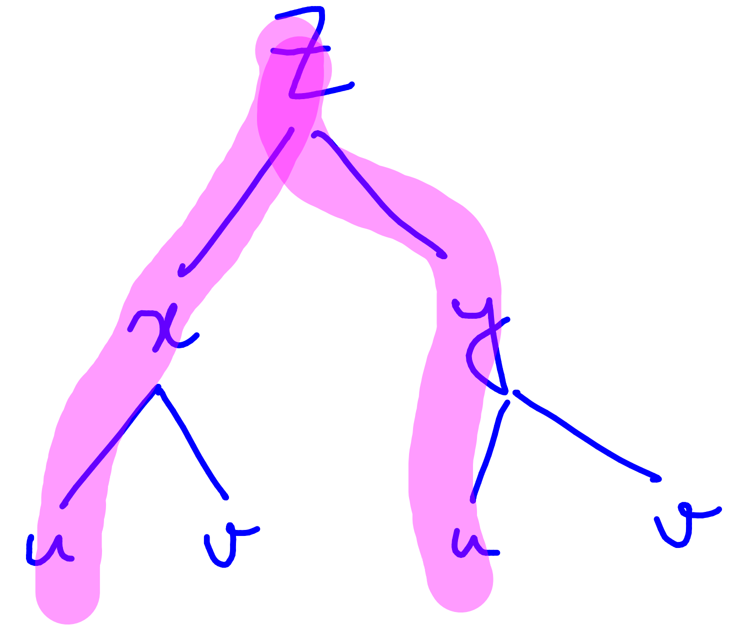


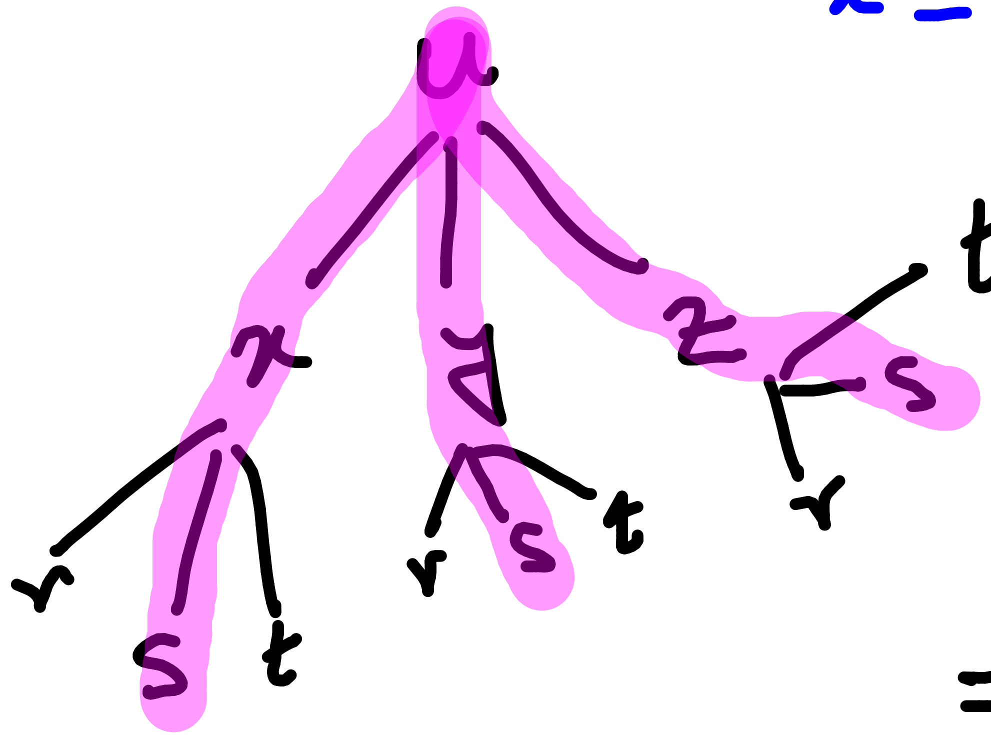
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

V EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

Ans: 192

$x = 2 \qquad y = 2 \qquad z = 0$



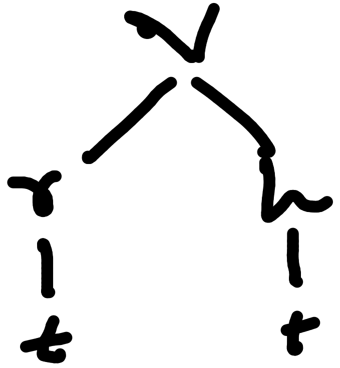
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

