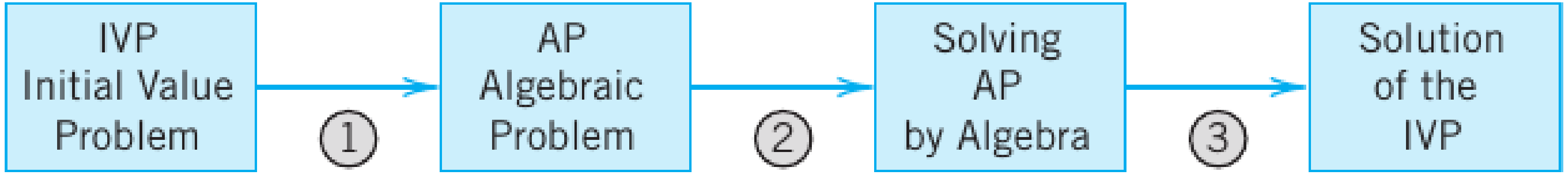
CHAPTER 6 Laplace Transforms

$\mathcal{L}(y)$ $\quad y = \mathcal{L}^{-1}(\mathcal{L}(y))$



solving for y

Fig. 113. Solving an IVP by Laplace transforms

Input : $f(t)$

Output : $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\, 1\, dt = \int_0^\infty e^{-st}\,dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty = \lim_{t\to\infty} -\frac{1}{s}\Big[\underbrace{e^{-st}}_{\to 0 \text{ only if } s>0} - 1\Big]$$

$$= \begin{cases} \frac{1}{s} & \text{if } s>0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathscr{L}(f)$.

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \begin{cases} \dfrac{1}{s-a} & \text{if } s > a \\ \text{undefined} & \text{if } s \leq a \end{cases}$$

e.g. $\mathcal{L}(e^{3t}) = \ ?? = \dfrac{1}{s-3} \quad (s>3)$

$\mathcal{L}(e^{-t}) = \dfrac{1}{s+1} \quad (s>-1)$

$$\mathcal{L}(t) = ?? = \int_0^\infty e^{-st}\, t\, dt = \frac{1}{s^2} \quad (s>0)$$

$$\mathcal{L}(t^2) = \int_0^\infty e^{-st} t^2\, dt$$

$$= \underbrace{\left| t^2 \frac{e^{-st}}{-s} \right|_0^\infty}_{=0 \text{ why ??}} + \underbrace{\frac{1}{s}\int_0^\infty 2t\, e^{-st}}_{= \frac{2}{s}\mathcal{L}(t) = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}} =$$

$$= \underbrace{\left| t^2 \frac{e^{-st}}{-s} \right|_0^\infty}_{\substack{=0 \\ \text{why ??}}} + \underbrace{\frac{1}{s}\int_0^\infty 2t\, e^{-st}}_{\frac{2}{s}\mathcal{L}(t) = \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3}} =$$

$$t = 0 \qquad \frac{t^2 e^{-st}}{-s} = 0$$

$$\lim_{t\to\infty} t^2 e^{-st} = \lim_{t\to\infty} \frac{t^2}{e^{st}} \quad \frac{\infty}{\infty} = \lim_{t\to\infty} \frac{2t}{s e^{st}} = \lim_{t\to\infty} \frac{2}{s^2 e^{st}}$$

$$= \frac{2}{\infty} = 0$$

$$\mathscr{L}(t^{n+1}) = \frac{n+1}{s}\mathscr{L}(t^n) = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

Proof ?!

