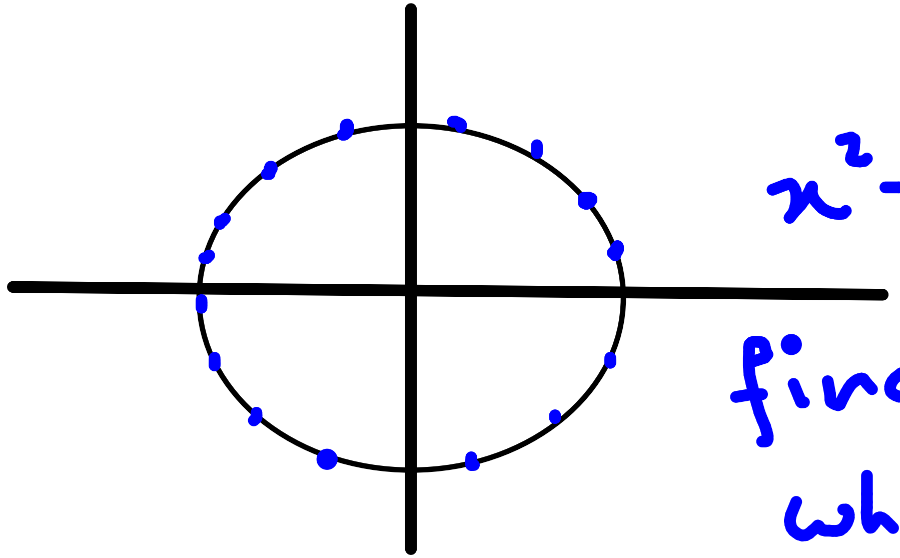
Ex: we are interested in maximizing a function $\boxed{f(x,y) = xy^2}$

over a level curve

$$x^2+y^2 = 1$$



$x^2+y^2=1$

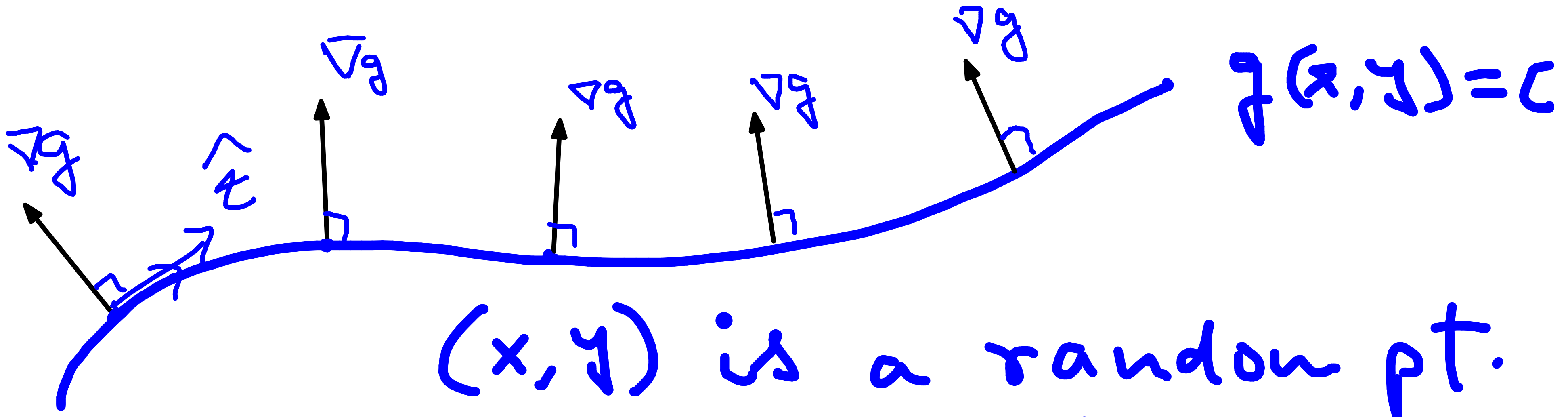
find (x,y) on the circle where $f(x,y) = xy^2$ takes the highest value

Lets look at a bigger picture

$$\boxed{\begin{array}{ll} \text{maximize} & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{array}}$$

In general $g(x,y) = c$ will represent some kind of curve.

