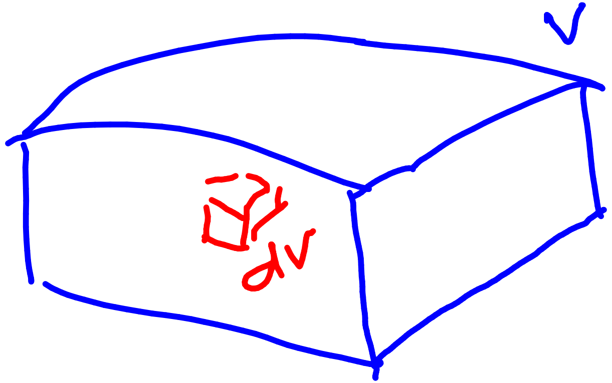
12.5 TRIPLE INTEGRALS



$\rho(x,y,z)$: density at point (x,y,z)

$$\iiint_V \underbrace{\rho(x,y,z)\, dV} \qquad dm = \rho\, dV = \text{local mass}$$

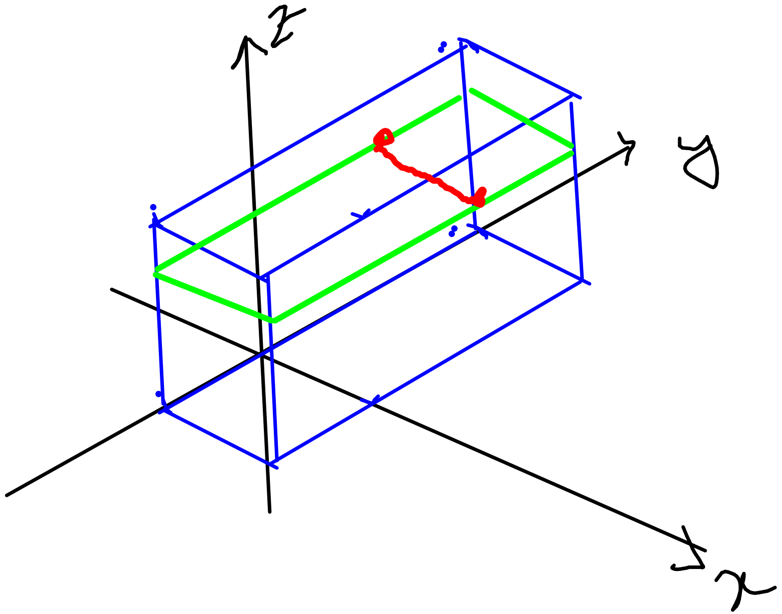
$$= \iiint_V dm = \text{total mass}$$

<u>Note</u>: Do 12.4 by yourself (Don't skip)

V EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

Sketch the domain



$$= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$$

start integrating from inner most integration

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} yz^2\,dy\,dz$$

V EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, \; -1 \leqslant y \leqslant 2, \; 0 \leqslant z \leqslant 3\}$$

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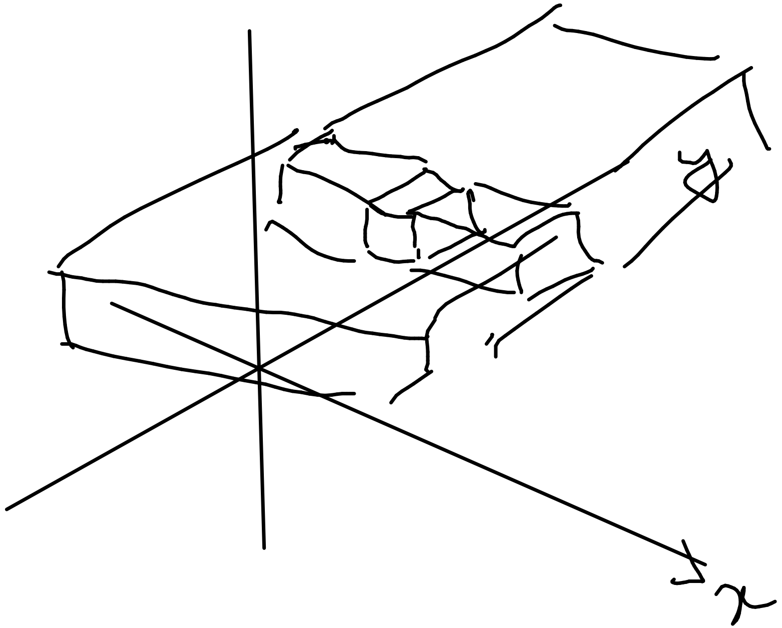

$$= \int_0^3 \int_{-1}^2 \left(\int_0^1 \underbrace{xyz^2 \, dx \Big) \, dy\,dz}$$

$$= \int_0^3 \int_{-1}^2 \underbrace{\left(\int_0^1 xyz^2 \, dx \right) \, dy\,dz}$$

$$= \frac{3}{4}\int_0^3 z^2\,dz = \frac{\cancel{3}}{4}\cdot\frac{1}{\cancel{3}}\cdot 27 = \frac{27}{4}$$

