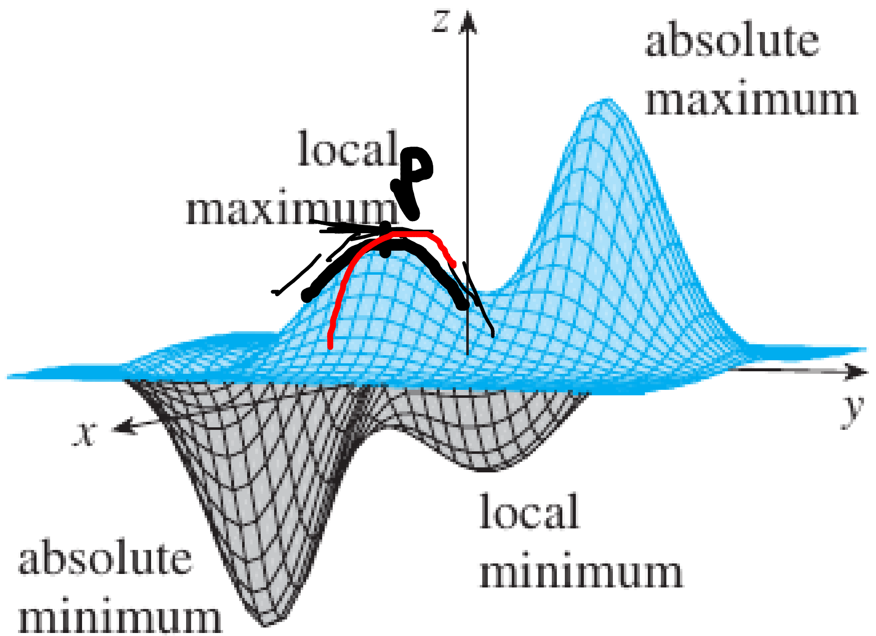
Q: at P,

$\frac{\partial f}{\partial x} > 0$, or $\frac{\partial f}{\partial x} < 0$,

$$\frac{\partial f}{\partial x} = 0$$

Q. at P $\frac{\partial f}{\partial y} = 0$



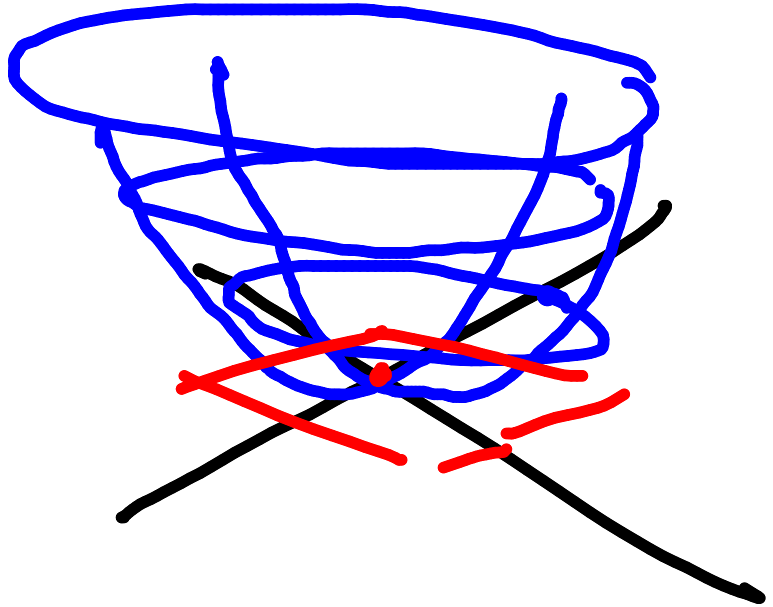


$$\frac{\sim}{x-1}$$

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

↳ pointy graph.

Q. find critical points of $f(x,y) = x^2+y^2$



simply solve: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$2x = 0$, $2y = 0$

critical point $= (0,0)$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. $=$

graph of $(x-1)^2 + (y-3)^2 + 4$

related to graph of x^2+y^2

find the critical points:

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$2x - 2 = 0$ $\qquad$ $2y - 6 = 0$

$x = 1$ $\qquad$ $y = 3$

→ 1 unit in x dir

→ 3 unit in y dir

→ 4 units in z dir

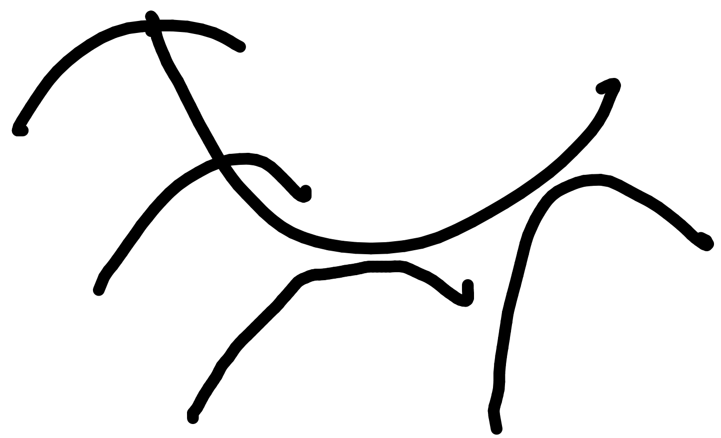
Q. $f(x,y) = x^2 - y^2$

find critical points.

