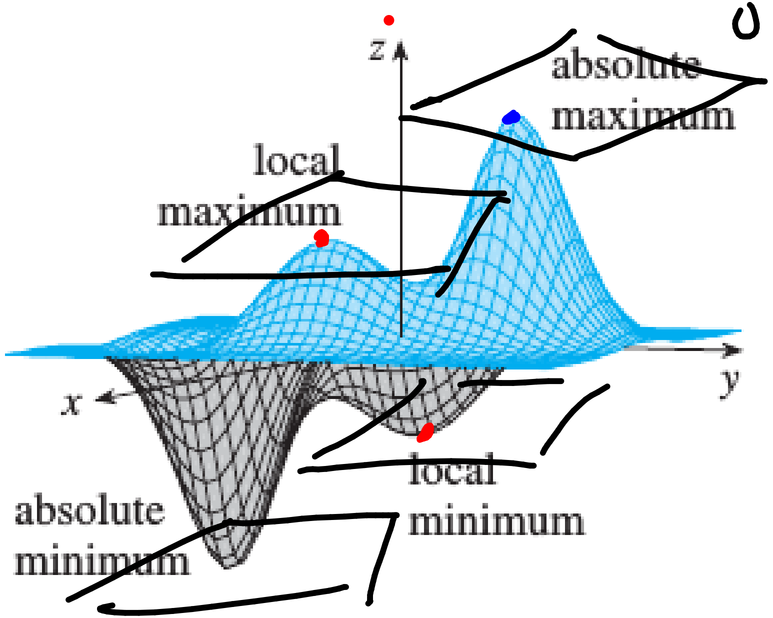
11.7 MAXIMUM AND MINIMUM VALUES

· Critical points

· classification of critical points

→ at local max/min points tangent plane will be horizontal



→ $z - z_0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

⇒ if (a,b) is a point of max/min

then $\boxed{f_x(a,b) = f_y(a,b) = 0}$

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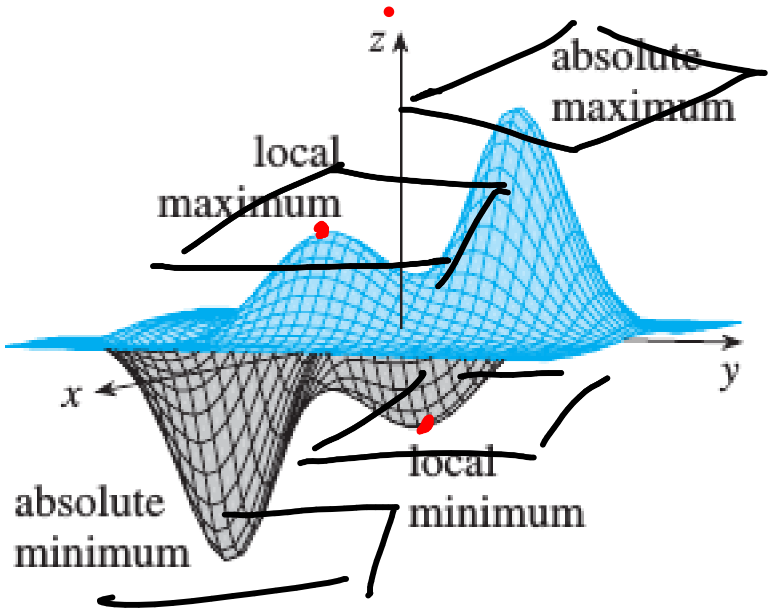
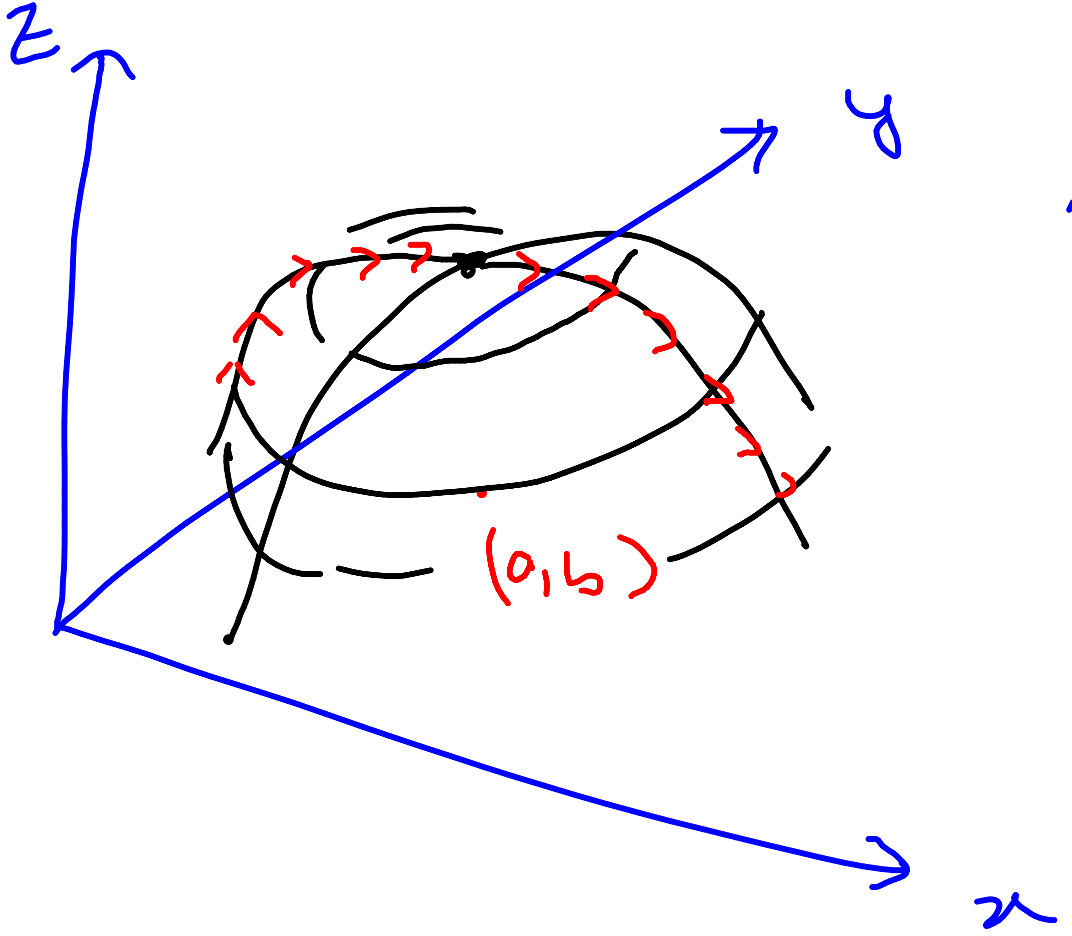


