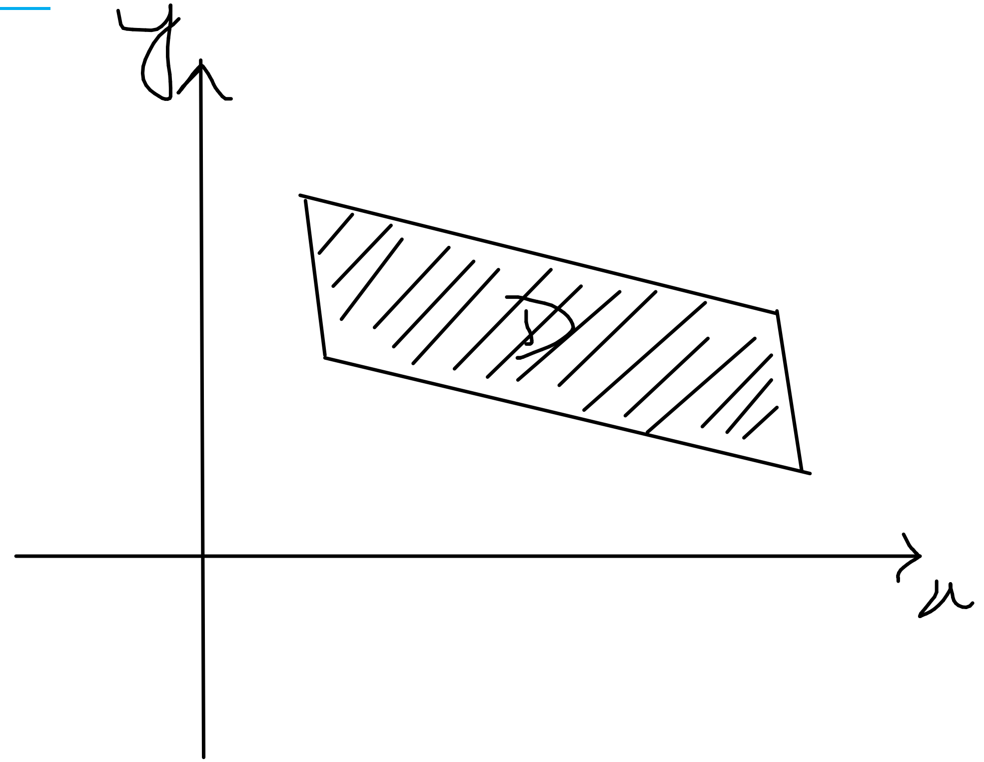
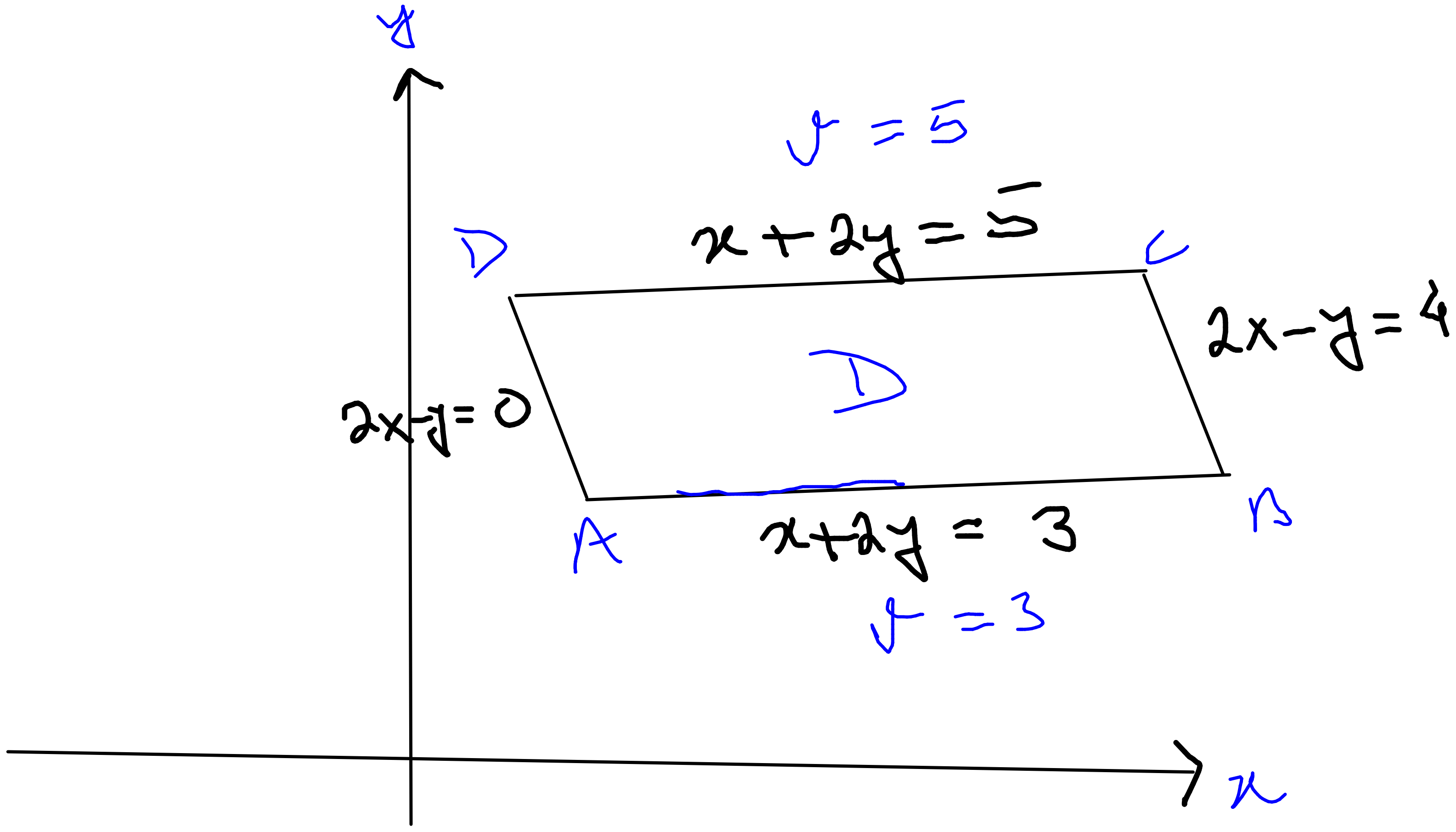
12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

→ $\iint_D f(x,y)\, dA$ — given



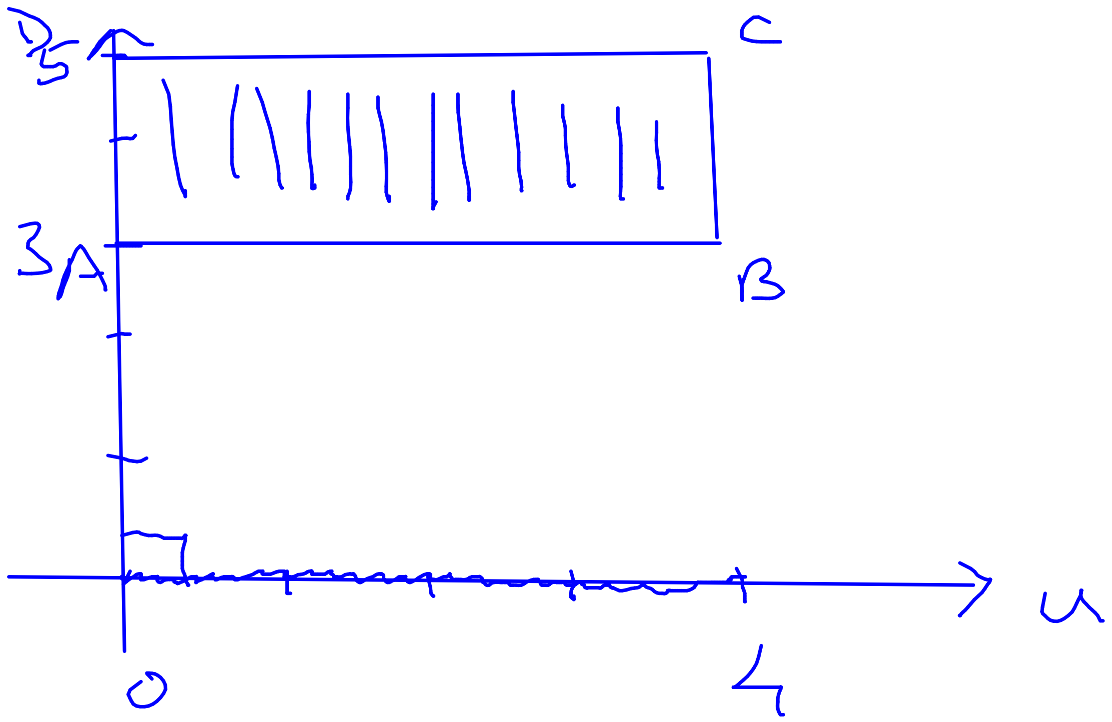
→ D : will be mildly complicated

→ D : will be simplified with change of variables



$u = 2x - y$

$v = x + 2y$



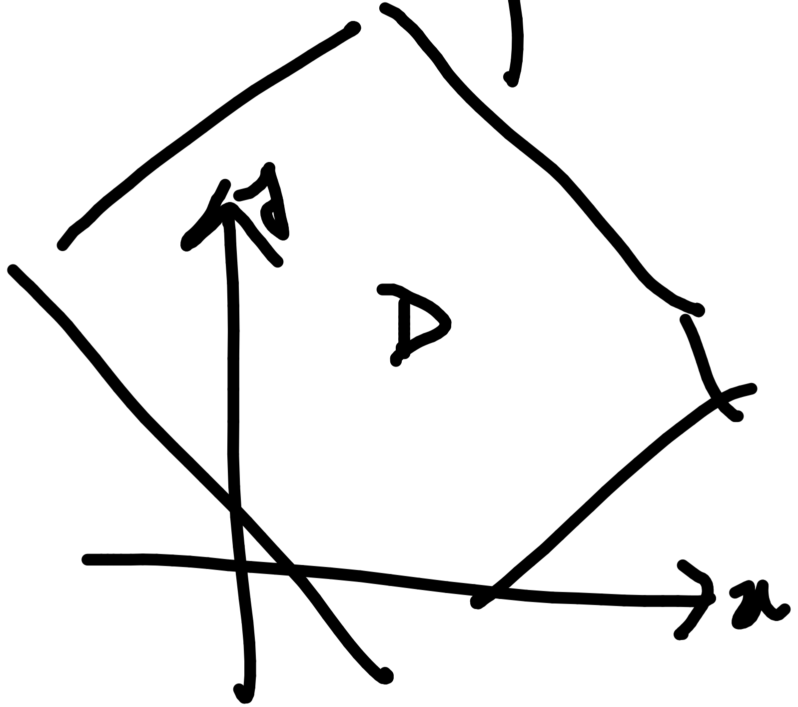
Find the Jacobian of the transformation.

