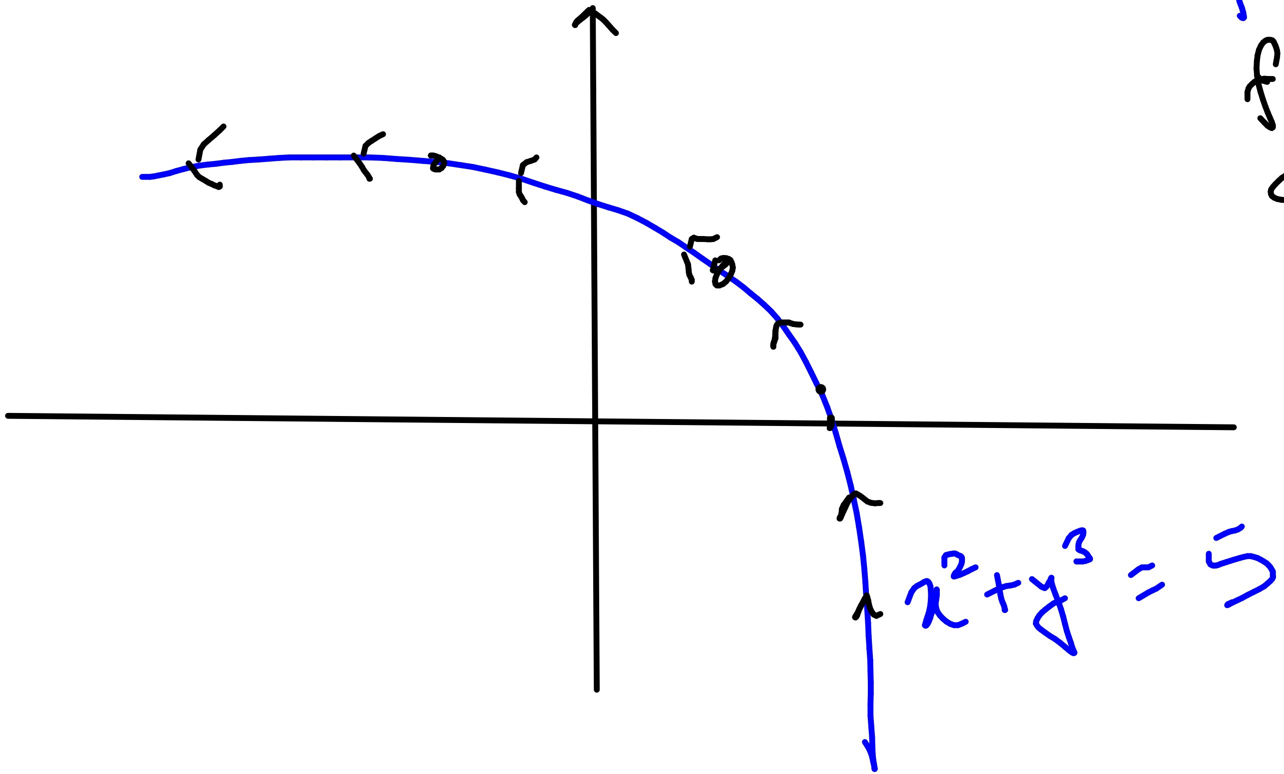
LAGRANGE MULTIPLIERS

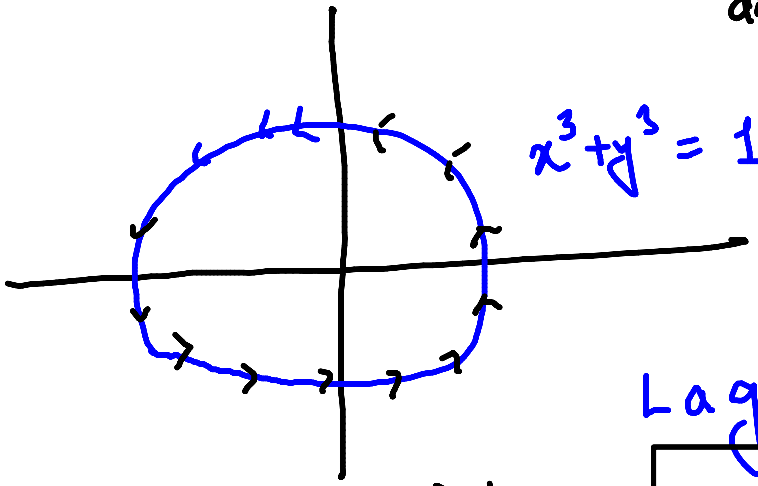


$f(x,y) = x+y$

find point on the curve $x^2+y^3=5$ where $f(x,y)=x+y$ is lowest.

