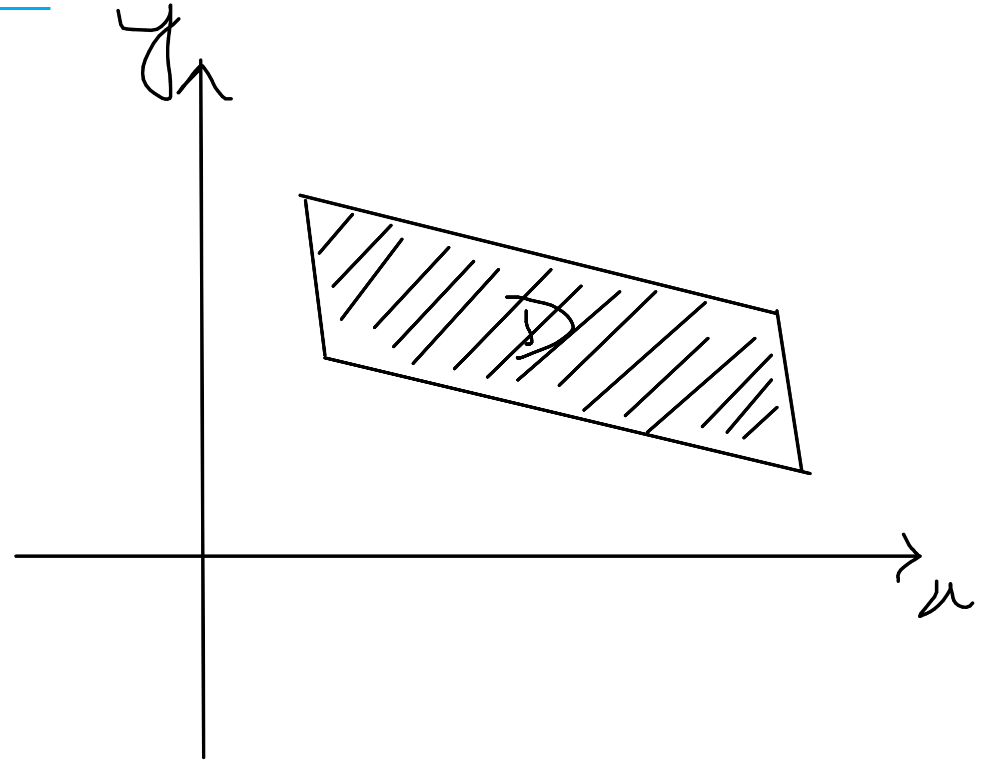
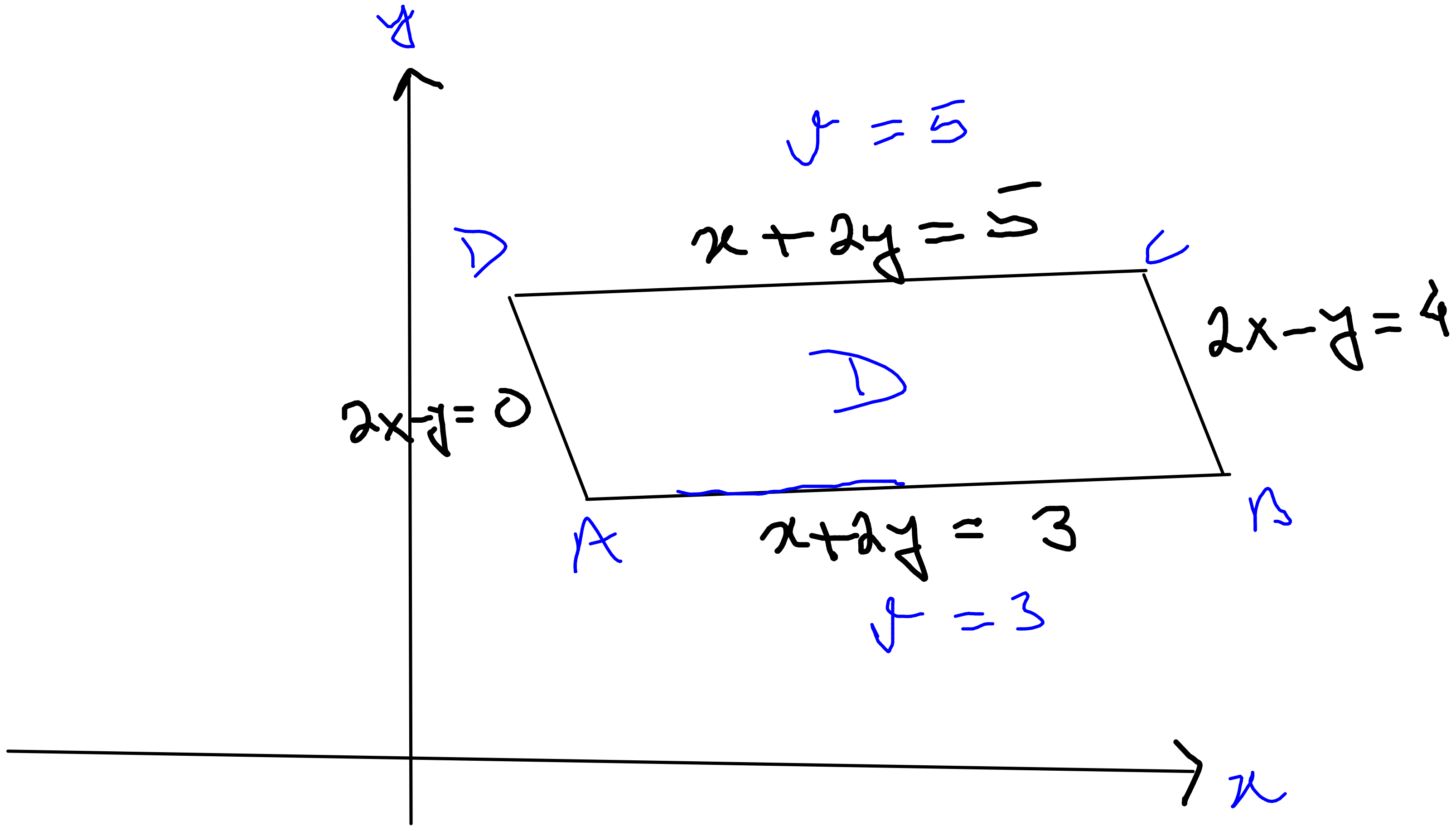
12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS



