

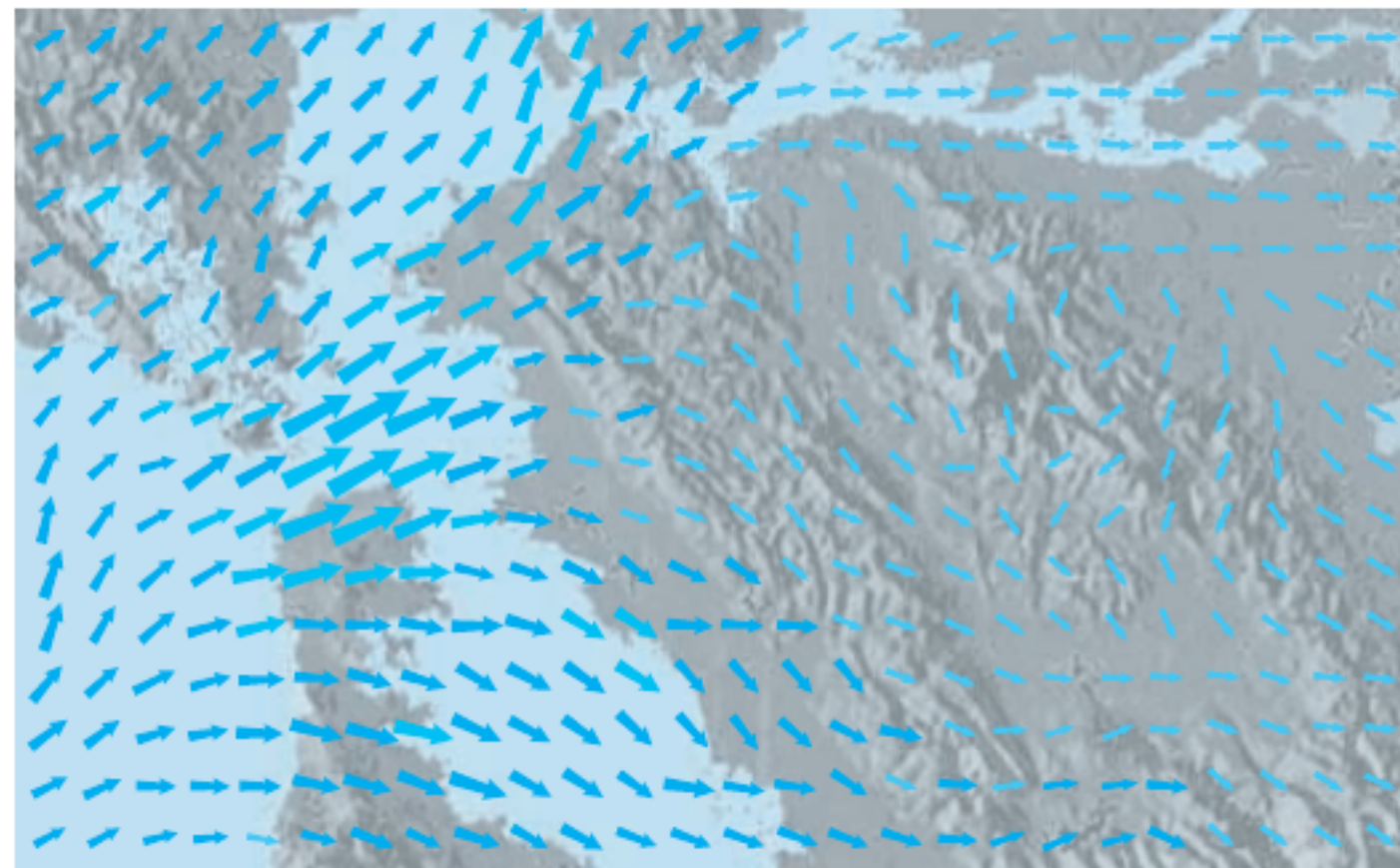
13

VECTOR CALCULUS

Started last time

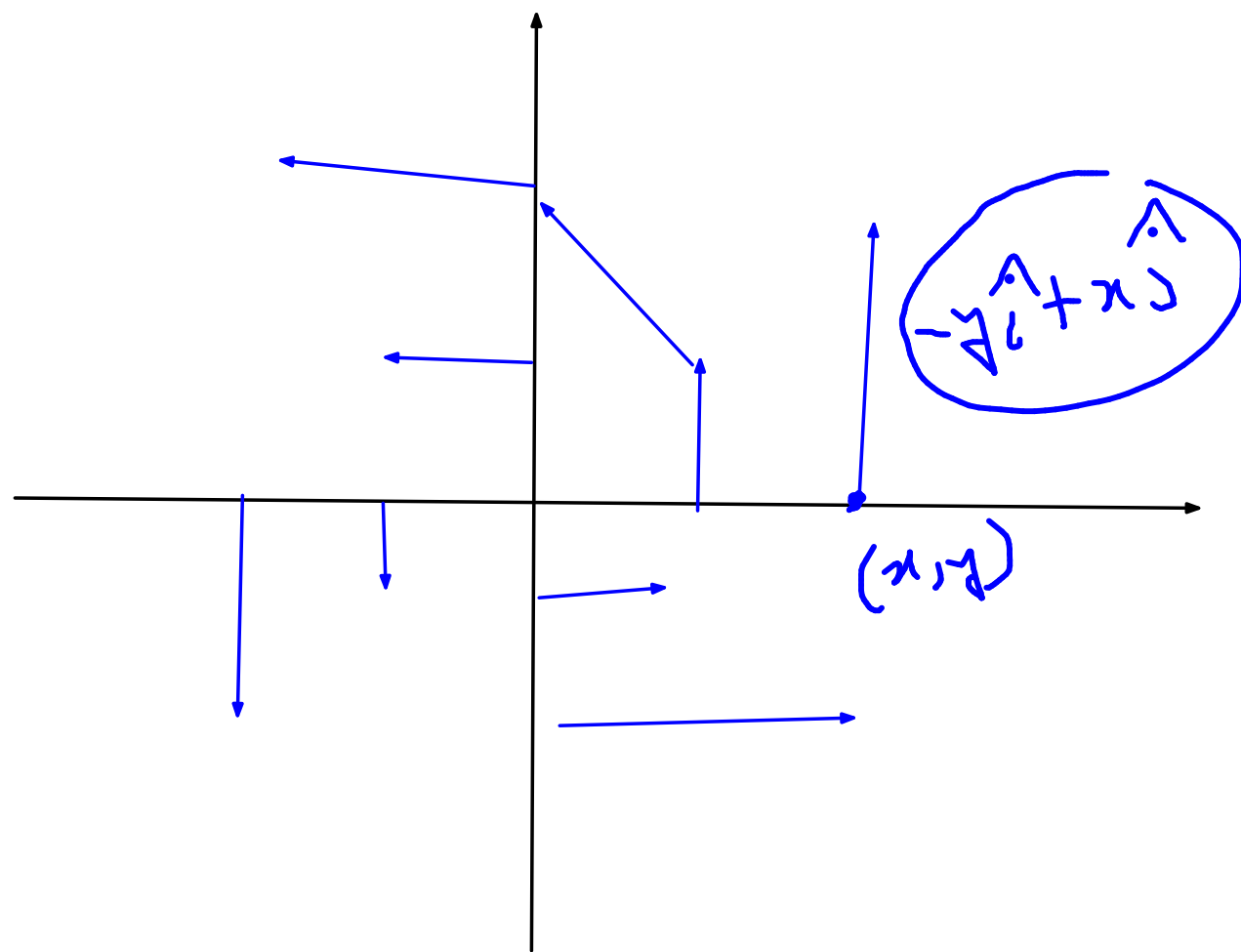
13.1

VECTOR FIELDS



Sketch the vector field \mathbf{F}

$$\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}.$$



Sketch the vector field \mathbf{F}

$$\mathbf{F}(x, y) = \frac{y \mathbf{i} - x \mathbf{j}}{\sqrt{x^2 + y^2}}$$

geogebra.org/m/QPE4PaDZ

Vector Field $\left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}} \right\rangle$

