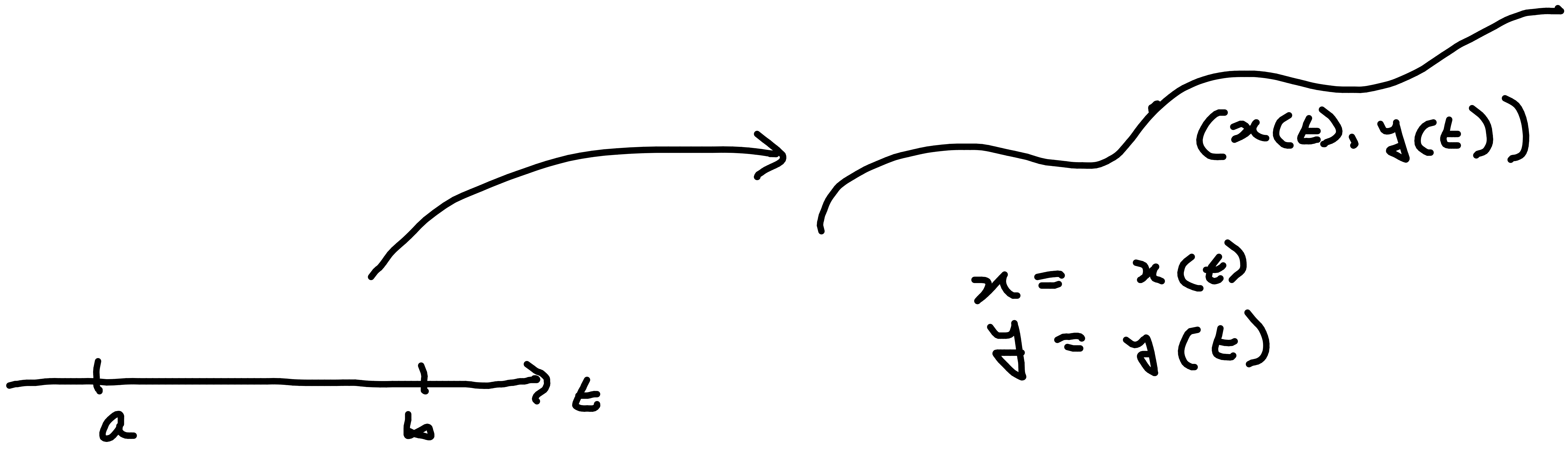
Recall parametric curves:



$$x = x(t)$$
$$y = y(t)$$

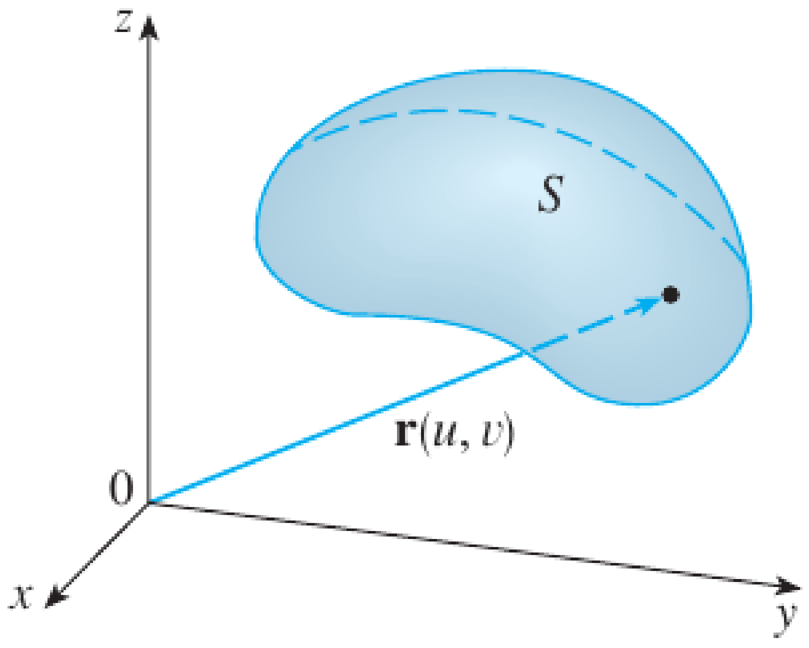
Parametric Surfaces

$$x = x(u,v)$$

$$y = y(u,v)$$

$$z = z(u,v)$$

13.6 PARAMETRIC SURFACES AND THEIR AREAS

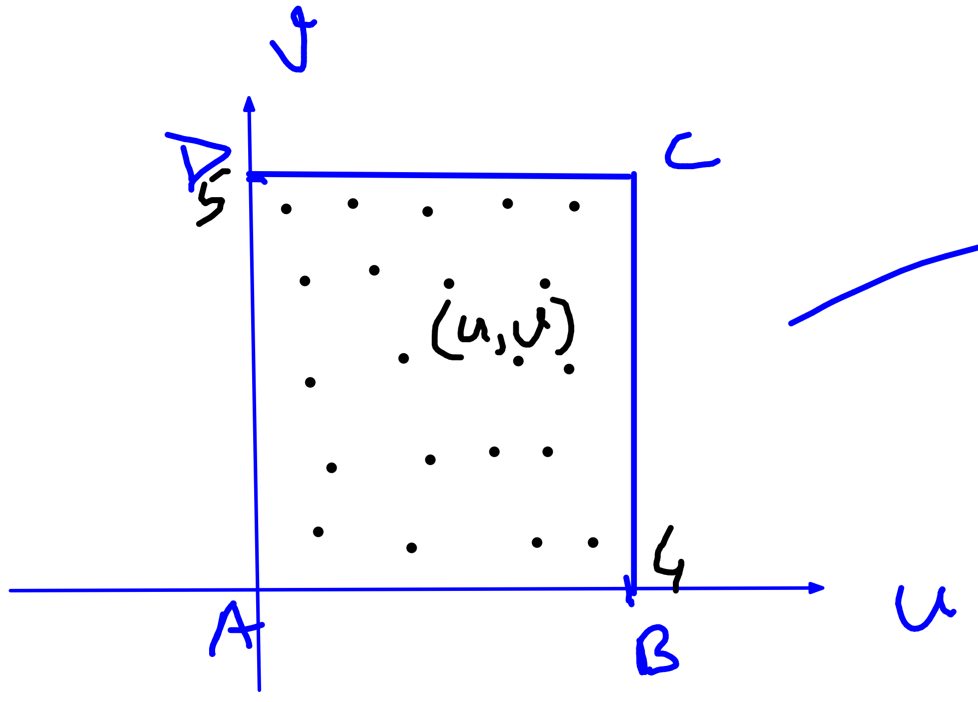


$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

Q.

