13.6 PARAMETRIC SURFACES AND THEIR AREAS

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \, \mathbf{i} + v \, \mathbf{j} + 2 \sin u \, \mathbf{k}$$

A bit of practice with matlab

→ LIVE SCRIPTS

EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

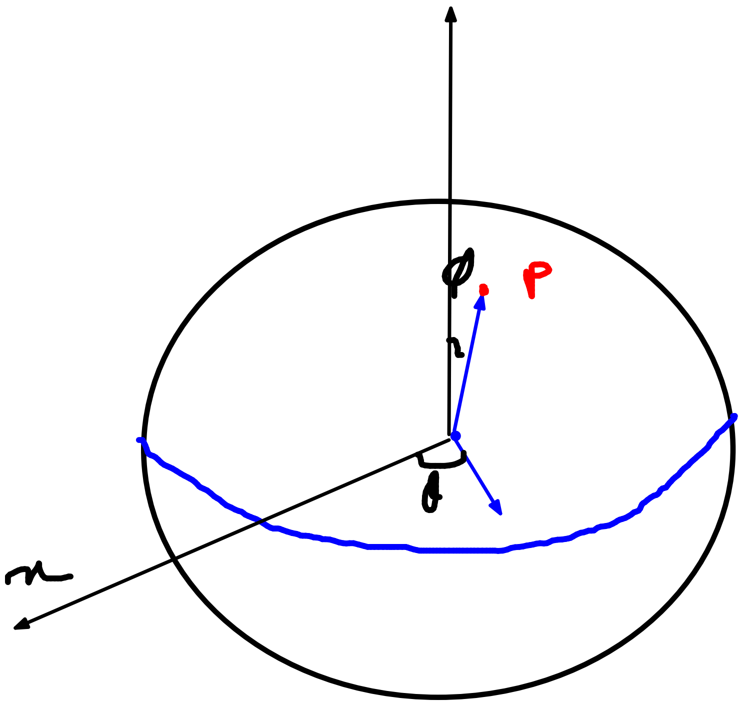
→ plot this in matlab

EXAMPLE 4 Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

$$x = a\cos\theta \sin\phi$$

$$y = a\sin\theta \sin\phi$$

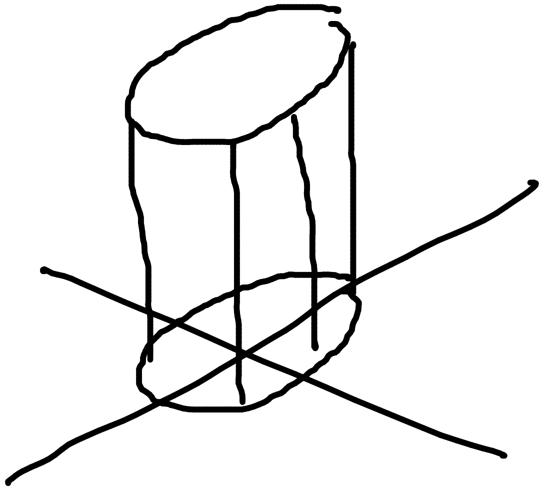
$$z = a\cos\phi$$



recall spherical coordinates

EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$



$x = 2\cos\theta$

$y = 2\sin\theta$

$z = z$

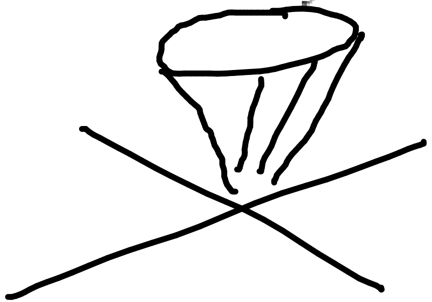
parameters

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 1$

EXAMPLE 6 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

Drawing ??