→ $\iint_D f(x,y)\,dA$ — given

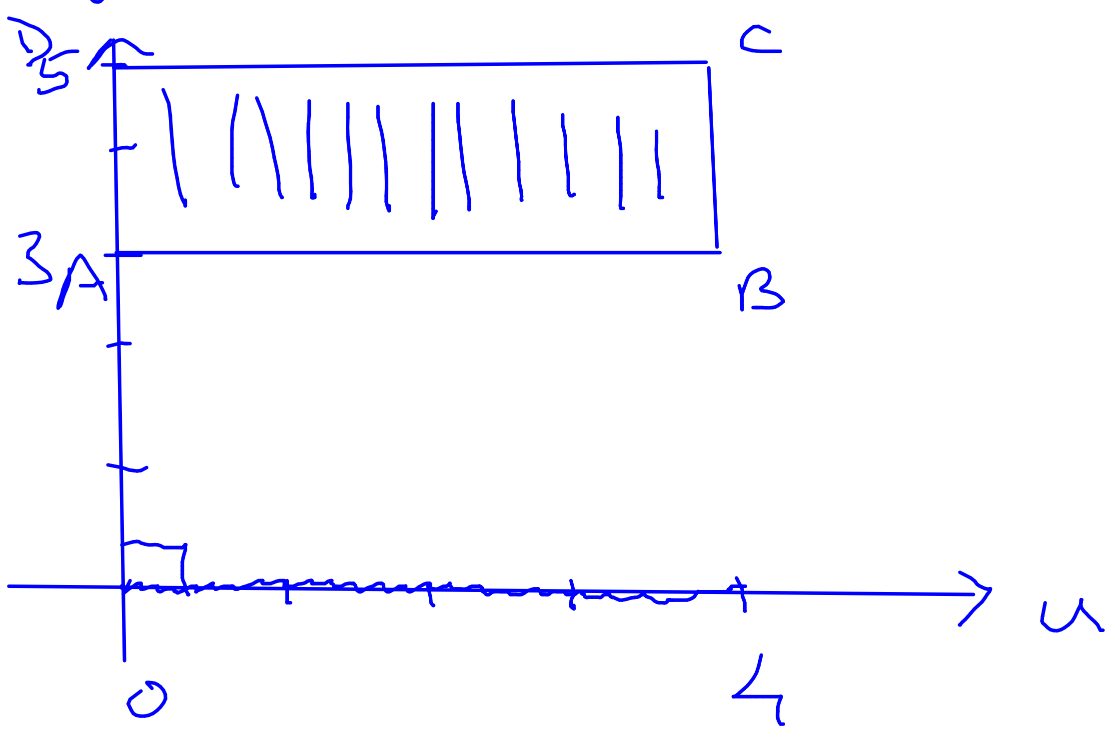
→ D : will be mildly complicated

→ D : will be simplified with change of variables



$u = 2x - y$

$v = x + 2y$



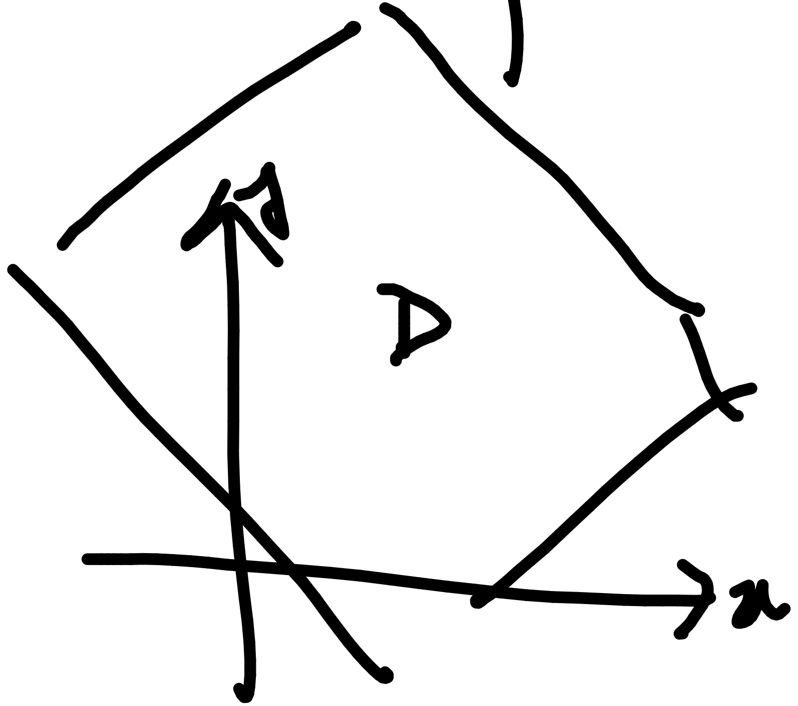
Find the Jacobian of the transformation.

$$x = u + 4v, \quad y = 3u - 2v$$

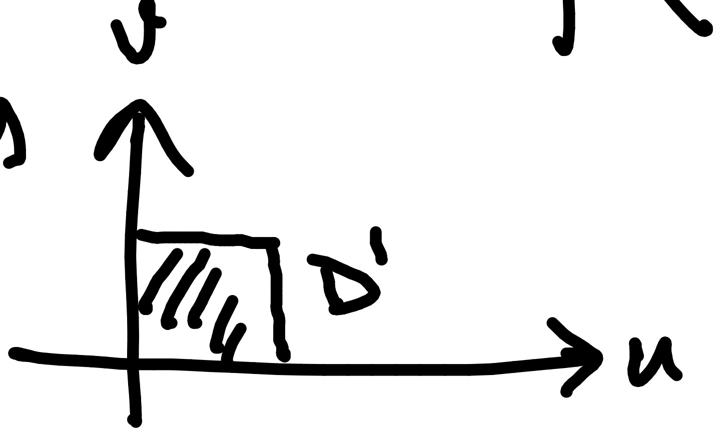
$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \text{abs}\left[\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right]$$

$$= \text{abs}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = |-14| = 14$$

$$\iint_D f(x,y)\,dx\,dy$$



D' ≈ 14 times smaller than D.



