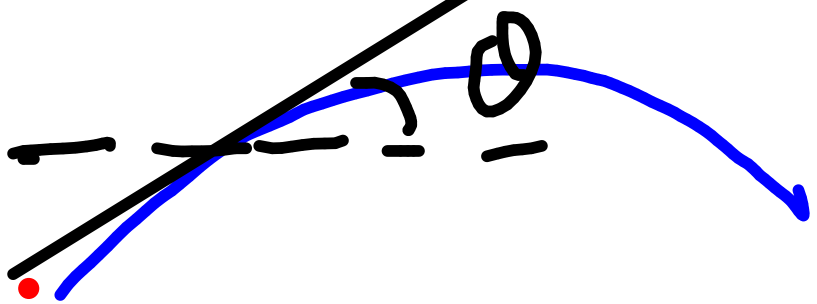


~~11.2 LIMITS AND CONTINUITY~~

Will do this later

$m = \tan\theta$

$= f'(x)$

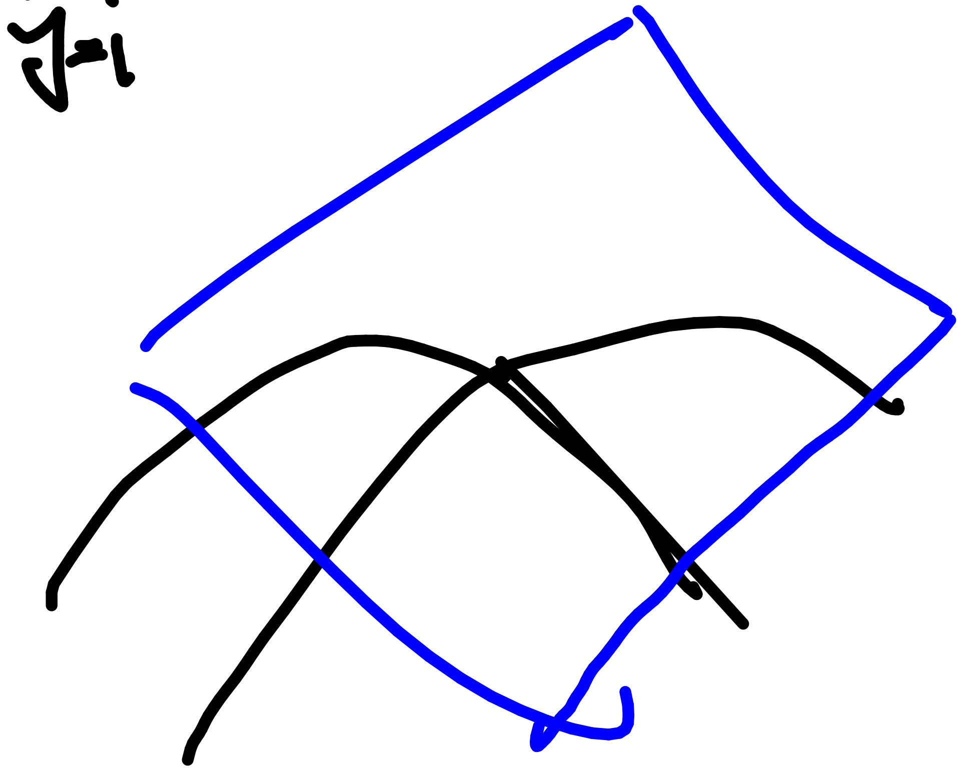
11.3 PARTIAL DERIVATIVES

& 11.4 Tangent Planes

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = -2x \Big|_{\substack{x=1 \\ y=1}} = -2$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = -4y \Big|_{\substack{x=1 \\ y=1}} = -4$$



$$f(x,y) = x^2 + y^3 \qquad (1,2)$$

$$f_y(1,2) = 3y^2 \Big|_{\substack{x=1 \\ y=2}} = 12$$

EXAMPLE 3 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{(-x)}{(1+y)^2}$$

f'

f''

