2.10 Solution by Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

→ solve the homogeneous part

$$y'' + p(x)y' + q(x)y = 0$$

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$$

→ $y_1 \quad y_2$

$$W = y_1 y_2' - y_2 y_1'$$

↓

Wronskian

→ $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

→ $\boxed{y_p = u_1 y_1 + u_2 y_2}$

where $u_1 = -\int \frac{y_2 r}{W} dx$ $u_2 = \int \frac{y_1 r}{W} dx$

2nd chapter :

$$y'' + a(x)\, y' + b(x)\, y = r(x)$$

→ $a(x), b(x)$: constants

→ $a = \frac{1}{x}$, $b = 1/x^2$

→ $\underbrace{r(x) = 0}_{\text{name: homogeneous}}$

$r(x) \neq 0$

non - homo

Q. Solve

$$y'' + 5y' + 2y = 0$$

Recall the steps:

→ solve auxilliary equations

→ $\lambda^2 + 5\lambda + 2 = 0$

λ: real distinct	λ real repeated	λ complex conjugate

Q. Solve

$$y'' + 5y' + 2y = x^2$$

⇒ solve the homogeneous eqn

$$y'' + 5y' + 2y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

⇒ general solution.

$$\boxed{y = c_1 y_1 + c_2 y_2 + \underbrace{y_p}_{\text{particular solution}}}$$

Q. Solve

$$y'' + 5y' + 2y = x^2$$

$$\boxed{y_p = a_0 + a_1 x + a_2 x^2}$$

find a_0, a_1, a_2

$r(x)$	y_p
x^m	$a_0 + \dots + a_m x^m$
e^{ax}	ce^{ax}
$\sin(\omega x)$	$A\sin\omega x + B\sin\omega x$

Q: Solve with variation of parameters:

$$y'' + 3y' + 2y = x^2$$

→ Solve the homogenous part.

$$y'' + 3y' + 2y = 0$$

$$y_1 = e^{-x} \qquad y_2 = e^{-2x}$$

↓

Wronskian

↳ $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

↳ $\boxed{y_p = u_1 y_1 + u_2 y_2}$

where $u_1 = -\int \frac{y_2 r}{W} dx \qquad u_2 = \int \frac{y_1 r}{W} dx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$u_1 = -\int \frac{y_2 r}{W}\,dx = -\int \frac{e^{-2x} x^2}{-e^{-3x}}\,dx = \int e^{x} x^2\,dx$$

$$= (x^2 - 2x + 2)e^{x}$$

$$u_2 = \int \frac{y_1 r}{W}\,dx = \int \frac{e^{-x} x^2}{-e^{-3x}}\,dx = -\int e^{2x} x^2\,dx$$

$$= -\frac{(2x^2 - 2x + 1)e^{2x}}{4}$$

(%i2) integrate(exp(x)·x^2,x);

(%o2) $(x^2 - 2x + 2)\,\%e^{x}$

(%i3) integrate(exp(2·x)·x^2,x);

(%o3) $\dfrac{(2x^2 - 2x + 1)\,\%e^{2x}}{4}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (x^2 - 2x + 2)\cancel{e^{x}}\,\cancel{e^{-x}} + -\frac{(2x^2 - 2x + 1)\cancel{e^{2x}}\,\cancel{e^{-2x}}}{4}$$

$$= \text{whatever}$$

$y'' + 9y = \sec 3x$

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

→ solution of the homogeneous part

→ $y_1 = \cos 3x \qquad y_2 = \sin 3x$

$$W = y_1 y_2' - y_2 y_1'$$

→ $W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$

→ $u_1 = -\int \frac{y_2 r}{W}\, dx$

$= -\int \frac{\sin 3x \sec 3x}{3}$

$u_2 = \int \frac{y_1 r}{W}\, dx$

$= \int \frac{\cos 3x \sec 3x}{3}\, dx$

$$u_1 = \frac{1}{9} \ln\left|\cos(3x)\right| \quad \Big| \quad u_2 = \frac{x}{3}$$

$$y_p = \frac{1}{9} \ln\left(\left|\cos 3x\right|\right) \cos 3x + \frac{x}{3} \sin 3x$$

$$y'' - 4y' + 5y = e^{2x}\csc x$$

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

$$W = y_1 y_2' - y_2 y_1'$$

do yourself

Idea of the Method.

End sem :

→ Option 1 : after the summer break

→ Option 2: online exam using blackboard, etc, or assignments

→ Option 3: make video presentation

→ making recording of solving given question

→ post it blackboard

→ peer evaluation