2.2 Homogeneous Linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0$$

characteristic equation (or *auxiliary equation*)

$$\lambda^2 + a\lambda + b = 0$$

$\lambda^2 + a\lambda + b = 0$

case ①

real & distinct roots

λ_1 , λ_2

general sol^n of ODE

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

case ②

real & repeated roots

λ

$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

case ③

complex roots

$\lambda = \alpha \pm i\beta$

$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$

Q. Solve: $y'' + y' + y = 0$

Char. Eqⁿ: $\lambda^2 + \lambda + 1 = 0$

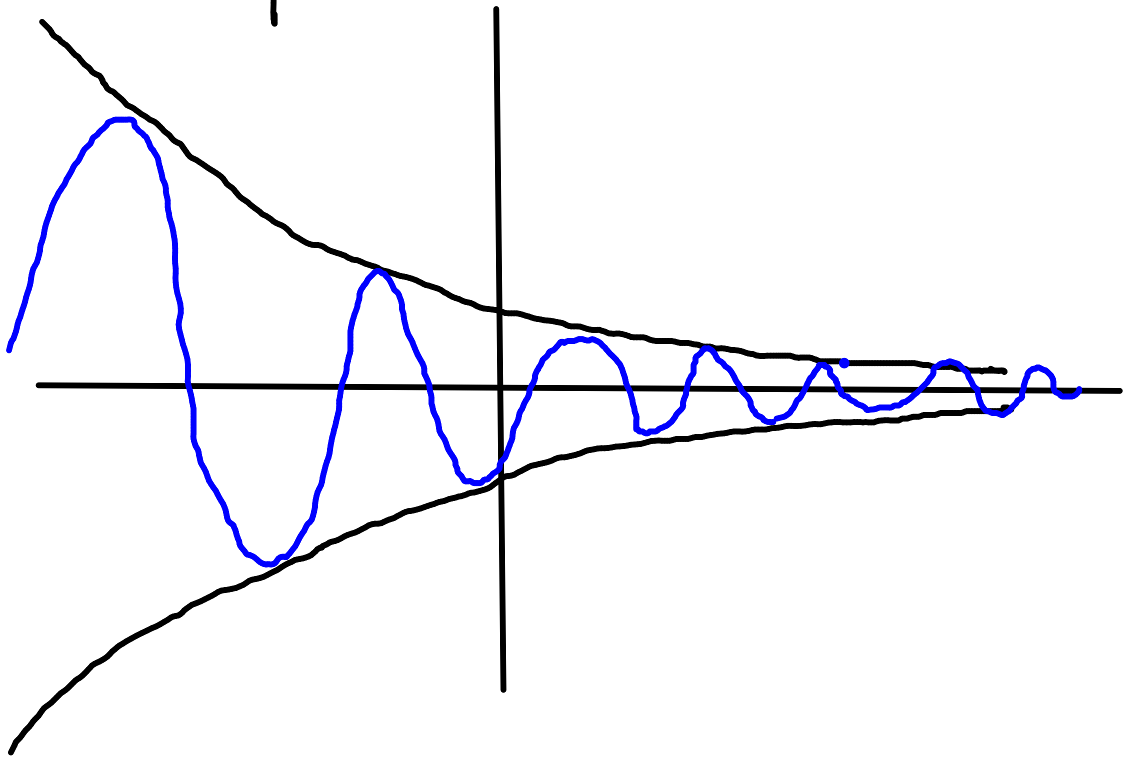
$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

Solⁿ: $y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

$$= A e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x + B\right)$$

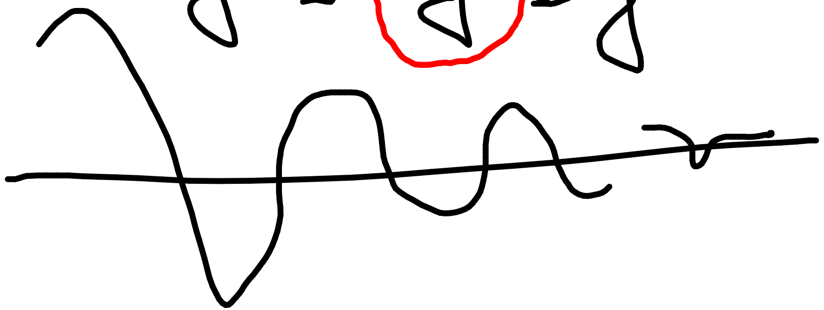
$$\cos\left(\frac{\sqrt{3}}{2}x + B\right)$$

