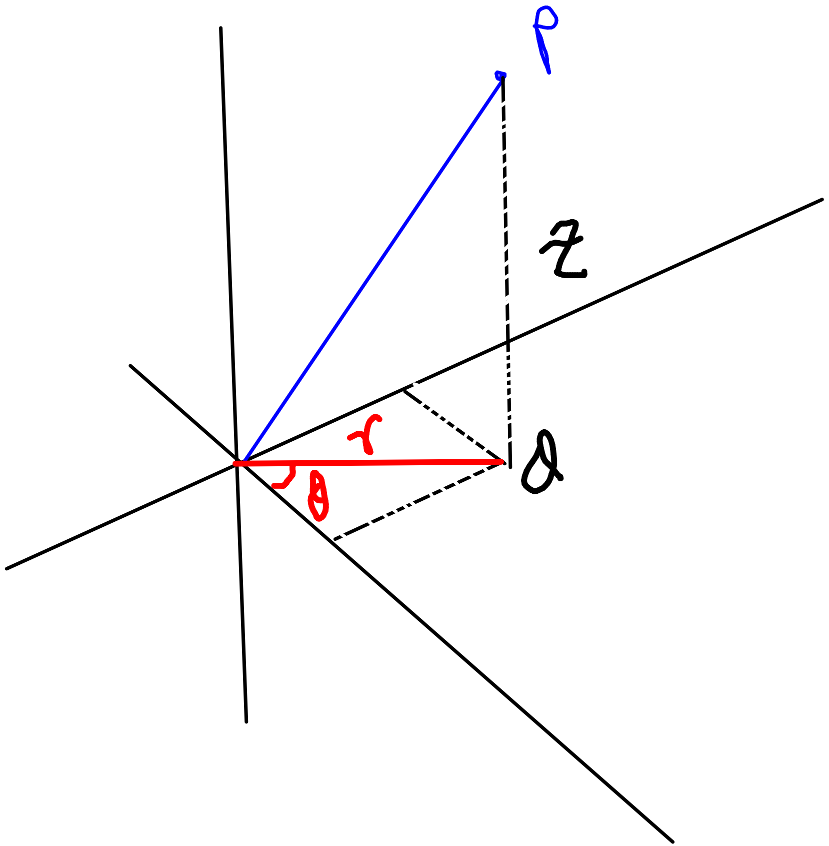
Section 12.6

## 12.6 TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Section 12.7

## 12.7 TRIPLE INTEGRALS IN SPHERICAL COORDINATES

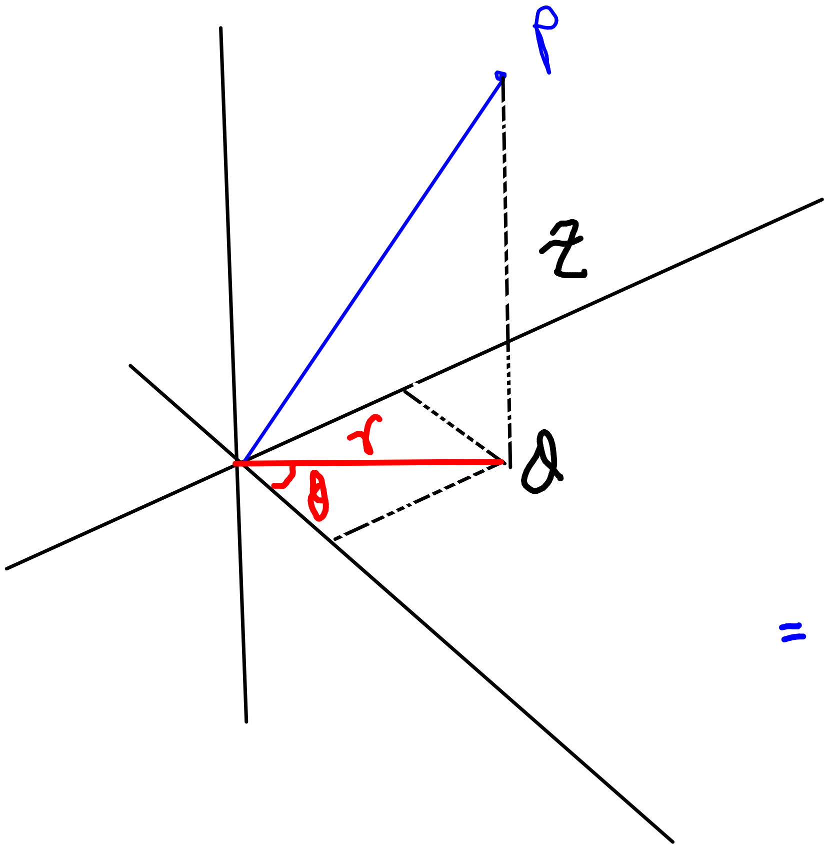
# CYLINDRICAL COORDINATES




$$(r, \theta, z) \qquad \left(5, \frac{\pi}{3}, 10\right)$$

$(r, \theta)$ = polar coordinates of projection of P on the xy plane