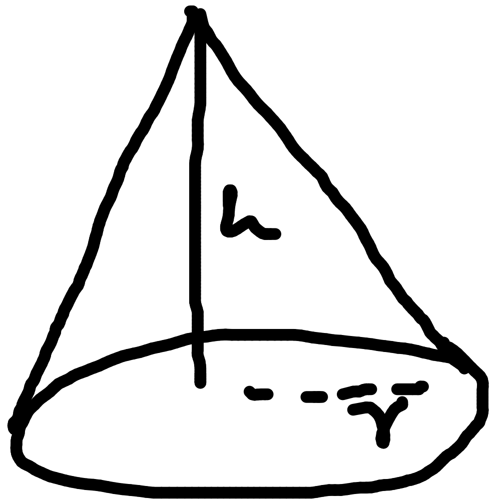
$$= 4x^3y(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + 3y^2z^2(r^2\sin t)$$

$$= 192$$

H.W. find $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial r}$

32. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?





$$\frac{dr}{dt} = 1.8$$

$$\frac{dh}{dt} = -2.5$$

Q: $\frac{dV}{dt} = ??$

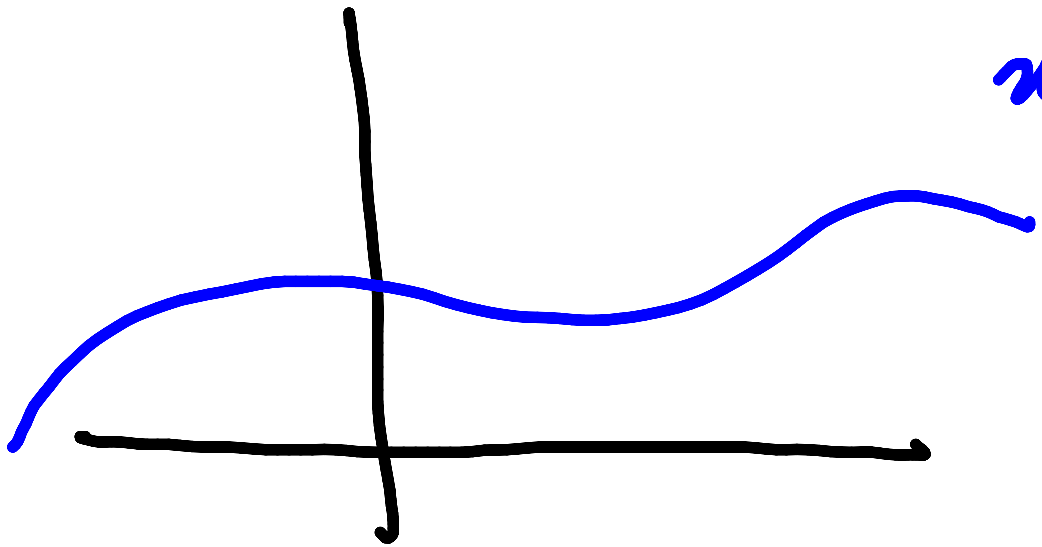
$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \left(\frac{2}{3}\pi r h\right)\frac{dr}{dt} + \left(\frac{\pi r^2}{3}\right)\frac{dh}{dt}$$