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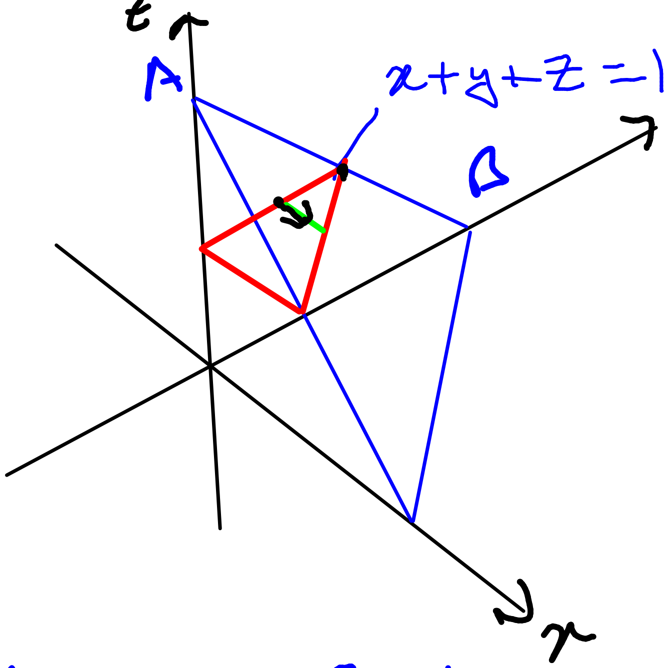
EXAMPLE 2 Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Sketch the region E



$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} z \; dx\, dy\, dz$$

$$= \int_?^? \int_?^? \int_?^? z \; dz\, dy\, dx$$

AB: $y+z=1$

$x=0$

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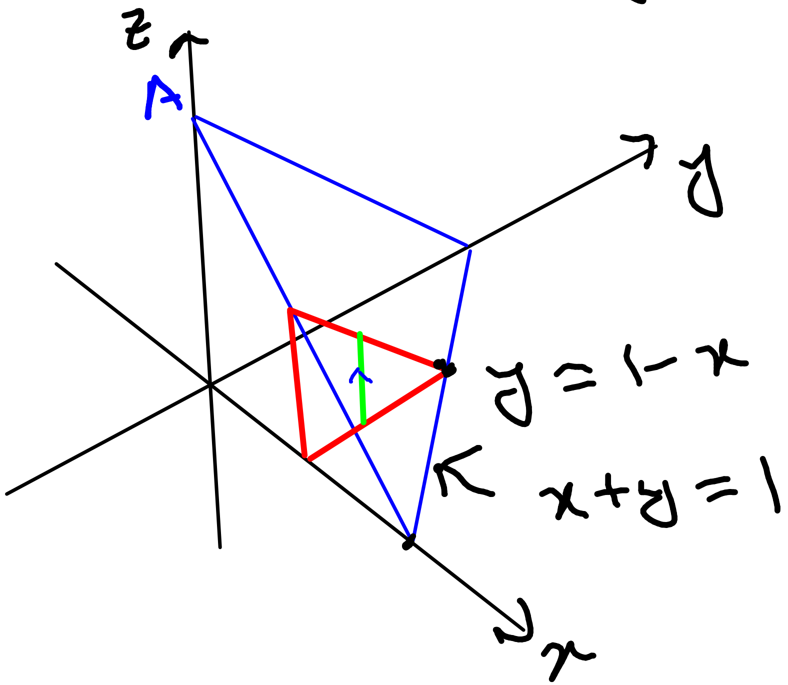
Sketch the region E



