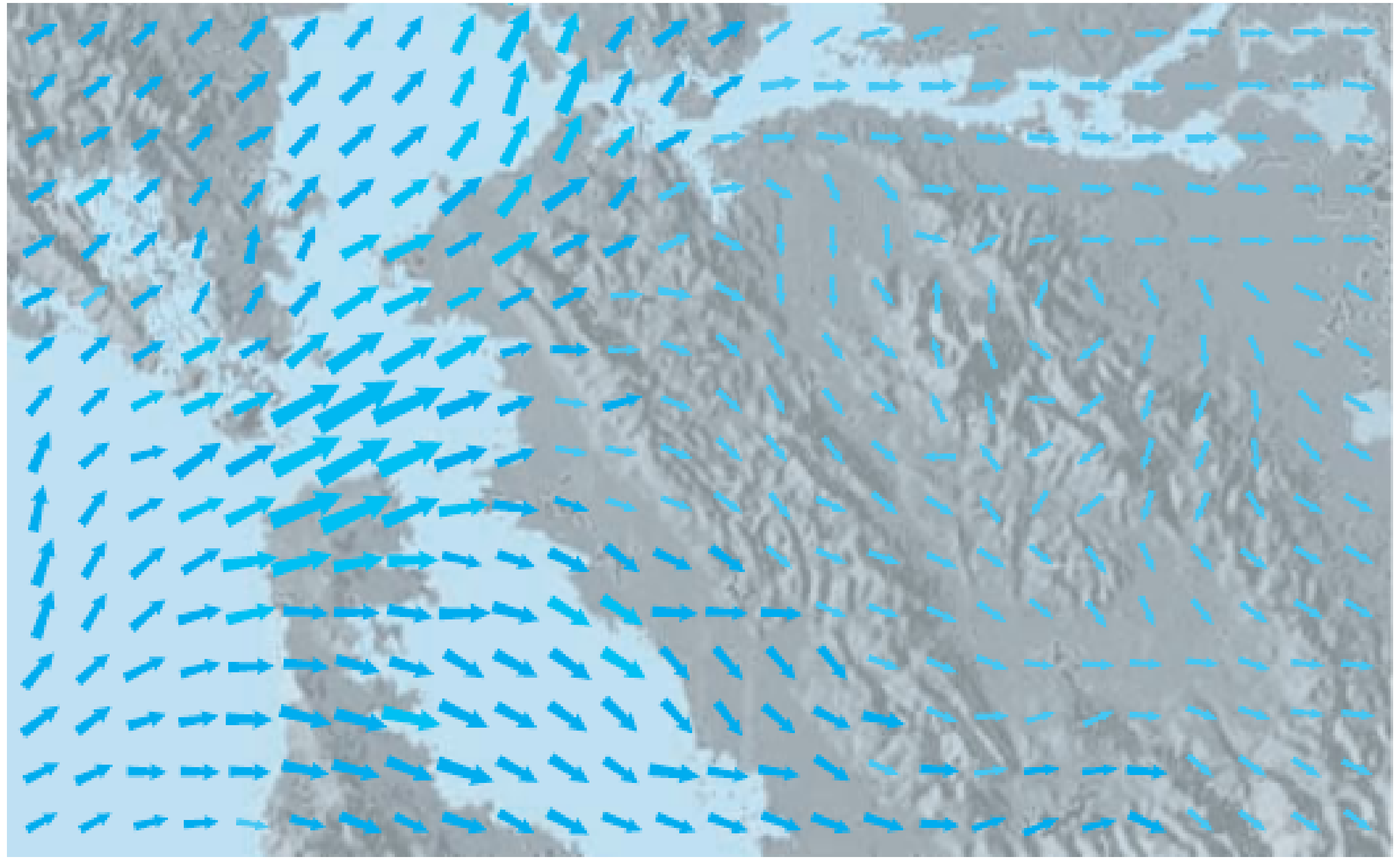
13 VECTOR CALCULUS

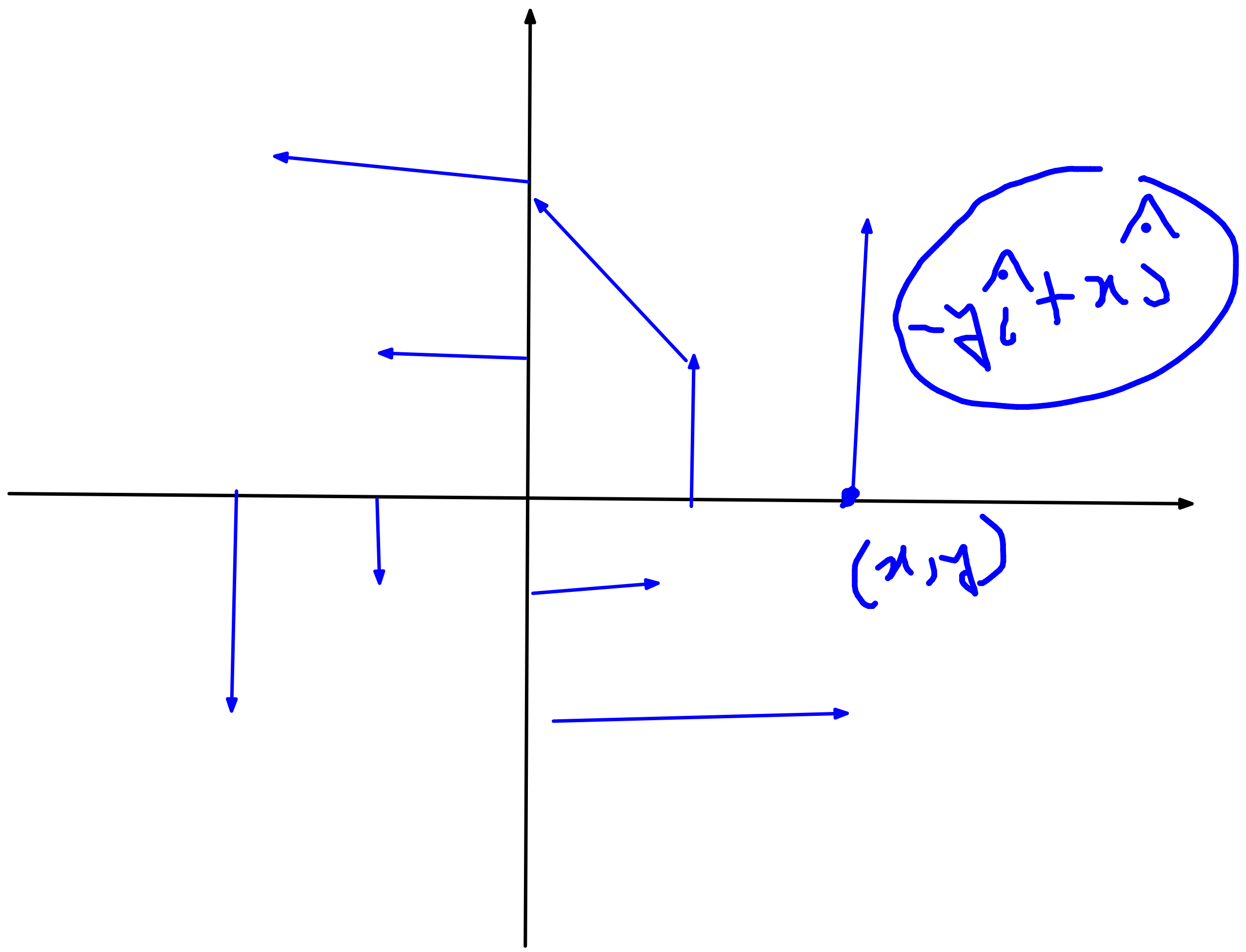
started last time

13.1 VECTOR FIELDS



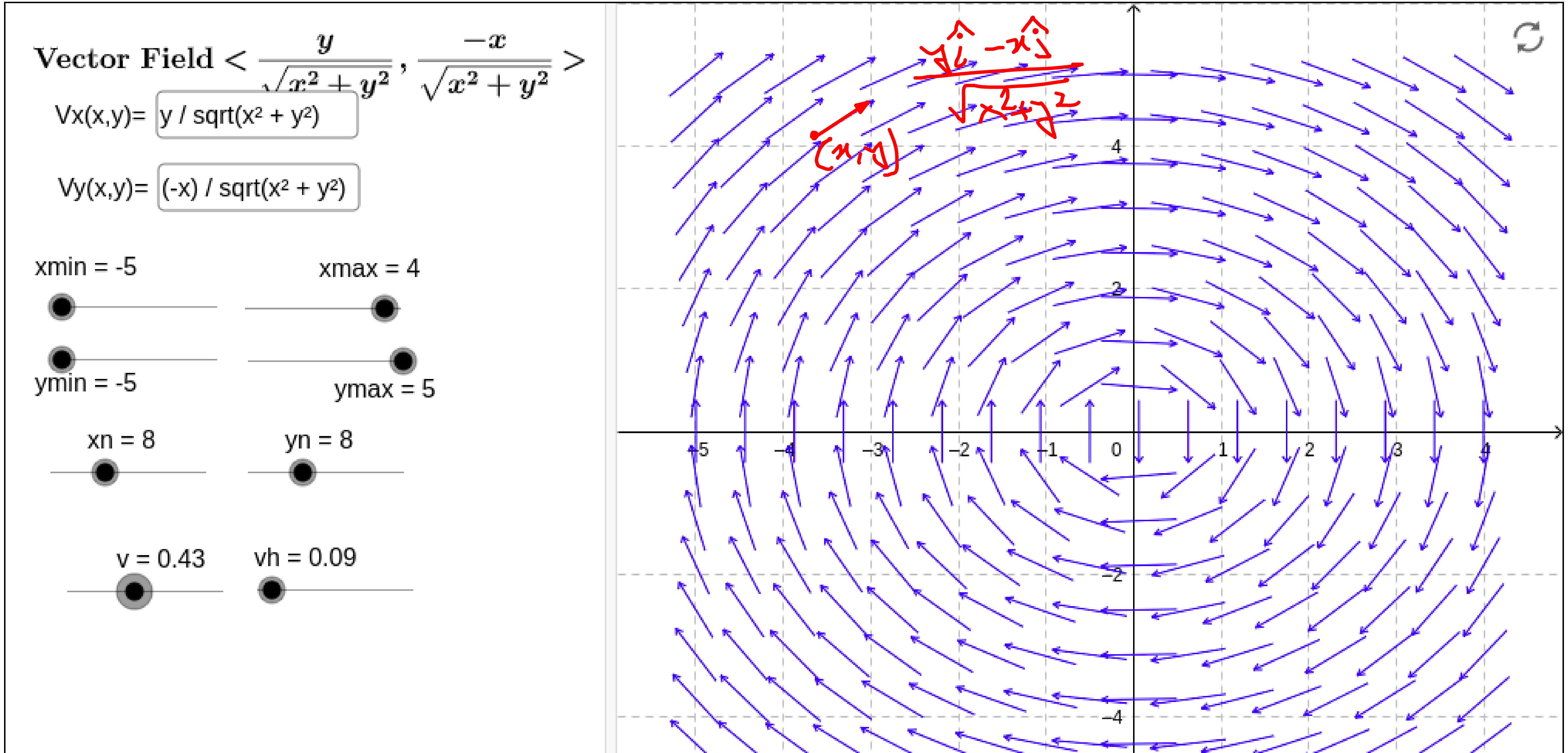
Sketch the vector field **F**

$$\mathbf{F}(\underbrace{x, y}) = -y\,\mathbf{i} + x\,\mathbf{j}.$$



Sketch the vector field **F** $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

geogebra.org/m/QPE4PaDZ

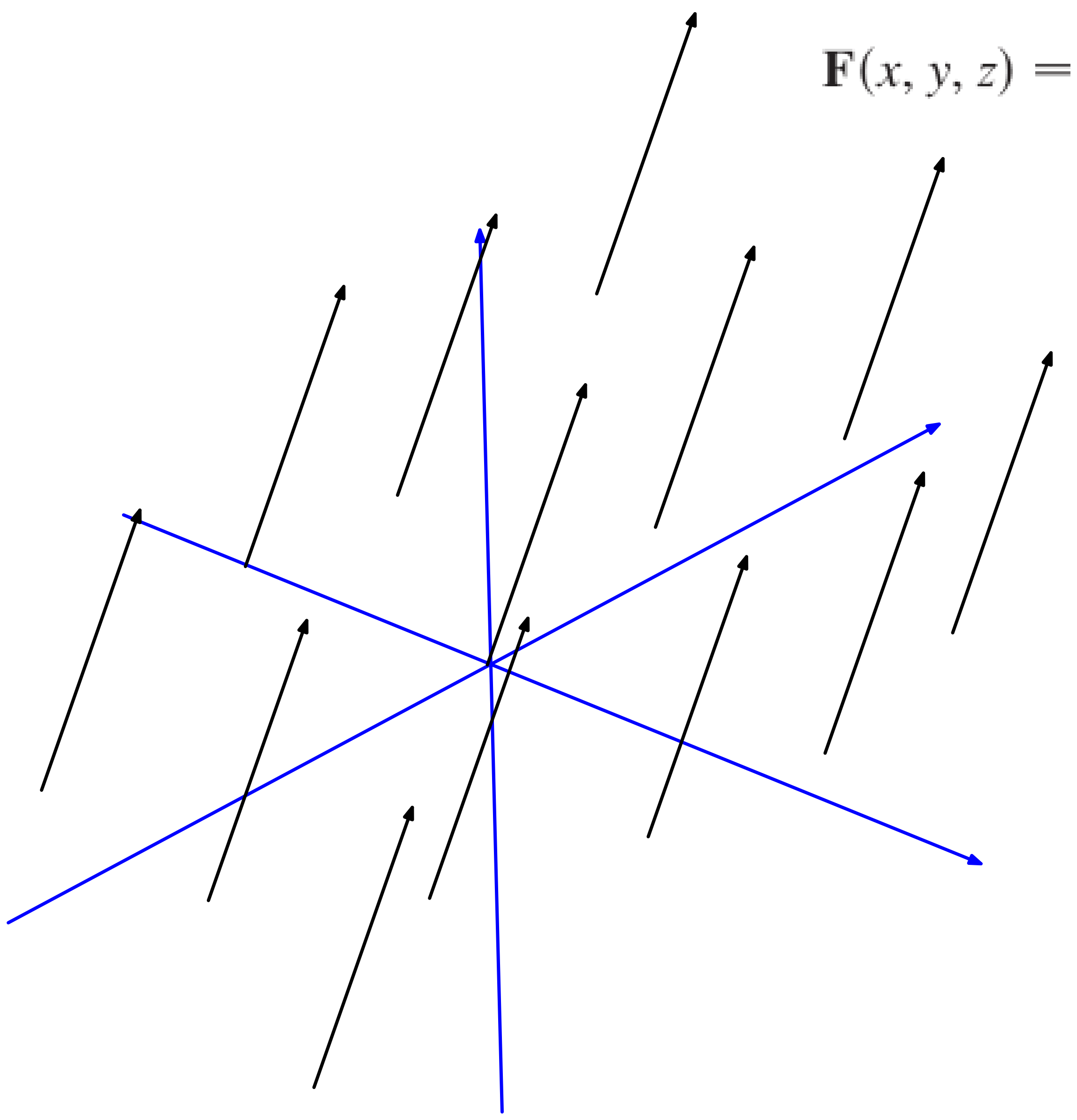


Sketch the vector field $\mathbf{F}$

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$$

Sketch the vector field **F**

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$$



Sketch the vector field $\mathbf{F}$

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$$

Vector Fields

(point in plane) $\longrightarrow$ vector

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x,y) = P\hat{i} + Q\hat{j}$$

$$= F_1\hat{i} + F_2\hat{j}$$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\vec{F}(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

