

2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$

undetermined
coefficients

[→ easier
→ range of
r(x) is small]

variation of
parameters

[→ slightly tedious
→ more general]

2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$



↳ step ① solve a corresponding homogeneous eqⁿ

$$y'' + py' + qy = 0$$

→ let solution y_h

↳ step ② find a particular solution

$$y'' + py' + qy = r$$

y_p

either by
→ undetermined coefficients
→ variation of parameter

C) final step : $y = y_h + y_p$ [general solution of E_r (*)]

→ Recall the ODE :

$$y'' + p y' + q y = r$$

→ $y_h'' + p y_h' + q y_h = 0$

→ $y_p'' + p y_p' + q y_p = r$

$$(y_h + y_p)'' + p (y_h + y_p)' + q (y_h + y_p) = r$$

End Sem :

May 7

→ mode : similar to mid term

→ syllabus : everything covered
this semester

So far we have been solving homogeneous 2nd order ODE

$$y'' + ay' + by = 0$$

now we will solve non-homogeneous 2nd order ODE

$$y'' + ay' + by = \delta(x) \quad \text{--- (X)}$$

Step 1 find a general solution of the corresponding homogeneous solⁿ:

y_h :

$$y_h'' + ay_h' + by_h = 0$$

Step 2 find a particular solⁿ : y_p which solves Eq (X)

↳ we will see how

Q. Does $y = y_h + y_p$ solve $E_2(*)$

Yes

$y = y_h + y_p$ is called general solution of $E_2(*)$

how to
find y_p

method of
undetermined
coefficients

Today

Variations of
parameters
Sec 2.10

Method of Undetermined Coefficients

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n ($n = 0, 1, \dots$)	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos \omega x$	$\left\{ K \cos \omega x + M \sin \omega x \right.$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left\{ e^{\alpha x} (K \cos \omega x + M \sin \omega x) \right.$
$ke^{\alpha x} \sin \omega x$	

Q. Solve: $y'' + y = 1$

→ Solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p using undetermined coefficients:

$$r(x) = 1$$

∴ our $r(x)$ is a constant

The table is suggesting that y_p will also be a constant

$$y_p = C$$

[C needs to be found]

∴ we substitute $y_p = C$ in the given ODE & solve for C

$$(C)' + (C) = 1$$

$$C = 1$$

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$ke^{\alpha x} \cos \omega x$	$\left\{ e^{\alpha x} (K \cos \omega x + M \sin \omega x) \right.$
$ke^{\alpha x} \sin \omega x$	

general solⁿ

$$y = C_1 \cos x + C_2 \sin x + 1$$

Q. Solve: $y'' + y = 5$

→ Solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p :

$$y_p = C$$

$$y_p = 5$$

will work

→

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
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$ke^{\alpha x} \sin \omega x$	

Q. Solve: $y'' + y = e^{-2x}$

→ solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p using undetermined coefficients

$$r(x) = e^{-2x}$$

Try $y_p = C e^{-2x}$, need to find C

→ plug in $y_p = C e^{-2x}$ in $y'' + y = e^{-2x}$

$$4C e^{-2x} + C e^{-2x} = e^{-2x} \Rightarrow y_p = \frac{1}{5} e^{-2x}$$

$$5C e^{-2x} = e^{-2x}$$

$$5C = 1$$

$$C = 1/5$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$k e^{\gamma x}$	$C e^{\gamma x}$
$k x^n$ ($n = 0, 1, \dots$)	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos \omega x$	$\left\{ K \cos \omega x + M \sin \omega x \right.$
$k \sin \omega x$	
$k e^{\alpha x} \cos \omega x$	$\left\{ e^{\alpha x} (K \cos \omega x + M \sin \omega x) \right.$
$k e^{\alpha x} \sin \omega x$	

general solution

$$y = C_1 \cos x + C_2 \sin x + \frac{e^{-2x}}{5}$$

EXAMPLE 1

Solve the initial value problem

$$(5) \quad y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5.$$

$$y_h = C_1 \cos x + C_2 \sin x$$

What should be y_p ?