# CYLINDRICAL COORDINATES




$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

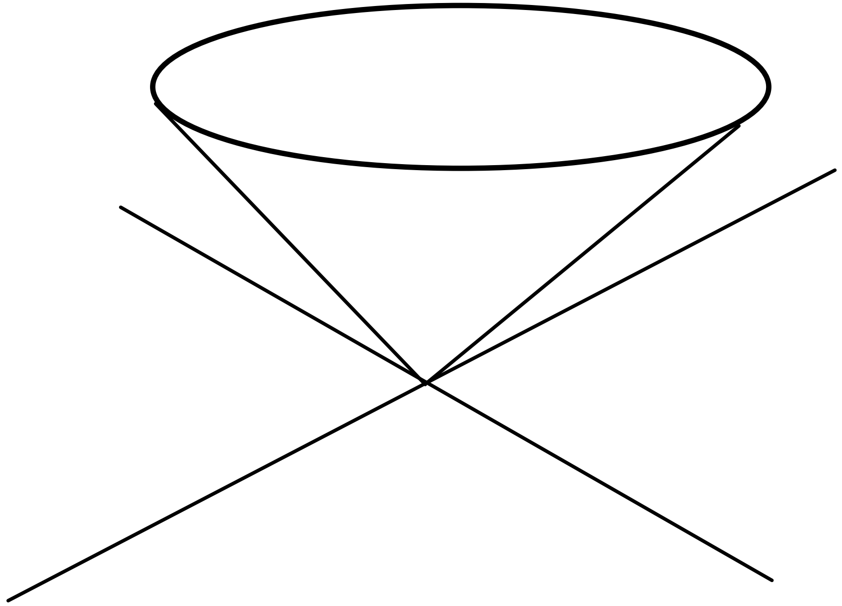
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\text{Jacobian} = \frac{\partial(x,y,z)}{\partial(r,\theta,z)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

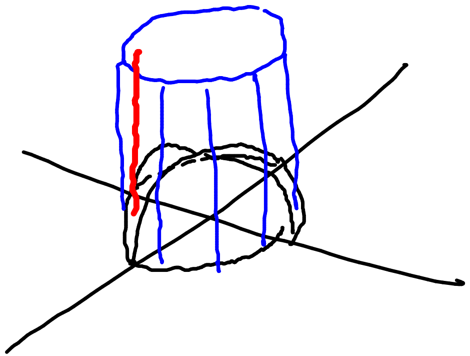
**EXAMPLE 2** Describe the surface whose equation in cylindrical coordinates is $z = r$.

