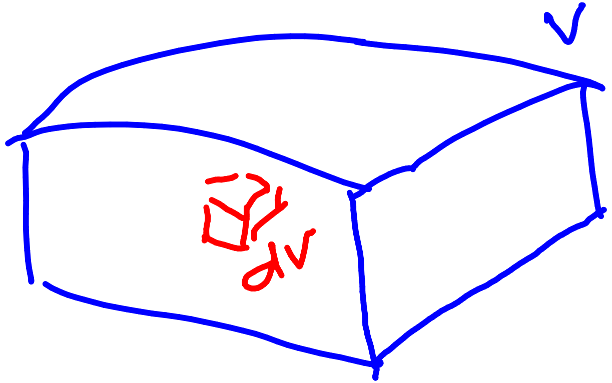
12.5 TRIPLE INTEGRALS



$\rho(x,y,z)$: density at point (x,y,z)

$dm = \rho \, dV$ = local mass

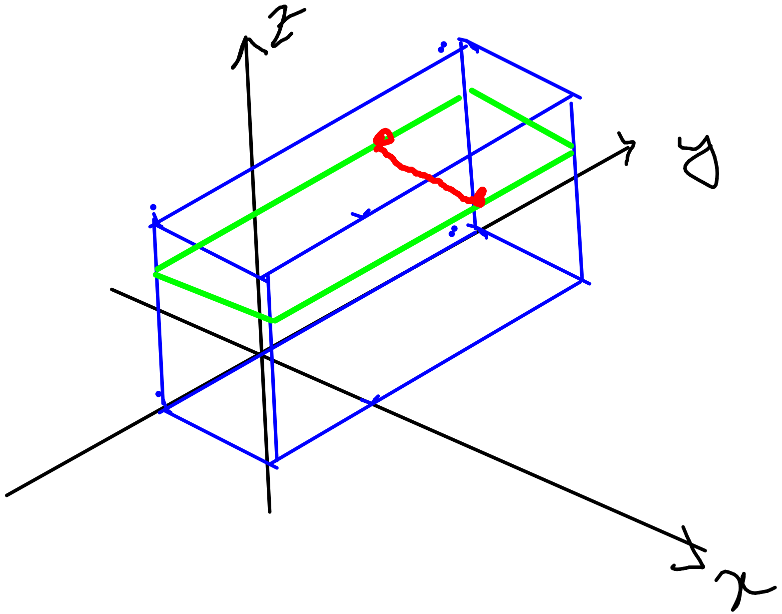
$$\iiint_V \underbrace{\rho(x,y,z)\, dV} = \iiint_V dm = \text{total mass}$$

Note: Do 12.4 by yourself (Don't skip)

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

Sketh the domain



$$= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz$$

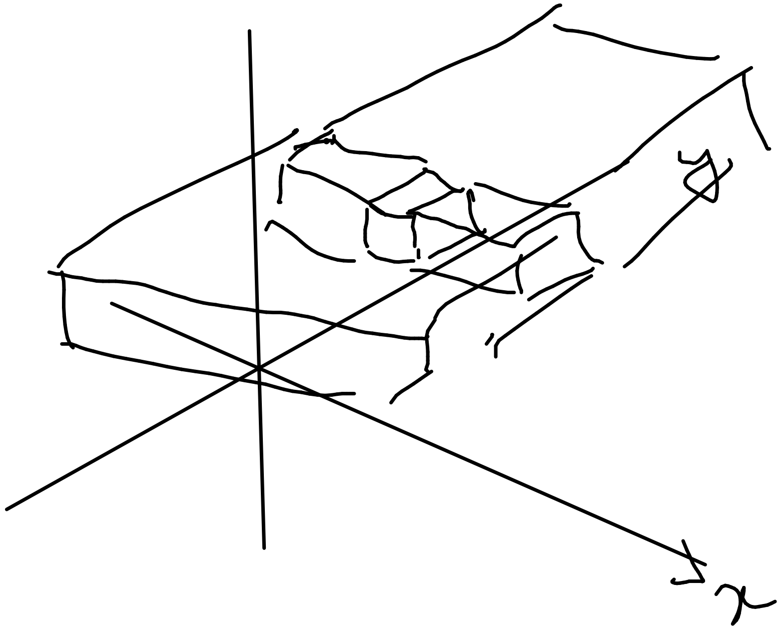
start integrating from inner most integration