GRADIENT FIELDS

| 13.1

later

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

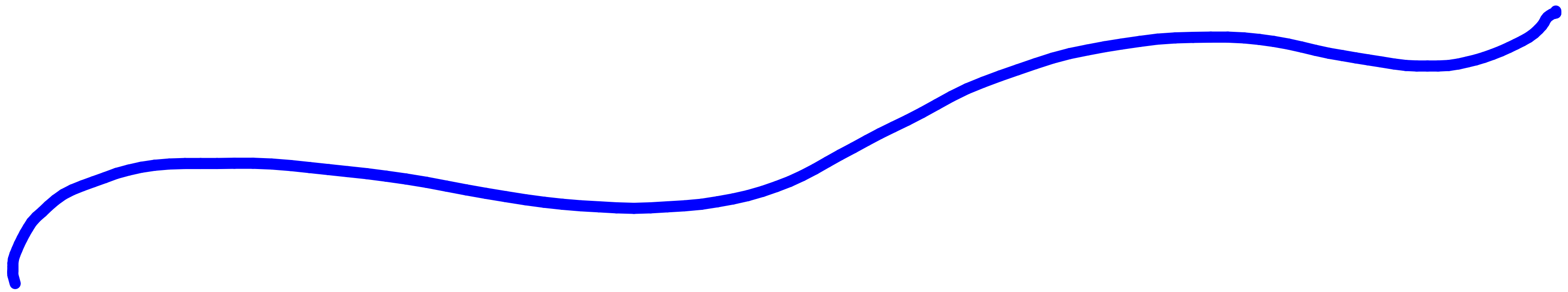
V EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

Recall position function from physics

→ a particle is moving in a path



→ $\vec{r}(t)$ = position of particle at time t

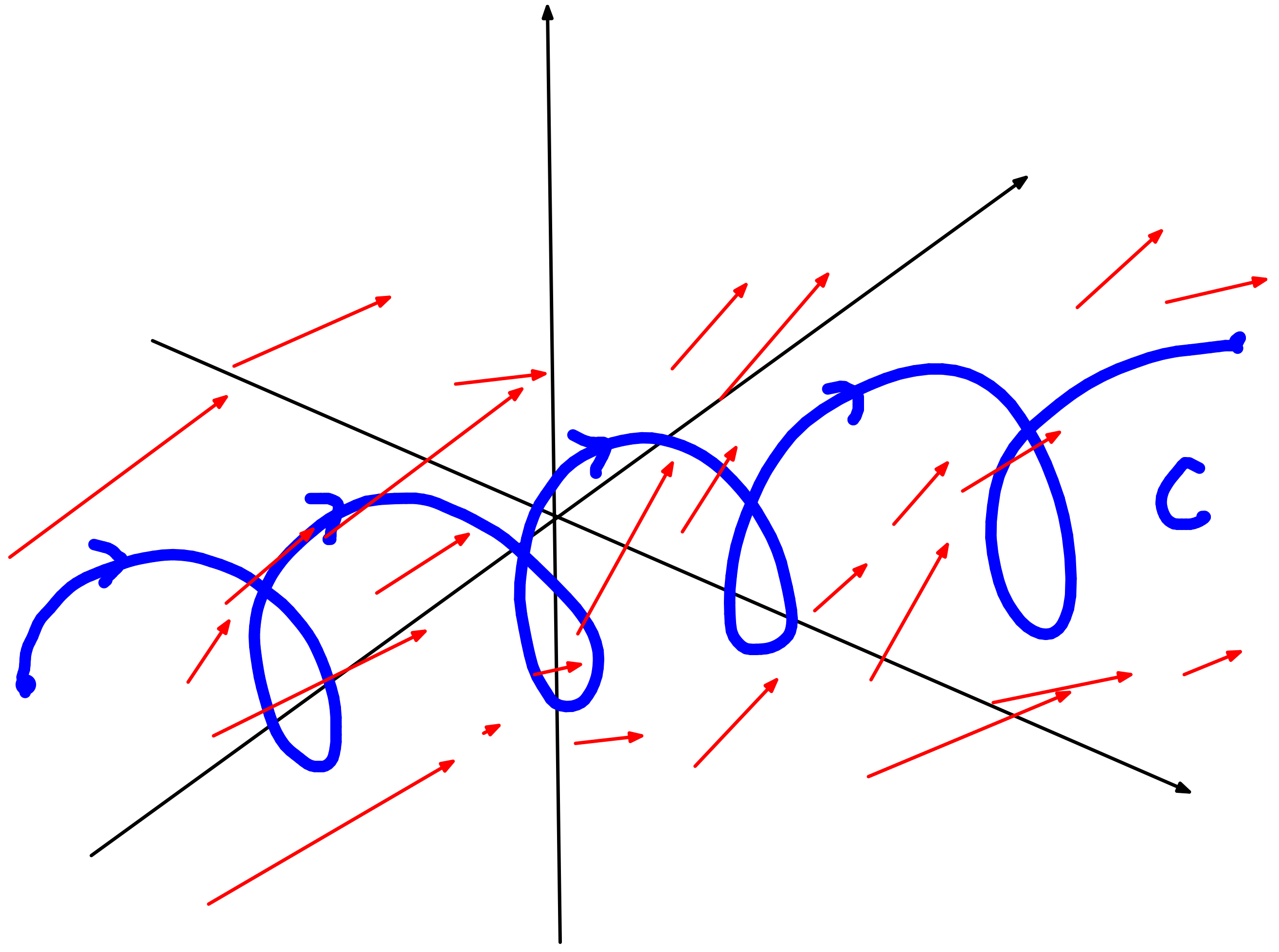
$= (x(t), y(t), z(t))$

→ Here we refer to these $\vec{r}(t)$ as curves.

& t is a parameter for the curve

We will focus for now:

Calculus on curves or paths.



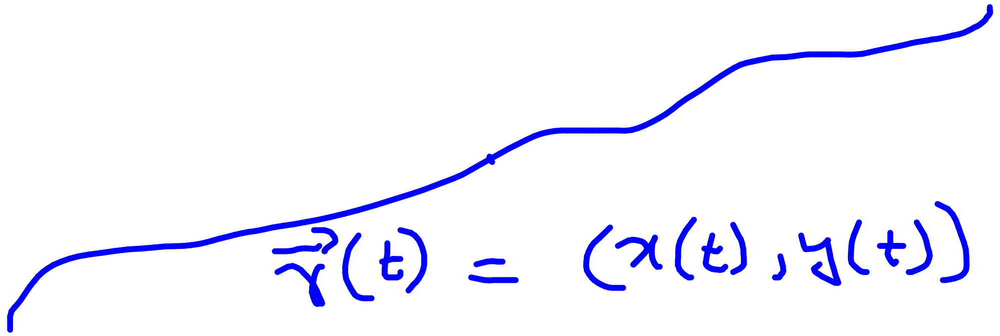
→ Length of the curve?

→ $\int_C f\,d\vec{r}$: integration of scalar function

→ $\int_C \vec{F}\cdot d\vec{r}$: integration of vector functions on curves.

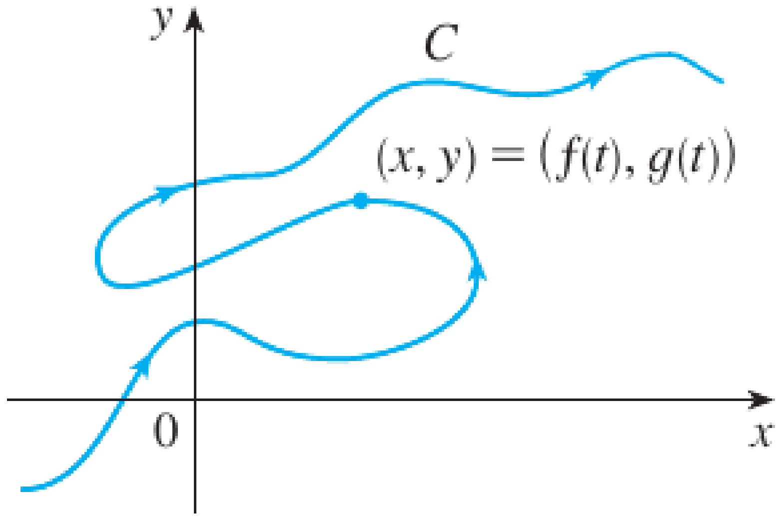
13.2 LINE INTEGRALS

next time



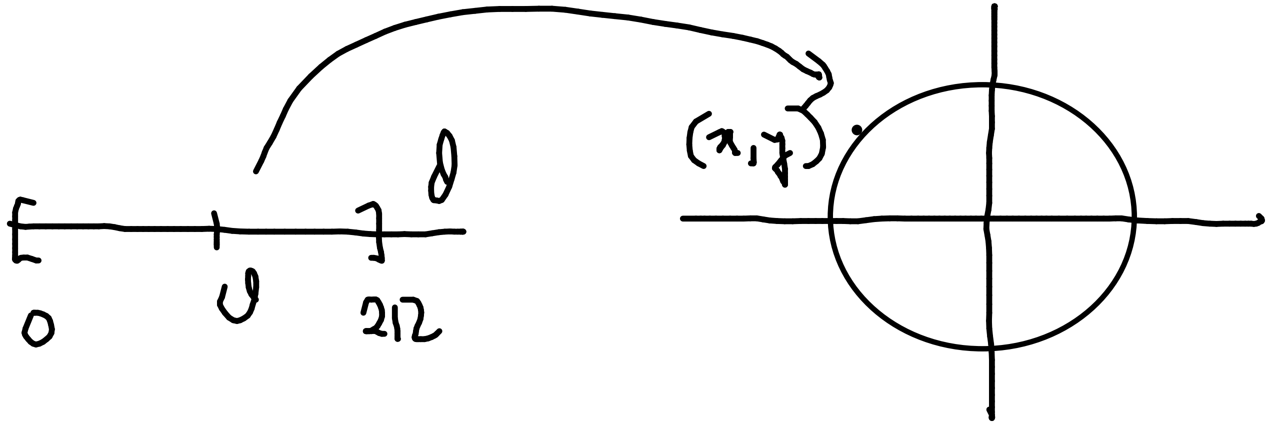
$\vec{r}(t) = (x(t), y(t))$

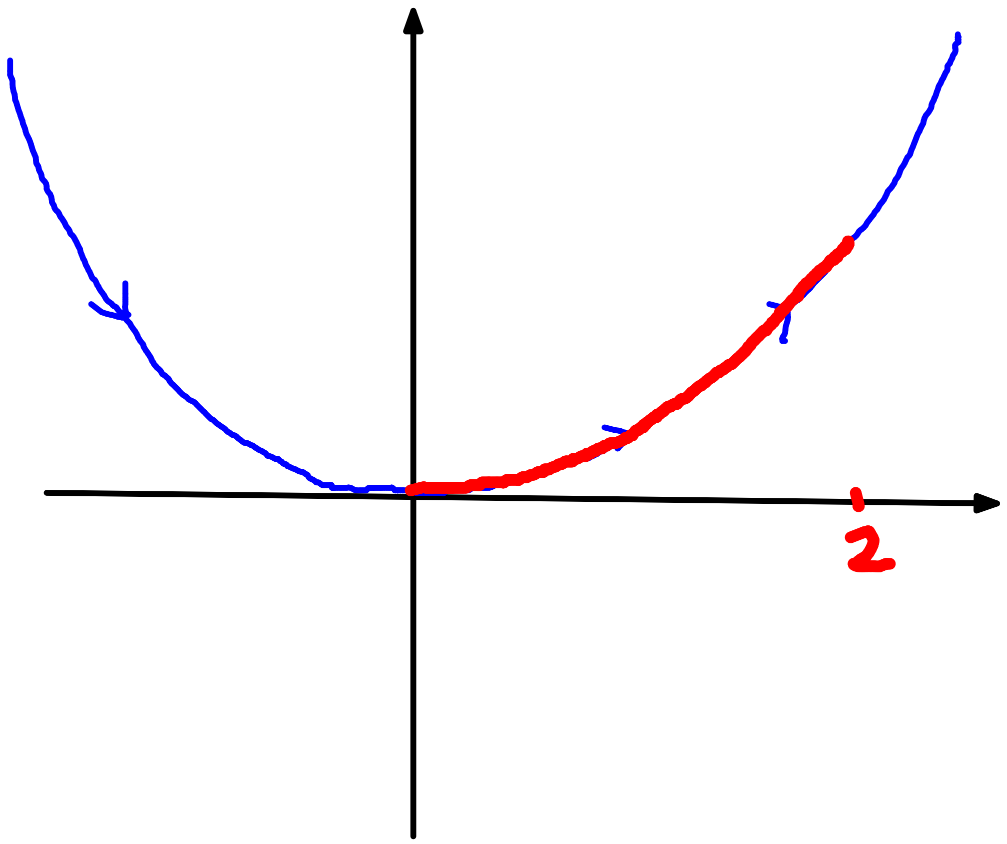
9.1 PARAMETRIC CURVES



:

$$\boxed{\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned} \qquad 0 \le \theta \le 2\pi}$$