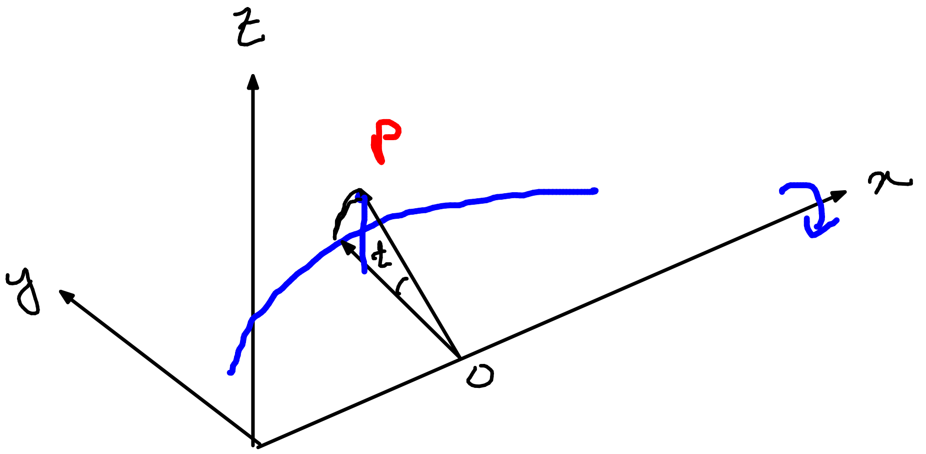


parametric eqn

$$x = x \qquad y = y \qquad z = 2\sqrt{x^2+y^2}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = 2r$$

$y = f(x)$



$OP = f(x)$

OP's y coordinate

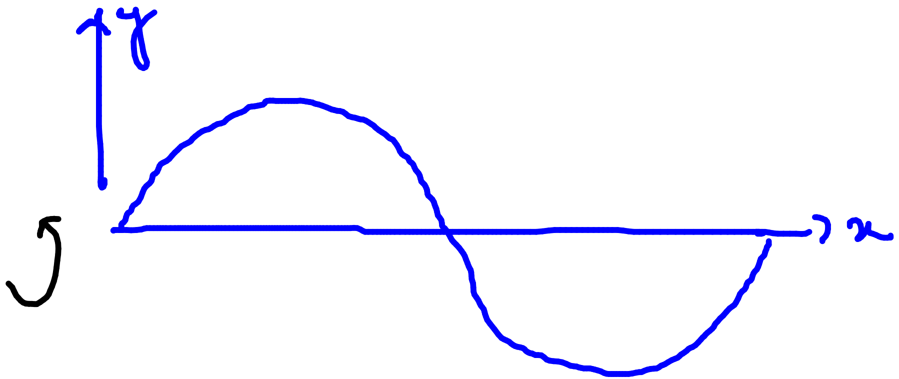
P: a point on the surface formed by revolving the graph of $y = f(x)$ about x axis

$\rightarrow$ x coordinate of P is x

y " " " $f(x)\cos(t)$

z " " " $f(x)\sin(t)$

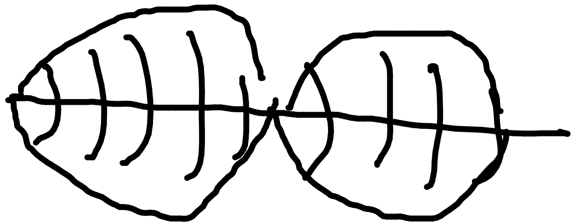
EXAMPLE 7 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leqslant x \leqslant 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