⇒ if (a,b) is a point of max/min

then $f_x(a,b) = f_y(a,b) = 0$



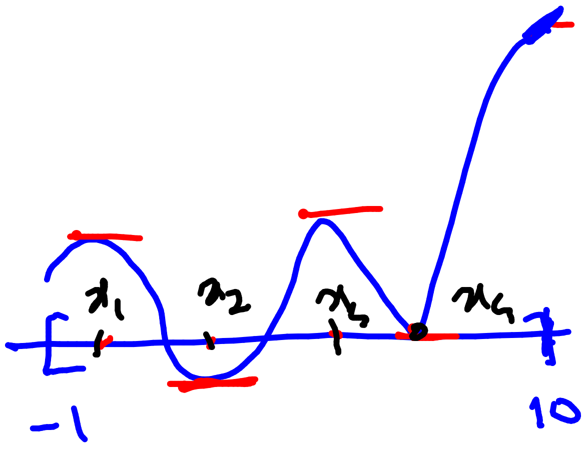
$f_x(a,b) = 0$

$f_y(a,b) = 0$

Recall questions like

$$f(x) = x^2 + \sin(x) + 2$$

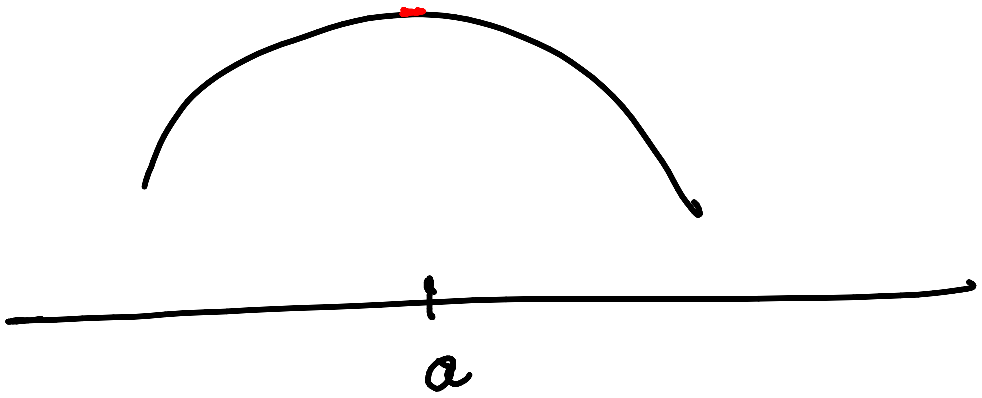
find the max / min



$\Rightarrow$ solve $f'(x) = 0$, $\Rightarrow x = x_1, x_2, x_3, x_4$

max value $=$ $\max\{f(x_1), f(x_2), f(x_3), f(x_4), f(a), f(b)\}$

min value $=$ $\min\{f(x_1), f(x_2), f(x_3), f(x_4), f(a), f(b)\}$



$f(x)$

$f'(a) = 0$

identify a as
a a local max
or a local min

f''

critical point

points in the domain of $f(x,y)$ where

$$f_x = 0 \quad \& \quad f_y = 0$$

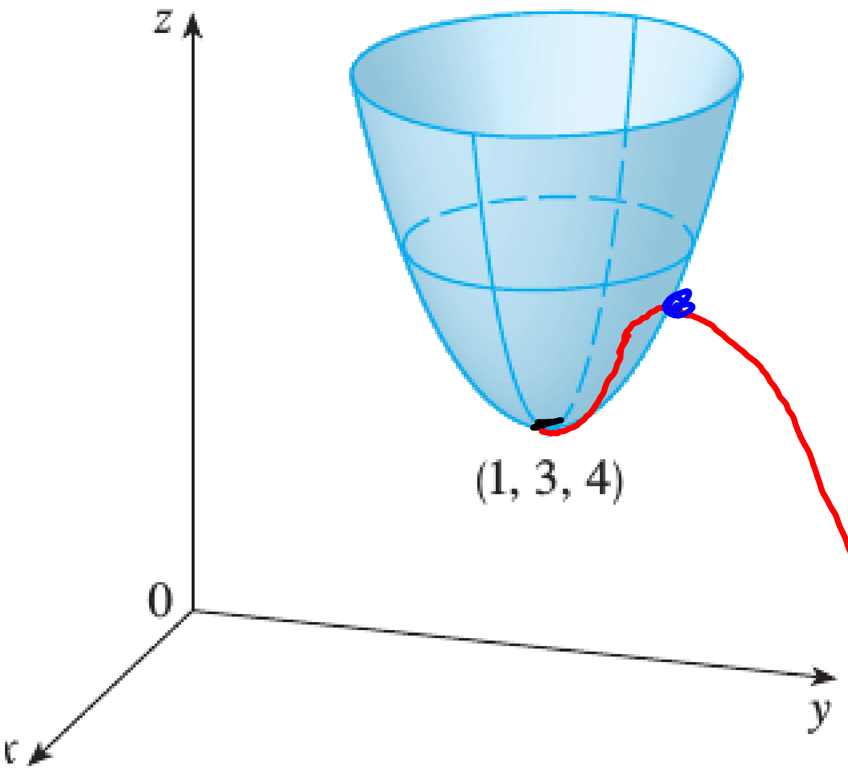
EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Q: find points of local min or max