$$y_p = K_0 + K_1 x + K_2 x^2$$

We need to find K_0, K_1, K_2 by

substituting y_p in $y'' + y = 0.001x^2$

$$K_2 x^2 + K_1 x + (K_0 + 2K_2) = 0.001x^2$$

→ matching the x, x^2 , & constants in LHS & RHS

$$K_2 = 0.001$$

$$K_1 = 0$$

$$K_0 = -0.002$$

Table 2.1 Method of Undetermined Coefficients

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$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
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γ	y_p
1	K_0
x	$K_0 + K_1 x$
x^2	$K_0 + K_1 x + K_2 x^2$
x^3	$K_0 + K_1 x + K_2 x^2 + K_3 x^3$

general solⁿ:

$$y = C_1 \cos x + C_2 \sin x - 0.002 + 0.001x^2$$

remaining work:

find c_1 & c_2 using

$$y(0) = 0$$

$$y'(0) = 1.5$$

↓ work

$$\rightarrow c_1 = 0.002, \quad c_2 = 1.5$$

EXAMPLE 2

Solve the initial value problem

$$(6) \quad y'' + 3y' + 2.25y = -10e^{-1.5x}, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\rightarrow y_h = ??$$

$$\lambda^2 + 3\lambda + 2.25 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{9 - 9}}{2} = -3/2$$

$$y_h = C_1 e^{-3/2 x} + C_2 x e^{-3/2 x}$$

$$\rightarrow y_p = C e^{-1.5x} \quad \text{should have worked but it's not.}$$

$$\text{trick: try } y_p = C x e^{-1.5x} \quad \text{X}$$

$$\text{try } y_p = C x^2 e^{-1.5x}$$

Table 2.1 Method of Undetermined Coefficients

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$$2.25 \left(y_p = C x^2 e^{-1.5x} \right)$$

$$+ 3 \left(y_p' = C 2x e^{-1.5x} - \frac{3C}{2} x^2 e^{-1.5x} \right)$$

$$+ \left(y_p'' = C 2 e^{-1.5x} + 2(2x) \left(-\frac{3}{2} \right) e^{-1.5x} + C x^2 \frac{9}{4} e^{-1.5x} \right)$$

$$-10 e^{-1.5x} = 2C e^{-1.5x}$$

$$C = -5$$

$$\Rightarrow \text{general sol}^n : y = C_1 e^{-1.5x} + C_2 x e^{-1.5x} - 5 x^2 e^{-1.5x}$$

then find C_1 & C_2 using $y(0) = 1$, $y'(0) = 0$

EXAMPLE 3

Solve the initial value problem

$$(7) \quad \boxed{y'' + 2y' + 0.75y = 2 \cos x - 0.25 \sin x + 0.09x}, \quad y(0) = 2.78, \quad y'(0) = -0.43.$$

→ Step ① $y'' + 2y' + 0.75y = 0$

$$\lambda^2 + 2\lambda + 0.75 = 0$$

$$\lambda = -0.5, -1.5$$

$$y_h = C_1 e^{-0.5x} + C_2 e^{-1.5x}$$

→ Step ② Find y_p

$$r(x) = 2 \cos x - 0.25 \sin x + 0.09x$$

$$\boxed{r_1(x) = 2 \cos x - 0.25 \sin x} \quad y_p \mid y_p'' + 2y_p' + 0.75y_p = 2 \cos x - 0.25 \sin x$$

$$r_2(x) = 0.09x \quad z_p \mid z_p'' + 2z_p' + 0.75z_p = 0.09x$$

Table 2.1 Method of Undetermined Coefficients

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EXAMPLE 3

Solve the initial value problem

(7) $y'' + 2y' + 0.75y = 2 \cos x - 0.25 \sin x + 0.09x, \quad y(0) = 2.78, \quad y'(0) =$

2a) find y_p : with $r(x) = 2 \cos x - 0.25 \sin x$

Try: $y_p = K \cos x + M \sin x$
 $2(y_p' = -K \sin x + M \cos x)$
 $y_p'' = -K \cos x - M \sin x$

$(2M - 0.25K) \cos x + (-0.25M - 2K) \sin x = 2 \cos x - 0.25 \sin x$

$2M - 0.25K = 2$
 $-0.25M - 2K = -0.25$

Partial y_p : $\sin x$

Table 2.1 Method of Undetermined Coefficients

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$$5 \cdot K = 2, -0.25 \cdot M - 2 \cdot K = -0.25$$

CALCULATE IT!