Q: Suppose we have a function $f(x,y) = xy^2$

domain $= \mathbb{R}^2$



$x^3+y^3 = 1$

Aim: find max/min $f(x,y)$

s.t. $x^3+y^3 = 1$

Lagrange multiplier way:

if (x,y) is a point of max/min

Solve:

$$\boxed{\begin{aligned} &x^3+y^3 = 1 \\ &\nabla(xy^2) = \lambda \nabla(x^3+y^3) \end{aligned}}$$

where λ is a new variable called Lagrange Multiplier i.e.

$$\begin{bmatrix} x^3 + y^3 = 1 \\ y^2 = \lambda\, 3x^2 \\ 2xy = \lambda\, 3y^2 \end{bmatrix}$$

solve this to get

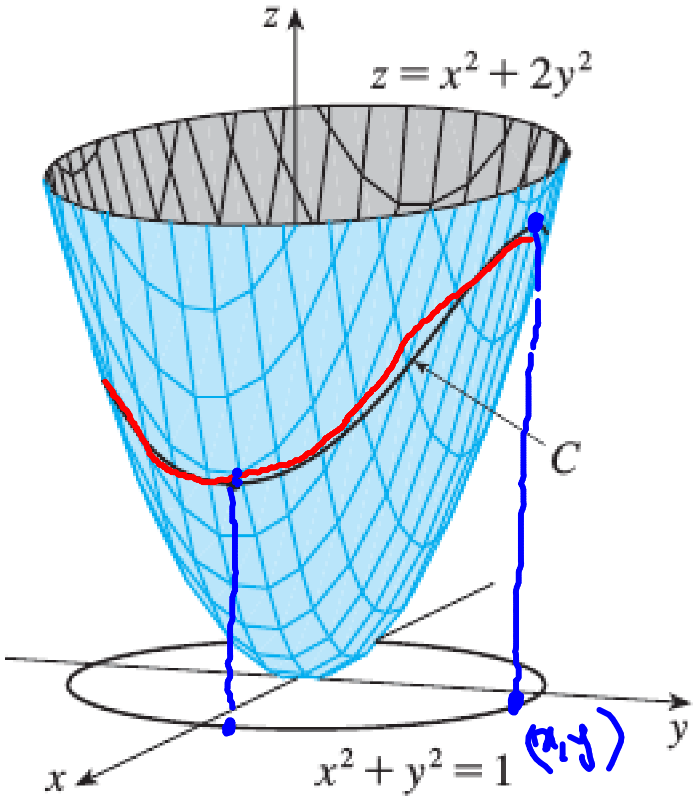
say

(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)

λ_1, λ_2, ..., λ_n

f_1 f_2 f_n

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.



Q: What equation we need to solve to find max/min points?

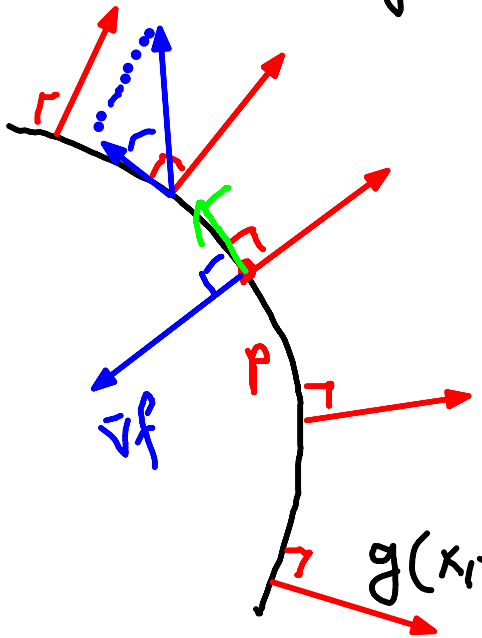
$x^2 + y^2 = 1$

$\nabla(x^2 + 2y^2) = \lambda \nabla(x^2 + y^2)$

$$\boxed{\begin{aligned} x^2 + y^2 &= 1 \\ 2x &= \lambda 2x \\ 4y &= \lambda 2y \end{aligned}}$$