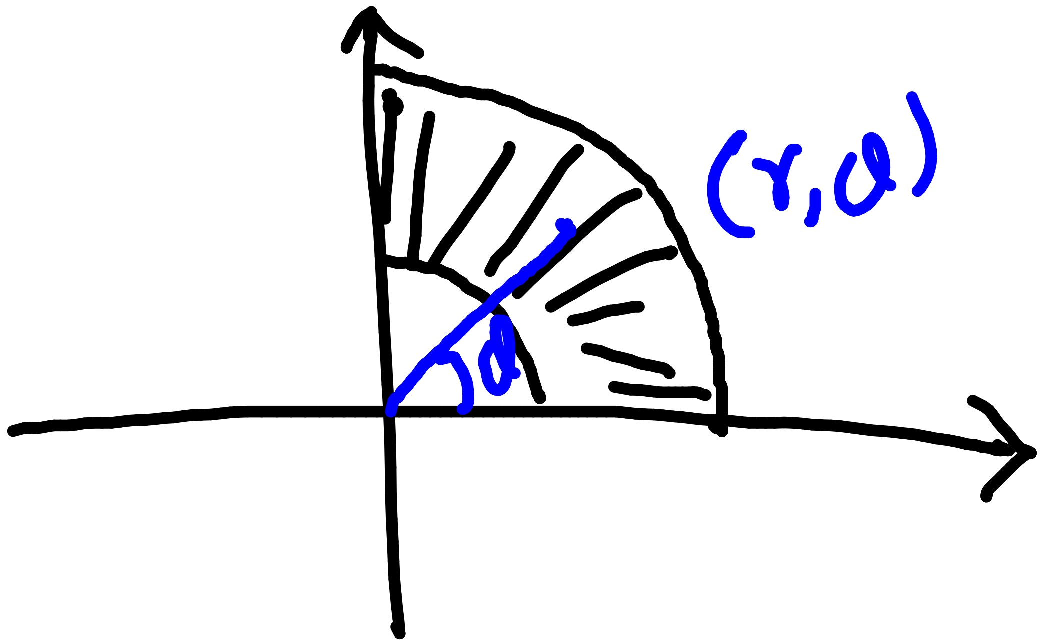
$$\iint$$

Find the Jacobian of the transformation.