$z = \sqrt{x^2+y^2}$



**V EXAMPLE 3** A solid $E$ lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of $E$.

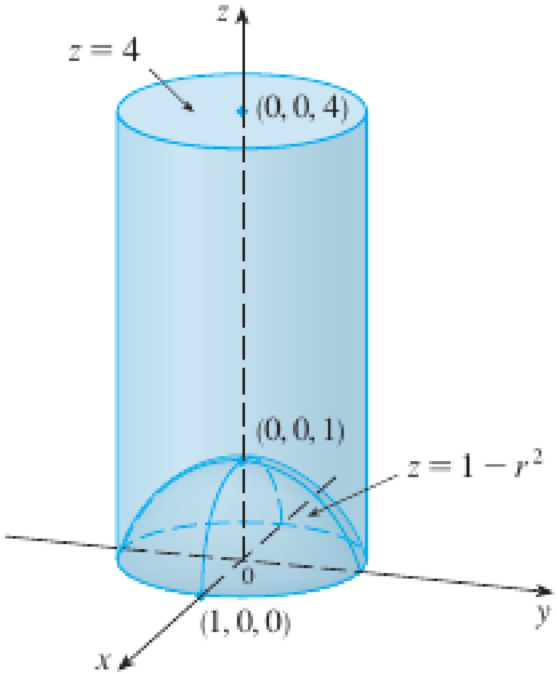
Sketch the domain



$\Big|\; z = 1 - r^2$

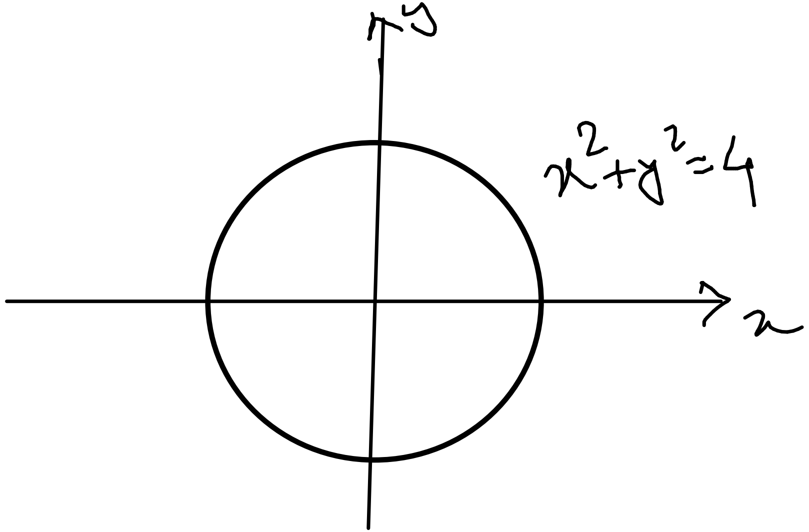
$$\int_0^1 \int_0^{2\pi} \int_{1-r^2}^{4} (\alpha r)\, r \, dz\, d\theta\, dr$$

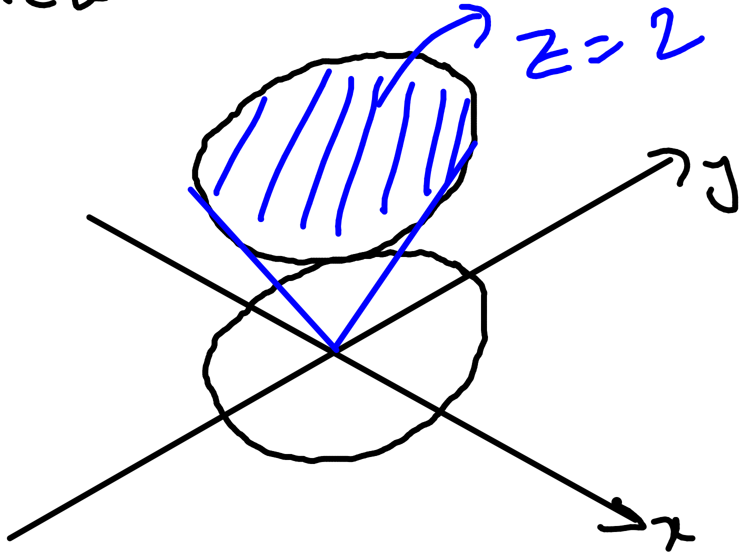
let density, $\rho = \alpha r$



**EXAMPLE 4** Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx. = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\, r\; dz\,dr\,d\theta$

Identify the region of integration and swithch to cylindrical coordinates.