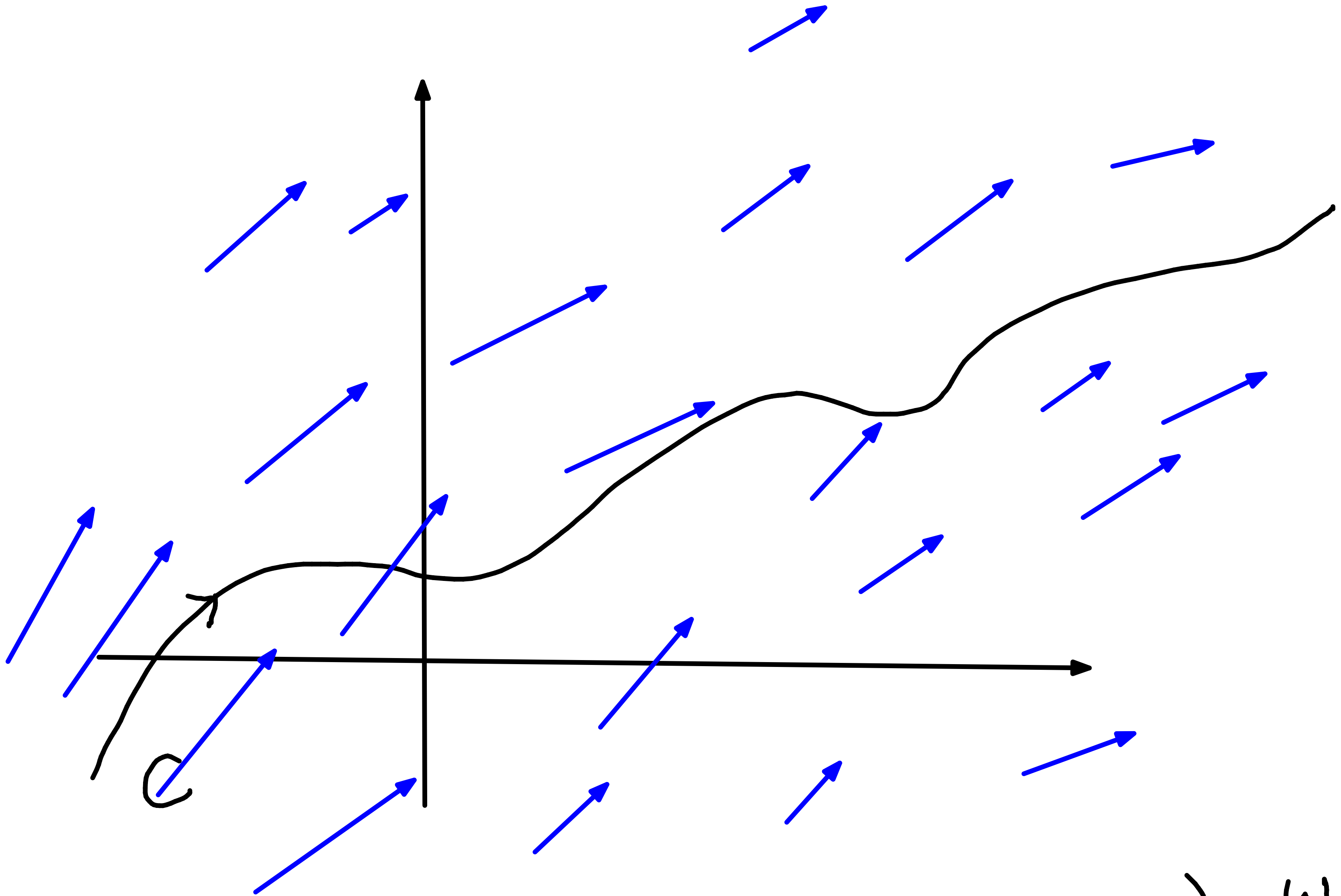


$y = x^2$

$$\left[\begin{array}{l} -\infty \le t \le \infty \\ x = t \\ y = t^2 \end{array}\right]$$

$$\boxed{\begin{array}{l} 0 \le t \le 2 \\ x = t \\ y = t^2 \end{array}} \longrightarrow$$

what curve ??



- Force field $\vec{F}$
- a particle is moving along a curve C

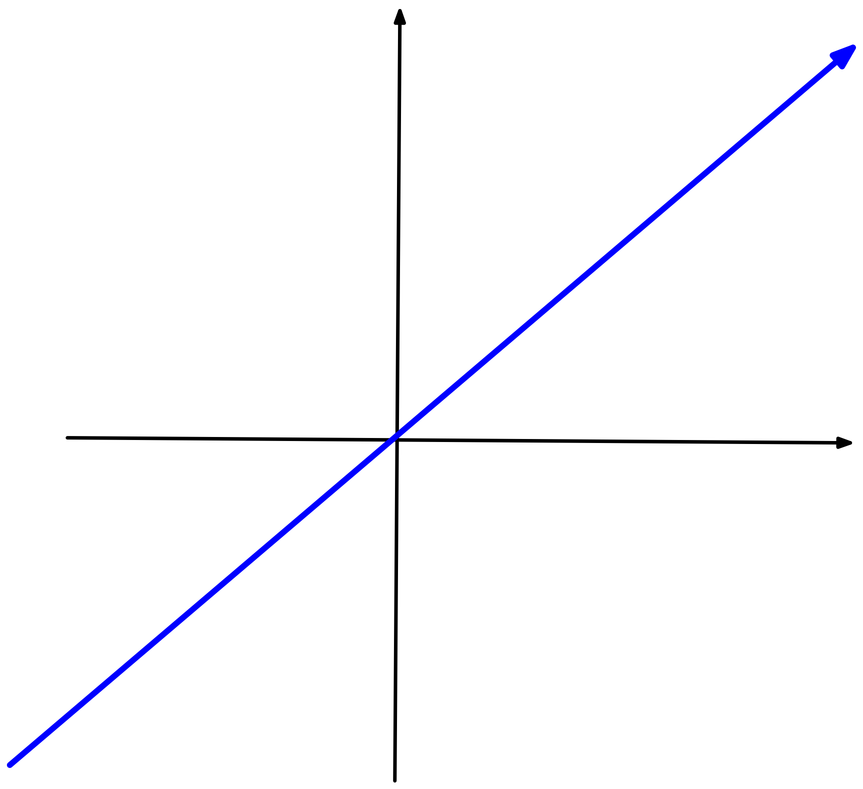
Q: find work done by $\vec{F}$ in moving the particle along the curve ??

→ we need a better precise description of whats a curve or path.