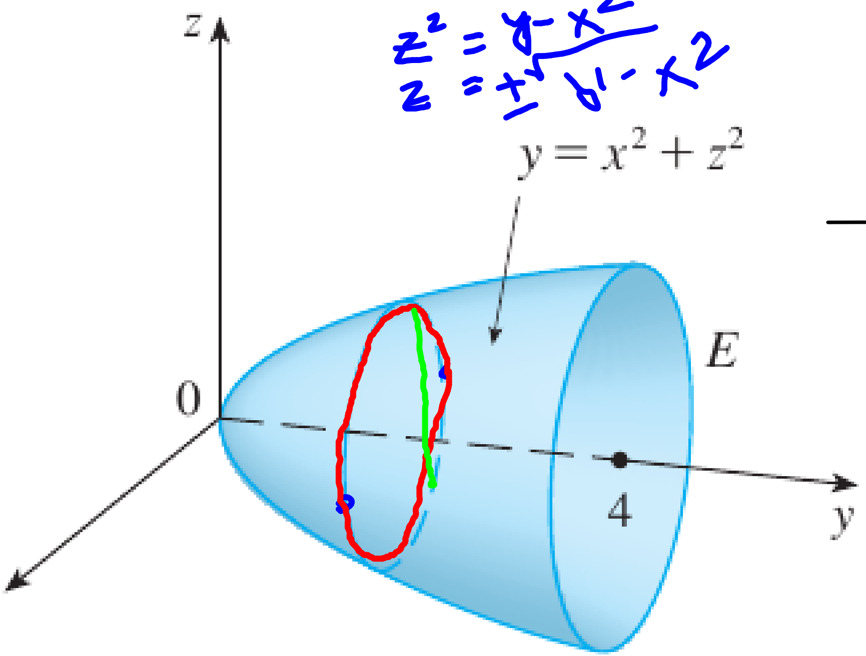
$$= \int_0^1 \int_0^{1-x} \int_0^{1-y-x} z \; dz\,dy\,dx$$