Why Lagrange Multipliers Works

maximizing $f(x,y)$, over the curve $g(x,y) = 5$



p: (x,y) f is taking a local max on the curve.

Q: Can you point the direction of ∇g & ∇f at p.

∇g is always $\perp$ to the curve

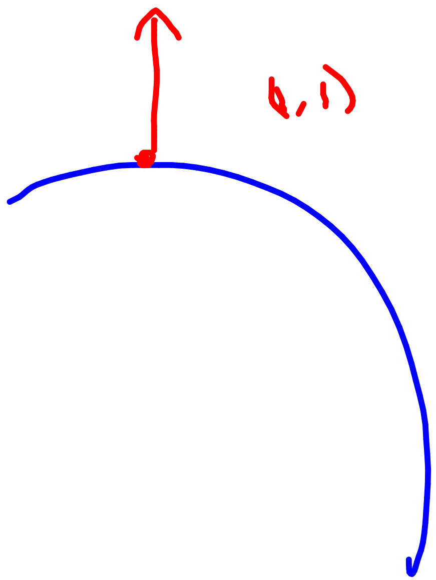
∇f is also $\perp$ to the curve at max/min points

p: is a point of max, $\hat{t}$: tangential at p

$$\boxed{\text{rate of change of } f \text{ along the curve at point } p} = \nabla f \cdot \hat{t} = 0$$

$$\Rightarrow \underbrace{\nabla f \perp \hat{t}}_{\text{at max}} \underbrace{\perp \nabla g}_{\text{always}}$$

$$\Rightarrow \boxed{\nabla f = \lambda \nabla g}$$



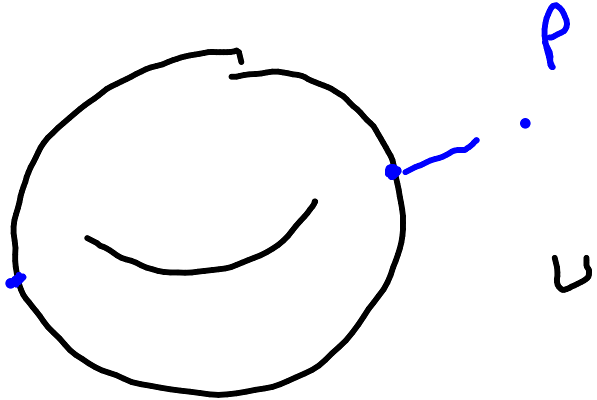
$$x^2+y^2 = 2$$

$$g(x,y) = x^2+y^2$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

$$\nabla g\big|_{(1,1)} = 2\hat{i} + 2\hat{j}$$

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.



maximize / minimize

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

s.t. $\quad x^2 + y^2 + z^2 = 4$

Using Lagrange multiplier : solve

$$x^2 + y^2 + z^2 = 4$$

$$\nabla\left((x-3)^2 + (y-1)^2 + (z+1)^2\right) = \lambda \nabla\left(x^2 + y^2 + z^2\right)$$

complete yourself.