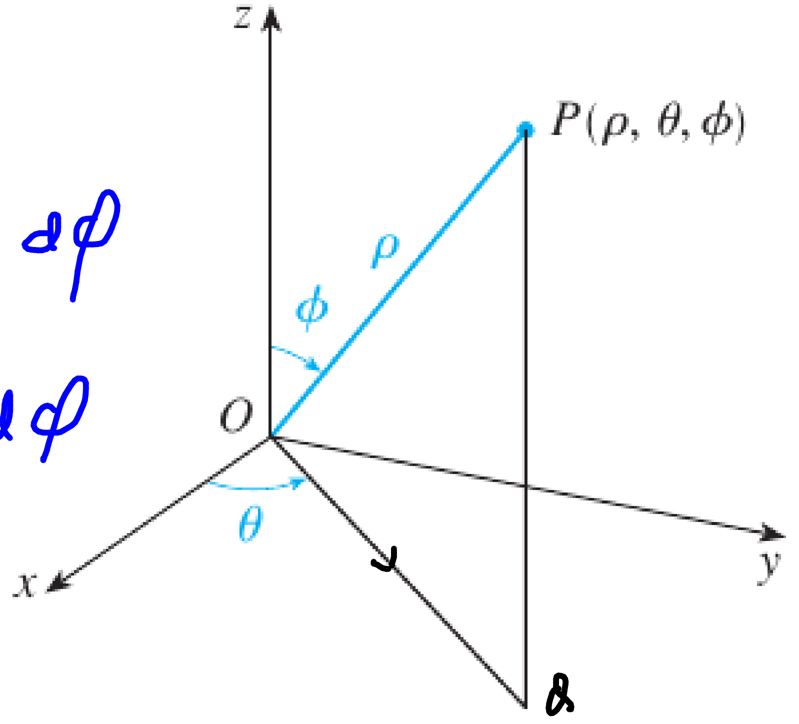

$z = \sqrt{x^2+y^2}$

$z = r$

## SPHERICAL COORDINATES

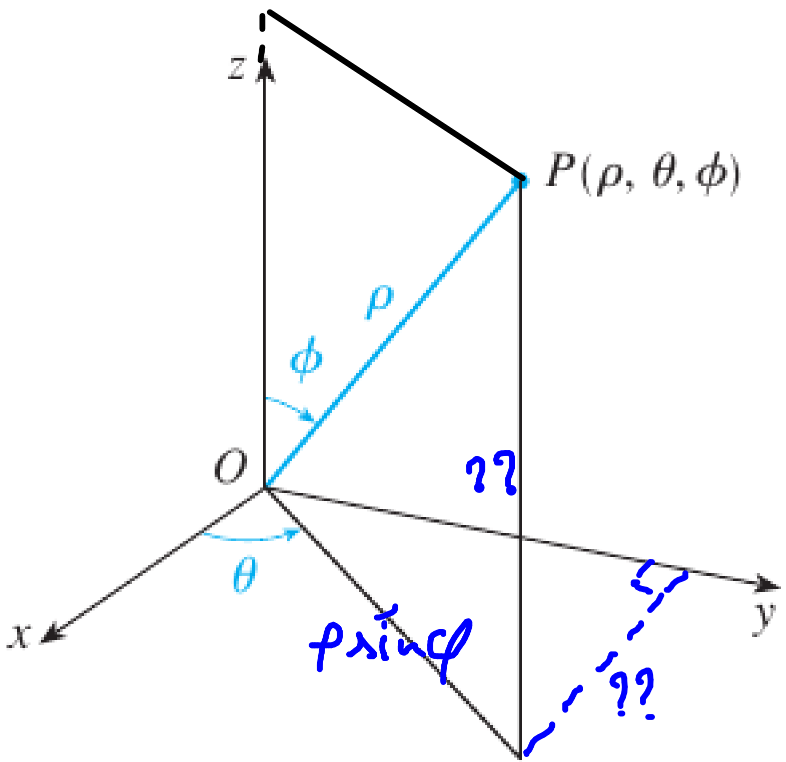
$$dx\,dy\,dz = (\ ???\ )\, d\rho\, d\theta\, d\phi$$

$$= \boxed{\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}}\, d\rho\, d\theta\, d\phi$$