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

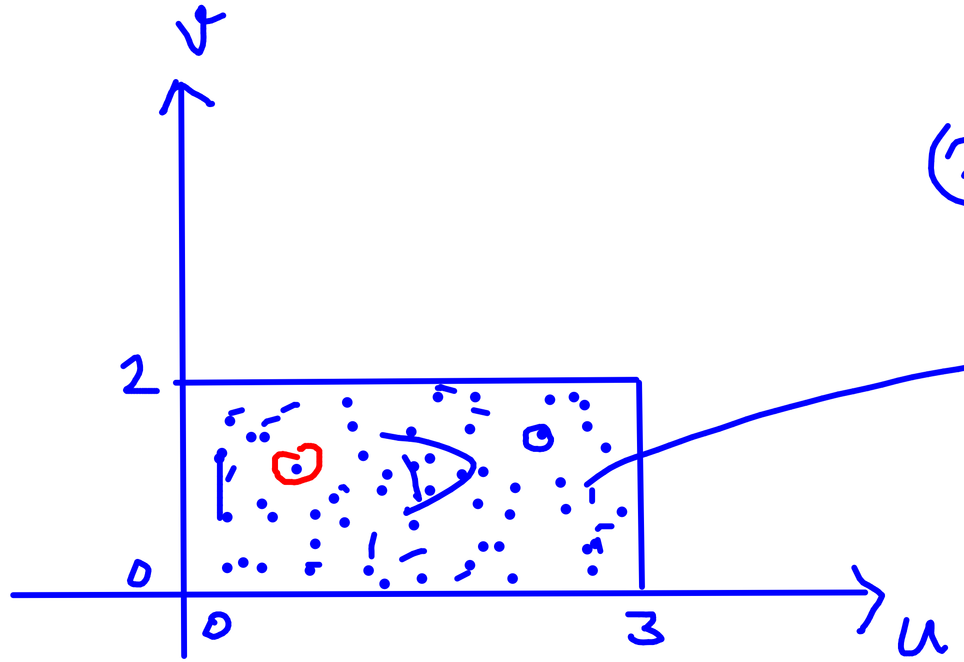


$$dx\,dy = r\,dr\,d\theta$$

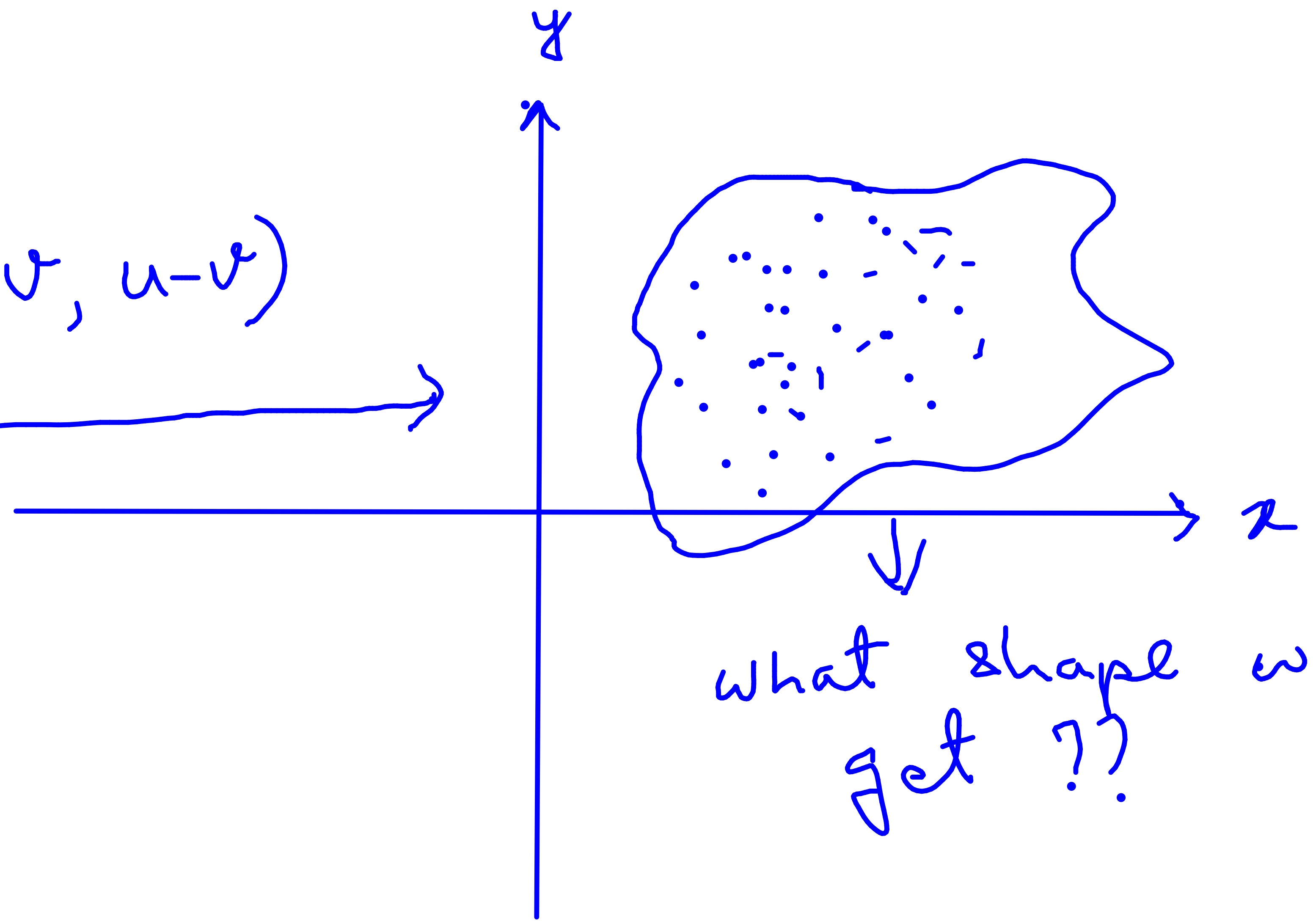
Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$?? shape??

$x = 2u + 3v,\ y = u - v$



$(x, y) = (2u+3v,\ u-v)$



what shape we get ??

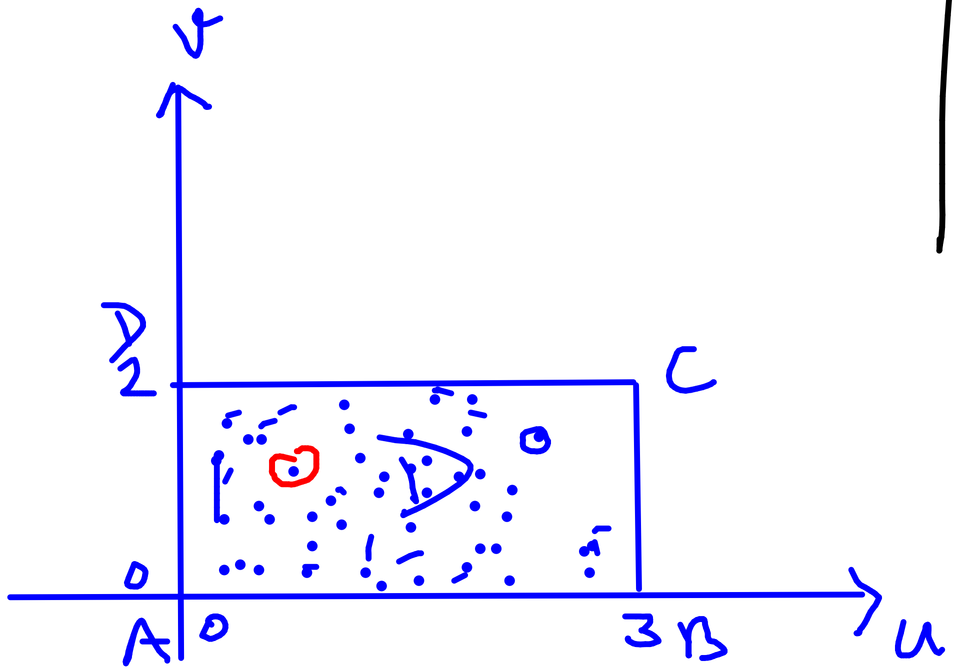
AB, $v = 0$, $0 \le u \le 3$

$\left.\begin{aligned} x &= 2u \\ y &= u \end{aligned}\right\}$ $x = 2y$

Find the image of the set S under the given transformation.