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1}\,dt$$

$$= \underbrace{\left| t^{n+1}\frac{e^{-st}}{-s}\right|_0^\infty}_{=0} + \frac{(n+1)}{s}\int_0^\infty e^{-st}t^n\,dt = \frac{n+1}{s}\mathcal{L}(t^n)$$

why ?? L'hopital's rule

$$= \frac{(n+1)}{s}\,\frac{n}{s}\,\mathcal{L}(t^{n-1})$$

$$= \frac{n+1}{s}\,\frac{n}{s}\,\frac{n-1}{s}\,\mathcal{L}(t^{n-2})$$

$$\mathscr{L}(t^{n+1}) = \frac{n+1}{s}\mathscr{L}(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

It is possible to find

$\mathcal{L}(t^a)$, where $a > 0$ & real

$\mathcal{L}(t^{1.5})$ e.t.c

But these require "Gamma functions"

Gamma Functions is not in syllabus

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

$$\mathcal{L}(\cos \Omega t) = \frac{s}{s^2 + \Omega^2} \qquad \mathcal{L}(\sin \Omega t) = \frac{\Omega}{s^2 + \Omega^2}$$

Table 6.1 **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	$\cancel{t^a \; (a \text{ positive})}$	$\cancel{\dfrac{\Gamma(a + 1)}{s^{a+1}}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

Euler

25–32 INVERSE LAPLACE TRANSFORMS

26. $\dfrac{5s+1}{s^2-25}$

find $f(t)$ s.t. $\mathcal{L}(f(t)) = \dfrac{5s+1}{s^2-25}$

$$\mathcal{L}^{-1}\left(\frac{5s+1}{s^2-25}\right) = \mathcal{L}^{-1}\left(\frac{A}{s-5}\right) + \mathcal{L}^{-1}\left(\frac{B}{s+5}\right),$$

$$= A e^{5t} + B e^{-5t}$$

$A = 13/5$

$B = 12/5$

25–32 INVERSE LAPLACE TRANSFORMS

27. $\dfrac{s}{L^2s^2 + n^2\pi^2}$

L, n are some const.

Recall

$$= \frac{1}{L^2}\left[\frac{s}{\frac{n^2\pi^2}{L^2} + s^2}\right]$$

$$\mathcal{L}(\cos\omega t) = \frac{s}{\omega^2 + s^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{L^2\left\{\frac{n^2\pi^2}{L^2} + s^2\right\}}\right] = \frac{1}{L^2}\mathcal{L}^{-1}\left(\frac{s}{\frac{n^2\pi^2}{L^2} + s}\right) = \frac{1}{L^2}\cos\left(\frac{n\pi}{L}t\right)$$

25–32 **INVERSE LAPLACE TRANSFORMS**

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

29. $\frac{12}{s^4} - \frac{228}{s^6}$

$$\mathcal{L}^{-1}\left(\frac{12}{s^4} - \frac{228}{s^6}\right)$$

$$= \frac{12}{3!}\mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) - \frac{228}{5!}\mathcal{L}^{-1}\left(\frac{5!}{s^6}\right)$$

$$= \frac{12}{3!}t^3 - \frac{228}{5!}t^5$$

First Shifting Theorem, s-Shifting

if $\mathcal{L}(f(t)) = F(s)$

$$\mathscr{L}\{e^{at}f(t)\} = F(s - a)$$

$$e^{at}f(t) = \mathscr{L}^{-1}\{F(s - a)\}$$

e.g. $\mathcal{L}(t^2) = ?? = \frac{2}{s^3}$

$$\mathcal{L}(e^{5t}t^2) = \frac{2}{(s-5)^3}$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(e^{10t} \cdot 1) = \frac{1}{s-10}$$

33–45 **APPLICATION OF s-SHIFTING** find the inverse transform.