→ find critical points

$f_x = 0$ $\qquad$ $f_y = 0$

$2x - 2 = 0$ $\qquad$ $2y - 6 = 0$

$x = 1$ $\qquad$ $y = 3$

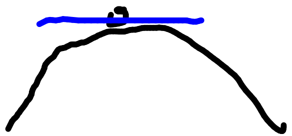


now check if it is a point of max/ min/ neither

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

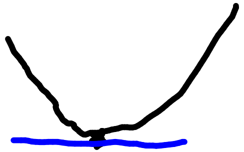
$$\underbrace{\det(H) = 4 \quad \& \quad f_{xx} = 2 > 0} \implies (1,3) \text{ is a point of local min}$$

Classification of critical points



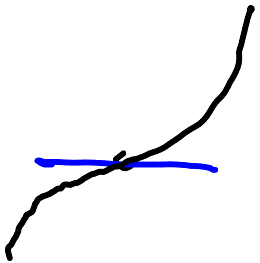
$f'' < 0$

f is concave down

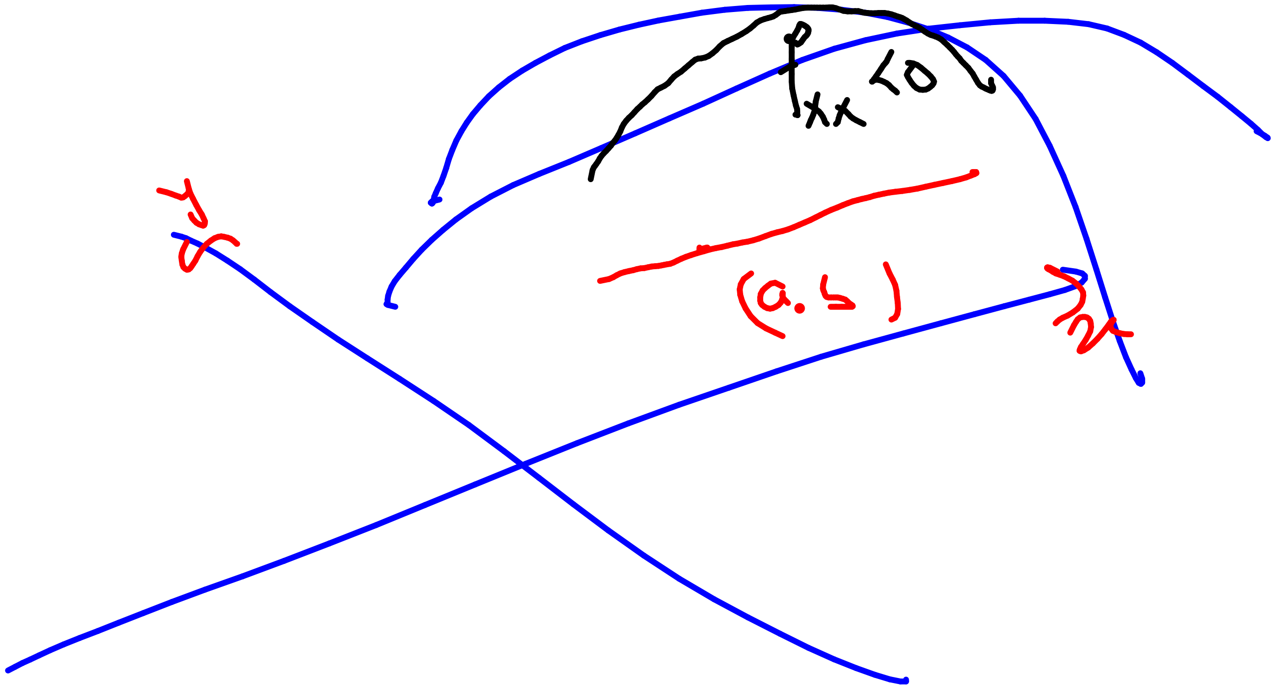


$f'' > 0$

f is concave up



$f'' = 0$



y
x
(a, b)
$f_{xx} < 0$

Classification of critical points (only applicable for two variable functions)

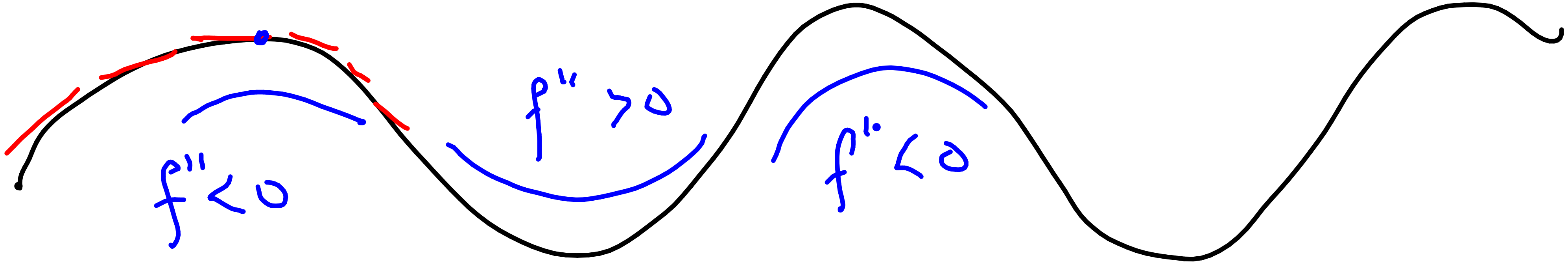
$f(x,y)$

Hessian Matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

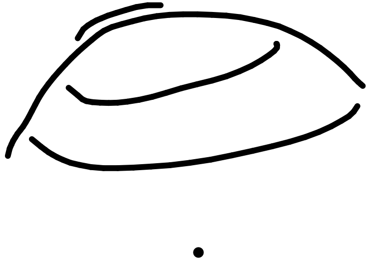
local max	local min	neither
$\det(H) > 0$	$\det(H) > 0$	$\det(H) < 0$
& $f_{xx} < 0$	& $f_{xx} > 0$	saddle point