$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

## SPHERICAL COORDINATES




Q: given we know the spherical coordinates of a point say $(\rho, \theta, \phi)$, find $x, y, z$.

$$x = (\rho \sin\phi) \cos\theta$$

$$y = (\rho \sin\phi) \sin\theta$$

$$z = \rho \cos(\phi)$$

$$x = (\rho \sin\phi)\cos\theta$$

$$y = (\rho \sin\phi)\sin\theta$$

$$z = \rho\cos(\phi)$$

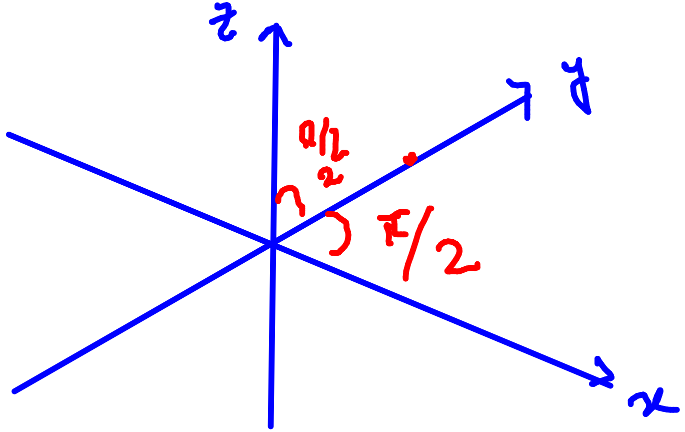
$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \rho^2 \sin\phi$$

$$dx\,dy\,dz \longleftrightarrow \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

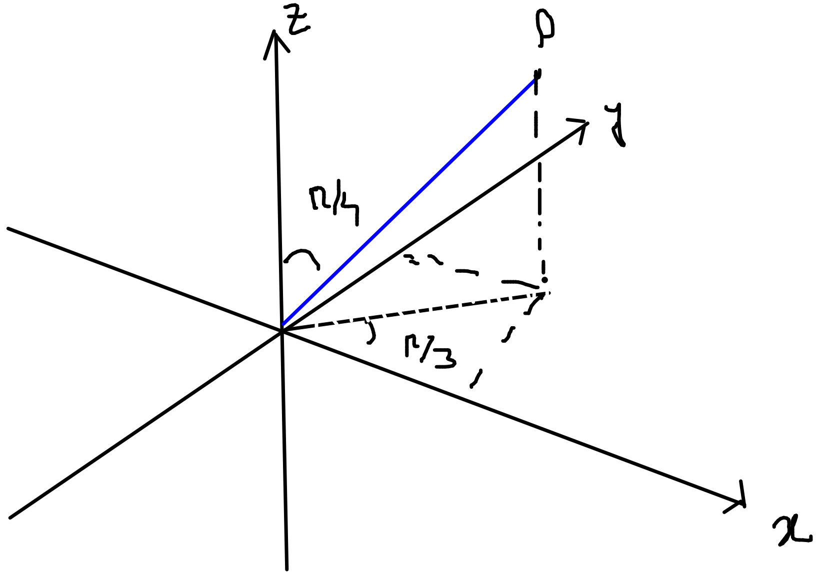
Locate the point in space given by the spherical coordinates

$\rightarrow (\rho, \theta, \phi) = (2, \pi/2, \pi/2)$



Locate the point in space given by the spherical coordinates

$\rightarrow (\rho, \theta, \phi) = (5, \pi/3, \pi/4)$



Q.