$[r = 120,\ h = 140]$

$$= 8160\pi$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

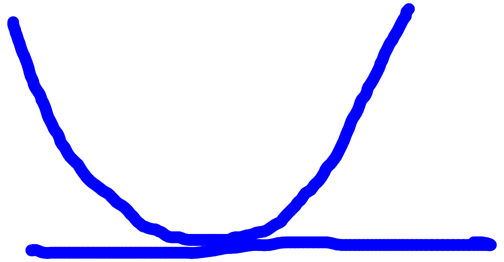


$x = \sqrt{1+t}$

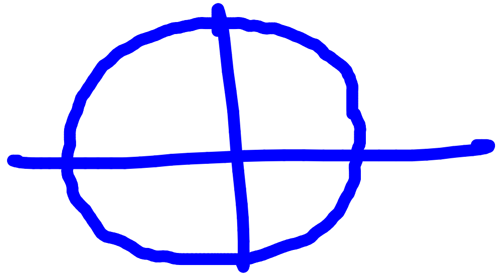
$y = 2 + \frac{1}{3}t$

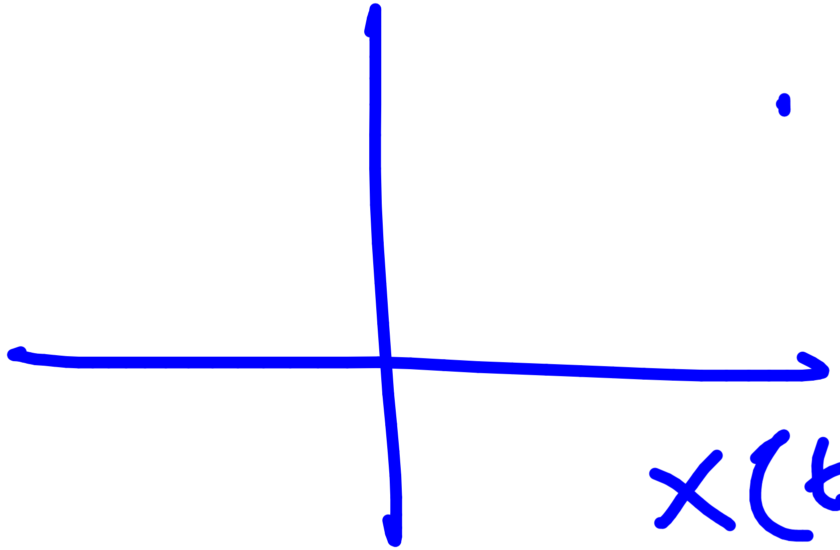
Paths: example

→ $x = t$, $y = t^2$



→ $x = \cos(t)$, $y = \sin(t)$





at time t the position of the bug is

$x(t) = \sqrt{1+t}$

$y(t) = 2 + \frac{1}{3}t$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

$$\begin{tikzcd} & T \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[d, no head] & & y \arrow[d, no head] \\ t & & t \end{tikzcd}$$

$t = 3$, $x = 2$, $y = 3$

$$\frac{dT}{dt}(t=3) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

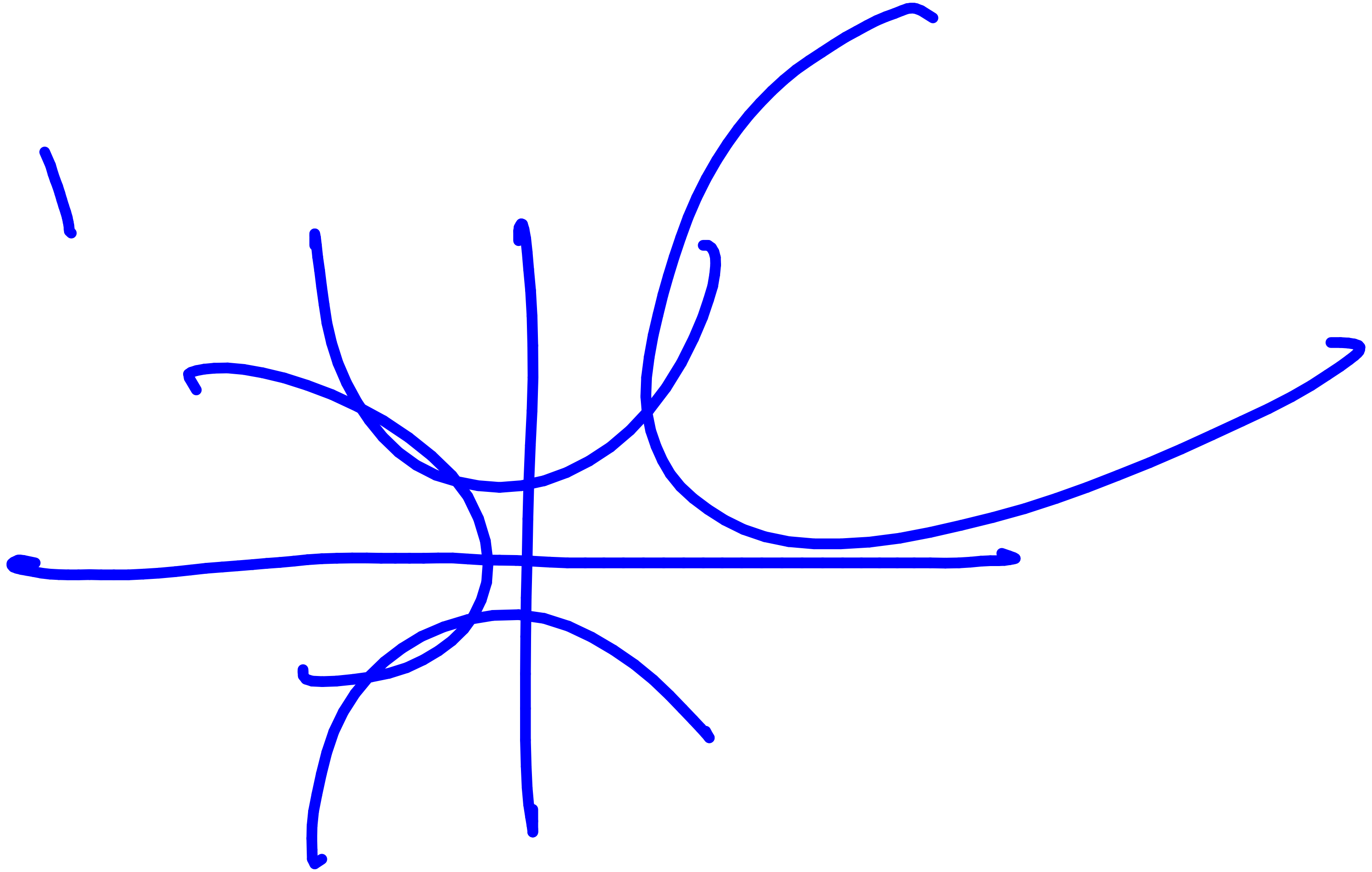
$$= \left(4\,\frac{1}{2\sqrt{1+t}} + 3\cdot\frac{1}{3}\right)\Bigg|_{t=3} = 2$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

