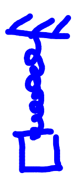
6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0).$$

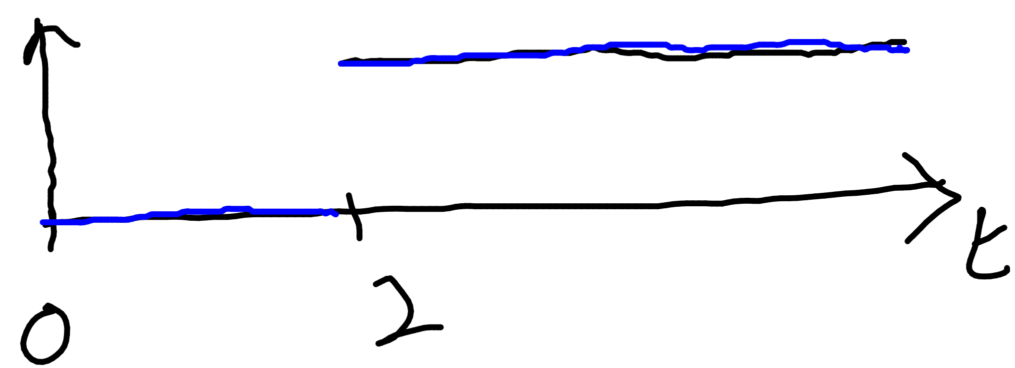


$my'' + cy' + ky = \underline{r(t)}$

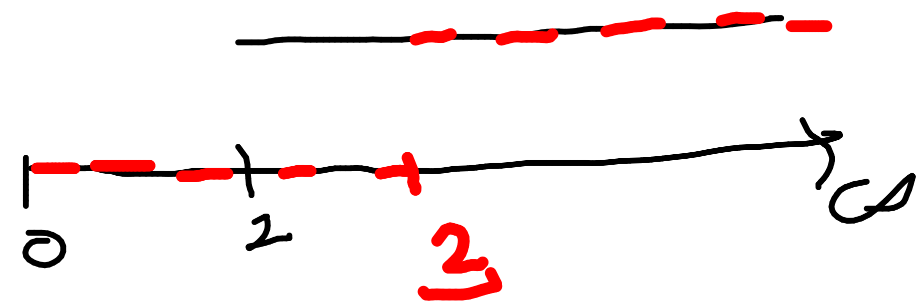
→ chapter 2 deals with only continuous $r(t)$