Solve

Lesson

Solve by Substitution



Let's solve your system by substitution.

$$2m - 0.25k = 2; -0.25m - 2k = -0.25$$

Show Step-By-Step

Answer:

$$k = 0 \text{ and } m = 1$$

Solve the initial value problem

(7) $y'' + 2y' + 0.75y = 2 \cos x - 0.25 \sin x + 0.09x$, $y(0) = 2.78$, $y'(0) = -0.43$.

find y_p : $y_p'' + 2y_p' + 0.75y_p = 0.09x$

Try $y_p = k_1x + k_0$

find k_1, k_2 by substituting y_p in Δ
comparing constants & x-coeffs
in LHS & RHS

$$k_1 = 0.12$$

$$k_0 = -0.32$$

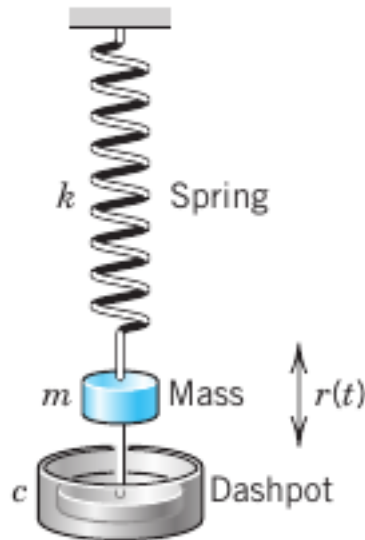
$$y_p = -0.32 + 0.12x$$

Find general solution

$$y = C_1 e^{-0.5x} + C_2 e^{-1.5x} + \sin x - 0.32 + 0.12x$$

→ find C_1, C_2 using initial conditions.

2.8 Modeling: Forced Oscillations. Resonance



$$my'' + cy' + ky = F_0 \cos \omega t.$$

<https://www.youtube.com/watch?v=XwIzBJIp1AA>

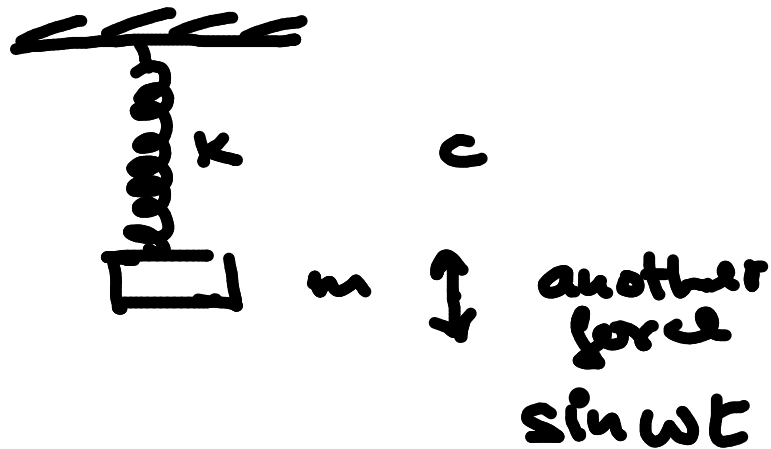
Resonance:

$$m\ddot{y} + c\dot{y} + ky = 0$$

→ assume $c=0$, $m=1$, $k=1$

$$\ddot{y} + y = \sin \omega t$$

↖ assume a new force
apart from gravity
& spring



→ $\sin \omega t$:
external force
→ frequency determined
by ω

Q. How will the general solution look like ??

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 \cos t + C_2 \sin t + y_p \end{aligned}$$

y_h : oscillatory solⁿ
frequency of y_h is
called internal frequency

→ if $\omega \neq 1$ [say $\omega = 2$]
 find y_p : $K \cos 2t + M \sin 2t$
 can you find K & M : $y'' + y = \sin 2t$
 $K=0, M=-1/3$
 $y_p = -\frac{1}{3} \sin 2t$
 general solⁿ
 $y = C_1 \cos t + C_2 \sin t - \frac{1}{3} \sin 2t$

not resonance

Resonance case: $[\omega=1]$ (internal frequency = frequency of external force)