$\boxed{\nabla g \cdot \hat{t} = 0}$ [using the info that $g = c$]

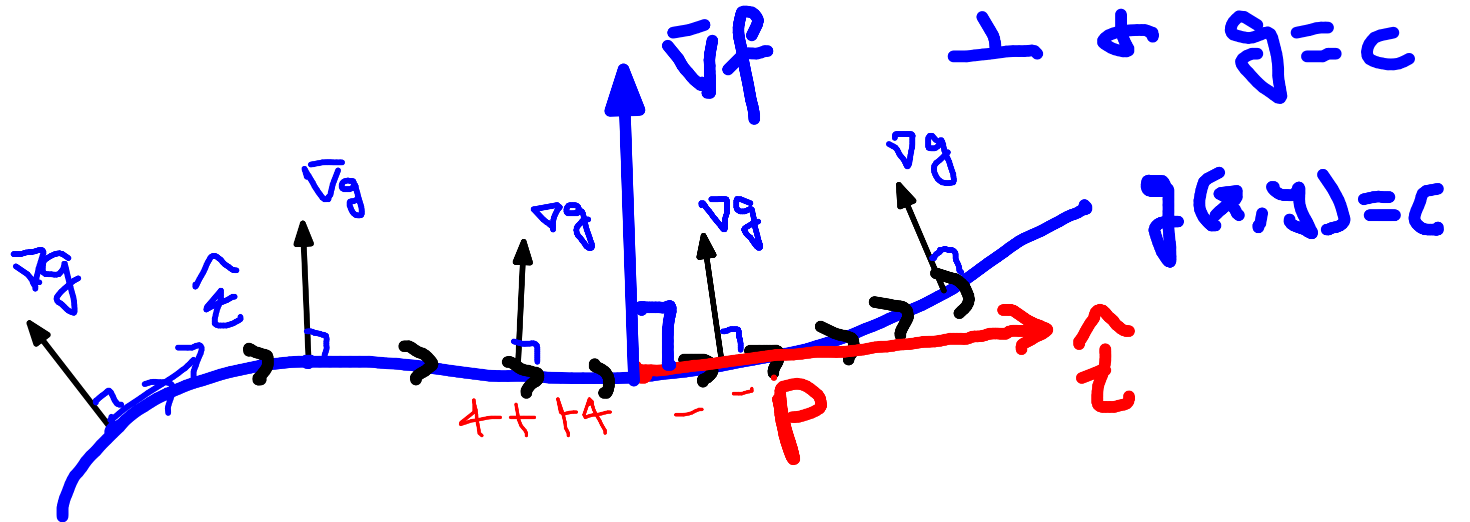


(x,y) is a random pt. on the curve

$$\nabla g(x,y) = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}$$

maximize $f(x,y)$

s.t. $g(x,y) = c$



note: ∇g is always $\perp$ to the curve $g = c$

let $f(x,y)$ has the higher value at point P on the curve.

Q: how will the directional derivative of f in the tangential direction change sign before & after P?

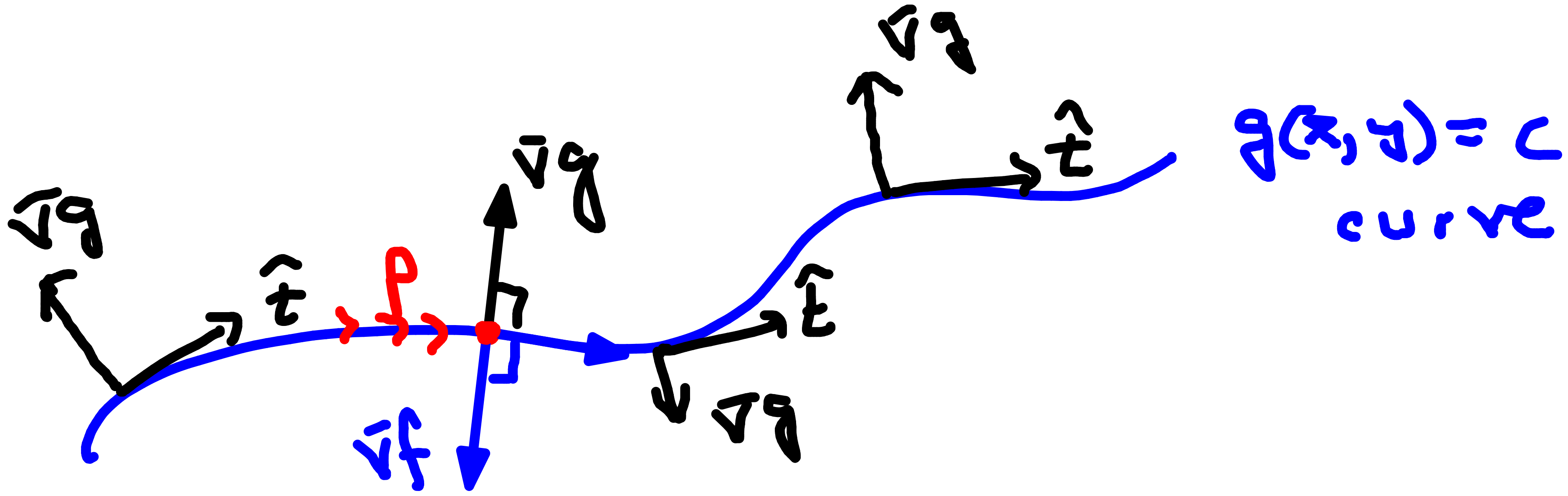
$$\nabla f \cdot \hat{t} \rightarrow + \text{ to } -\text{ve}$$

