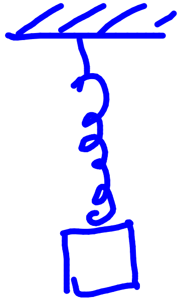
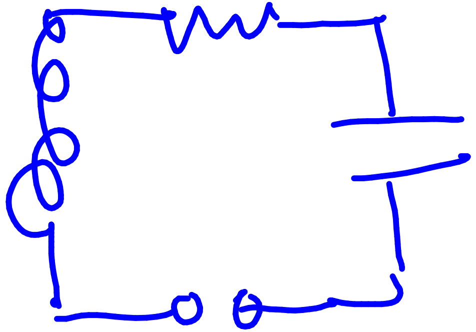
2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$





Method of Undetermined Coefficients

(this section)

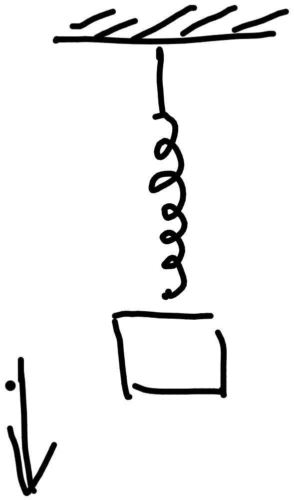
Later: Method of variation of parameters

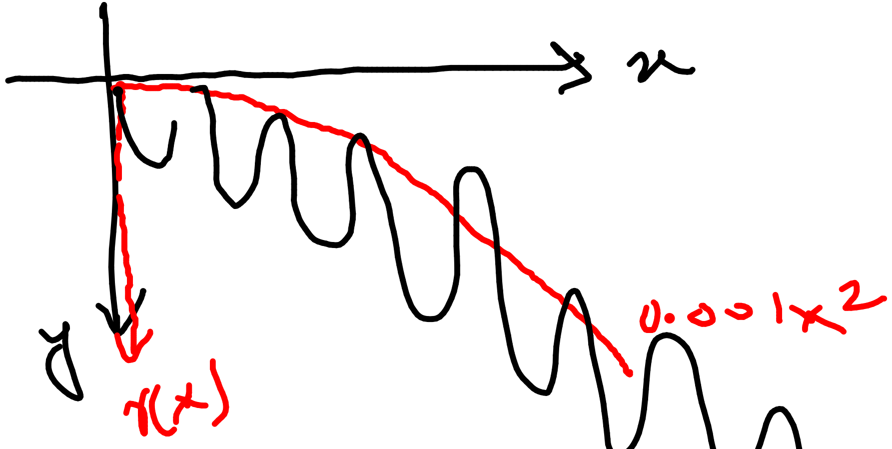
EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

external force

guess the graph of $y(x)$





EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ solve the corresponding homogeneous eqn

$y'' + y = 0$, get $y_h = c_1 y_1 + c_2 y_2$

→ Guess a formula y_p which solves

$y'' + y = 0.001x^2$ [y_p will not have any arbit constants]

→ Final solution: $y = \underbrace{c_1 y_1 + c_2 y_2}_{y_h} + y_p$

EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ solve the corresponding homogeneous eq^n

$y'' + y = 0$, get $y_h = c_1 y_1 + c_2 y_2$

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ guess a formula y_p which satisfies

$$y'' + y = 0.001x^2$$

→ seems like $y_p = a_0 + a_1 x + a_2 x^2$ will work.

→ find a_0, a_1, a_2 by solving $y_p'' + y_p = 0.001x^2$ & comparing constants, x, x^2 terms in LHS & RHS