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$2x = 0$ $\qquad$ $-2y = 0$

0 $\qquad$ 0

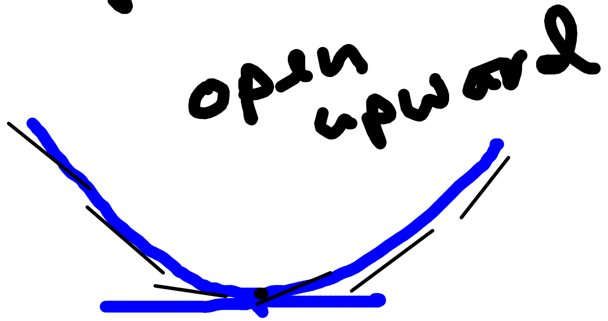


neither max or min

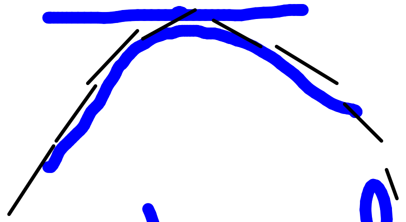
$\rightarrow$ saddle point

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

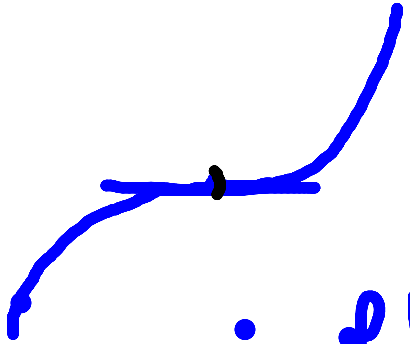
Recall one variable function criteria for classification of critical points



local min



local max



inflection pt.

$f'' > 0$

$\iff f'$ is increasing

$\iff f''$ is open upward

$f'' < 0$

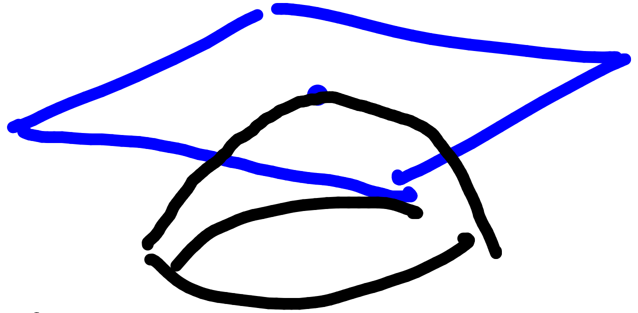
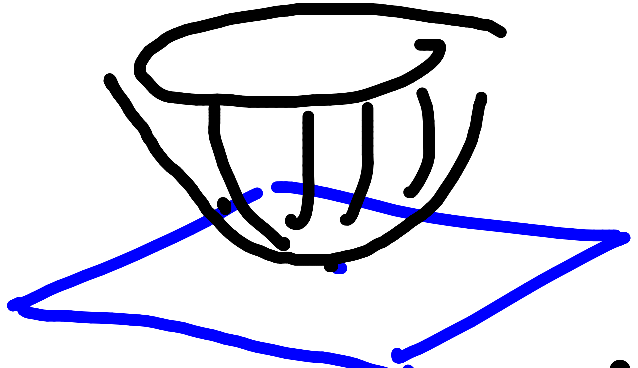
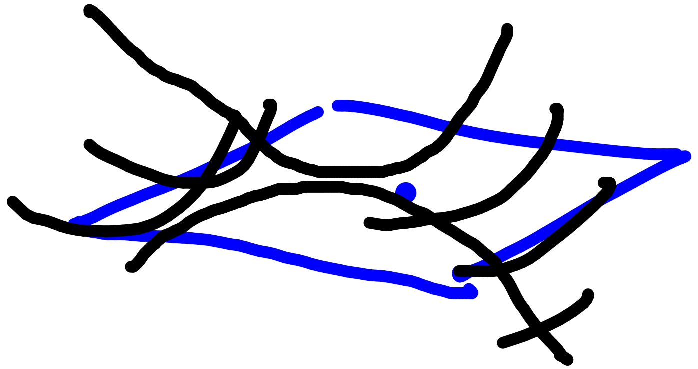
$\iff f'$ is decreasing