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} y z^2\, dy\, dz$$

V EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

Sketch the domain



$$= \int_0^3 \int_{-1}^2 \left(\int_0^1 xyz^2 \, dx \right) dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \left(\int_0^1 xyz^2 \, dx \right) dy\,dz$$

$$= \frac{3}{4}\int_0^3 z^2\,dz = \frac{\cancel{3}}{4}\cdot\frac{1}{\cancel{3}}\cdot 27 = \frac{27}{4}$$

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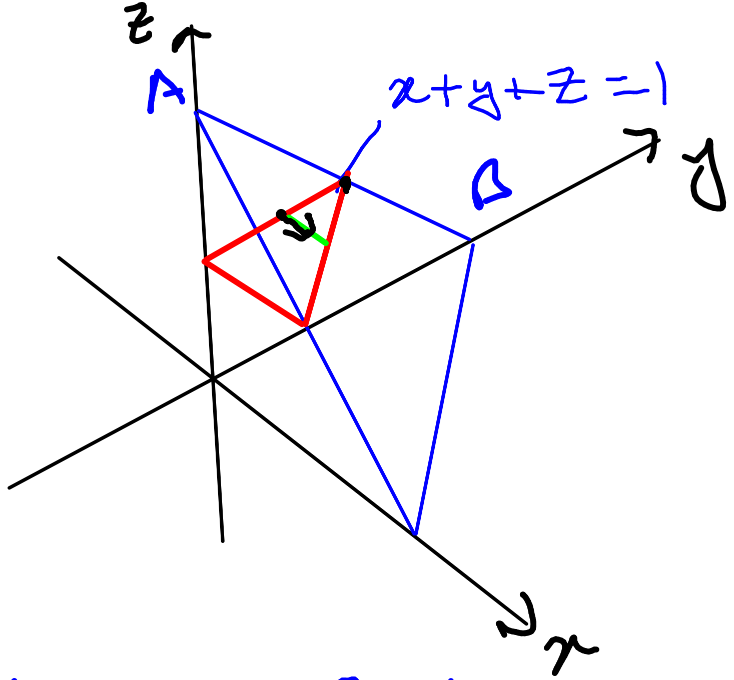
EXAMPLE 2 Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Sketch the region E



$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} z \quad dx\,dy\,dz$$

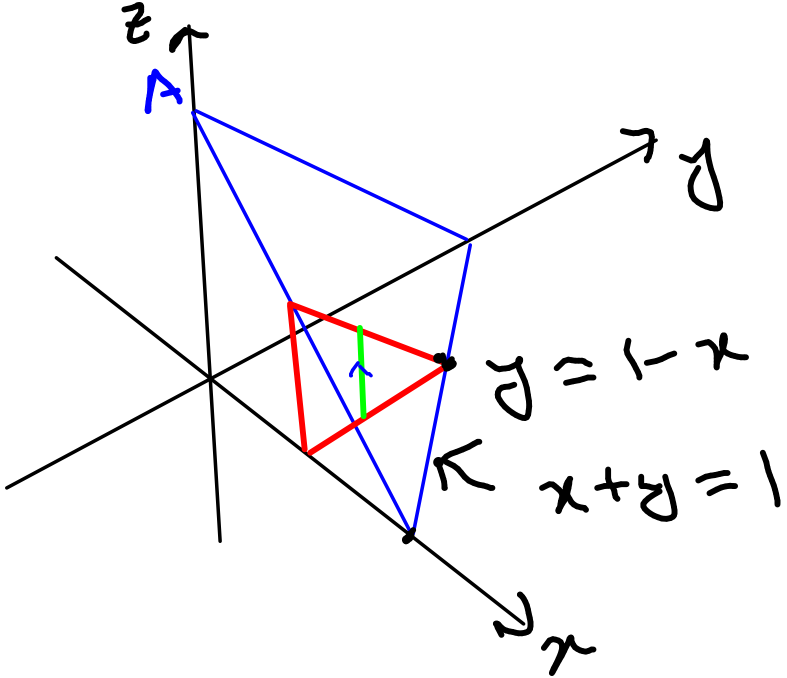
$$= \int_?^? \int_?^? \int_?^? z \quad dz\,dy\,dx$$