$$x^2 + y^2 + z^2 = 1$$

$$2(x-3) = \lambda 2x$$

$$2(y-1) = \lambda 2y$$

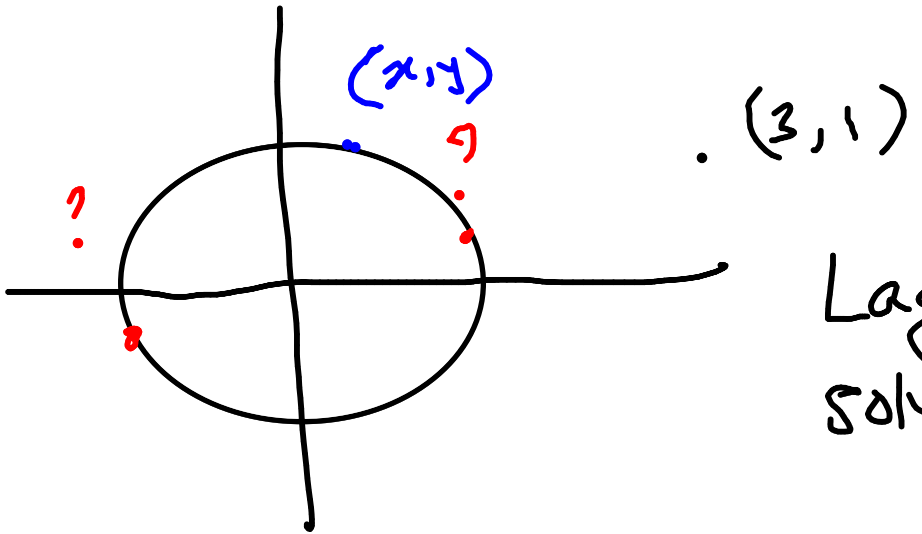
$$2(z+1) = \lambda 2z$$

EXAMPLE 4 Find the points on the circle ~~sphere~~ $x^2 + y^2 = 4$ that are closest to and farthest from the point $(3, 1)$.

maximize & minimize

$$f(x,y) = (x-3)^2 + (y-1)^2$$

s.t. $x^2 + y^2 = 4$



Lagrange multipliers:

solve:

$$x^2 + y^2 = 4$$

$$\nabla\left((x-3)^2 + (y-1)^2\right) = \lambda \nabla\left(x^2 + y^2\right)$$

$x^2+y^2=4$

$\cancel{2}(x-3)=\alpha\,\cancel{2}x$

$\cancel{2}(y-1)=\alpha\,\cancel{2}y$

$x=3/(1-\alpha)$

$y=1/1-\alpha$

Rough

$$\left(\frac{3}{1-\alpha}\right)^2+\left(\frac{1}{1-\alpha}\right)^2=4$$

$$\frac{10}{4}=(1-\alpha)^2$$

Ans: closest point & farthest point

$\left(\frac{6}{\sqrt{10}},\frac{2}{\sqrt{10}}\right)$ $\left(-\frac{6}{\sqrt{10}},-\frac{2}{\sqrt{10}}\right)$

$$(1-\lambda) = \pm \frac{\sqrt{10}}{2}$$

$$\lambda = 1 \pm \frac{\sqrt{10}}{2}$$

Check!

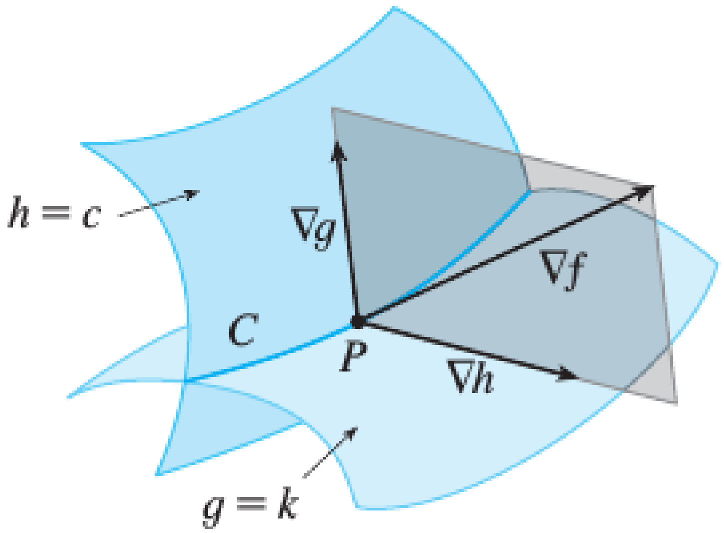
$$\lambda = 1 + \frac{\sqrt{10}}{2} \qquad x = 6/\sqrt{10}, \quad y = 2/\sqrt{10}$$

