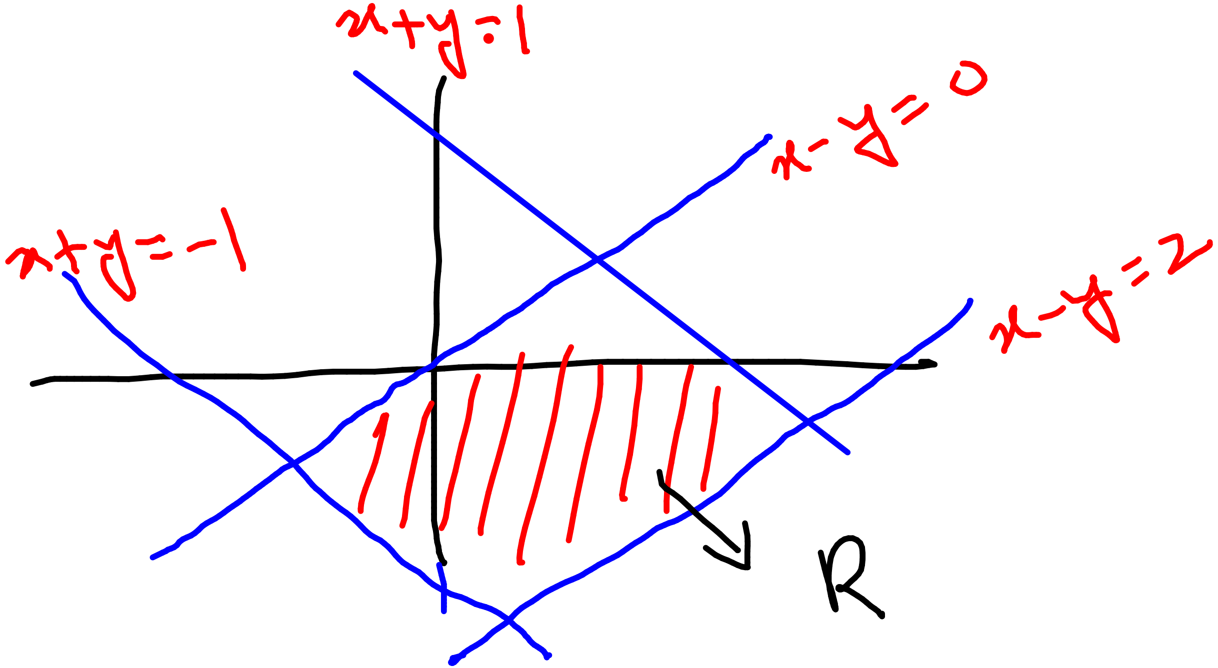
12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

Analog of u subsitution for multivariable integration.

Suppose our region of integration is a parallelogram like

$-1 \le x+y \le 1$, $0 \le x-y \le 2$



$$u = x+y \qquad x = (u+v)/2$$
$$v = x-y \qquad y = (u-v)/2$$

$$\int_0^2 \int_{-1}^1 ???? \, du \, dv$$

area of region R

$$= \iint_R 1\,dx\,dy = \int_0^2 \int_{-1}^1 (\text{Jacobian})\,du\,dv$$

$$= \int_0^2 \int_{-1}^1 \frac{1}{2}\,du\,dv = 2$$

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \begin{vmatrix} 1/2 & 1/2 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left| -\frac{1}{2} \right| = \frac{1}{2}$$

$$\partial(x,y) = \frac{1}{2}\,\partial(u,v)$$

$$dx\,dy = \frac{1}{2}\,du\,dv$$

7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Similarly (in 3d)

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

9 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \underbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}_{\text{Jacobian}} du\, dv$$

19. $\displaystyle\iint_R \frac{x - 2y}{3x - y}\, dA$, where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

15. $\iint_R xy\,dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $\quad x = u/v, \; y = v$