AB: $y+z=1$

$x=0$

EXAMPLE 2 Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Sketch the region E



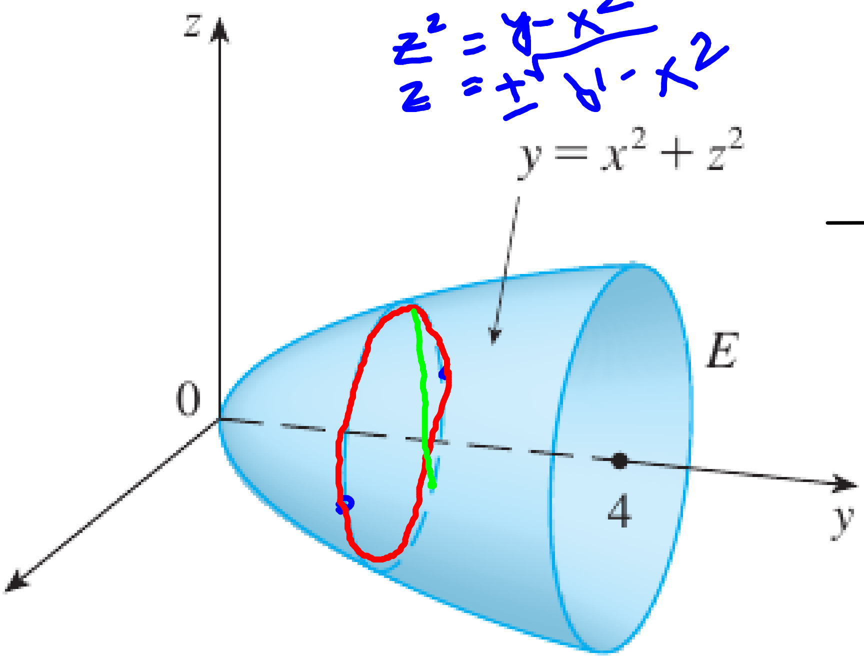
$$= \int_0^1 \int_0^{1-x} \int_0^{1-y-x} z \; dz\,dy\,dx$$