Recall one variable calculus



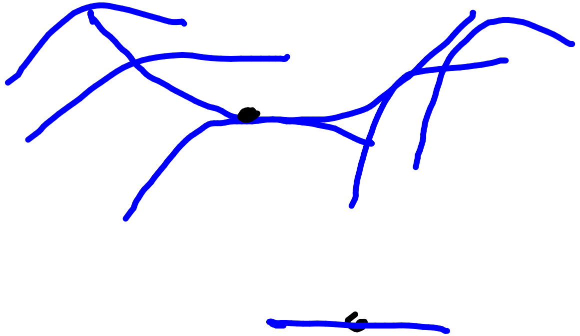
f' is decreasing

Classification of critical points



f is concave down

f is concave up



neither

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

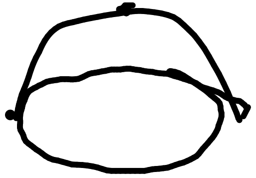
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Hessian matrix

max

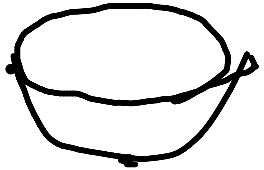


concave down

$D > 0$

$f_{xx} < 0$

min

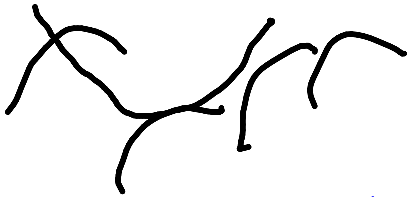


concave up

$D > 0$

$f_{xx} > 0$

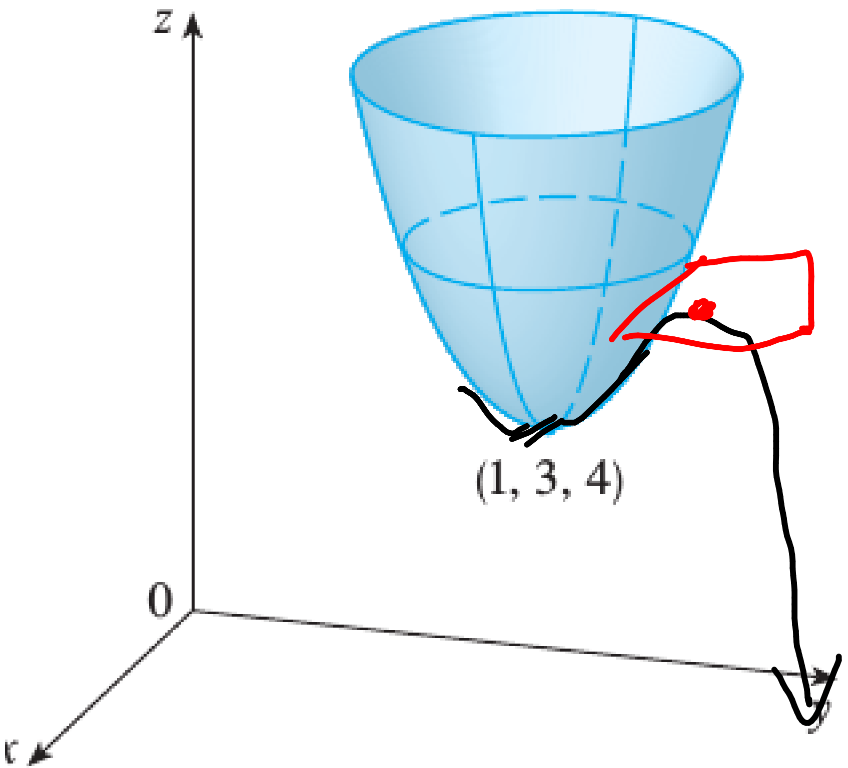
neither



neither

$D < 0$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Q: find points of local min or max



$f_x = 0$

$2x - 2 = 0$

$x = 1$

$f_y = 0$

$2y - 6 = 0$

$y = 3$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\underbrace{\det(H) > 0 \ \& \ f_{xx} > 0}_{\Rightarrow (1,3) \text{ is a min point}}$$

& (1,3) is also an absolute min

1. Suppose (1, 1) is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

(b) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$

a) $|H| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7 > 0$

$f_{xx} = 4 > 0$

f concave up $\Rightarrow$ (1,1) is a local min

b) $|H| = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} = -1 < 0$

(1,1) is a saddle point

Q. $f(x,y) = x^2 - y^2$ find & classify critical points

$f_x = 0$ $f_y = 0$

$2x = 0$ $2y = 0$

$x = 0$ $y = 0$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

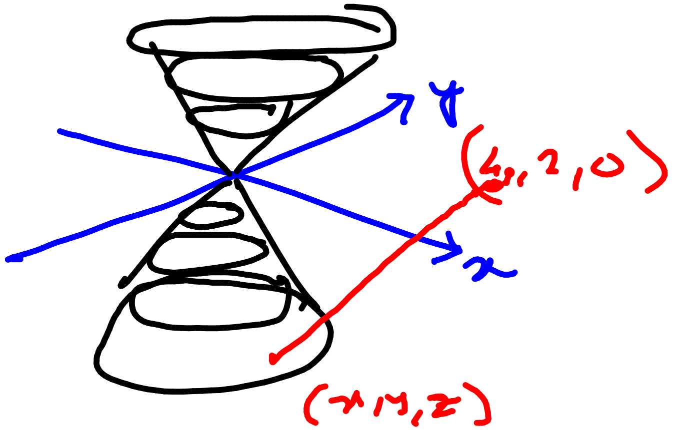
$|H| = -4 < 0$

Saddle point

33. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

s.t. (x,y,z) belongs to the surface $z^2 = x^2 + y^2$



solve it

use $z^2 = x^2 + y^2$

minimize $f = (x-4)^2 + (y-2)^2 + x^2 + y^2$

find critical points

$f_x = 2(x-4) + 2x = 0$ | $f_y = 2(y-2) + 2y = 0$

$\Rightarrow \quad x = 2$ | $y = 1$

$(2,1)$: critical point

check for max/min $\quad |H| = \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 > 0$