$$e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x + B\right)$$



$\rightarrow \quad y'' + y' + y = 0$

$y'' = -y' - y$



$y'' + y = 0 \qquad y'' = -y$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y = c_1 \cos x + c_2 \sin x$

$= A \cos(x + B)$

Chapter 2

we will study ODEs of the form

$$a(x)\,y'' + b(x)\,y' + c(x)\,y = r(x)$$

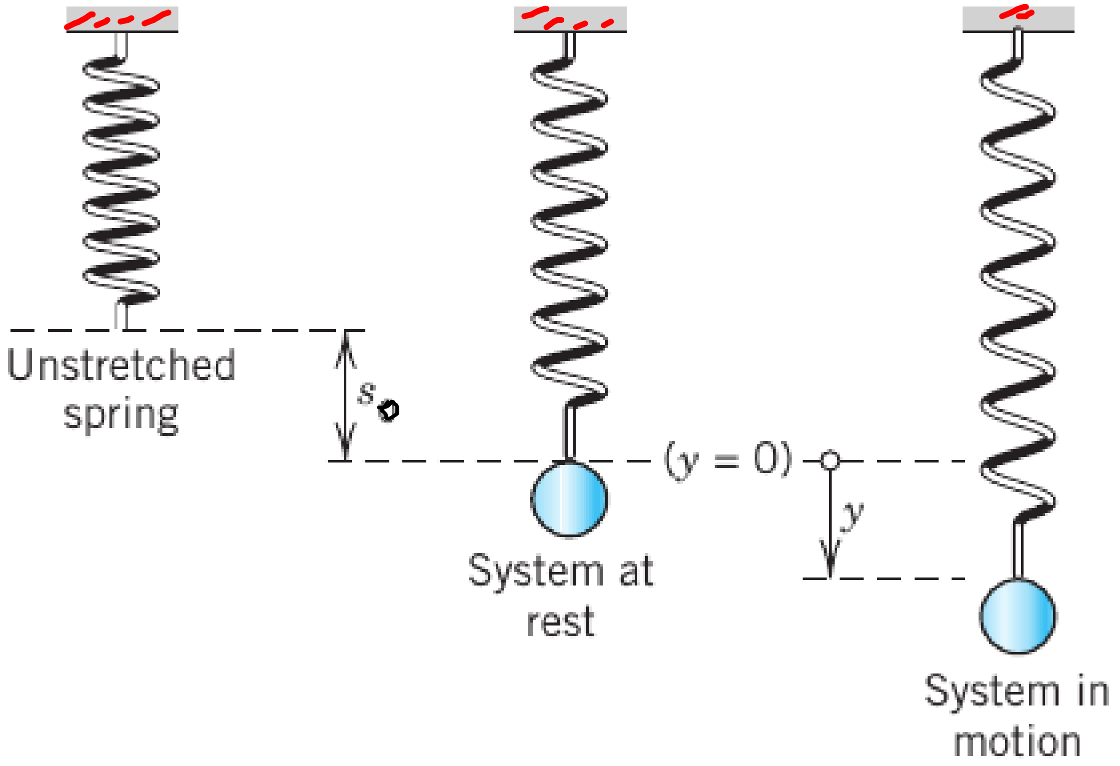
Q: an example where we encounter this type of ODE.