$$x = u + 4v, \quad y = 3u - 2v$$

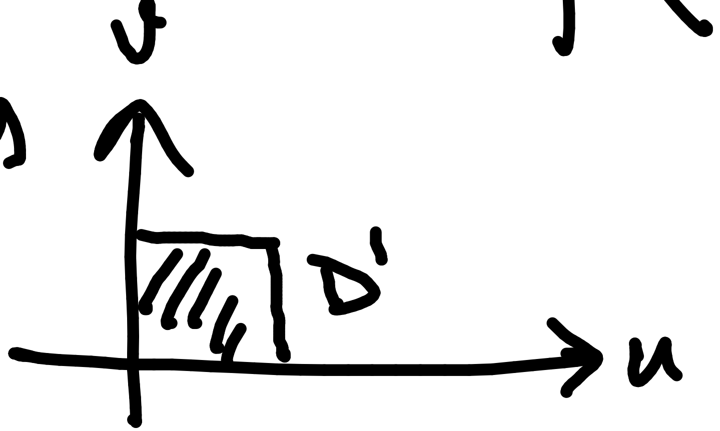
$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \text{abs}\left[\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right]$$

$$= \text{abs}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = |-14| = 14$$

$$\iint_D f(x,y)\,dx\,dy$$



D' ≈ 14 times smaller than D.

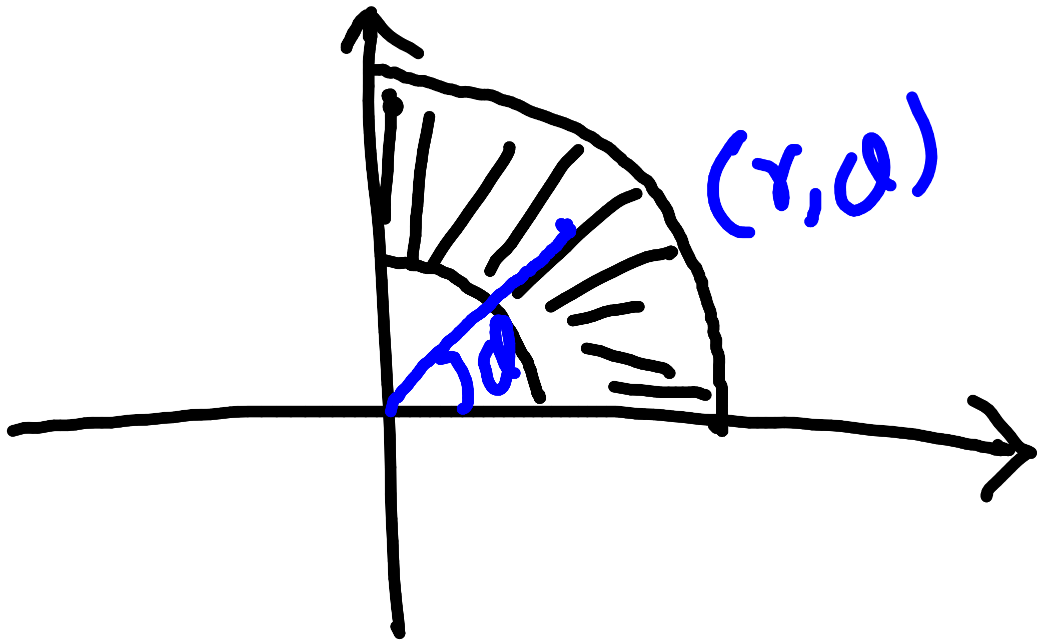


$$\iint$$

Find the Jacobian of the transformation.

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$



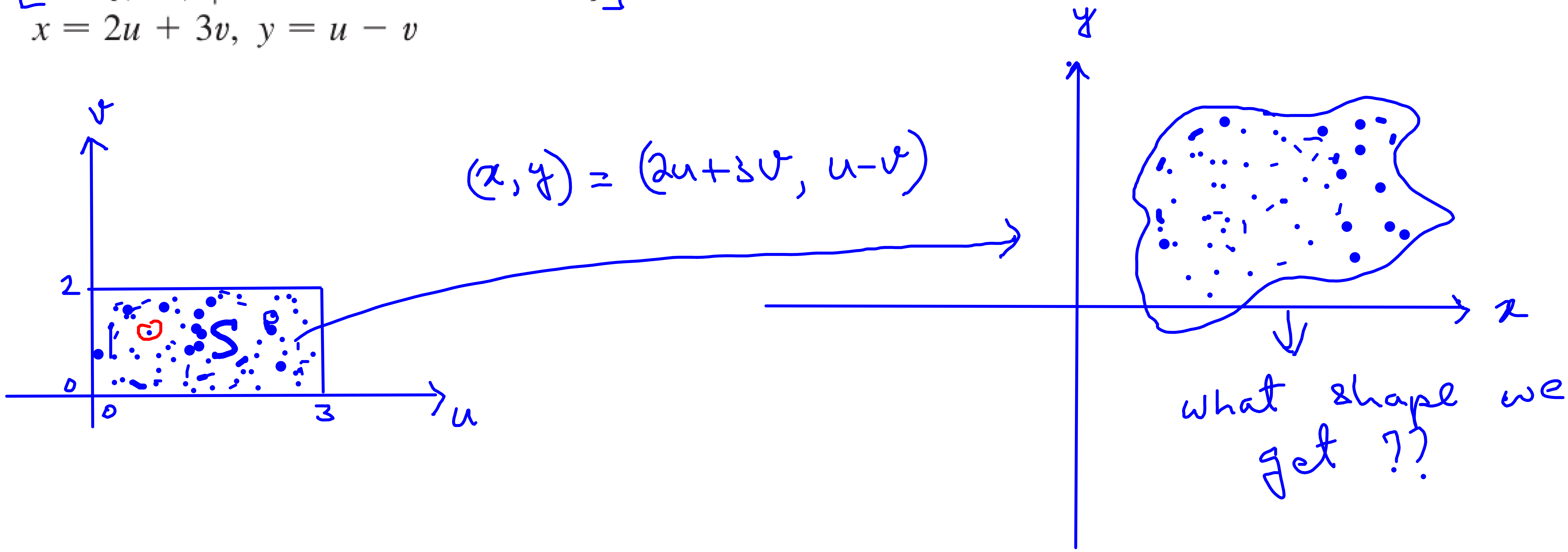
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \underline{\underline{r}}$$

$$dx\,dy = r\,dr\,d\theta$$

Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; ?? shape??

$x = 2u + 3v,\ y = u - v$



Find the image of the set S under the given transformation.

$\boxed{S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};}$?? shape??

