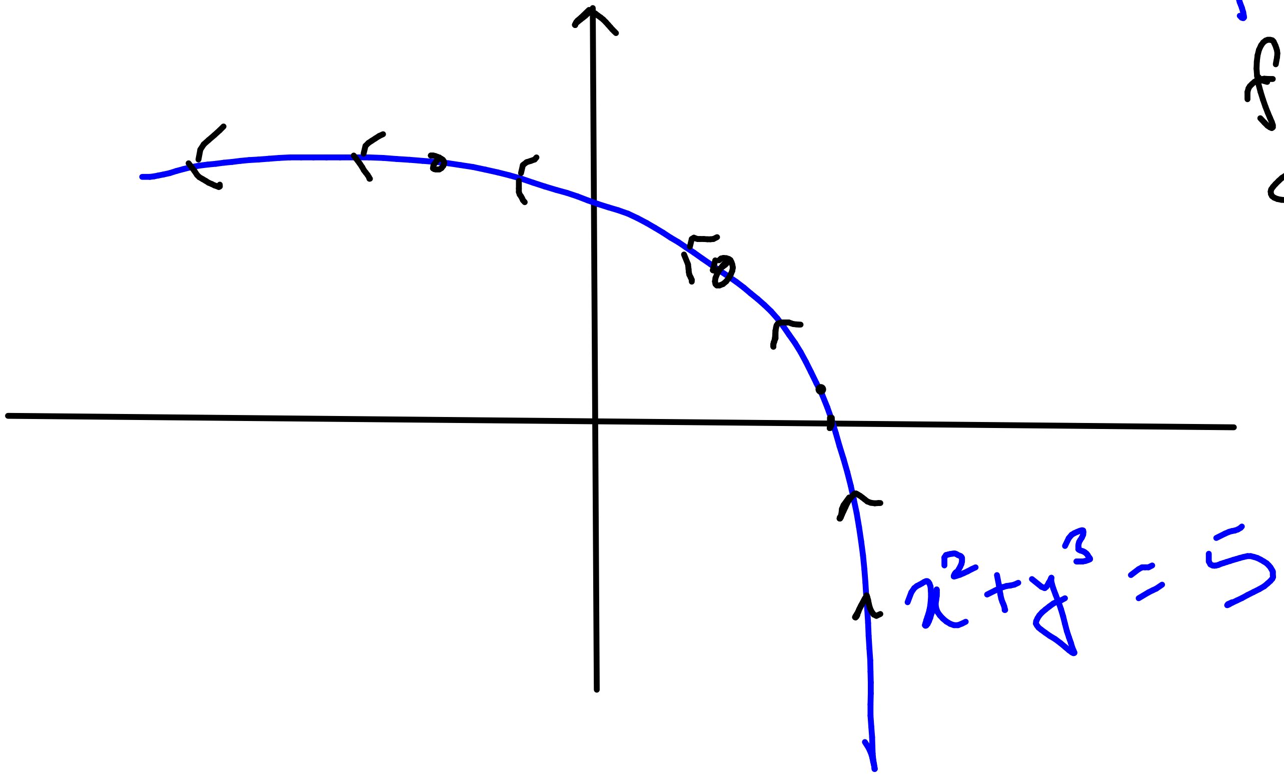
LAGRANGE MULTIPLIERS

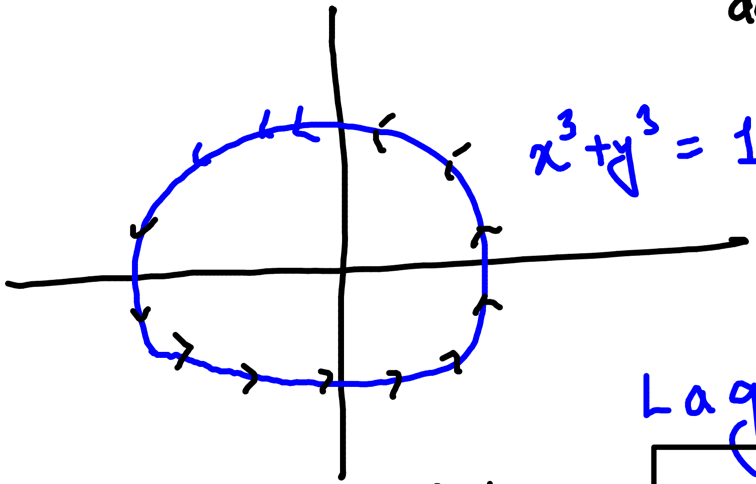


$f(x,y) = x+y$

find point on the curve $x^2+y^3=5$ where $f(x,y)=x+y$ is lowest.