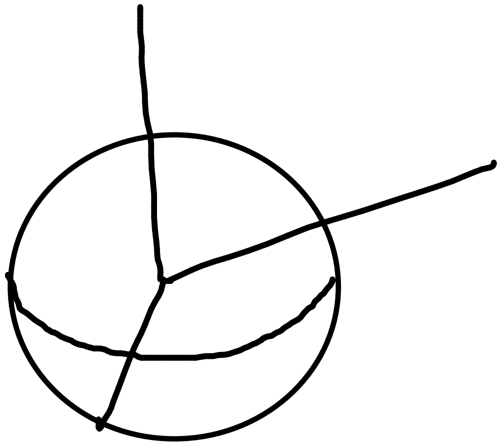
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz$$

switch to spherical coordinates.

$$\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{\infty} f(\qquad)\,\rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

**V EXAMPLE 3** Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where $B$ is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$



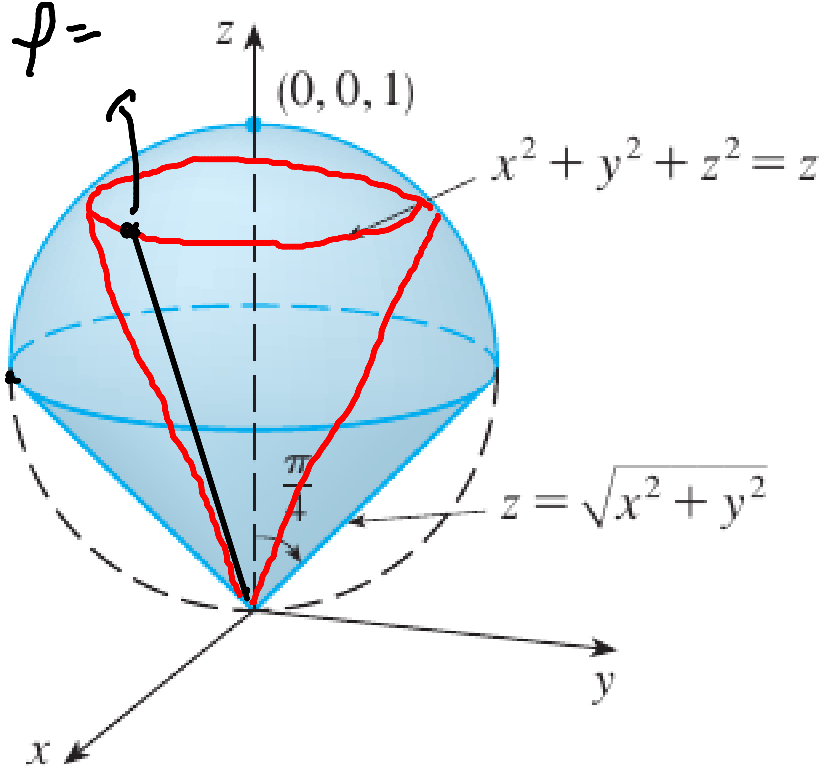
$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \underline{e}^{(\rho^3)}\, \rho^2 \sin\phi \quad d\rho \quad d\theta \quad d\phi$$

**V EXAMPLE 4** Use spherical coordinates to find the volume of the solid that lies above the cone $\underbrace{z = \sqrt{x^2 + y^2}}_{\phi = \pi/4}$ and below the sphere $\underbrace{x^2 + y^2 + z^2 = z}$. (See Figure 9.)

$$\iiint 1 \, dV$$

$$\rho^2 = \rho \cos\phi$$

$$\rho = \cos\phi$$




$$0 \le \phi \le \pi/4$$

$$0 \le \theta \le 2\pi$$

$$0 \le \rho \le \cos\phi$$

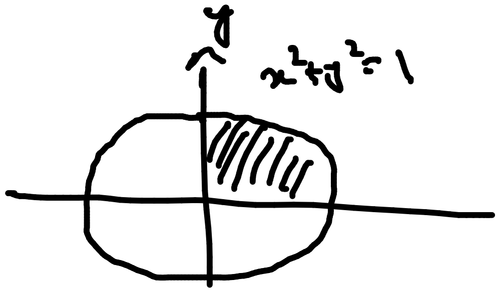
$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