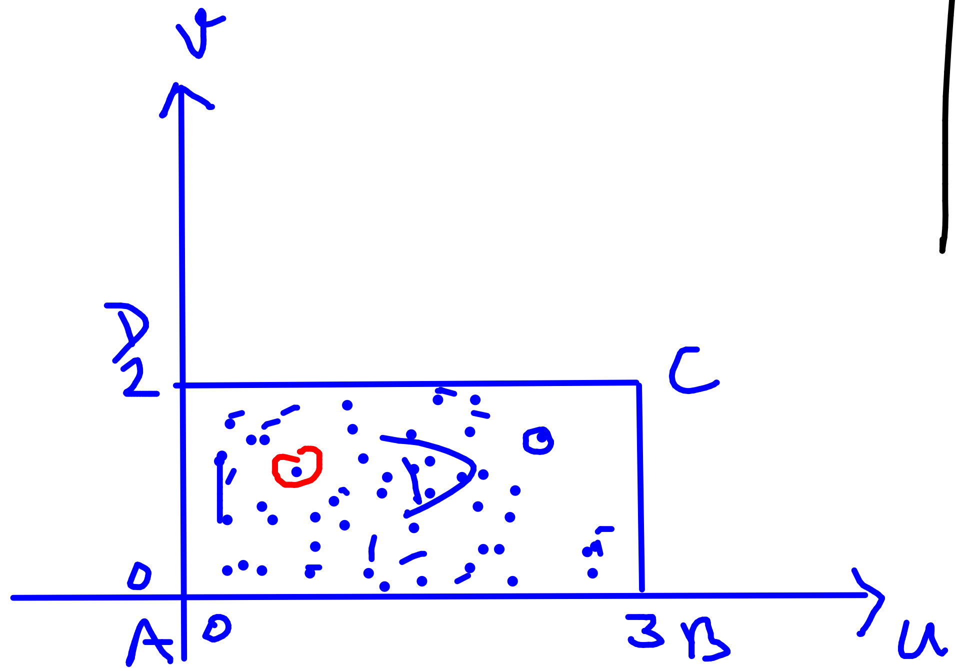
$x = 2u + 3v,\ y = u - v$

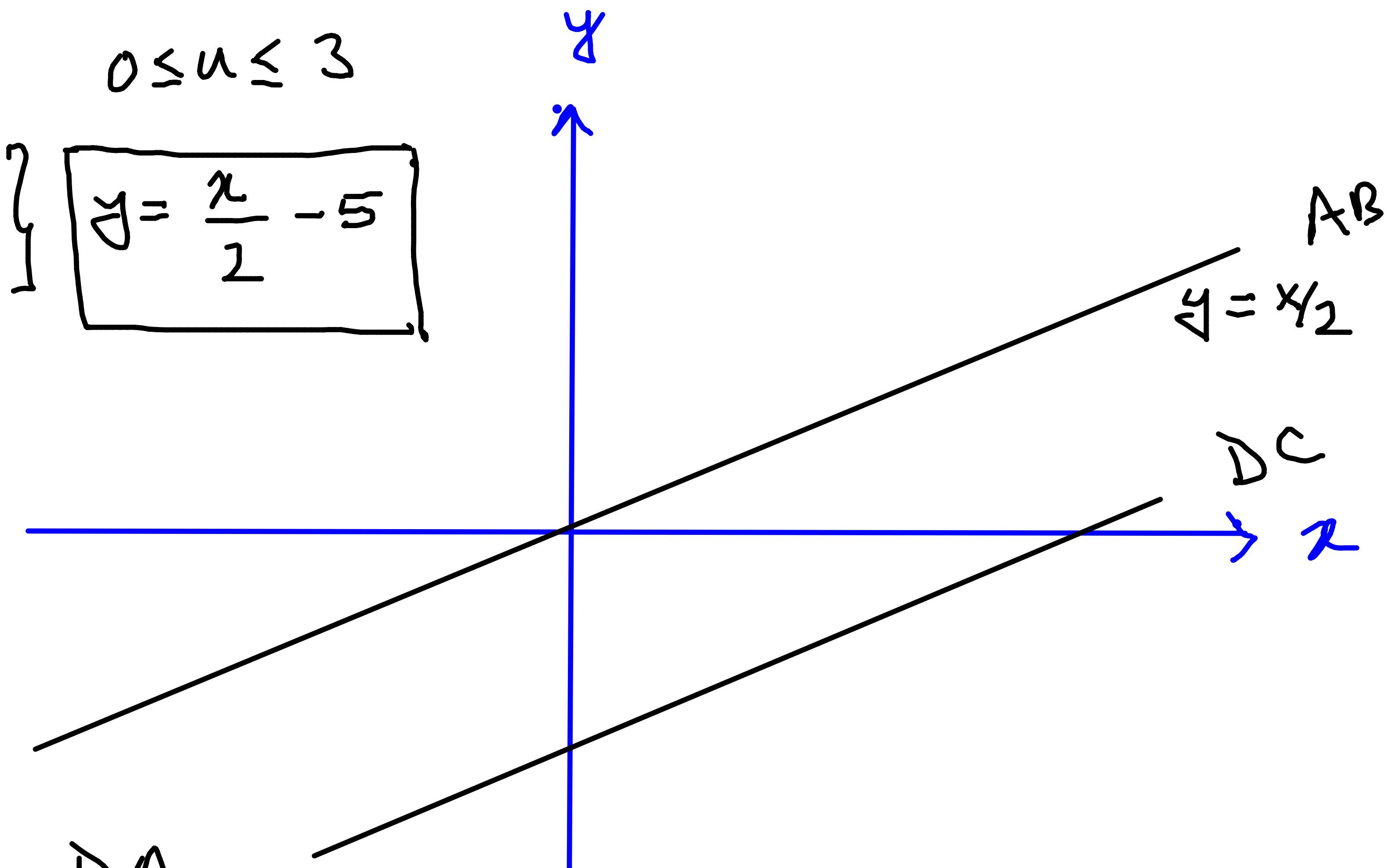
$AB,\ v = 0,\ 0 \le u \le 3$

$\left.\begin{array}{l} x = 2u \\ y = u \end{array}\right\}$ $x = 2y$

$DC,\ v = 2 \quad 0 \le u \le 3$

$\left.\begin{array}{l} x = 2u + 6 \\ y = u - 2 \end{array}\right\}$ $\boxed{y = \frac{x}{2} - 5}$





strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given transformation.

$\boxed{S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};}$?? shape??

$x = 2u + 3v,\ y = u - v$

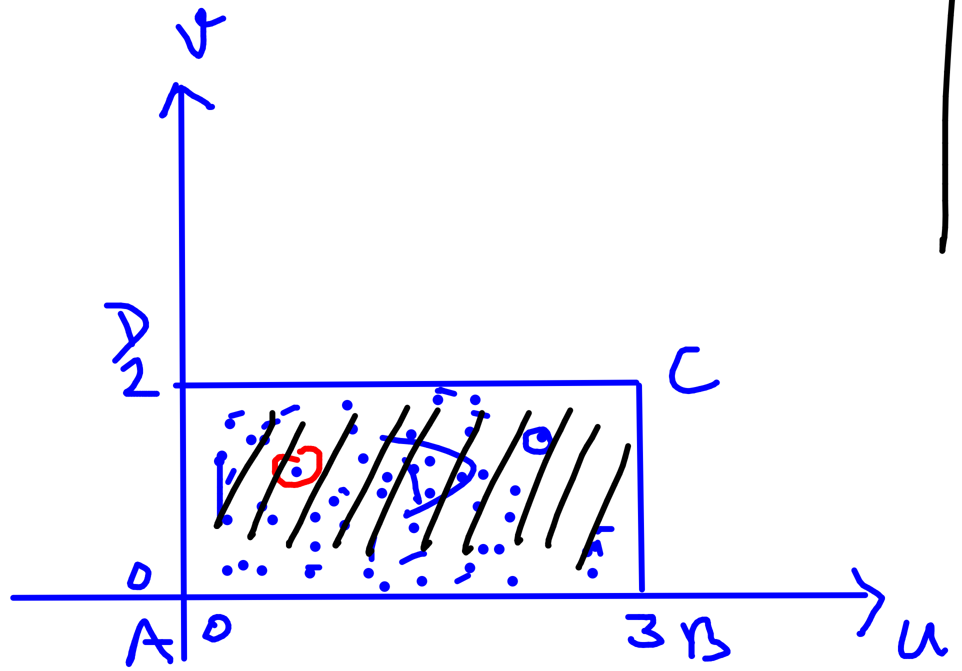
AD : $u = 0$, $0 \le v \le 2$

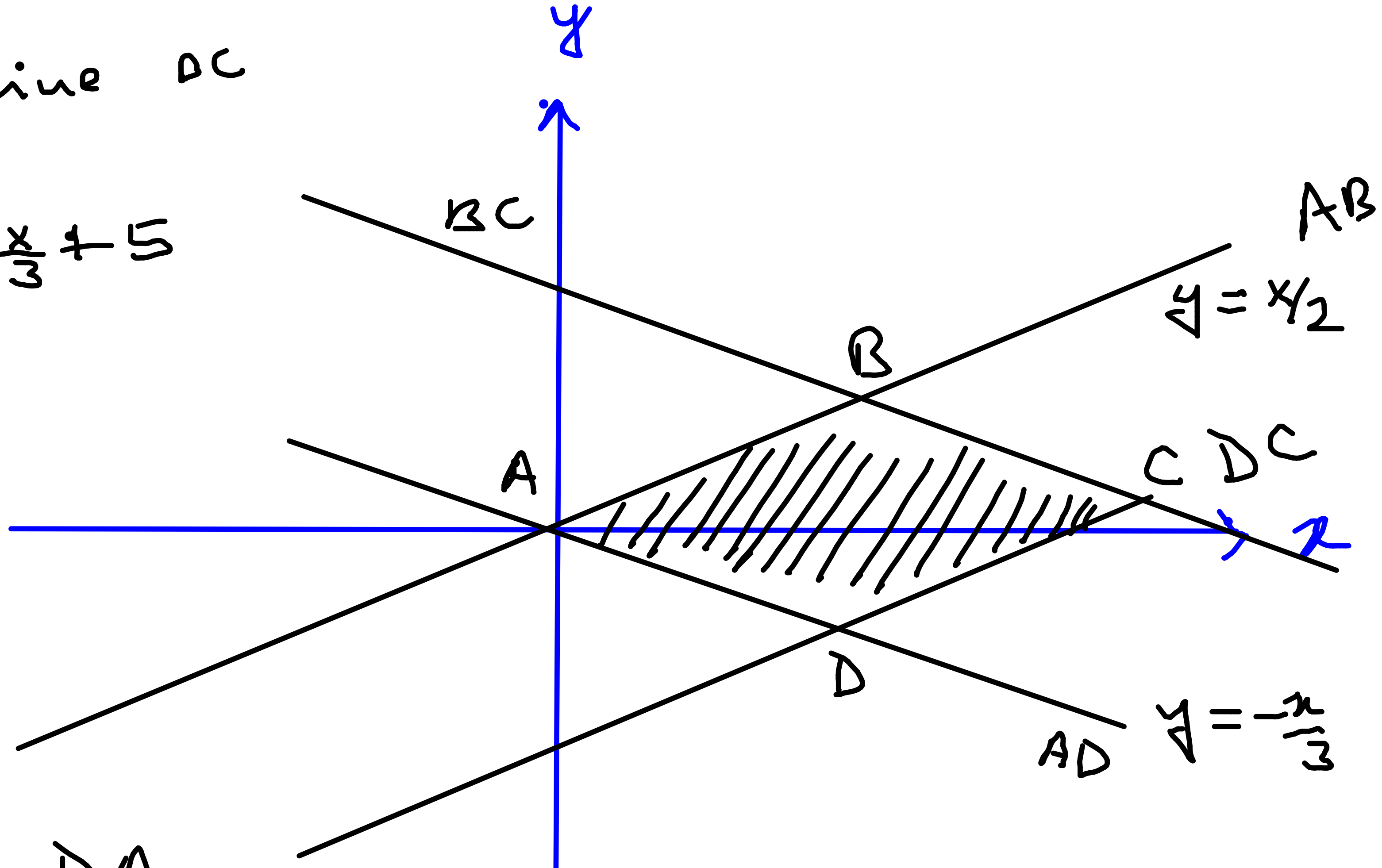
$x = 3v$, $y = -v$ $\quad x = -3y$, $y = -\frac{1}{3}x$