Q. Suppose we have a function $f(x,y) = xy^2$

domain $= \mathbb{R}^2$



$x^3+y^3 = 1$

Aim: find max/min $f(x,y)$

s.t. $x^3+y^3 = 1$

Lagrange multiplier way.

Solve:

$$\boxed{\begin{array}{c} x^3+y^3 = 1 \\ \nabla(xy^2) = \lambda \nabla(x^3+y^3) \end{array}}$$

where λ is a new variable called Lagrange Multiplier

if (x,y) is a point of max/min

i.e.

$$\begin{bmatrix} x^3 + y^3 = 1 \\ y^2 = \lambda\, 3x^2 \\ 2xy = \lambda\, 3y^2 \end{bmatrix}$$

solve this to get

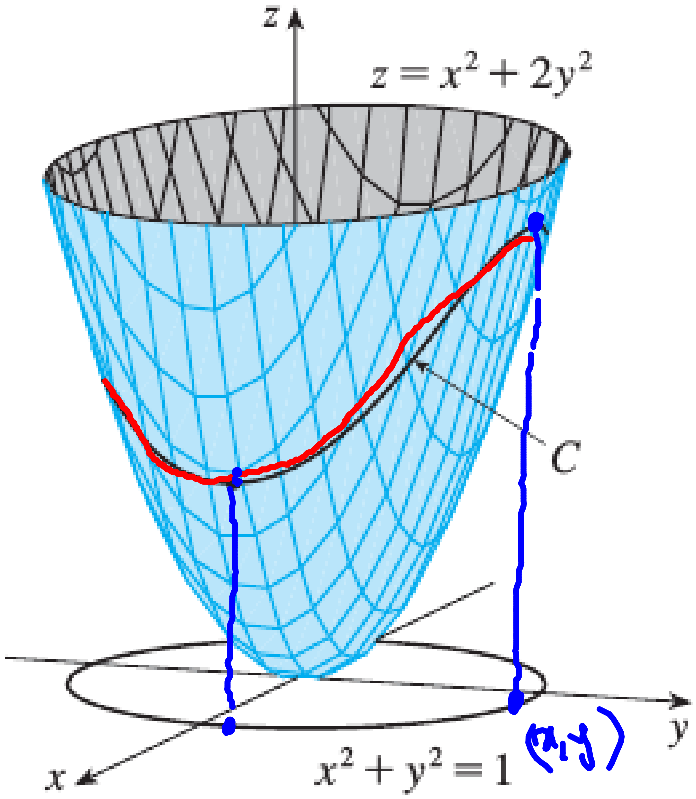
say

(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)

λ_1, λ_2, ..., λ_n

f_1 f_2 f_n

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.



Q: What equation we need to solve to find max/min points?

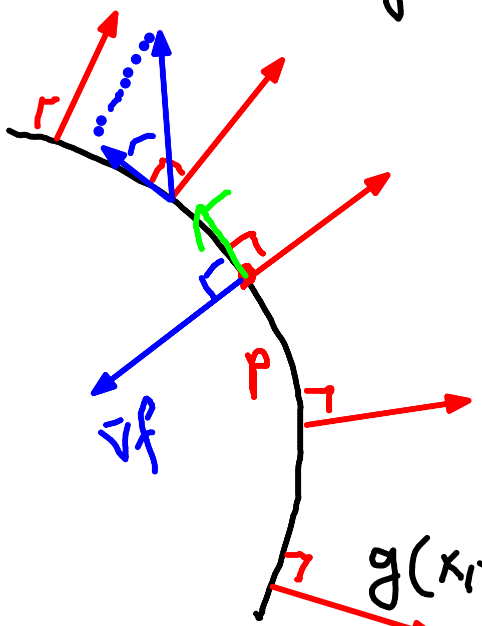
$x^2 + y^2 = 1$

$\nabla(x^2 + 2y^2) = \lambda \nabla(x^2 + y^2)$

$$\boxed{\begin{aligned} x^2 + y^2 &= 1 \\ 2x &= \lambda 2x \\ 4y &= \lambda 2y \end{aligned}}$$

Why Lagrange Multipliers Works

maximizing $f(x,y)$, over the curve $g(x,y)=5$



p: (x,y) f is taking a local max on the curve.

Q: Can you point the direction of ∇g & ∇f at p.