$-6\pi \leq t \leq 6\pi$

$x = t$

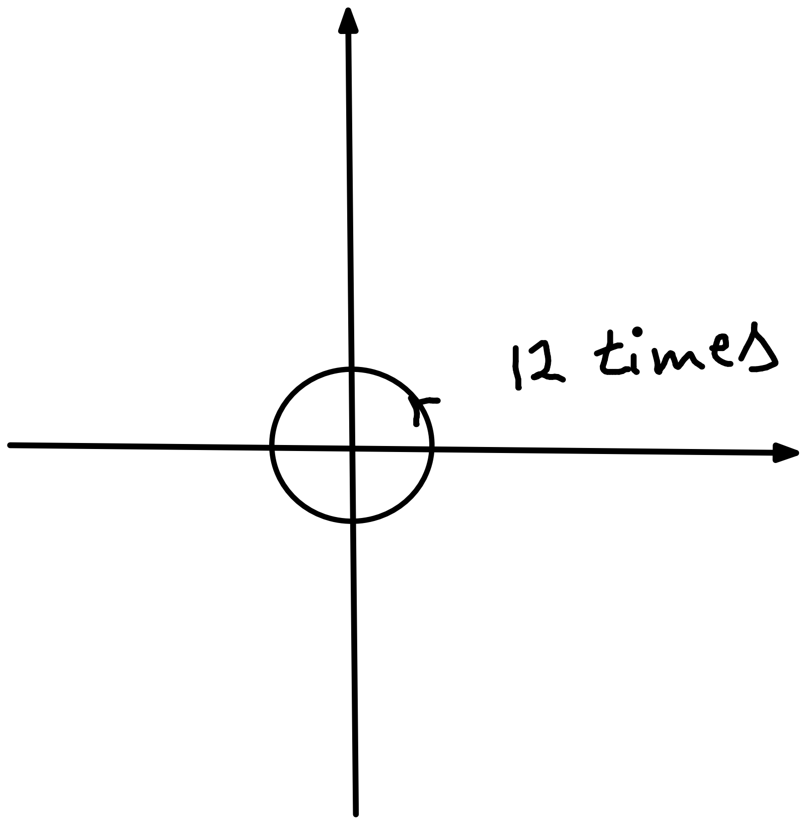
$y = t$



$-6\pi \leq t \leq 6\pi$

$x = 0.1\cos(2t)$

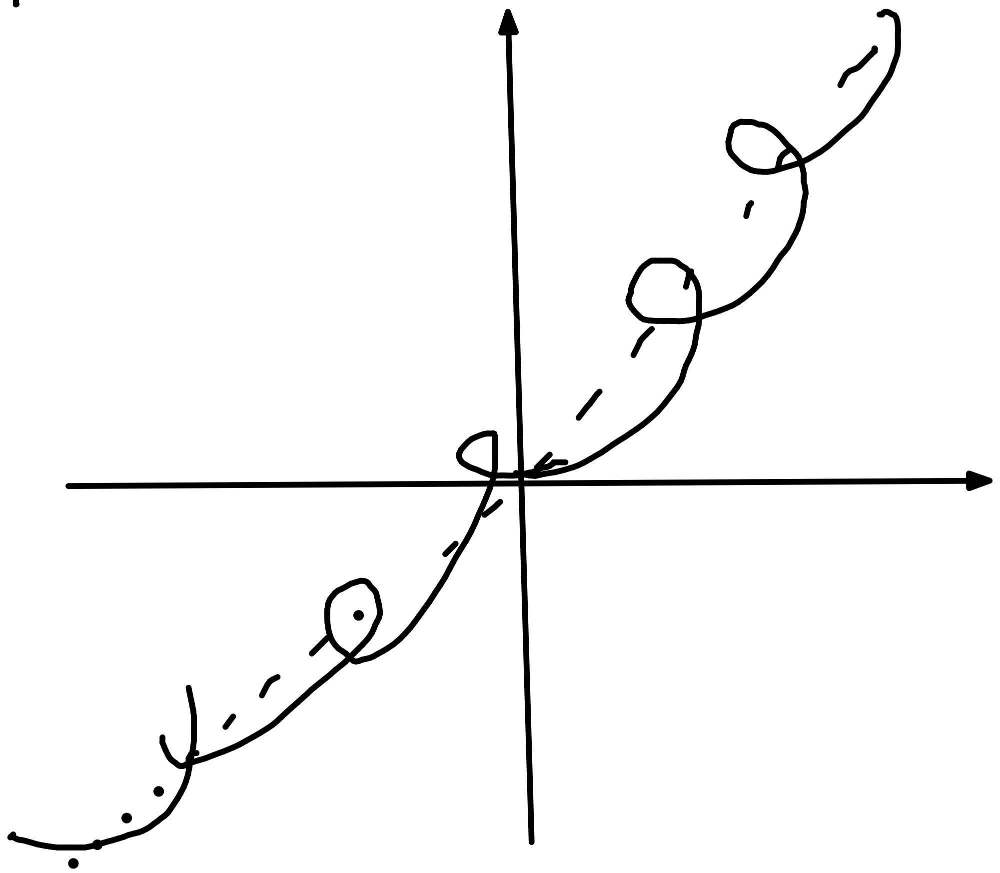
$y = 0.1\sin(2t)$

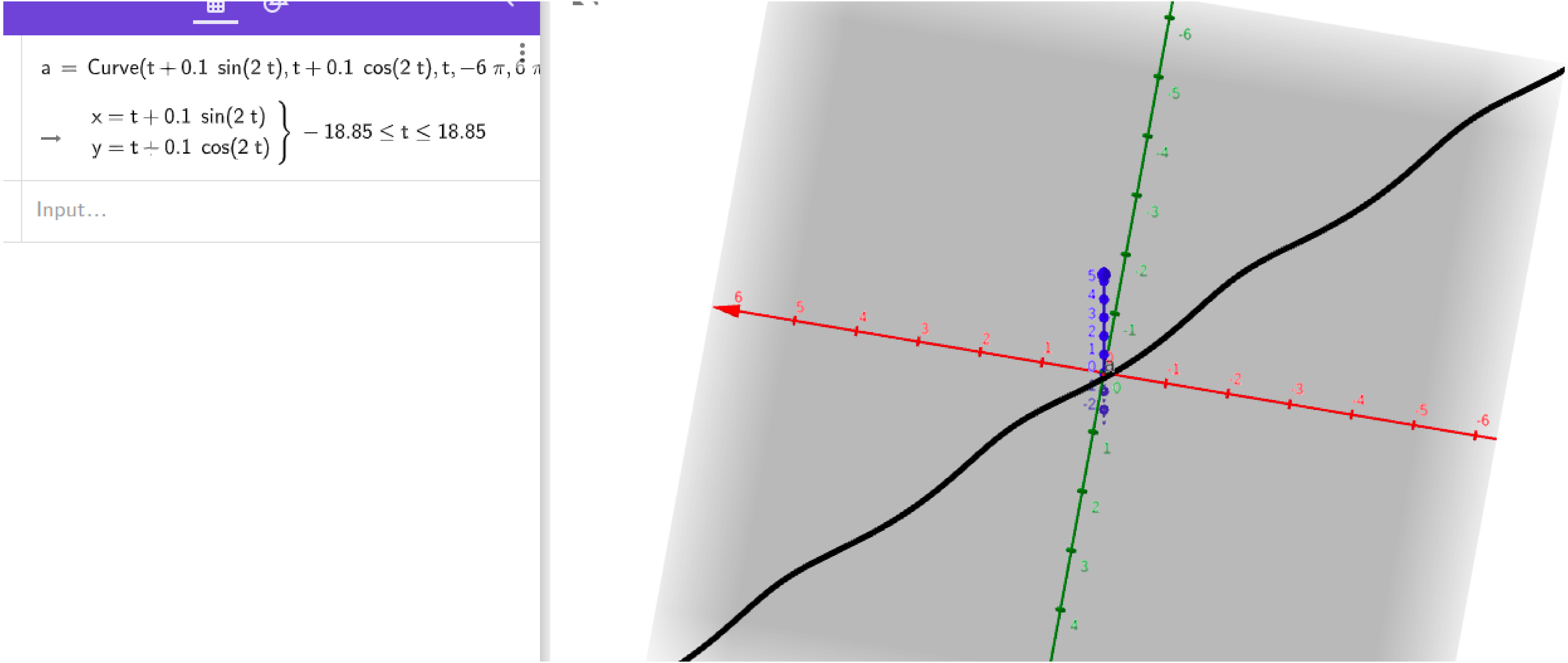


$-6\pi \leq t \leq 6\pi$

$x = t + 0.1\cos(2t)$

$y = t + 0.1\sin(2t)$

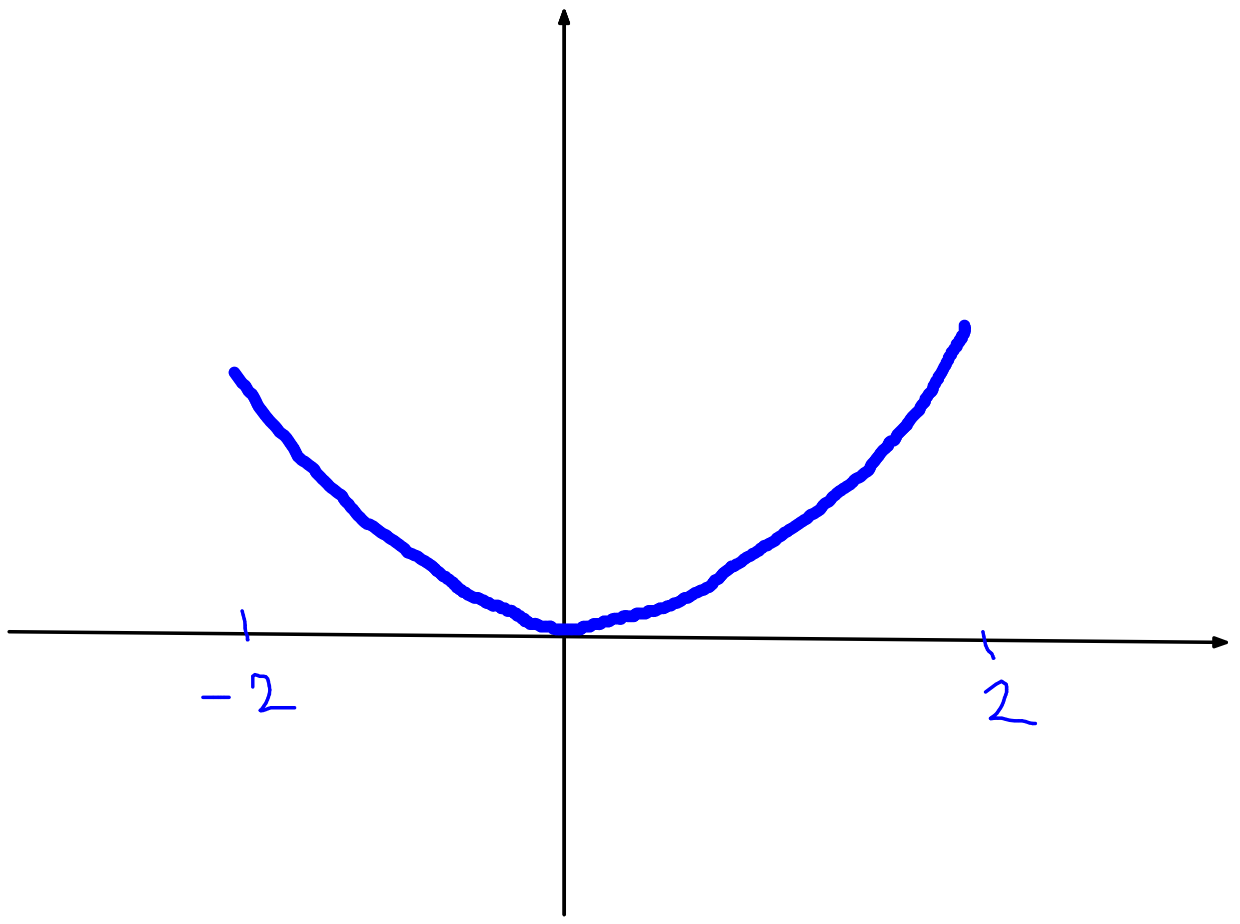




a = Curve(t + 0.1 sin(2 t), t + 0.1 cos(2 t), t, −6 π, 6 π
x = t + 0.1 sin(2 t)
y = t + 0.1 cos(2 t)
− 18.85 ≤ t ≤ 18.85
Input…

Q. Sketch the path

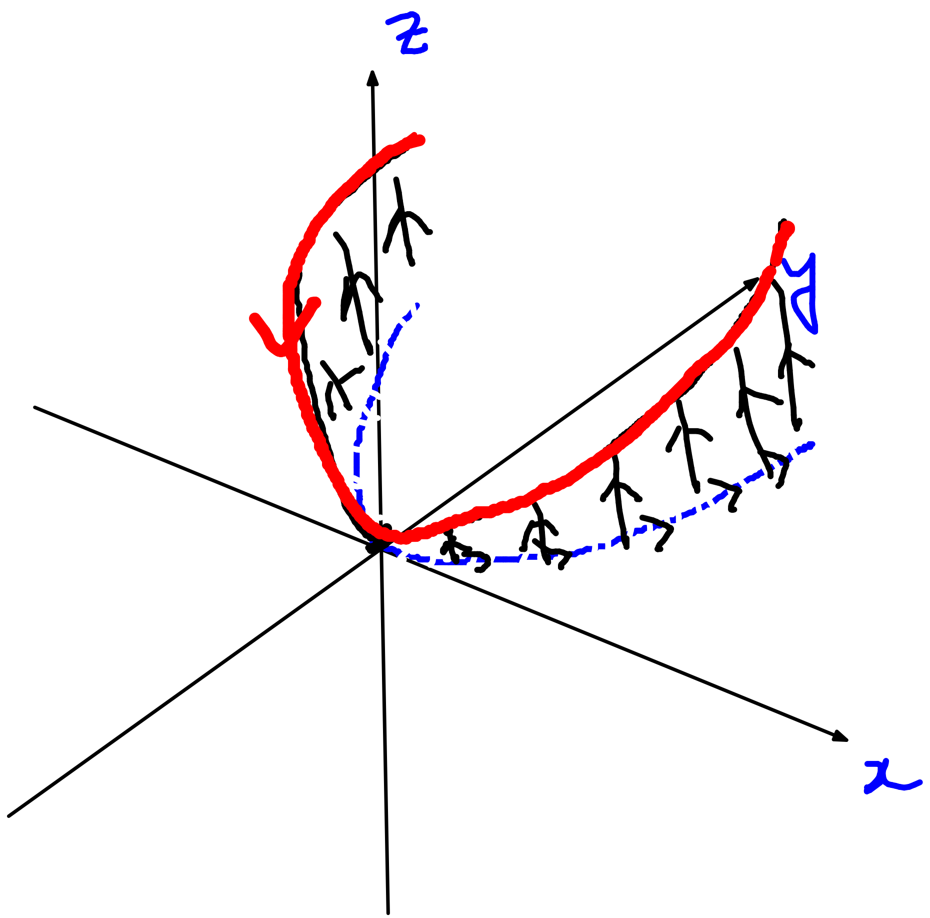
$$x = t \quad , \quad y = t^2 \quad , \quad -2 \le t \le 2$$