$\& \ f_{xx} = 4 > 0$

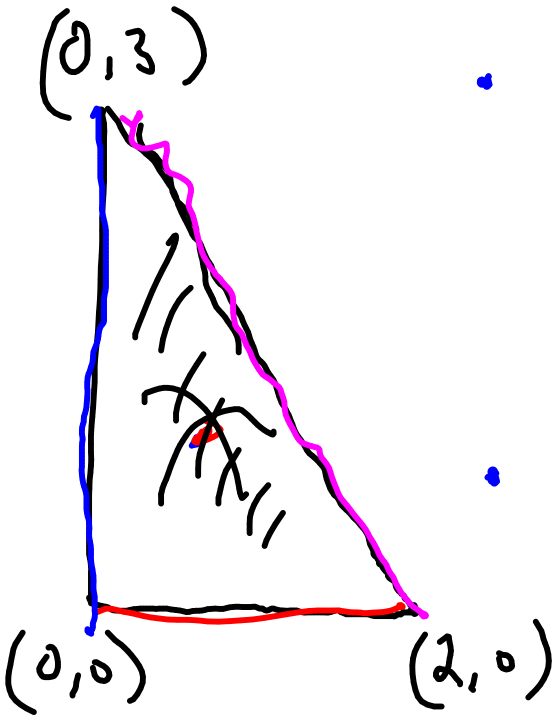
$(2,1)$ is a point of local min

but also absolute min (why??)

Ans: The point on the cone $z^2 = x^2 + y^2$ closest to $(4,2,0)$ is $(2,1,\sqrt{5})$ & $(2,1,-\sqrt{5})$

Q. Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$



- max/min points can be in the interior of the domain

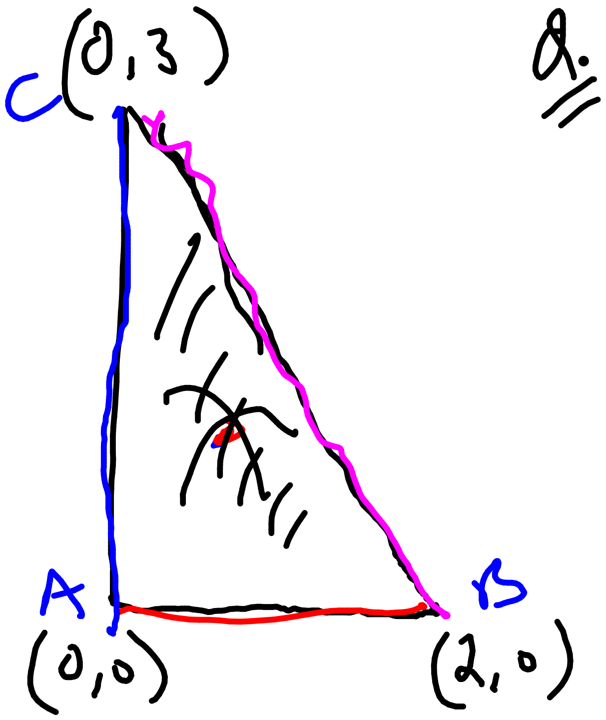
  $f_x = 0$ & $f_y = 0$

- max/min points can be in the boundary

Q:

Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices (0, 0), (2, 0), and (0, 3)



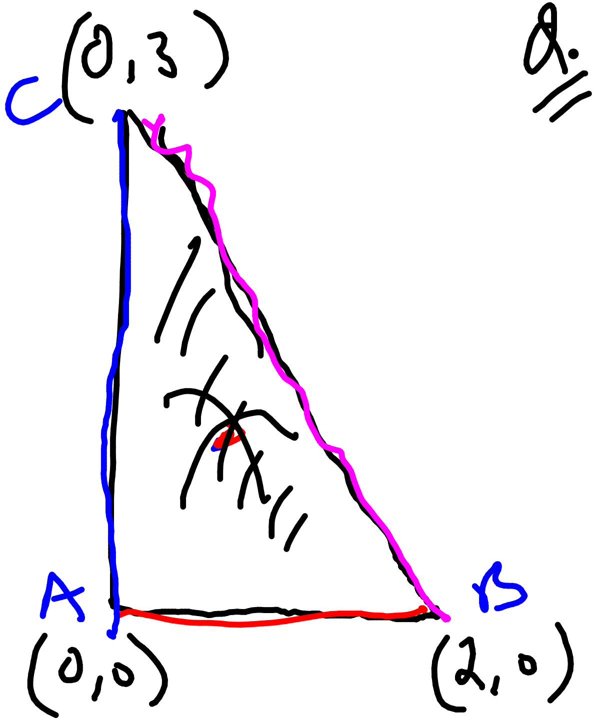
Q. Does f has a max/min point in the interior ??

$f_x = 0$ & $f_y = 0$

$(0,0)$ not in interior

Q. Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$



Q. Does f has a max/min point on the boundary segment AB

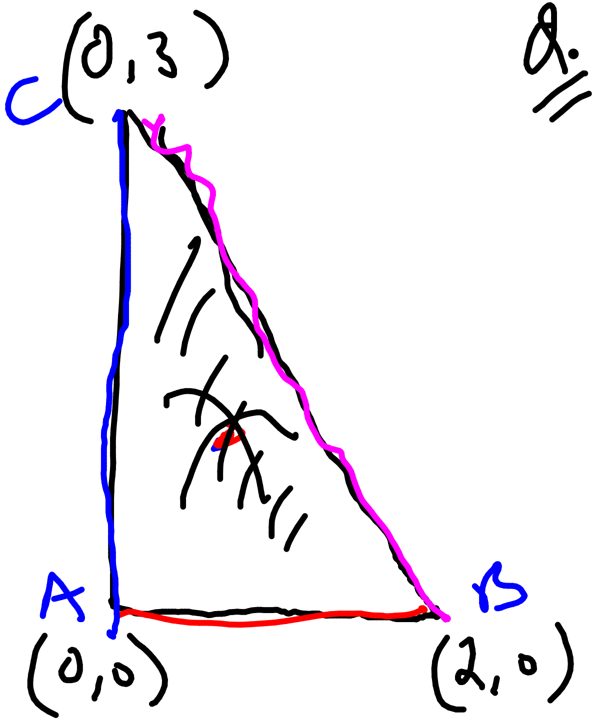
$f\big|_{AB} = 1+4x^2 \quad , \quad 0 \le x \le 2$

min	max
$x=0$ $\quad f=1$	$x=2$ $\quad f=17$
$y=0$	$y=0$

Q.

Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$



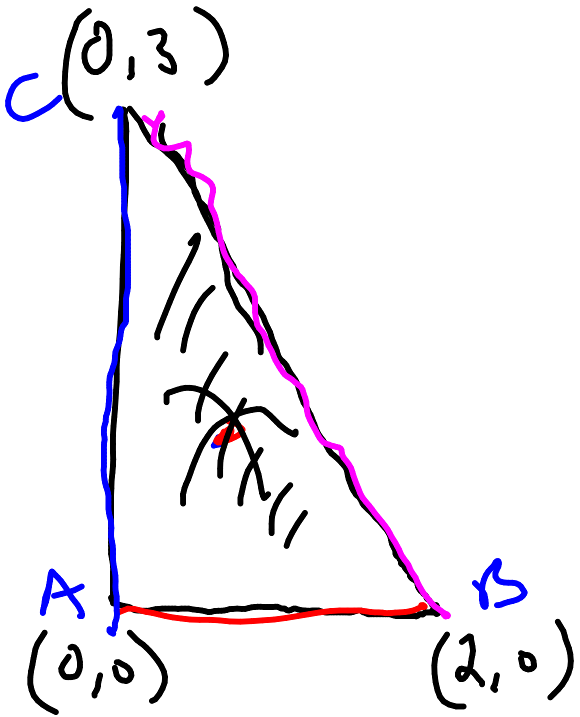
Q. Does f has a max/min point on the boundary segment AC

$$f\Big|_{AC} = 1 - 5y^2 \qquad 0 \le y \le 3$$

min	max
$y = 3$, $x = 0$: $f = -44$	$y = 0$, $x = 0$: $f = 1$

Q.
Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices (0, 0), (2, 0), and (0, 3)



Q. Does f has a max/min point on the boundary segment BC,

$3x + 2y = 6$

$y = 3 - \frac{3}{2}x$

$$f\big|_{BC} = 1 + 4x^2 - 5\left(3 - \frac{3}{2}x\right)^2$$

$0 \le x \le 2$

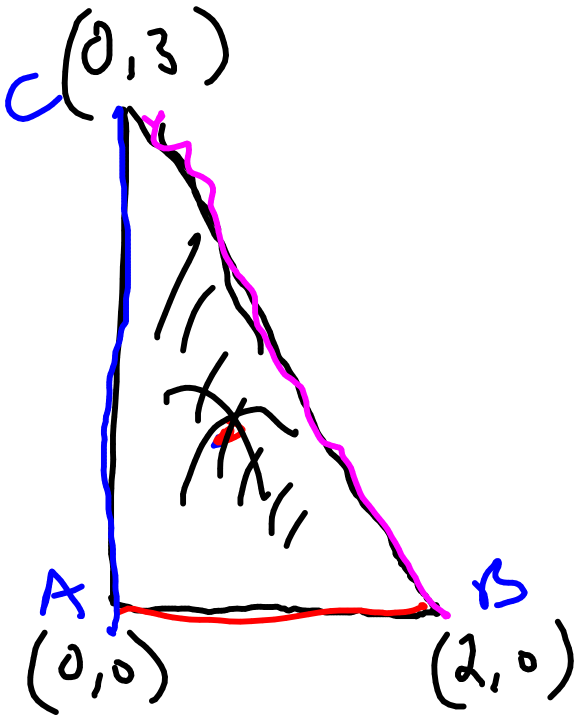
max at $x = 2$

min at $x = 0$

Q.

Find the absolute maximum and minimum values of f on the set D.

$f(x, y) = 1 + 4x^2 - 5y^2$, D is the closed triangular region with vertices (0, 0), (2, 0), and (0, 3)



max at B : 17

min at C = -44

maximize & minimize this

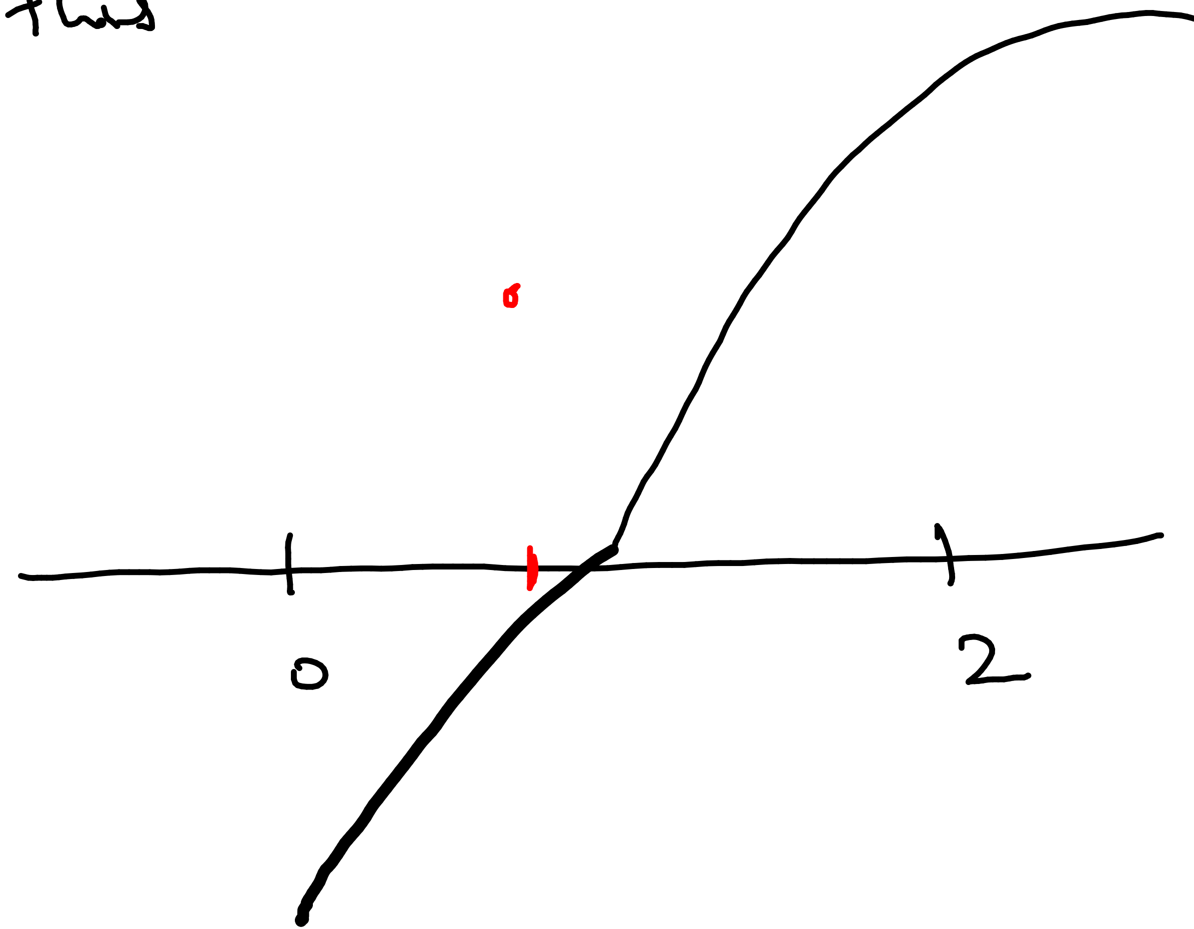
$$f|_{BC} = 1+4x^2 - 5\left(3 - \frac{3x}{2}\right)^2$$