$f^{(''')}$

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE 6 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6xy^2$$

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

• f^n should be defined in a nbd of (a,b)

■ Alexis Clairaut was a child prodigy in mathematics, having read l'Hospital's textbook on calculus when he was ten and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure*, which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

$$\frac{x-1}{?} = \frac{y-2}{?} = \frac{z-2}{?} = t$$

$x = 1 + t(?)$

$y = 2 + t(?)$ | $y = 2$

$z = 2 + t(?)$

$\frac{\partial z}{\partial x}(1,2,2) = ??$

$$\frac{\partial}{\partial x}\left(4x^2 + 2y^2 + z^2 = 16\right)$$

$$8x + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$8 + 4\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -2$$

If we were only in zx plane

$z-2=-2(x-1) \rightarrow$ plane

$y=2 \rightarrow$ another plane

$\rightarrow$ H.W. write parametric eqⁿ for the intersection of above two planes.

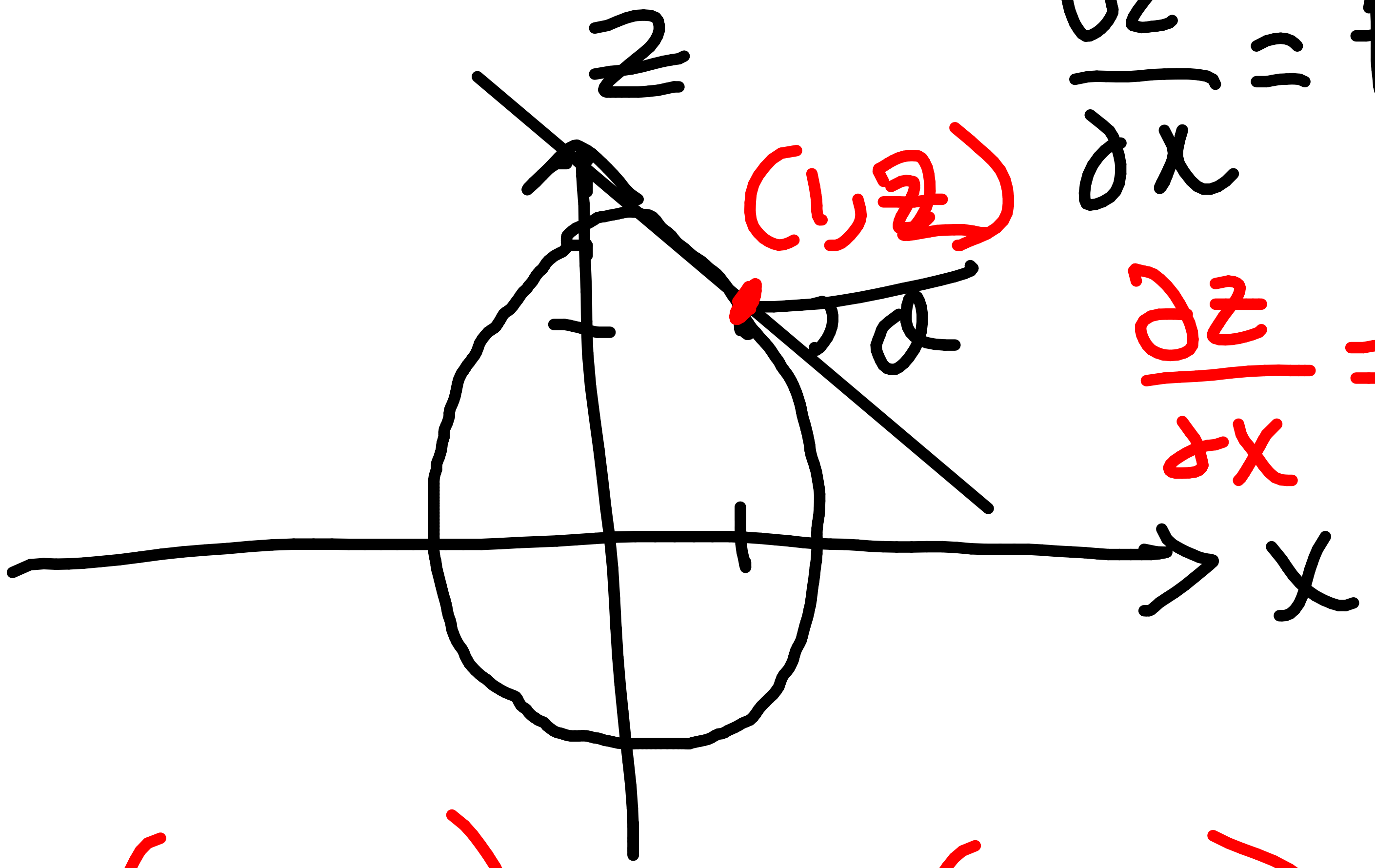
$4x^2+2y^2+z^2=16$

$(1,2,2)$

$\boxed{y=2}$

$4x^2+8+z^2=16$

$4x^2+z^2=8$



$\frac{\partial z}{\partial x}=\tan\theta$

$\frac{\partial z}{\partial x}=-2$

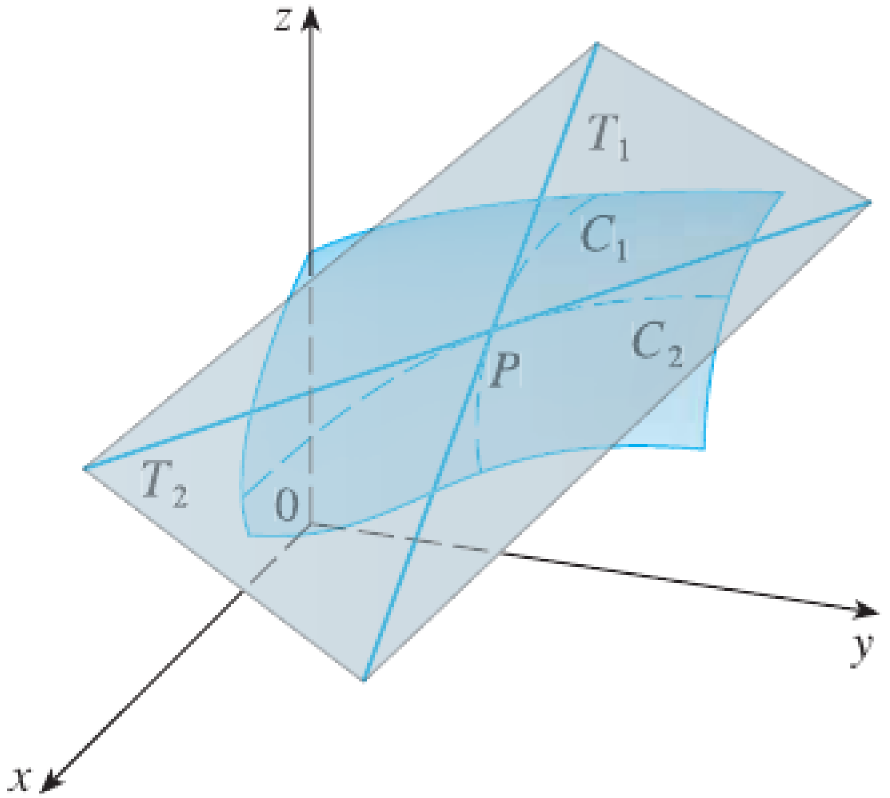
$(z-2)=-1(x-1)$

Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use a computer to graph f.
(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.
(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.
(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.
(e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS



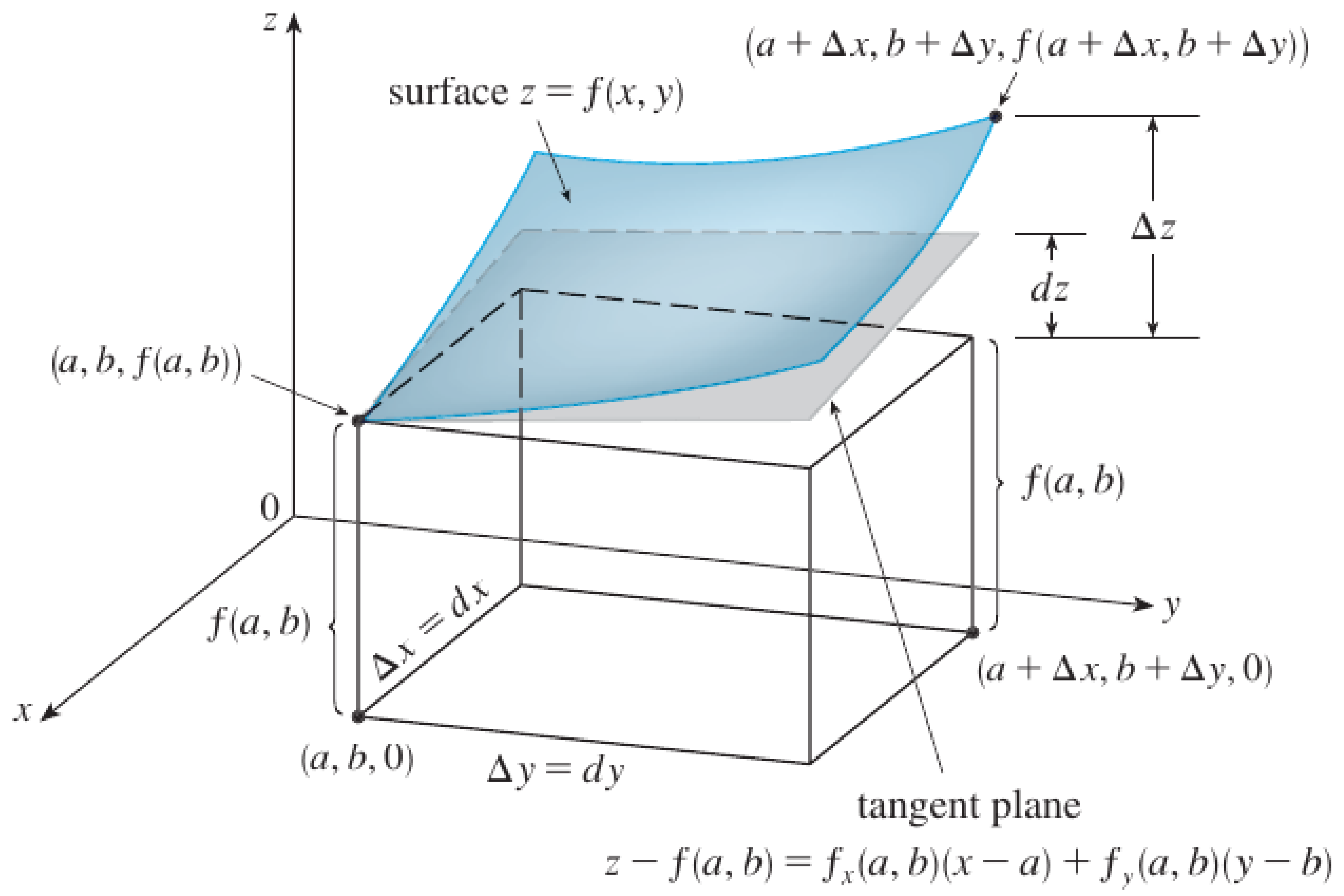
V EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

DIFFERENTIALS

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$



EXAMPLE 3

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.