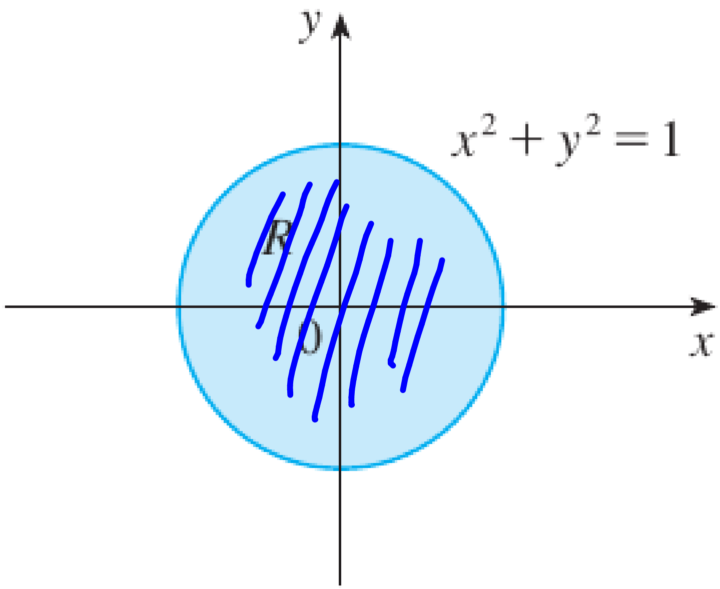
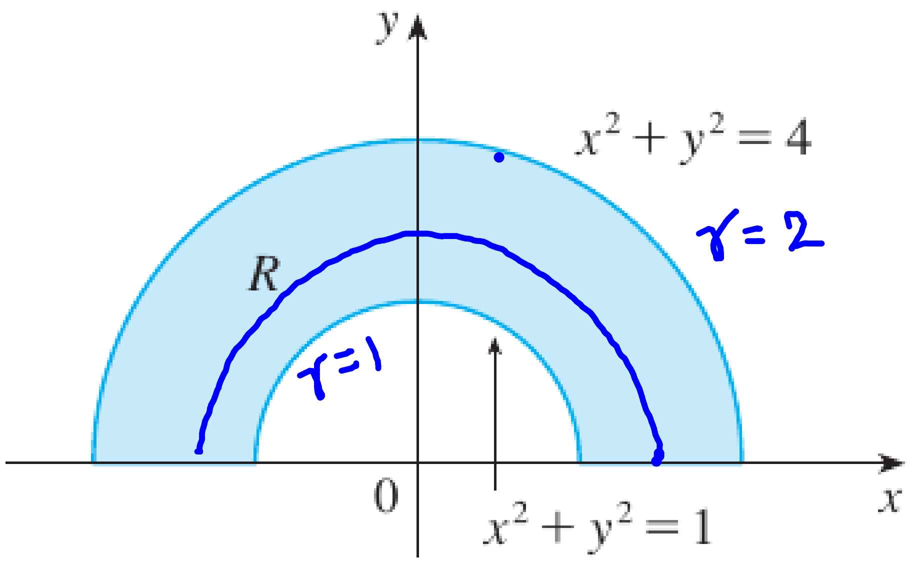
12.3 DOUBLE INTEGRALS IN POLAR COORDINATES