$r(t) = \sin(t)$

$$u(t-2) = \begin{cases} 0 & , \ t < 2 \\ 1 & , \ t > 2 \end{cases}$$

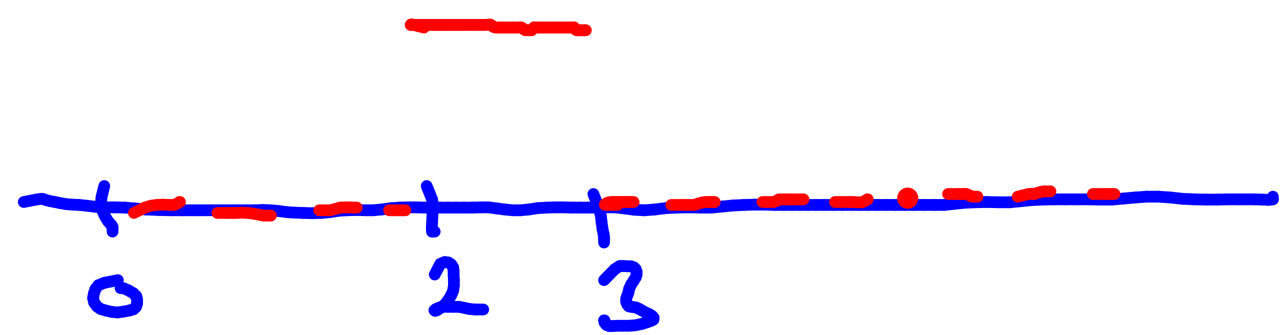


$u(t-2)$



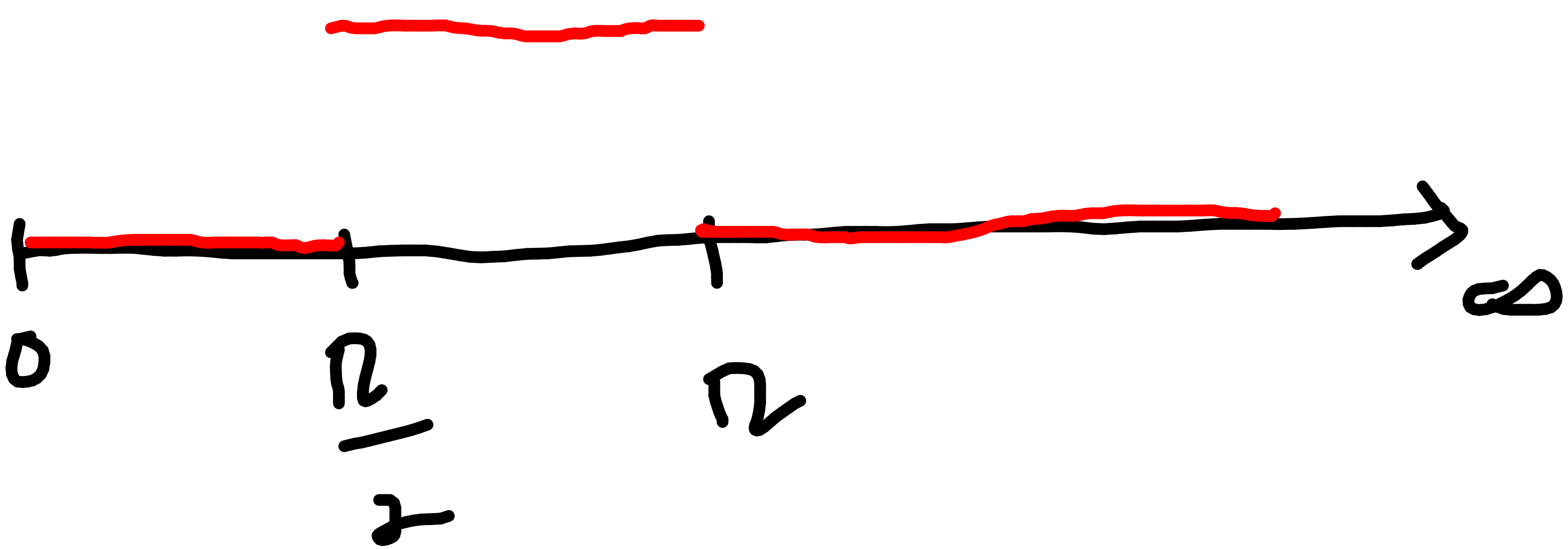
$u(t-3)$

$u(t-2) - u(t-3)$

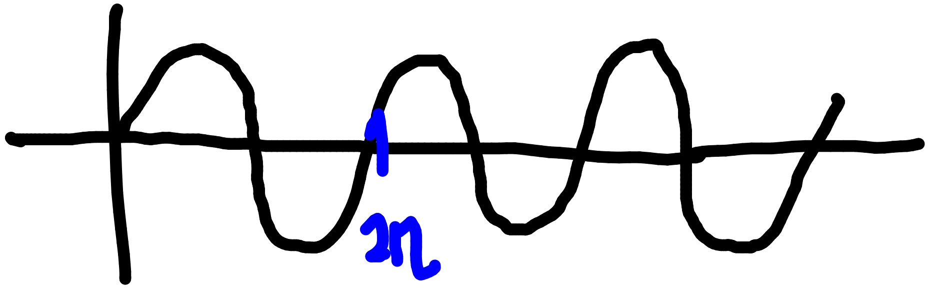


