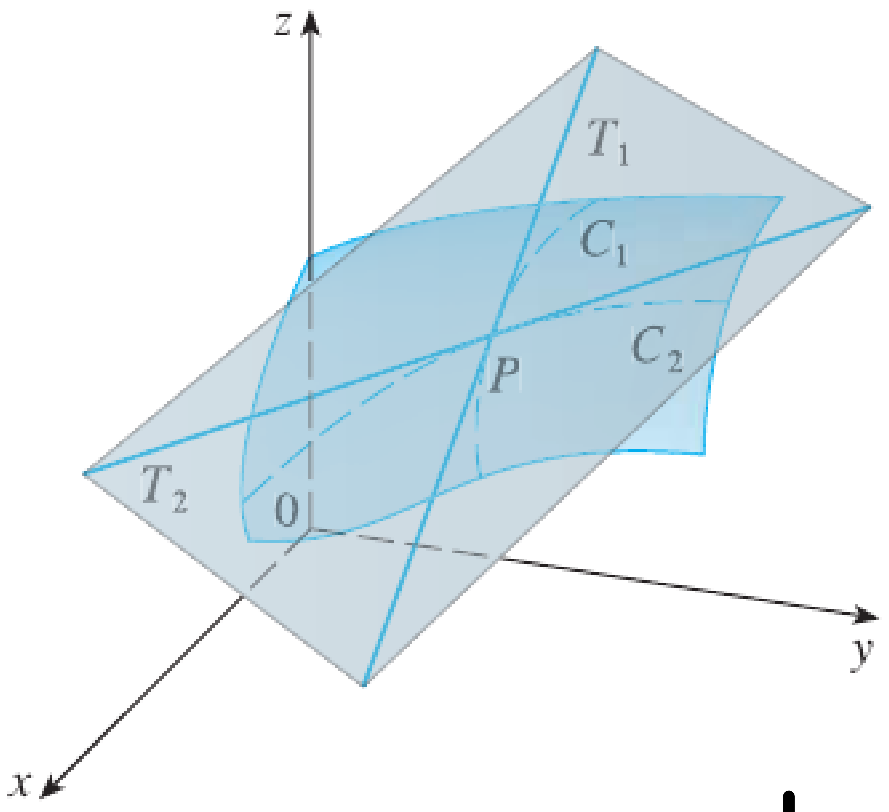
→ applications
→ Linearization
→ Differentials



