Last time:

2nd order linear ODE with constant coeff

$$ay'' + by' + cy = r(x)$$

→ we have learned how to solve a homogeneous 2nd order ODE

$$ay'' + by' + cy = 0$$

steps for solving $ay'' + by' + cy = 0$

$\longrightarrow$ auxilliary eqn

$$a\lambda^2 + b\lambda + c = 0$$

real distinct roots	real & repeated roots	roots are complex
$\lambda_1 \quad \lambda_2$	λ	$\lambda = \alpha \pm i\beta$
$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Initial Value Problem in the Case of Distinct Real Roots

Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad \underbrace{y(0) = 4, \qquad y'(0) = -5.}$$

use these to get c_1, c_2

aux eqn: $d^2 + d - 2 = 0$

$(d+2)(d-1) = 0$

$d = -2, 1$

general soln: $y = c_1 e^{-2x} + c_2 e^{x}$

$y' = -2c_1 e^{-2x} + c_2 e^{x}$

$c_1 + c_2 = 4$

$-2c_1 + c_2 = -5$

$c_1 = 3$

$c_2 = 1$

→ Particular soln:

$$\boxed{y = 3e^{-2x} + e^{x}}$$

Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

$\lambda^2 + \lambda + \frac{1}{4} = 0$

$4\lambda^2 + 4\lambda + 1 = 0$

$(2\lambda + 1)^2 = 0$

$\lambda = -\frac{1}{2}$

general solⁿ:

$y = c_1 e^{-x/2} + c_2 x e^{-x/2} \Big|_{x=0} = 3$

$y' = -\frac{c_1}{2} e^{-x/2} + c_2 \left[-\frac{x}{2} e^{-x/2} + e^{-x/2} \right] \Big|_{x=0} = -3.5$

$c_1 = 3$

$c_2 = -2$

Complex Roots. Initial Value Problem

Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

$$\lambda = \frac{-0.4 \pm \sqrt{(0.4)^2 - 4\cdot(9.04)}}{2} = \frac{-0.4 \pm \sqrt{-36}}{2}$$