d. Create a switch which is on only in the interval $[\pi/2, \pi]$

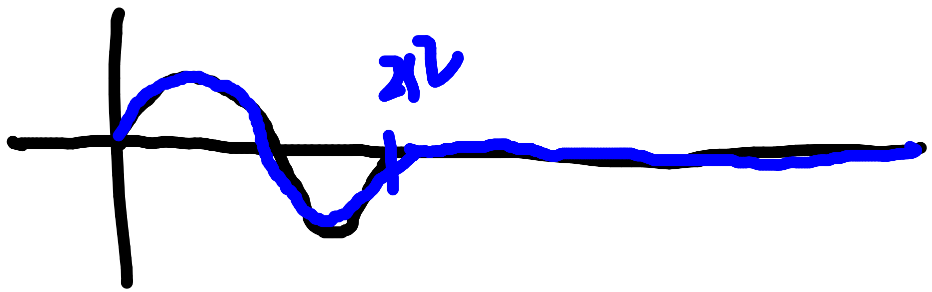
$$u(t-\pi/2) - u(t-\pi)$$



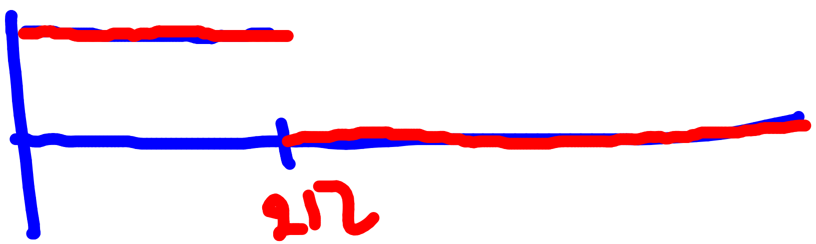
Q.



$\sin(t)$

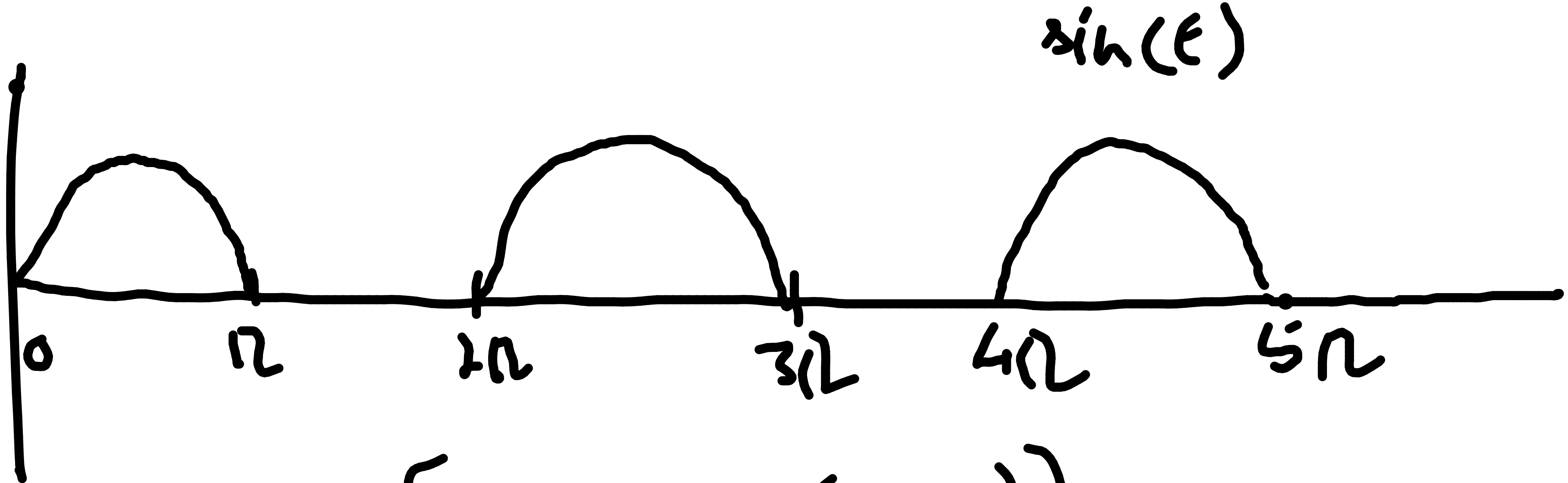


$\sin(t)\left[u(t) - u(t-2\pi)\right]$



$u(t) - u(t-2\pi)$

Q:



$$\sin(t)\left[u(t) - u(t-\pi)\right]$$

$$+ \sin(t)\left[u(t-2\pi) - u(t-3\pi)\right]$$

$$+ \left[\underline{\sin}(t)\left[u(t-4\pi) - u(t-5\pi)\right]\right]$$

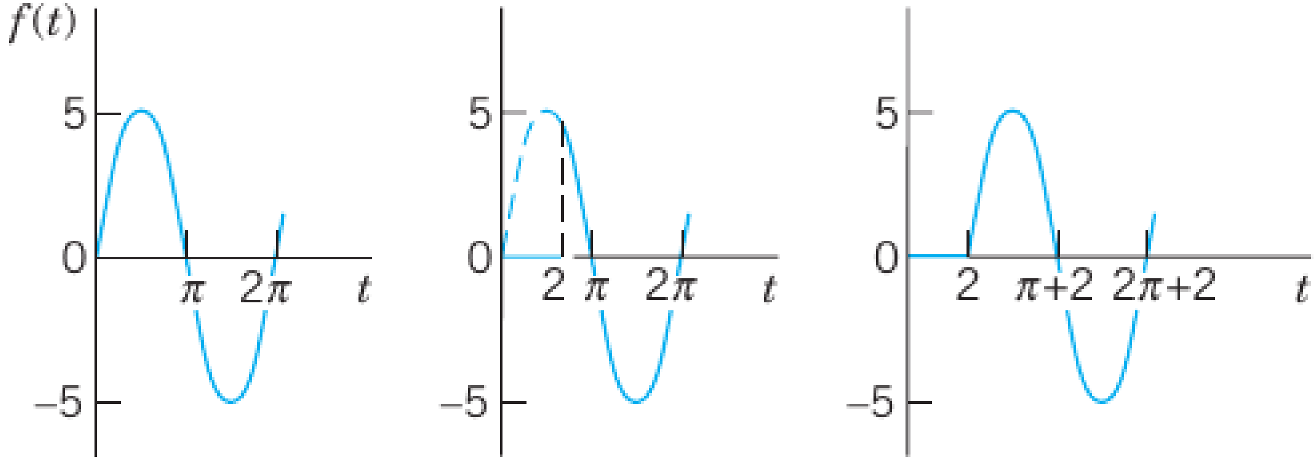
$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st}\, u(t-a)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left. \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= -\frac{1}{s}\Big[\underbrace{0}_{\text{for } s>0} - e^{-sa} \Big] = \frac{e^{-as}}{s}$$

$$[s>0]$$



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin (\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

EXAMPLE 1

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \quad t > \frac{1}{2}\pi. \end{cases}$$

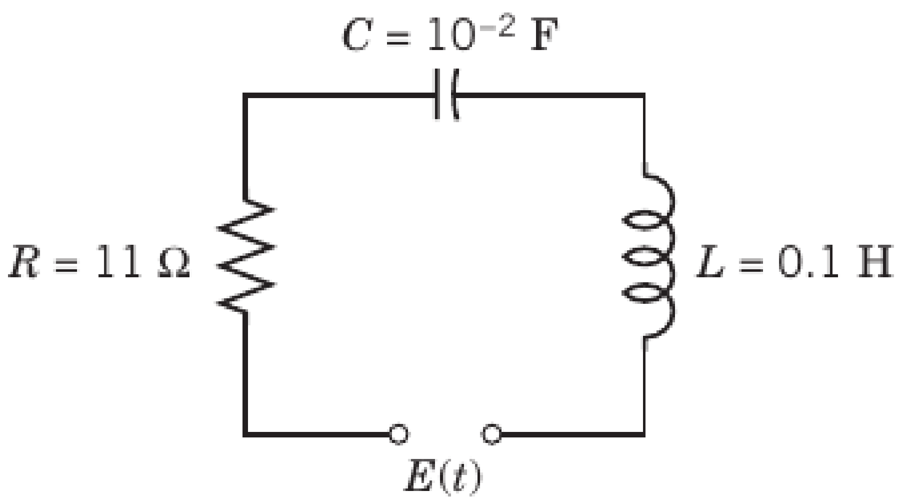
$$f(t) = 2\left(u(t) - u(t-1)\right) + \frac{1}{2}t^2\left[u(t-1) - u\left(t-\frac{\pi}{2}\right)\right] + \cos(t)\left[u\left(t-\pi/2\right)\right]$$

EXAMPLE 4

Find the response (the current) of the *RLC*-circuit in Fig. 125, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t) = 100 \sin 400t \quad \text{if } 0 < t < 2\pi \qquad \text{and} \qquad E(t) = 0 \text{ if } t > 2\pi$$

and current and charge are initially zero.