∇g is always $\perp$ to the curve

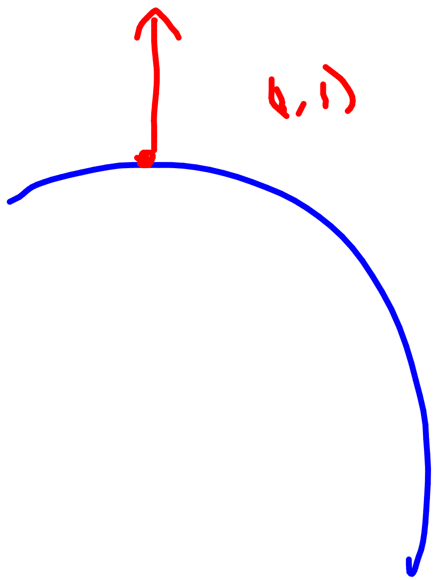
∇f is also $\perp$ to the curve at max/min points

p: is a point of max, $\hat{t}$: tangential at p

$$\boxed{\text{rate of change of } f \text{ along the curve at point } p} = \nabla f \cdot \hat{t} = 0$$

$$\Rightarrow \underbrace{\nabla f \perp \hat{t}}_{\text{at max}} \underbrace{\perp \nabla g}_{\text{always}}$$

$$\Rightarrow \boxed{\nabla f = \lambda \nabla g}$$



$$x^2+y^2 = 2$$

$$g(x,y) = x^2+y^2$$

$$\nabla g = 2x\,\hat{i} + 2y\,\hat{j}$$

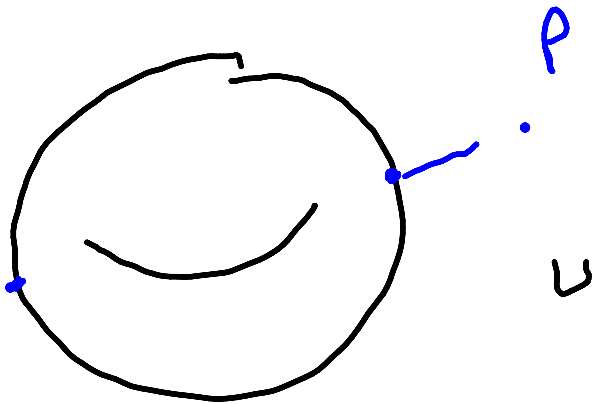
$$\nabla g\Big|_{(1,1)} = 2\hat{i} + 2\hat{j}$$

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

maximize / minimize

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

s.t. $\quad x^2 + y^2 + z^2 = 4$



Using Lagrange multiplier : solve

$$x^2 + y^2 + z^2 = 4$$

$$\nabla\left((x-3)^2 + (y-1)^2 + (z+1)^2\right) = \lambda \nabla\left(x^2+y^2+z^2\right)$$

complete yourself.

$$x^2 + y^2 + z^2 = 1$$

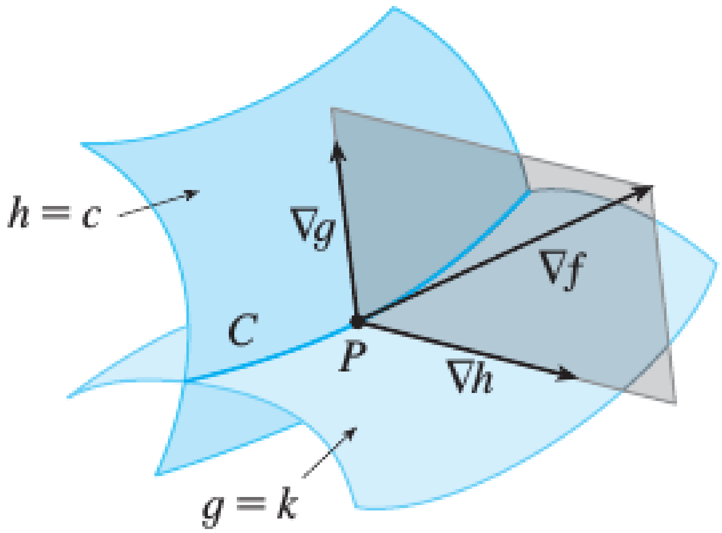
$$2(x-3) = \lambda 2x$$

$$2(y-1) = \lambda 2y$$

$$2(z+1) = \lambda 2z$$

EXAMPLE 4 Find the points on the ~~sphere~~ circle $x^2 + y^2 = 4$ that are closest to and farthest from the point $(3, 1)$.

TWO CONSTRAINTS



$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

V EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

1–15 ▪ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35$$

1–15 ■ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4$$