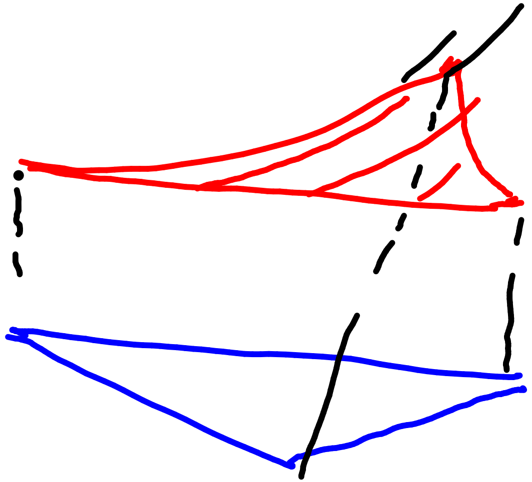
$0 \le x \le 2$

$$g = -44 + 45x - \frac{29}{4}x^2$$

$$g' = 45 - \frac{29}{2}x$$

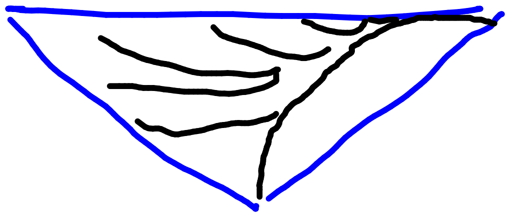
$$x = \frac{90}{29}$$





$f_x \neq 0$ & $f_y \neq 0$

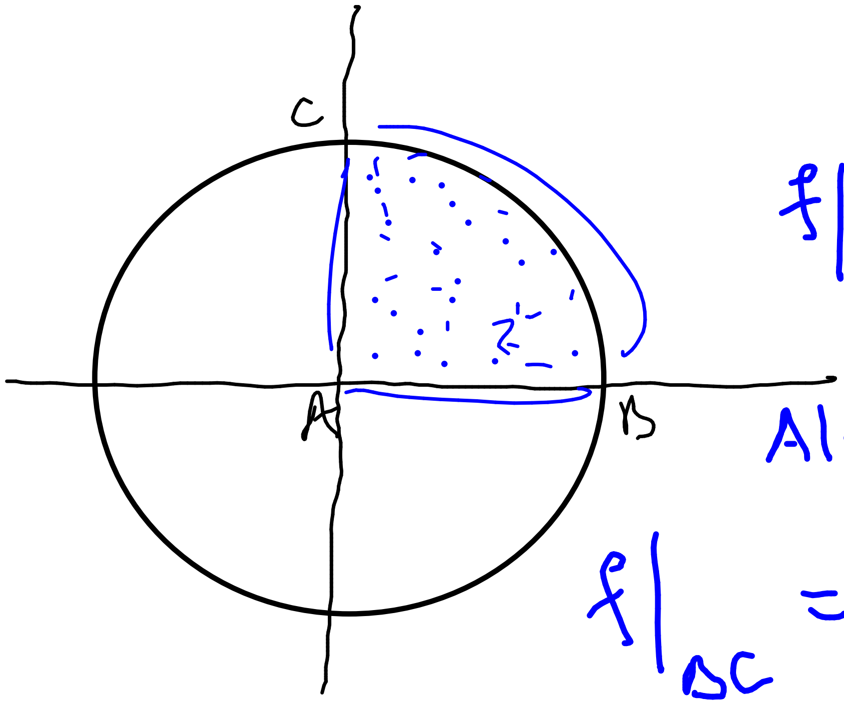
& max point at boundary



max/min may not happen at corners

d. Find the absolute maximum and minimum values of f on the set D.

$$f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0, x^2 + y^2 \leqslant 3\}$$



