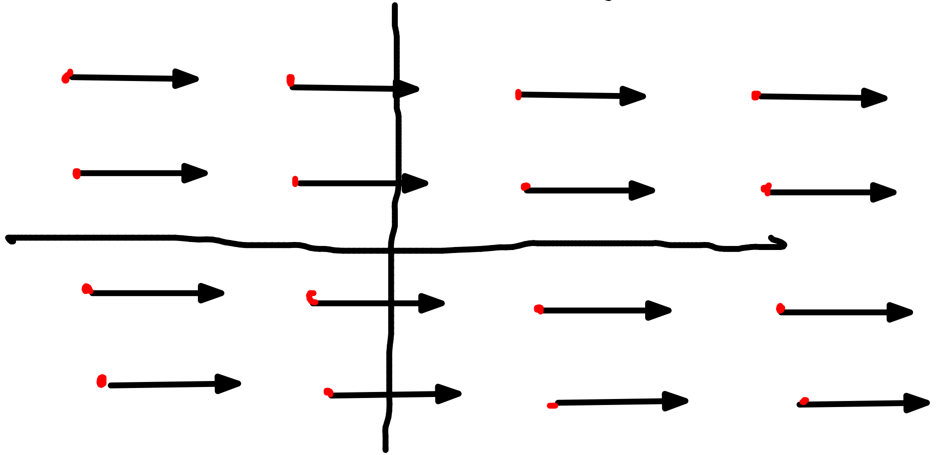
13.1 VECTOR FIELDS

functions whose range set are vector sets

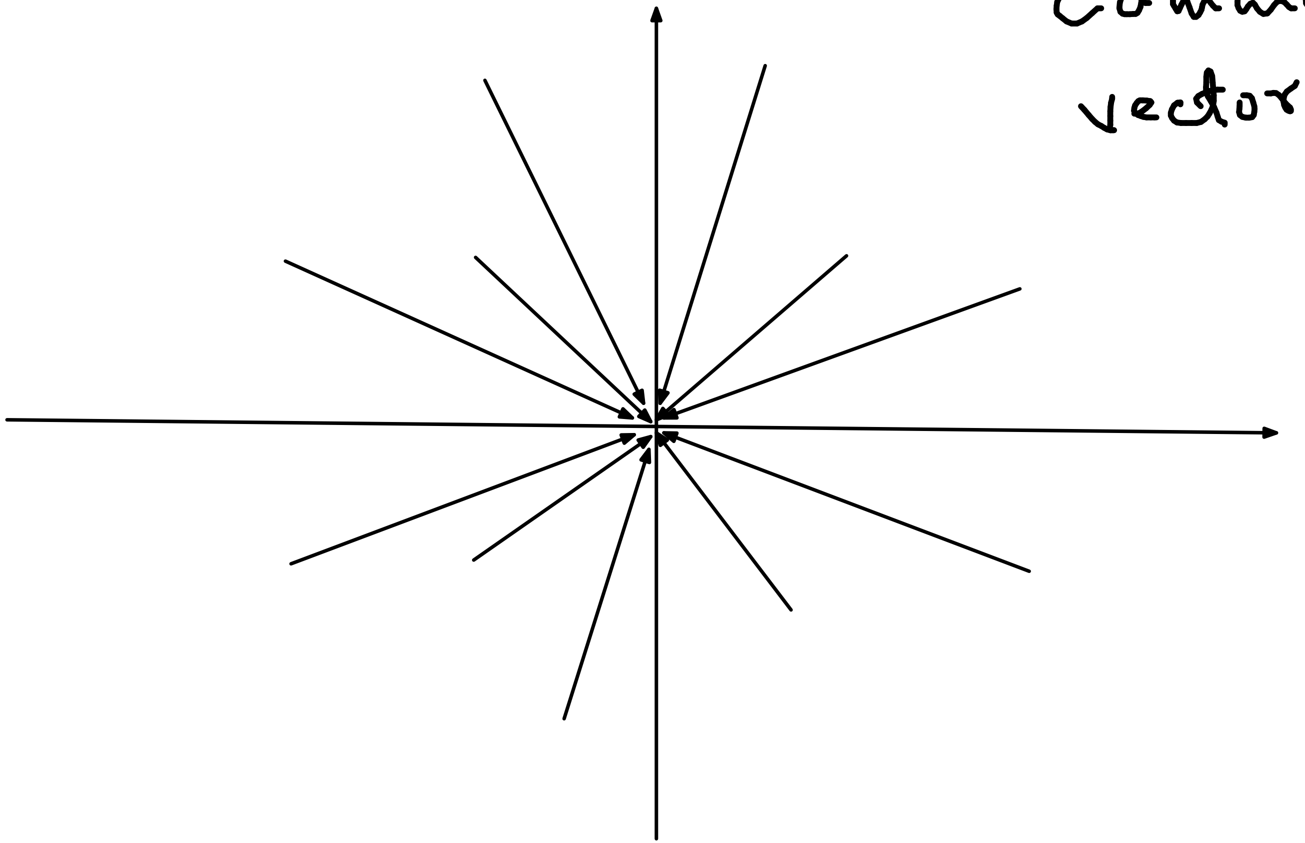
e.g. $\vec{F}(x,y) = \hat{i}$

VF $\mathbb{R}^2 \to \mathbb{R}^2$

maps a point in $\mathbb{R}^2$ to a 2 dimensional vector



$$\vec{F}(x,y) = -x\hat{i} - y\hat{j}$$



command for plotting vector fields in matlab/octave

"quiver"