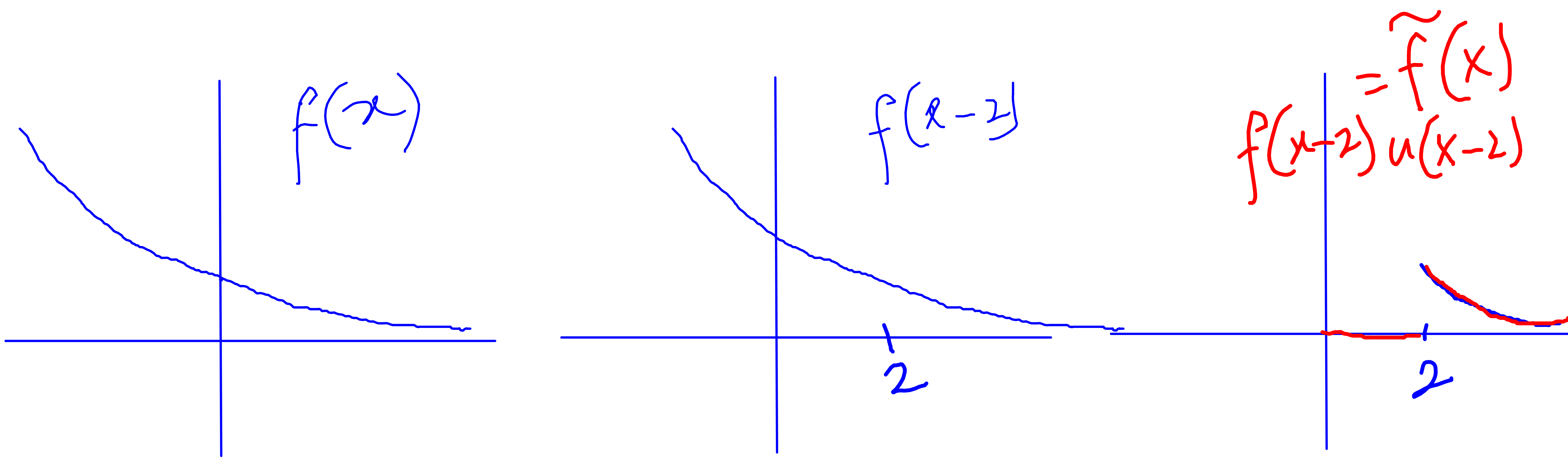


yourself

Recall shifted function

$f(x) = e^{-x}$ $\qquad a = 2$

$\tilde{f}(x) = f(x-2)\,u(x-2)$ $\qquad \Big| \qquad \mathcal{L}\big(f(x-2)\,u(x-2)\big) = \dfrac{e^{-2s}}{(s+1)}$



Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the "**shifted function**"

$$(3) \qquad \tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. *That is, if* $\mathcal{L}\{f(t)\} = F(s)$, *then*

$$\textbf{(4)} \qquad \boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).}$$

Or, if we take the inverse on both sides, we can write

$$\textbf{(4*)} \qquad \boxed{f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.}$$