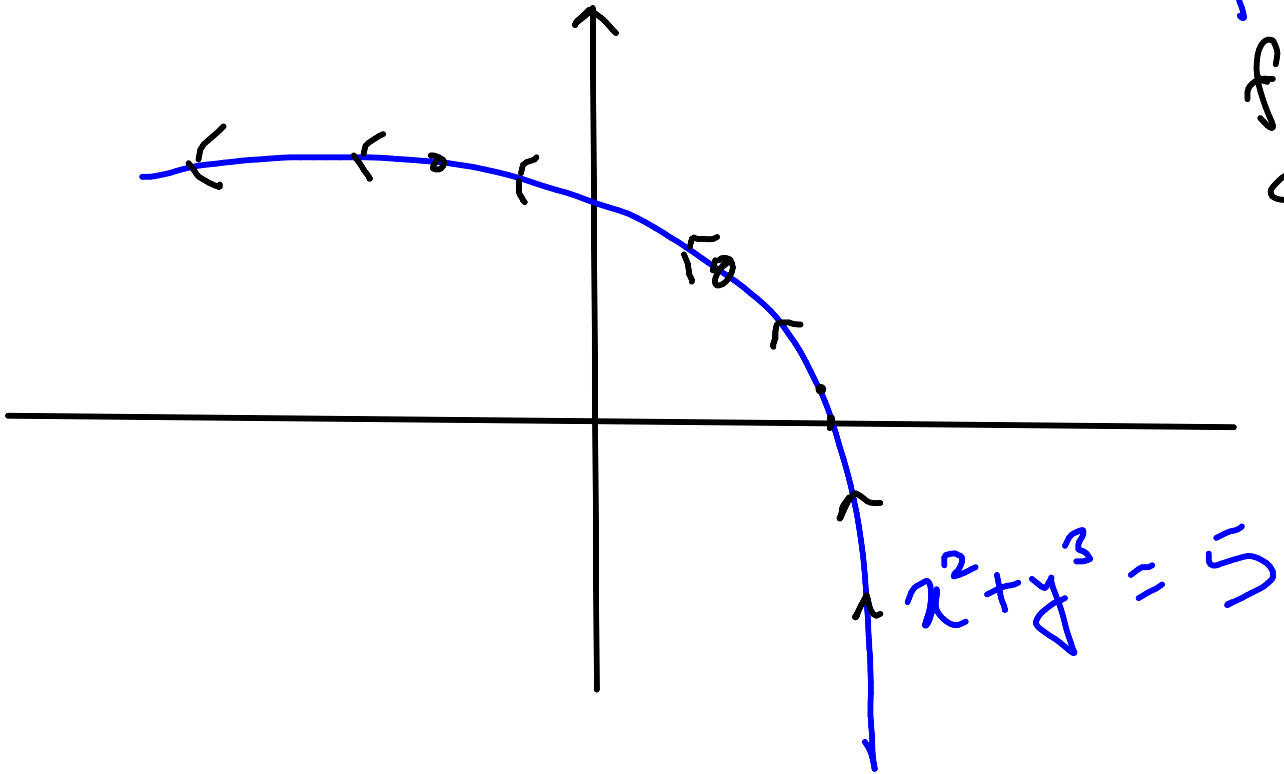
BC : $x^2 + y^2 = 3$

$f\big|_{BC} = x(3 - x^2)$, $0 \le x \le \sqrt{3}$

Alternatively.

$f\big|_{BC} = (\sqrt{3})^3 \cos\theta \sin^2\theta$, $0 \le \theta \le \pi/2$

LAGRANGE MULTIPLIERS

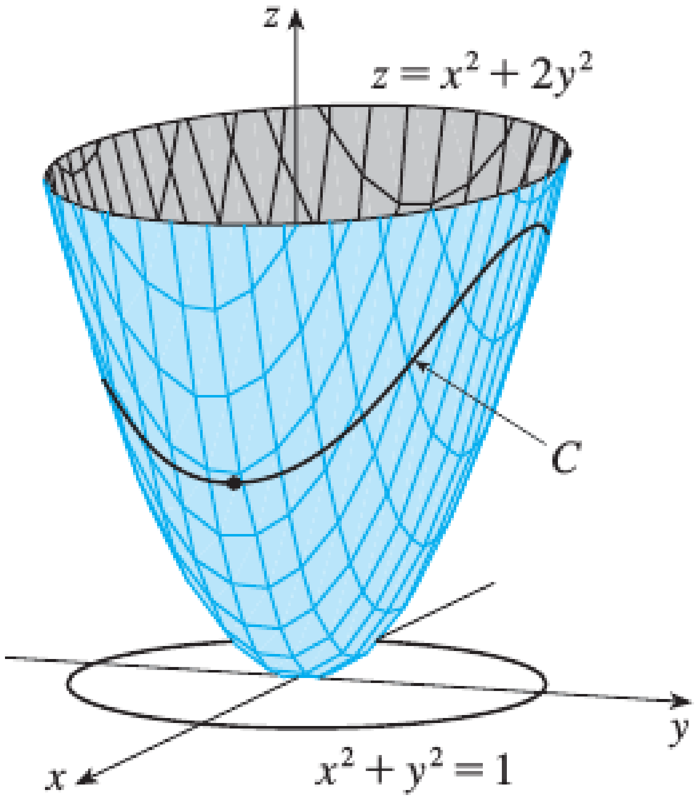


$f(x,y) = x+y$

find point on the curve $x^2+y^3=5$ where $f(x,y)=x+y$ is lowest.

next time

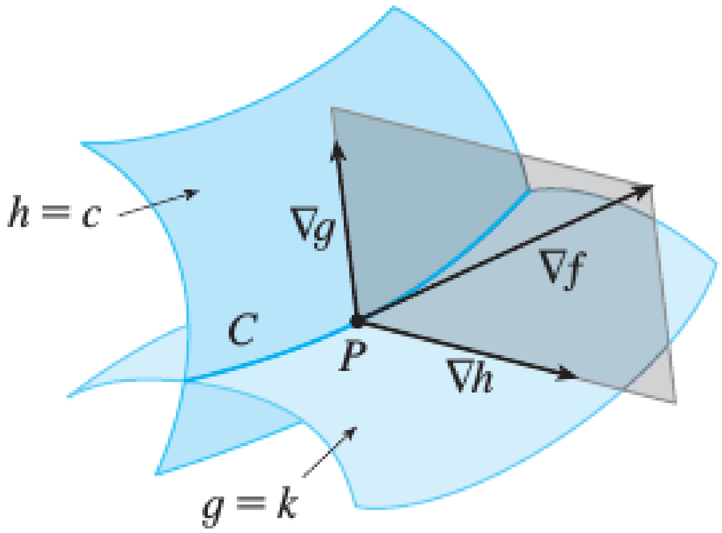
EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.



EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

EXAMPLE 4 Find the points on the circle ~~sphere~~ $x^2 + y^2 \quad = 4$ that are closest to and farthest from the point $(3, 1 \quad)$.

TWO CONSTRAINTS



$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

V EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35$$

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4$$