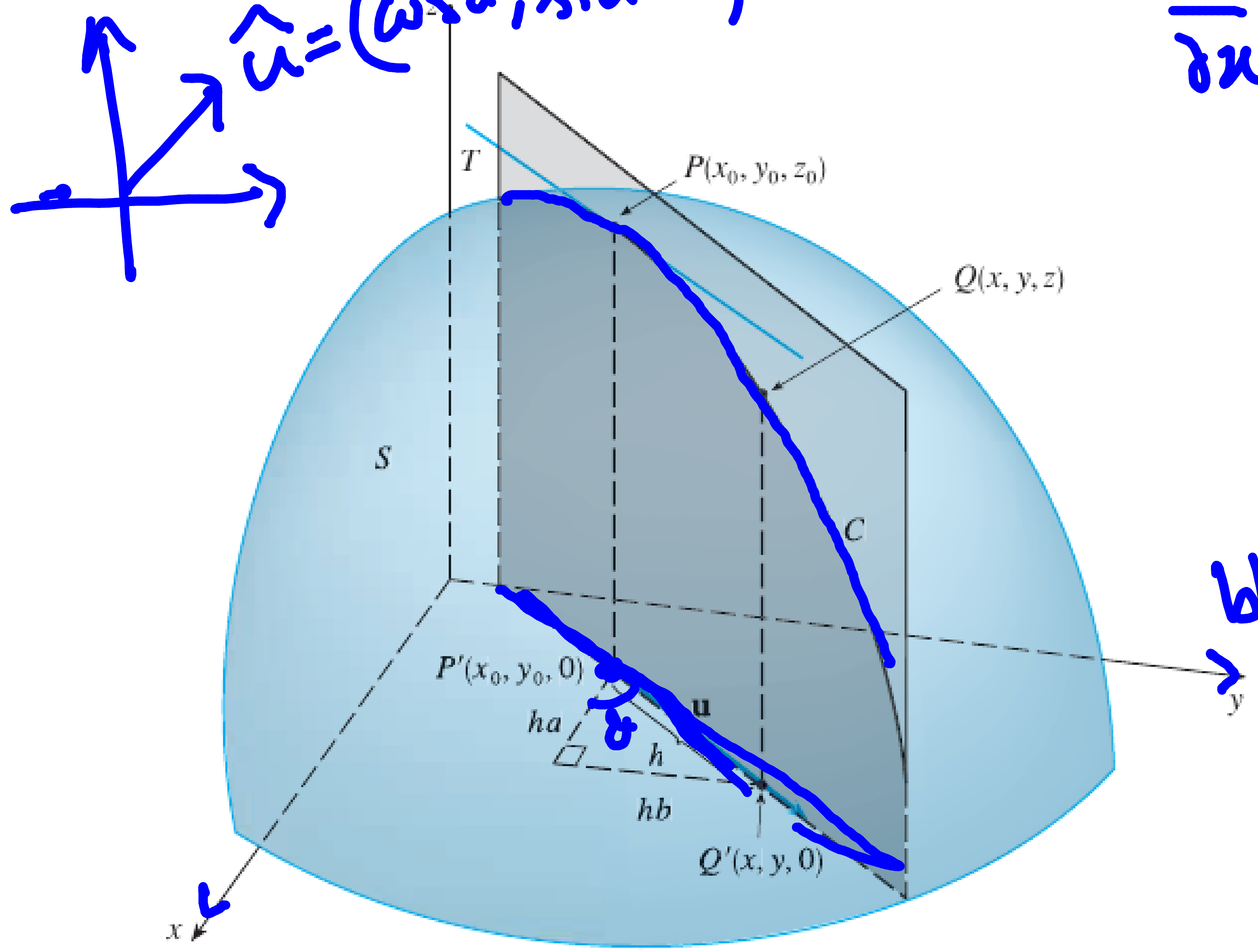
$x = \sqrt{1+t} \rightarrow t = x^2 - 1$