$\rightarrow \quad 2a_2 + (a_0 + a_1 x + a_2 x^2) = 0.001 x^2$

$\rightarrow \quad a_2 = 0.001$

$a_1 = 0$

$2a_2 + a_0 = 0$

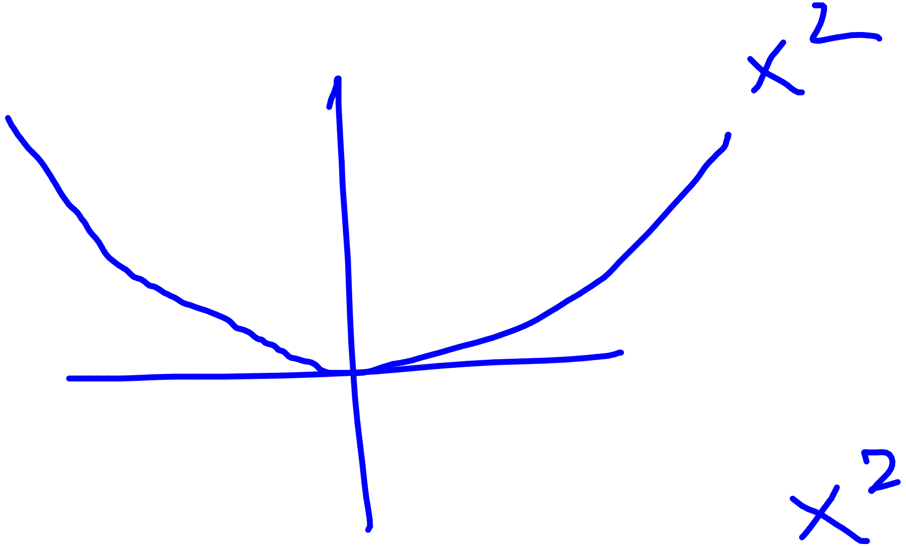
$a_0 = -0.002$

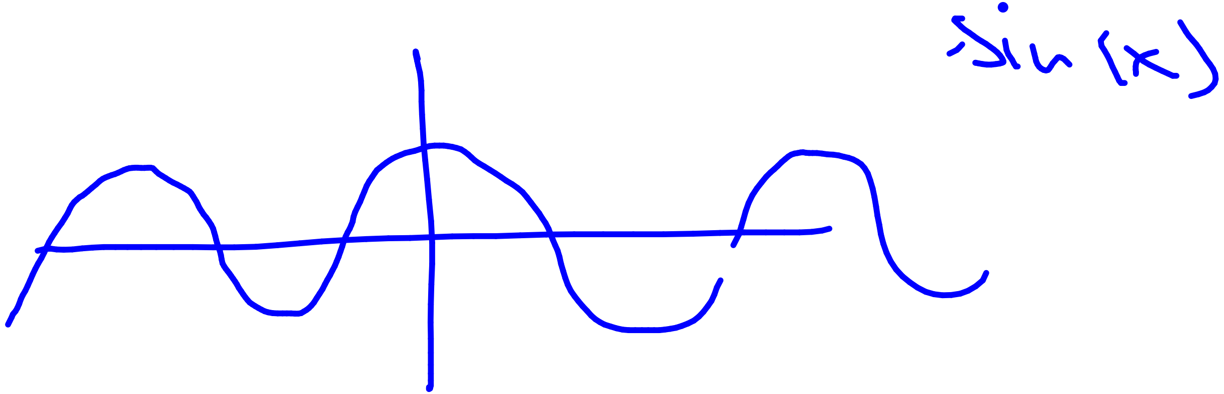
$\rightarrow \quad y_p = -0.002 + 0.001 x^2$

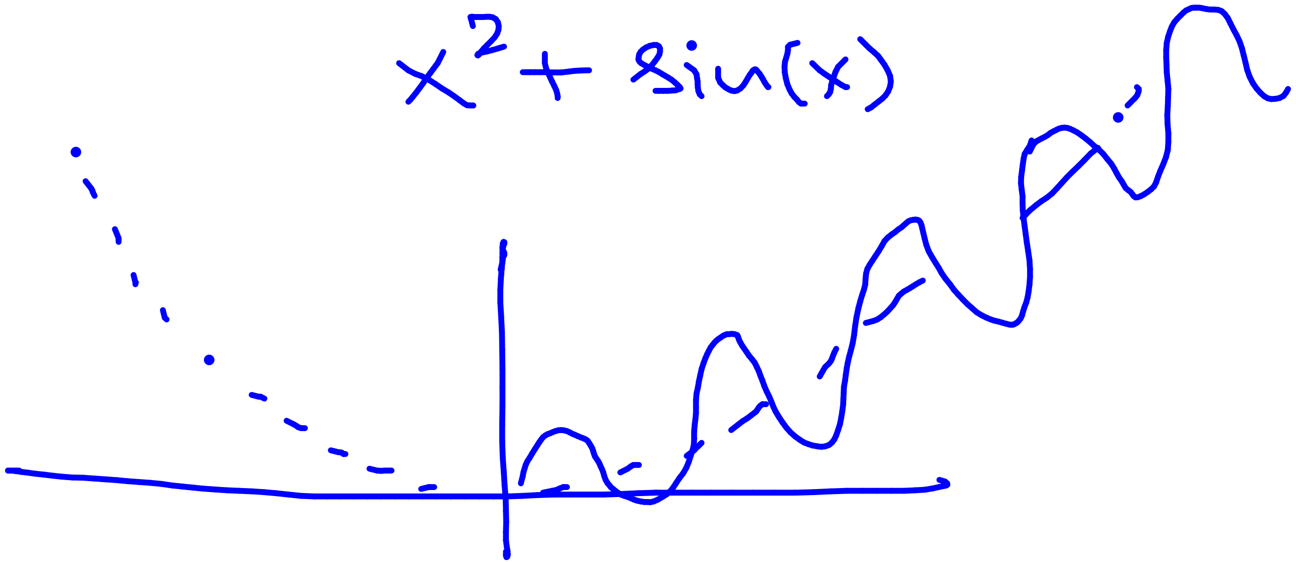
general solution:

$$y = c_1 \cos x + c_2 \sin x - 0.002 + 0.001 x^2$$

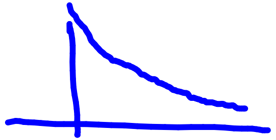
H.W. find c_1, c_2 using other conditions $y(0) = 0$, $y'(0) = 1.5$


x^2


$\sin(x)$


$x^2 + \sin(x)$

EXAMPLE 2



$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

→ Solve the homogeneous part

$$y'' + 3y' + 2.25y = 0$$

$$\lambda^2 + 3\lambda + 2.25 = 0 \qquad \lambda = \frac{-3 \pm \sqrt{9-9}}{2} = \frac{-3}{2}$$

$$y_h = e^{-3/2\,x}(c_1 x + c_2)$$

→ Let's find y_p, try $y_p = Ce^{-1.5x}$

find C

will not work (why)

$\rightarrow$ try $y_p = cxe^{-1.5x}$

will not work either (why)

$\rightarrow$ try $y_p = cx^2e^{-1.5x}$

$$y_p' = 2cxe^{-1.5x} - 1.5cx^2e^{-1.5x}$$

$$y_p'' = 2ce^{-1.5x} - 3cxe^{-1.5x} - 3cxe^{-1.5x} + 2.25cx^2e^{-1.5x}$$

$$= 2ce^{-1.5x} - 6cxe^{-1.5x} + 2.25cx^2e^{-1.5x}$$

$$y_p'' + 3y_p' + 2.25\,y_p = -10\,e^{-1.5x}$$

$$2C\,\cancel{e^{-1.5x}} = -10\,\cancel{e^{-1.5x}}$$

$$C = -5$$

$\rightarrow$ general solution

$$y = e^{-1.5x}(c_1 x + c_2) + \underbrace{(-5)x^2 e^{-1.5x}}$$

EXAMPLE 3

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) =$$

$y_p = A\cos x + B\sin x + C + Dx$

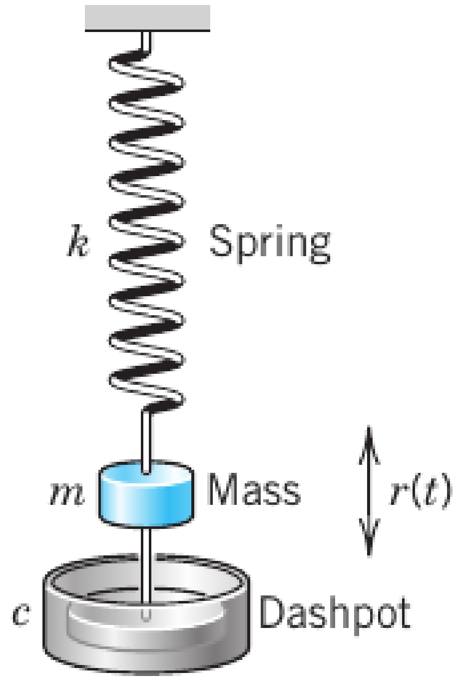
find y_p by matching cos(x), sin(x), constant & x terms in LHS & RHS.

$$y'' + 5y' + 4y = 10e^{-3x}$$

$$y'' + 3y' + 2y = 12x^2$$

$$y'' - 9y = 18 \cos \pi x$$

2.8 Modeling: Forced Oscillations



$$my'' + cy' + ky = r(t).$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the nonhomogeneous ODE

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2}$$

Its solution will reveal facts that are fundamental in engineering mathematics and allow us to model resonance.