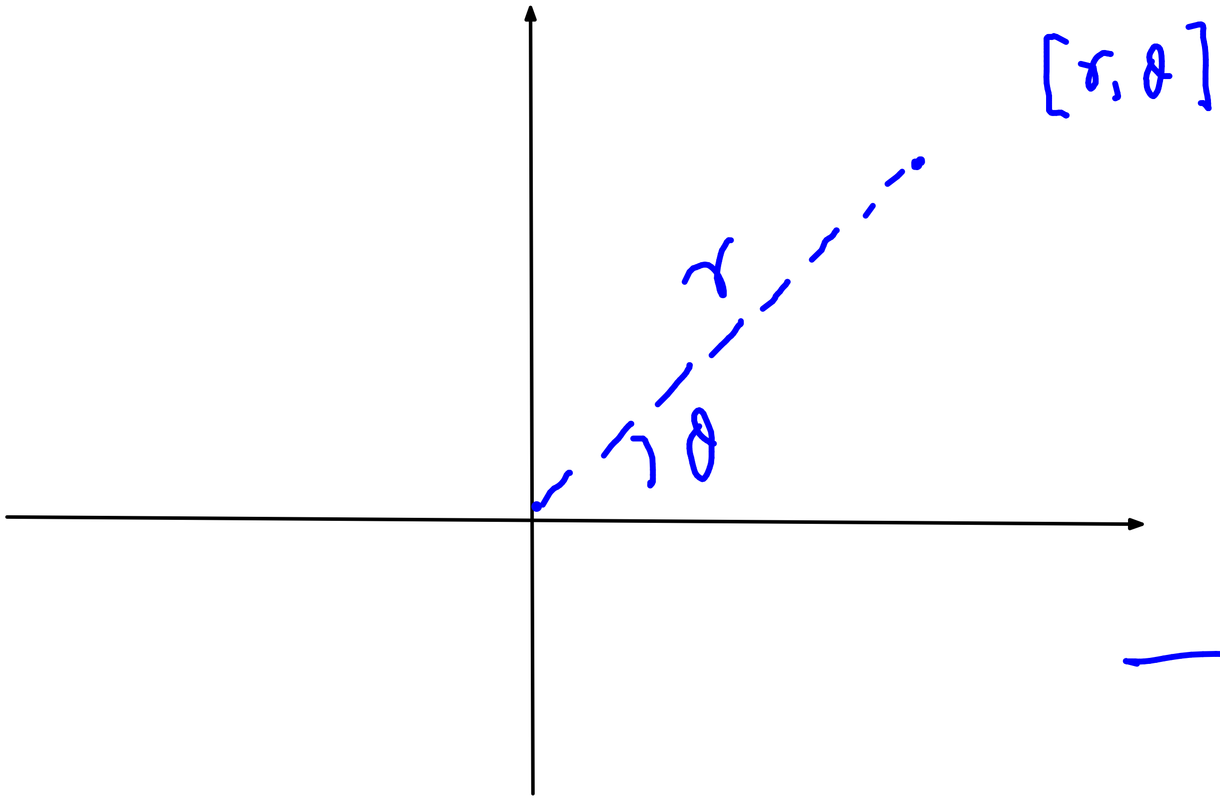
(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