Q: if P is the max point, $\nabla f \cdot \hat{t}$ at $P = 0$

i.e. at the max point P, ∇f is also perpendicular to the curve maximize $f(x,y)$

s.t. $g(x,y) = c$

→ $\hat{t}$: a generic tangent vector on the curve



→ $\nabla g \cdot \hat{t} = 0$ at all point on the curve $g=c$

→ in general $\nabla f \cdot \hat{t} \neq 0$

→ if at the max point P, $\nabla f \cdot \hat{t} = 0$

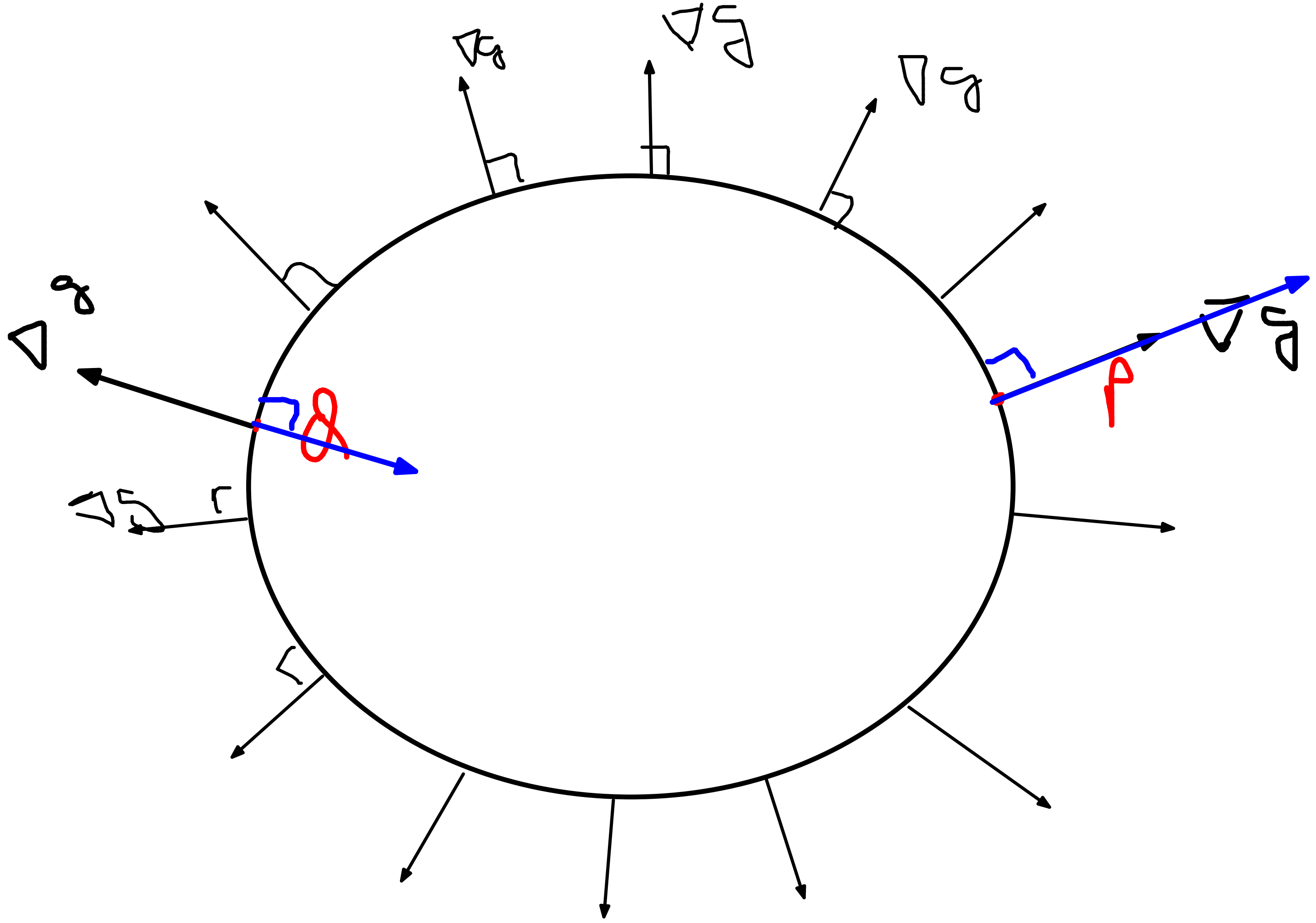
→ This leads to the defining eqn for the point P

at P: we must have $\nabla f = \lambda \nabla g$

(α: a scalar typically called Lagrange multiplier)

Q. maximize $f(x,y) = xy^2$

s.t. $\underbrace{x^2+y^2}_{g} = 1$



if P & Q are extremum

$\nabla f = \lambda g$ at P & Q.