(nearest)

$$\lambda = 1 - \frac{\sqrt{10}}{2} \qquad x = -6/\sqrt{10}, \quad y = -2/\sqrt{10}$$

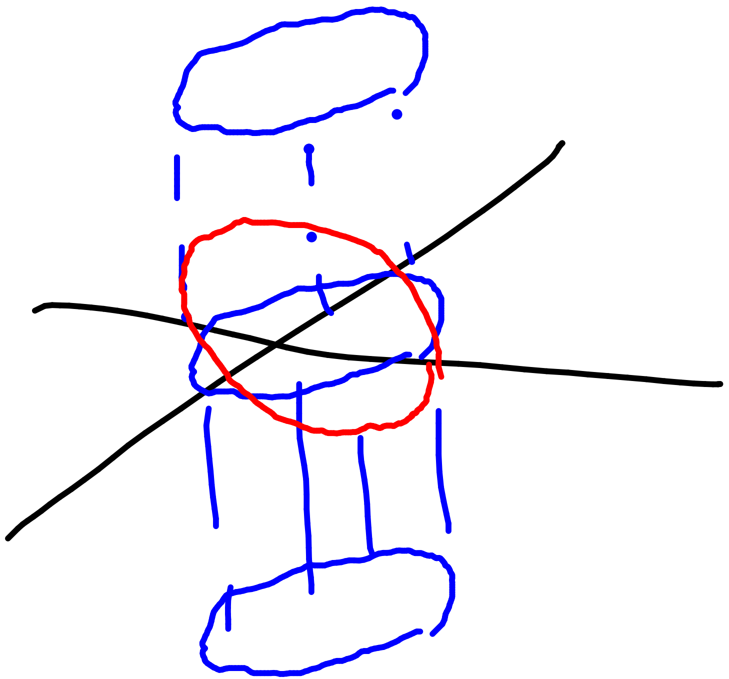
(farthest)

TWO CONSTRAINTS



$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

V EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.