$$V_x(x, y) = y / \sqrt{x^2 + y^2}$$

$$V_y(x, y) = (-x) / \sqrt{x^2 + y^2}$$

xmin = -5

xmax = 4

ymin = -5

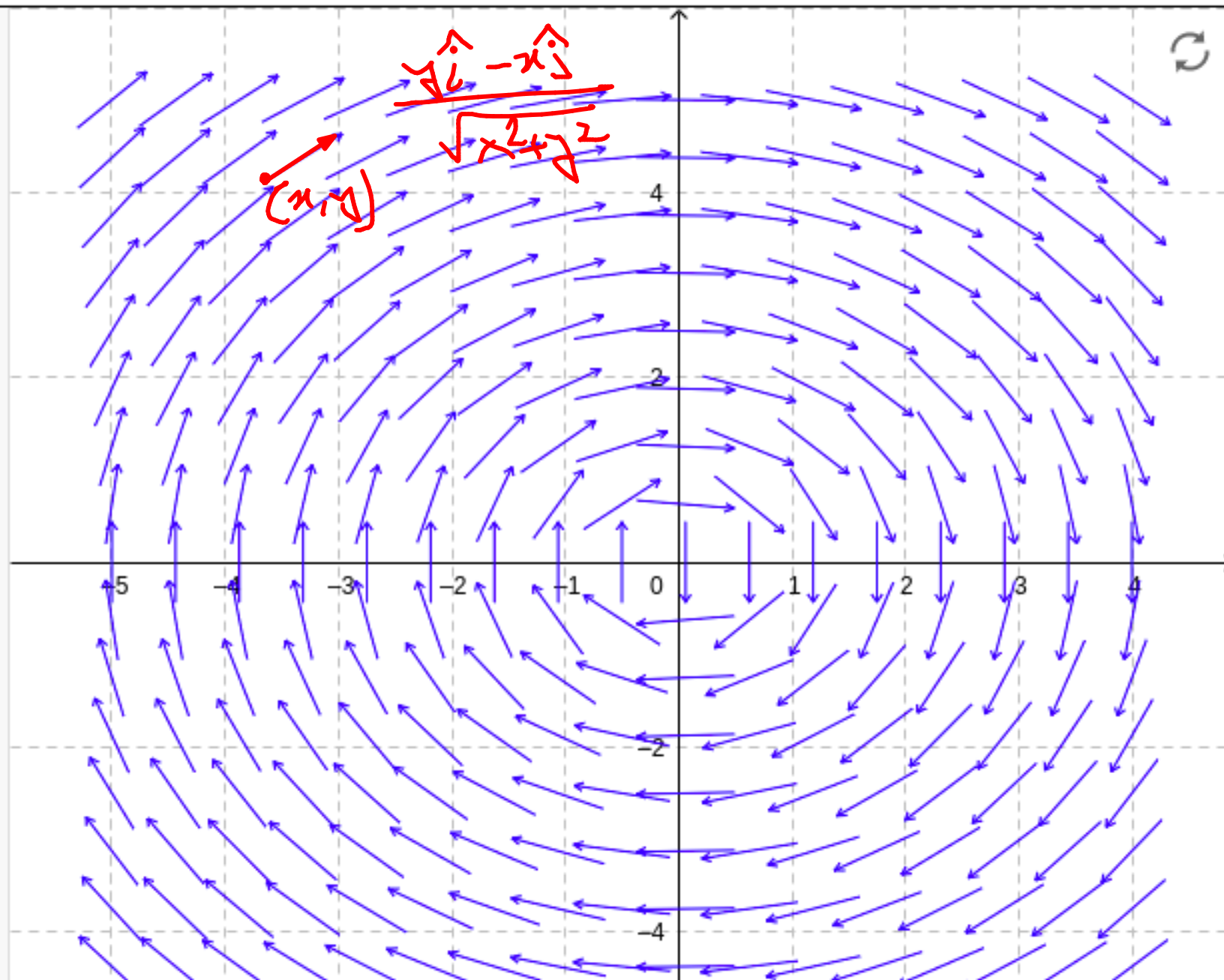
ymax = 5

xn = 8

yn = 8

v = 0.43

vh = 0.09

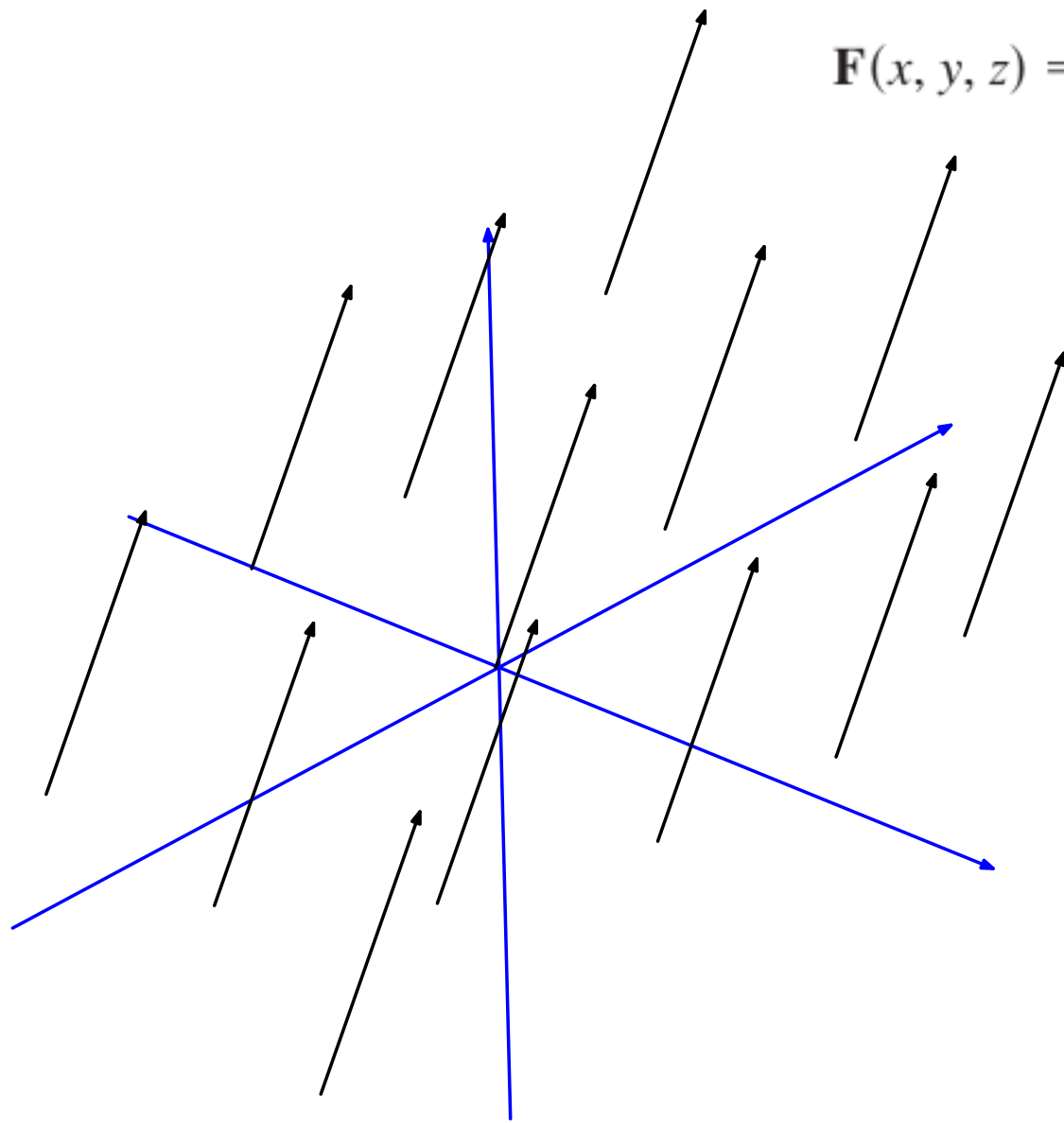


Sketch the vector field \mathbf{F}

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + x \mathbf{j}$$

Sketch the vector field \mathbf{F}

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$



Sketch the vector field \mathbf{F}

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$$

Vector Fields

(point
in plane) \rightarrow vector

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\begin{aligned}\vec{F}(x, y) &= P\hat{i} + Q\hat{j} \\ &= F_1\hat{i} + F_2\hat{j}\end{aligned}$$

$$\mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\begin{aligned}\vec{F}(x, y, z) &= P\hat{i} + Q\hat{j} + R\hat{k} \\ &= F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\end{aligned}$$

GRADIENT FIELDS

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (or $\text{grad } f$) is defined by

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}$$

Therefore, ∇f is really a vector field on \mathbb{R}^2 and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on \mathbb{R}^3 given by

$$\nabla f(x, y, z) = f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k}$$

13.1
later

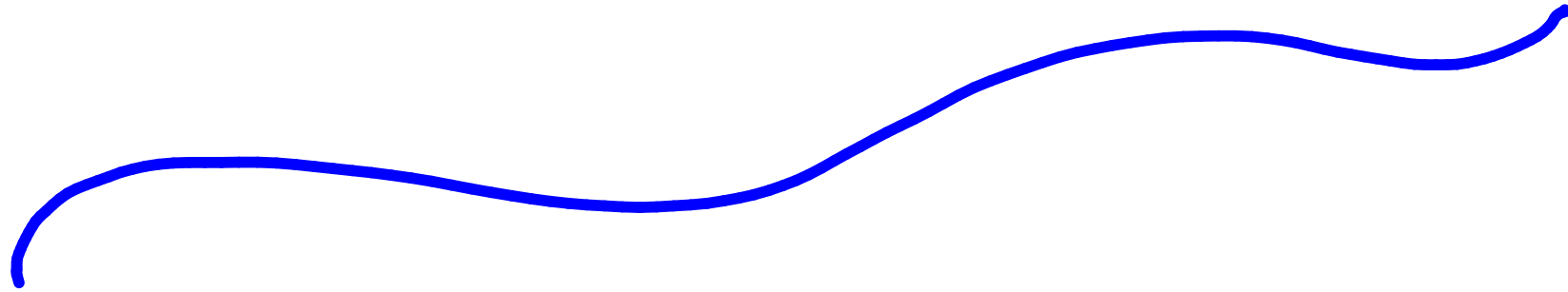
V EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f . How are they related?

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for \mathbf{F} .

A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$.
If it is at position $(2, 1)$ at time $t = 3$, estimate its location
at time $t = 3.01$.