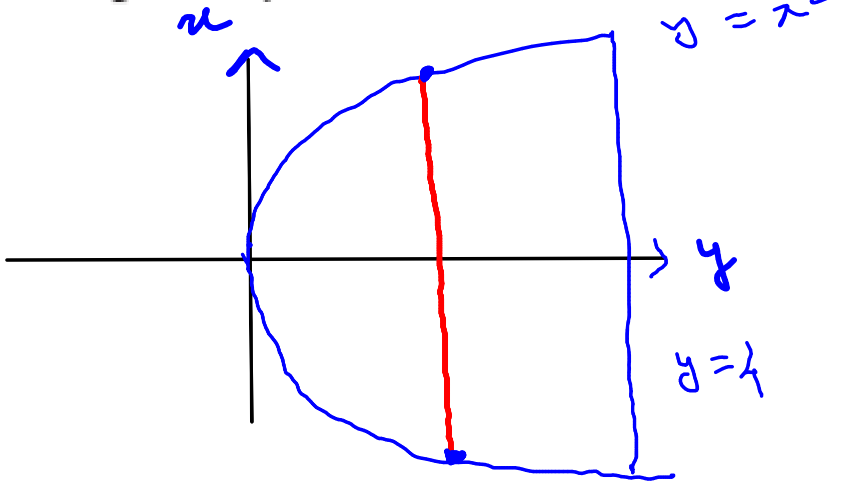
$$= \frac{1}{24}$$

V EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

$z^2 = y - x^2$

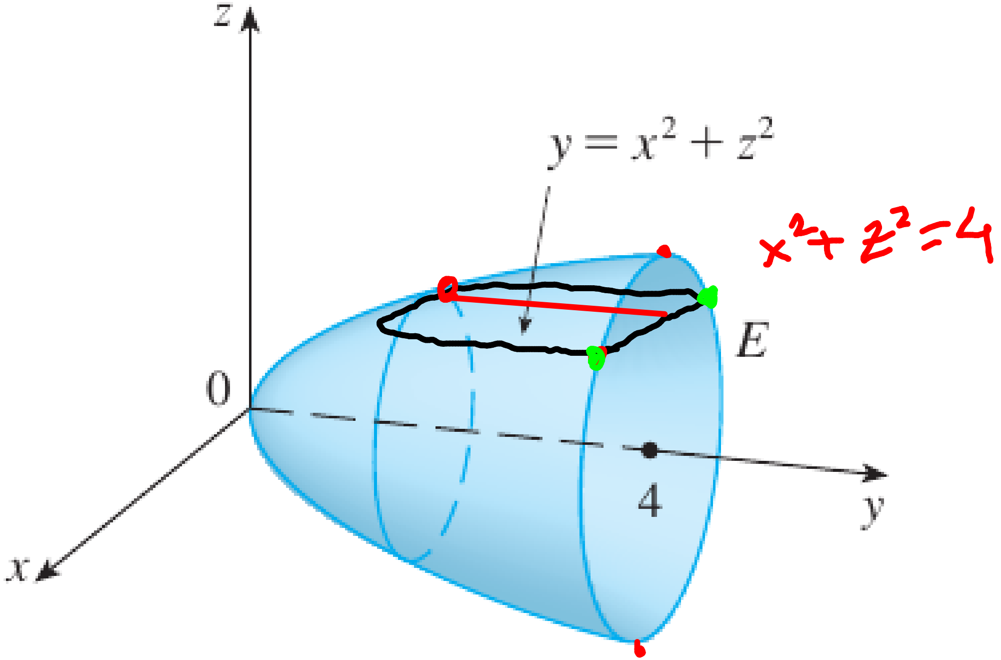
$z = \pm\sqrt{y - x^2}$

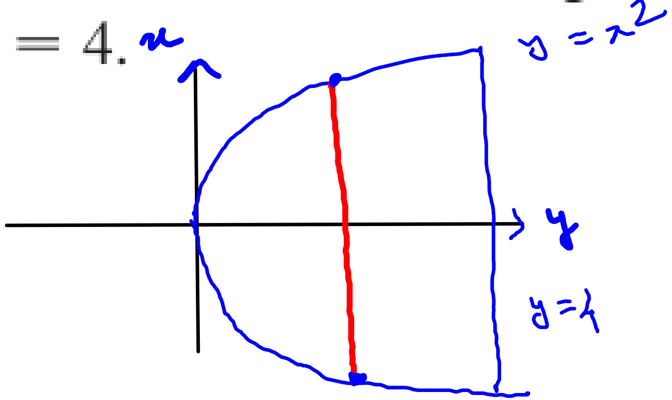




$$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\; dz\, dx\, dy$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.



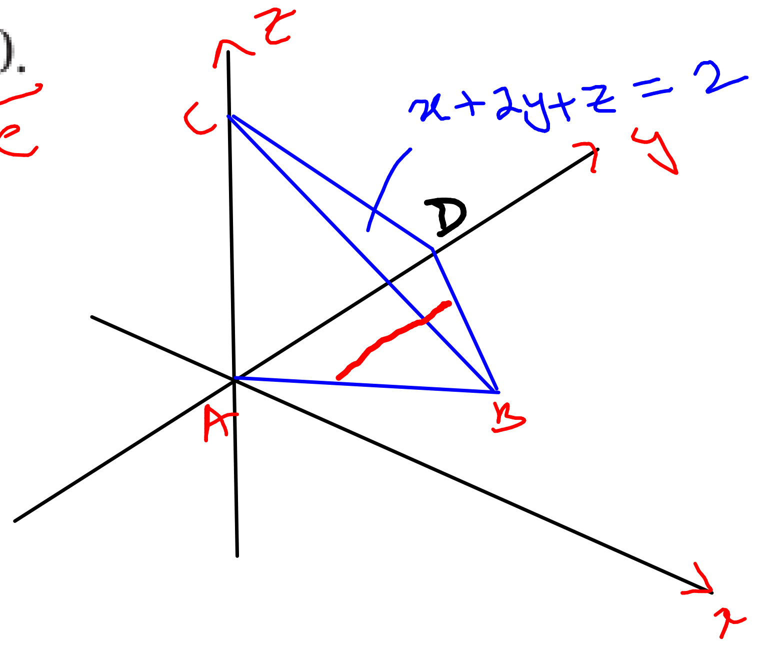


$$= \int_{-2}^{2} \int_{-\sqrt{4-z}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\, dx\, dz$$

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $\underbrace{z = 0}_{\text{base}}$.