$f(x,y)$ $\quad$ $P = (a, b, z_0)$

$z_0 = f(a, b)$

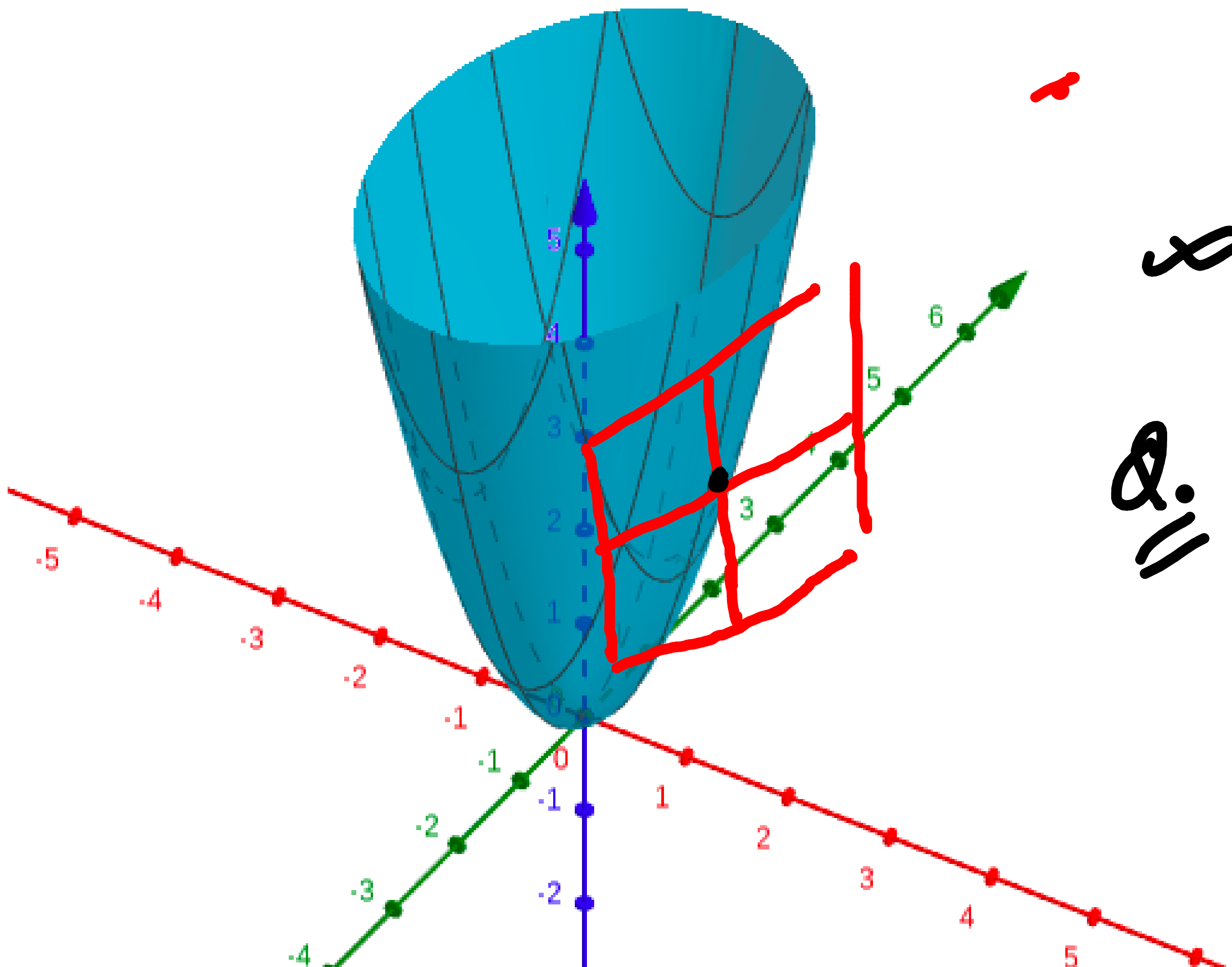
$$z - z_0 = A(x-a) + B(y-b)$$

where $A = \frac{\partial f}{\partial x}(a,b)$, $B = \frac{\partial f}{\partial y}(a,b)$

V EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Q: is $(1,1,3)$ a point on the graph??

Q: give general eqn of planes passing through $(1,1,3)$??



$\propto \quad A(x-1) + B(y-1) + C(z-3) = 0$

Q: if $C = 0$??