Recall position function from physics

→ a particle is moving in a path

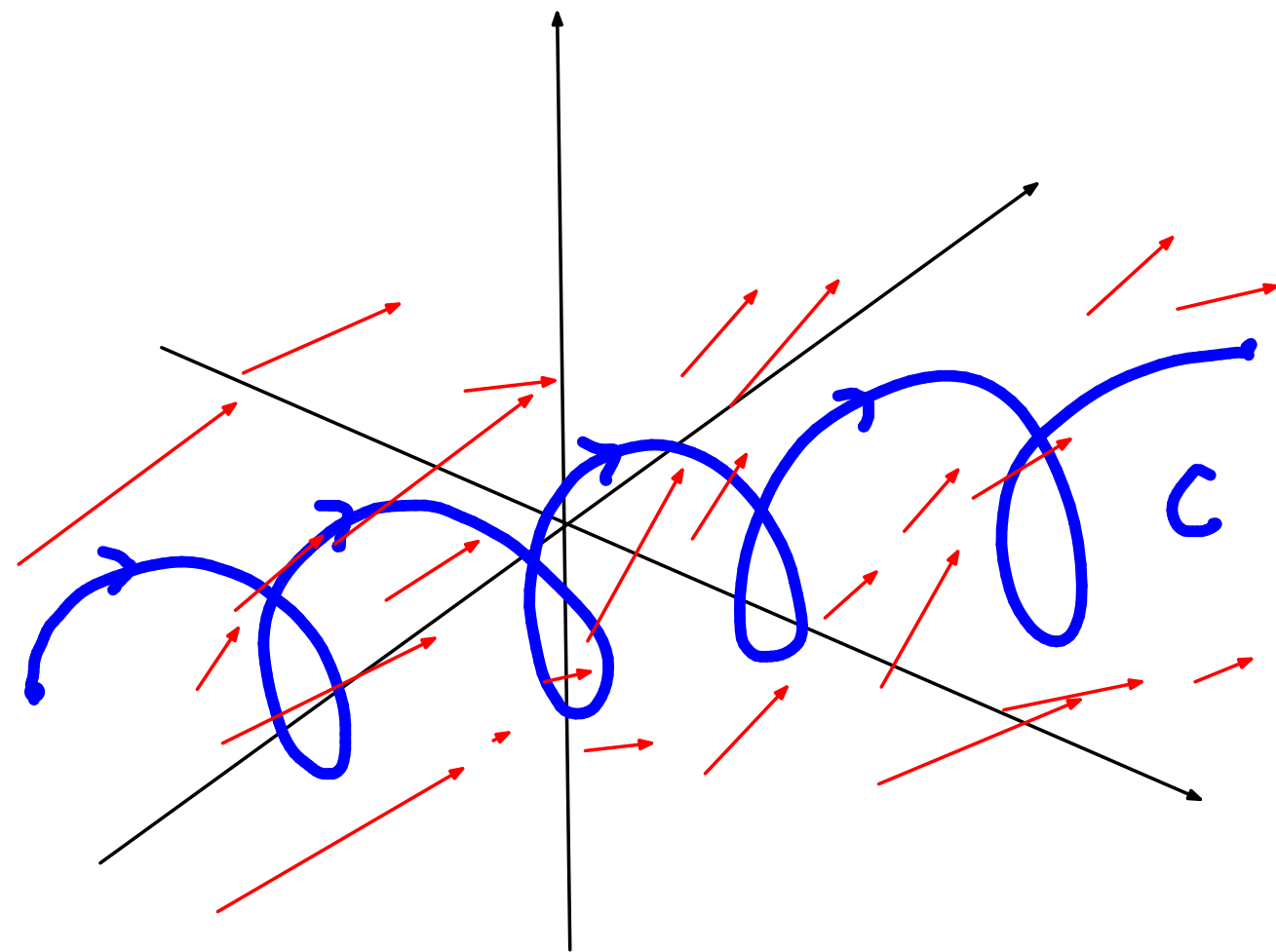


→ $\vec{r}(t)$ = position of particle at time t
= $(x(t), y(t), z(t))$

→ Here we refer to these $\vec{r}(t)$ as curves.
& t is a parameter for the curve

We will focus for now:

Calculus on curves or paths.



→ Length of the curve?

→ $\int_C f d\vec{r}$: integration of scalar functions

→ $\int_C \vec{F} \cdot d\vec{r}$: integration of vector functions on curves.

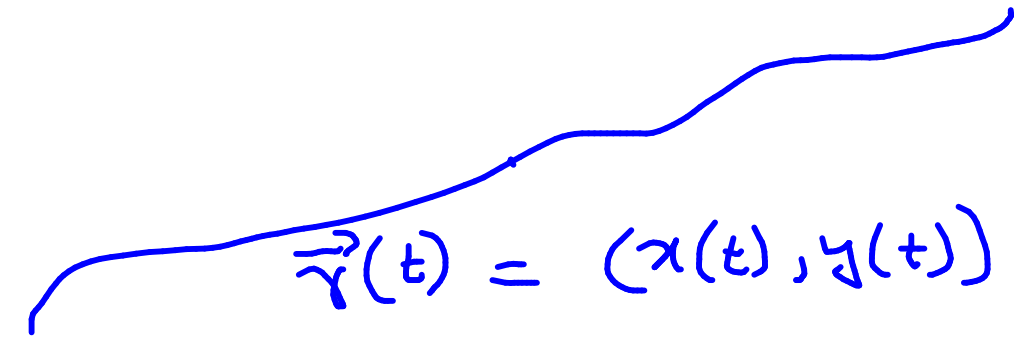
13.2

LINE INTEGRALS

]

next

time



9.1

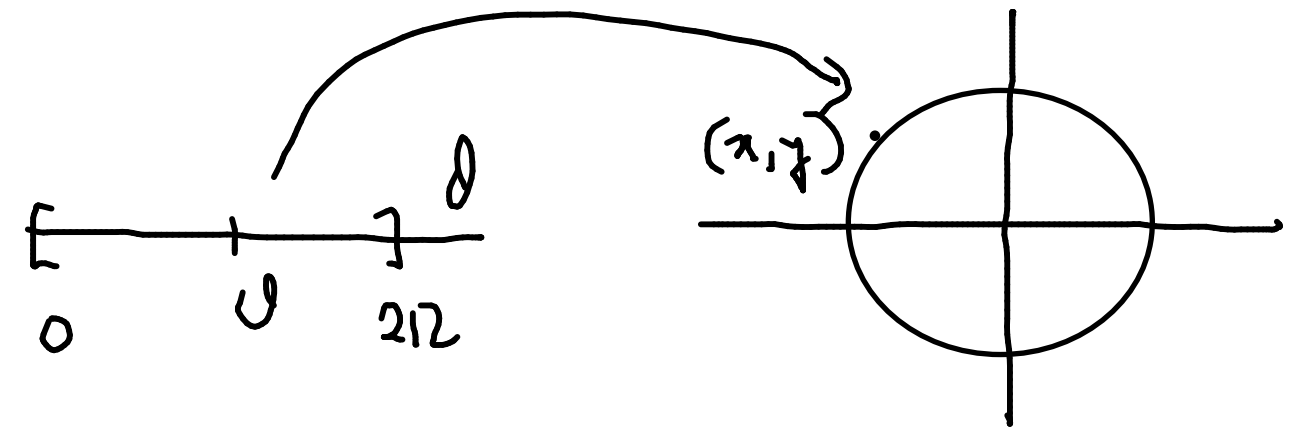
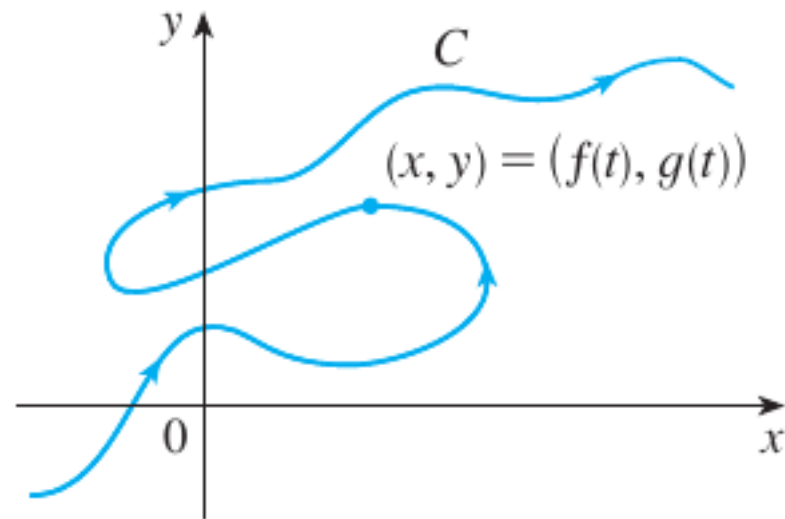
PARAMETRIC CURVES

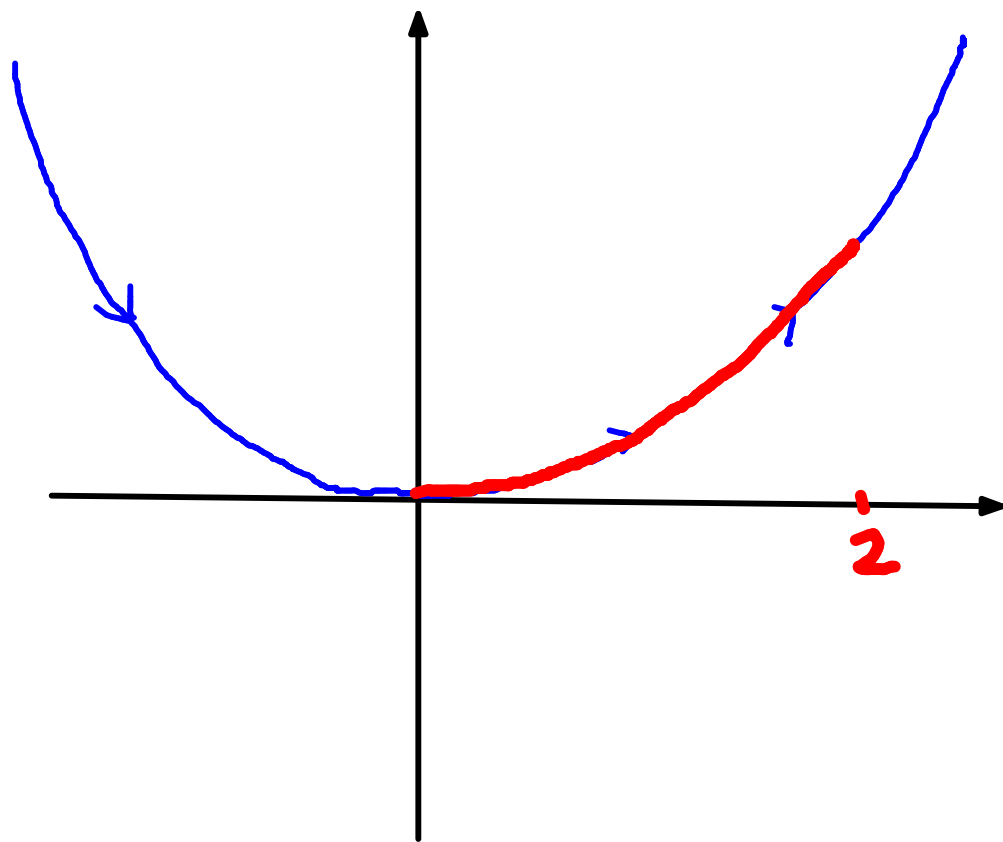
:

$$x = \cos \theta$$

$$0 \leq \theta \leq 2\pi$$

$$y = \sin \theta$$



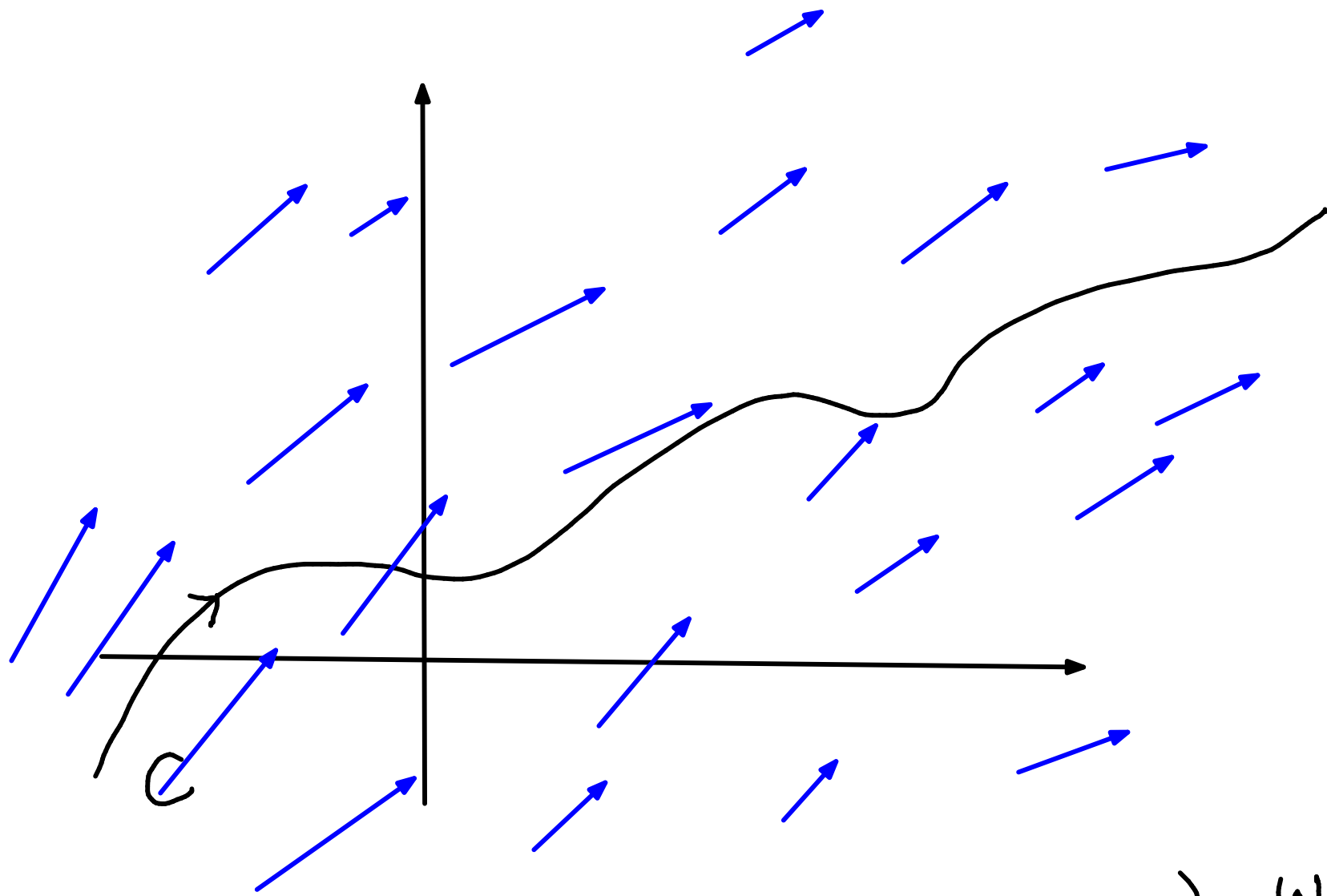


$$y = x^2$$

$$\begin{bmatrix} -\infty \leq t \leq \infty \\ x = t \\ y = t^2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \leq t \leq 2 \\ x = t \\ y = t^2 \end{bmatrix}$$

what curve ??



- Force field \vec{F}
- a particle is moving along a curve C

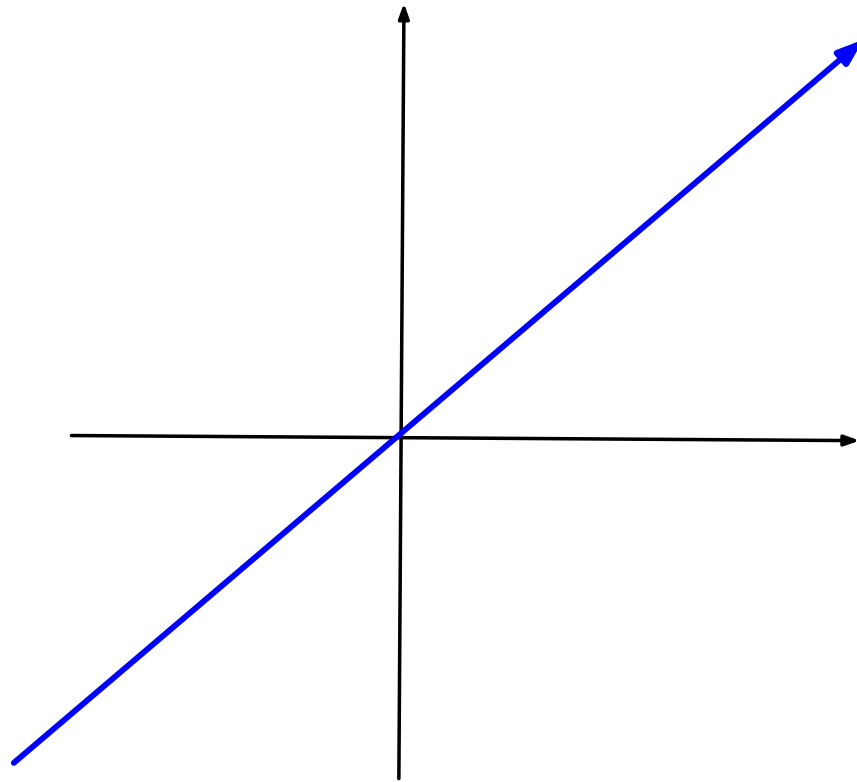
Q: find work done by \vec{F} in moving the particle along the curve ??

→ we need a better precise description of what's a curve or path.