- mass-spring system
- LCR circuits
-

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega,$ $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

2.4 Modeling of Free Oscillations of a Mass–Spring System



$y(t)$: position of mass from equilibrium positi

$$my'' = \text{net force on the mass}$$

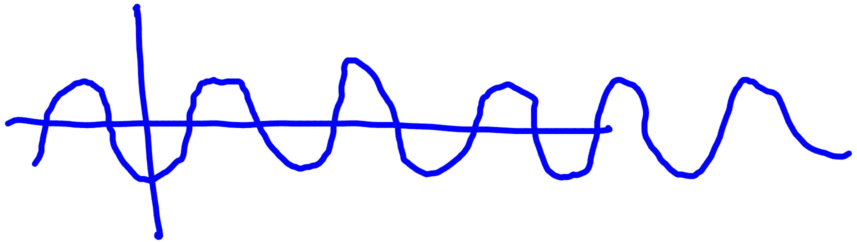
$$= \underbrace{mg}_{\text{gravity}} \underbrace{- k(s_0 + y)}_{\text{spring}} \underbrace{- cy'}_{\text{damping}}$$

$$my'' + cy' + ky = \underbrace{mg - kS_0}_{=0 \text{ (why??) at equilibrium}}$$

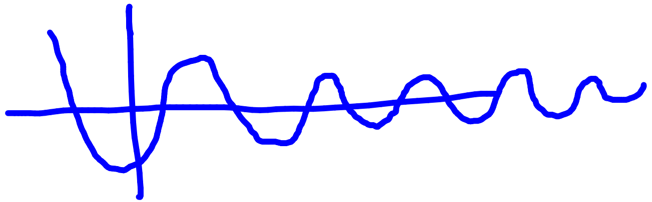
$$\rightarrow \quad \boxed{my'' + cy' + ky = 0}$$

$$my'' + cy' + ky = 0$$

→ if $c = 0$

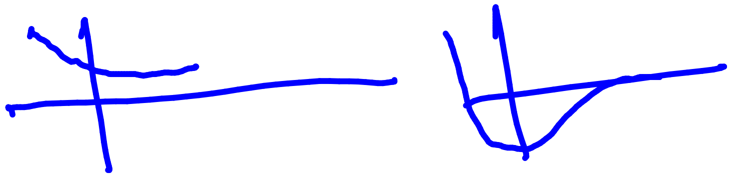


→ $c > 0$ but not too big



we need complex roots

→ $c > 0$ very big

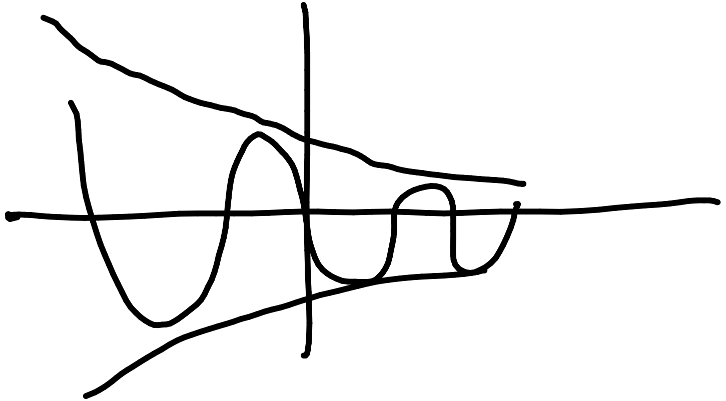


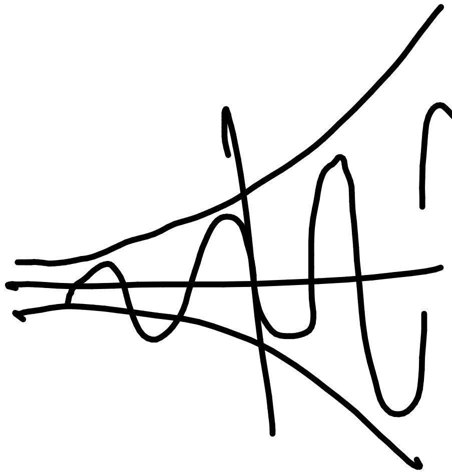
$$m\lambda^2 + c\lambda + k = 0$$

under damped system $\Leftrightarrow$ complex roots

$\Leftrightarrow \quad c^2 - 4mk < 0$

$\Leftrightarrow \quad c < \sqrt{4mk}$





8. Archimedian principle. This principle states that the buoyancy force equals the weight of the water displaced by the body (partly or totally submerged).