$y = 2 + \frac{1}{3}t$

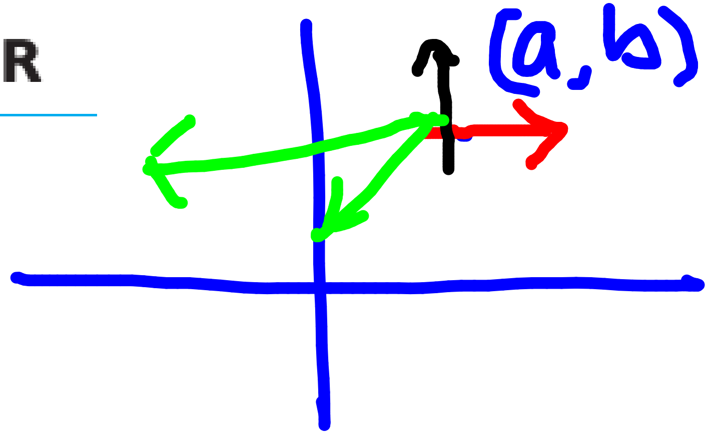
$y = 2 + \frac{(x^2-1)}{3}$



$\hat{u} = (\cos\theta, \sin\theta)$



$\dfrac{\partial f}{\partial x} =$ $\dfrac{\partial f}{\partial y}$



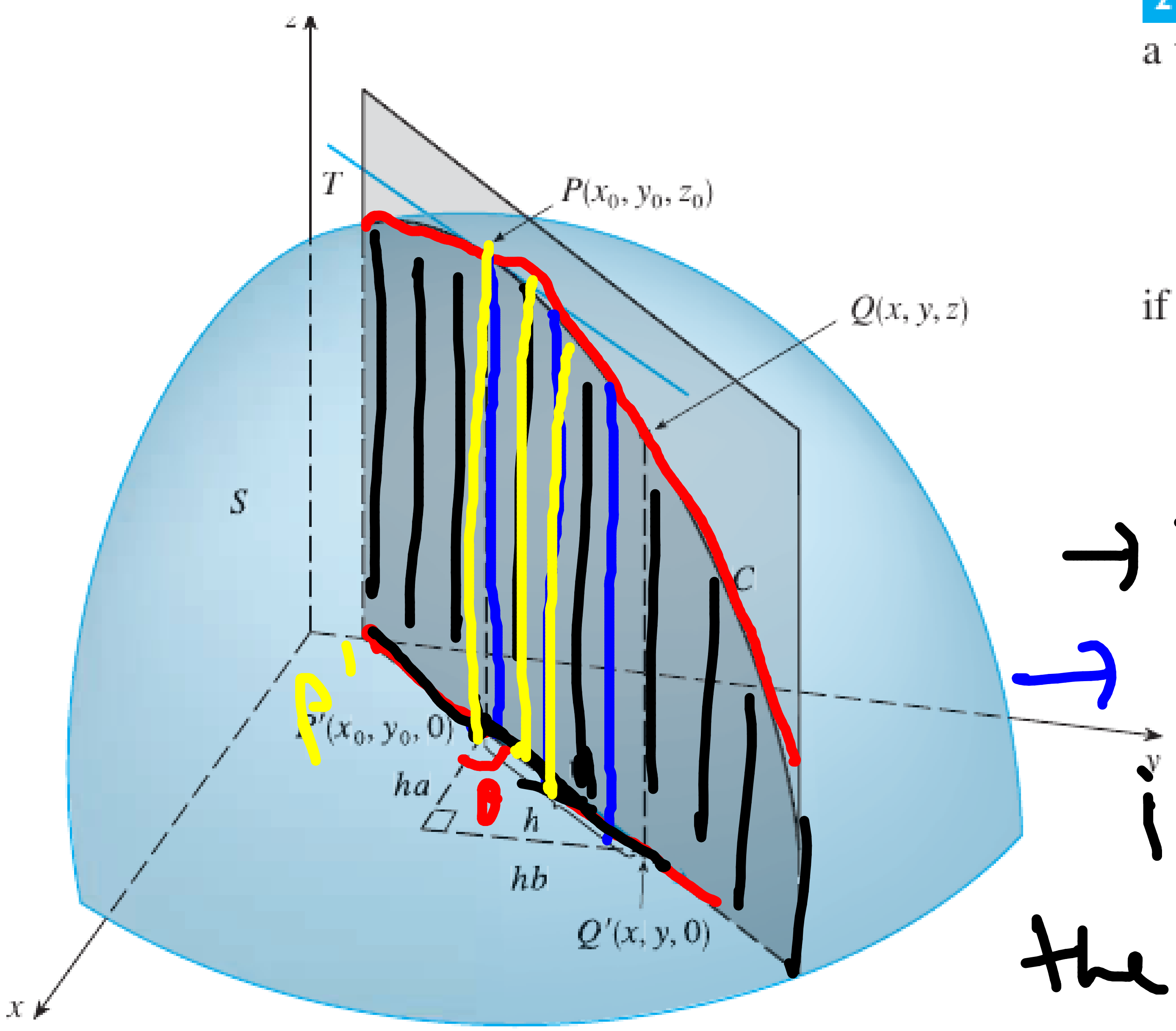
2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