maximize $f(x,y,z) = x + 2y + 3z$

s.t. $x^2 + y^2 = 1$

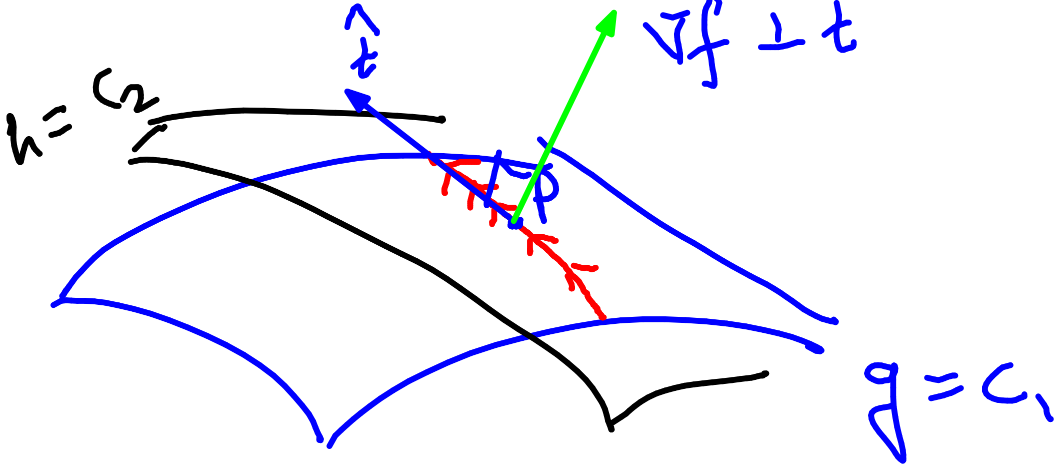
$x - y + z = 1$

$$\text{maximize} \quad f(x,y,z) = x+2y+3z$$

$$\text{s.t.} \quad x^2+y^2=1 \iff g(x,y,z)=c_1.$$

$$x-y+z=1 \iff h(x,y,z)=c_2$$

$\nabla f \perp \hat{t}$



∇g : is $\perp$ surface $g=c_1$

∇h : is $\perp$ surface $h=c_2$

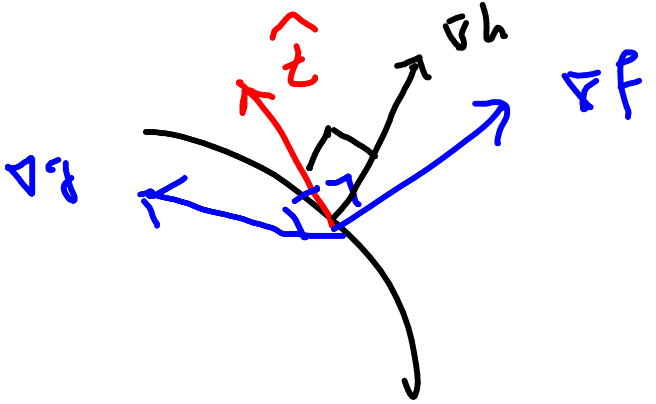
$$\nabla f \cdot \hat{t}\Big|_p = 0 \quad (\because p \text{ is a max point})$$

$$\nabla f \perp \hat{t} \text{ at } p$$

Notice: at point p :

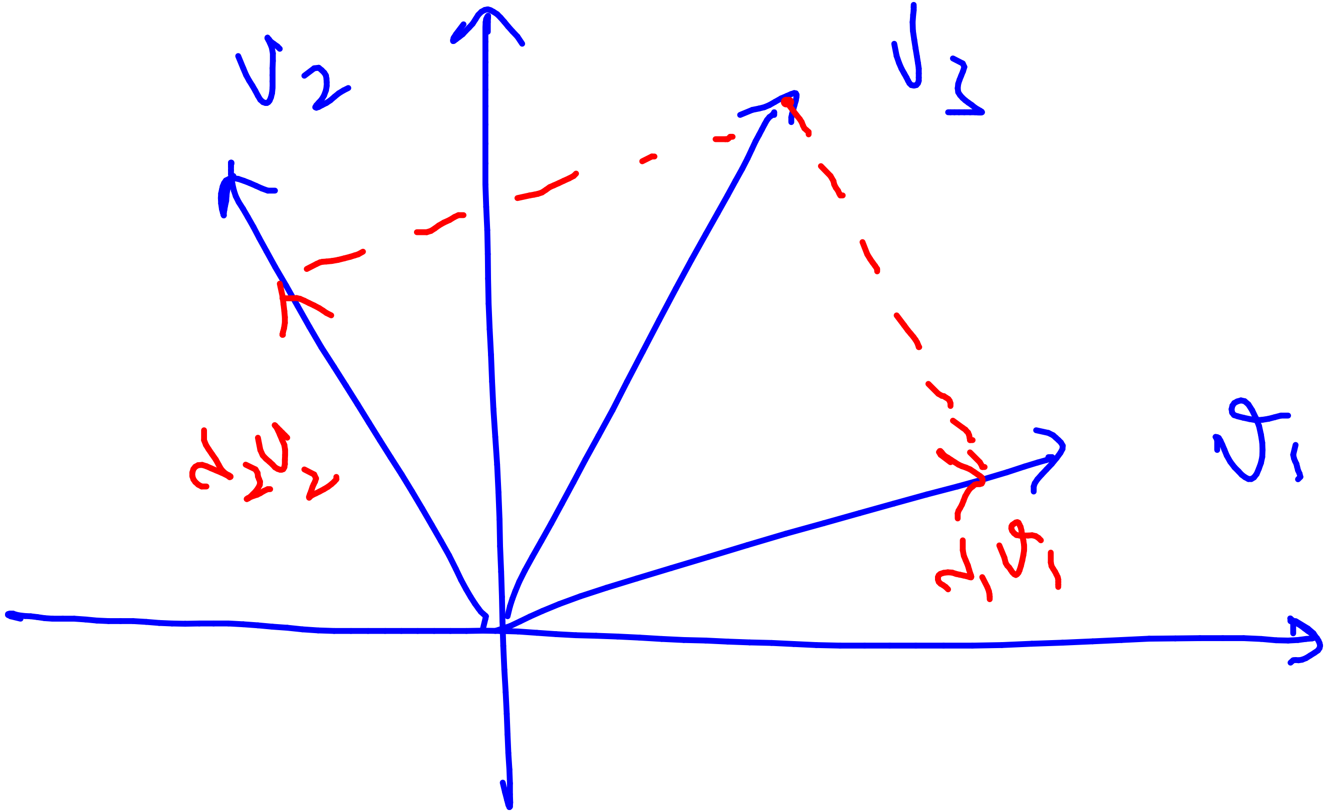
s $\left.\begin{array}{l}\nabla g \\ \nabla h \\ \nabla f\end{array}\right\}$ all perpendicular to $\hat{t}$