$0 \le u \le 4$

$0 \le v \le 5$

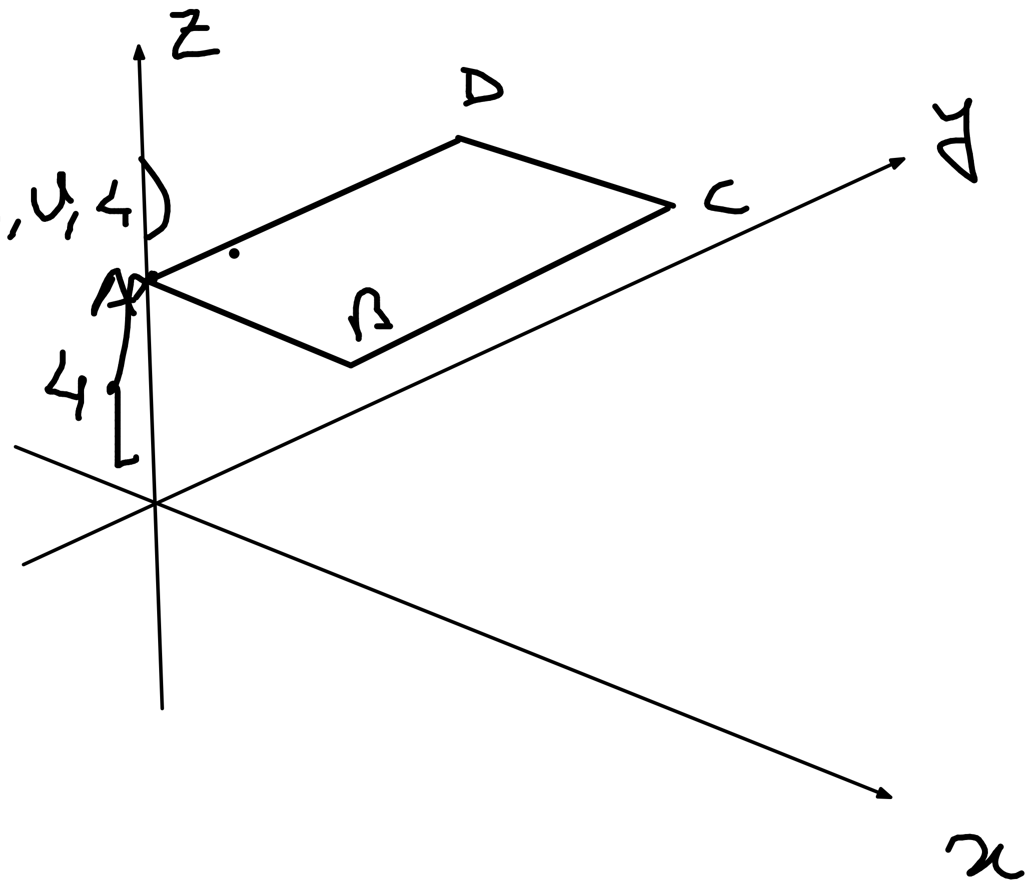


$x = u$

$y = v$

$z = 4$

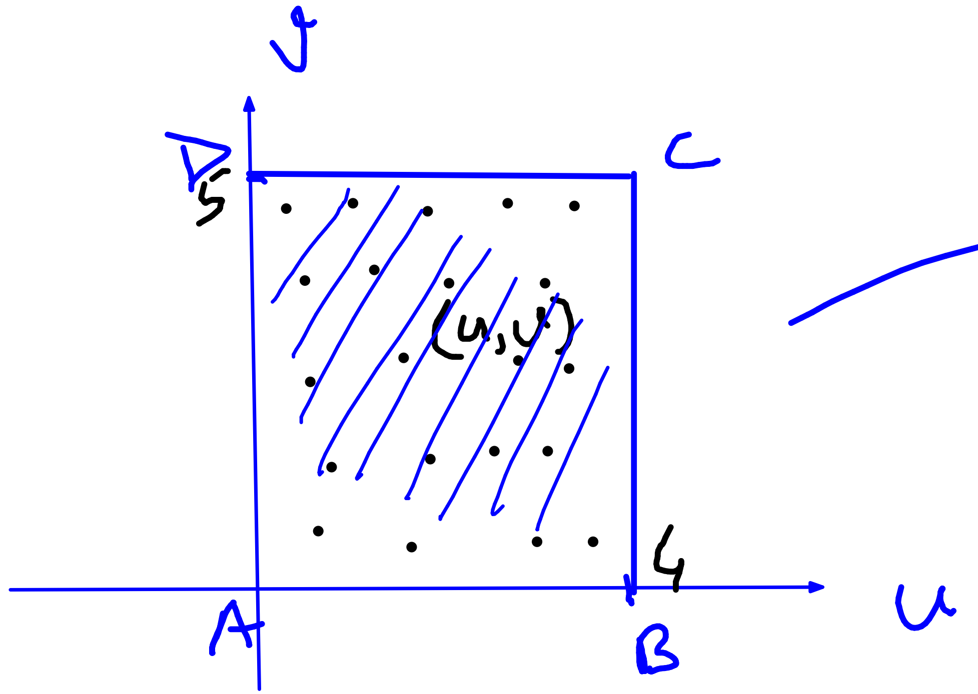
$(x,y,z) = (u,v,4)$



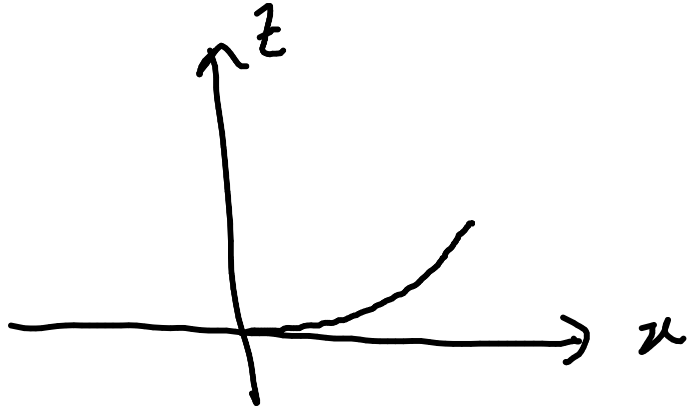
Q.