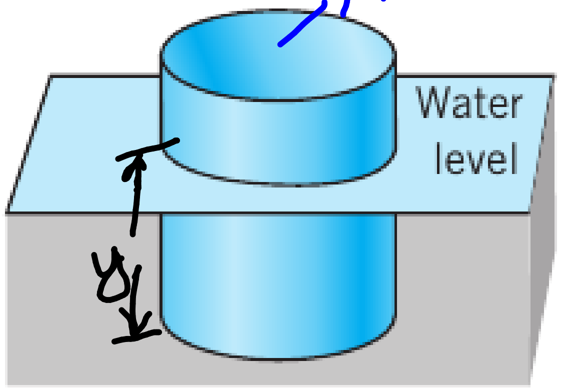
The cylindrical buoy of diameter 60 cm in Fig. 43 is floating in water with its axis vertical. When depressed downward in the water and released, it vibrates with period 2 sec. What is its weight?

ρ = density of water

$$my'' = mg - Ay\rho g$$

OK but non homogeneous

→ A: cross area



→ how to make it homogeneous

let y_0 be s.t. $mg = Ay_0\rho g$

let $y(t)$: be depth of lower edge from y_0.

$$m y'' = \cancel{m g} - \rho A (\cancel{y_0} + y) g$$

$$\rightarrow \quad m y'' + (A \rho g) y = 0$$

$$\rightarrow \quad y'' + \left(\frac{A \rho g}{m}\right) y = 0 \qquad \text{SHM}$$

$$y(t) = c_1 \cos\left(\sqrt{\frac{A\rho g}{m}}\, t + c_2\right)$$

$$\text{period} := \frac{2\pi}{\sqrt{\frac{A\rho g}{m}}} = 2 \quad (\text{given})$$

$\rightarrow$ solve for m

Bonus H.W. do this experiment

and tell me if this calculation predicts right weight.

(b) Flat spring (Fig. 45). The harmonic oscillations of a flat spring with a body attached at one end and horizontally clamped at the other are also governed by (3). Find its motions, assuming that the body weighs 8 nt (about 1.8 lb), the system has its static equilibrium 1 cm below the horizontal line, and we let it start from this position with initial velocity 10 cm/sec.

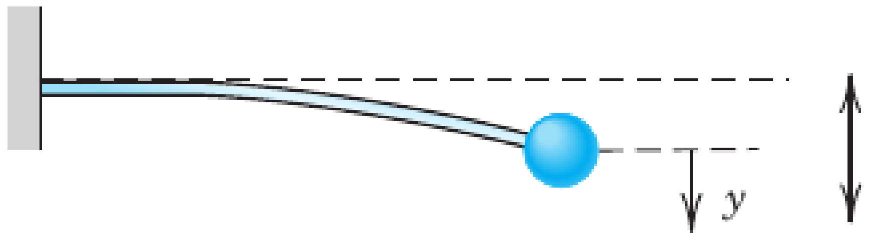


Fig. 45. Flat spring

(c) Torsional vibrations (Fig. 46). Undamped torsional vibrations (rotations back and forth) of a wheel attached to an elastic thin rod or wire are governed by the equation $I_0\theta'' + K\theta = 0$, where θ is the angle measured from the state of equilibrium. Solve this equation for $K/I_0 = 13.69 \text{ sec}^{-2}$, initial angle $30°(= 0.5235 \text{ rad})$ and initial angular velocity $20° \text{ sec}^{-1} (= 0.349 \text{ rad} \cdot \text{sec}^{-1})$.

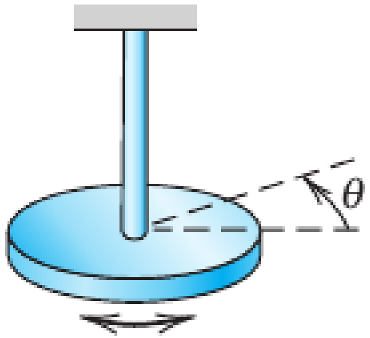


Fig. 46. Torsional vibrations