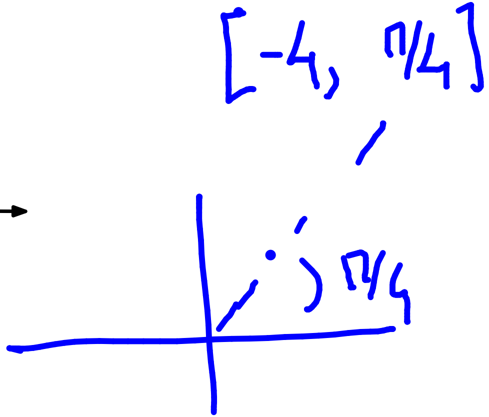


(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

$-1 \leq r \leq 1 \qquad 0 \leq \theta \leq \pi$

Polar Coordinates





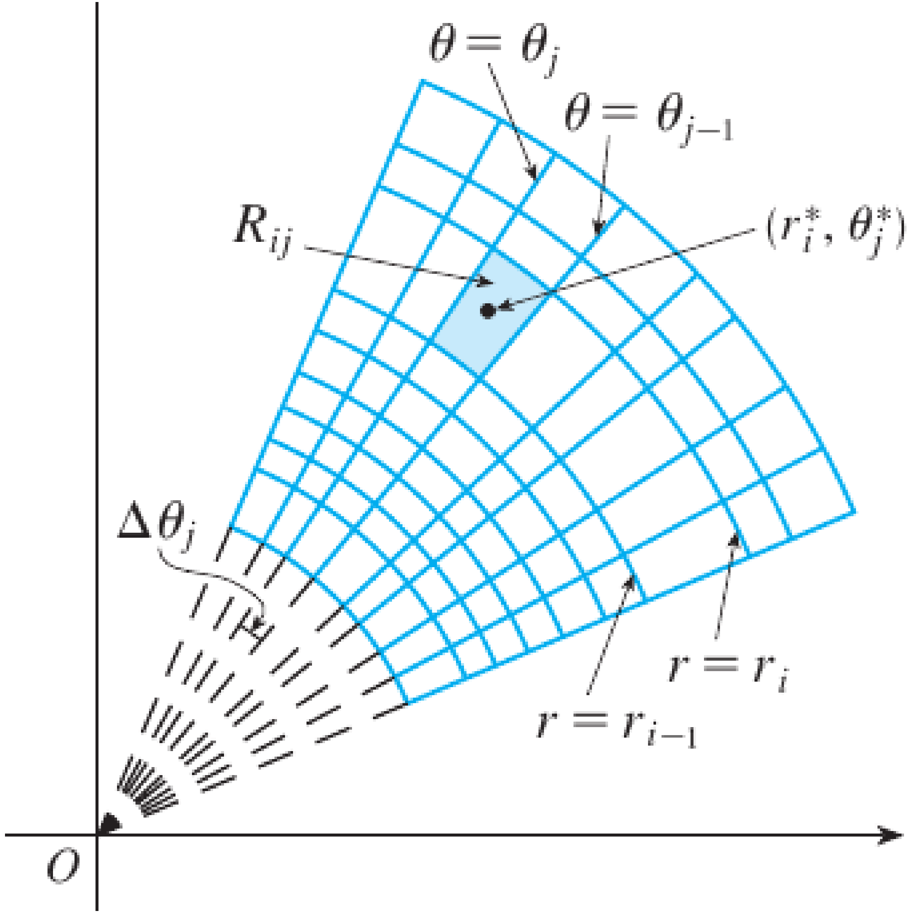
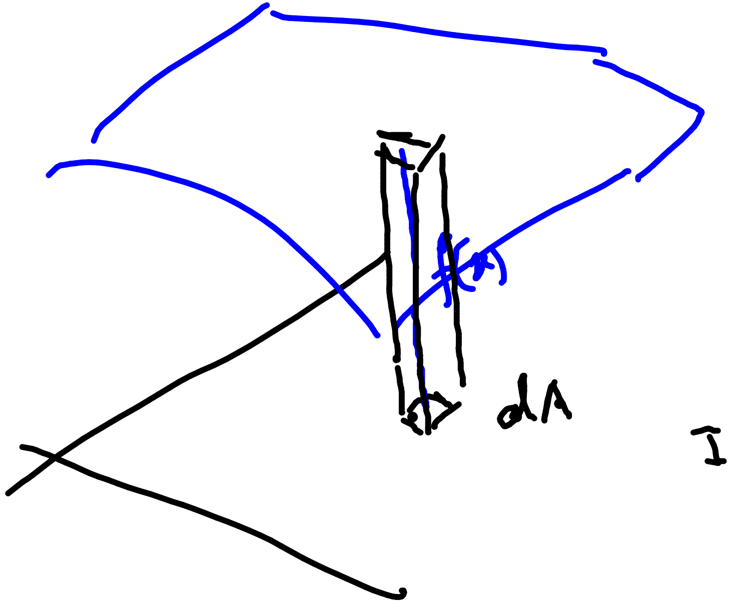