$$\frac{6}{(s+1)^3}$$

$$\mathcal{L}^{-1}\left(\frac{6}{s^3}\right) = 3t^2$$

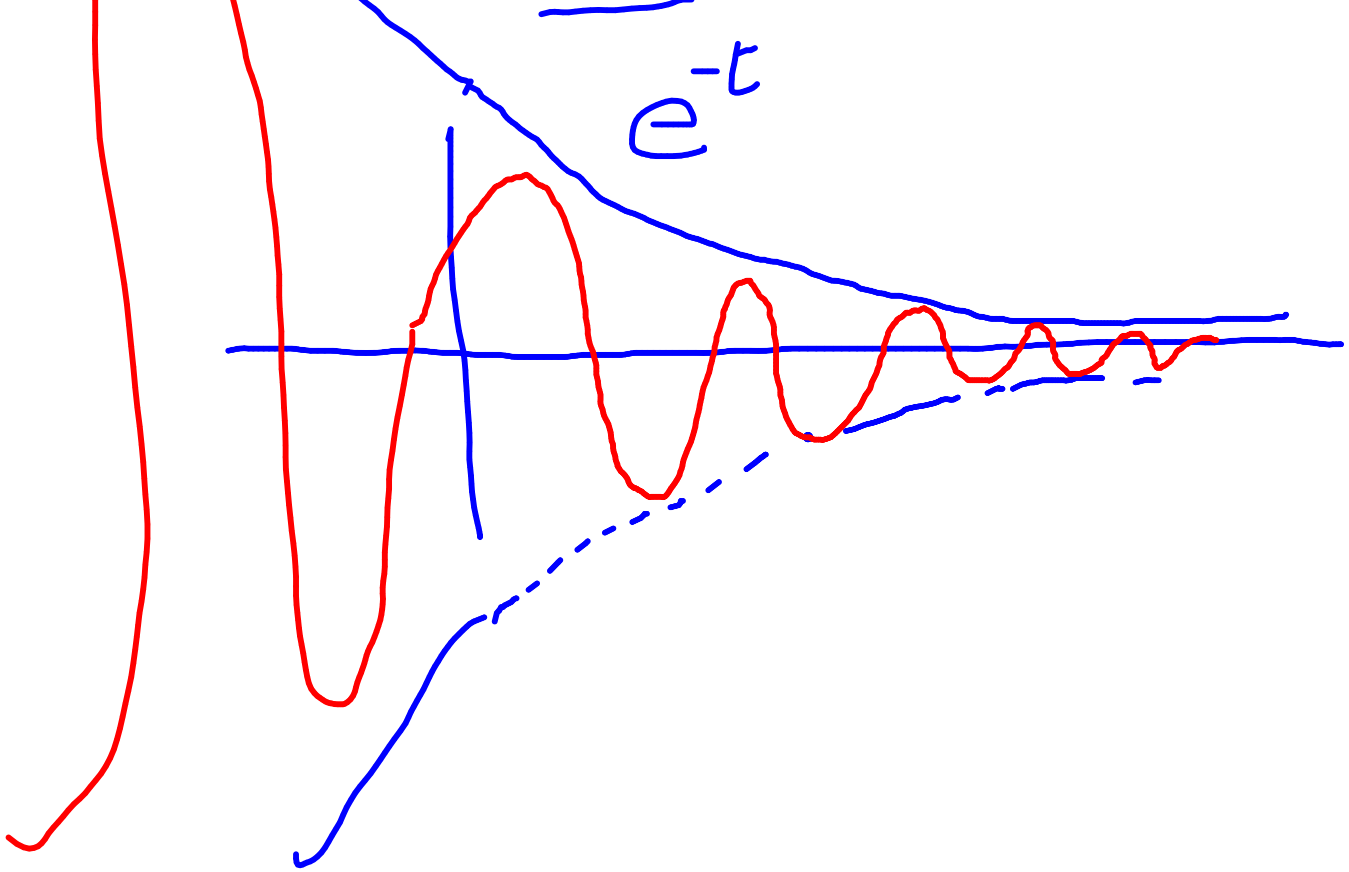
$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(3t^2) = \frac{6}{s^3}$$

$$\mathcal{L}(e^{-t}3t^2) = \frac{6}{(s+1)^3}$$

8.

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$



33–45 APPLICATION OF *s*-SHIFTING

find the inverse transform.

$$\frac{4}{s^2 - 2s - 3}$$

33–45 **APPLICATION OF *s*-SHIFTING** find the inverse transform.

$$\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}$$

6.2 Transforms of Derivatives and Integrals. ODEs

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

EXAMPLE 4 **Initial Value Problem: The Basic Laplace Steps**

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

EXAMPLE 5 Solve the initial value problem

$$y'' + y' + 9y = 0. \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

EXAMPLE 6 Shifted Data Problems

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$