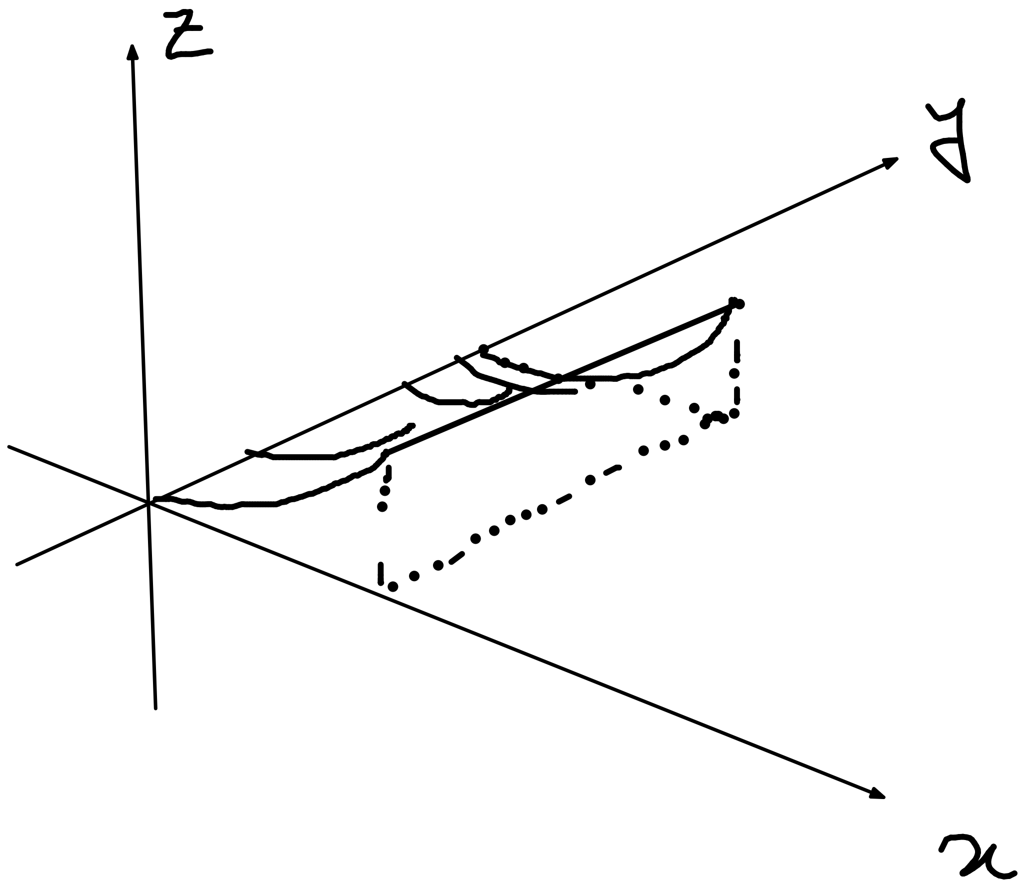
$0 \le u \le 4$

$0 \le v \le 5$



$x = u$
$y = v$
$z = u^2$

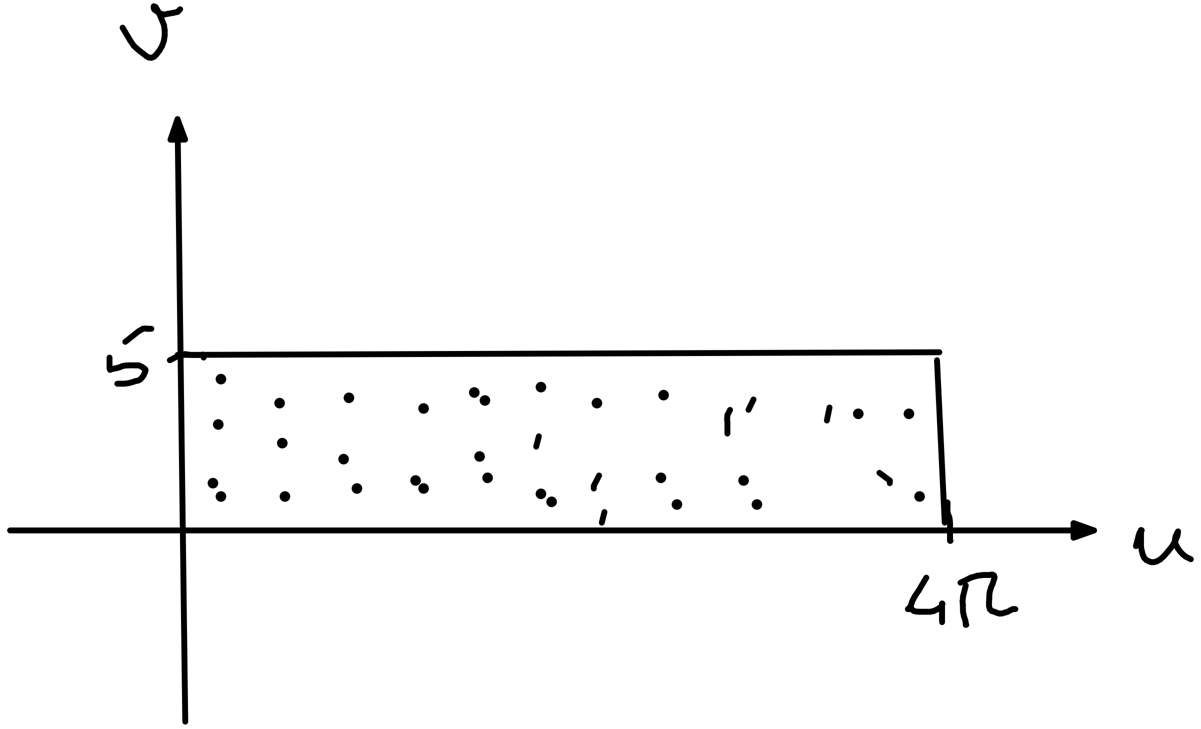




8.