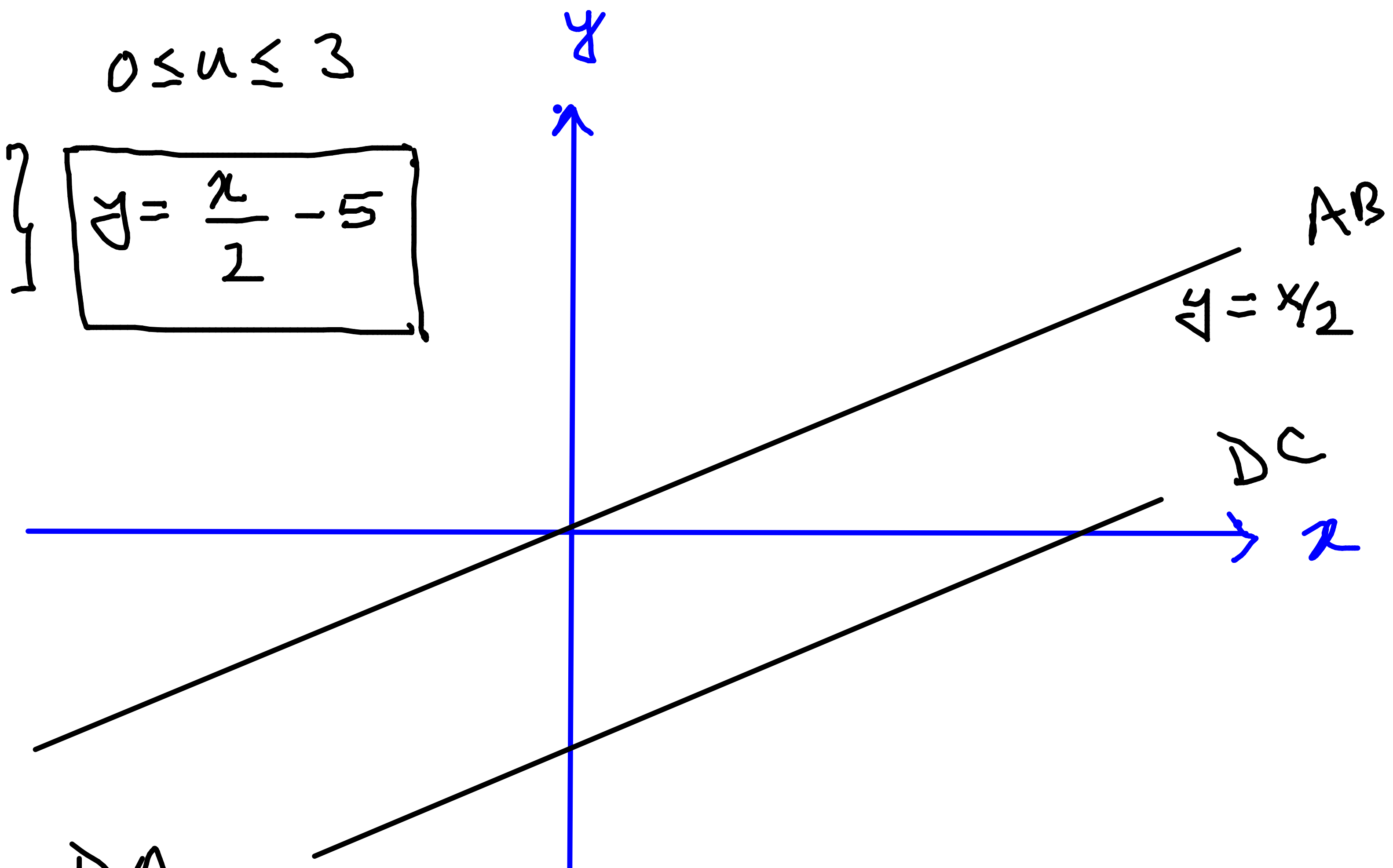
$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; ?? shape??

$x = 2u + 3v,\ y = u - v$



DC, $v = 2$, $0 \le u \le 3$

$\left.\begin{aligned} x &= 2u + 6 \\ y &= u - 2 \end{aligned}\right\}$ $\boxed{y = \frac{x}{2} - 5}$



strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given transformation.

$\boxed{S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};}$?? shape??