blue = graph of $f(x,y)$ =

11.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR



2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

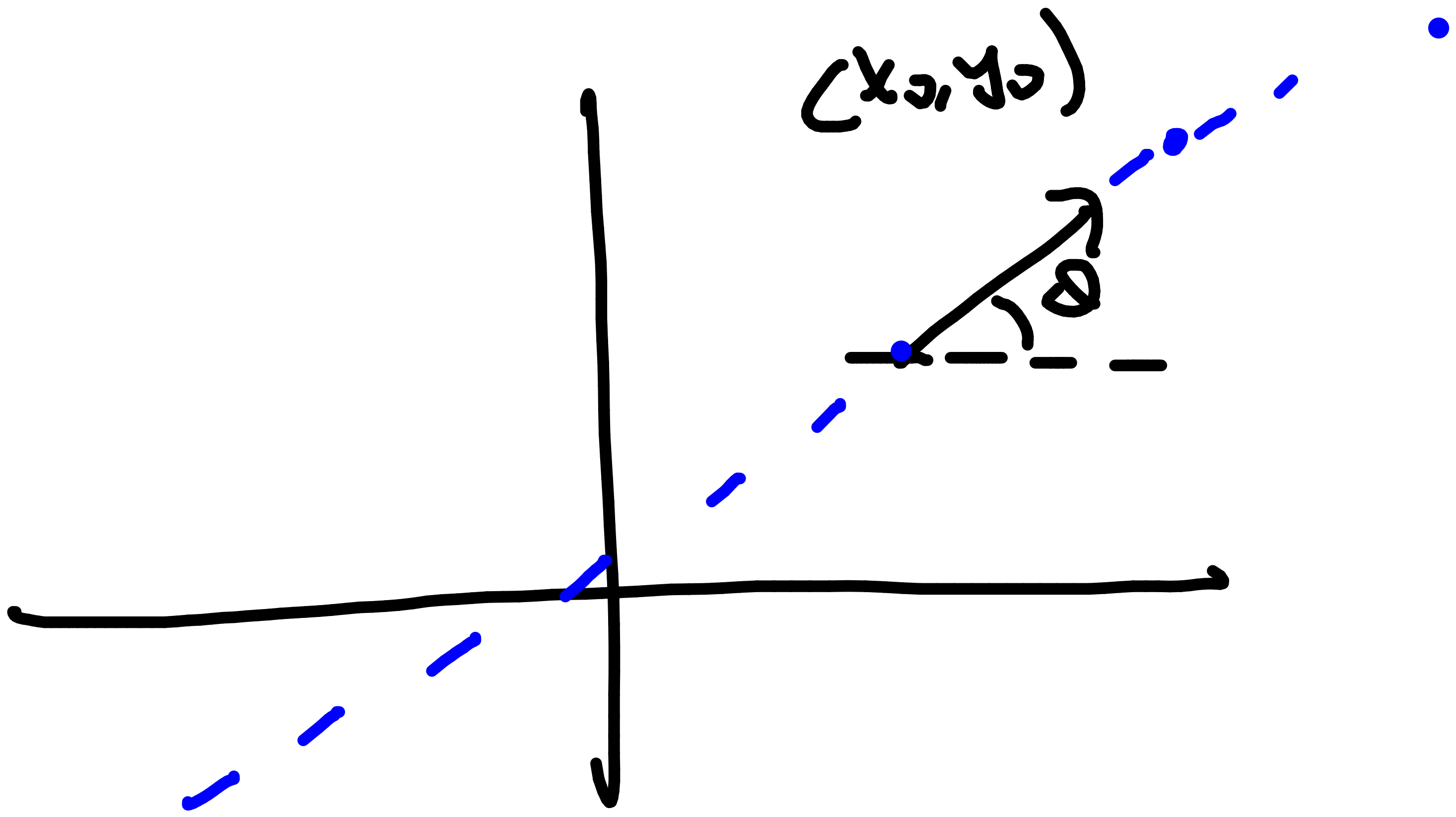
$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

graph of $f(x,y)$

→ Directional derivative of
→ $f(x,y)$, at $P'(x_0, y_0)$
in the direction $\hat{u}$, is
the rate of change you observe

in f when we start moving from p' in the direction of $\hat{u}$.