$0 \le u \le 4\pi$

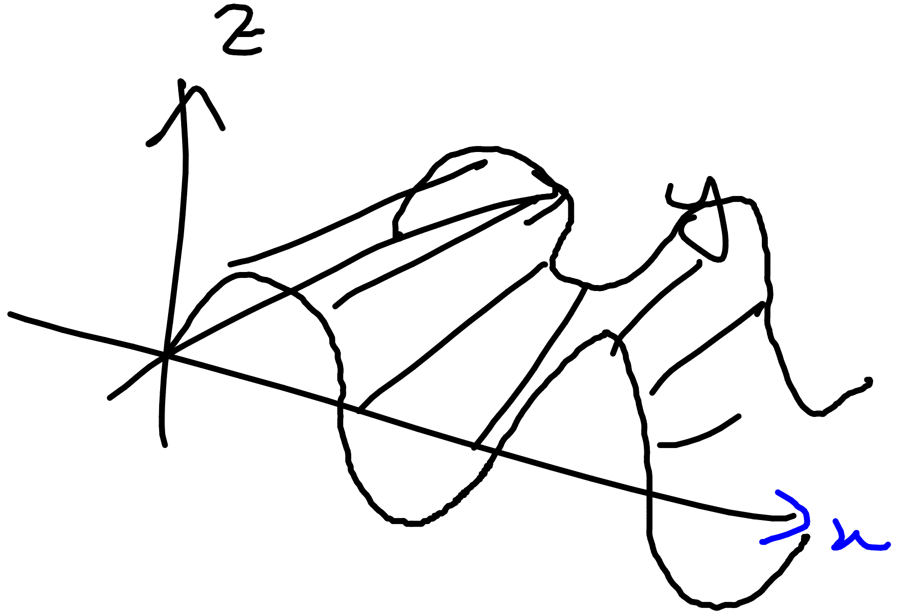
$0 \le v \le 5$

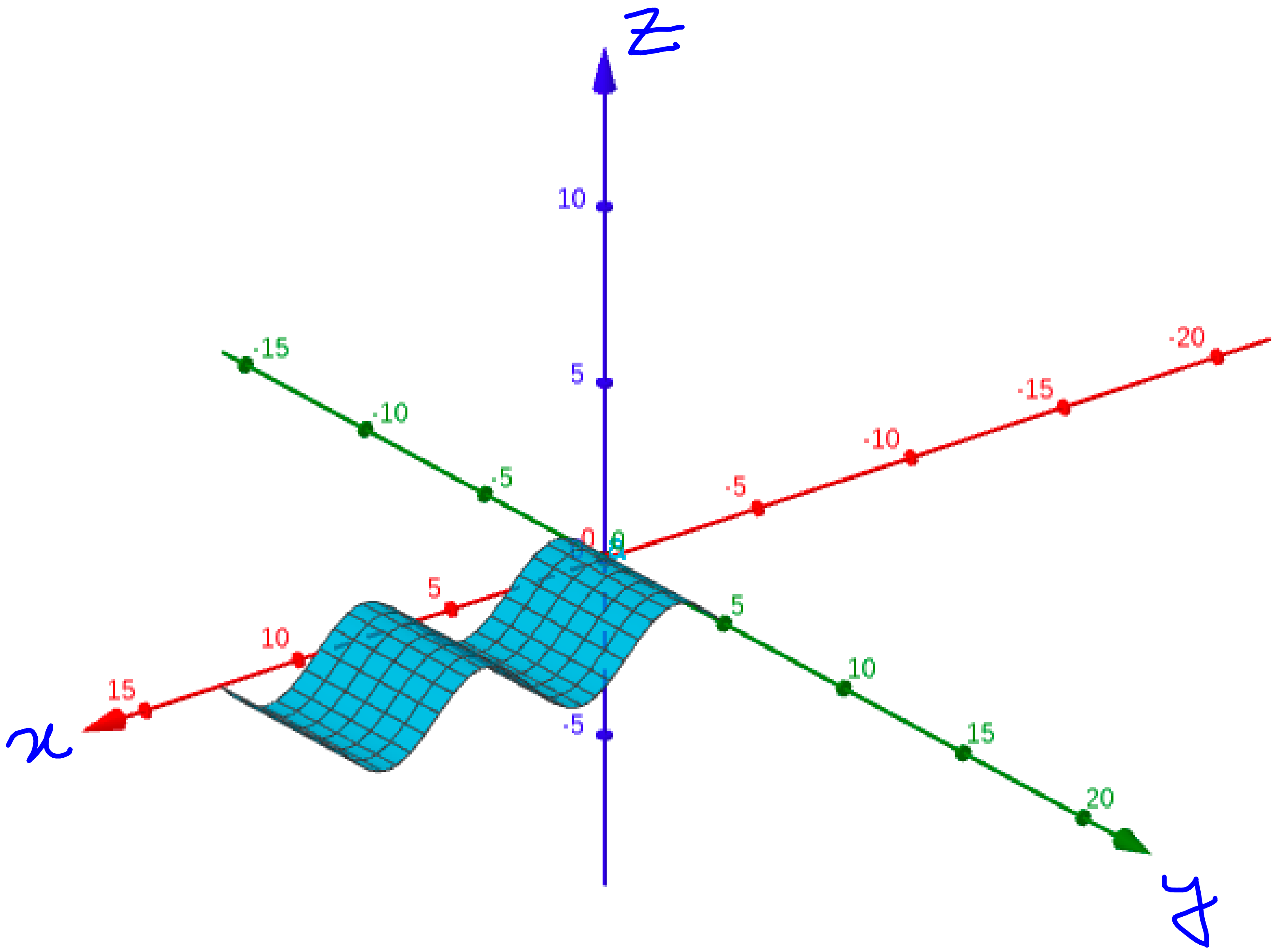


$x = u$

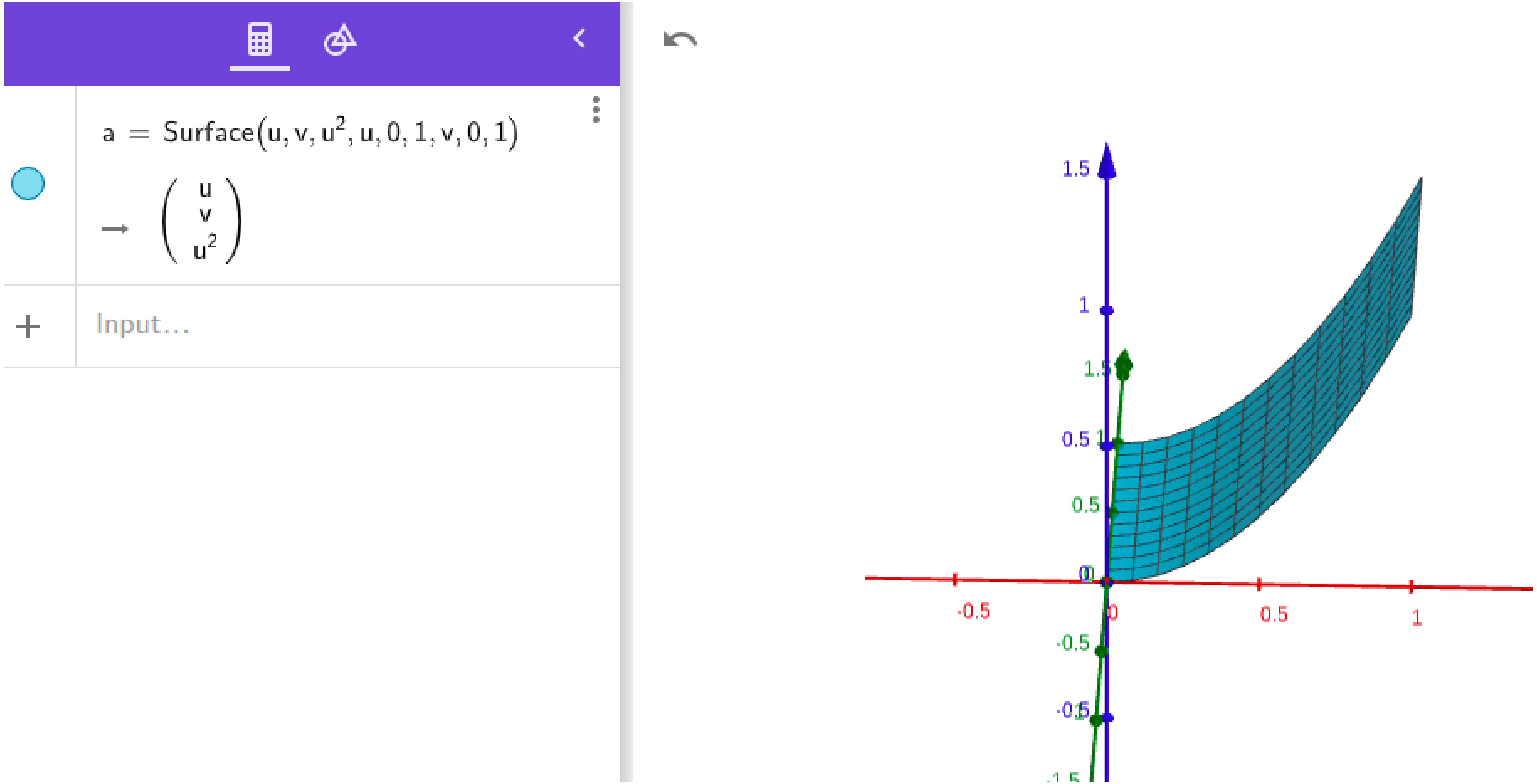
$y = v$

$z = \sin(u)$



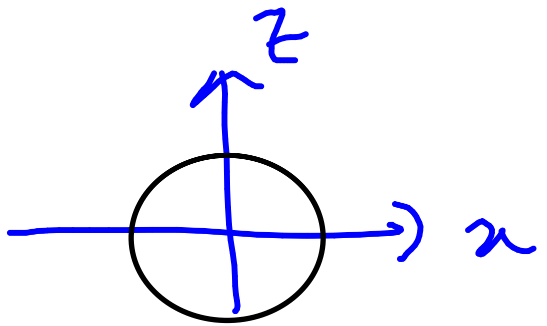


z
x
y
10
5
-5
15
10
5
0
-5
-10
-15
-20
-15
-10
-5
5
10
15
20


a = Surface(u, v, u^2, u, 0, 1, v, 0, 1)
→ (u, v, u^2)
Input…
1.5
1
0.5
-0.5
0.5
1

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = \underbrace{2 \cos u}_{x}\, \mathbf{i} + \underbrace{v}_{y}\, \mathbf{j} + \underbrace{2 \sin u}_{z}\, \mathbf{k}$$