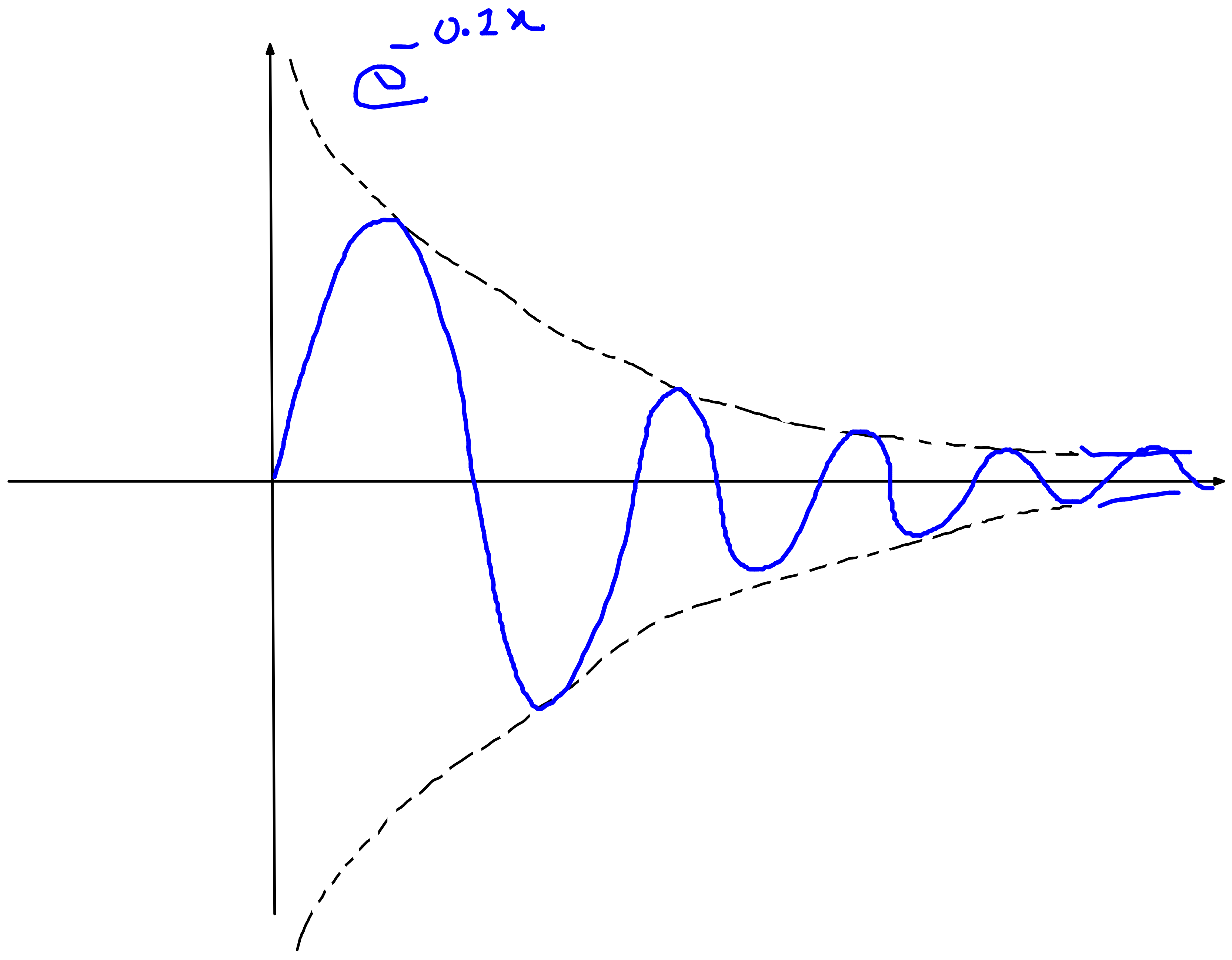
$$= -0.2 \pm 3i$$

$$y = e^{-0.2x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

$$y(0) = 0 \rightarrow c_1 = 0$$

$$y'(0) = 3 \rightarrow 3c_2 - 0.2c_1 = 3 \Big] \Rightarrow c_2 = 1$$

$$y = \underbrace{e^{-0.2x}} \sin 3x \qquad \text{graph ??}$$



Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

2.4 Modeling of Free Oscillations of a Mass–Spring System

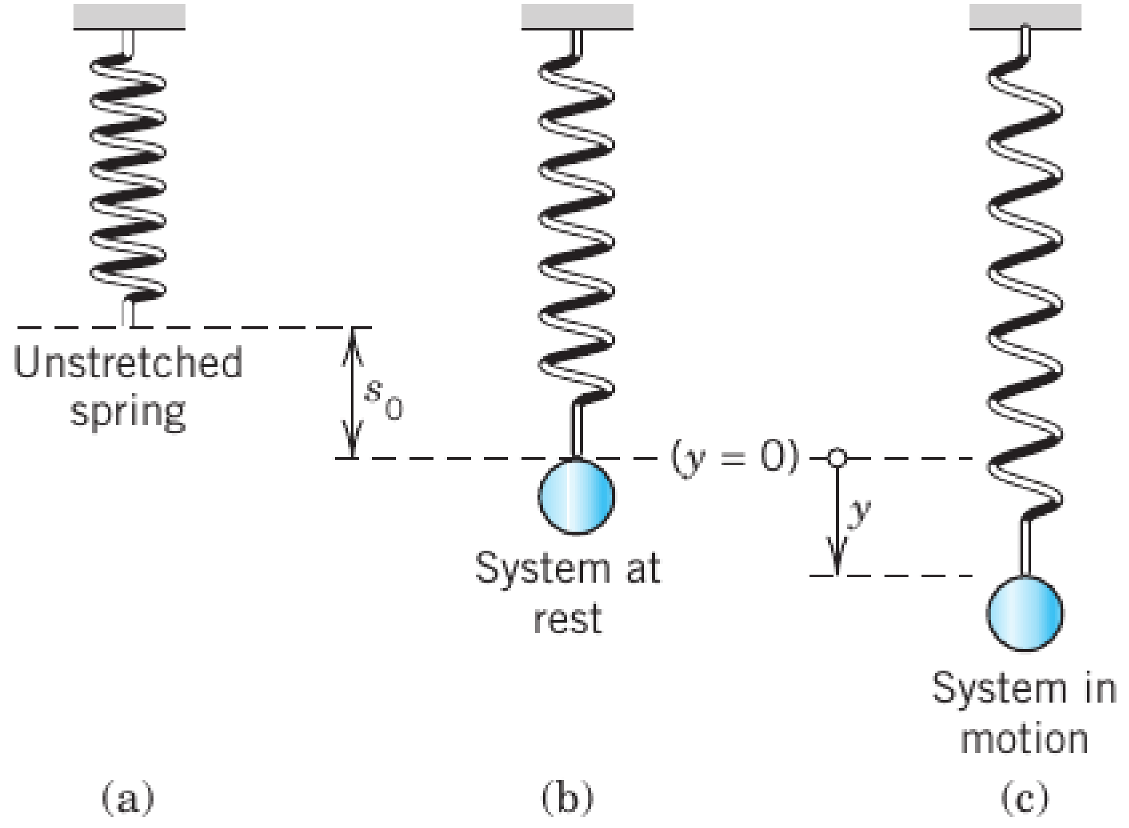


Fig. 33. Mechanical mass–spring system

Detailed application & analysis of mechanical vibration using 2nd order ODEs

2.4 Modeling of Free Oscillations of a Mass–Spring System

↓ downward direction is positive dir

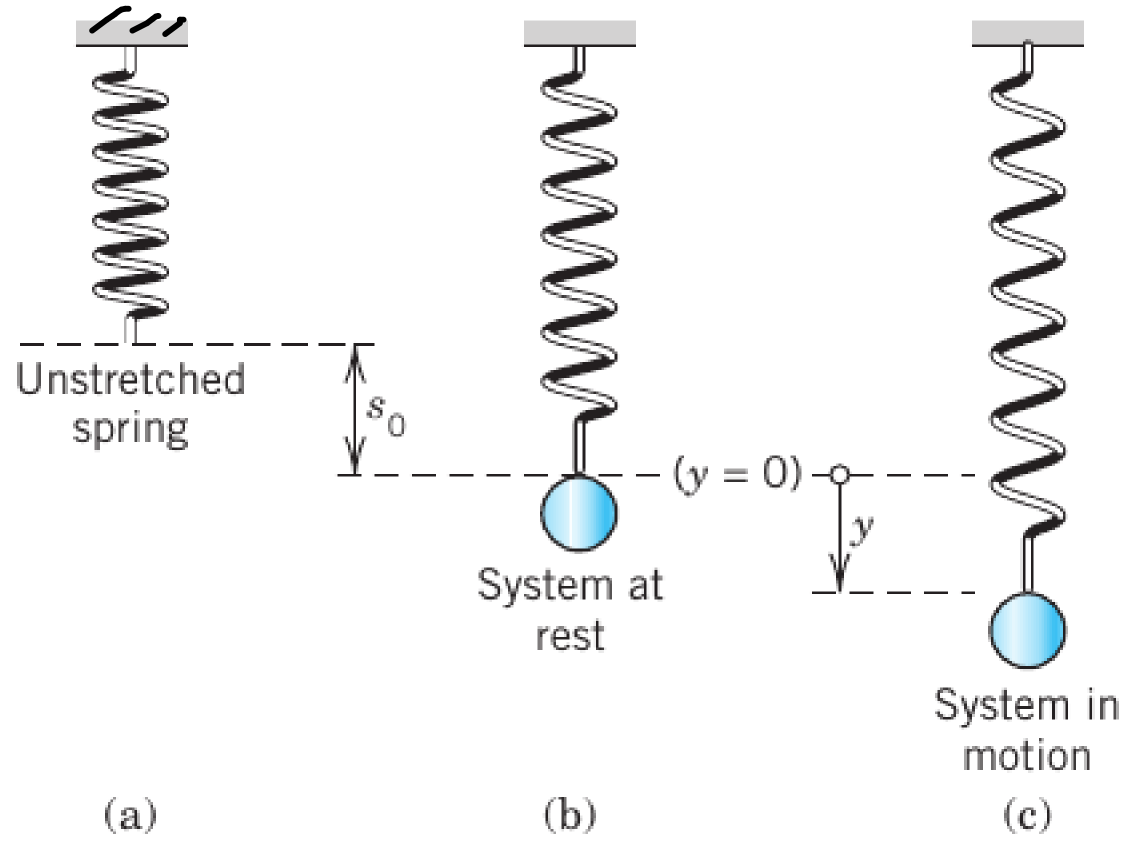


Fig. 33. Mechanical mass–spring system

$y(t)$: position of the body from equilibrium position at time t

↓ eqn for $y(t)$

$m\dfrac{d^2y}{dt^2}$ = net force on the body at time t