$$-6\pi \leq t \leq 6\pi$$

$$x = t$$

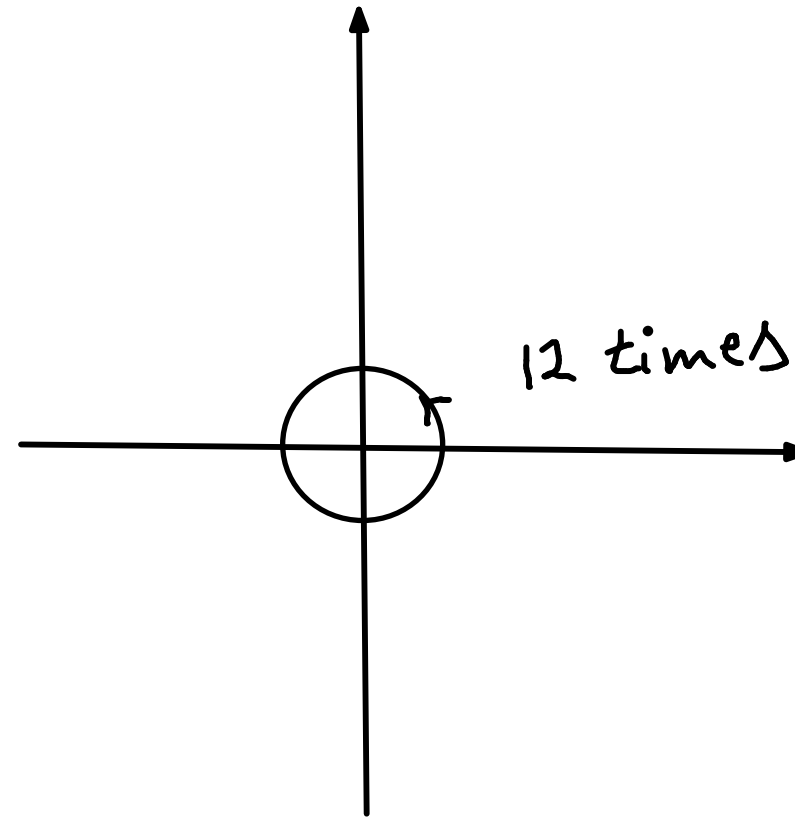
$$y = t$$



$$-6\pi \leq t \leq 6\pi$$

$$x = 0.1 \cos(2t)$$

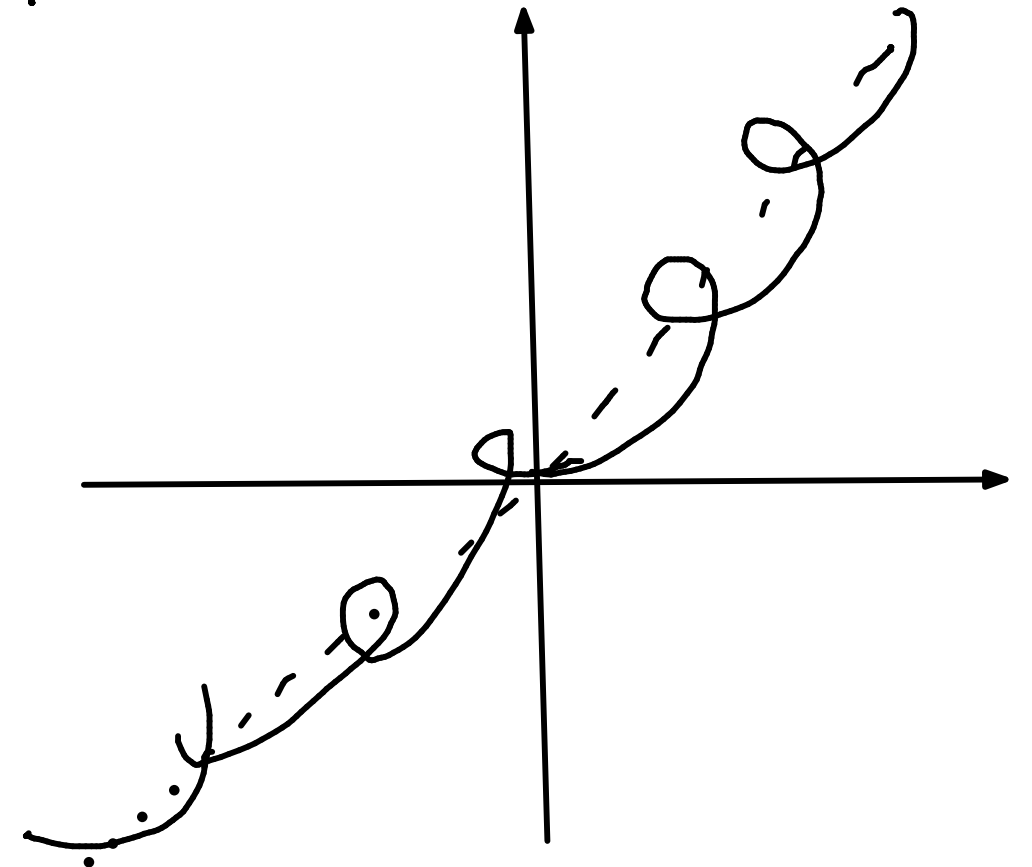
$$y = 0.1 \sin(2t)$$



$$-6\pi \leq t \leq 6\pi$$

$$x = t + 0.1 \cos(2t)$$

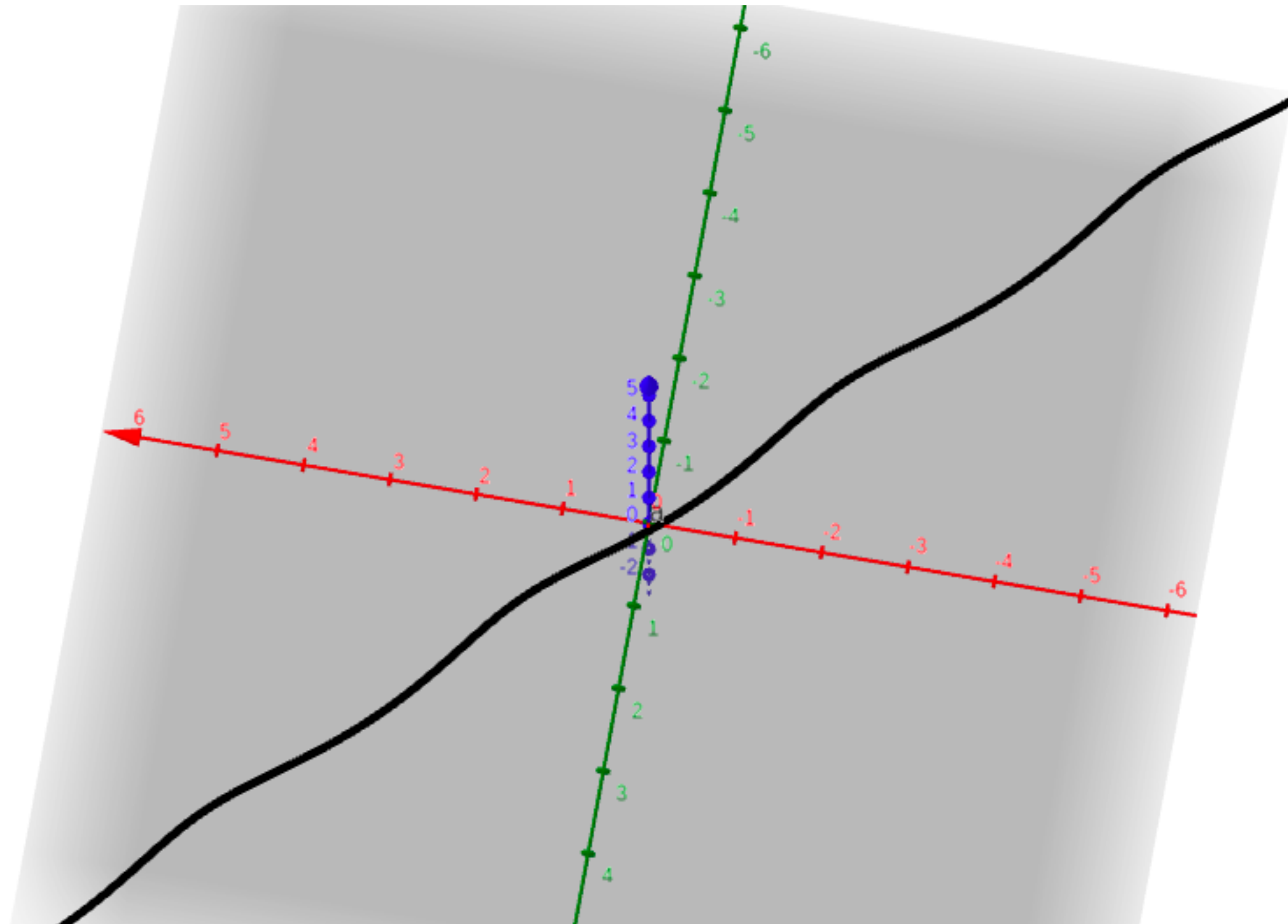
$$y = t + 0.1 \sin(2t)$$



$a = \text{Curve}(t + 0.1 \sin(2t), t + 0.1 \cos(2t), t, -6\pi, 6\pi)$

$$\rightarrow \left. \begin{array}{l} x = t + 0.1 \sin(2t) \\ y = t + 0.1 \cos(2t) \end{array} \right\} -18.85 \leq t \leq 18.85$$

Input...

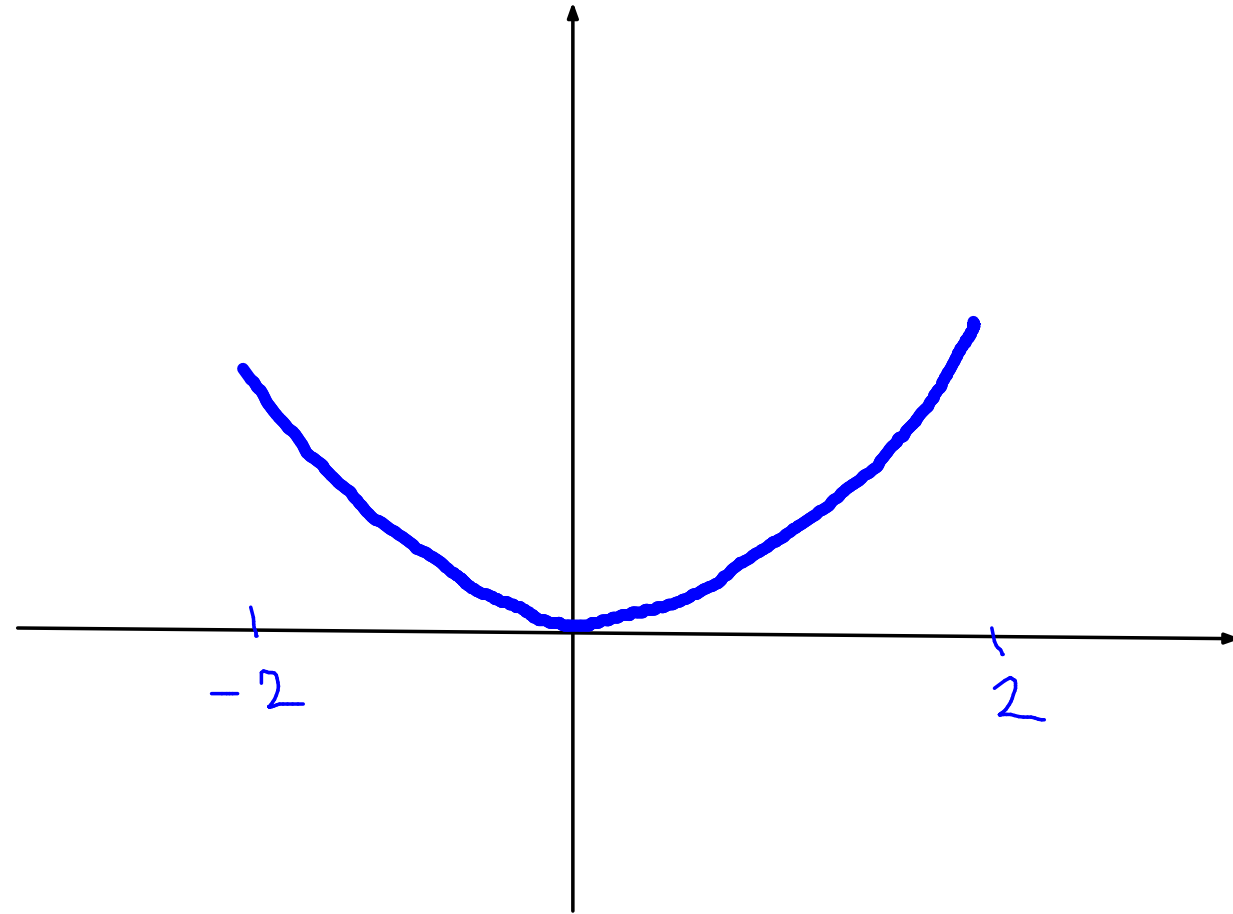


Q.