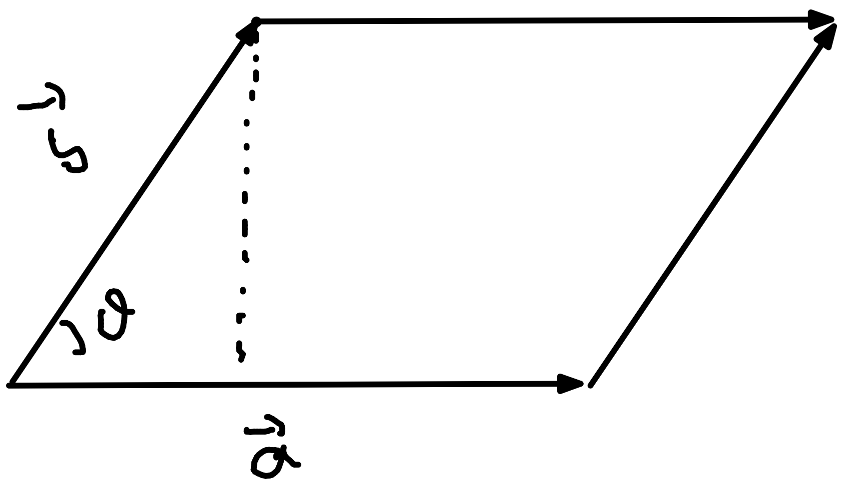


$$x = x$$

$$y = \sin(x)\cos(t)$$

$$z = \sin(x)\sin(t)$$



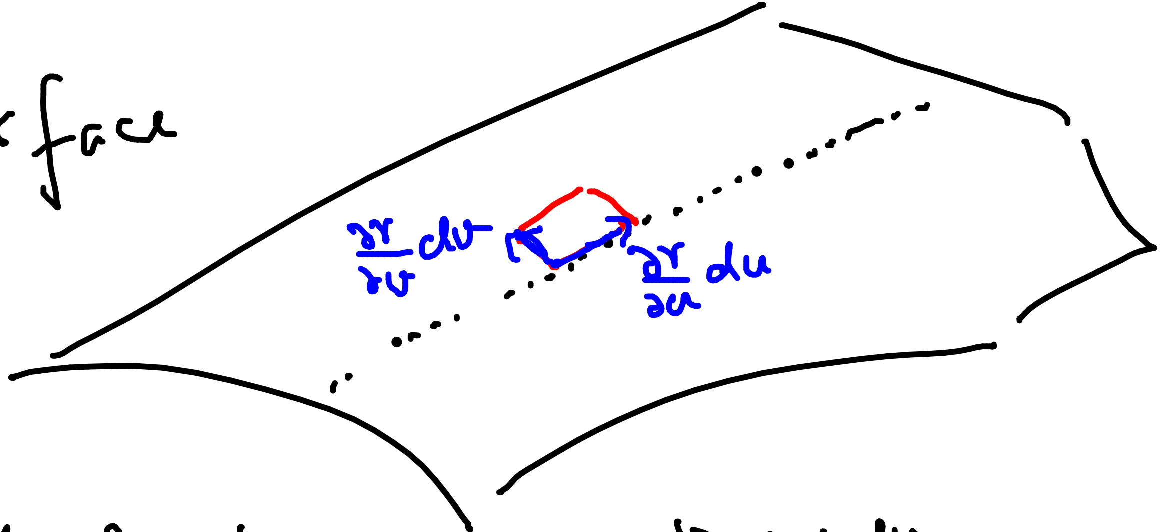


$$\text{area} = |\vec{a} \times \vec{b}|$$

$$= |a|\ |b|\ \sin\theta$$

Area of parametric surfaces?

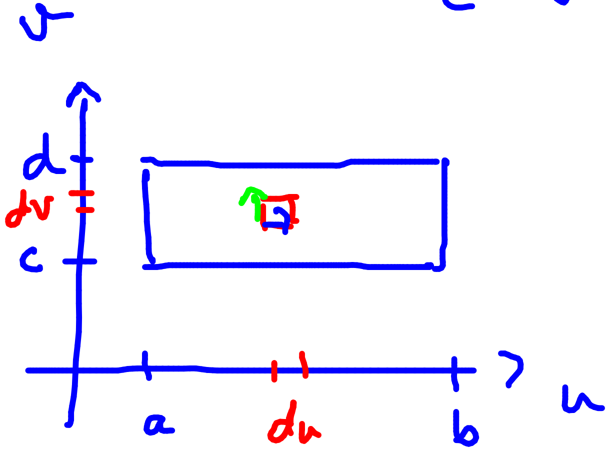
Suppose we have a surface



$$r(u,v) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$a \le u \le b$$

$$c \le v \le d$$



→ the effect of change in u to u + du

$$\frac{\partial r}{\partial u} du$$

→ the effect of change in v to v + dv

$$\frac{\partial r}{\partial v} dv$$

Area on the surface swiped by changing u to $u+du$ & v to $v+dv$

$$= \left| \frac{\partial r}{\partial u} du \times \frac{\partial r}{\partial v} dv \right|$$