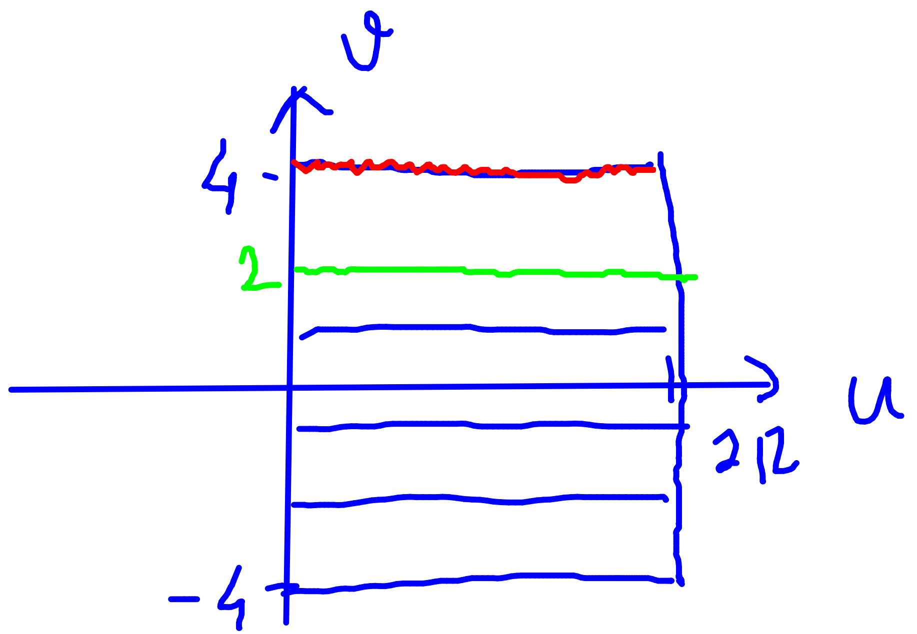
$x = 2\cos u$

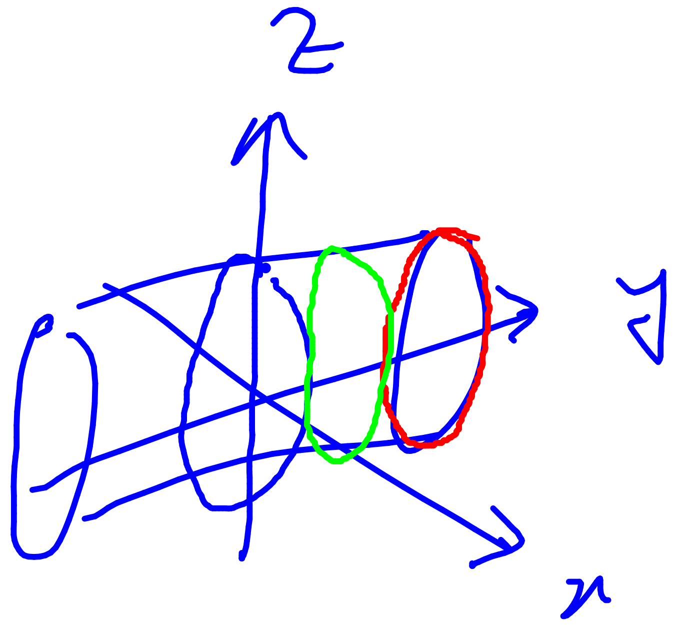
$y = v$

$z = 2\sin u$

$0 \le u \le 2\pi$

$-4 \le v \le 4$

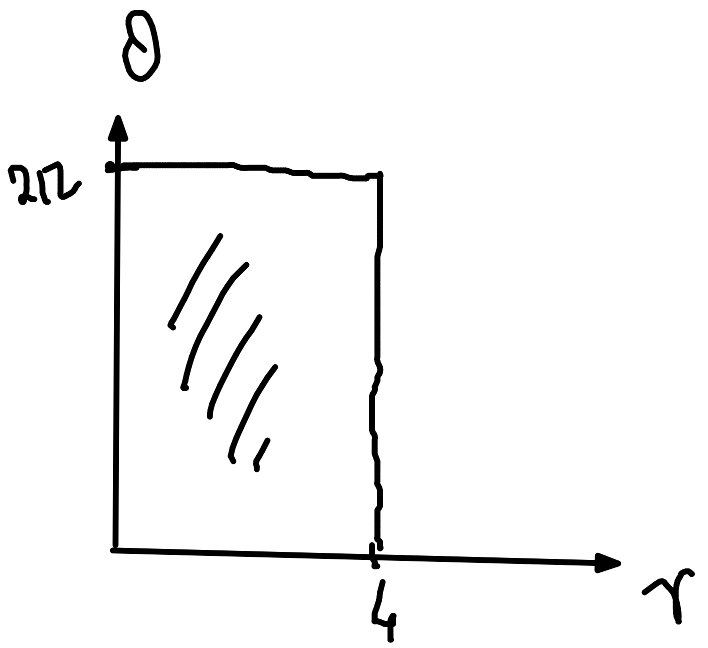




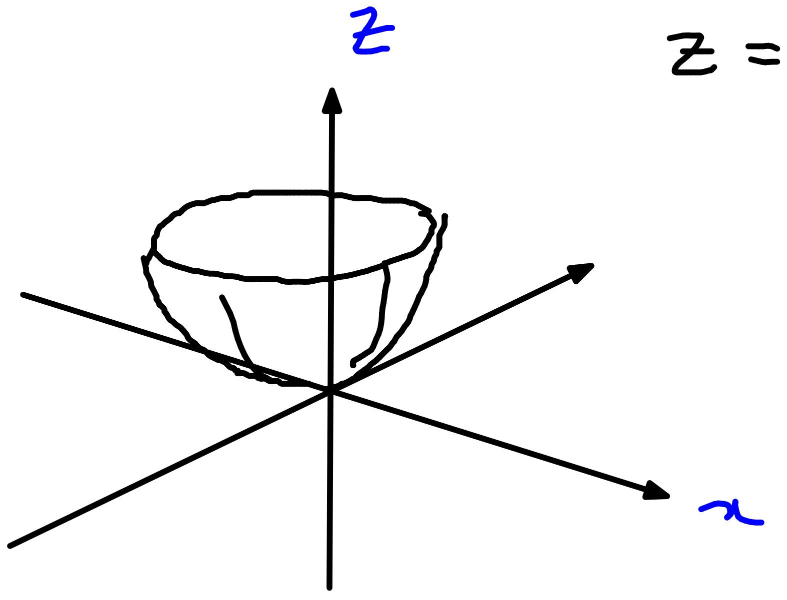
EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

d.



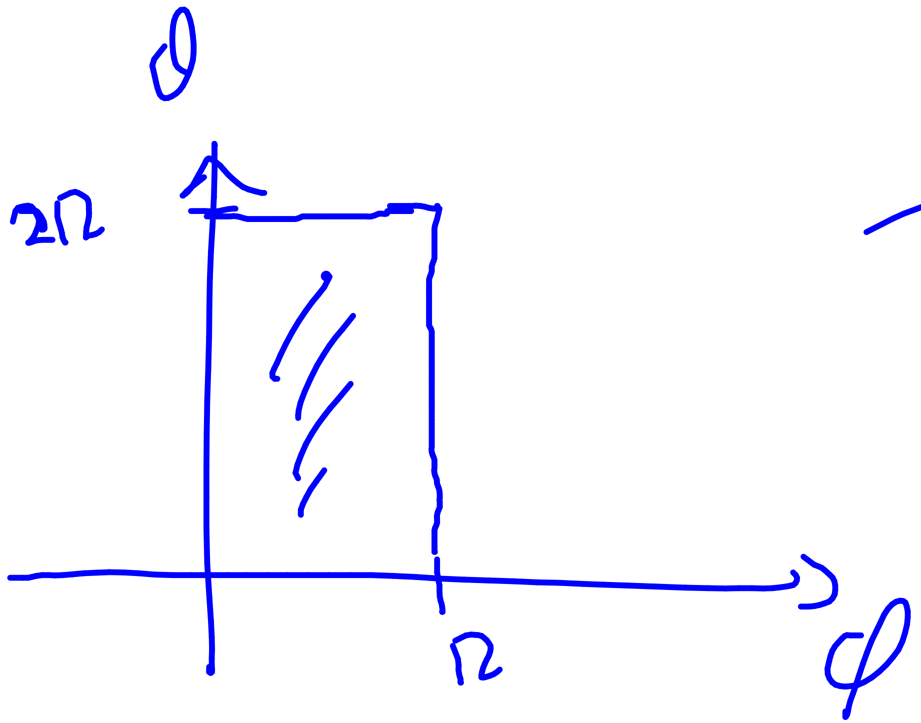
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r^2$$



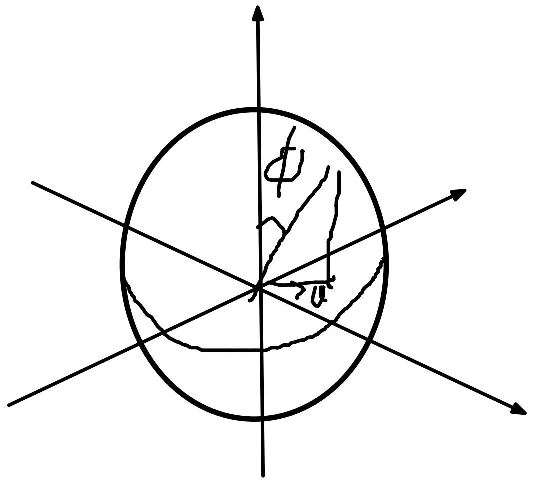
$$z = x^2 + y^2$$

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 4 Find a parametric representation of the sphere $x^2 + y^2 + z^2 = 3^2$

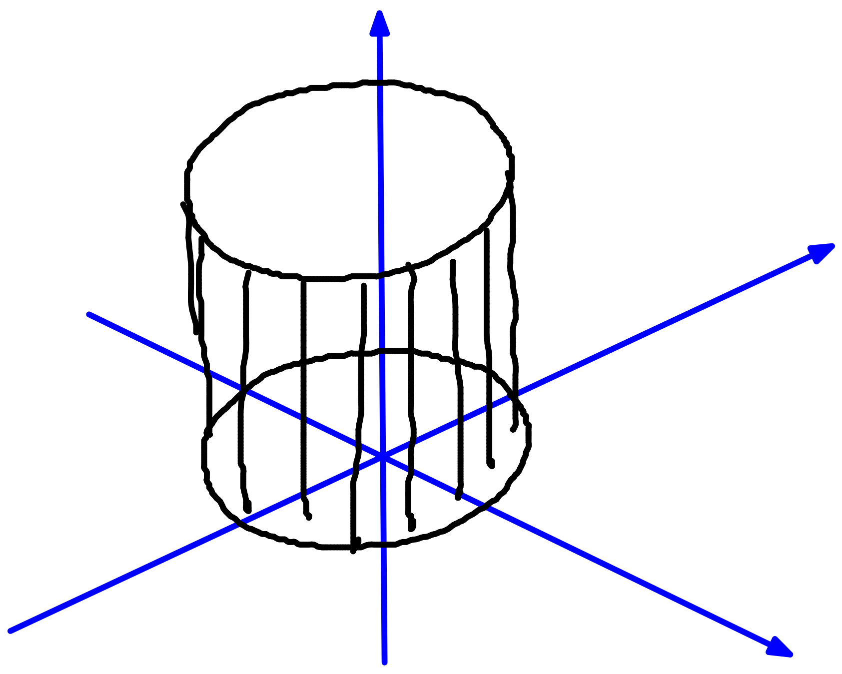


$$\boxed{\begin{aligned} x &= 3 \sin\phi \cos\theta \\ y &= 3 \sin\phi \sin\theta \\ z &= 3 \cos\phi \end{aligned}}$$



EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$



$x = 2\cos\theta$

$y = 2\sin\theta$

$z = z$

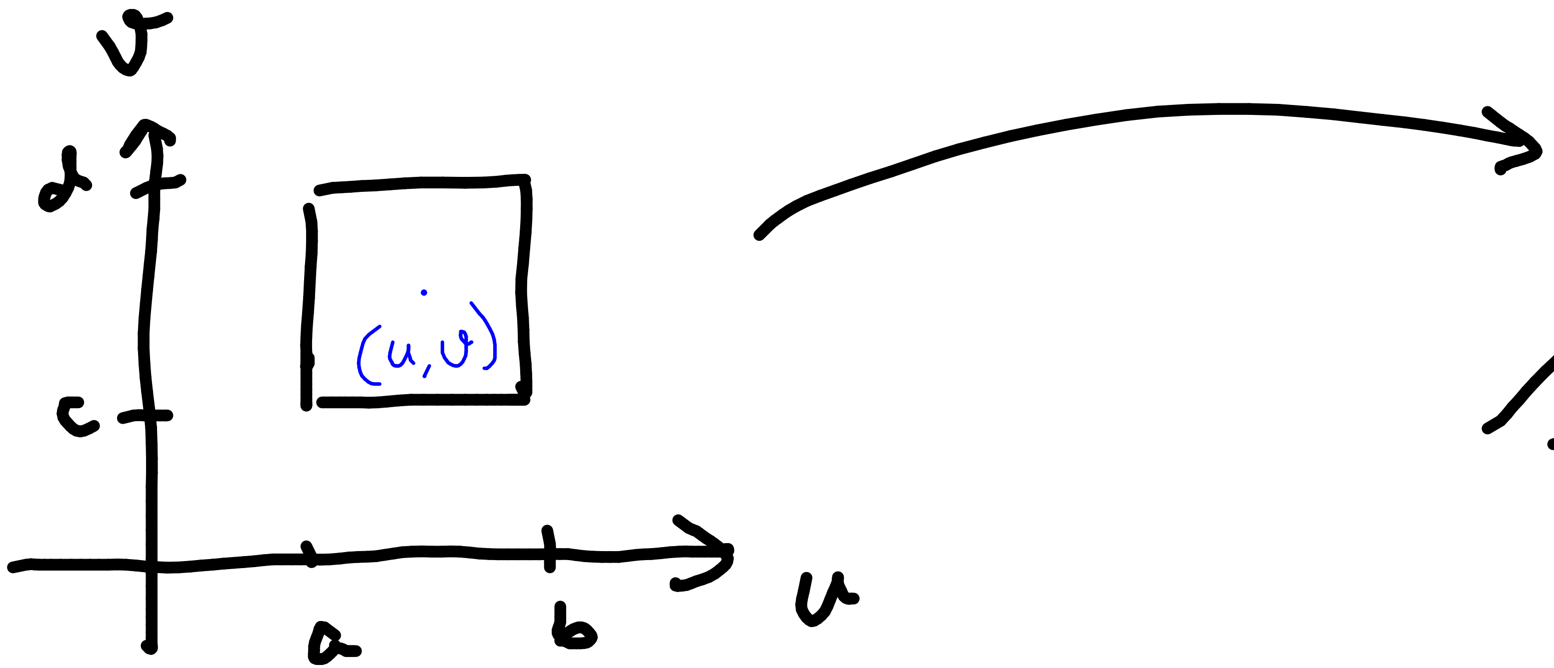
range for θ & z

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 1$

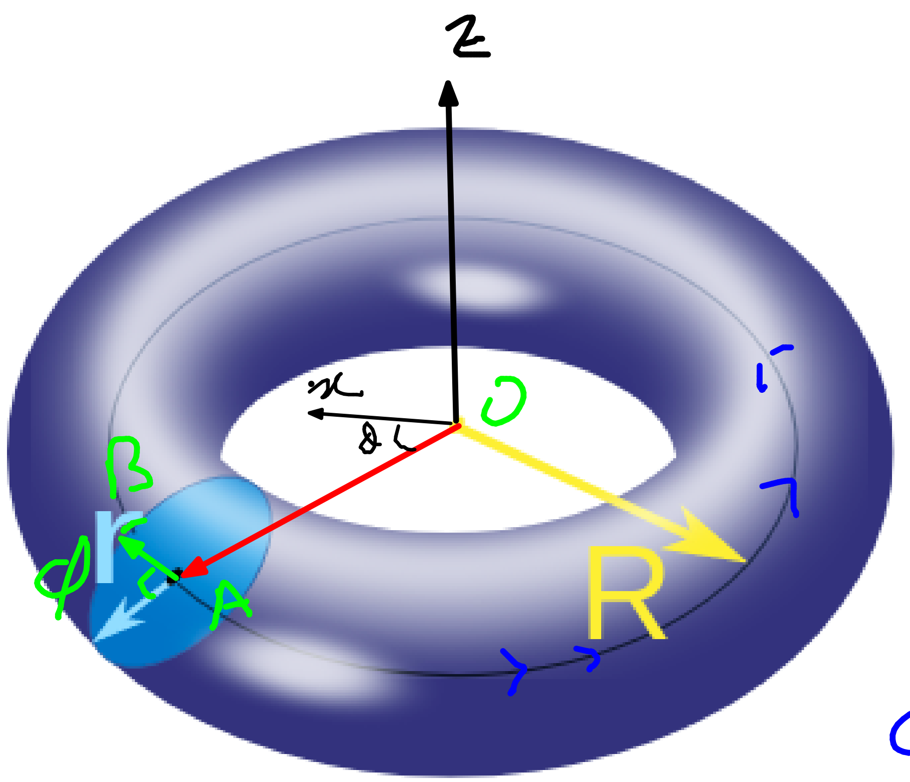
Parametric Surfaces:

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$



$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

Q. find a parametric representation for a torus



ϕ : angle between the vector $\vec{AB}$ & $\vec{OA}$

$0 \le \phi \le 2\pi$

$0 \le \theta \le 2\pi$

Q: find coordinates of the point B: in terms of r, R, θ, ϕ

$x = ?$

$y = ?$

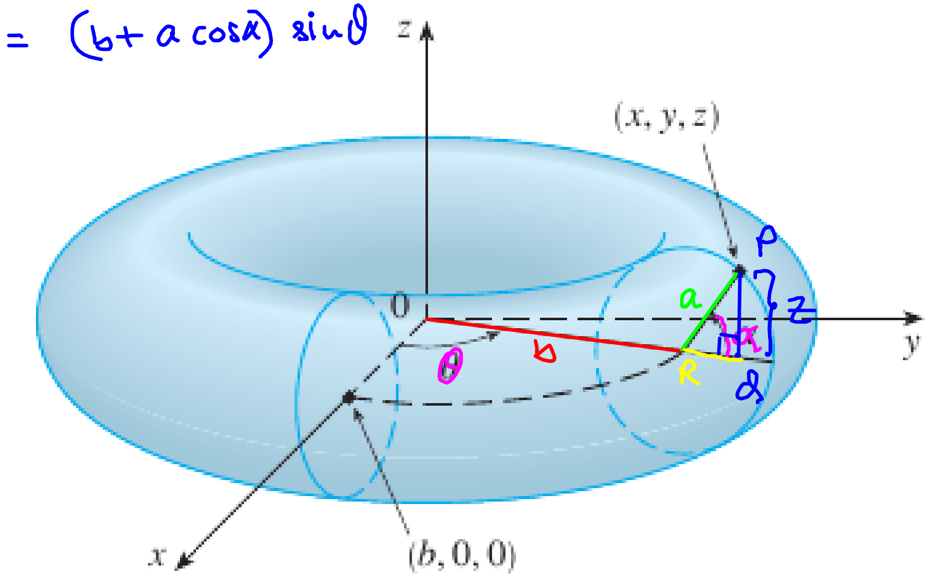
$z = ?$

H. W.

$x = OQ \cos\theta = (b + a\cos\alpha)\cos\theta$

$y = OQ \sin\theta = (b + a\cos\alpha)\sin\theta$

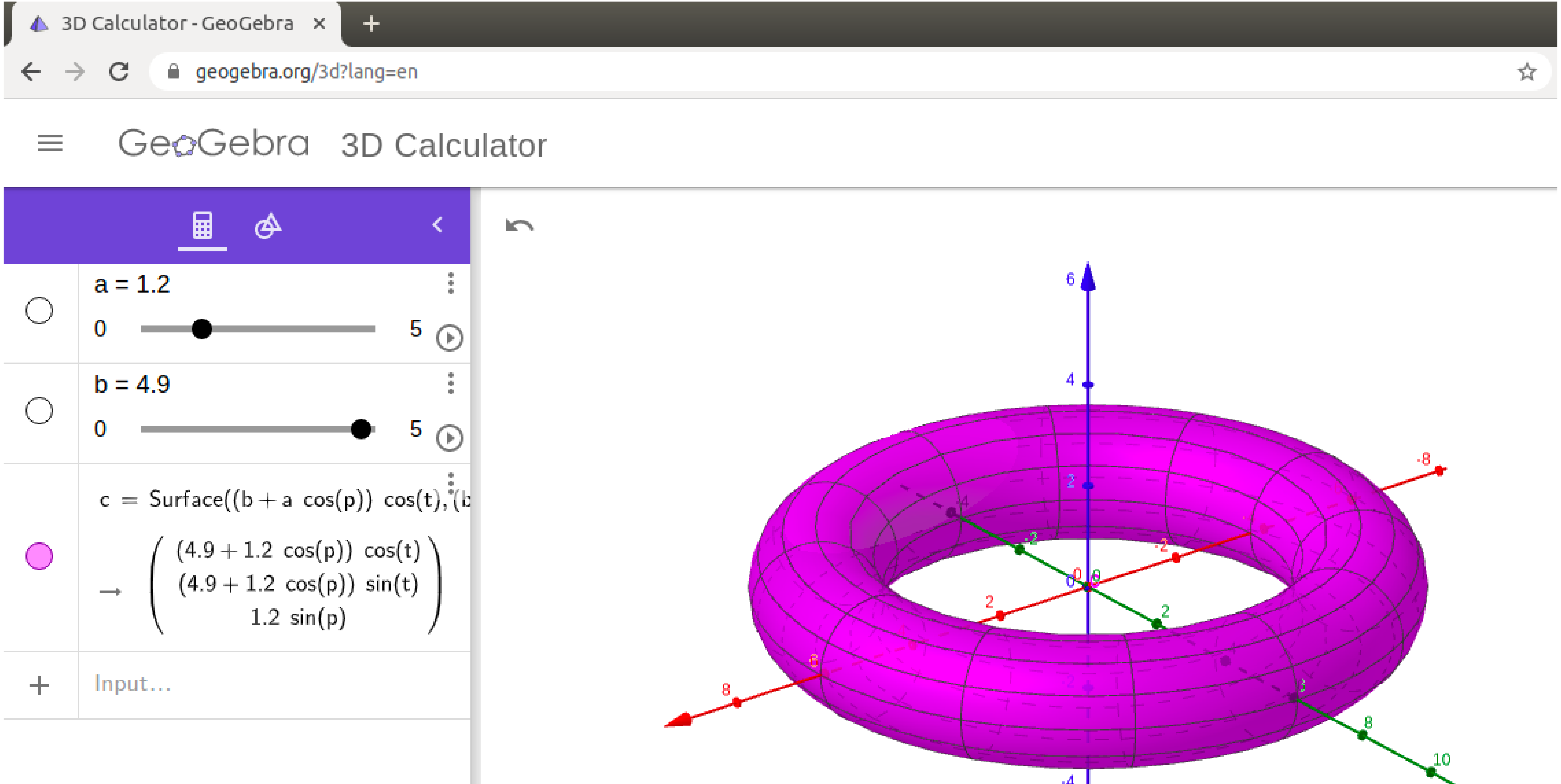
$z = a \sin\alpha$



Torus: radius : b

radius of cross sectional circle → a

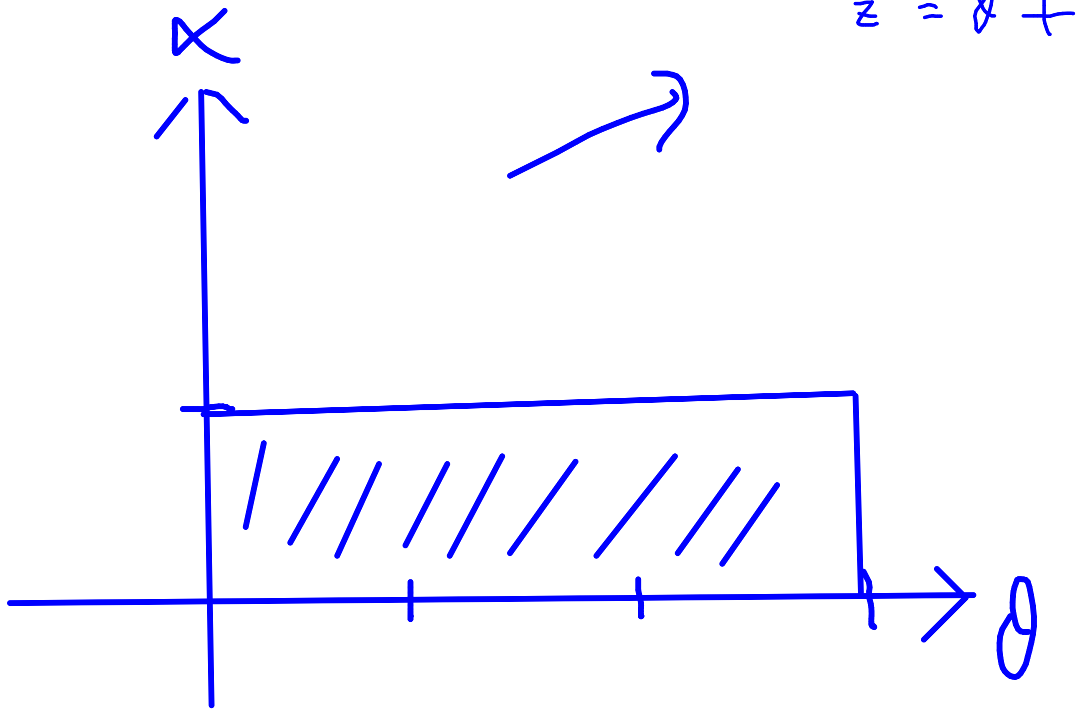
(x, y, z) : will be uniquely identified if we know θ & α.

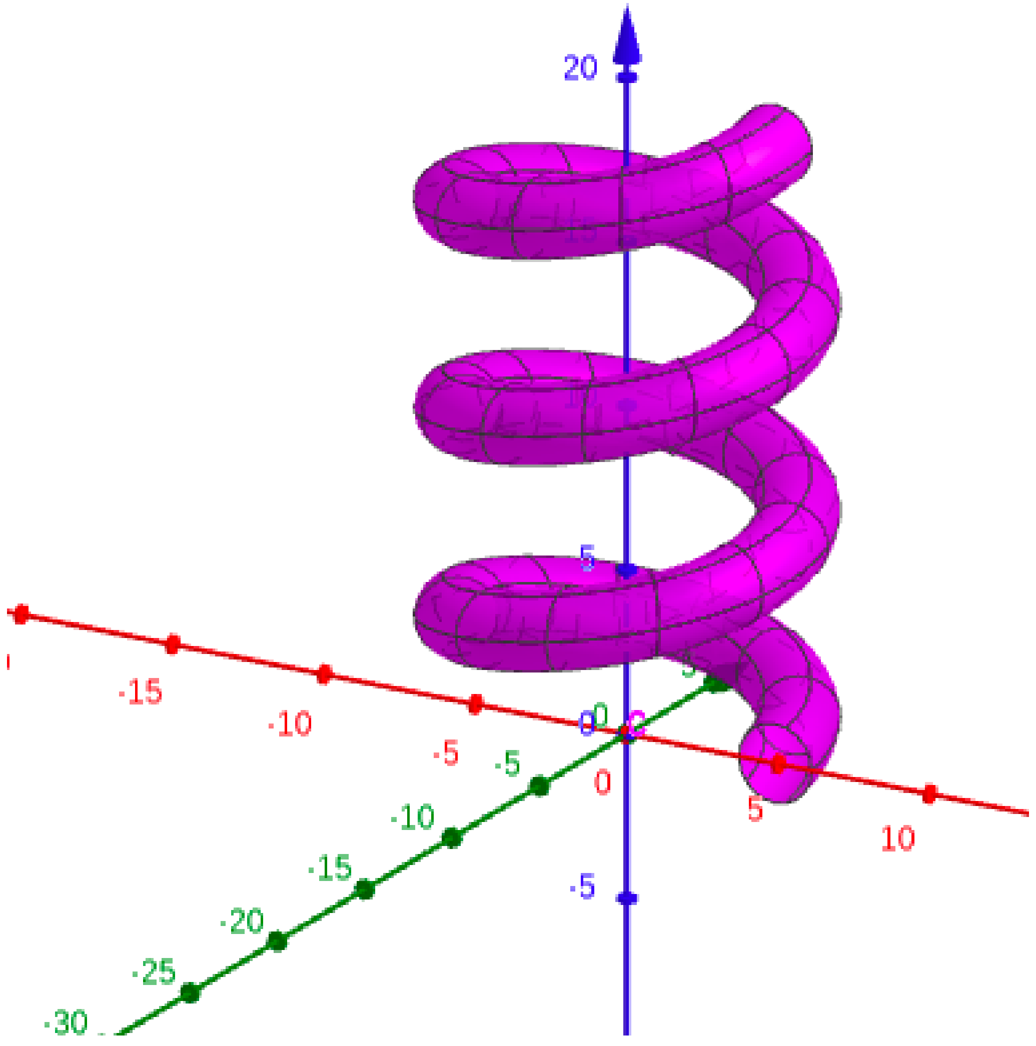

3D Calculator - GeoGebra
geogebra.org/3d?lang=en
GeoGebra 3D Calculator
a = 1.2
0
5
b = 4.9
0
5
c = Surface((b + a cos(p)) cos(t), (b
(4.9 + 1.2 cos(p)) cos(t)
(4.9 + 1.2 cos(p)) sin(t)
1.2 sin(p)
Input…
6
4
2
-8
-2
2
8
10

