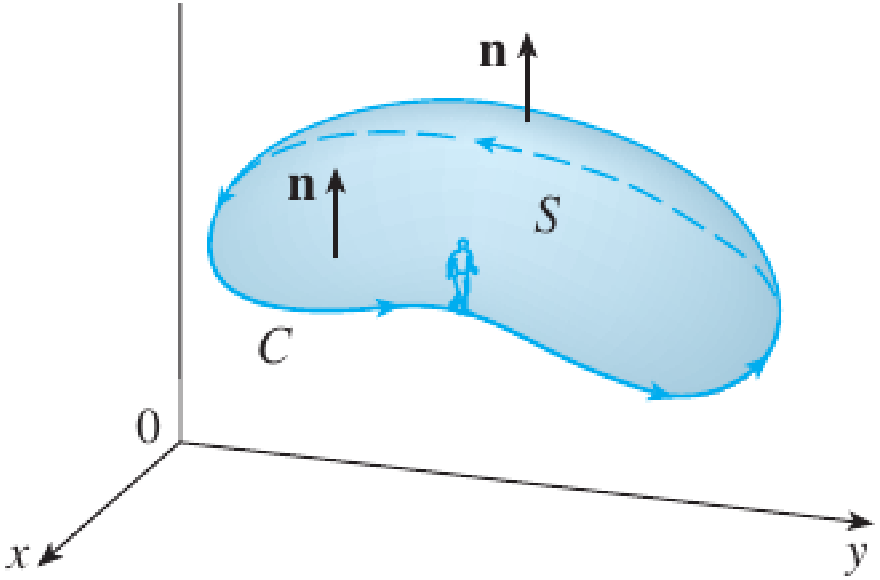
13.8 STOKES' THEOREM

13.9 THE DIVERGENCE THEOREM

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

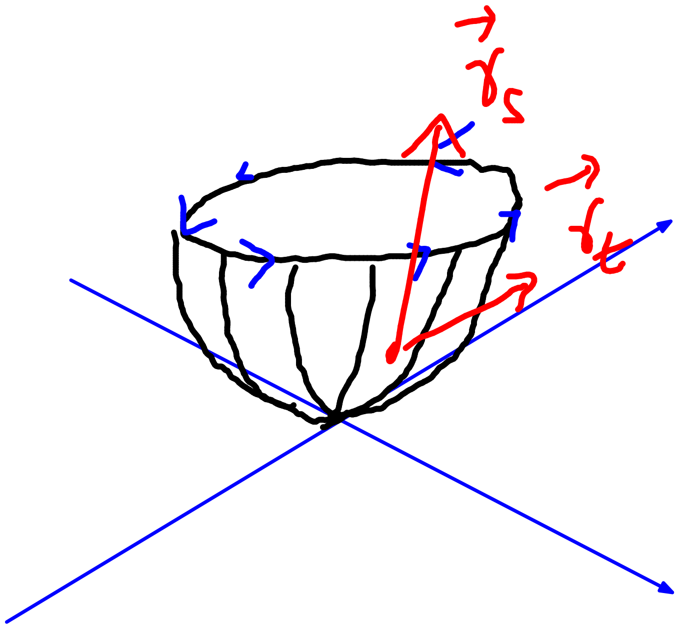
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



Verify that Stokes' Theorem is true for the given vector field **F** and surface S.

$\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



boundary parametric eq^n

$\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + \hat{k}$ $0 \le t \le 2\pi$

Surface parametric eq^n

$\vec{r}(s,t) = s\cos t\,\hat{i} + s\sin t\,\hat{j} + s^2\,\hat{k}$

$0 \le s \le 1$
$0 \le t \le 2\pi$

LHS

```
syms t
x = cos(t);
y = sin(t);
z = 1;
r = [x,y,z];
F = [y^2, x, z^2];
lhs = int(sum(F.*diff(r,t)),t,0,2*
```