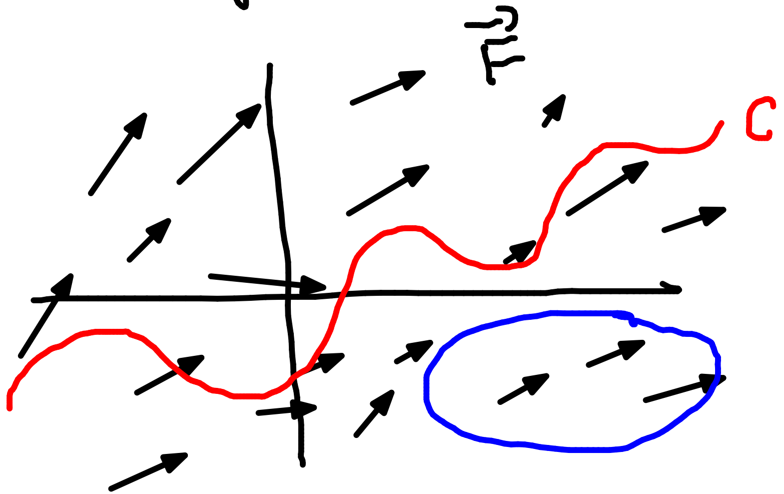
$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$$

Force field

$$\vec{v}(x,y,z) = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$$

Velocity field

Preview of the chapter



work done by $\vec{F}$ on moving a particle along the given path C

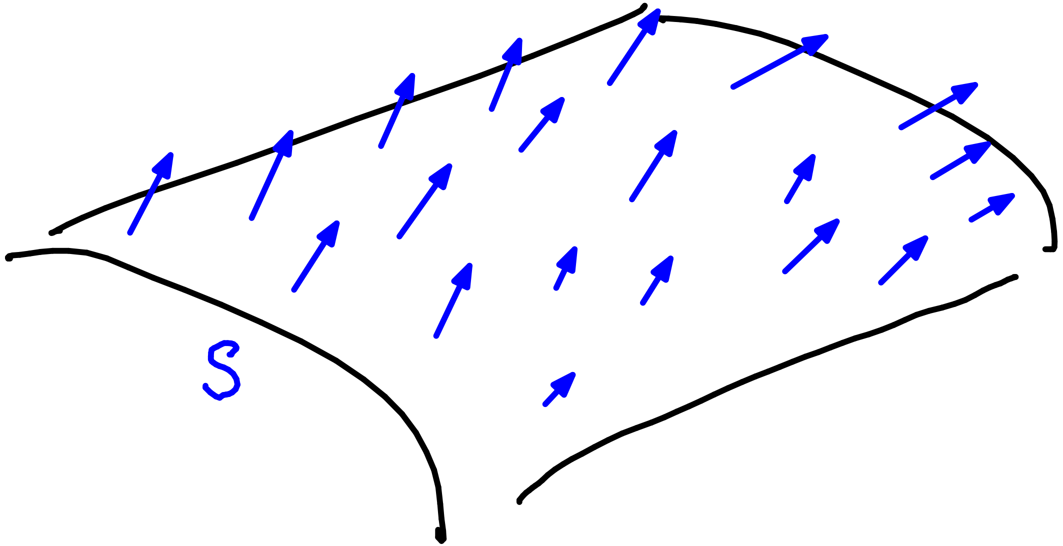
$$\int_C \vec{F}\cdot d\vec{r}$$

- Green's theorem
- Stokes theorem

simplification in $\int_C \vec{F}\cdot d\vec{r}$ if C is a closed loop

- Conservative Vector Fields

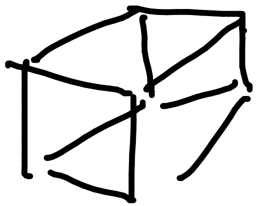
Later half of Chapter 13



$$\iint_S \vec{F} \cdot dA$$

flux of vector fields

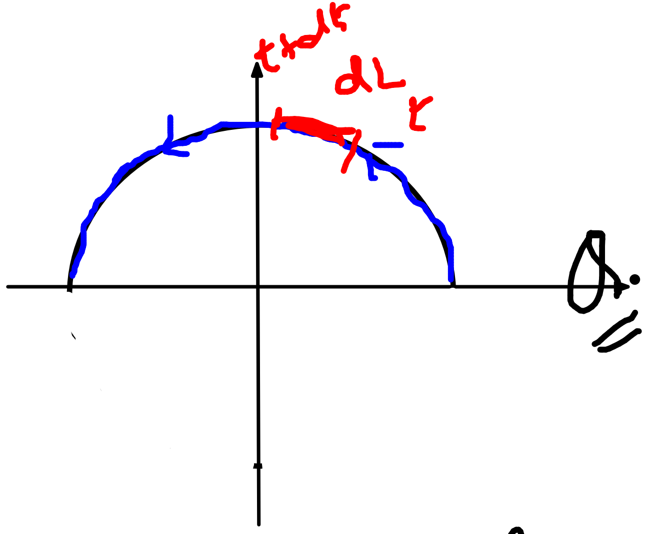
→ Divergence theorem:



13.2 LINE INTEGRALS

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

$f = 2 + x^2 y$: mass per unit length at point (x, y)



$$\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$$

$$0 \le t \le \pi$$

$$\int_C (2 + x^2 y)\, ds = \text{mass of } C$$

Can you find length of C ?!

$$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

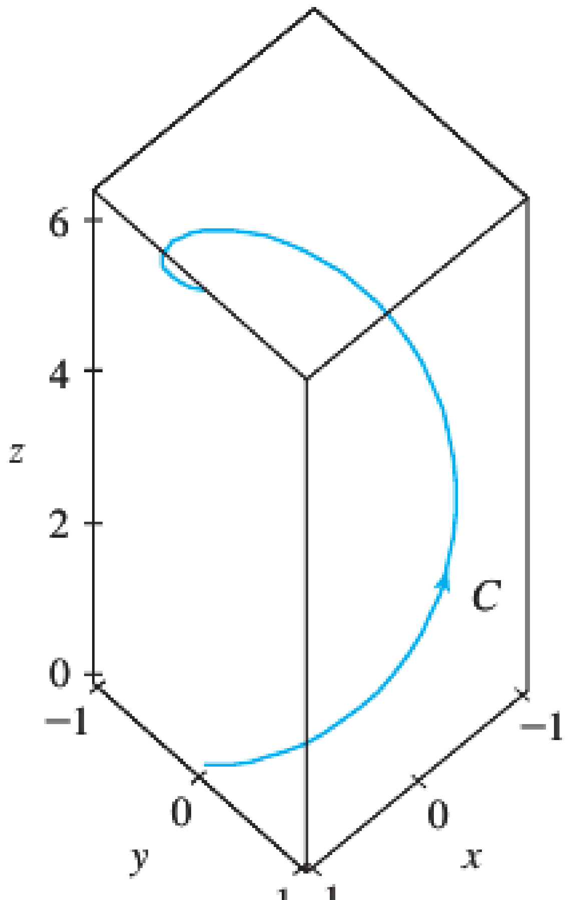
$$dm = (\text{density})\, dL$$

$$= (2+x^2y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Total mass $\quad m = \int_0^{\pi} dm = \int_0^{\pi} (2+x^2y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\,dt = \text{whatever} \checkmark$$

EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)



$$\int_C y \sin z \, ds = \int_0^{2\pi} y \sin z \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

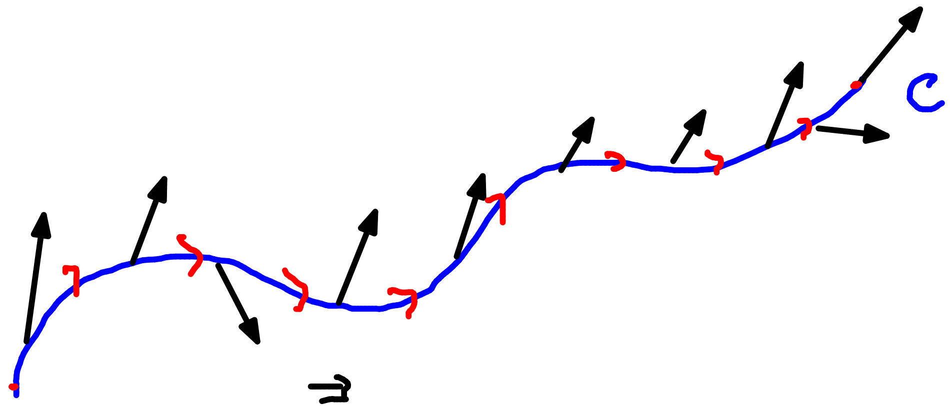
$$= \int_0^{2\pi} \sin^2(t) \sqrt{2} \, dt = \text{whatever}\checkmark$$

Aim for Today:

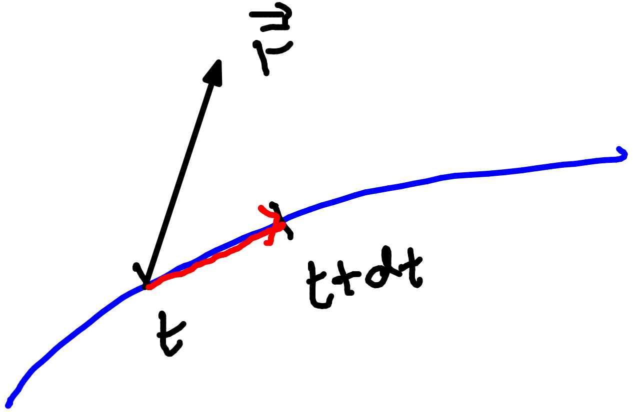
finish 13.3

LINE INTEGRALS OF VECTOR FIELDS

$\vec{F}$ integrate $\vec{F}$ along C



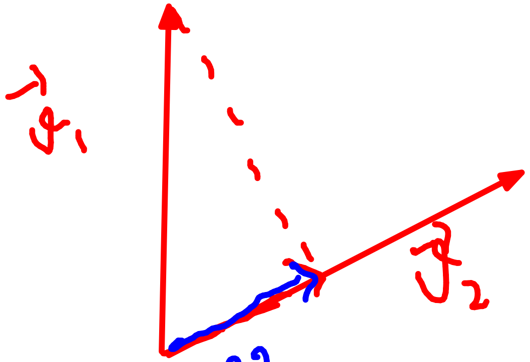
$$\boxed{\int_C \vec{F}\cdot d\vec{r}} = \text{work done by } \vec{F} \text{ in moving a particle along the curve}$$



$W = \vec{F}\cdot\vec{d} = \left(\text{component of } \vec{F} \text{ in the direction of displacement}\right) \times \left(\text{distance travelled}\right)$

$dW = \left(\text{tangential component of } \vec{F}\right) dL$

tangential Component of $\vec{F} = \vec{F} \cdot \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



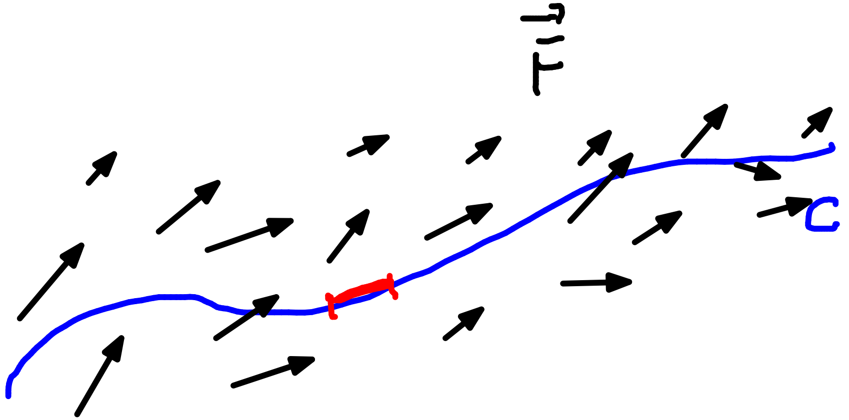
??

$= \vec{v}_1 \cdot \dfrac{\vec{v}_2}{|\vec{v}_2|}$

$$dW = \vec{F} \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \cdot \cancel{|\vec{r}\,'(t)|}\, dt$$

$$= \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$W = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$$



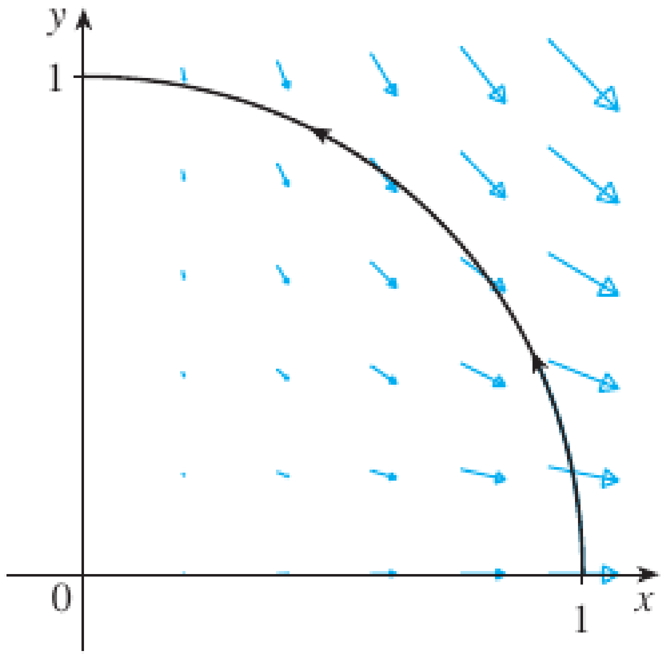
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$a \le t \le b$$

$$\vec{F}(x,y,z) = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C dW = \int_C \underbrace{\vec{F}\,\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}}_{\text{component of } F \text{ in the tangential dir}} \underbrace{\cancel{|\vec{r}\,'(t)|}\,dt}_{dL} = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.



$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi/2} \left(\cos^2 t, -\cos t \sin t\right) \cdot \left(-\sin t, \cos t\right) dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \; dt = -\frac{2}{3}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt = \frac{27}{28}$$

13.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

f: potential f^n

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

whats the theorem??