the tangent plane becomes vertical

V EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

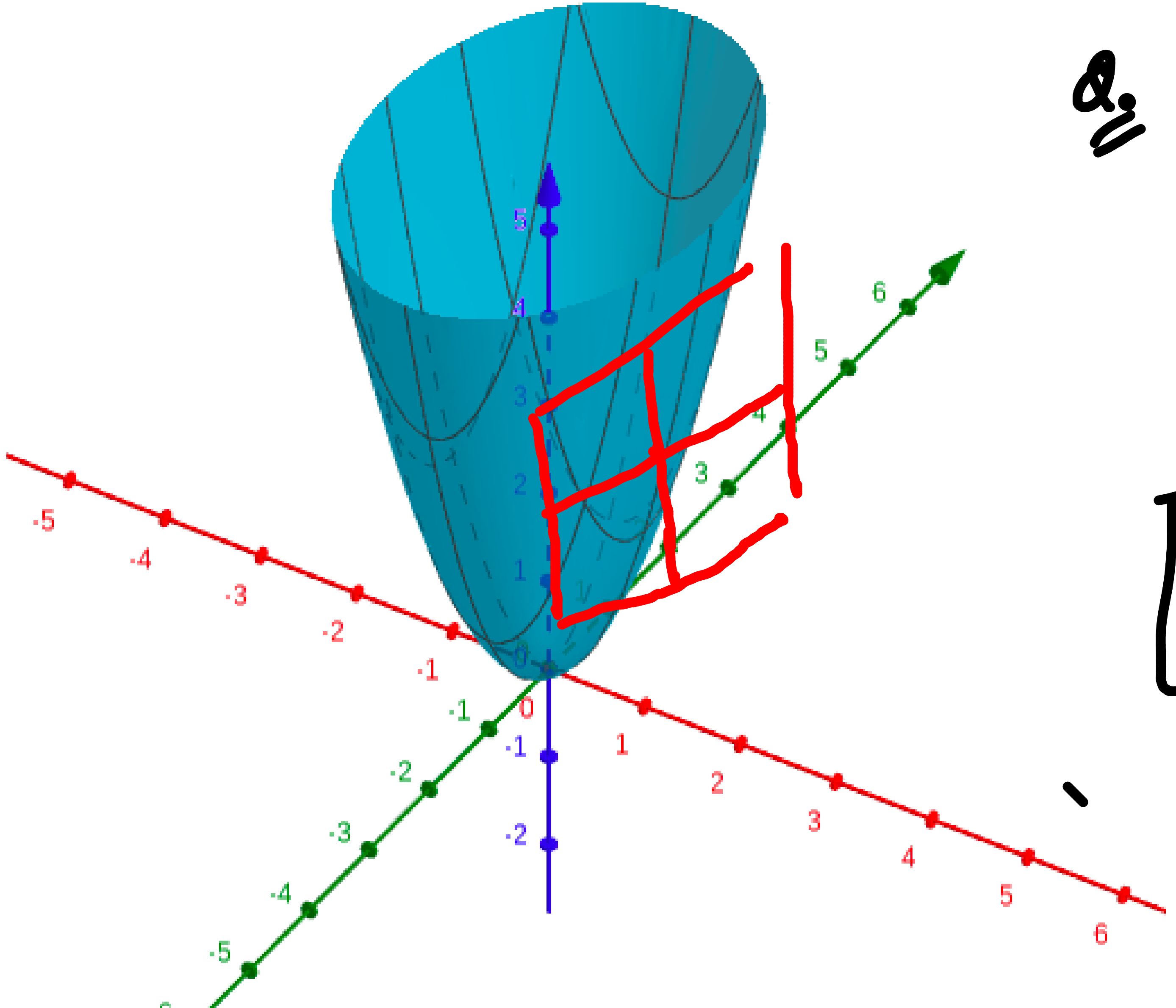
$A(x-1) + B(y-1) + C(z-3) = 0$

Q: if $C = 0$??

the tangent plane becomes vertical

$$\boxed{z-3 = -\frac{A}{C}(x-1) + \left(-\frac{B}{C}\right)(y-1)}$$

$$\boxed{z-3 = A(x-1) + B(y-1)}$$



$A = \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1}}$ $B = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}}$

$z = 2x^2 + y^2$

$\frac{\partial z}{\partial x} = 4x$

$\frac{\partial z}{\partial y} = 2y$

$\frac{\partial z}{\partial x}(1,1) = 4 = A$

$\frac{\partial z}{\partial y}(1,1) = 2 = B$

$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$

Q. Find the eqn of tangent plane

$$f(x,y) = x \sin(y-x)$$

$$a = 2, \quad b = 1$$

$$z - \underbrace{(-2\sin(1))}_{\substack{z_0 \\ = f(a,b)}} = \underbrace{(-\sin(1) - 2\cos(1))}_{\frac{\partial f}{\partial x}(2,1)}(x-2) + \underbrace{2\cos(1)}_{\frac{\partial f}{\partial y}(2,1)}(y-1)$$