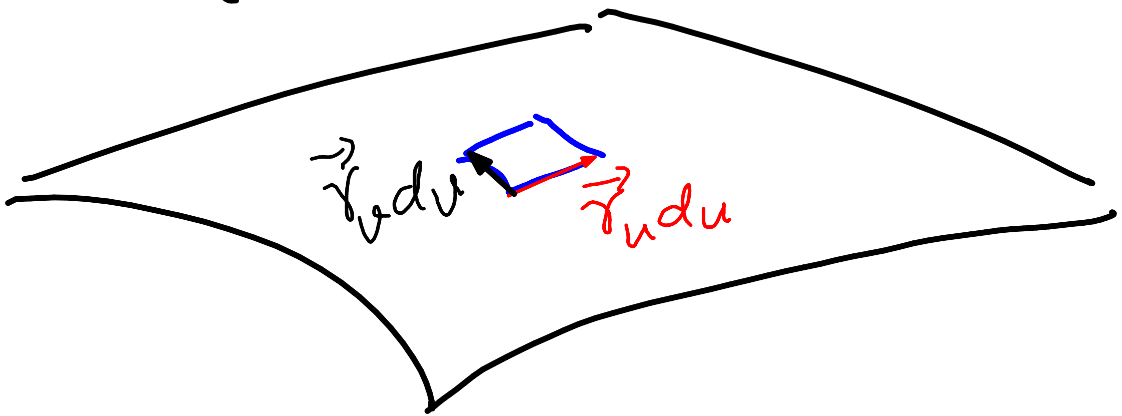
$$\boxed{dA = \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du \, dv}$$

$$A = \iint dA = \int_c^d \int_a^b \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du \, dv$$

Surface area of Parametric Surfaces

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$|\vec{r}_u \times \vec{r}_v|\,du\,dv$$



$\vec{r}_u\,du$

$\vec{r}_v\,dv$

$$dA = |\vec{r}_u du \times \vec{r}_v dv|$$

$$= |\vec{r}_u \times r_v|\,du\,dv$$

6 **DEFINITION** If a smooth parametric surface S is given by the equation

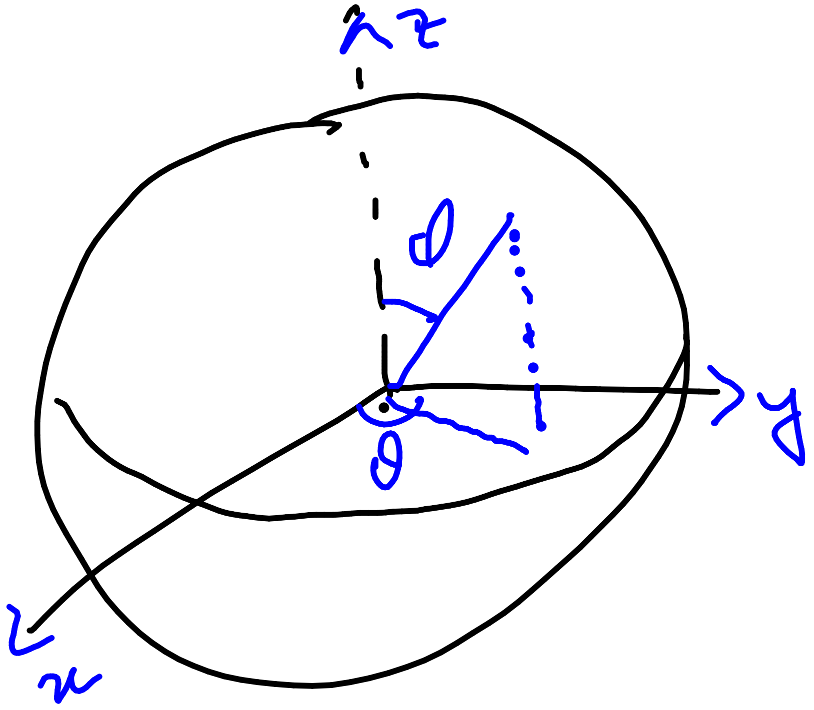
$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|\, dA}$$

where $\qquad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\,\mathbf{i} + \dfrac{\partial y}{\partial u}\,\mathbf{j} + \dfrac{\partial z}{\partial u}\,\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\,\mathbf{i} + \dfrac{\partial y}{\partial v}\,\mathbf{j} + \dfrac{\partial z}{\partial v}\,\mathbf{k}$

EXAMPLE 9 Find the surface area of a sphere of radius a.