Sketch the path

$$x = t, \quad y = t^2,$$

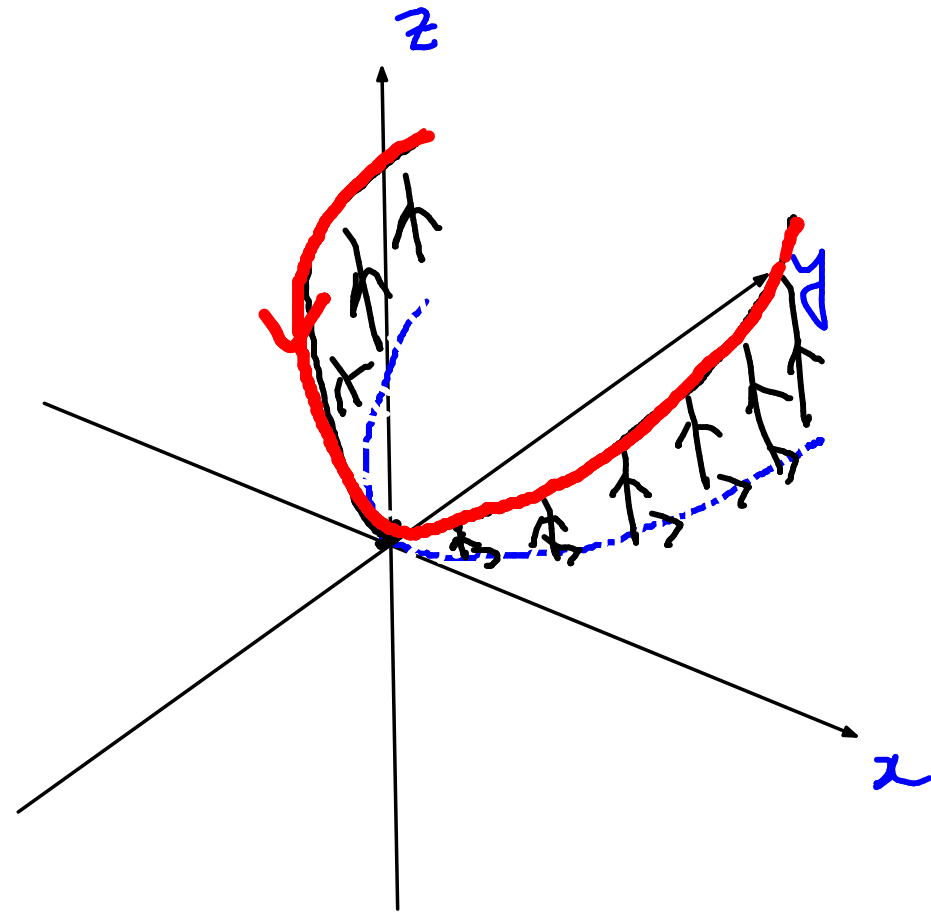
$$-2 \leq t \leq 2$$



Q.

Sketch the path

$$\underbrace{x = t, \quad y = t^2}_{y = x^2}$$






$$z = t^2$$

$$-2 \leq t \leq 2$$

$$t = 0$$

$$(x, y, z) = (0, 0, 0)$$

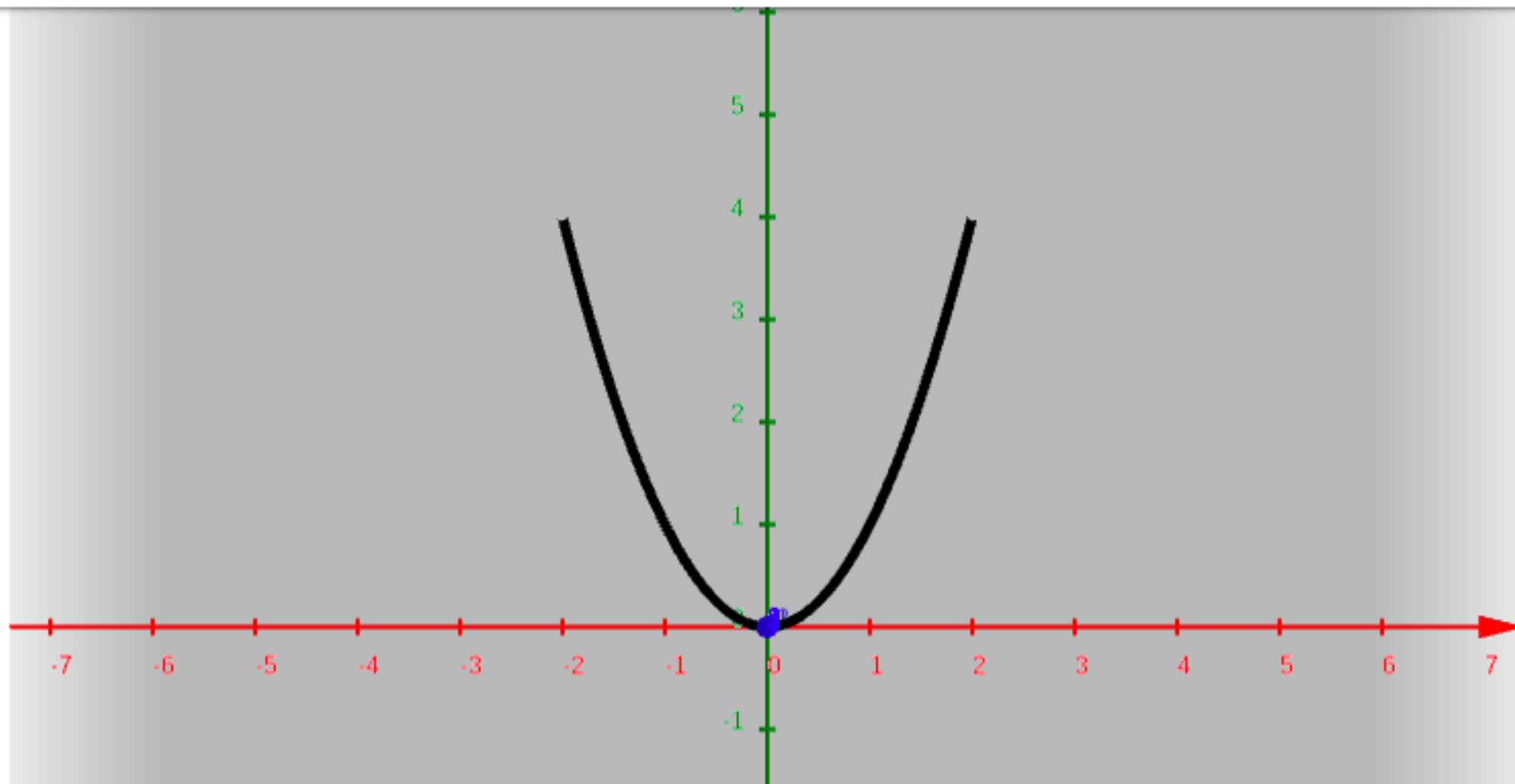
GeoGebra 3D Calculator

$a = \text{Curve}(t, t^2, t^2, t, -2, 2)$

$\rightarrow \left. \begin{array}{l} x = t \\ y = t^2 \\ z = t^2 \end{array} \right\} -2 \leq t \leq 2$

Input...

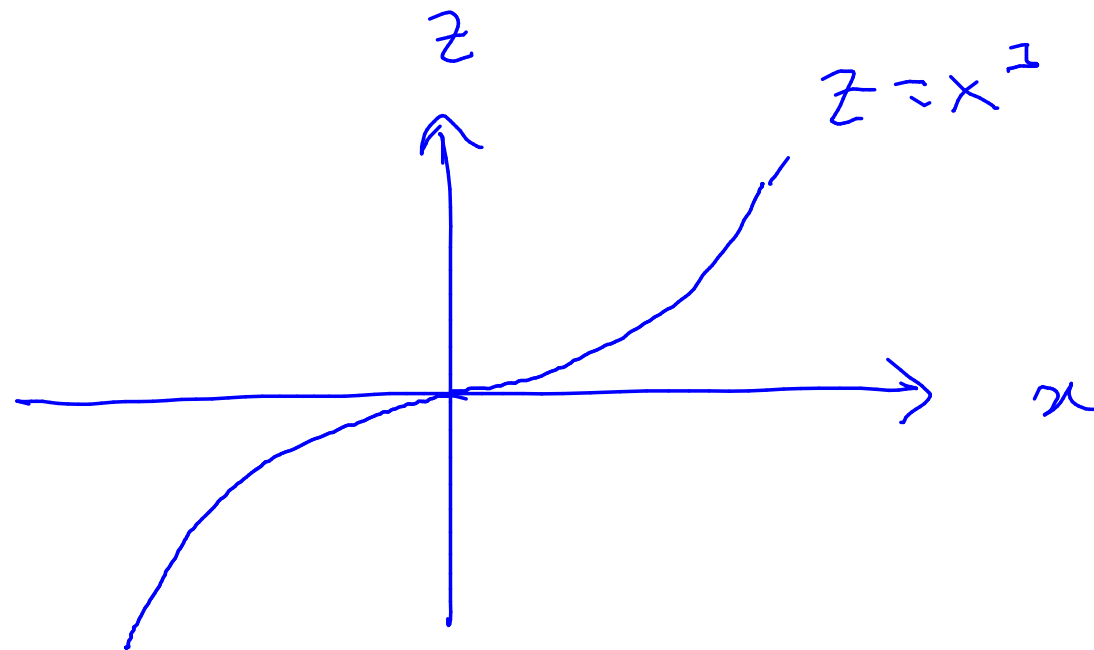
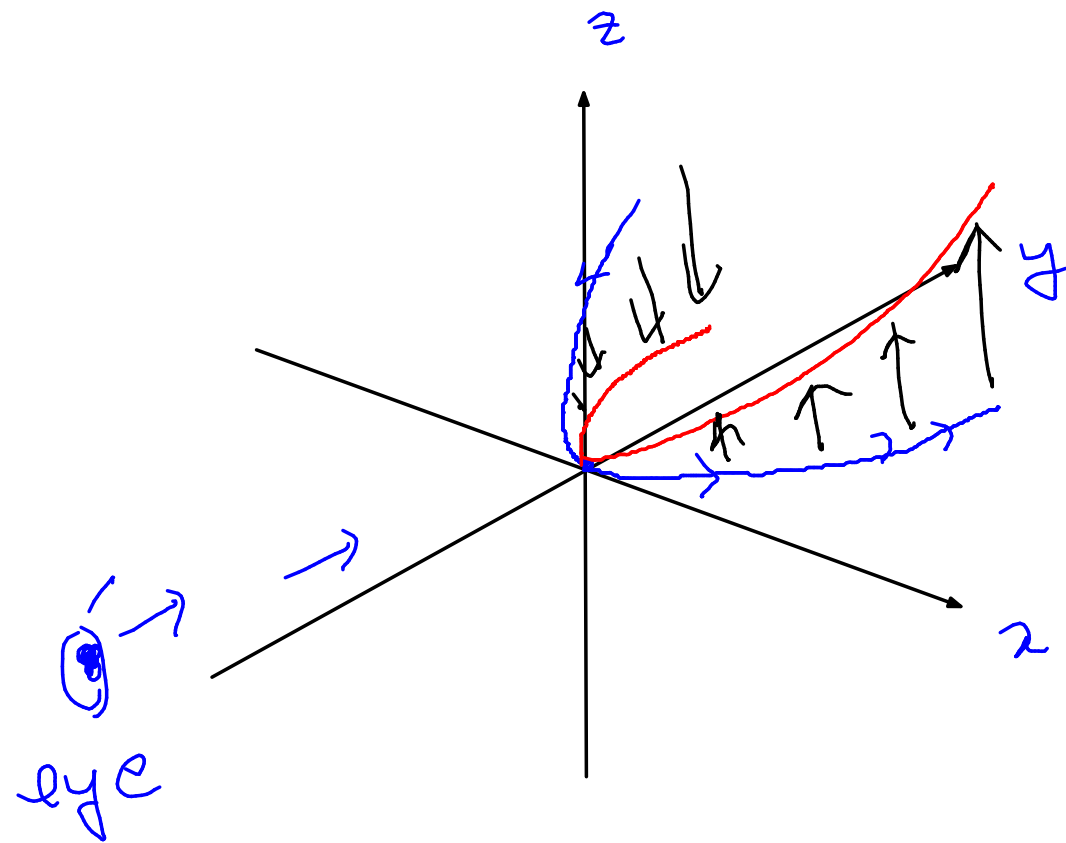