Q. Find the eq^n of tangent plane

$$f(x,y) = x\sin(y-x)$$

$$, a=2 \quad b=1$$

Recall the formula for the tangent plane

$$z - z_0 = A(x-a) + B(y-b)$$

where $z_0 = f(a,b)$

$A = \frac{\partial f}{\partial x}(a,b)$

$B = \frac{\partial f}{\partial y}(a,b)$

$z_0 = f(2,1) = 2\sin(-1) = -2\sin(1)$

$\frac{\partial f}{\partial x} = -x\cos(y-x) + \sin(y-x)$

$A = -2\cos(-1) + \sin(-1)$

$$= -2\cos(1) - \sin(1)$$

$$\frac{\partial f}{\partial y} = x\cos(y-x)$$

$$B = 2\cos(1-2) = 2\cos(1)$$

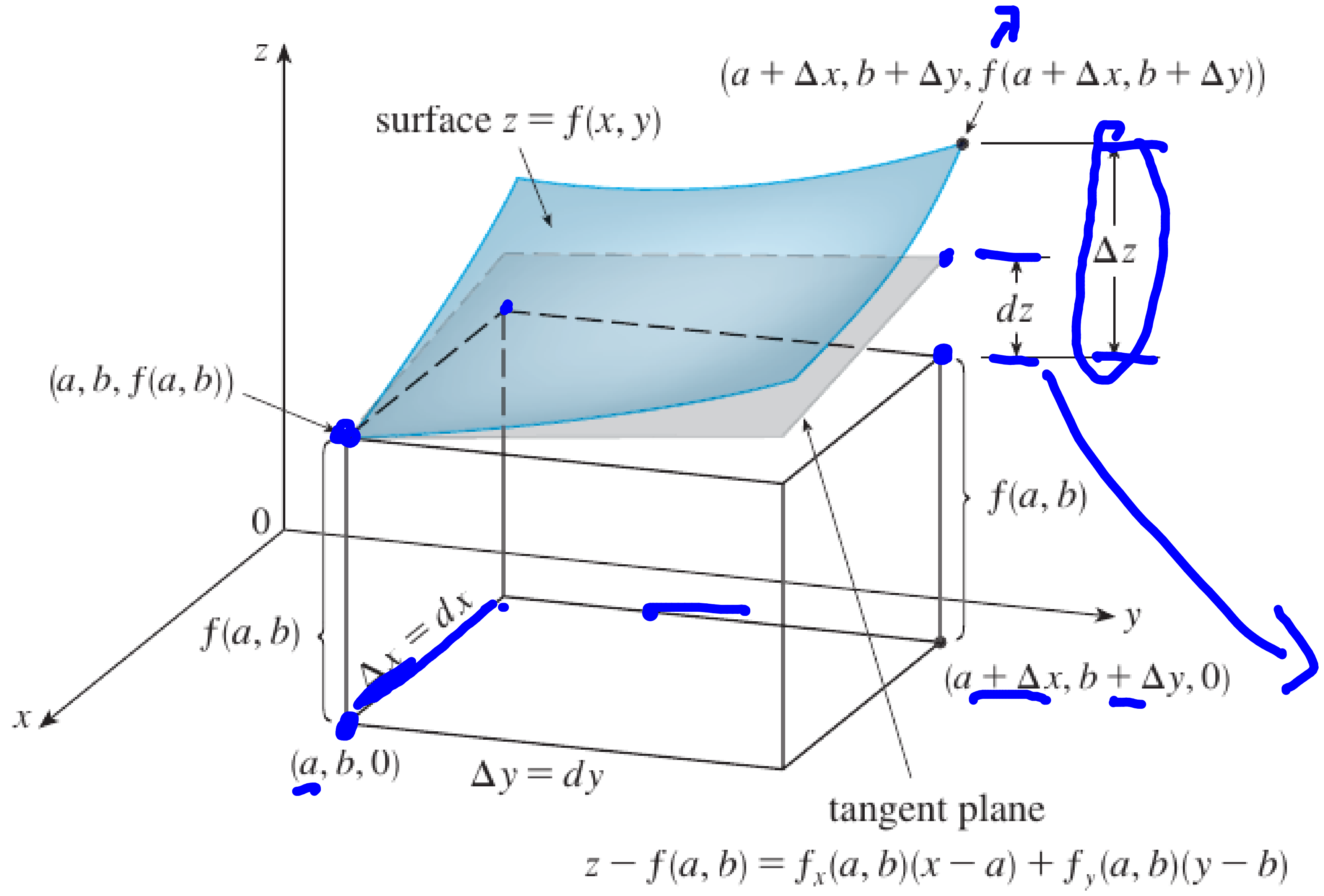
<u>Rule:</u> To find the tangent plane of $z=f(x,y)$ at point (a,b,c), the eqn:

$$z-c = A(x-a) + B(y-b)$$

where $A = \frac{\partial z}{\partial x}(a,b)$, $B = \frac{\partial z}{\partial y}(a,b)$

DIFFERENTIALS

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



→ Suppose we are given a formula $f(x,y)$

→ suppose we are at point $x = a$, $y = b$

→ $z = f(a,b)$

$dz \approx$ approximate change in f

= exact change in the tangent plane

→ in Differentials :

Q: how much change will happen in $f(x,y)$

as $x \to a \to a + \Delta x$

$y \to b \to b + \Delta y$

→ Use the tangent plane to approximate the change in f for small changes in x & y

Recall the eqn of the tangent plane

$$\boxed{z - f(a,b) = \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(\underline{y-b})}$$

→ Aim: as x changes from a to $a+\Delta x$
& y changes from b to $b+\Delta y$
find the corresponding change in z

→ simply substitute $x = a+\Delta x$, $y = b+\Delta y$

$$\underbrace{z - f(a,b)}_{\text{change in } z} = \frac{\partial f}{\partial x}(a,b)\,\underbrace{\Delta x}_{\text{change in } x} + \frac{\partial f}{\partial y}(a,b)\,\underbrace{\Delta y}_{\text{change in } y}$$

$$\underbrace{z - f(a,b)} = \frac{\partial f}{\partial x}(a,b)\underbrace{(x-a)} + \frac{\partial f}{\partial y}(a,b)\underbrace{(y-b)}$$

Q: find z for $x = a$, $y = b$.

$$z = f(a,b)$$

Q: find z for $x = a + \Delta x$, $y = b + \Delta y$

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y$$

notice: the change in z as $x: a \to a + \Delta x$,
$y: b \to b + \Delta y$

change in z in tangent plane $=$ $\boxed{dz = \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y}$

$$\underbrace{z - f(a,b)}_{\text{change in } z} = \frac{\partial f}{\partial x}(a,b) \underbrace{\Delta x}_{\text{change in } x} + \frac{\partial f}{\partial y}(a,b) \underbrace{\Delta y}_{\text{change in } y}$$

Differentials

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

$\rightarrow$ estimate of actual change

EXAMPLE 3

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

$$\frac{\partial z}{\partial x} = 2x + 3y$$

$$\frac{\partial z}{\partial y} = 3x - 2y$$

S: $z = x^2 + 3xy - y^2$

estimate dz at $x = 2$ & $y = 3$

for $dx = 0.05$ $dy = -0.04$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + 3y) dx + (3x - 2y) dy$$

$$= 13(0.05) + (0)(-0.04)$$

$$= 0.65$$

& actual change $\Delta z = f(2.05, 2.96) - f(2,3) = 0.64$

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

1–6 ▪ Find an equation of the tangent plane to the given surface at the specified point.

$$z = y\cos(x - y), \quad (2, 2, 2)$$

11–14 ▪ Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

$$f(x, y) = x\sqrt{y}, \quad (1, 4)$$

30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.