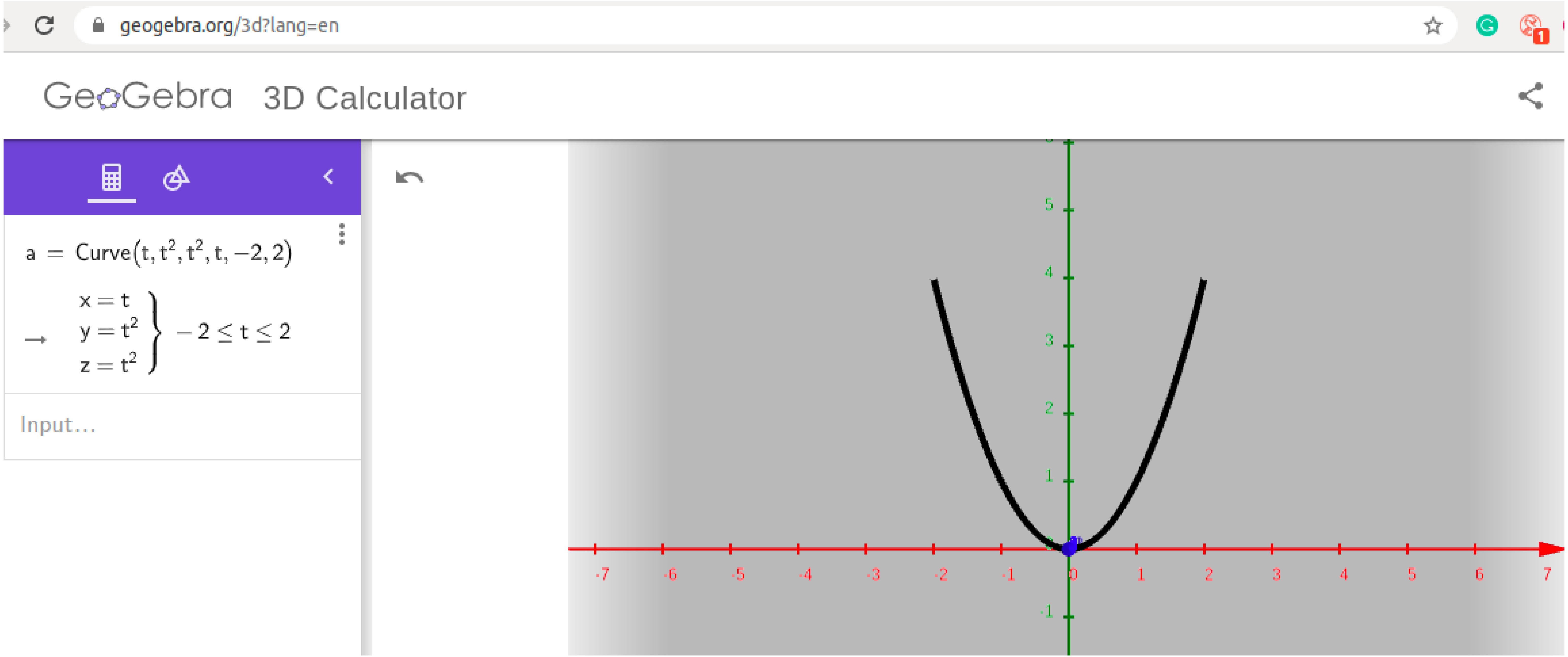
Q. Sketch the path

$$x = t \quad , \quad \underbrace{y = t^2}_{y = x^2} \quad , \quad z = t^2$$

$$-2 \le t \le 2$$



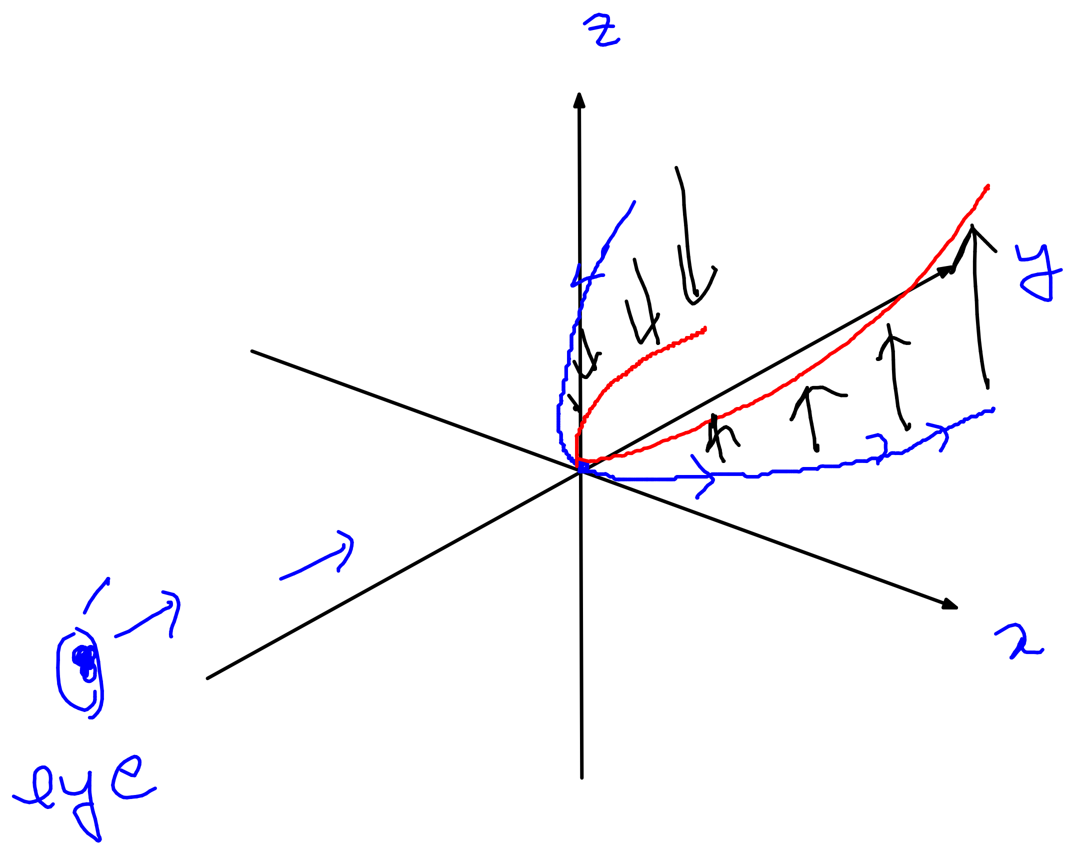
$t = 0 \qquad (x, y, z) = (0, 0, 0)$

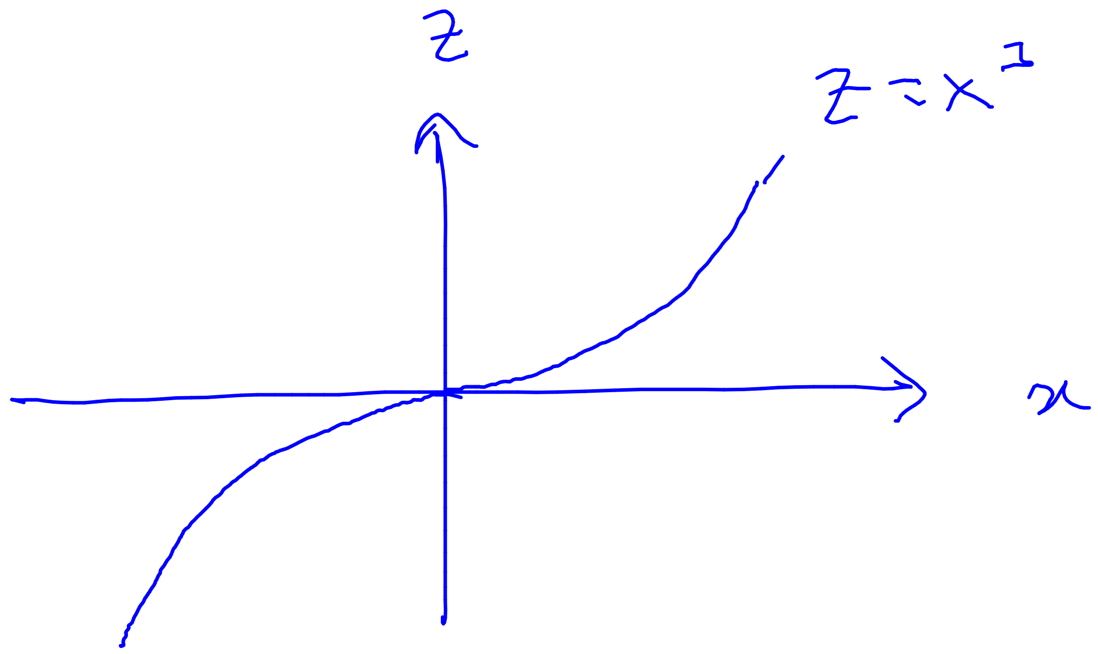

geogebra.org/3d?lang=en
GeoGebra
3D Calculator
$a = \mathrm{Curve}(t, t^2, t^2, t, -2, 2)$
$\left.\begin{array}{l} x = t \\ y = t^2 \\ z = t^2 \end{array}\right\} -2 \le t \le 2$
Input…
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
5 4 3 2 1 -1