Volume $= \iiint_T dV$

sketch these planes & the enclosed volume



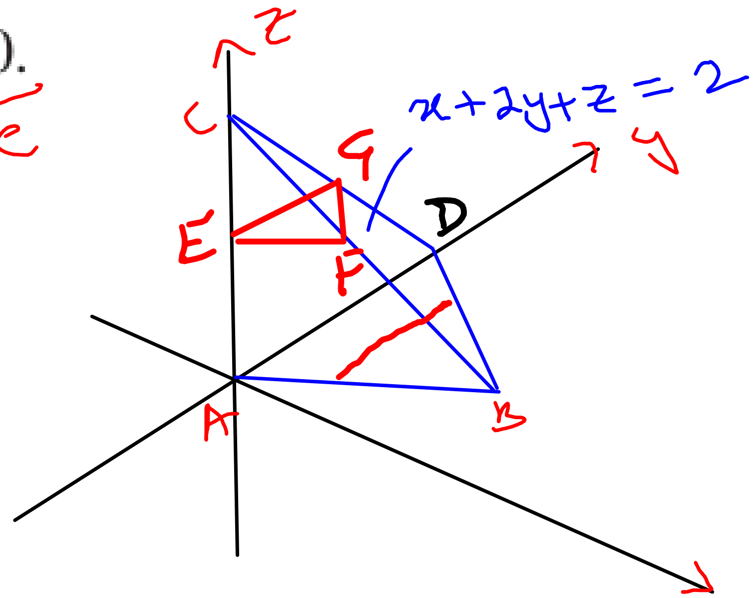
$$\int_0^1 \int_{x/2}^{\left(\frac{2-x}{2}\right)} \int_0^{(2-x-2y)} dz\, dy\, dx$$

$$= \frac{1}{3}$$

ABC : $x = 2y$

D $\left|\begin{array}{l} x + 2y = 2 \\ x = 2y \\ y = 1/2 \ , \ x = 1 \end{array}\right.$

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $\underbrace{z = 0}_{\text{base}}$.

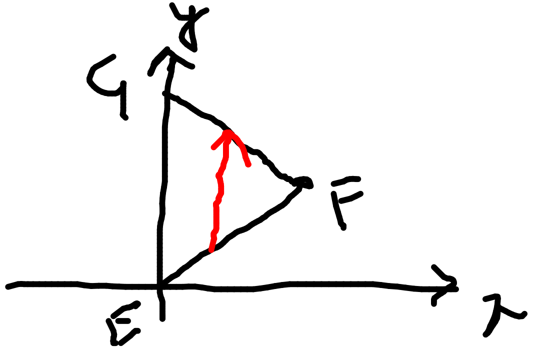


Volume $= \iiint_V dv$

sketch these planes & the enclosed volume

$$\int_0^2 \int_0^{1-\frac{z}{2}} \int_{\frac{x}{2}}^{(2-x-z)/2} dy\,dx\,dz$$

$= 1/3$



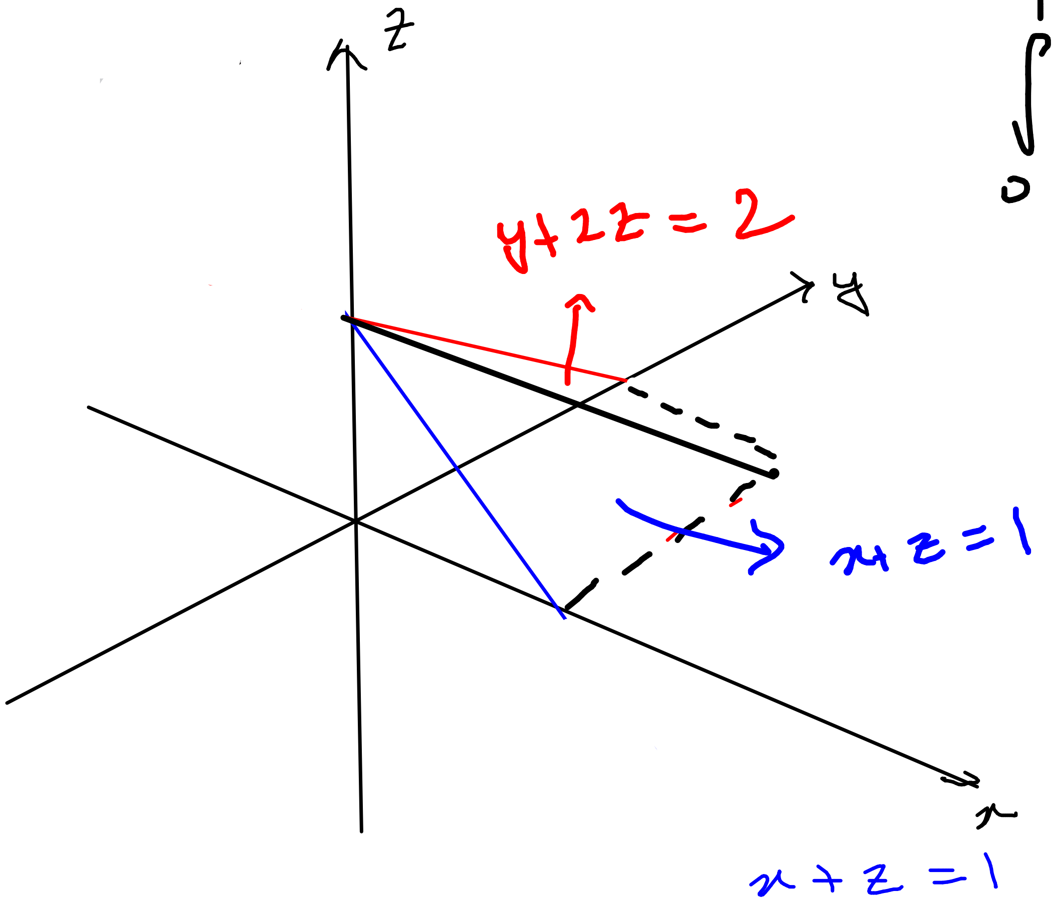
F : lies on the planes $x = 2y$, $x + 2y + z = 2$