similarly: line DC

$x + 3y = 15$

$y = -\frac{x}{3} + 5$





strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ y = u - v$$

Find the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

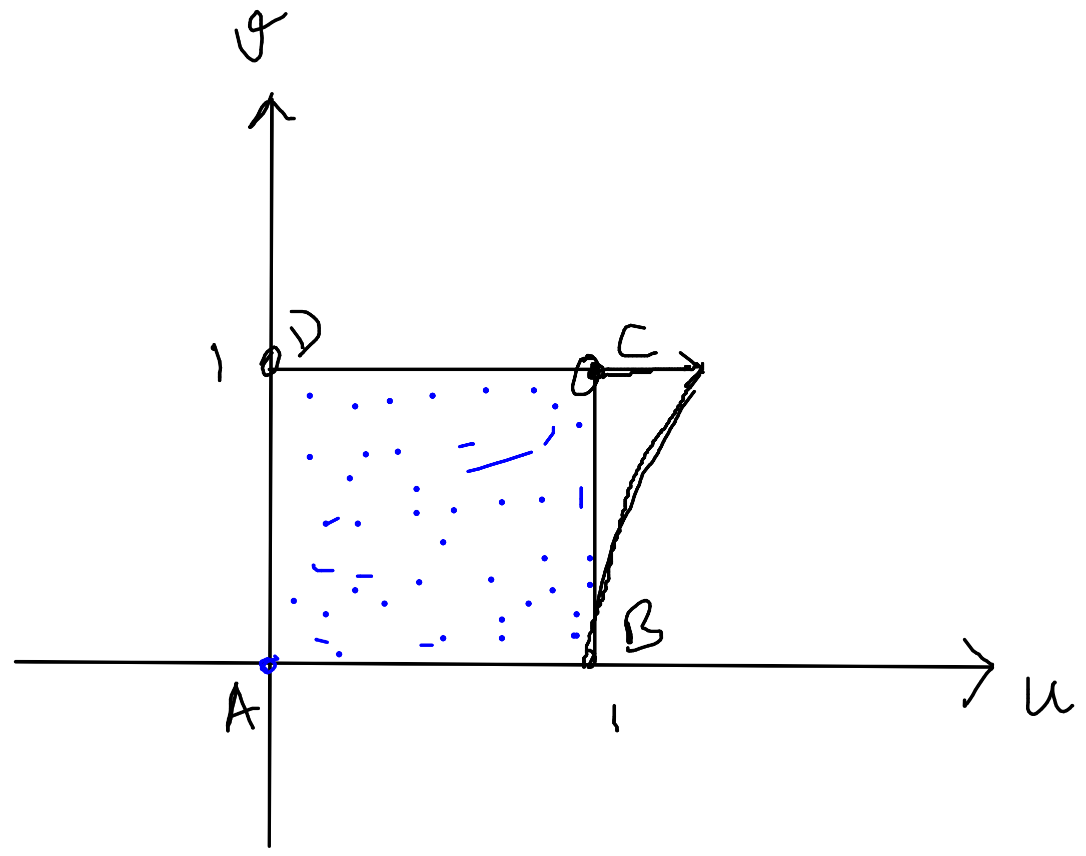
$$dx\,dy = J\ du\,dv$$

$$= \left| \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \right| = |-5|$$

$$= 5$$

Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v, \; y = u(1 + v^2)$



AB

$v = 0$

$\left.\begin{array}{l} x = 0 \\ y = u \end{array}\right\} \quad \begin{array}{l} x = 0 \\ 0 \le y \le 1 \end{array}$

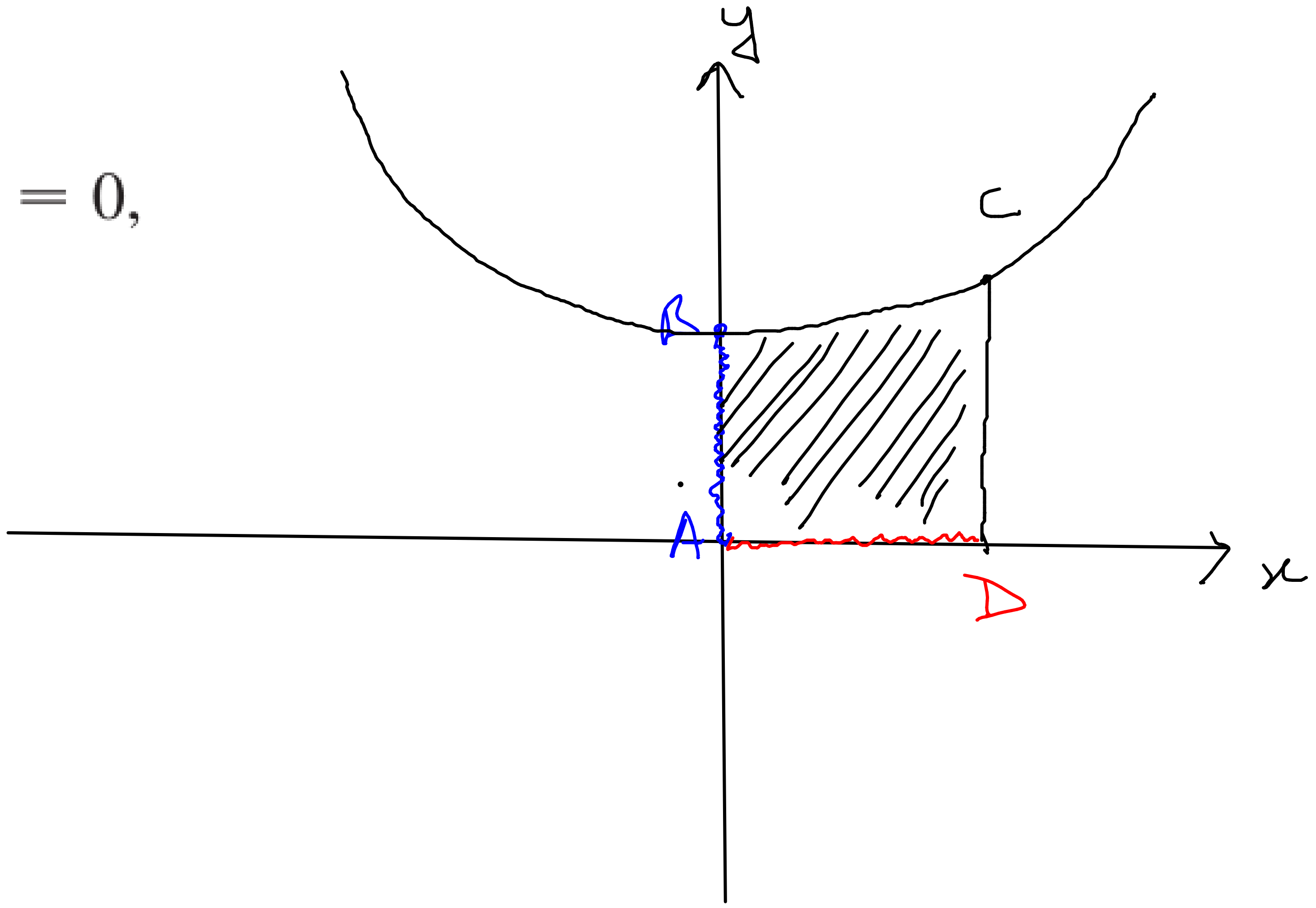
AD $\quad u = 0 \quad 0 \le v \le 1$

$\left.\begin{array}{l} x = v \\ y = 0 \end{array}\right\} \quad \begin{array}{l} y = 0 \\ 0 \le x \le 1 \end{array}$

BC : $\quad u = 1 \; , \; 0 \le v \le 1$

$\left.\begin{array}{l} x = v \\ y = 1 + v^2 \end{array}\right\} \quad \begin{array}{l} y = 1 + x^2 \\ 0 \le x \le 1 \end{array}$

DC



Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

Find the Jacobian

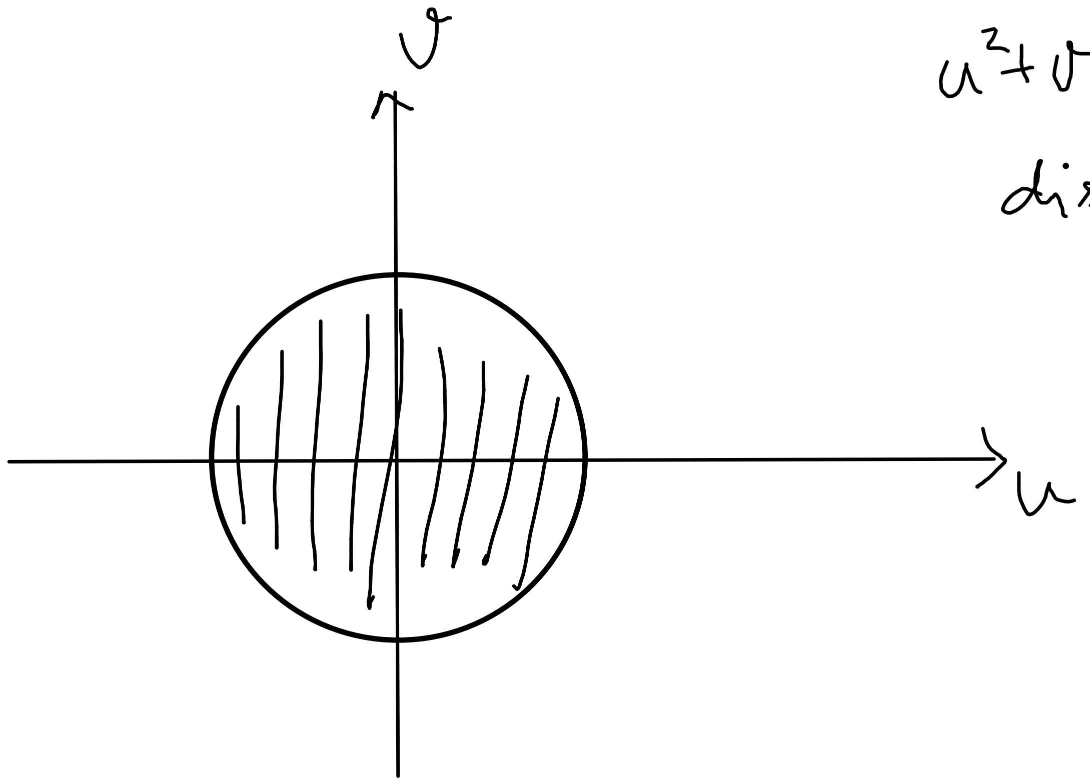
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} =$$

$$= 1+v^2$$

Find the image of the set S under the given transformation.