Q. Sketch

$$x = t, \quad y = t^2, \quad z = t^3$$

$$-2 \le t \le 2$$





$$z = y^{3/2}$$

between

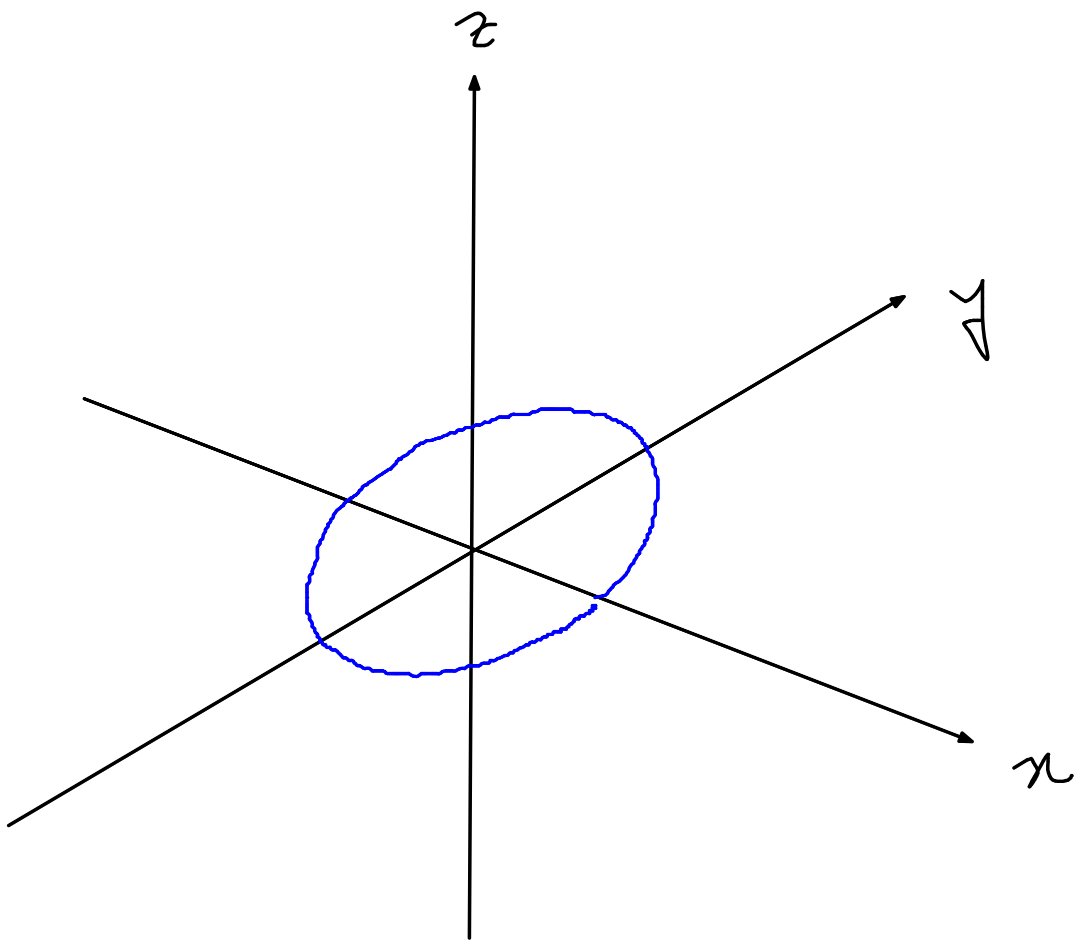
$z = y$

$z = y^2$

Q. Sketch

$x = \cos(t)$ $y = \sin(t)$ $z = 0$

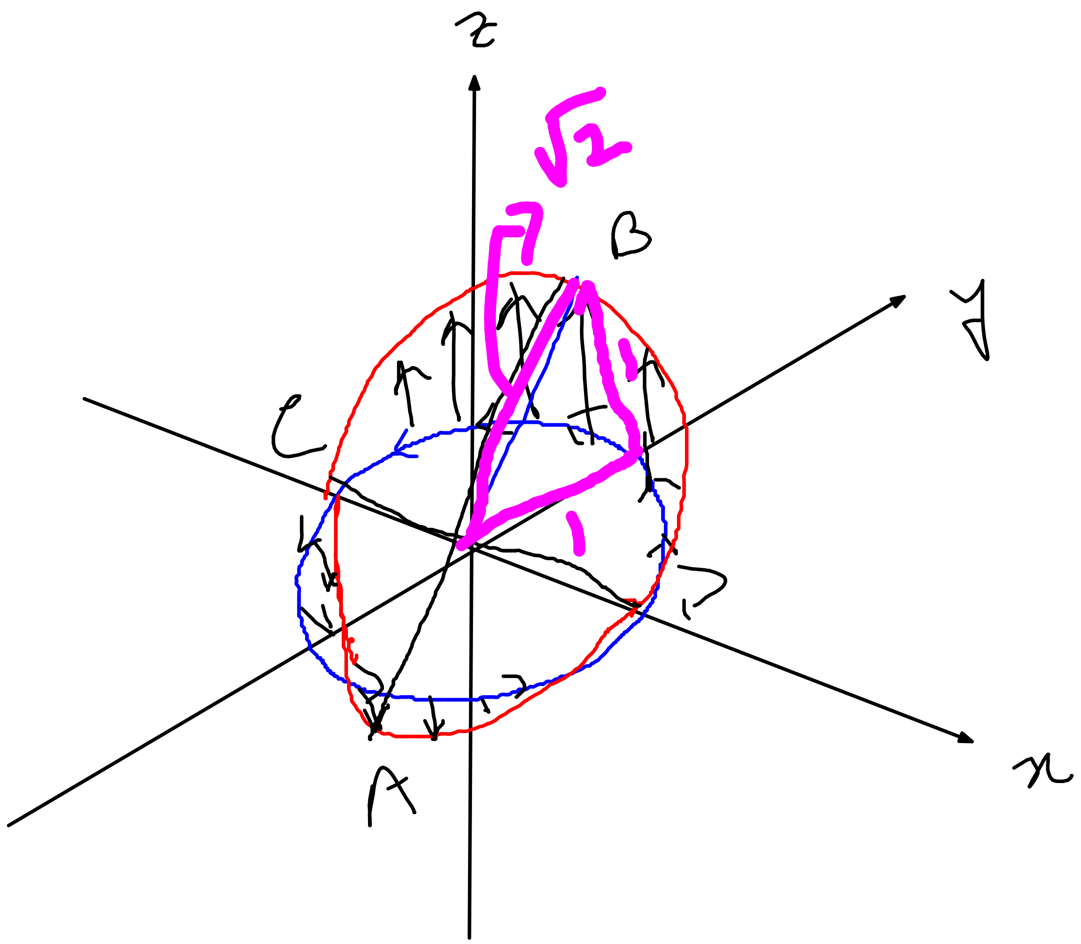
$0 \le t \le 6\pi$



Q. Sketch

$x = \cos(t)$ $\quad$ $y = \sin(t)$ $\quad$ $z = \sin(t)$

$0 \leq t \leq 6\pi$



$AB \overset{??}{=} CD$

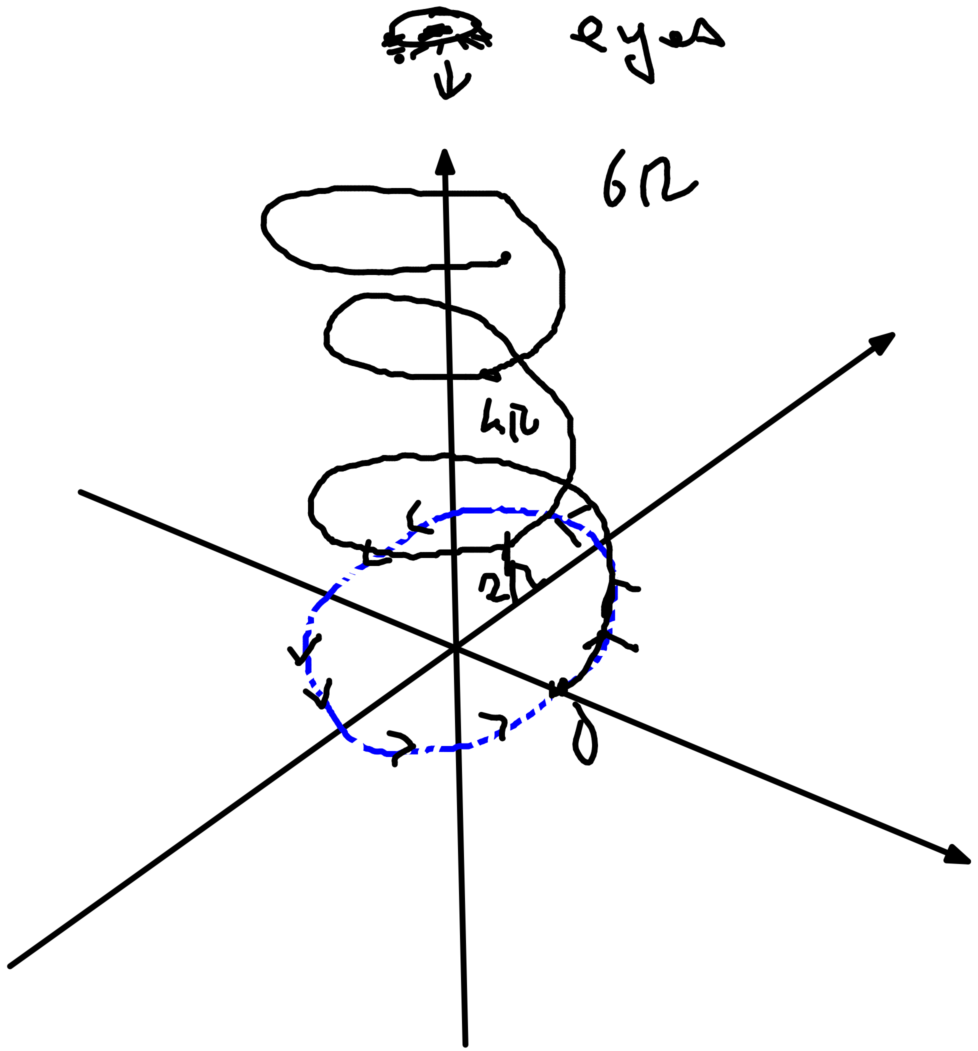
$2\sqrt{2}$ $\quad$ 2

ellipse

Q. Sketch

$x = \cos(t)$ $\quad y = \sin(t)$, $\quad z = t$

$0 \le t \le 6\pi$



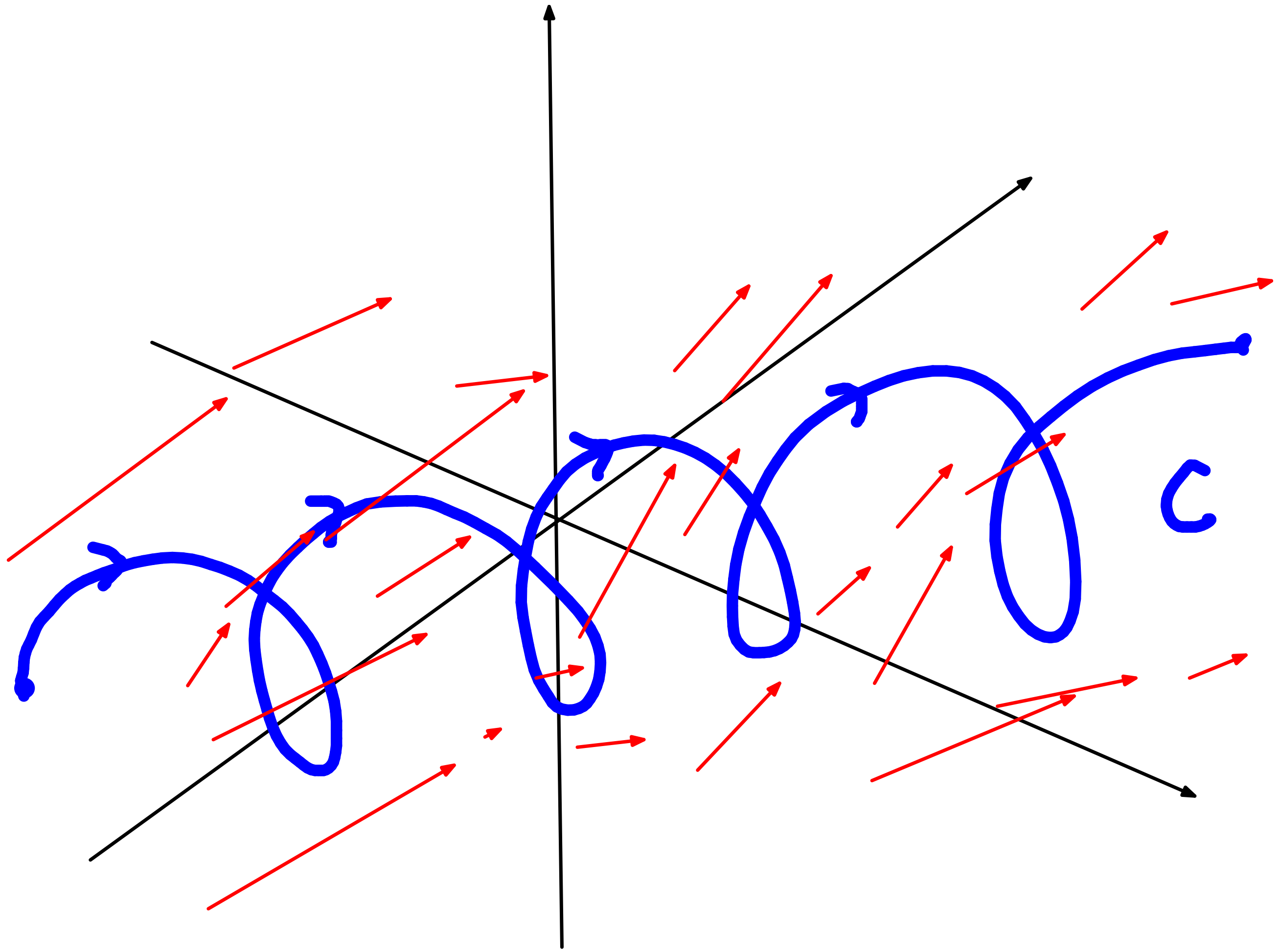
Calculus on curves

— next time

We will focus for now:

next time

Calculus on curves or paths.



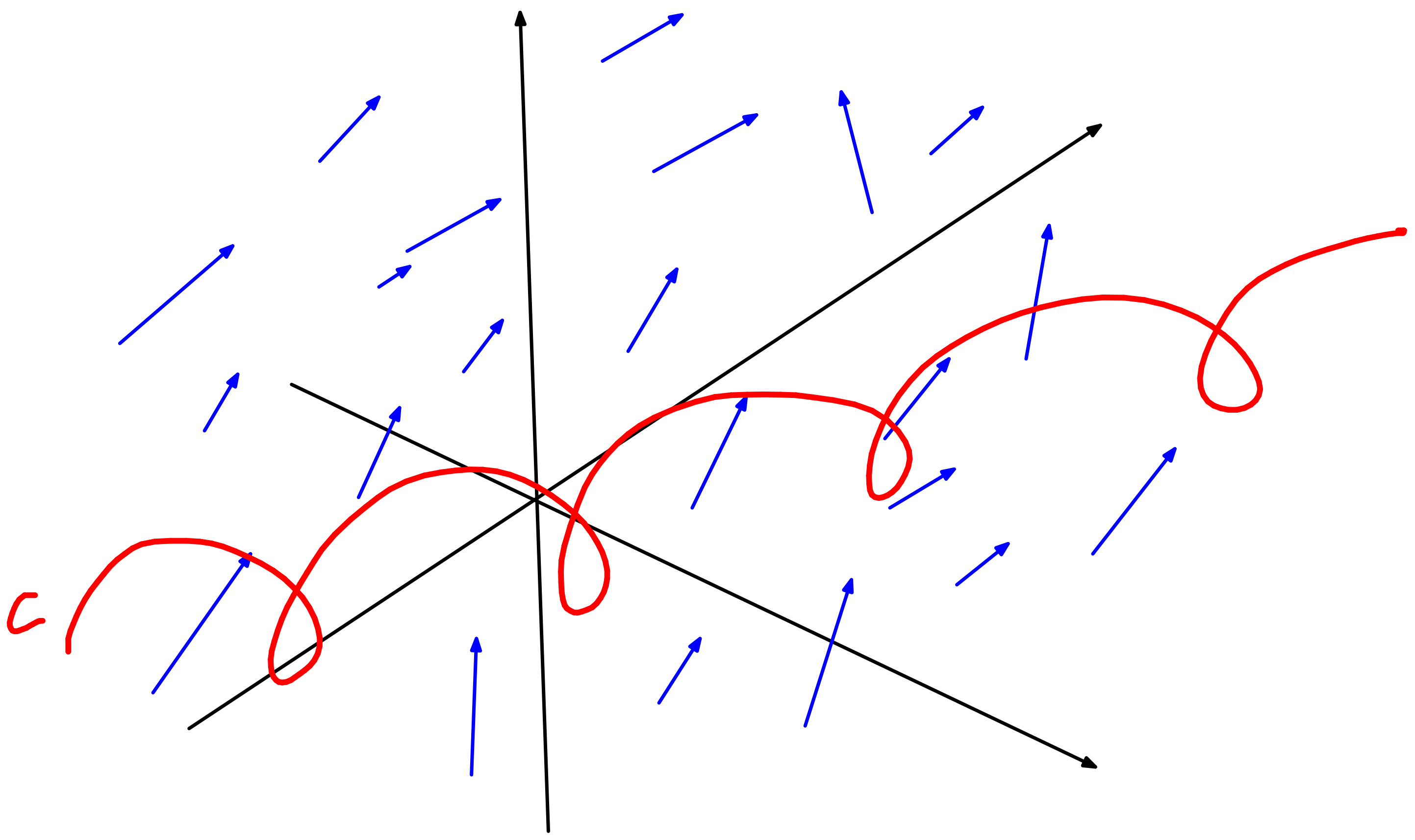
→ Length of the curve?

→ $\int_C f\,d\vec{r}$: integration of scalar function

→ $\int_C \vec{F}\cdot d\vec{r}$: integration of vector functions on curves.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$