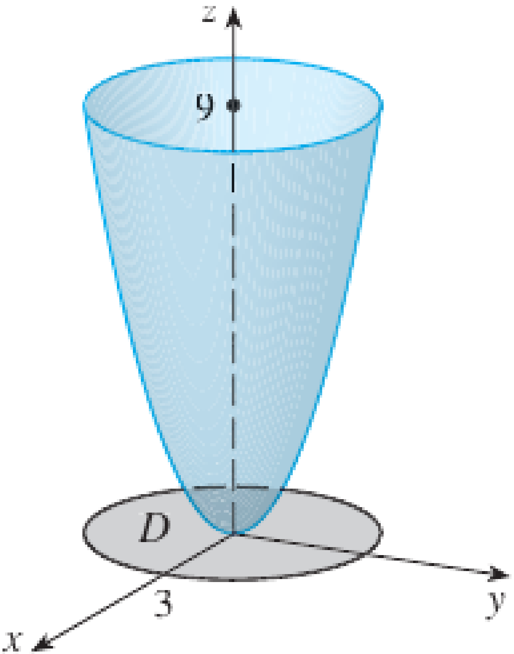


$x = a\cos\theta \sin\phi$

$y = a\sin\theta \sin\phi$

$z = a\cos\phi$

V EXAMPLE 10 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



$$x = s \sin t$$
$$y = s \cos t \qquad 0 \le s \le 3$$
$$z = s^2 \qquad 0 \le t \le 2\pi$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\text{area} = \iint |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

13.5 CURL AND DIVERGENCE

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

rotation
$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \ldots$$

expansion & contraction
$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$

3 THEOREM If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

verify now
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{0}$$

$$\underbrace{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)}_{=0 \text{ (using Clairaut's thm)}}\hat{i} + (\quad)\hat{j} + (\quad)\hat{k}$$

$$\vec{F} = \nabla f \iff \vec{F} \text{ is conservative}$$

Note: This theorem can be used to check if a vector field is conservative or not.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

check curl $(\vec{F}) \neq 0$

$$= \begin{pmatrix} -2y - xy \\ x \\ yz \end{pmatrix} \neq 0$$

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.") Theorem 4 is the three-dimensional version of Theorem 13.3.6. Its proof requires Stokes' Theorem and is sketched at the end of Section 13.8.

4 THEOREM If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

V EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

curl(F) = 0,

& F is defined

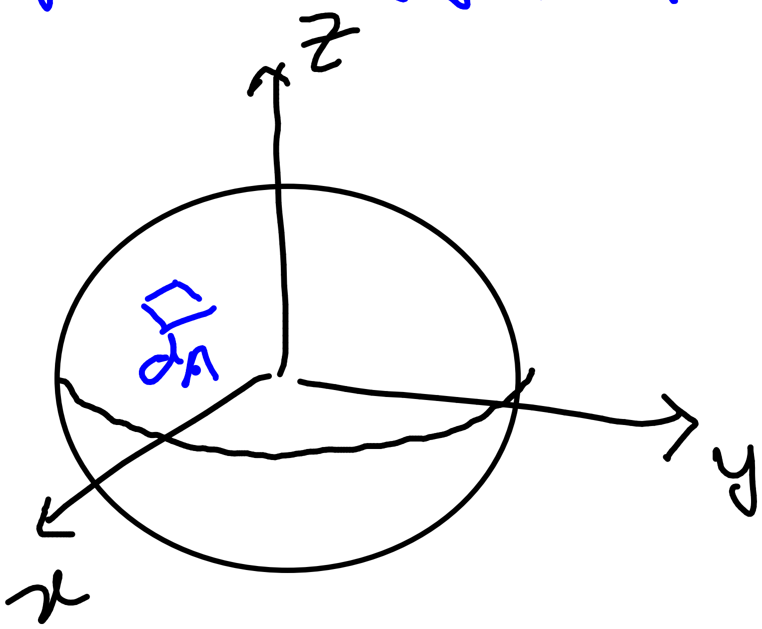
everywhere

$\Rightarrow$ F is conservative.

b) H.W.

Q.

$r = x\hat{i} + y\hat{j} + z\hat{k}$



$x^2+y^2+z^2 = 1$

$\boxed{dA = |\vec{r}_u \times \vec{r}_v| \, du\, dv}$

$f(x,y,z) = z+1$

is the material density (mass/area) on the ball

find the mass of the ball.