$f(x,y,z)$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\int_C \vec{F}\cdot d\vec{r}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j}$$

$$= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

for some scalar valued function $f(x, y)$

$$\vec{F} = xy\,\hat{i} + \sin(5+2x)\,\hat{j}$$

11–16 ■ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$x^4y^4/4$

11. $\mathbf{F}(x, y) = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}$, $\quad 0 \leqslant t \leqslant 1$

find $f(x,y)$

$\nabla f = \vec{F}$

i.e $\dfrac{\partial f}{\partial x} = x^3y^4$ $\qquad \underline{\dfrac{\partial f}{\partial y} = x^4y^3}$

$\Downarrow$

$f = \dfrac{x^4y^4}{4} + h(y)$

$\dfrac{\partial f}{\partial y} = x^4y^3 + h'(y) = x^4y^3$

$h'(y) = 0$

$h =$ constant

$f(x,y) = \dfrac{x^4y^4}{4}$

or

$f(x,y) = \dfrac{x^4y^4}{4} + 10$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{end point of } C) - f(\text{starting point of } C)$$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

11. $\mathbf{F}(x, y) = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}$, $\quad 0 \leqslant t \leqslant 1$

$\vec{F} = P\,\hat{i} + Q\,\hat{j} = \nabla f$

$\Rightarrow \quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$= f(1,2) - f(0,1) = 4 - 0 = 4$$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$f(x,y,z)$

13. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

$\frac{\partial f}{\partial x} = yz \;\Big|\; \frac{\partial f}{\partial y} = xz \;\Big|\; \frac{\partial f}{\partial z} = xy + 2z$

$f = xyz + h(y,z)$

$\frac{\partial f}{\partial y} = \cancel{xz} + \frac{\partial h}{\partial y} = \cancel{xz}$

$h = h(z)$

$\frac{\partial f}{\partial z} = \cancel{xy} + h'(z) = \cancel{xy} + 2z$

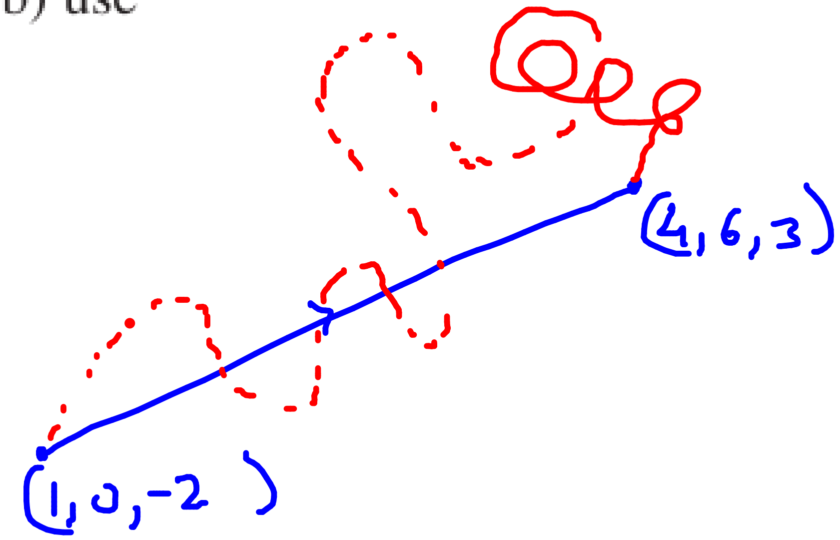
$h'(z) = 2z$

$h(z) = z^2$

$f = xyz + z^2$ will work $\Leftarrow$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

13. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$



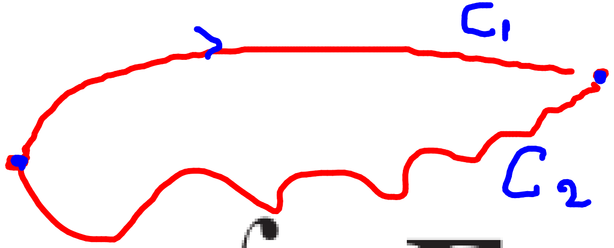
$f = xyz + z^2$

will work

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(4,6,3) - f(1,0,-2)$$

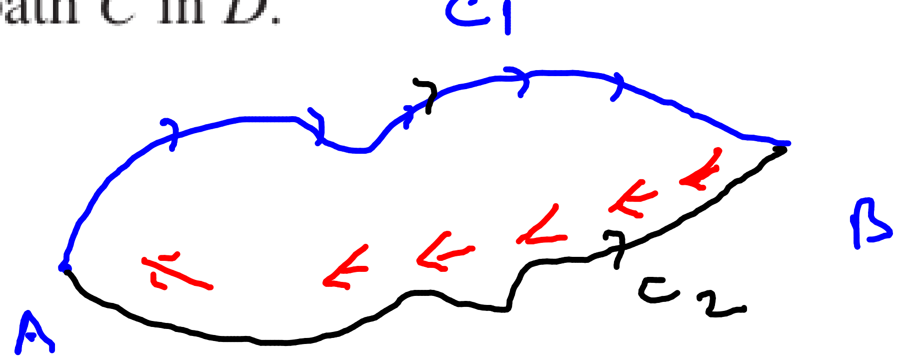
$$= 77$$

INDEPENDENCE OF PATH



$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

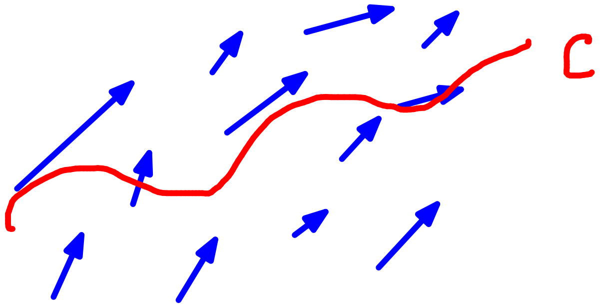
3 THEOREM $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.



$$\int_{C_1} \vec{F} \cdot d\vec{r} \overset{??}{=} \int_{C_2} \vec{F} \cdot d\vec{r}$$

Recall main points of Chapter 13 so far

$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$



$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$$

· Fundamental theorem of line integration??

if $\vec{F} = \nabla f$, for some $f(x,y,z)$

then $$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{final point in } C) - f(\text{initial point in } C)$$

Corollary: if $\vec{F} = \nabla f$ the work done by $\vec{F}$ on any closed curve will be 0.

Definition: $\vec{F}$ is conservative if work done by $\vec{F}$ in any closed loop is zero.

Q: if $\vec{F} = \nabla f$ for some $f(x,y,z)$, will $\vec{F}$ be conservative??

Yes, because $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \nabla f \cdot d\vec{r}$ = f(final point) − f(starting point)

$= 0$

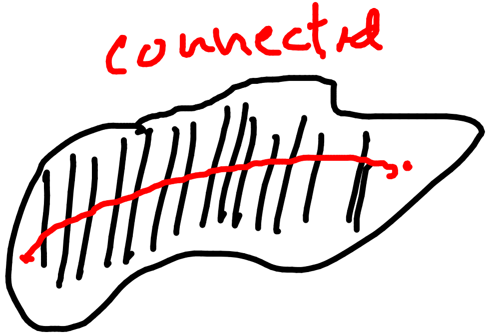
Q: if $\vec{F}$ is known to be conservative, does $\vec{F}$ have to be a gradient of some function f. ??

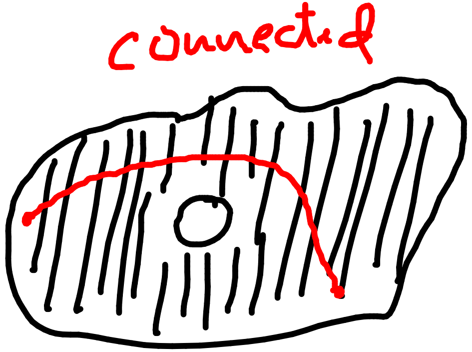
Ans: Almost always Yes

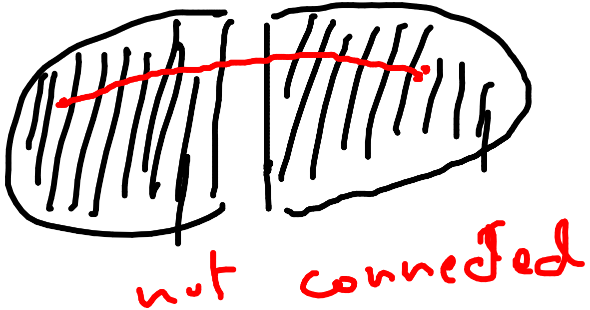
$\vec{F}$ simply needs to be

→ continuous

→ on a connected domain







given
$$\int_{C_1 \cup (-C_2)} \vec{F} \cdot d\vec{r} = 0 \qquad \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \quad \int_{C_1} \vec{F} \, d\vec{r} = \int_{C_2} \vec{F} \cdot dr$$

4 THEOREM Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

- Previously: if $\vec{F} = \nabla f$ then the work done is path independent

- Theorem is other way round:
  if the work done is independent of path then $\vec{F}$ must necessarily be ∇f for some f, provided $\vec{F}$ is continuous in the domain

5 THEOREM If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Q: if $\vec{F}$ is conservative, then why $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\boxed{P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}}$$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial^2 f}{\partial y\,\partial x} \\ \frac{\partial Q}{\partial x} &= \frac{\partial^2 f}{\partial x\,\partial y} \end{aligned}\right\} \text{ same ??}$$