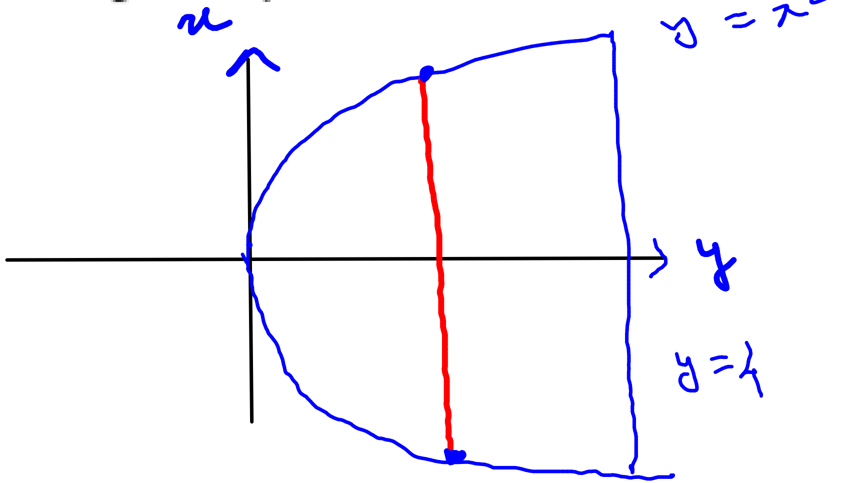
$$= \frac{1}{24}$$

V EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

$z^2 = y - x^2$

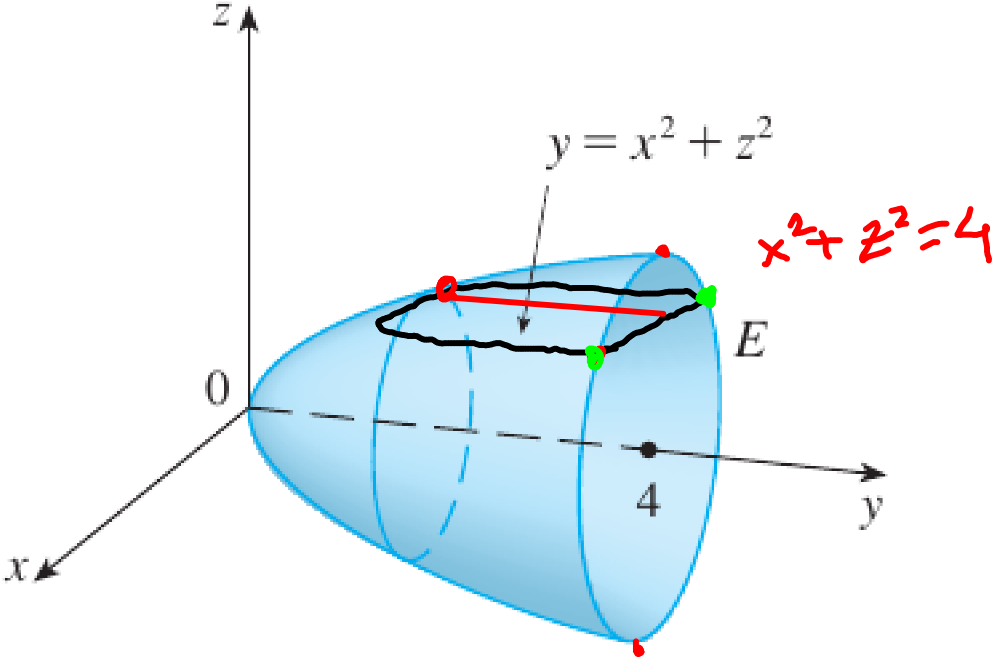
$z = \pm\sqrt{y - x^2}$

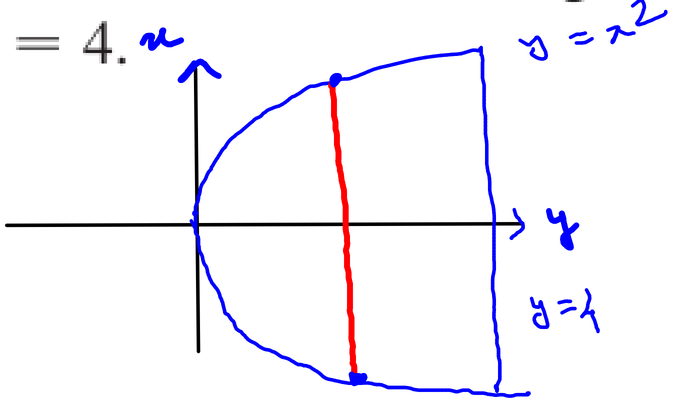




$$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\; dz\, dx\, dy$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