FIGURE 4 Dividing R into polar subrectangles

$$\Delta A_{ij} = r_i^* \, \Delta r_i \, \Delta\theta_j$$

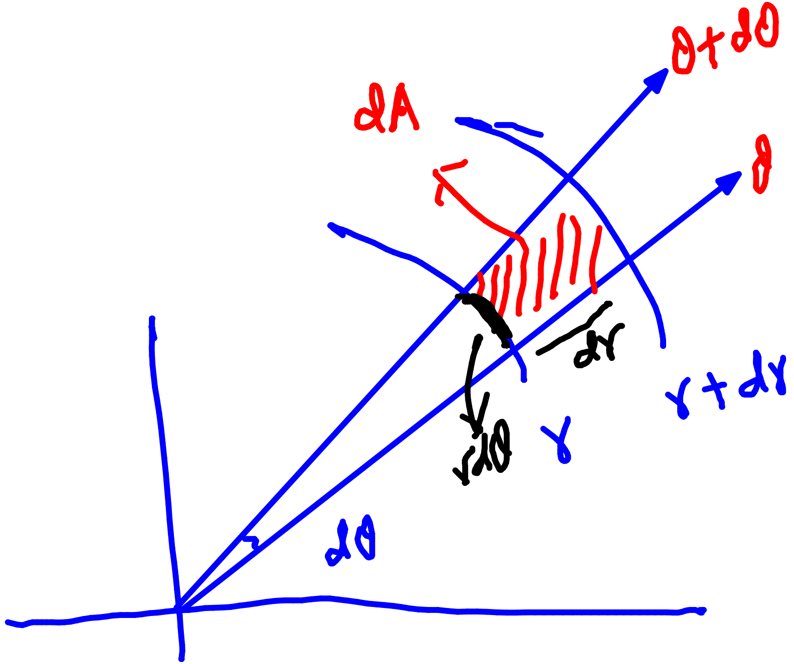


$$\iint_D f(x,y)\, dA$$

$f(x,y)\, dA$

In xy coordinates

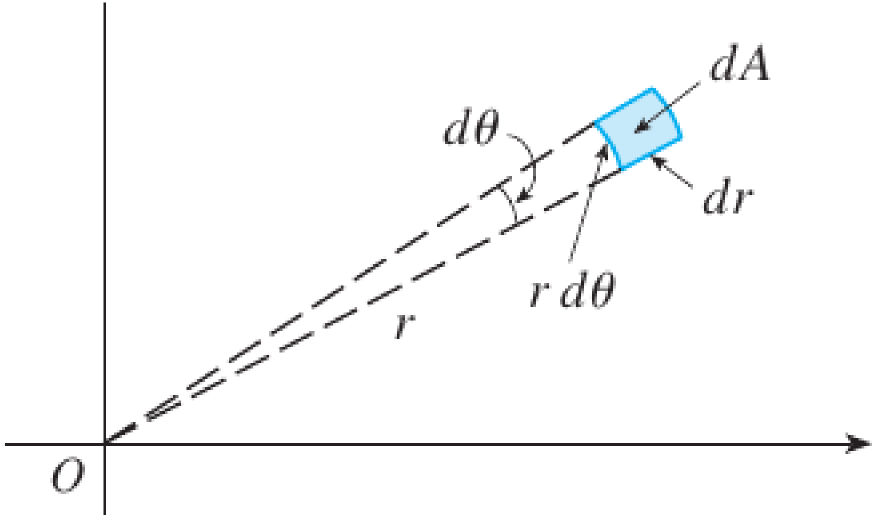
$dA = dx\, dy$



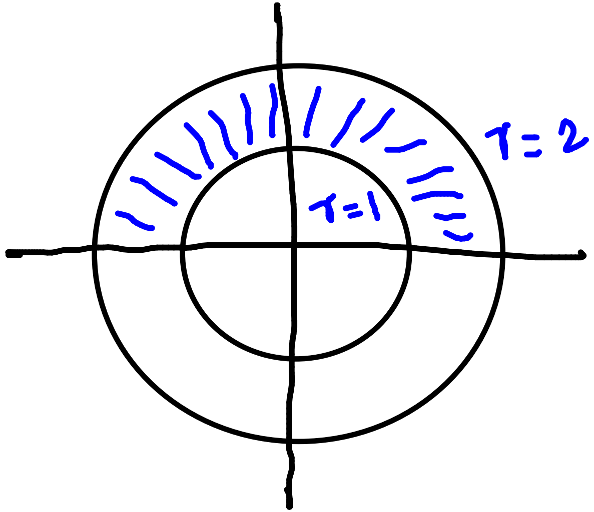
$$\boxed{dA \approx r\, d\theta\, dr}$$

2 CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\boxed{\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$



EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

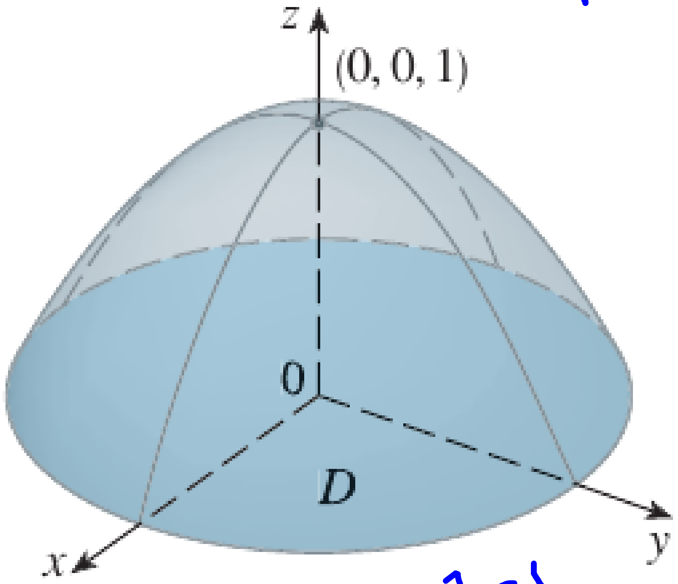


$$\int_0^{\pi}\int_1^{2}\left[3r\cos\theta + 4(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \frac{15\pi}{2}$$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Set up the integration in polar coordinates