$x = 2u + 3v,\ y = u - v$

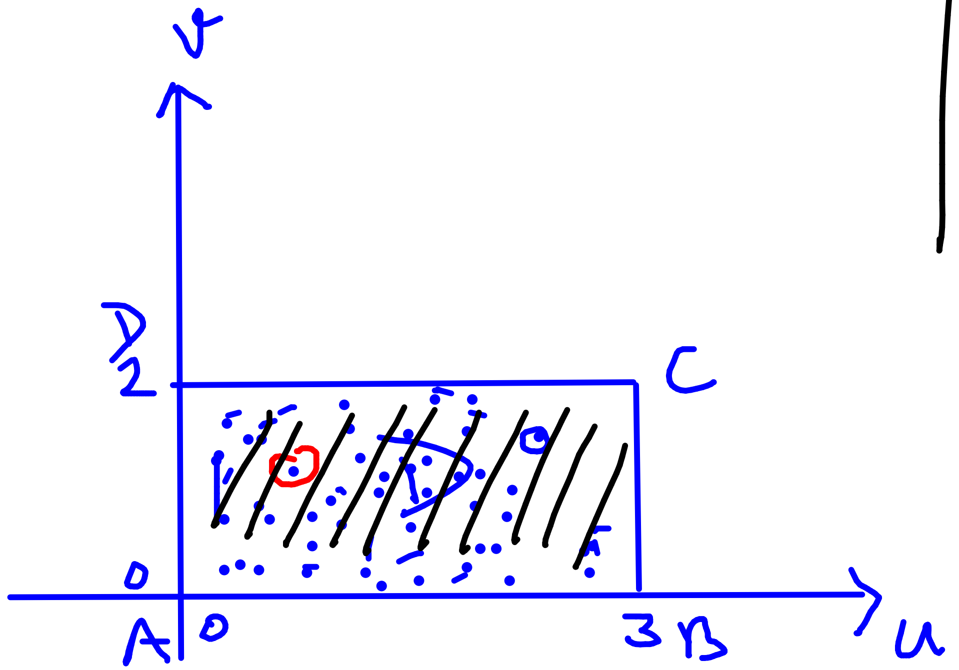
AD : $u = 0$, $0 \le v \le 2$

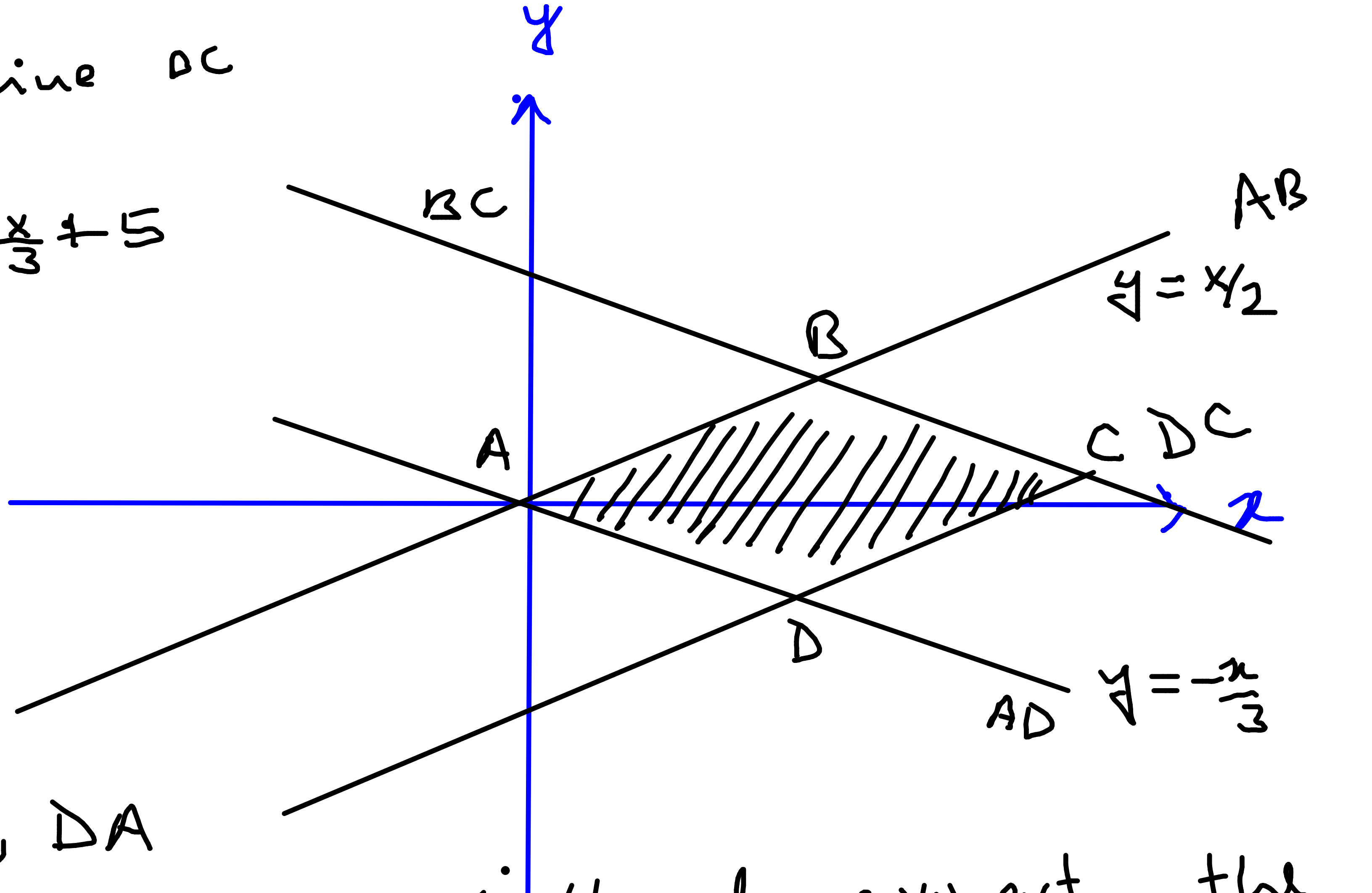
$x = 3v$, $y = -v$ → $x = -3y$, $y = \frac{-1}{3}x$