$$x = \rho \cos\theta = (b + a\cos\alpha)\cos\theta$$

$$y = \rho \sin\theta = (b + a\cos\alpha)\sin\theta$$

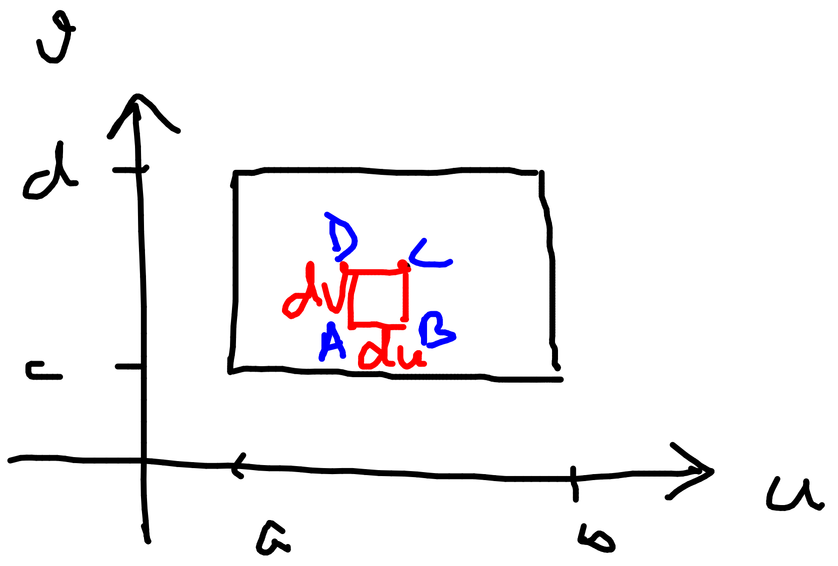
$$z = \theta + a\sin\alpha$$

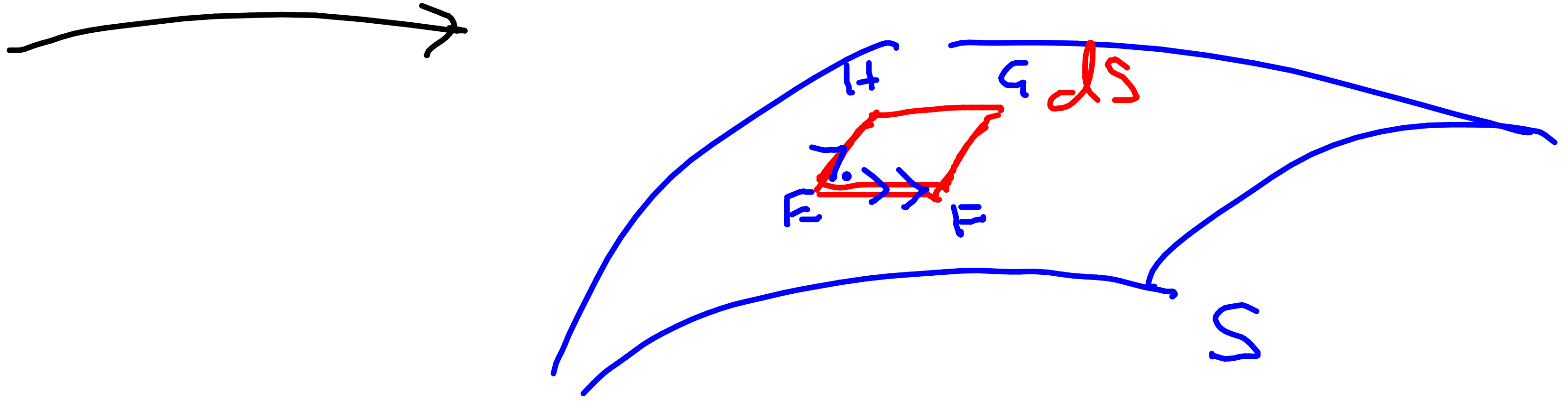




Next task: find area of parametric surfaces:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$





Q: keeping v constant & changing u to $u+du$ how much distance we are covering in surface

$$EF \sim \frac{\partial \vec{r}}{\partial u}\,du$$

$$EH \sim \frac{\partial \vec{r}}{\partial v}\,dv$$

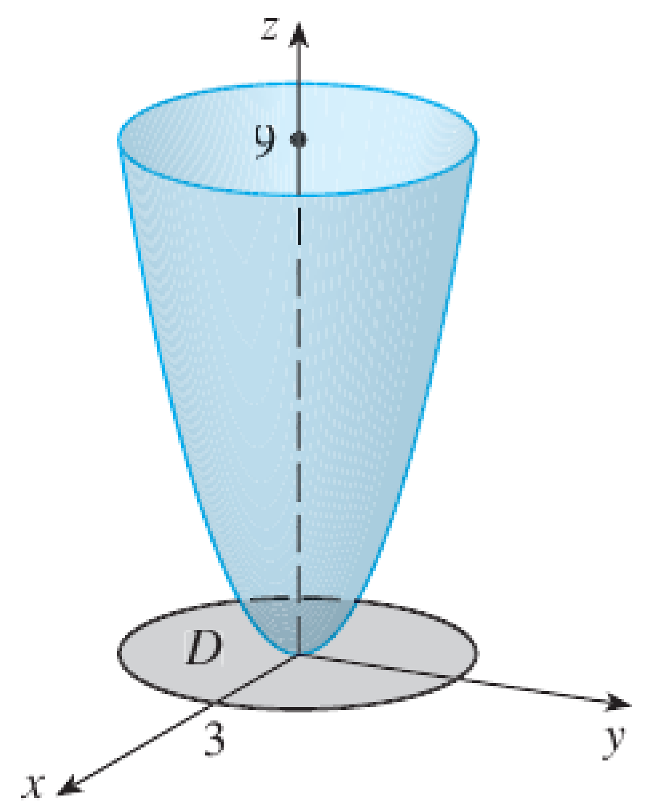
$$\text{area}(S) = \iint_S dS$$

$$dS = \left|\overrightarrow{EF} \times \overrightarrow{EH}\right|$$

$$dS = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right| = \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}_{\text{think this like Jacobian}} du\, dv$$

$$\text{Total area} = \int_c^d \int_a^b \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

V EXAMPLE 10 Find the area of the part of the paraboloid $\underbrace{z = x^2 + y^2}$ that lies under the plane $z = 9$.



$$\vec{r}(s,\theta) = s\cos\theta\,\hat{i} + s\sin\theta\,\hat{j} + s^2\,\hat{k}$$

$$0 \le s \le 3$$

$$0 \le \theta \le 2\pi$$

$$\text{area} = \int_0^{2\pi}\int_0^{3} \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| ds\, d\theta$$

$$\frac{\partial \vec{r}}{\partial s} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 2s\,\hat{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -s\sin\theta\,\hat{i} + s\cos\theta\,\hat{j} + 0\,\hat{k}$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2s \\ -s\sin\theta & s\cos\theta & 0 \end{vmatrix} = -2s^2\cos\theta\,\hat{i} - 2s^2\sin\theta\,\hat{j} + s\,\hat{k}$$

$$\left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{4s^4 + s^2}$$

$$\text{area} = \int_0^{2\pi}\int_0^{3} \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial \theta} \right| ds\, d\theta = \int_0^{2\pi}\int_0^{3} s\sqrt{4s^2+1} \; ds\, d\theta = \text{whatever}$$

$$= \frac{\pi}{6}\left(37^{2/3} - 1\right)$$

Find the area of the surface.

The part of the plane $3x + 2y + z = 6$ that lies in the first octant

Find the area of the surface.

The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

Find the area of the surface.

The part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, and $z = 1$