$\iff f$ is open down ward

$f'' = 0$

$f(x,y)$ critical point $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

$D > 0$		$D < 0$
$\frac{\partial^2 f}{\partial x^2} = f_{xx} < 0$	$f_{xx} > 0$	
		
local max	local min	neither

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$f(x,y) = x^2 \sin(y)$

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

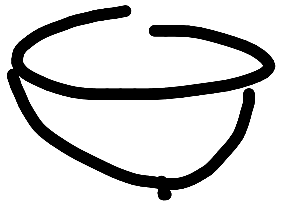
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

classification of critical points into local max / min / saddle point

Q: classify critical points of

$f = x^2 + y^2$



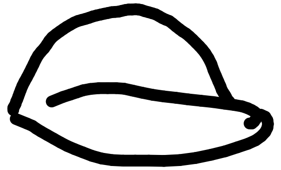
critical point $= (0,0)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$D > 0$ & $f_{xx} > 0$

$\Rightarrow (0,0)$ is a point of local min

$f = -x^2 - y^2$



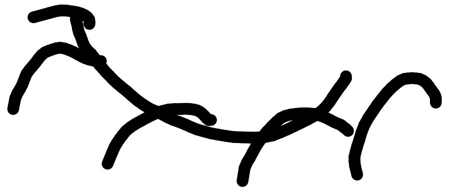
critical point $= (0,0)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$D > 0$ & $f_{xx} < 0$.

$\Rightarrow (0,0)$ is a point of local max

$f = x^2 - y^2$



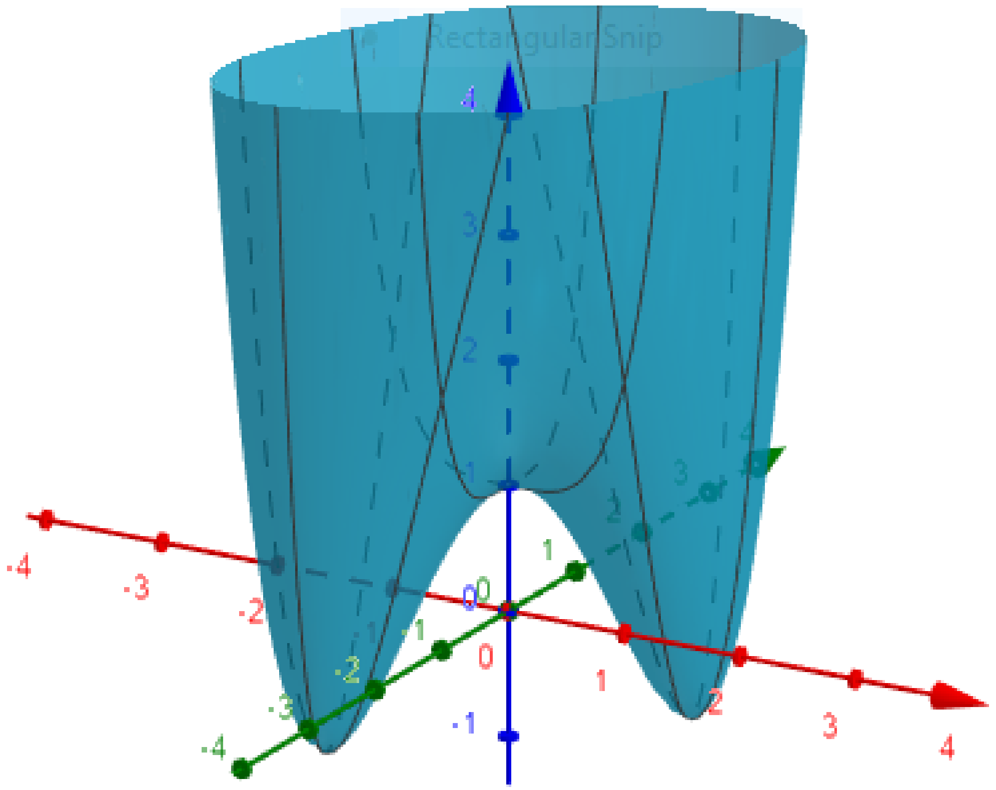
critical point $= (0,0)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

$(0,0)$ is a saddle point.

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

& check your ans on Geogebra



$f_x = 0$

$4x^3 - 4y = 0$

$x^3 - y = 0$

$y^3 - x = 0$

$f_y = 0$

$4y^3 - 4x = 0$

$y = x^3$

$(x^3)^3 - x = 0$

$x^9 - x = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

x	$y = x^3$	D	f_{xx}	
0	0	$-16 < 0$	whatever	saddle pt
-1	-1	$128 > 0$	$12 > 0$	local min
1	1	$128 > 0$	$12 > 0$	local min

$$x(x^8 - 1) = 0$$

$x = 0$ & $x^8 - 1 = 0$

$x^8 = 1$

$x = \pm 1$

V EXAMPLE 4 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

V EXAMPLE 5 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

33. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$9x^2 + 36y^2 + 4z^2 = 36$$