$dm = f(x,y,z)\, dA =$

$m = \iint dm =$

13.7 SURFACE INTEGRALS

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v)) \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv}_{du\, dv}$$

EXAMPLE 1 Compute the surface integral $\iint_S x^2\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

we need parametric eqn for surface

$x = \cos\theta \sin\phi$

$y = \sin\theta \sin\phi$

$z = \cos\phi$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\iint_S x^2\,ds = \int_0^{\pi}\int_0^{2\pi} (\cos\theta \sin\phi)^2 \,|\vec{r}_\phi \times \vec{r}_\theta|\, d\theta\, d\phi$$

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

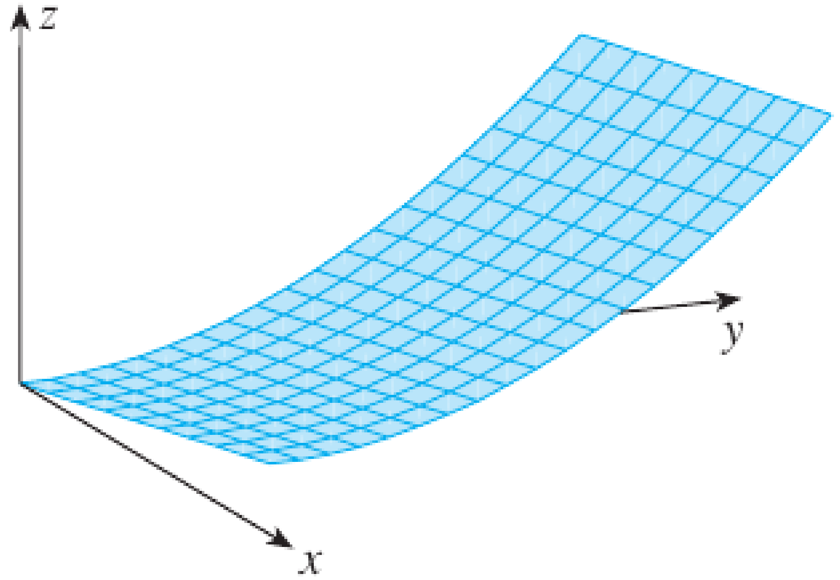
```
syms u v

x = u
y = v
z = u + v^2

r = [x,y,z]
ru = diff(r,u);
rv = diff(r,v);

c = cross(ru,rv);
m = norm(c);

s = int(int(y*m,u,0,1),v,0,2)
```