find y_p : in $y'' + y = \sin t$
 Try $y_p = K \cos t + M \sin t$

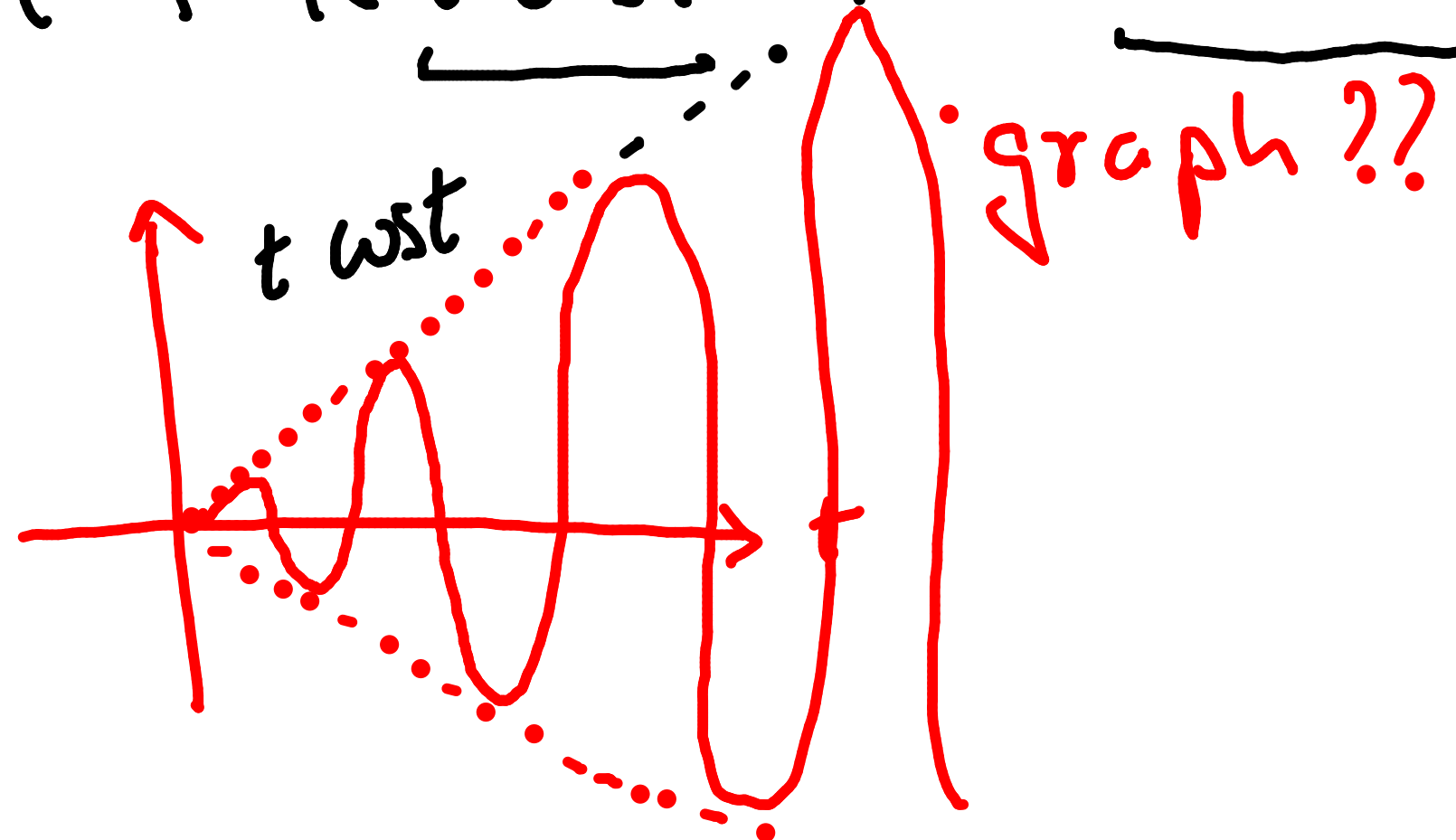
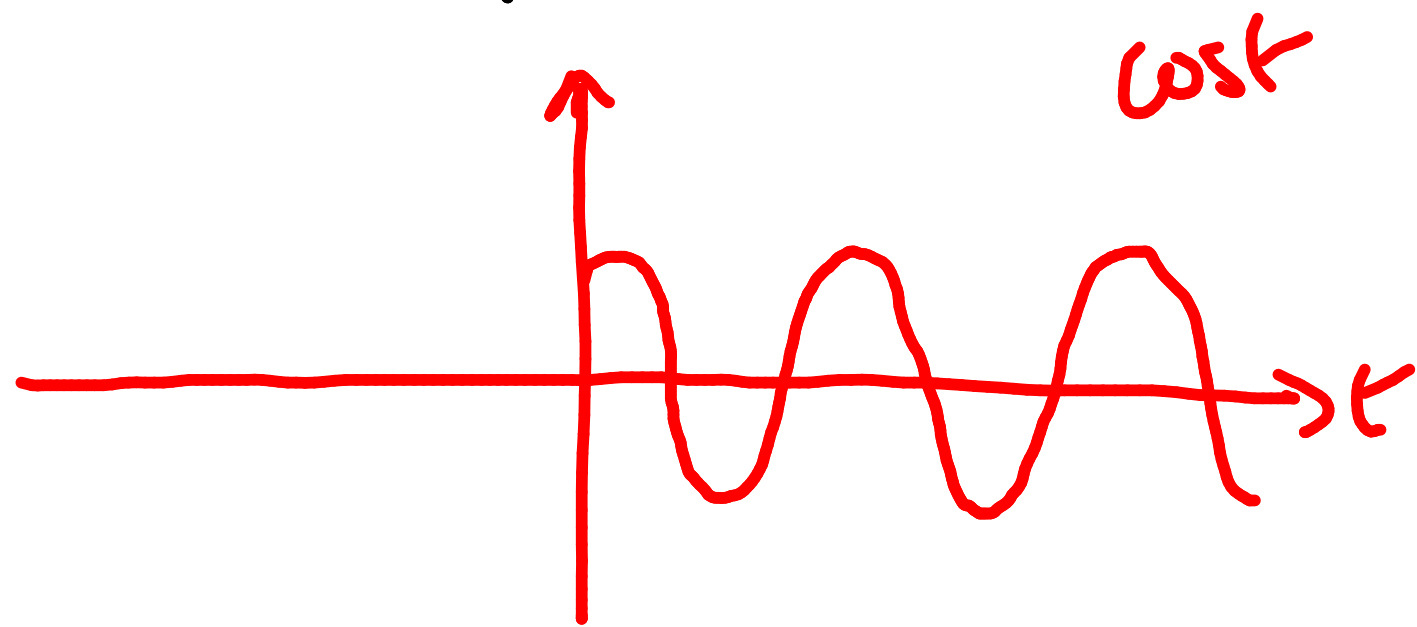
won't work