$\int_C ds$: find length of curve

$\int_C f.ds$: integration of scalar f^n function

$\int_C \vec{F}.d\vec{r}$: work done by $\vec{F}$

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

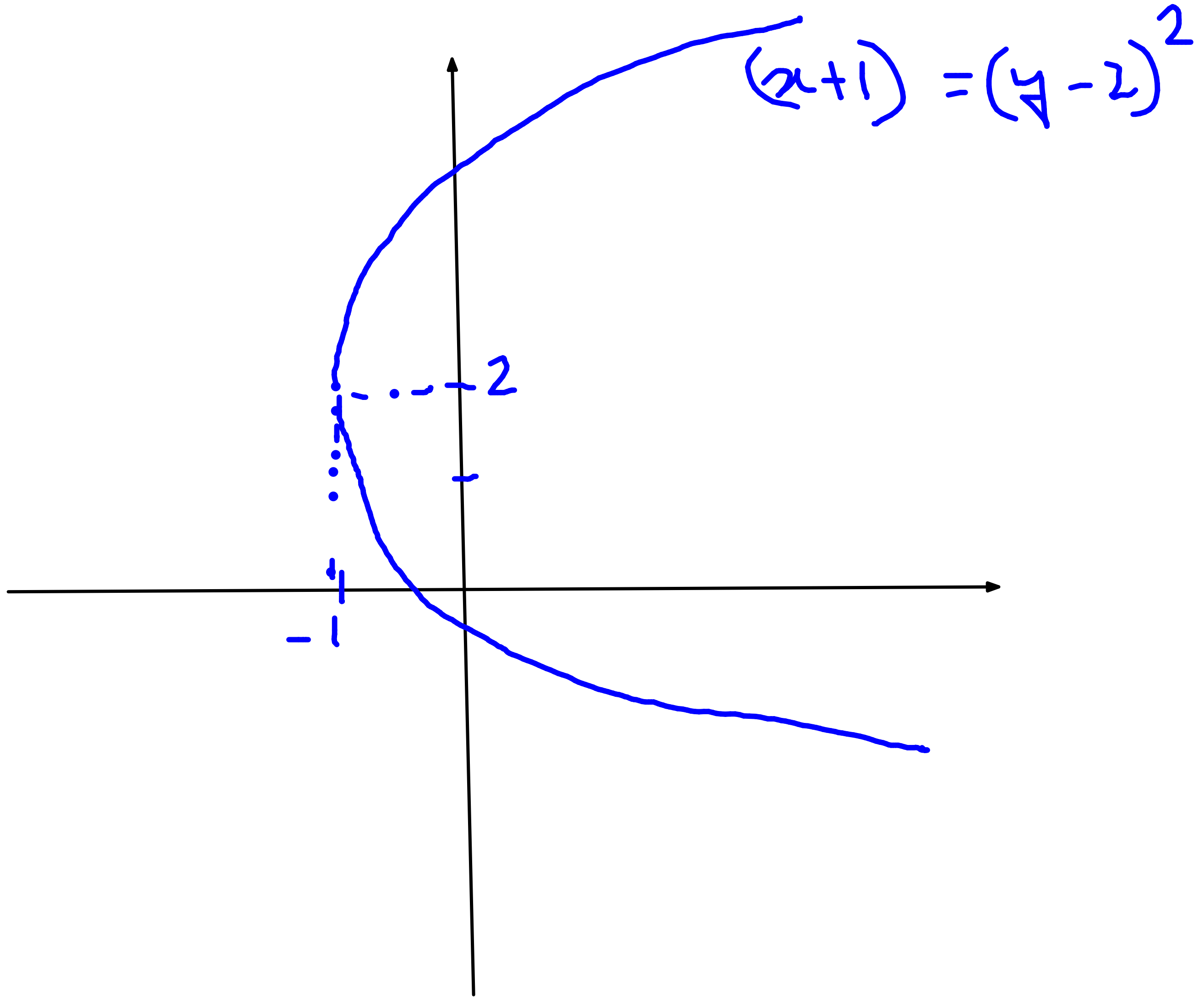
$x = t^2 - 2t$

$y = t + 1$

?? Sketch the shape

$x = (y-1)^2 - 2(y-1)$

$(x+1) = (y-2)^2$



$$x = t + 2\sin 2t$$
$$y = t + 2\cos 5t$$

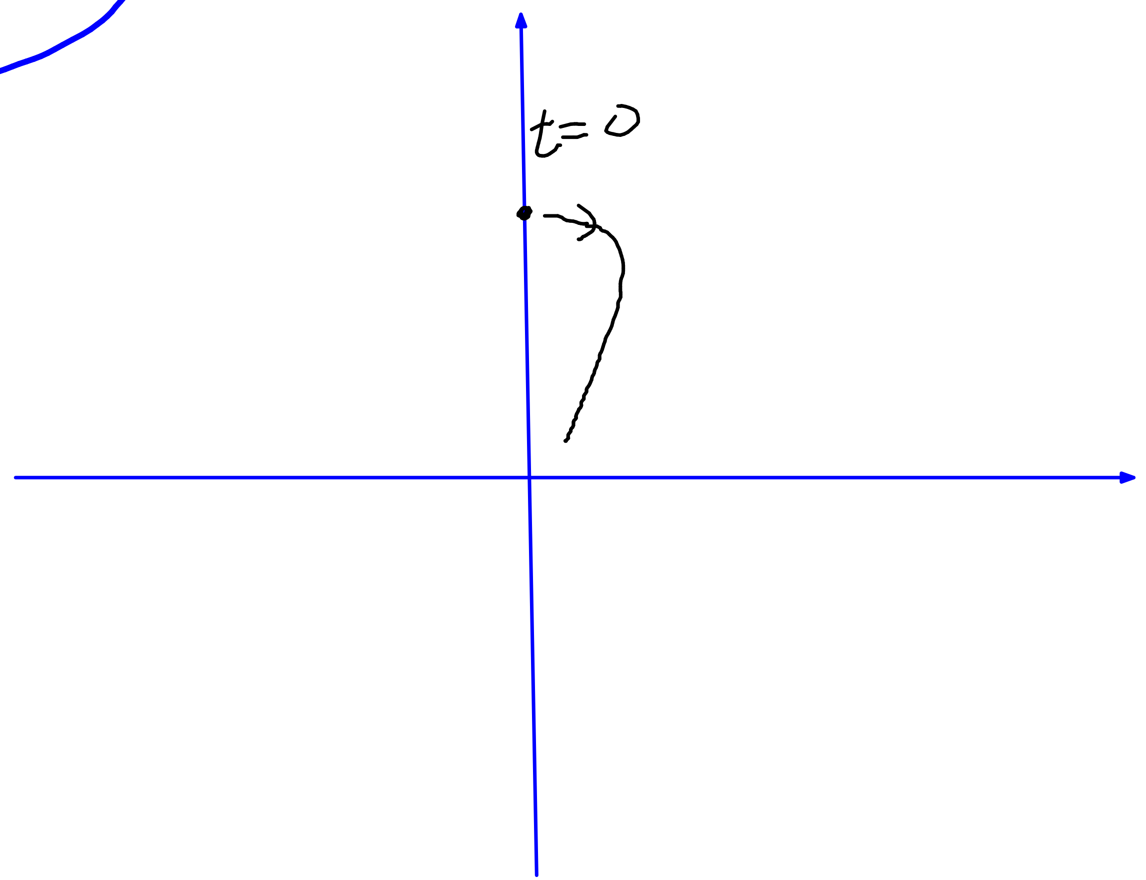
Sketch by intuition first

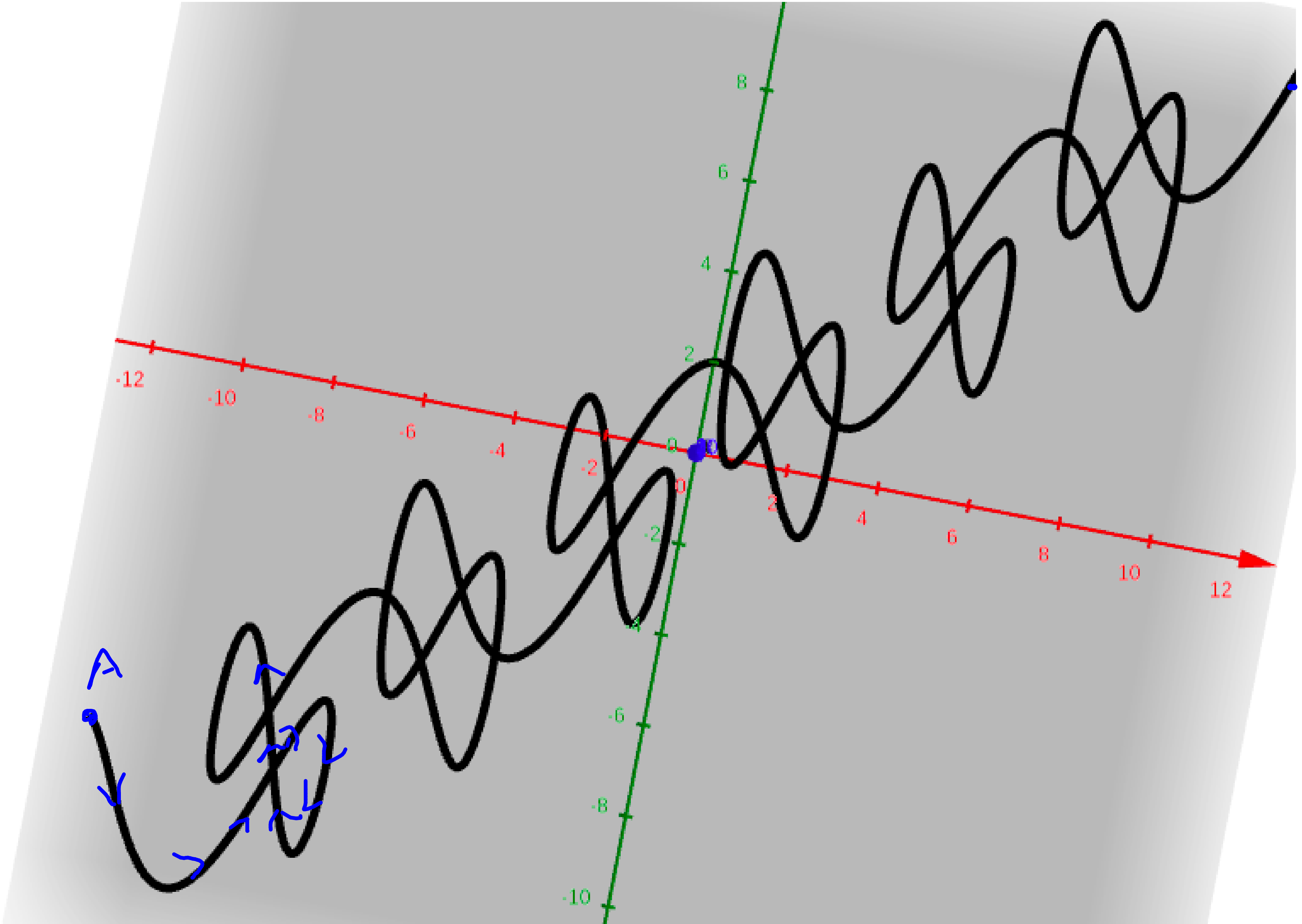
$x = t$
$y = t$

$+$

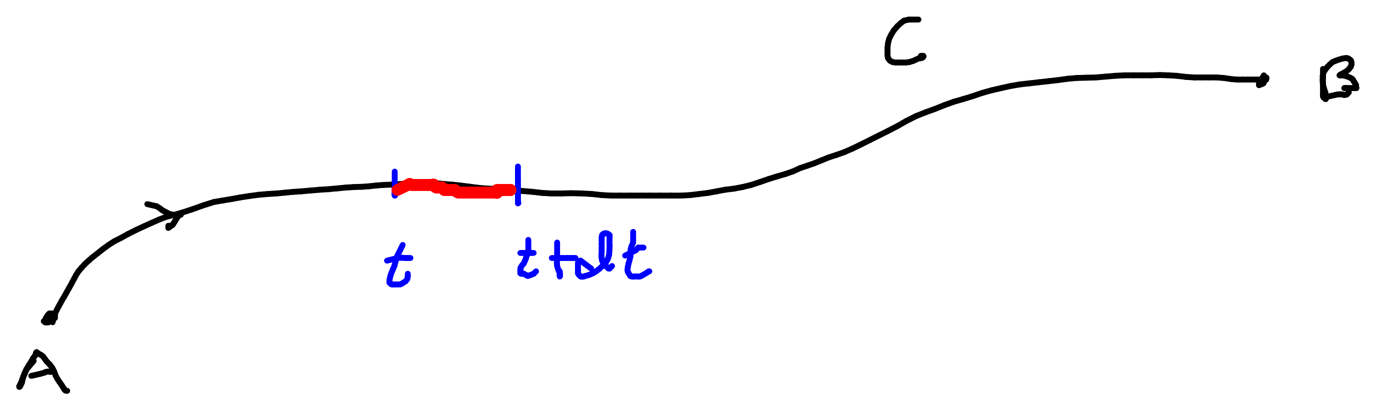
$x = 2\sin 2t$
$y = 2\cos 5t$

$t = 0$
$\pi/6$
$\pi/4$
$\pi/3$
$\pi/2$




A

Q.



parametric eq^n

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$a \le t \le b$

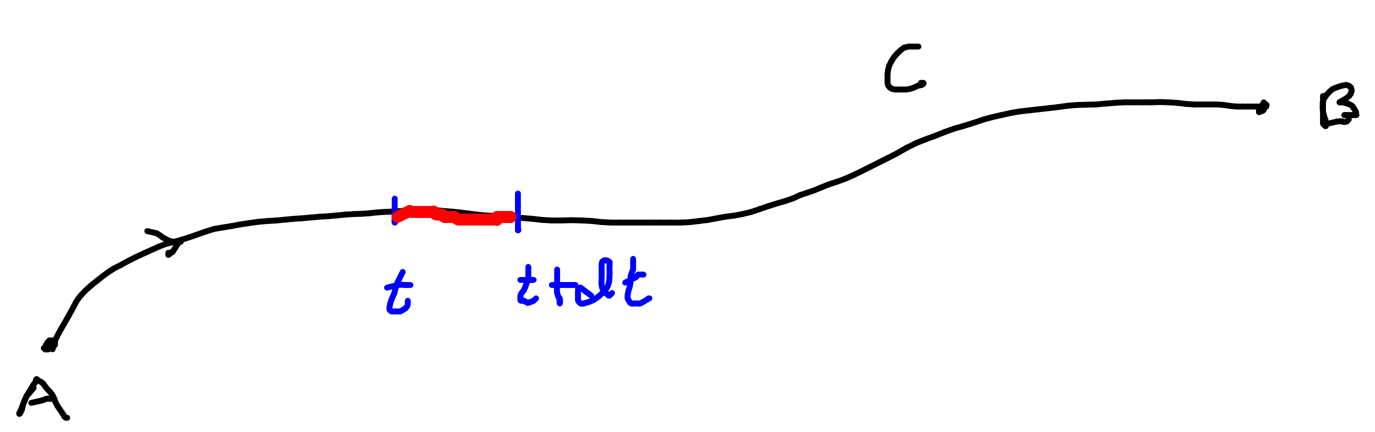
is there a distance formula to measure the length of the curve ??

L : total length

dL : distance travelled in t to $t+dt$; & dt is small enough to assume that speed was constant in the interval $(t, t+dt)$

$dL = (\text{speed at time } t)\, dt$

Q.



parametric eq^n

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$a \le t \le b$$

is there a distance formula to measure the length of the curve ??

L : total length

dL : distance travelled in t to $t+dt$; & dt is small enough to assume that speed was constant in the interval $(t, t+dt)$

$$dL = \underbrace{(\text{speed at time } t)}_{??}\, dt$$

$$= |\vec{r}\,'(t)| \; dt$$

$$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

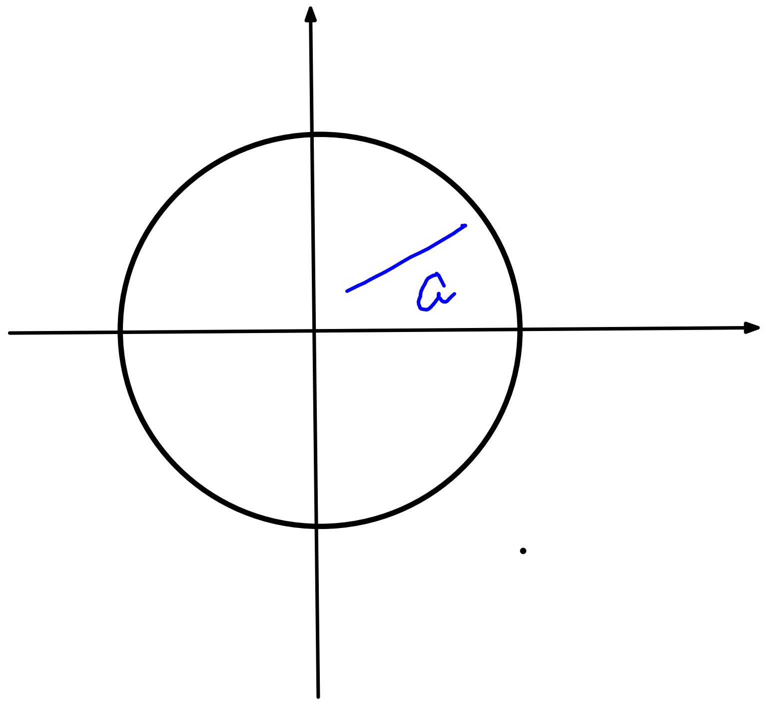
position $\quad \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

velocity $\quad \vec{r}\,'(t) = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j}$

speed $\quad |\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$L = \int dL = \int_a^b (\text{speed}) \, dt = \int_a^b |\vec{r}\,'(t)| \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Q.



circumference = $2\pi a$

find the length of path

$x = a\cos t \qquad 0 \le t \le 2\pi$

$y = a\sin t$

$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

$|\vec{r}\,'(t)| = a$ = speed

$$L = \int_0^{2\pi} |\vec{r}(t)|\,dt = \int_0^{2\pi} a\,dt = 2\pi a$$

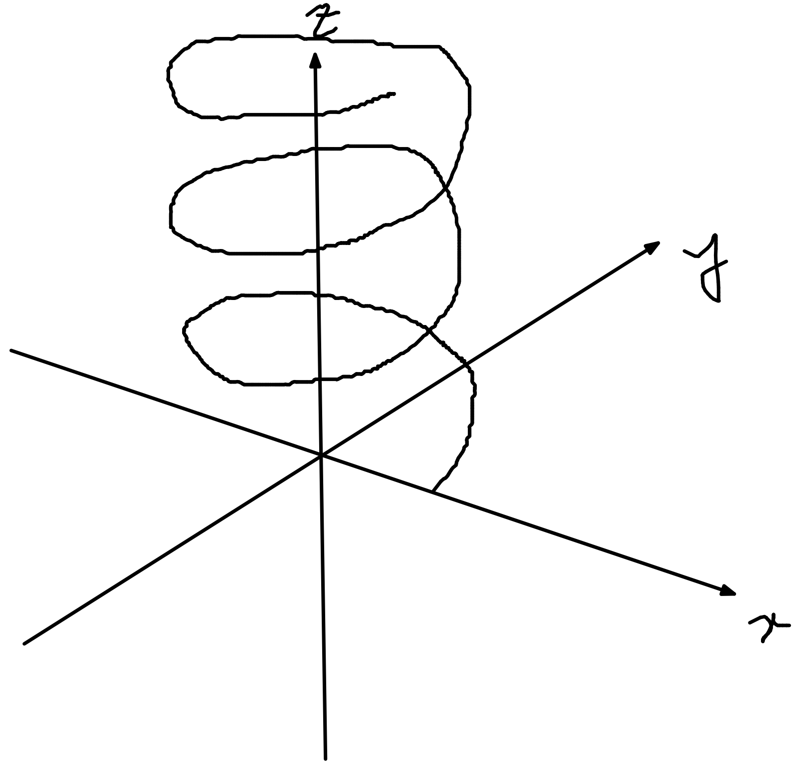
Q. Recall this curve

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = t$$

$0 \le t \le 6\pi$

→ sketch

→ find length



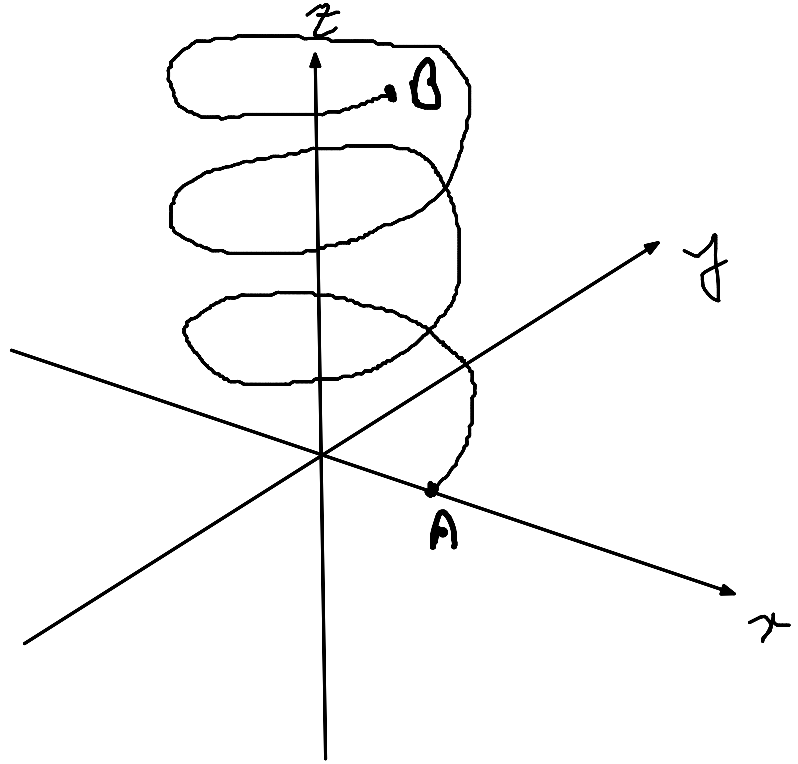
$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{2}$$