similarly: line DC

$x + 3y = 15$

$y = -\frac{x}{3} + 5$

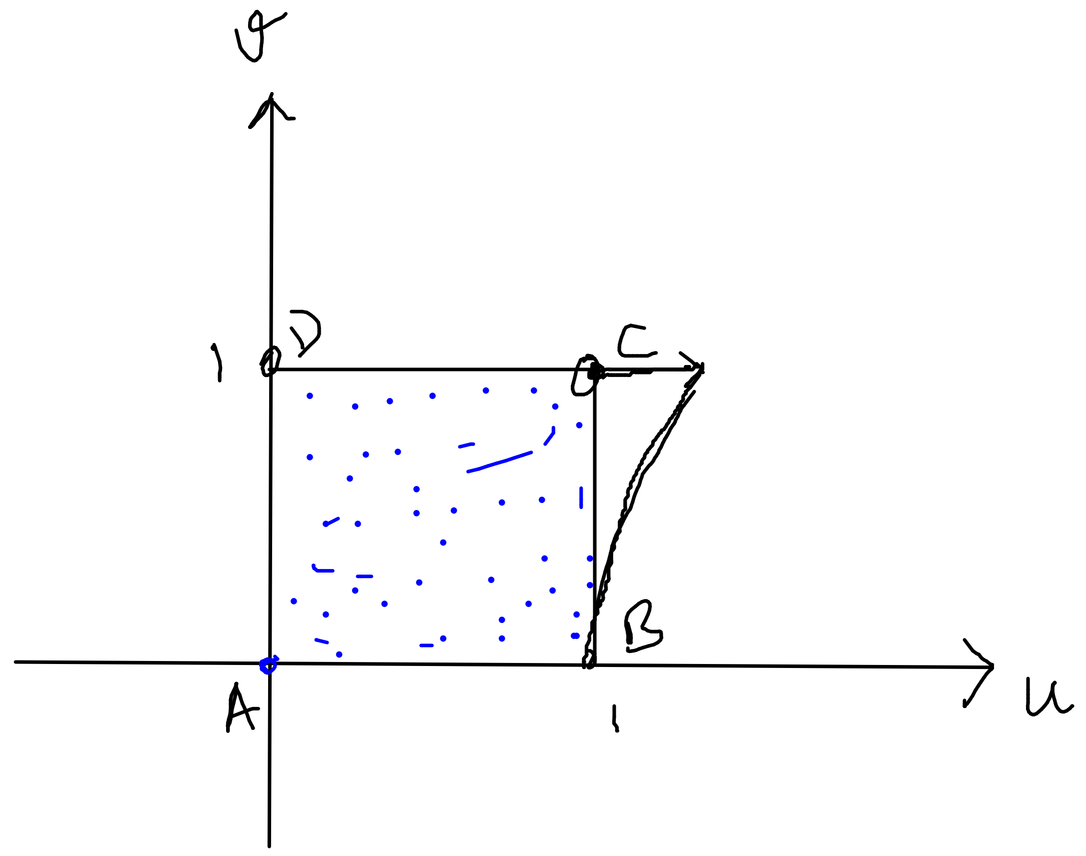




strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$



AB

$v = 0$

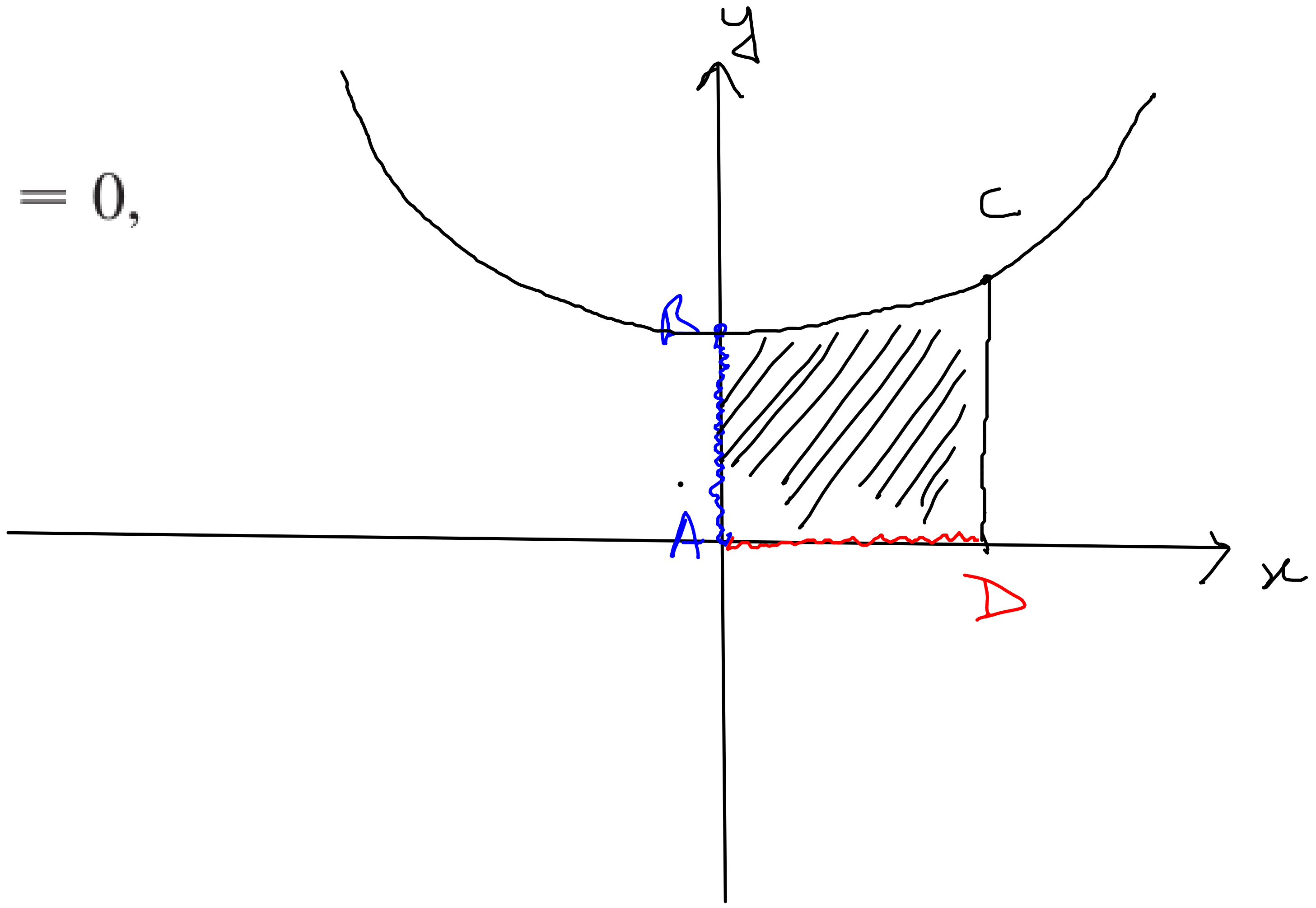
$\left.\begin{array}{l} x = 0 \\ y = u \end{array}\right\}$ $x = 0$, $0 \le y \le 1$

AD $u = 0$ $0 \le v \le 1$

$\left.\begin{array}{l} x = v \\ y = 0 \end{array}\right\}$ $y = 0$, $0 \le x \le 1$

BC : $u = 1$, $0 \le v \le 1$

$\left.\begin{array}{l} x = v \\ y = 1 + v^2 \end{array}\right\}$ $y = 1 + x^2$, $0 \le x \le 1$