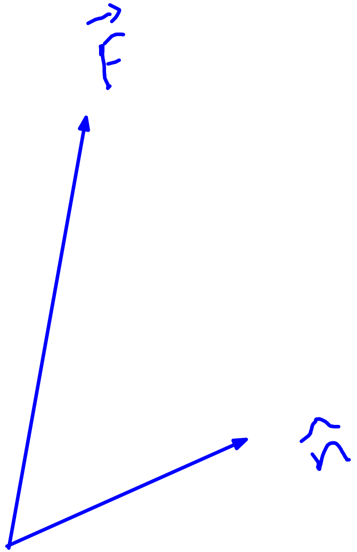
$$= \frac{13\sqrt{2}}{3}$$

$x = u$ $\qquad 0 \le u \le 1$

$y = v$ $\qquad 0 \le y \le 2$

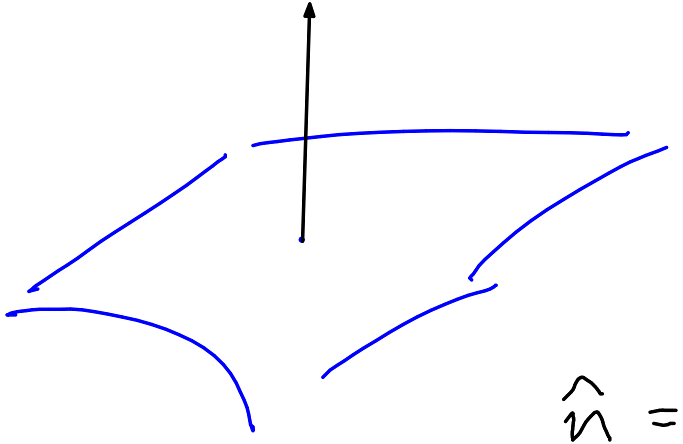
$z = u + v^2$





$$F_n = \vec{F} \cdot \hat{n}$$

$\vec{r}$



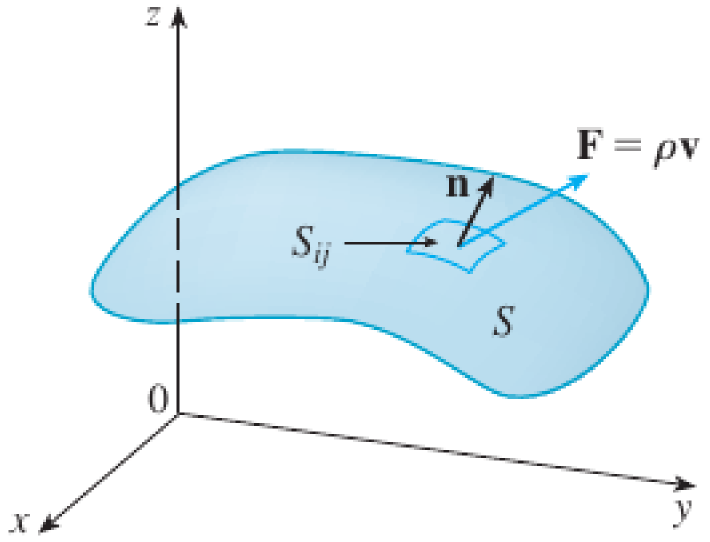
$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

SURFACE INTEGRALS OF VECTOR FIELDS

8 DEFINITION If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This integral is also called the **flux** of **F** across S.



This integral is also called the **flux** of **F** across S.

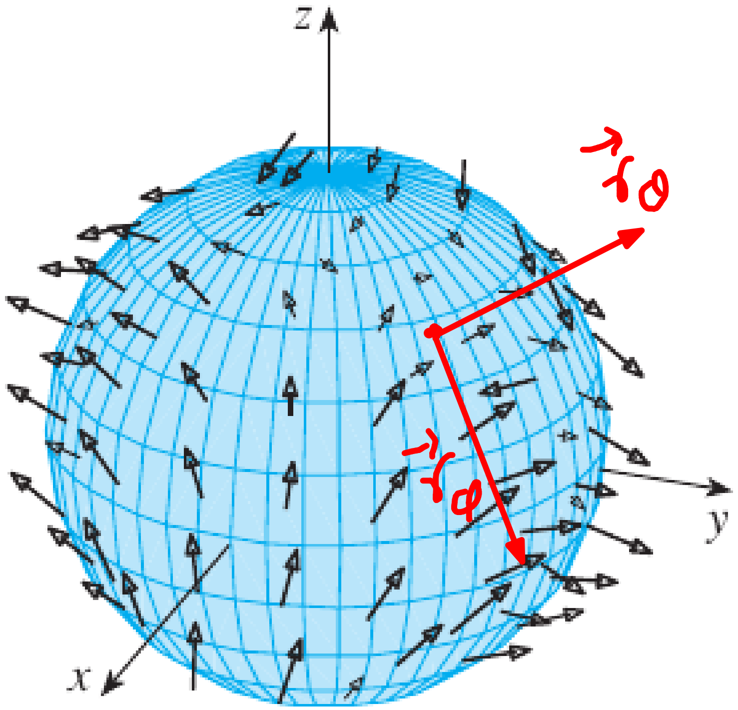
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,\underline{dS} \quad = \iint_D \left[\mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\cancel{|\mathbf{r}_u\times\mathbf{r}_v|}}\right]\cancel{|\mathbf{r}_u\times\mathbf{r}_v|}\; du\,dv$$

$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA}$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the outward unit sphere $x^2 + y^2 + z^2 = 1$.

$$\vec{r} = (\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi)$$

outward direction $\Leftrightarrow$ $\vec{r}_\phi \times \vec{r}_\theta$



$$\iint_S \vec{F}\cdot d\vec{s} = \int_0^{\pi}\int_0^{2\pi} \vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta)\, d\theta\, d\phi$$

```
x = cos(t)*sin(p);
y = sin(t)*sin(p);
z = cos(p);

r = [x,y,z];

rp = diff(r,p);   % differentioan
rt = diff(r,t);
c = simplify(cross(rp,rt));

F = [z,y,x];

dotProduct = simplify(sum(F.*c))

flux = int(int(dotProduct,p,0,pi),t,0,2*pi)
```

$$= \frac{4\pi}{3}$$