Try $y_p = tK \cos t + tM \sin t$

Let's assume $y_p = k t \cos t + m t \sin t$ works

general solⁿ:

$$y = C_1 \cos t + C_2 \sin t + \underbrace{k t \cos t}_{t \cos t} + \underbrace{m t \sin t}_{\text{graph??}}$$



2.9 Modeling: Electric Circuits

next
time

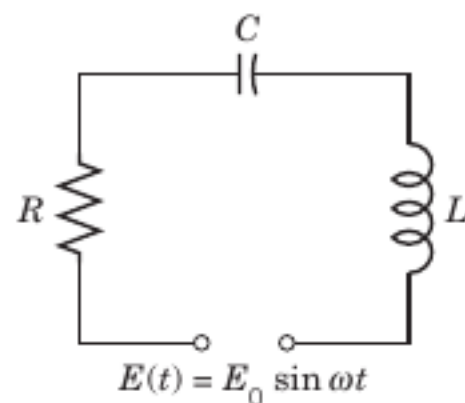


Fig. 61. RLC-circuit

Name	Symbol	Notation	Unit	Voltage Drop
Ohm's Resistor		R Ohm's Resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L \frac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

RLC-Circuit

Find the current $I(t)$ in an RLC -circuit with $R = 11 \, \Omega$ (ohms), $L = 0.1 \, \text{H}$ (henry), $C = 10^{-2} \, \text{F}$ (farad), which is connected to a source of EMF $E(t) = 110 \sin(60 \cdot 2\pi t) = 110 \sin 377 t$ (hence $60 \, \text{Hz} = 60 \, \text{cycles/sec}$, the usual in the U.S. and Canada; in Europe it would be $220 \, \text{V}$ and $50 \, \text{Hz}$). Assume that current and capacitor charge are 0 when $t = 0$.