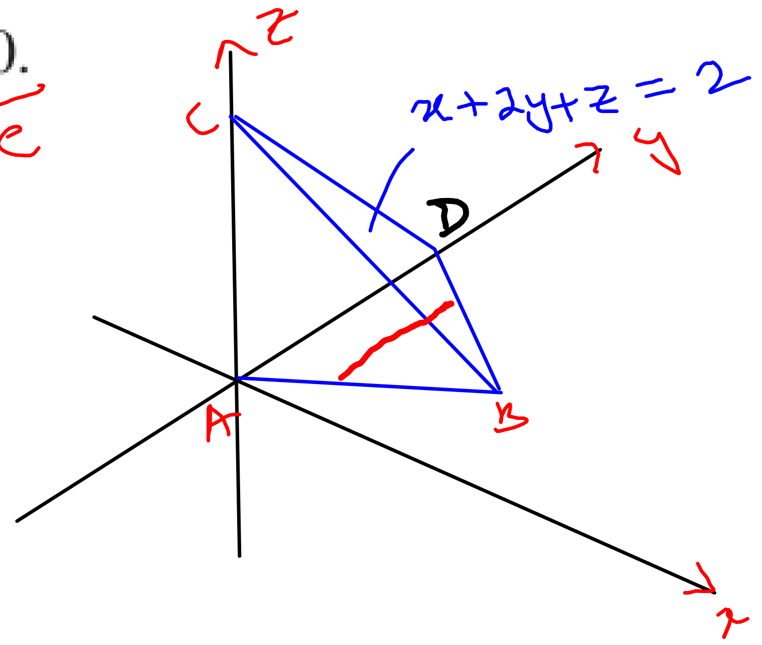




$$= \int_{-2}^{2} \int_{-\sqrt{4-z}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\, dx\, dz$$

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $\underbrace{z = 0}_{\text{base}}$.

Volume $= \iiint_V dV$



sketch these planes & the enclosed volume

$$\int_0^1 \int_{x/2}^{\left(\frac{2-x}{2}\right)} \int_0^{(2-x-2y)} dz\, dy\, dx$$

ABC : $x = 2y$

D $\begin{cases} x + 2y = 2 \\ x = 2y \end{cases}$

$y = 1/2$, $x = 1$

25–26 ▪ Sketch the solid whose volume is given by the iterated integral.

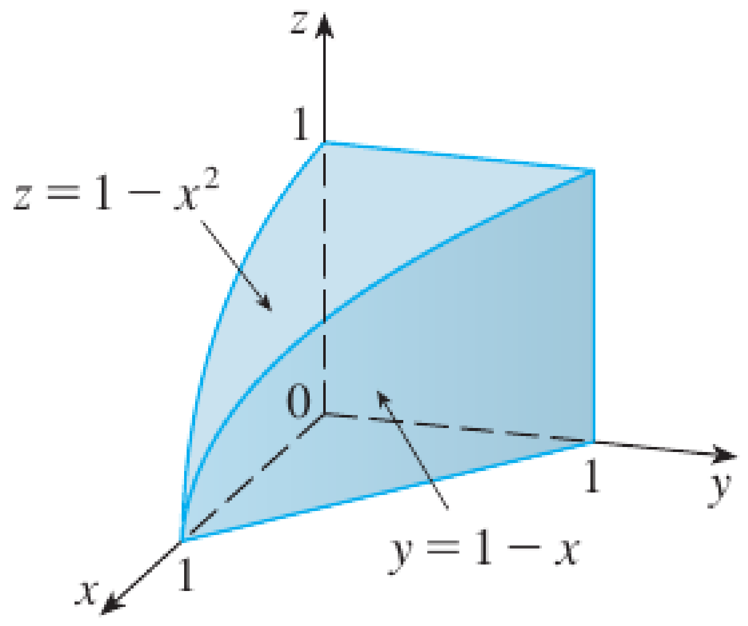
25. $\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$

26. $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

32. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

9 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$