$x^2+y^2=1$

$r=1$

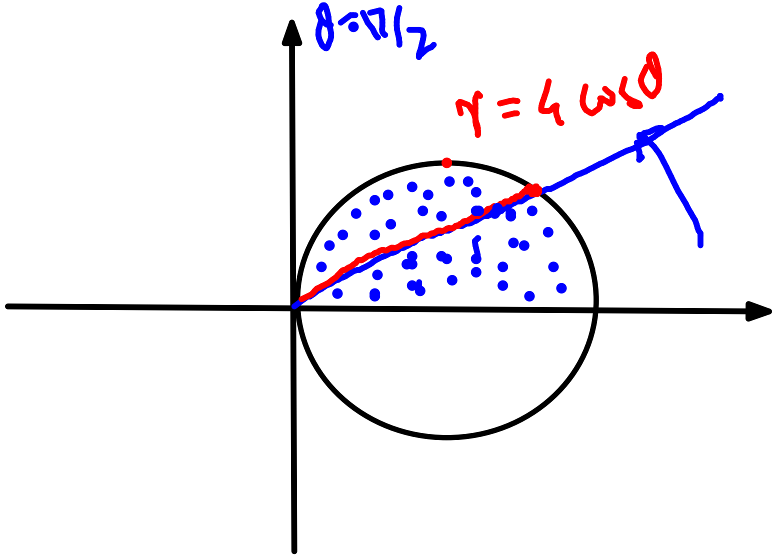
$$\iint_D (1-x^2-y^2)\,dA \quad \text{where } D=\{(x,y) \mid x^2+y^2\le 1\}$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

Q.11 Sketch the region whose area is given by the integral and evaluate the integral.

$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta \quad = \quad \int_0^{\pi/2} 8\cos^2\theta \; d\theta$$

$\theta = 0$, $\theta = \pi/2$ $\quad r = 0$



$r = 4\cos\theta$

$r^2 = 4r\cos\theta$

$x^2+y^2 = 4x$

$(x-2)^2+y^2 = 4$

Evaluate the iterated integral by converting to polar coordinates.

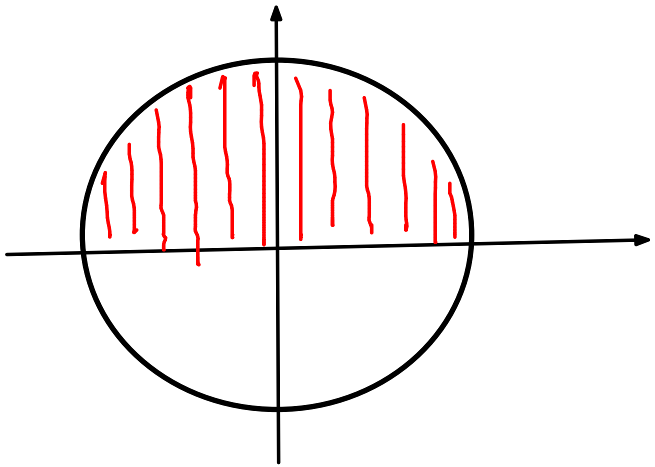
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$$