$$S \text{ is the disk given by } u^2 + v^2 \leqslant 1; \quad x = au, \; y = bv$$

for simplicity, assume $a = 3$, $b = 2$



$u^2+v^2 \leq 1$

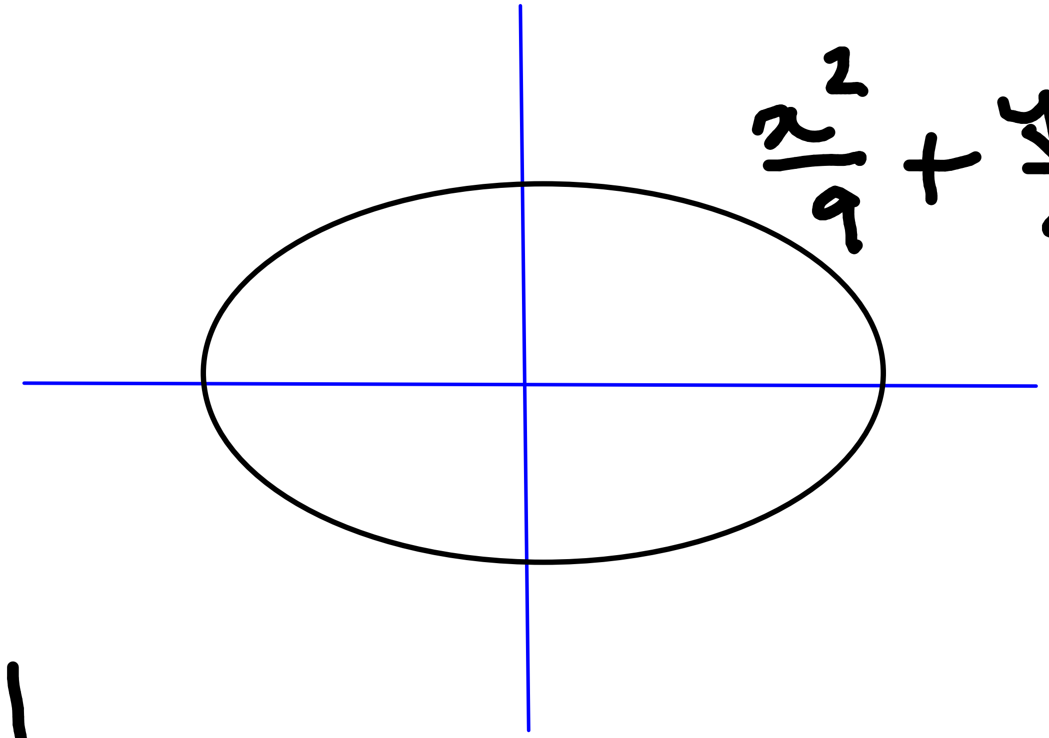
disc

$x = 3u$

$y = 2v$

$u^2+v^2 = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

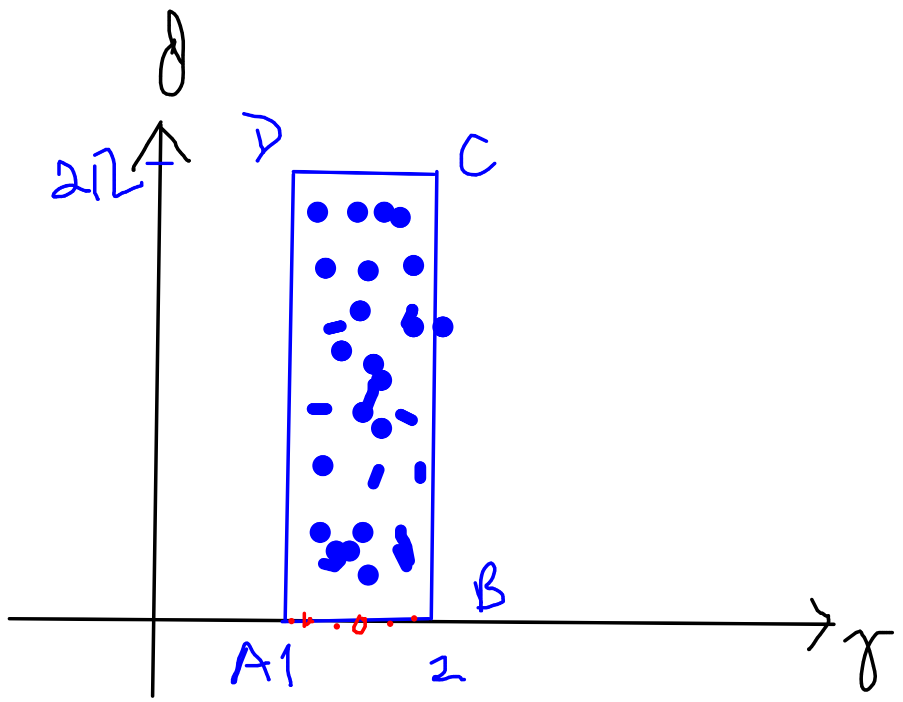


$\frac{x^2}{9} + \frac{y^2}{4} = 1$

Q. Find the image of S under the given transformation.

$$S = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$



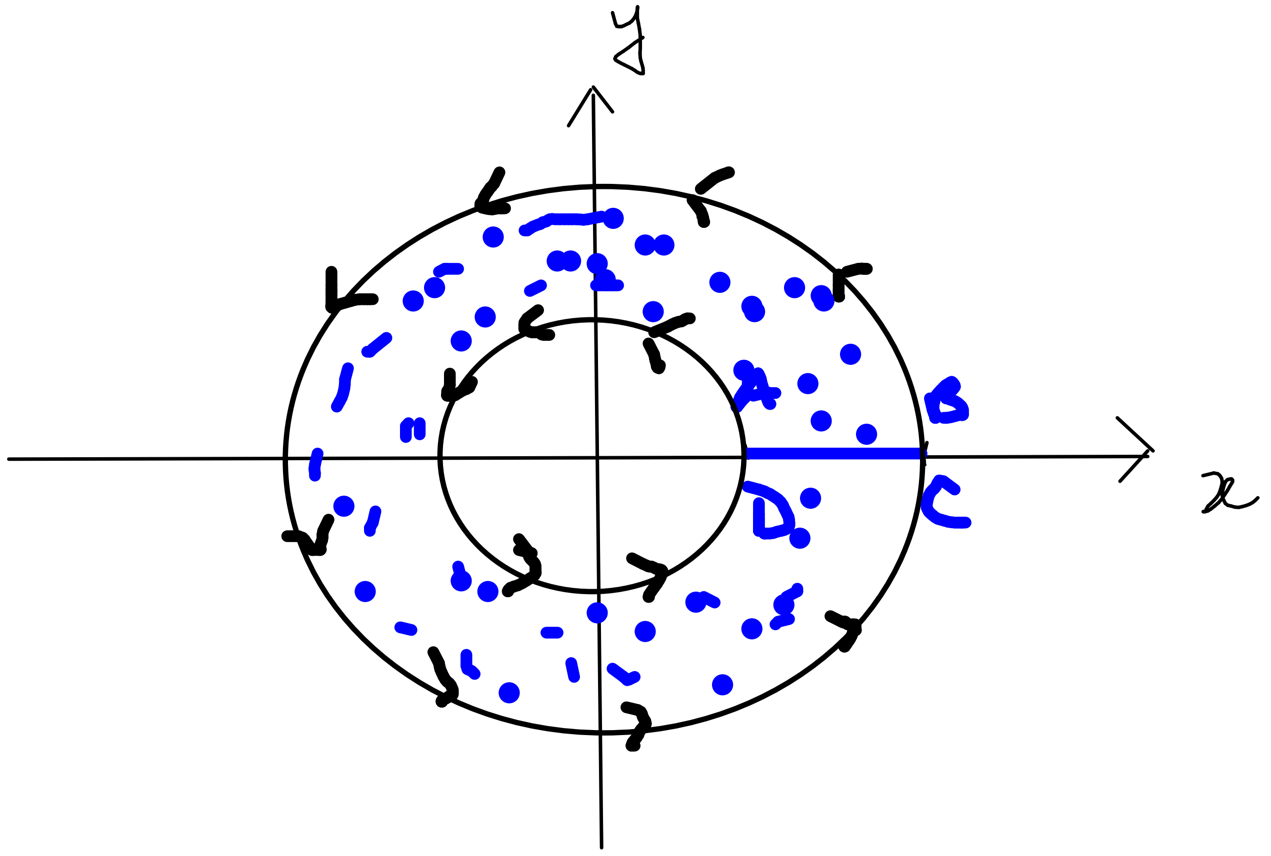
AB

$\theta = 0, \ 1 \le r \le 2$

BC

$r = 2, \ 0 \le \theta \le 2\pi$

$x = 2\cos\theta, \ y = 2\sin\theta$



DC, $\theta = 2\pi$, $1 \le r \le 2$

$x = r\cos 2\pi$, $y = r\sin 2\pi$

$x = r$, $y = 0$

$0 \le x \le 2$

AD:

$r = 1$ $\quad 0 \le \theta \le 2\pi$

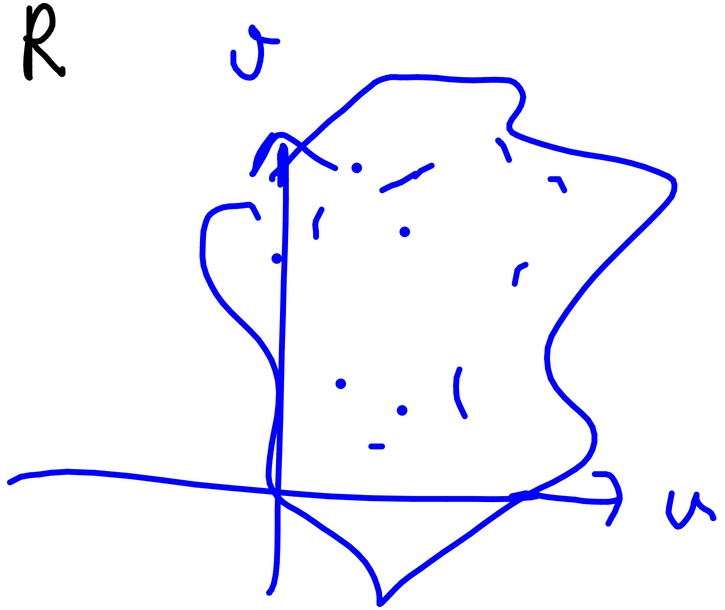
$x = \cos\theta$ $\quad y = \sin\theta$

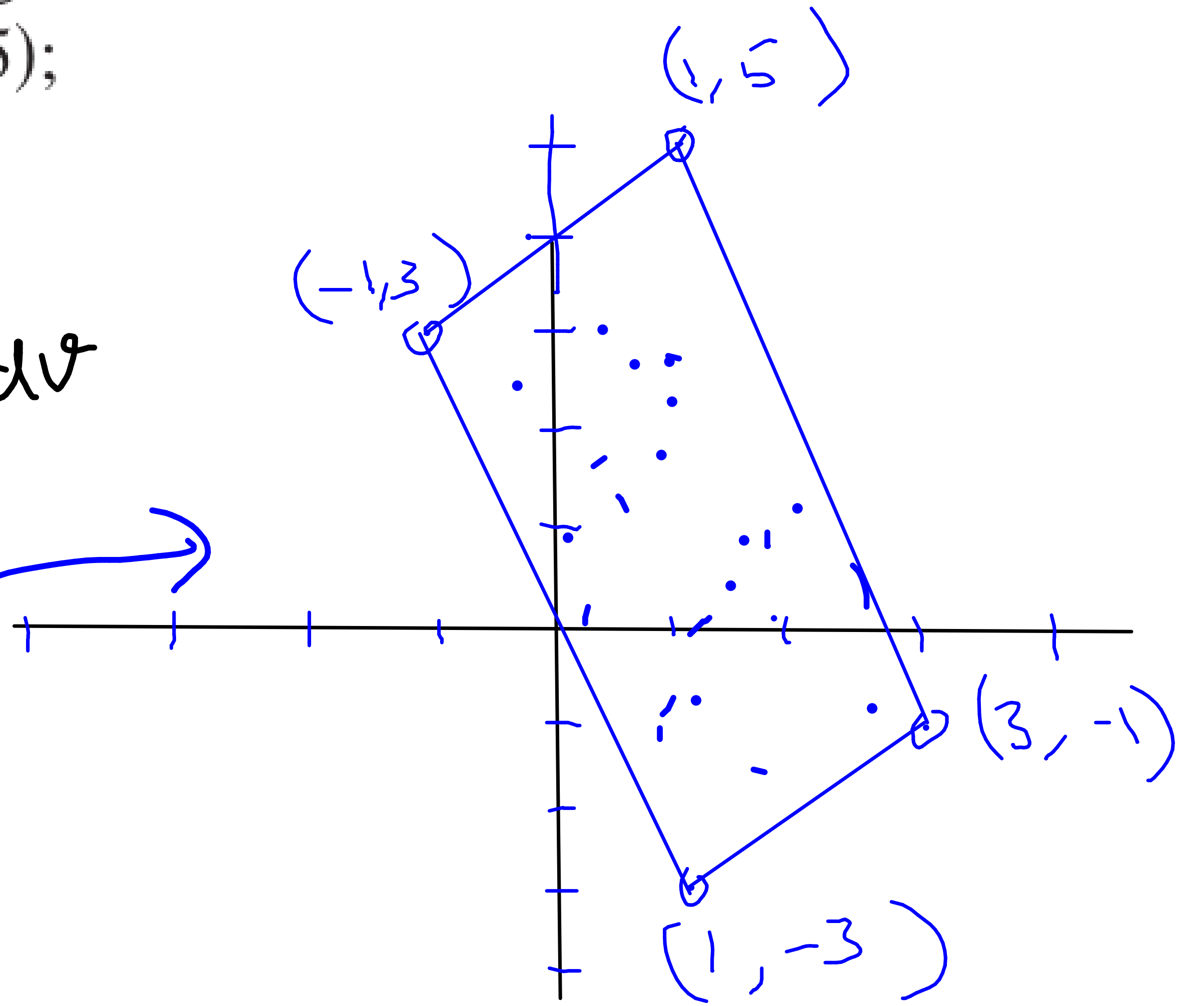
Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;

$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\,dA = \iint_{??}^{??} (\;??\;)\,du\,dv$$





Use the given transformation to evaluate the integral.

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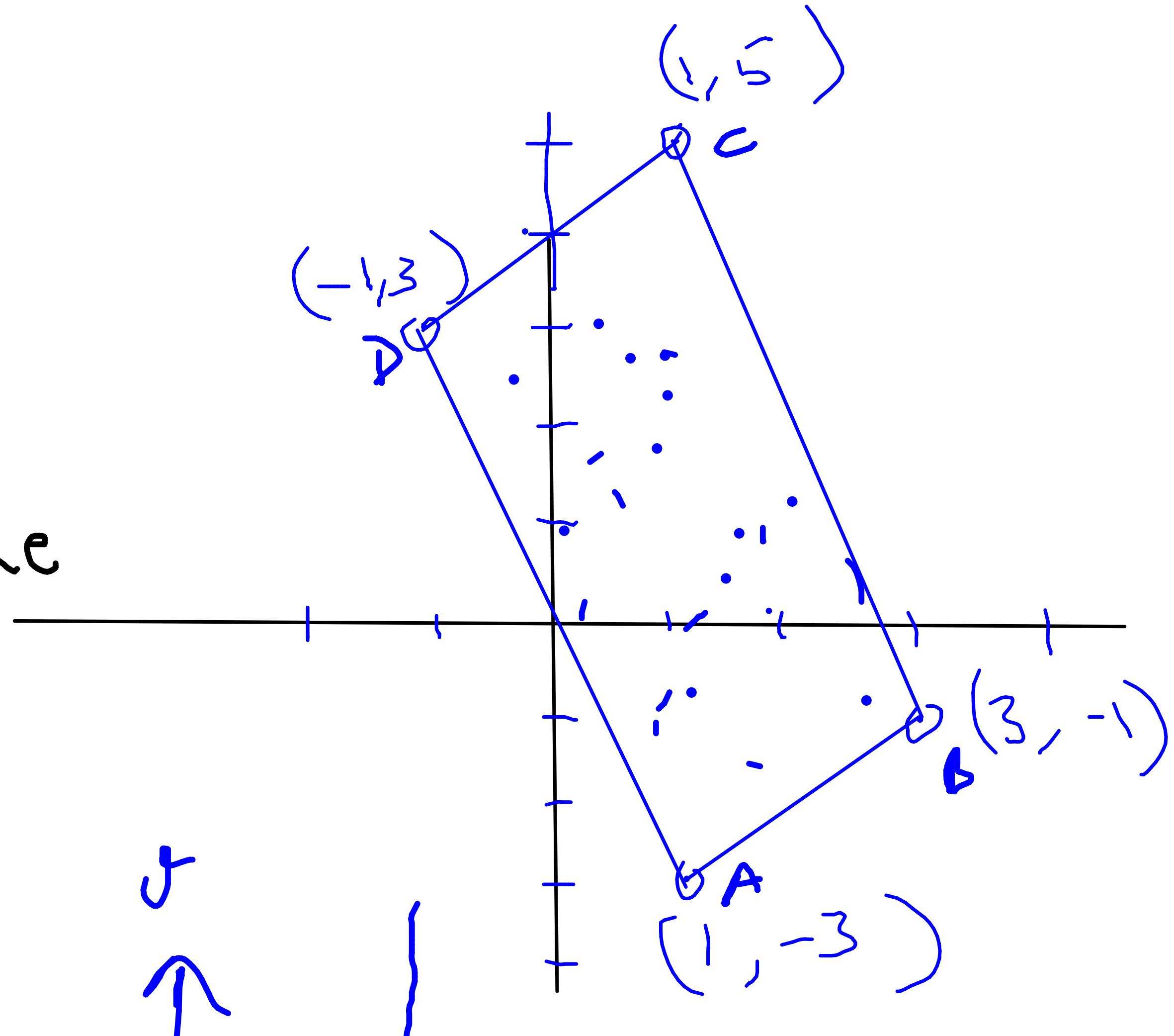


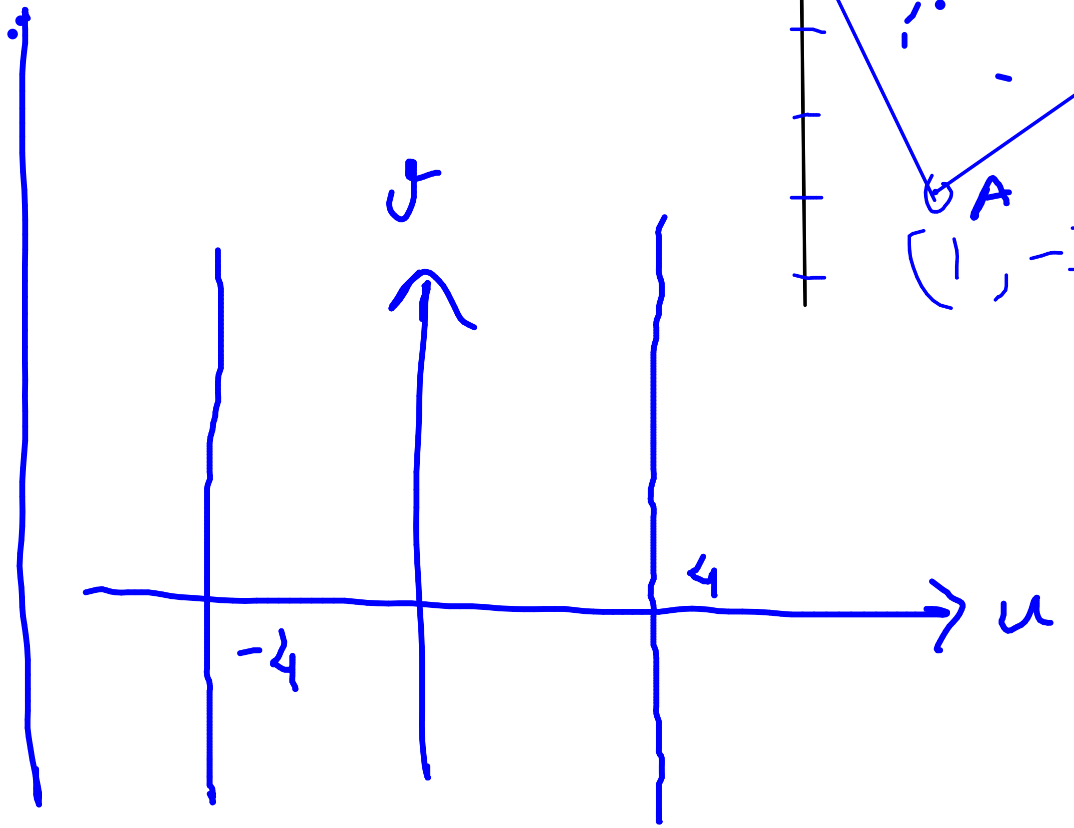
find out the shape of ABCD in the uv-plane

→ AB: eqn of AB in xy variable.

$$x - y = 4$$

$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = 4$$

$$u = 4$$



→ DC:

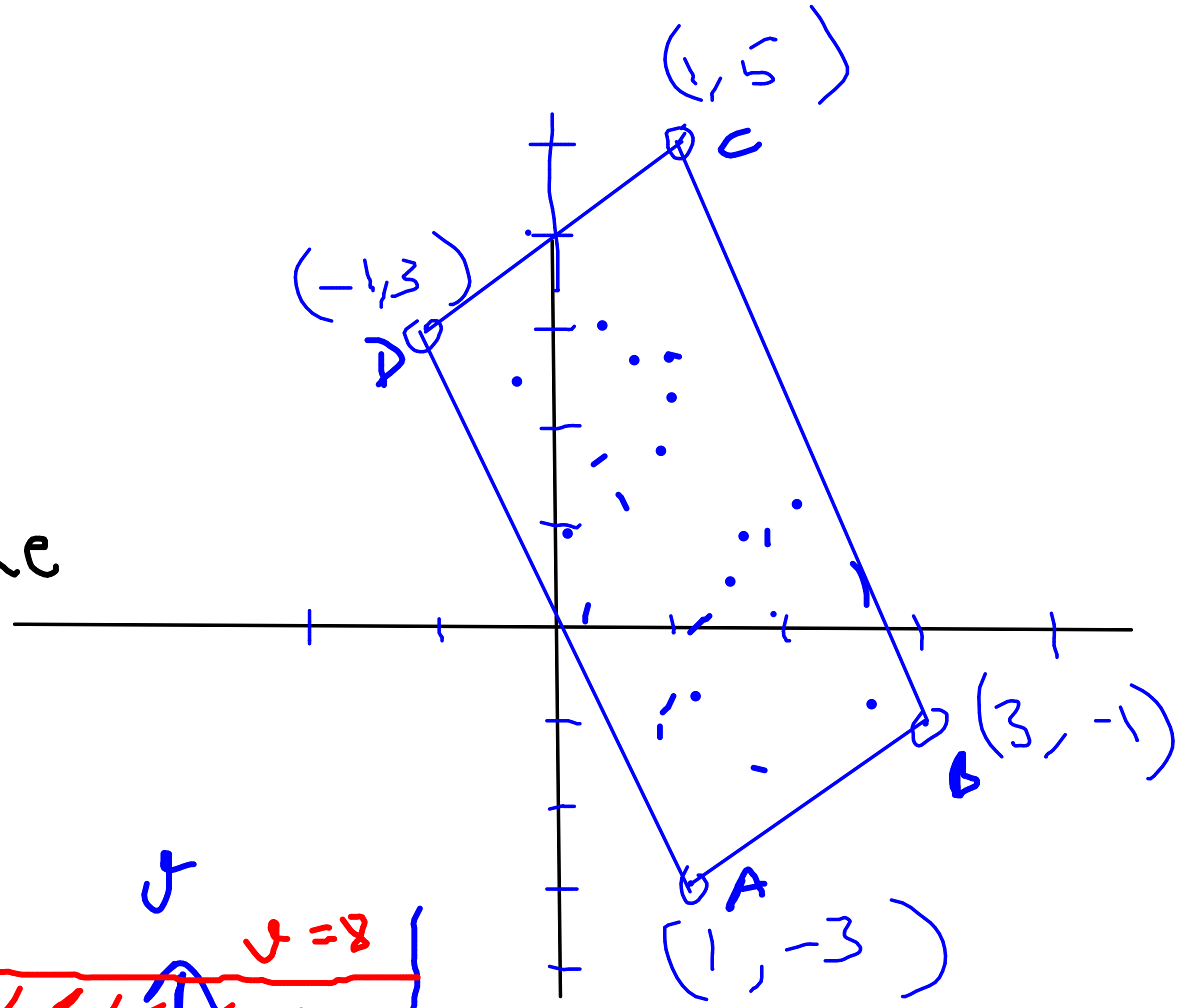
$$x - y = -4$$

$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = -4$$

$$u = -4$$

Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$



find out the shape of ABCD in the uv-plane

$\rightarrow$ AD, $\quad y = -3x$

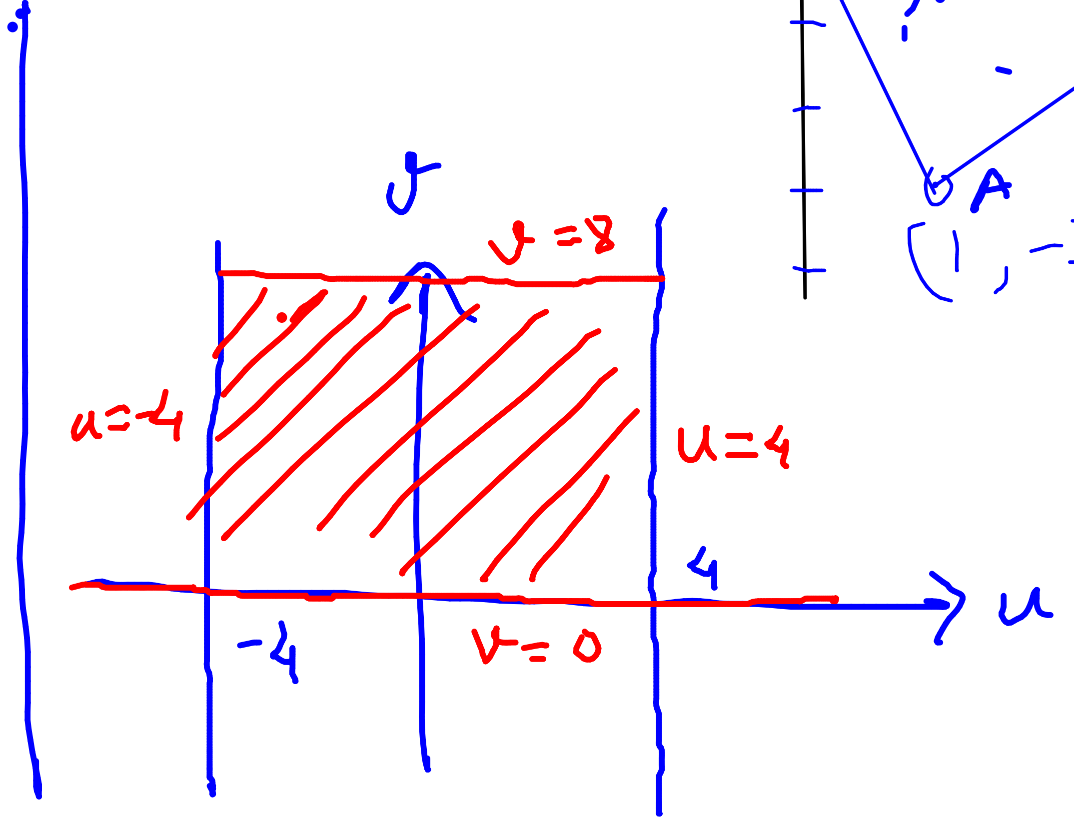
$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)$$

$$v = 0$$

$\rightarrow$ BC, $\quad y = -3x + 8$

$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v) + 8$$

$$v = 8$$



Use the given transformation to evaluate the integral.

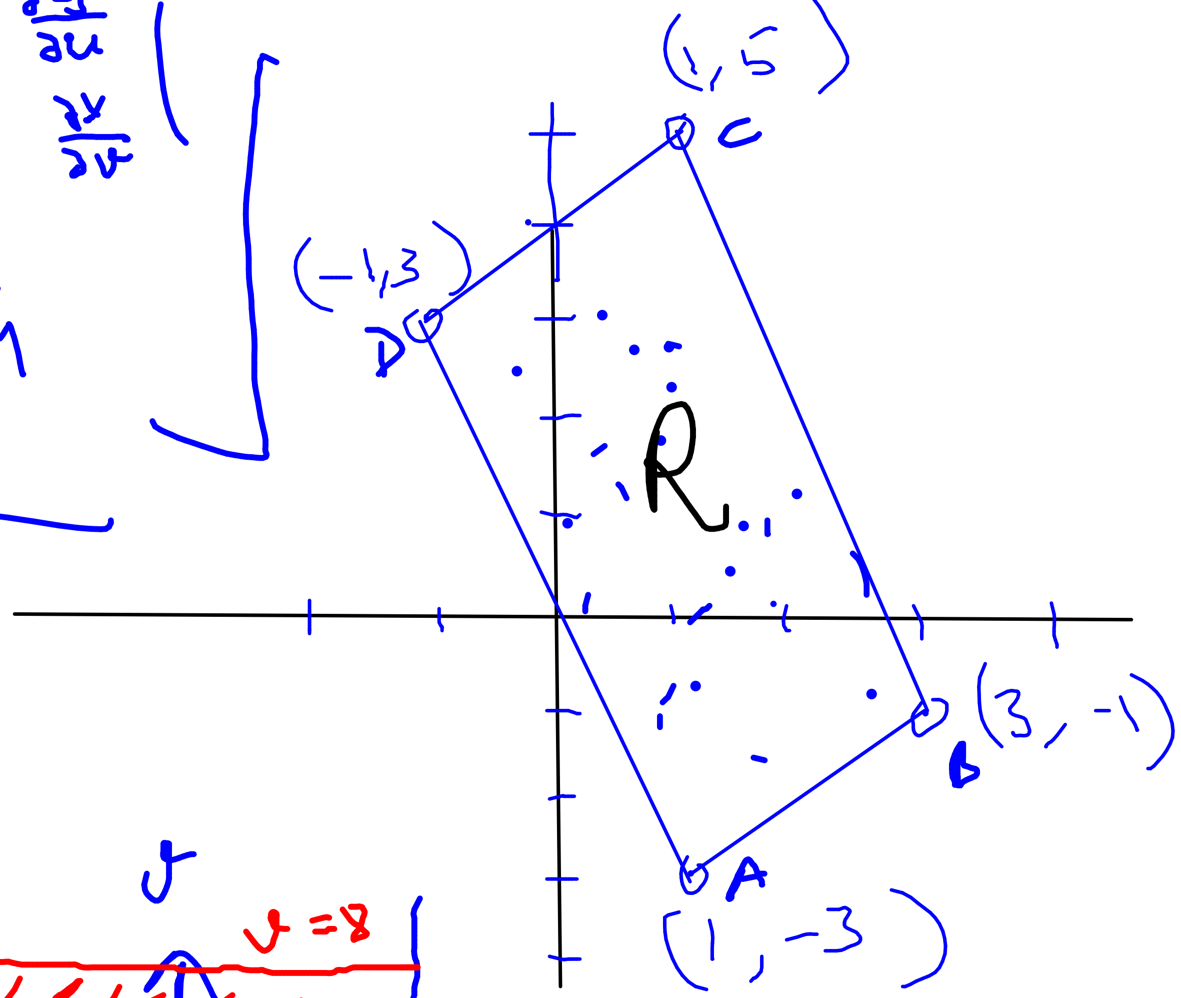
$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