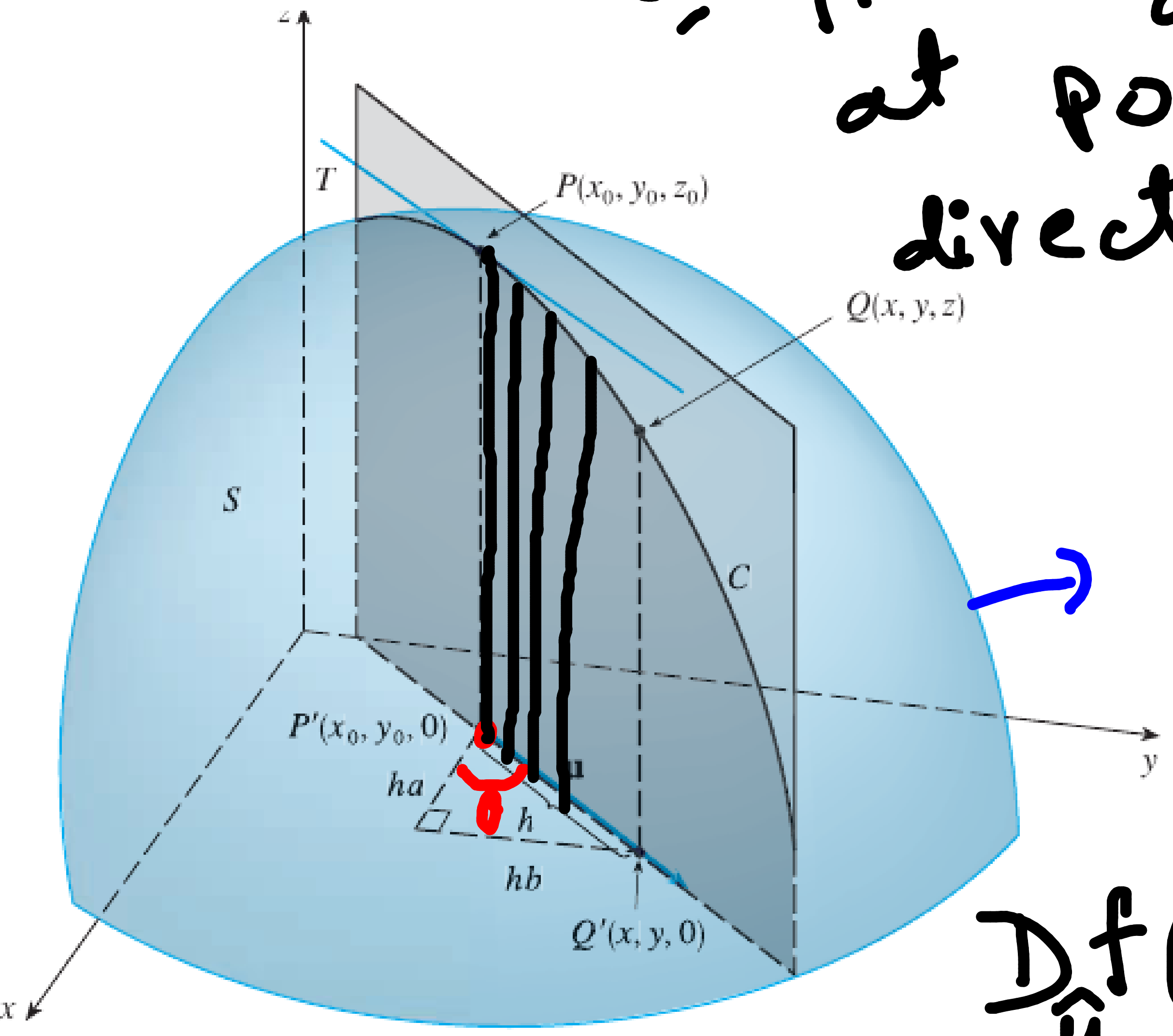


$$(x_0, y_0) + h(\cos\theta, \sin\theta)$$

$$= (x_0 + h\cos\theta,\ y_0 + h\sin\theta)$$

Q: find directional derivative of $f(x,y)$ at point (x_0,y_0) in the direction $\hat{u} = (\cos\theta, \sin\theta) = (a, b)$