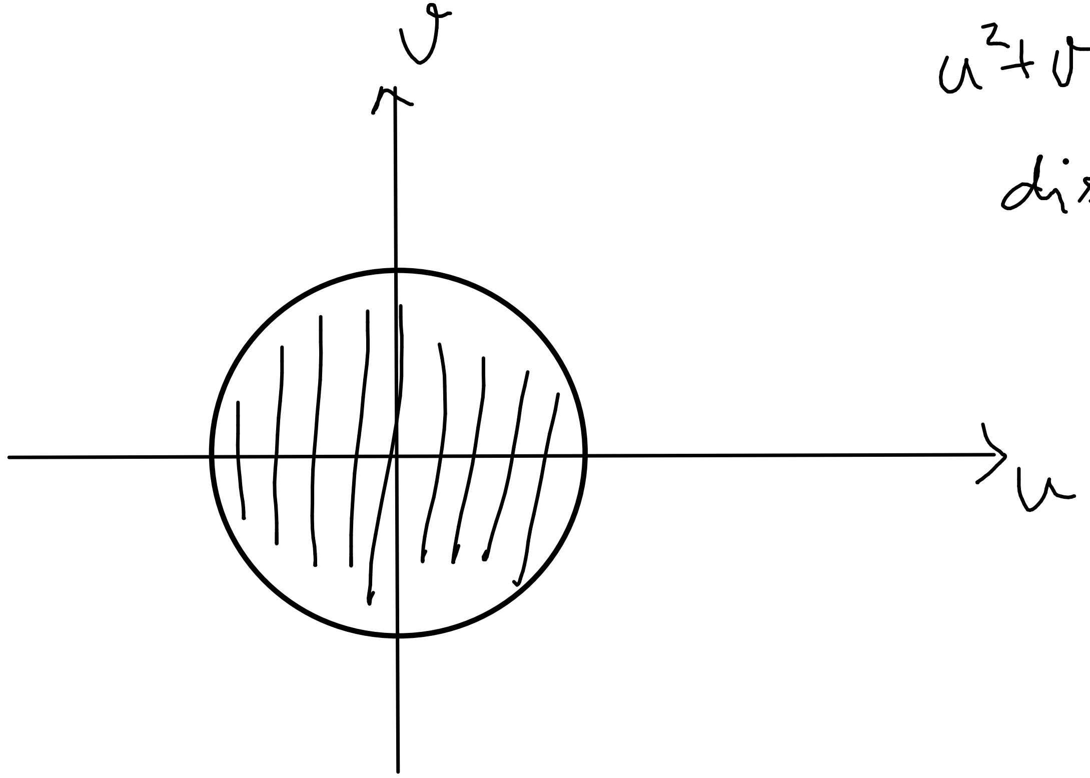


DC

Find the image of the set S under the given transformation.

$$S \text{ is the disk given by } u^2 + v^2 \leqslant 1; \quad x = au, \; y = bv$$

for simplicity, assume $a = 3$, $b = 2$



$u^2+v^2 \leq 1$

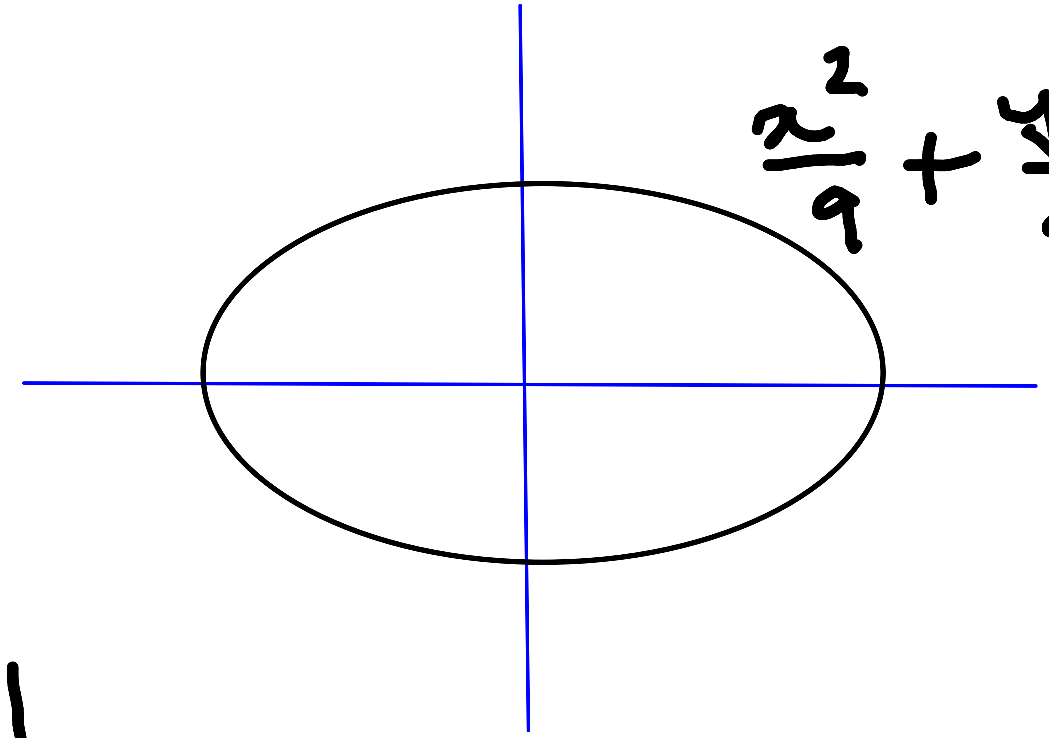
disc

$x = 3u$

$y = 2v$

$u^2+v^2=1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

Use the given transformation to evaluate the integral.

$\iint_R x^2\,dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $\quad x = 2u,\ y = 3v$

Use the given transformation to evaluate the integral.

$\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $x = 2u + v$, $y = u + 2v$

Evaluate the integral by making an appropriate change of variables.

$$\iint_R \frac{x - 2y}{3x - y}\, dA,$$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

Evaluate the integral by making an appropriate change of variables.

$\iint_R e^{x+y}\, dA$, where R is given by the inequality $|x| + |y| \leqslant 1$