Ans

= π

```
F = [y^2, x, z^2]
cF = curl(F)
```

RHS

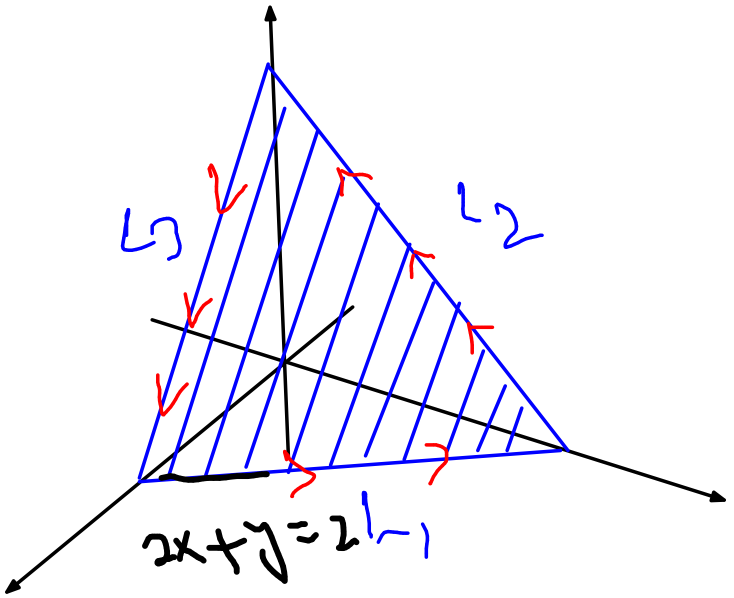
```
% problem 1
% rhs

syms s t
x = s*cos(t);
y = s*sin(t);
z = s^2;
r = [x,y,z];
cF = [0, 0, 1 - 2*y];
c = simplify(cross(diff(r,s),diff(r,t))

rhs = int(int(sum(c.*cF),t,0,2*pi),s,0,1
```

Verify that Stokes' Theorem is true for the given vector field **F** and surface S.

$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward



boundary integration

$$= \int_{L_1} \vec{F}\cdot d\vec{r} + \int_{L_2} \vec{F}\cdot d\vec{r} + \int_{L_3} \vec{F}\cdot d\vec{r}$$

parametric eqn for surface

$x = x$

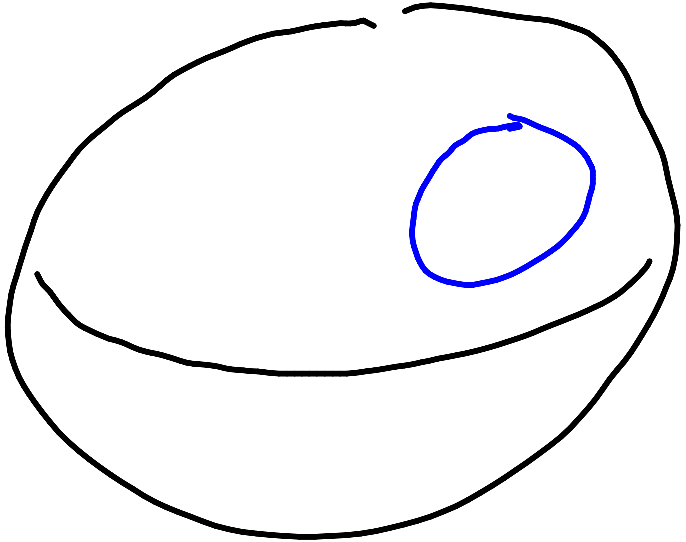
$y = y$

$z = 2 - 2x - y$

$0 \le x \le 1$

$0 \le y \le 2 - 2x$

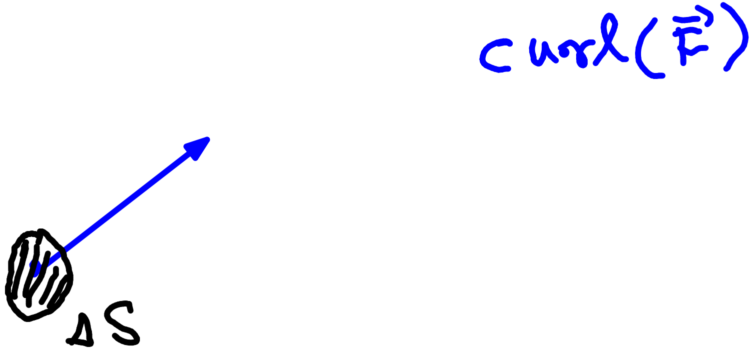
15. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.



the boundary of a sphere is empty set.

$$\int_{\emptyset} \vec{F} . dr = 0$$

Q. Physical interpretation of curl$(\vec{F})$.



$$\boxed{\iint_{\Delta S} \operatorname{curl}(\vec{F}) \cdot ds = \oint_{\partial S} \vec{F} \cdot dr}$$

$$\operatorname{curl}(\vec{F})\,|\Delta S| = \oint_{\partial S} \vec{F} \cdot dr$$

$\Delta S \perp \operatorname{curl}(\vec{F})$

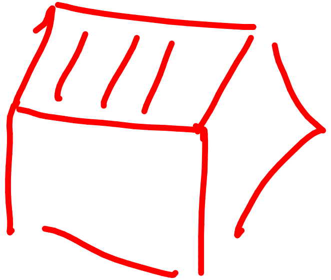
& ΔS is so small s.t. curl$(\vec{F})$ is constant.

$$\boxed{\operatorname{curl}(\vec{F}) = \lim_{|\Delta S| \to 0} \frac{\oint_{\partial S} \vec{F} \cdot dr}{|\Delta S|}}$$

kind of rotational energy density $\Leftarrow$

THE DIVERGENCE THEOREM Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$



next time

V EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

V EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)