$[a^2+b^2=1]$



→ graph of $f(x,y)$

value of f on the line at h distance from p'

$$D_{\hat{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f\big((x_0,y_0)+h(\cos\theta,\sin\theta)\big) - f(x_0,y_0)}{h}$$

skip proof

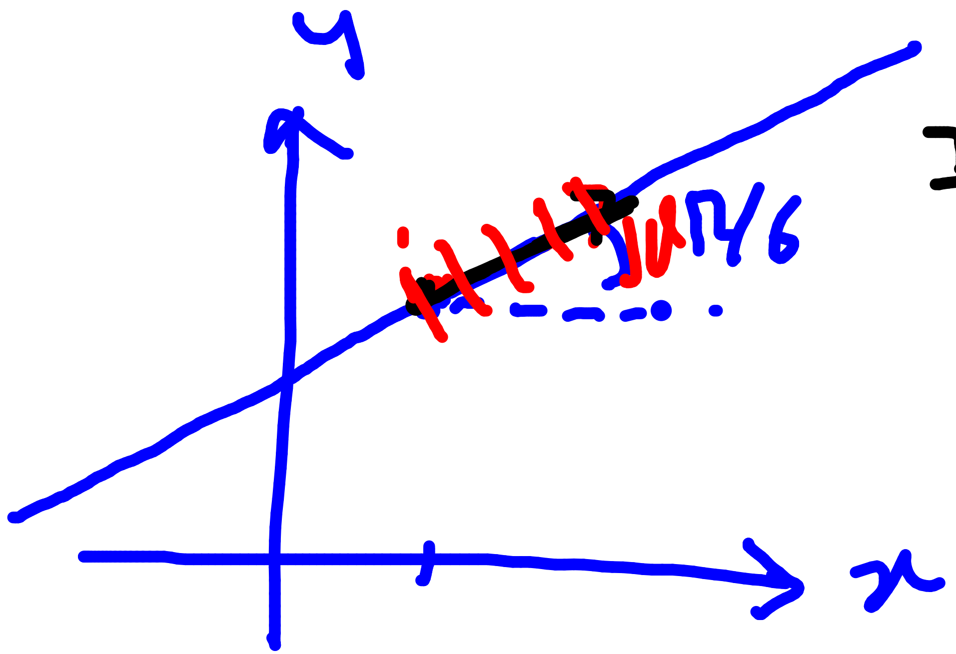
Theorem:

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

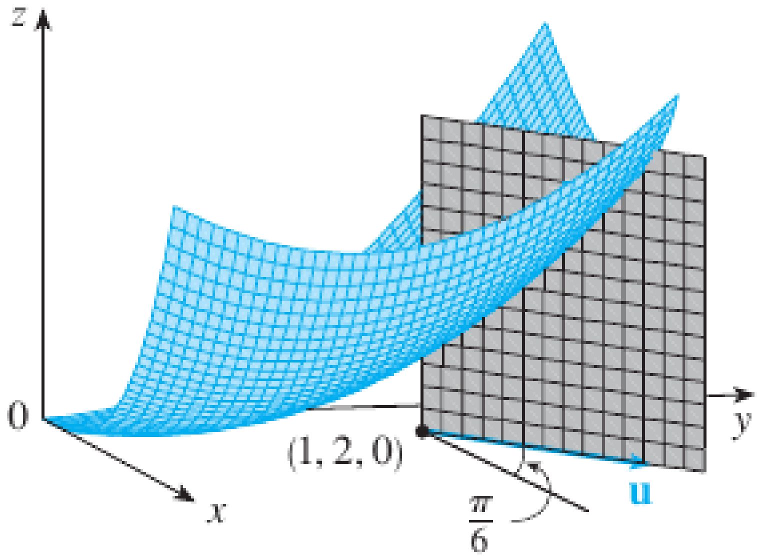
3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

EXAMPLE 1 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?



$$D_{\hat{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$



$$\boxed{\frac{\partial f}{\partial x} = 3x^2 - 3y = 3 - 6 = -3}$$

$$\hat{u} = \cos(\pi/6)\,\hat{i} + \sin(\pi/6)\,\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$\boxed{\frac{\partial f}{\partial y} = -3x + 8y = -3 + 16 = 13}$$

$$D_{\hat{u}} f(1,2) = (-3)\cos(\pi/6) + 13\sin(\pi/6) = (13 - 3\sqrt{3})/2$$

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

EXAMPLE 4 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

MAXIMIZING THE DIRECTIONAL DERIVATIVE

EXAMPLE 5

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?