$$x + 2 \cdot \frac{x}{2} + z = 2$$

$$2x + z = 2$$

$$x = \frac{(2 - z)}{2}$$

25–26 ▪ Sketch the solid whose volume is given by the iterated integral.



$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$$

$y + 2z = 2$

$\frac{y}{2} + z = 1$

Simply plot the surfaces

$x = 0$

$x = 1$

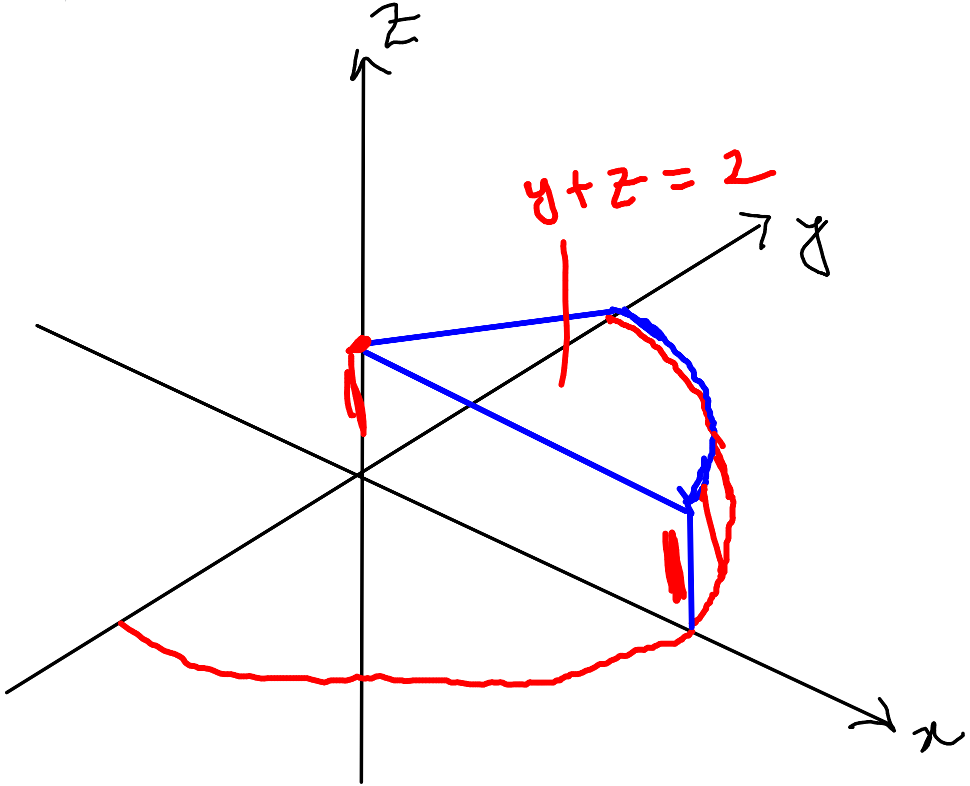
$z = 0$

$z = 1 - x$

$y = 0$

$y = 2 - 2z$

25–26 ▪ Sketch the solid whose volume is given by the iterated integral.



$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$$

$z = 2-y$

$y+z=2$

$x = 4-y^2$

32. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.