say. $f(t) = t$

$F(s) = \frac{1}{s^2}$

$a = 5$

$\tilde{f}(t) = (t-5)u(t-5)$

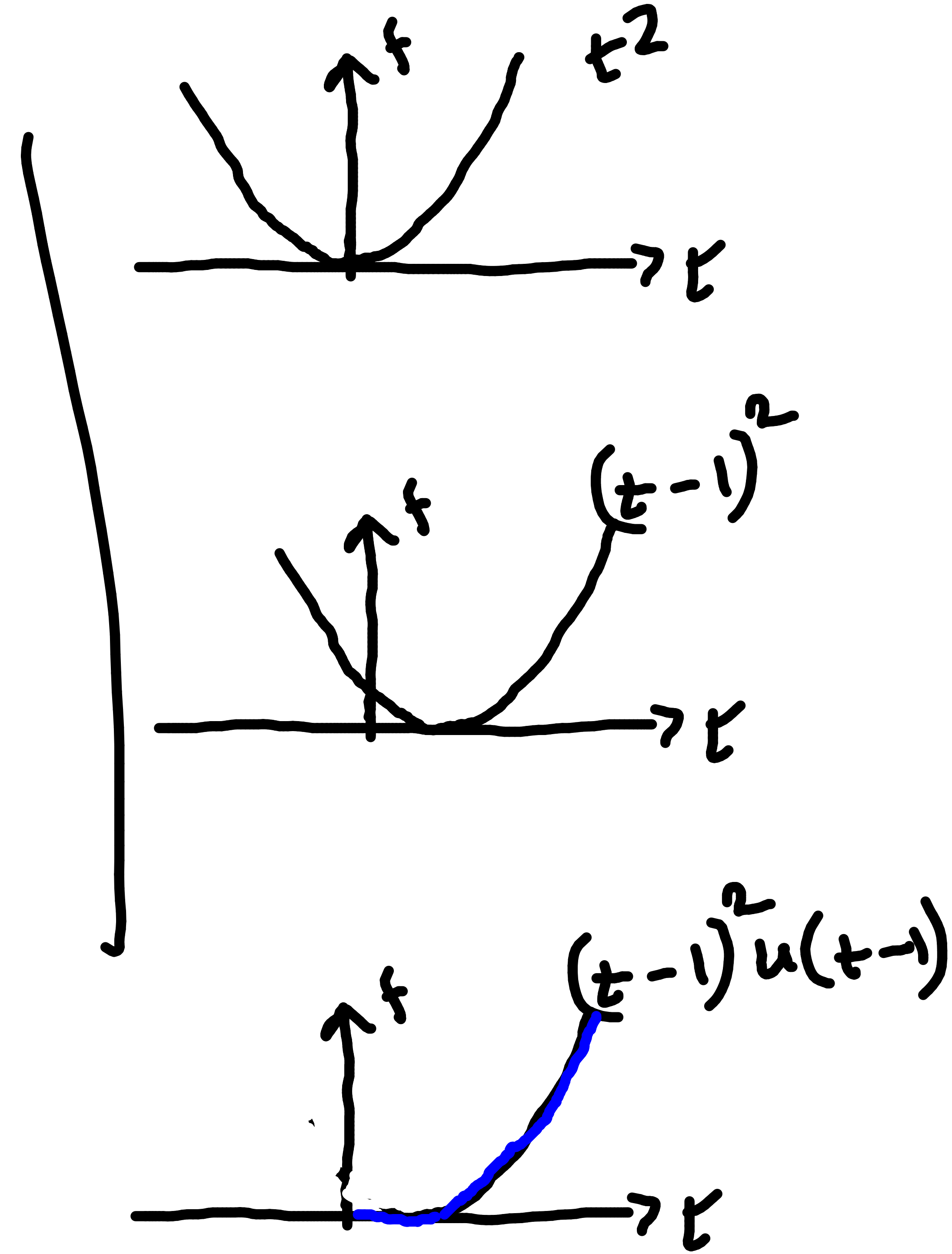
$\mathcal{L}(\tilde{f}) = \frac{e^{-5s}}{s^2}$

"shifted function"