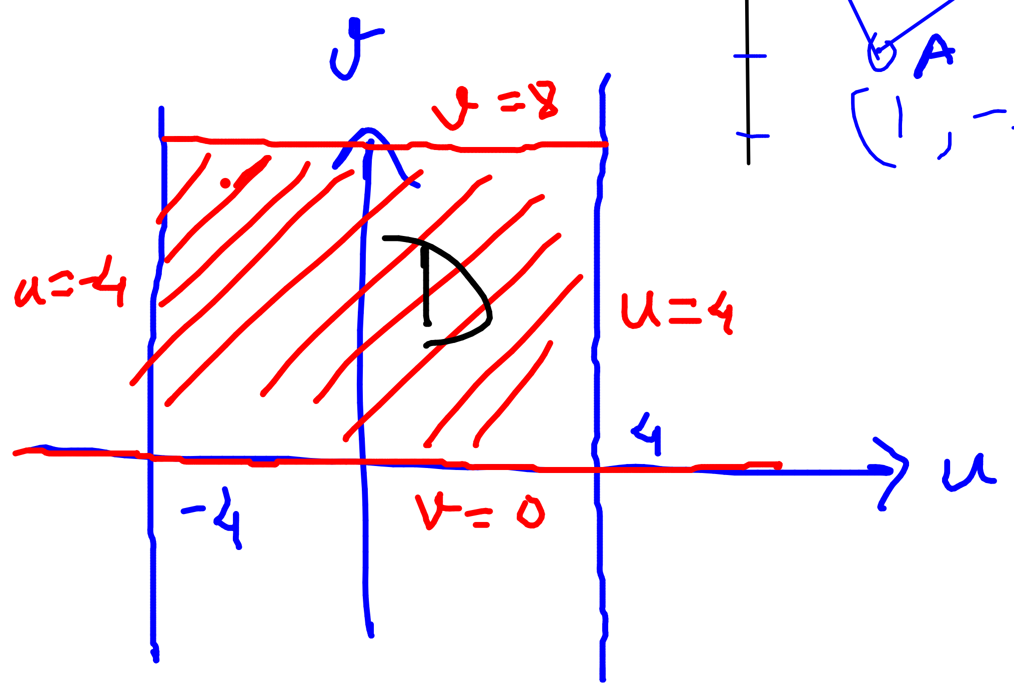
$$4x+8y = 4\cdot\frac{1}{4}(u+v) + 8\cdot\frac{1}{4}(v-3u)$$
$$= 3v-5u$$

$$\iint_R (4x+8y)\, dR = \iint_D (3v-5u)\,(\text{Jacobian})\, dD$$

$$= \int_0^8 \int_{-4}^{4} (3v-5u)\left(\frac{1}{4}\right) du\, dv$$
$$= 192$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$
$$= 1/4$$





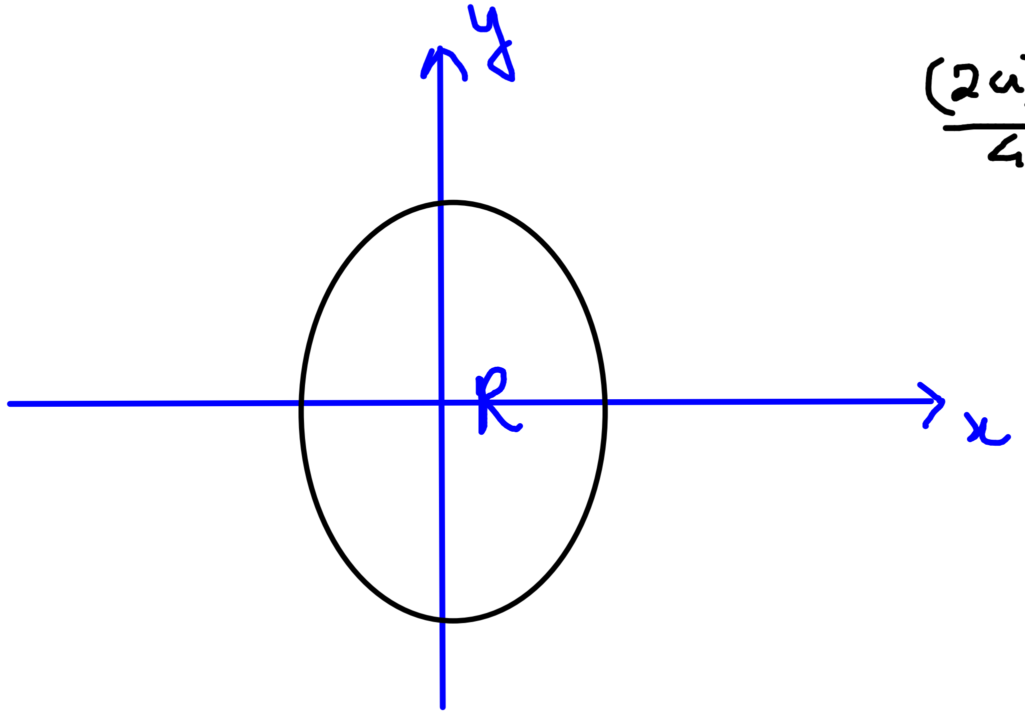
Use the given transformation to evaluate the integral.

$\iint_R x^2\,dA$, where R is the region bounded by the ellipse $9x^2+4y^2=36$; $x=2u$, $y=3v$

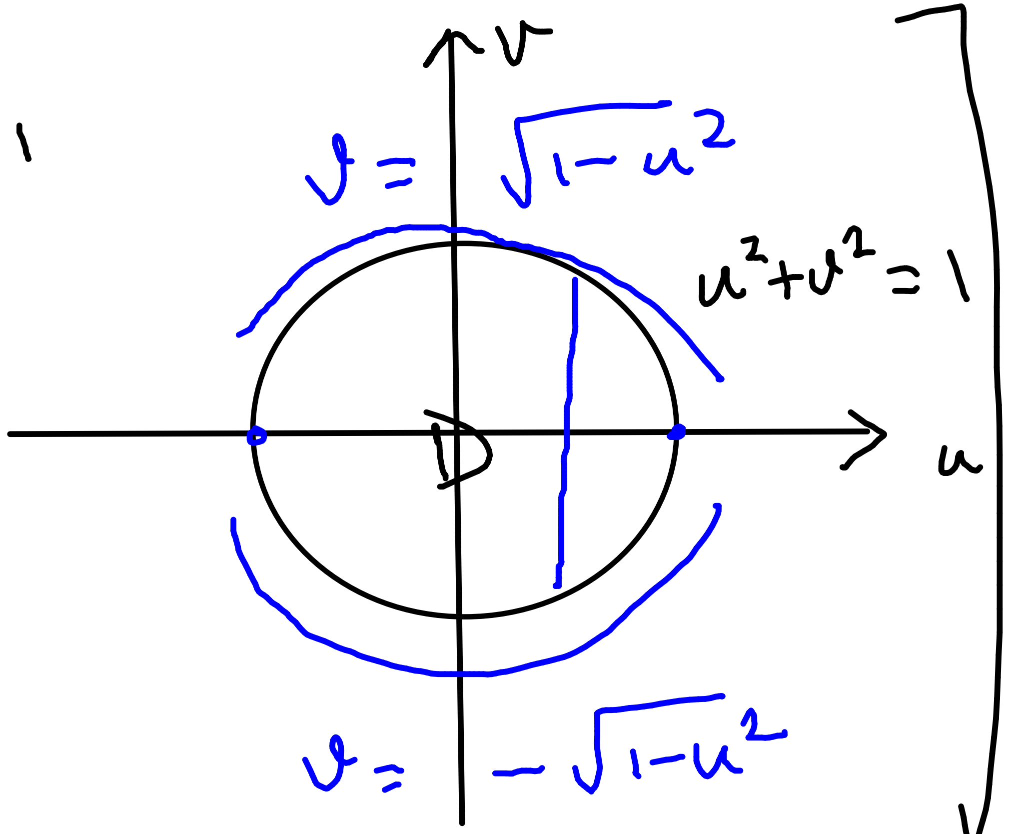
$$\iint_R x^2\,dA = \iint_D (2u)^2\cdot(\text{Jacobian})\,dD$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\frac{(2u)^2}{4}+\frac{(3v)^2}{9}=1$$



Jacobian = 6



$$= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 4u^2\cdot 6\,dv\,du$$

$$= \underline{\qquad\qquad}$$

complete it.

Use the given transformation to evaluate the integral.

$\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $x = 2u + v$, $y = u + 2v$

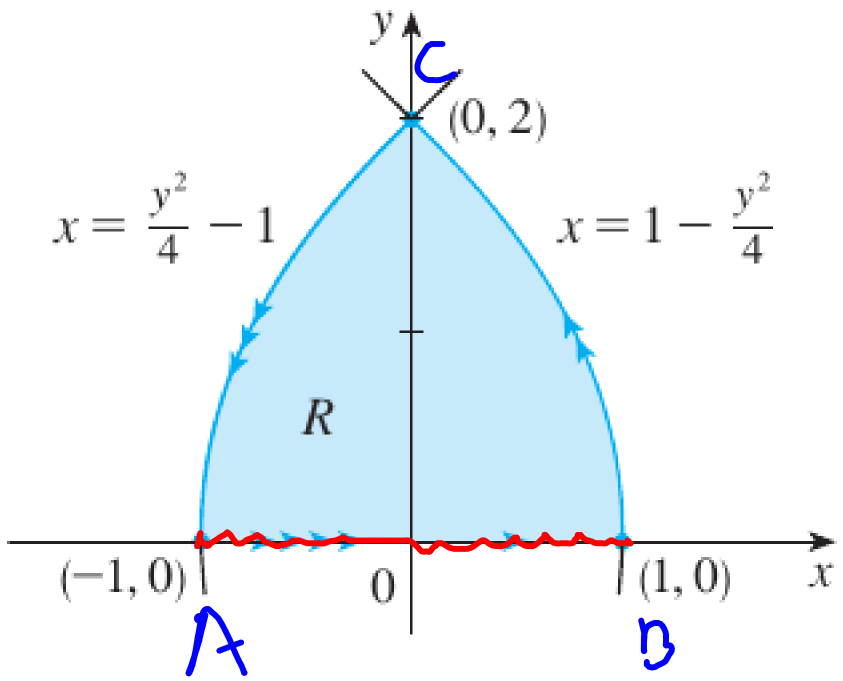
Evaluate the integral by making an appropriate change of variables.

$$\iint_R \frac{x - 2y}{3x - y}\, dA,$$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

Evaluate the integral by making an appropriate change of variables.

$\iint_R e^{x+y}\,dA$, where R is given by the inequality $|x| + |y| \leqslant 1$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.



BC :

$u = \pm 1$

OB $y = 0$, $x \geq 0$

$u = 0$ or $v = 0$
$[u^2 \geq v^2]$ $\Rightarrow$

AC

$y^2 = 4 + 4x$

$u^2 v^2 = 1 + u^2 - v^2$

$1 + u^2 - v^2 - u^2 v^2 = 0$

$\cancel{(1+u^2)}(1 - v^2) = 0$

$v = \pm 1$