∇g, ∇f, ∇h are all in same plane



$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

→ extra equ for two constraints



$$v_3 = \alpha_1 v_1 + \alpha_2 v_2$$

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

maximize $f = x + 2y + 3z$

s.t. $x - y + z = 1$

$x^2 + y^2 = 1$

if $P = (x, y, z)$ is the max point, it satisfies

$x - y + z = 1$

$x^2 + y^2 = 1$

$\nabla f = \lambda \nabla (x - y + z) + \mu \nabla (x^2 + y^2)$

i.e.

$$x - y + z = 1$$

$$x^2 + y^2 = 1$$

$$1 = \lambda + 2x\mu$$

$$2 = -\lambda + 2\mu y$$

$$3 = \lambda$$

$$\lambda = 3$$

$$\mu = \pm \sqrt{29}/2$$

$$(x, y) = \left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right) \wedge z = ??$$

$$(x, y) = \left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right), z = ??$$

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35$$

if $P = (x, y, z)$ is a max/min point

$x^2 + y^2 + z^2 = 35$

$\nabla f = \lambda \nabla (x^2 + y^2 + z^2)$

$x^2 + y^2 + z^2 = 35$

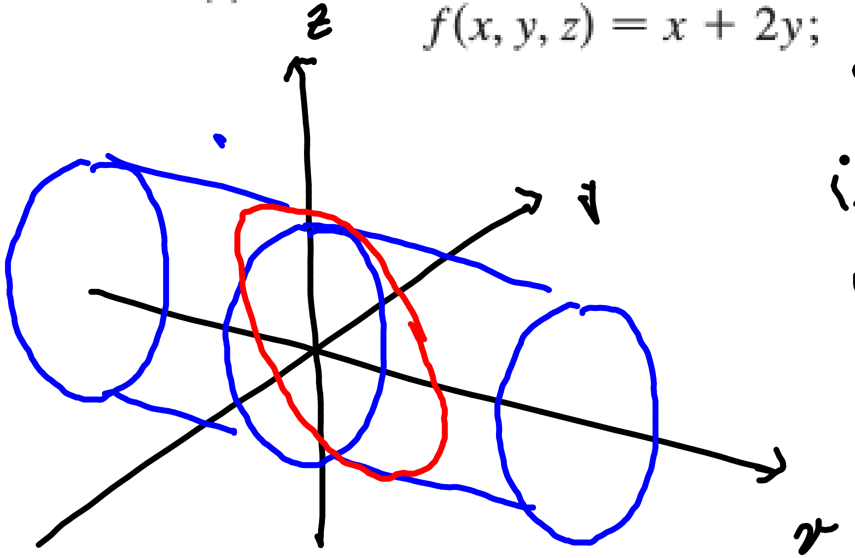
$2 = \lambda 2x$

$6 = \lambda 2y$

$10 = \lambda 2z$

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = x + 2y; \quad \underbrace{x + y + z = 1, \quad y^2 + z^2 = 4}$$



if $P = (x, y, z)$ is a max/min point

it must satisfy

$$x + y + z = 1$$

$$y^2 + z^2 = 4$$

$$\nabla f = \lambda \nabla (x+y+z) + \mu \nabla (y^2+z^2)$$

$$1 = \lambda$$

$$2 = \lambda + 2y\mu$$

$$0 = \lambda + 2z\mu$$

$$x + y + z = 1$$

$$y^2 + z^2 = 4$$