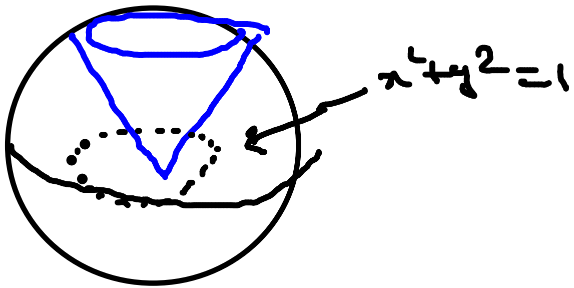
**35–36** ▪ Evaluate the integral by changing to spherical coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx$$

$z = \sqrt{x^2+y^2}$
cone

$z = \sqrt{2-x^2-y^2}$
$x^2+y^2+z^2 = 2$




Circle of intersection of cone & sphere

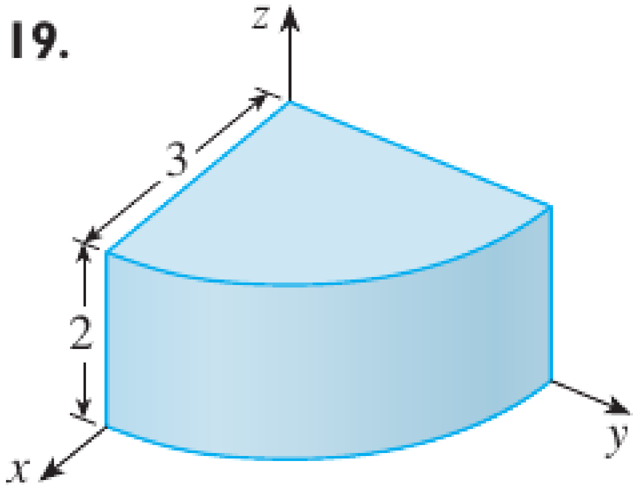
$\sqrt{x^2+y^2} = \sqrt{2-x^2-y^2}$

$x^2+y^2 = 1$

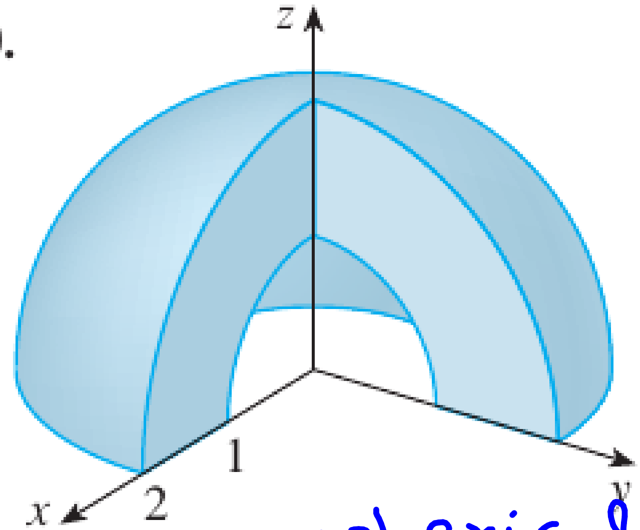
$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