$m\dfrac{d^2y}{dt^2}$ = gravity + spring force + damping

$$m\frac{d^2y}{dt^2} = mg - k(y+s_0) - c\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = \underbrace{mg - kS_0}_{=0} \quad \text{why ??}$$

S_0 is the equilibrium strech

we see that $y(t)$ satisfied a homogenous 2nd order ODE

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0}$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

how much minimum damping you need to avoid oscillation

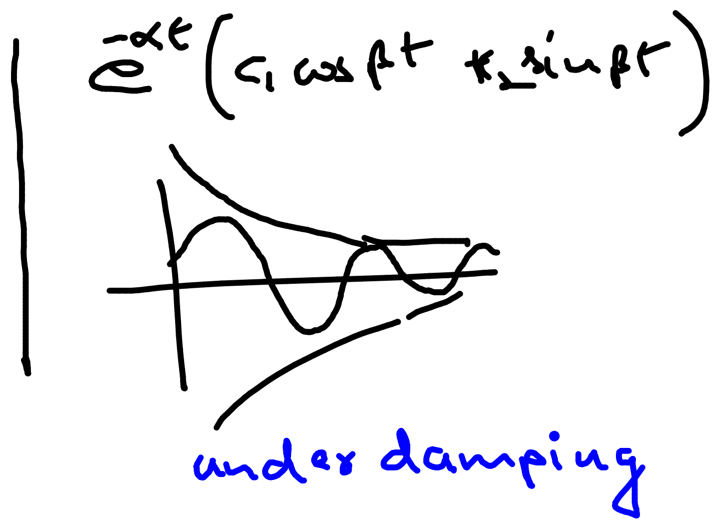
→ aux eqn: $m\alpha^2 + c\alpha + K = 0$

$$\alpha = \frac{-c \pm \sqrt{c^2 - 4mK}}{2m}$$

$c^2 > 4mK$

if $c^2 < 4mK$

critical damping



Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Q: is it over damped / underdamped / critical damped case

$m = 1 \qquad c = 1 \qquad k = 0.25$

$c^2 = 4mk$ critical damping

Q: verify the behaviour in the simulation app.

EXAMPLE 2 **The Three Cases of Damped Motion**