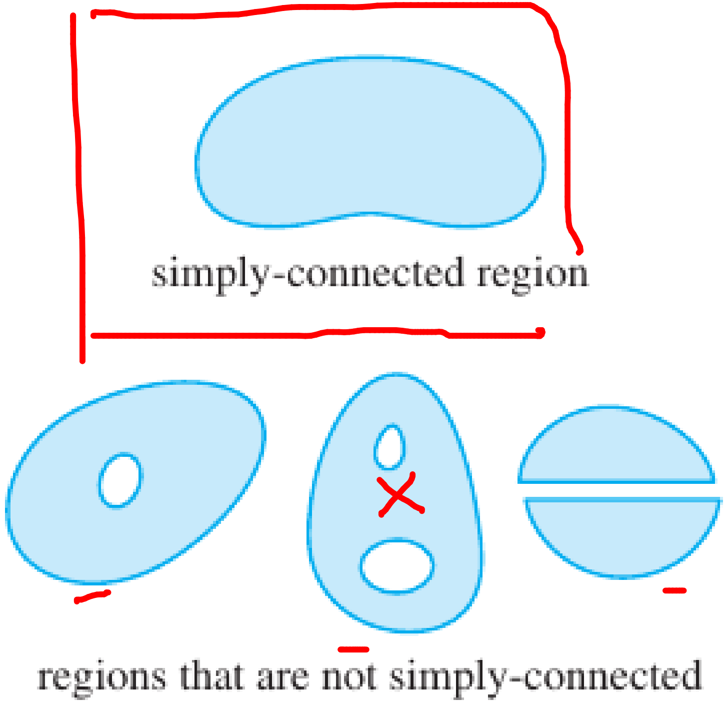
Recall Clairaut's thm

6 THEOREM Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

note: converse of previous theorem



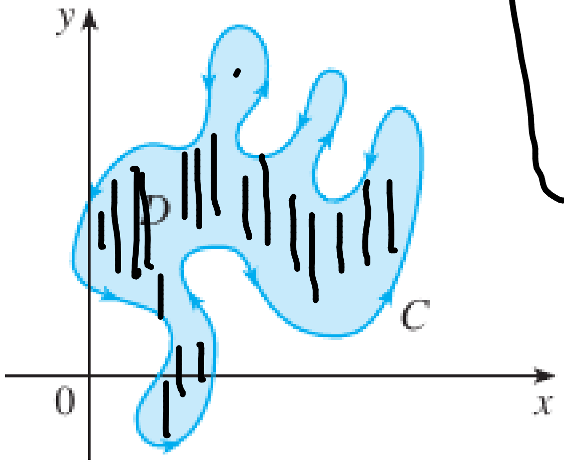
simply connected domain $\Leftrightarrow$ no hole

Green's theorem

- boundary integration gets switched to area integration

$\vec{F} = P\hat{i} + Q\hat{j}$

not necessarily conservative



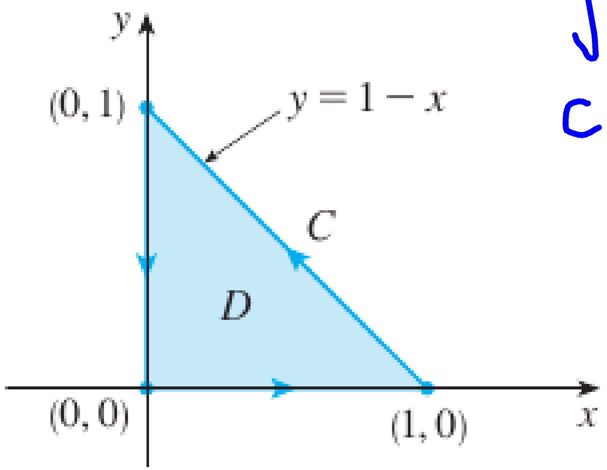
(a) Positive orientation

domain should be in the left if we are moving on the domain

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

$\vec{F} = x^4\,\hat{i} + xy\,\hat{j}$

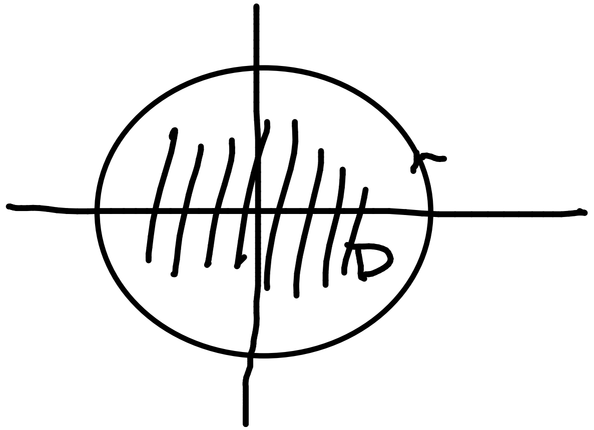


$$\int_C x^4\,dx + xy\,dy = \int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (y - 0)\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{6}$$

EXAMPLE 2 Evaluate $\oint_C \underbrace{(3y - e^{\sin x})}_{P}\, dx + (\underbrace{7x + \sqrt{y^4 + 1}}_{Q})\, dy$, where C is the circle $x^2 + y^2 = 9$.