$y = \sqrt{9-x^2}$

$x^2+y^2 = 9$

→ Sketch the region of integration

→ then set up the integration in polar coordinates



$$= \int_{0}^{\pi}\int_{0}^{3} \sin(r^2)\, r\, dr\, d\theta$$

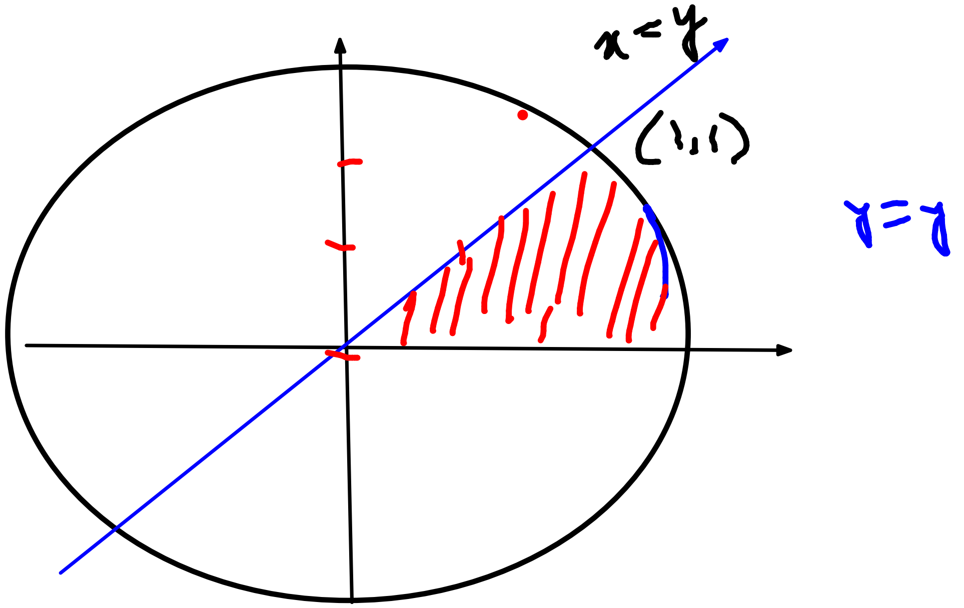
$$= \pi\left(\frac{1}{2} - \frac{\cos(9)}{2}\right)$$

Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^1 \int_{y=x}^{\sqrt{2-y^2}=x} (x + y)\, dx\, dy =$$

→ sketch the region of integration

→ set it up in polar coordinates



$x = y$ | $x = \sqrt{2-y^2}$

$\theta = \pi/4$ | $x^2+y^2 = 2$

$$= \int_0^{\pi/4}\int_0^{\sqrt{2}} r(\cos\theta + \sin\theta)\, r\, dr\, d\theta$$

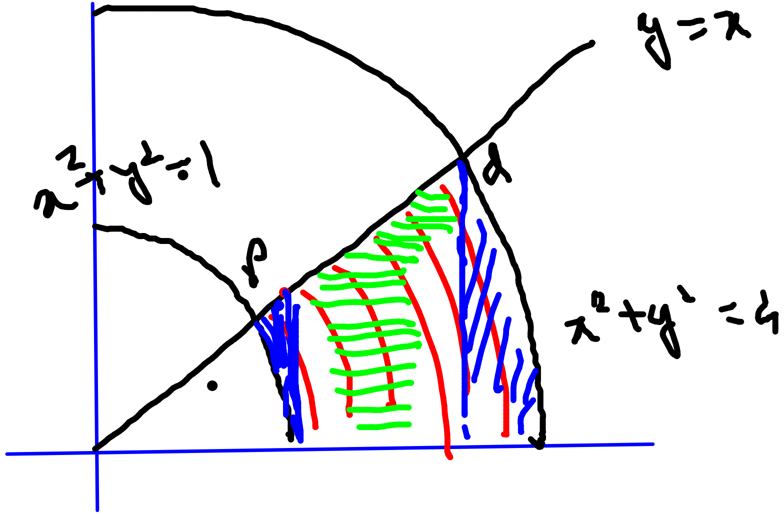
29. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$d = \left(\sqrt{2}, \sqrt{2}\right)$

into one double integral. Then evaluate the double integral.



$$= \int_0^{\pi/4}\int_1^2 r^2(\sin\theta\ \cos\theta)\ r\,dr\,d\theta$$

$$= 15/16$$