Q. Sketch

$$x = t, \quad y = t^2$$

$$z = t^3$$

$$-2 \leq t \leq 2$$



$$z = y^{3/2}$$

between

$$z = y$$

$$z = y^2$$

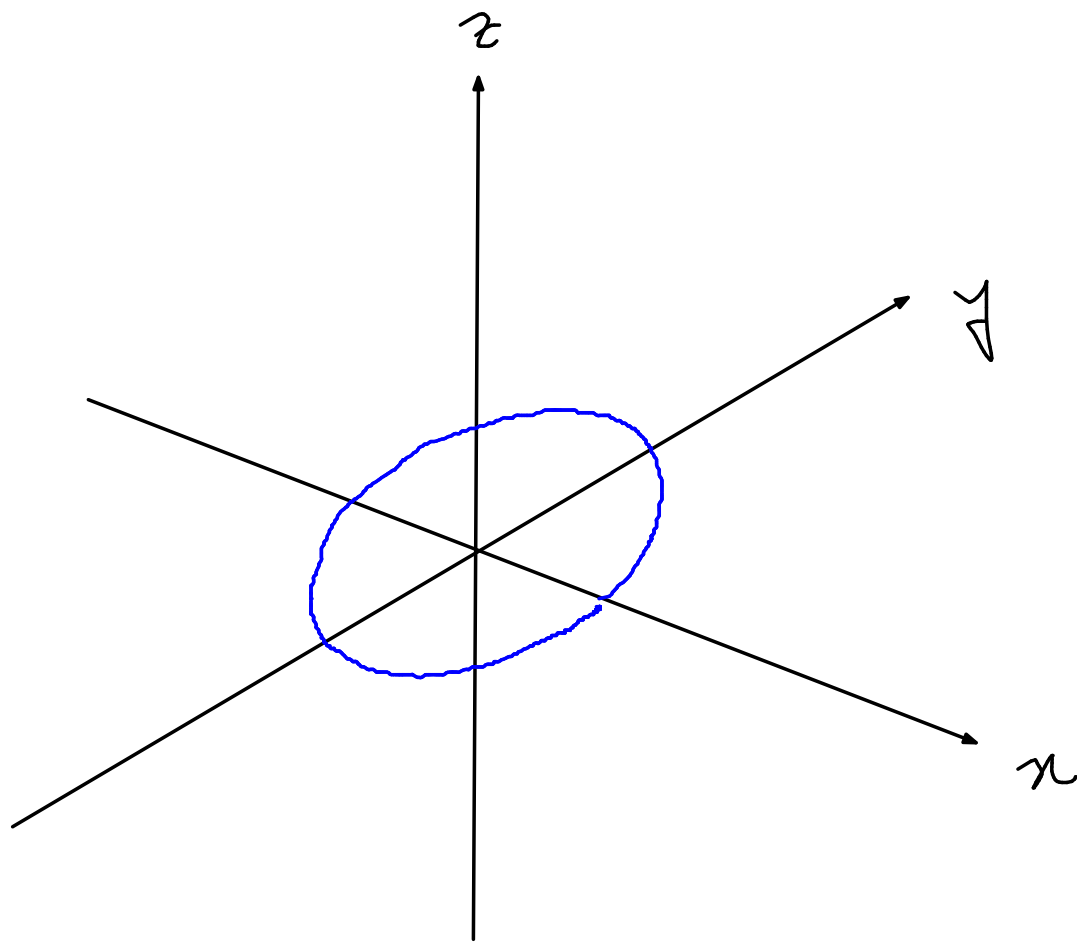
Q. Sketch

$$x = \cos(t)$$

$$y = \sin(t)$$

$$z = 0$$

$$0 \leq t \leq 6\pi$$



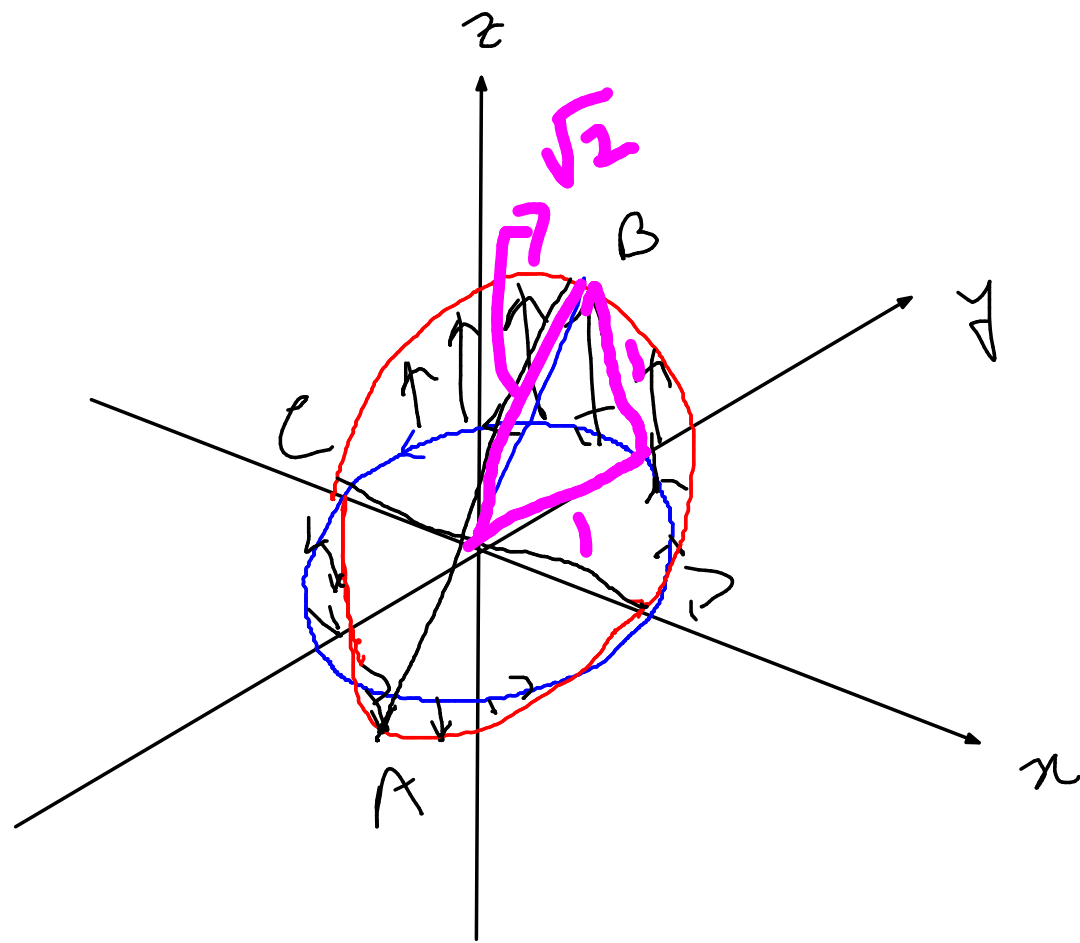
Q. Sketch

$$x = \cos(t)$$

$$y = \sin(t)$$

$$z = \sin(t)$$

$$0 \leq t \leq 6\pi$$



$$AB \stackrel{??}{=} CD$$

$$2\sqrt{2} \quad 2$$

ellipse

Q.