$$L = \int_0^{6\pi} \sqrt{2}\,dt = 6\pi\sqrt{2}$$

Q: Recall this curve

$$x = \cos(t) \qquad 0 \le t \le 6\pi$$
$$y = \sin(t)$$
$$z = t$$



→ think of AB as a wire

→ material used to make this wire in non-uniform

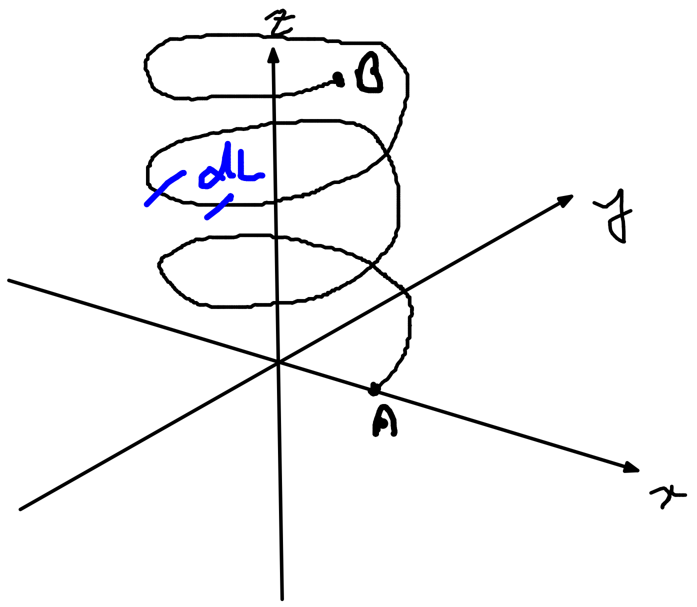
→ $f(x,y,z)$ represents linear density (mass per unit length)

→ let $f(x,y,z) = z$

→ Q: find the mass of wire AB

Q: Recall this curve

$$x = \cos(t) \qquad 0 \le t \le 6\pi$$
$$y = \sin(t)$$
$$z = t$$



→ think of AB as a wire

→ material used to make this wire in non-uniform

→ $f(x,y,z)$ represents linear density (mass per unit length)

→ let $f(x,y,z) = z$

→ Q: find the mass of wire AB

dL ~ so small that we can assume that density is constant

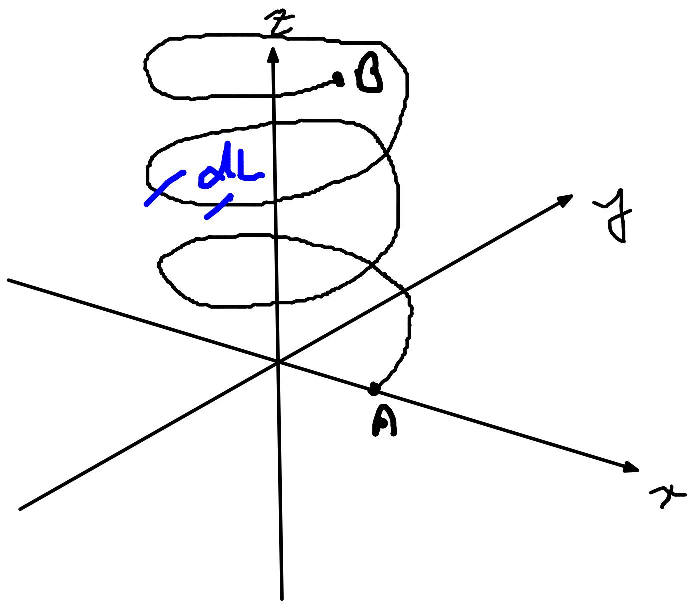
dm = mass for dL

$$dm = f\, dL$$

$$\text{Total mass} = \int dm = \int f\, dL = \int_a^b f \underbrace{|\vec{r}\,'(t)|\, dt}_{dL}$$

Q: Recall this curve

$x = \cos(t)$
$y = \sin(t)$ $\quad 0 \le t \le 6\pi$
$z = t$



- → think of AB as a wire
- → material used to make this wire in non-uniform
- → $f(x,y,z)$ represents linear density (mass per unit length)
- → let $f(x,y,z) = z$
- → Q: find the mass of wire AB

$$dm = z\, dL$$

$$m = \int dm = \int z\, dL = \int_0^{6\pi} \underbrace{z}_{t}\ \underbrace{(\text{speed})}_{\sqrt{2}}\, dt$$

$$= \int_0^{6\pi} t\sqrt{2}\, dt = 18\sqrt{2}\pi^2$$

a: Evaluate the line integral, where C is the given curve.

$$\int_C y\, ds, \quad C: x = t^2,\ y = t,\ 0 \leqslant t \leqslant 2$$

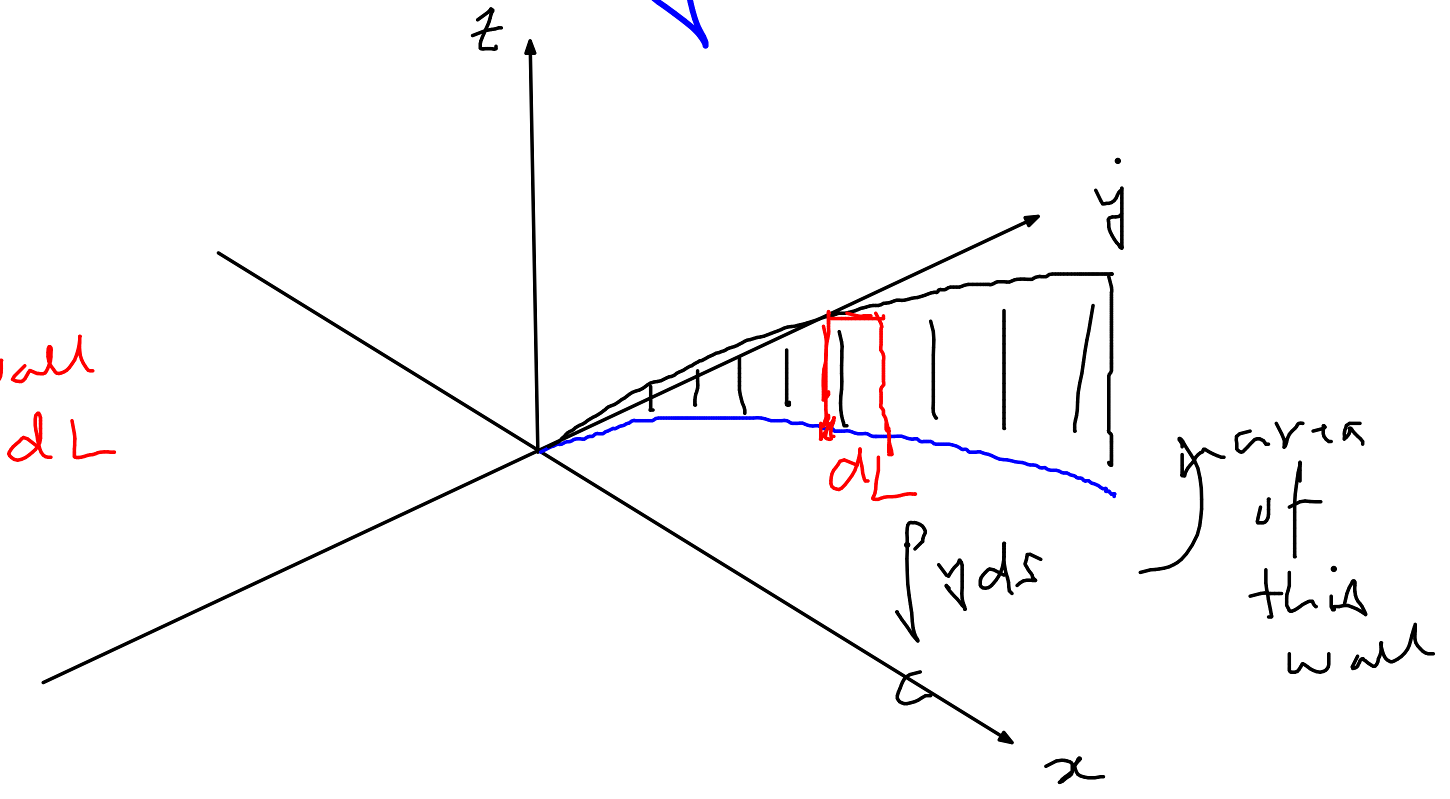
→ sketch the curve

→

$x = t^2 \quad y = t \quad 0 \leq t \leq 2$

$f = y$: linear density

$f\,dL$ = area of wall above dL



$$= \int_0^2 t \sqrt{(2t)^2 + 1^2}\, dt$$

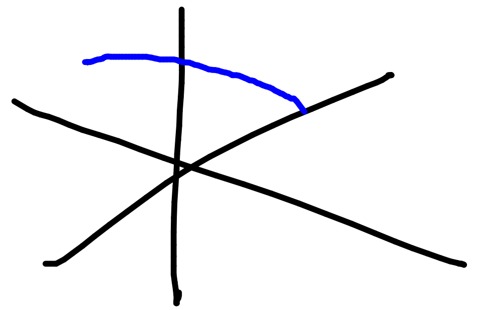
$$= \frac{17\sqrt{17} - 1}{12}$$

V EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

Evaluate the line integral, where C is the given curve.

$$\int_C xy^3\, ds,$$
$$C: x = 4\sin t,\ y = 4\cos t,\ z = 3t,\ 0 \leqslant t \leqslant \pi/2$$

Sketch ??



$$\vec{r}(t) = 4\sin(t)\,\hat{i} + 4\cos(t)\,\hat{j} + 3t\,\hat{k}$$

$$\text{speed} = \sqrt{16+9} = 5$$

$$\int_C xy^3\, ds = \int_0^{\pi/2} 4\sin t \cdot (4\cos t)^3\, 5\, dt = ?? = 320$$

LINE INTEGRALS OF VECTOR FIELDS

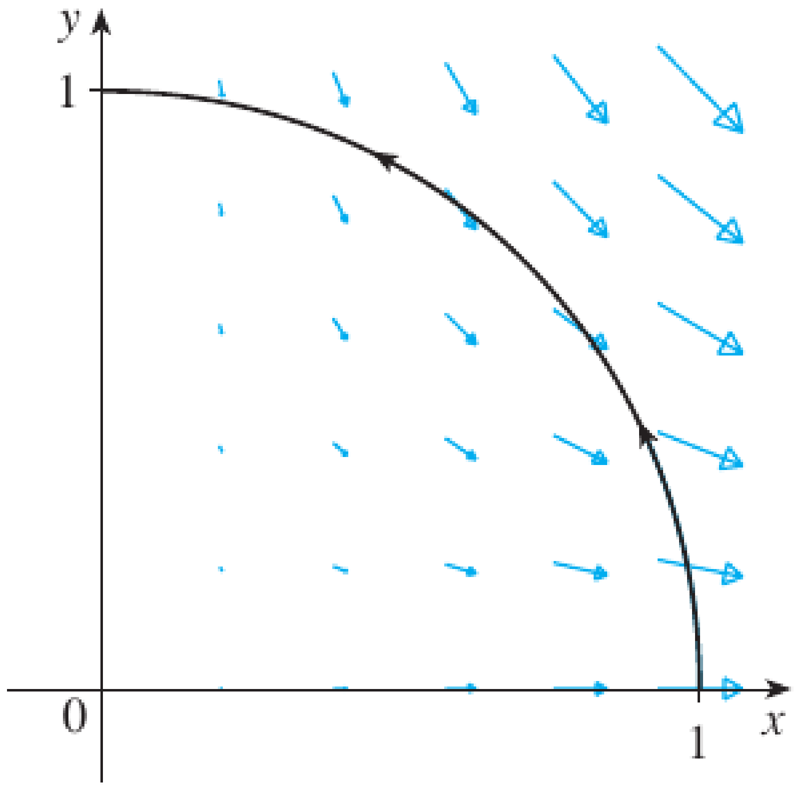
13 DEFINITION Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

next time.

$\rightarrow$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.



EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$