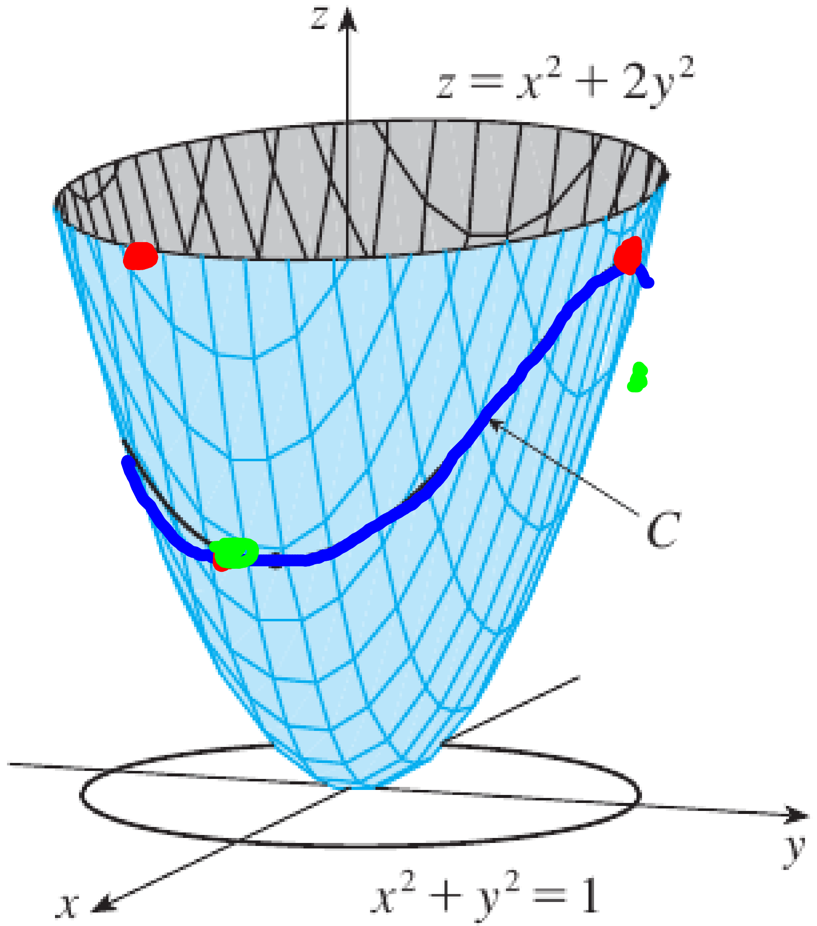
$\rightarrow y^2\hat{i} + 2xy\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$

Eq^ns for extreme points

$$\left.\begin{aligned} y^2 &= 2\lambda x \\ 2xy &= 2\lambda y \\ x^2+y^2 &= 1 \end{aligned}\right]$$

solve this to get (x,y) as your max/mi points

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $\underbrace{x^2 + y^2}_{g} = 1$.



According to Lagrange Multipliers we know that at max/min points

$$\boxed{\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= c \end{aligned}}$$

$$2x\hat{i} + 4y\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$$

$$x^2 + y^2 = 1$$

$$2x = \lambda 2x$$

$$4y = \lambda 2y$$

$$x^2 + y^2 = 1$$

$$2x = \alpha 2x$$

$$x = \alpha x$$

$$x - \alpha x = 0$$

$$x(1-\alpha) = 0$$

$$\Downarrow$$

$$x = 0 \quad \text{or} \quad \alpha = 1$$

$$\left.\begin{array}{l} x = 0 \\ x^2+y^2 = 1 \end{array}\right\} \quad y^2 = 1$$

$$y = \pm 1$$

$$\Downarrow$$

$$4y = \alpha 2y$$

$$4(\pm 1) = \alpha 2(\pm 1)$$

$$4 = 2\alpha$$

$$\alpha = 2$$

$$\boxed{x = 0 \ , \ y = \pm 1 \ , \ \alpha = 2}$$

$$\left.\begin{array}{l} \alpha = 1 \\ 4y = \alpha 2y \end{array}\right\} \quad 4y = 2y \Rightarrow y = 0$$

$$x^2 + y^2 = 1$$

$$\Downarrow$$

$$x = \pm 1$$

$$\boxed{x = \pm 1, \quad y = 0, \quad \lambda = 1}$$

(x, y)	λ	f
$(0, 1)$	2	2
$(0, -1)$	2	2
$(1, 0)$	1	1
$(-1, 0)$	1	1

max points

min points

$$f(x, y) = x^2y; \quad x^2 + 2y^2 = 6$$