**19–20** ▪ Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

**19.**




**20.**




cylindrical

$0 \le z \le 2$

$0 \le r \le 3$

$0 \le \theta \le \pi/2$

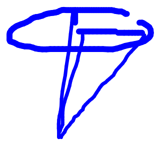
spherical

$1 \le \rho \le 2$

$\pi/2 \le \theta \le 2\pi$

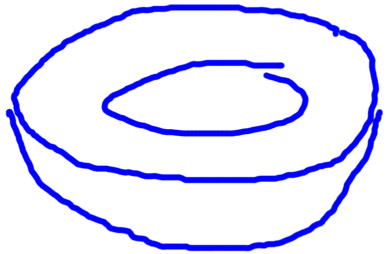
$0 \le \phi \le \pi/2$

**17–18** ▪ Sketch the solid whose volume is given by the integral and evaluate the integral.



**17.** $\displaystyle\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$

**18.** $\displaystyle\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$

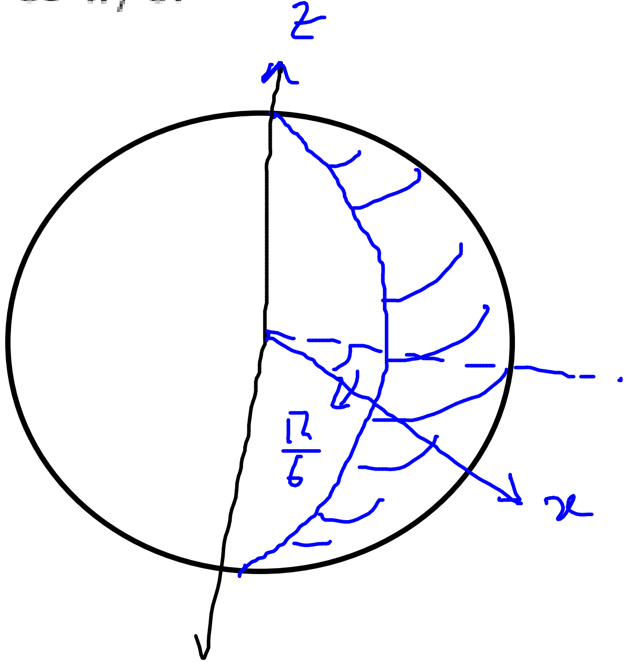


Use cylindrical or spherical coordinates, whichever seems more appropriate.

31. Find the volume and centroid of the solid $E$ that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

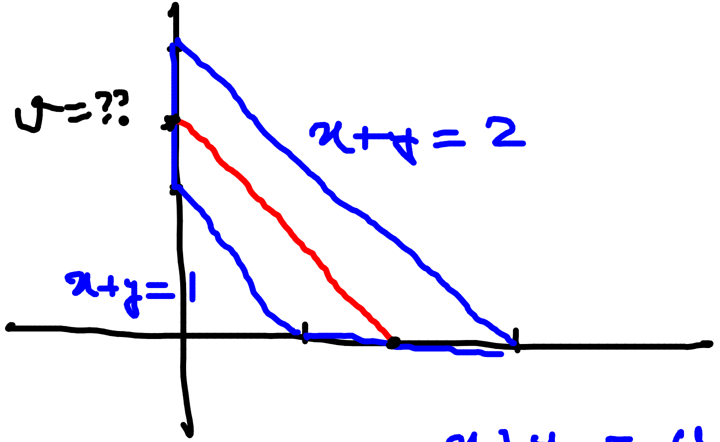
Use cylindrical or spherical coordinates, whichever seems more appropriate.

**32.** Find the volume of the smaller wedge cut from a sphere of radius $a$ by two planes that intersect along a diameter at an angle of $\pi/6$.




$$\int_0^a \int_0^{\pi} \int_0^{\pi/6} \rho^2 \sin\phi \; d\theta \, d\phi \, d\rho$$

**21.** $\displaystyle\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where $R$ is the trapezoidal region with vertices (1, 0), (2, 0), (0, 2), and (0, 1)




$u = x+y$

$v =$ choose from $[y,\ x-y,\ x]$

picking $v = y$

$x+y = u$
$y = u$
$v = u$

$$\int_1^2 \int_0^u \cos\left(\frac{???}{u}\right) \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| dv\, du$$

**22.** $\iint_R \sin(9x^2 + 4y^2)\, dA$, where $R$ is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$