$$\vec{F} = (3y - e^{\sin x})\,\hat{i} + (7x + \sqrt{y^4+1})\,\hat{j}$$

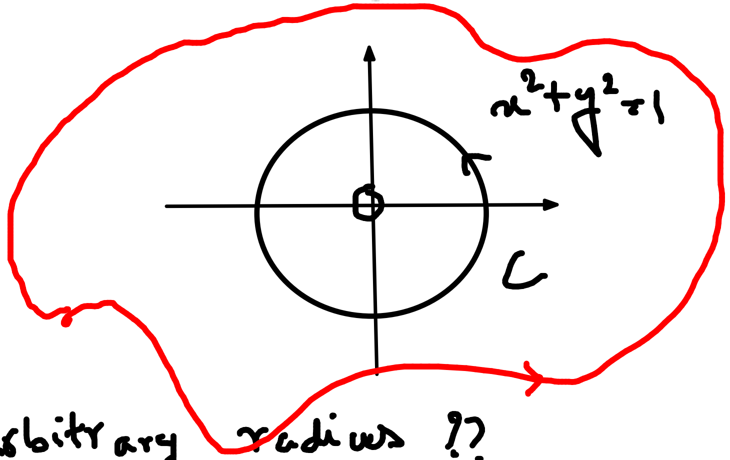


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (7-3)\, dA = 4\iint_D dA$$

$$= 4\pi 9 = 36\pi$$

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Q: is $\vec{F}$ conservative ??



Q: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$

Q: same as above but C is of arbitrary radius ??

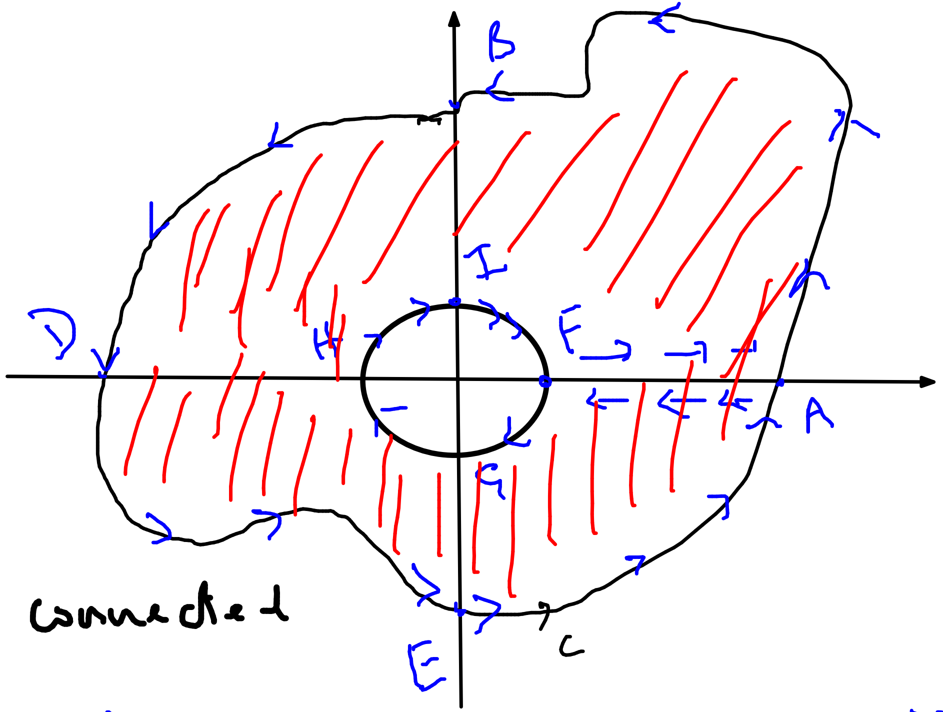
$\vec{r}(t) = (a\cos t,\ a\sin t)$

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$$P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$

provided the area bounded by C is simply connected (no hole in the domain)



Region enclosed by $\tilde{C}$ is simply connected

$$\oint_{\tilde{C}} \vec{F} \cdot d\vec{r} = 0$$

$$\underbrace{E \to A \to F \to G \to H \to I \to F \to A \to B \to D \to E}_{\tilde{C}}$$

$$\oint_{\tilde{C}} \vec{F}\cdot d\vec{r} = \oint_{C} \vec{F}\cdot d\vec{r} + \underbrace{\oint_{C} \vec{F}\cdot d\vec{r}}_{-2\pi} + \underbrace{\int_{AFA} \vec{F}\cdot d\vec{r}}_{=0} = 0$$

$$\oint_{C} \vec{F}\cdot d\vec{r} = 2\pi$$

Proof of fundamental theorem of Line Integrals.

1–4 ▪ Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C xy^2\,dx + x^3\,dy$,
C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3)

13–16 ▪ Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

13. $\mathbf{F}(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$,
C consists of the arc of the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

Show that $\mathbf{F}$ is conservative and use this fact to evalu ate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

$$\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j},$$
$$C\colon \mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j},\ \ 0 \leqslant t \leqslant 1$$

Show that $\mathbf{F}$ is conservative and use this fact to evalu ate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

$\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

13–14 ▪ Show that $\mathbf{F}$ is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.