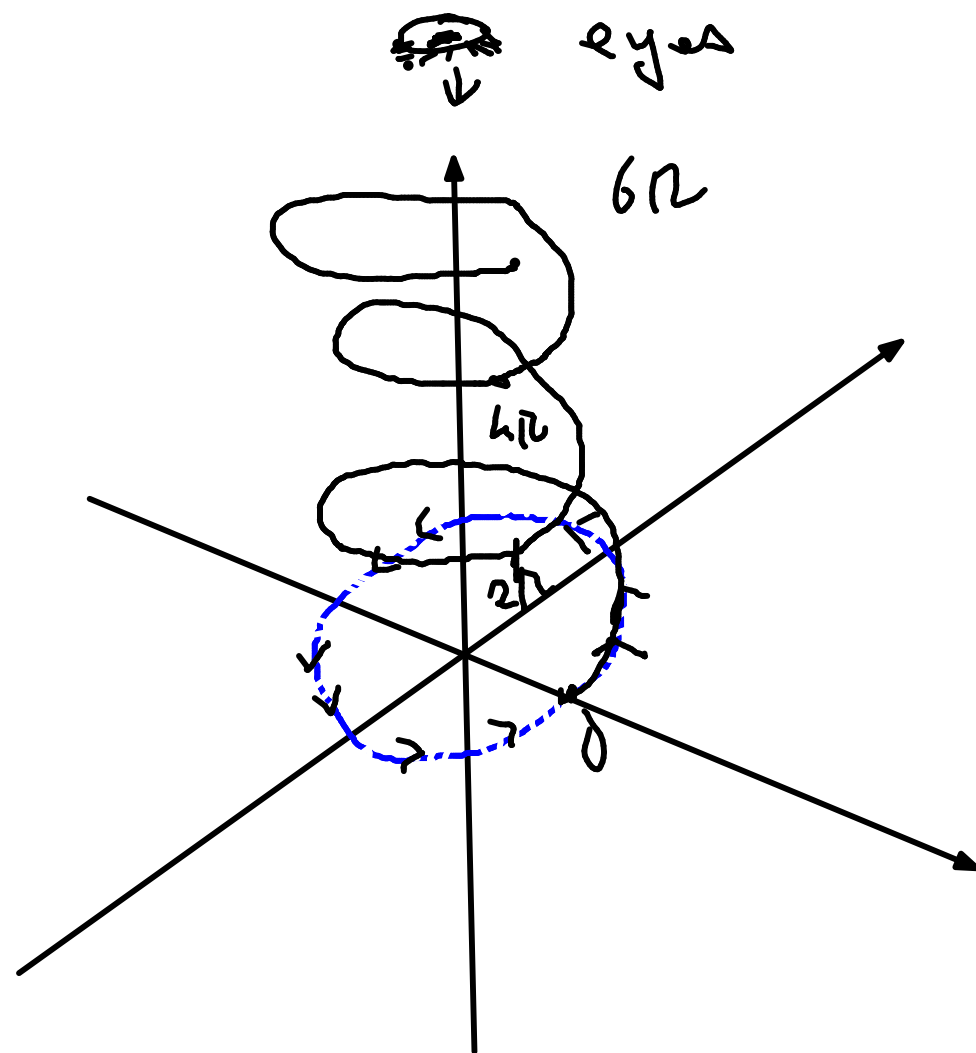
Sketch

$$x = \cos(t)$$

$$y = \sin(t)$$

$$z = t$$

$$0 \leq t \leq 6\pi$$



Calculus on curves

— next time

We will focus for now:

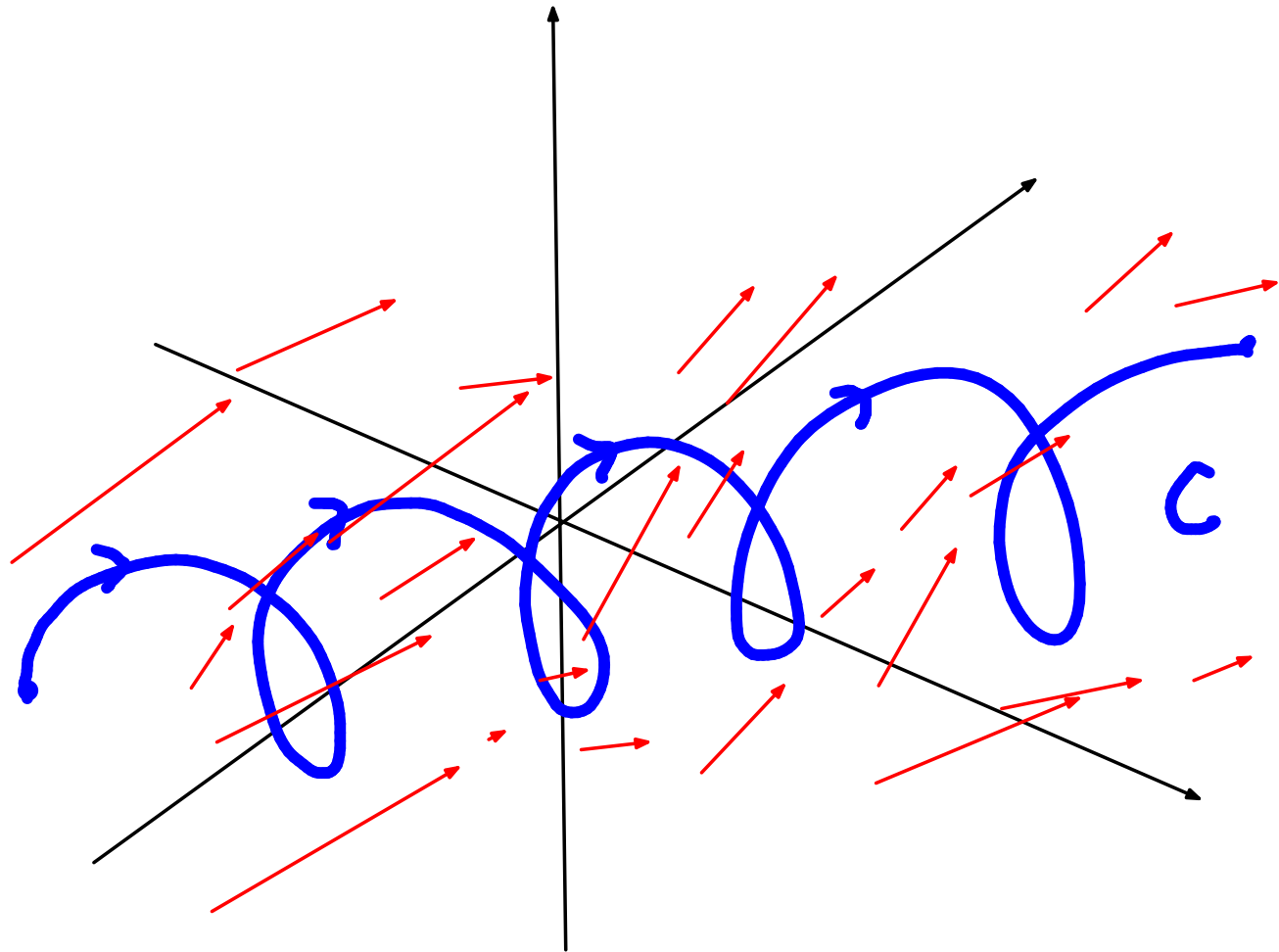
Calculus on curves or paths.

next time

→ Length of the curve?

→ $\int_C f d\vec{r}$: integration of scalar functions

→ $\int_C \vec{F} \cdot d\vec{r}$: integration of vector functions on curves.



EXAMPLE I Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \quad y = t + 1$$

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

$$x = t + 2 \sin 2t$$

$$y = t + 2 \cos 5t$$

Evaluate the line integral, where C is the given curve.

$$\int_C y \, ds, \quad C: x = t^2, \, y = t, \, 0 \leq t \leq 2$$

V EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. (See Figure 9.)

Evaluate the line integral, where C is the given curve.

$$\int_C xy^3 \, ds,$$

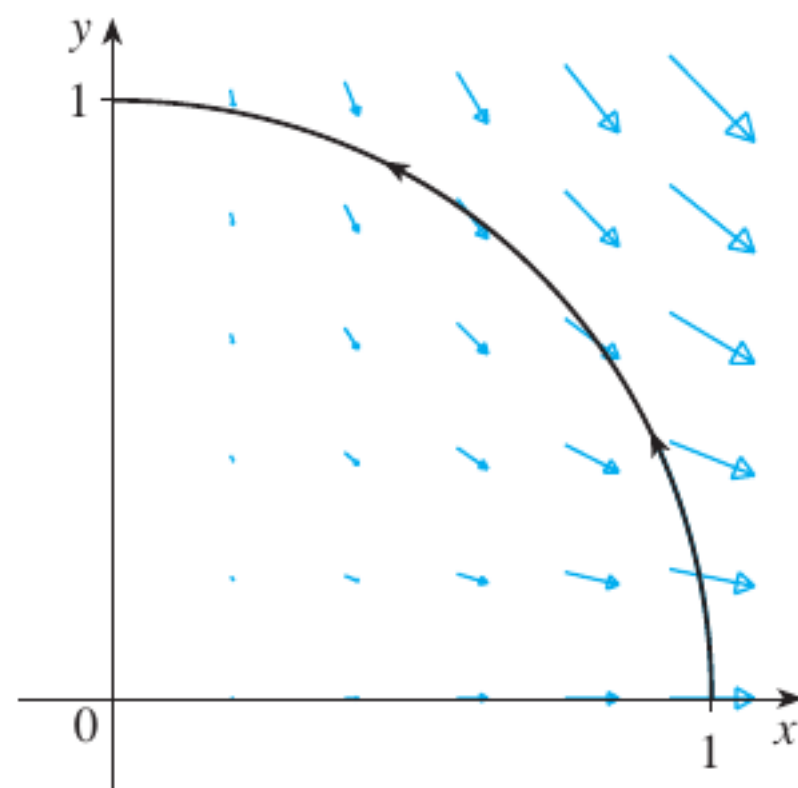
$$C: x = 4 \sin t, \, y = 4 \cos t, \, z = 3t, \, 0 \leq t \leq \pi/2$$

LINE INTEGRALS OF VECTOR FIELDS

13 DEFINITION Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of \mathbf{F} along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} - xy \mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $0 \leq t \leq \pi/2$.



EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ and C is the twisted cubic given by

$$x = t \quad y = t^2 \quad z = t^3 \quad 0 \leq t \leq 1$$