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$f(t) = t^2$, $a = 1$

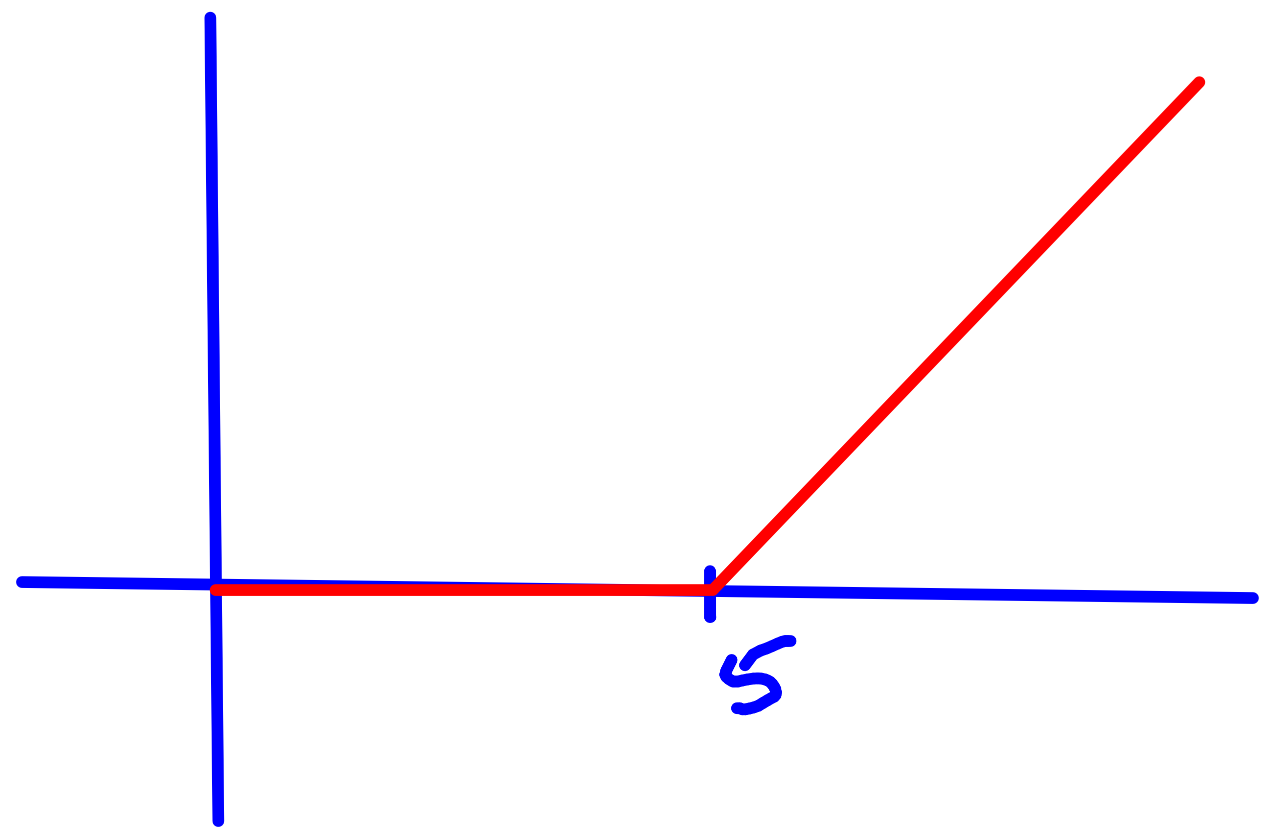
$\tilde{f}(t) = f(t-1)\,u(t-1)$



$$\tilde{f}(t) = f(t-a)\,u(t-a)$$

$f(t) = t$ $\qquad a = 5$

sketch $\tilde{f}(t)$



$$f(t) = \sin(t)$$

$$\boxed{\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}}$$

$$\mathcal{L}\left(\sin(t)\, u(t-2\pi)\right) = ??$$

$$= \mathcal{L}\left(\sin(t-2\pi)\, u(t-2\pi)\right)$$

$$= \frac{e^{-2\pi s}}{s^2+1}$$

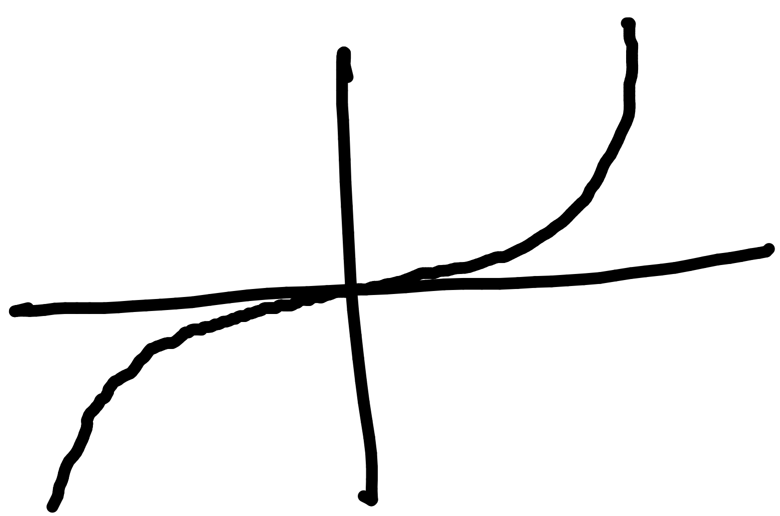
INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

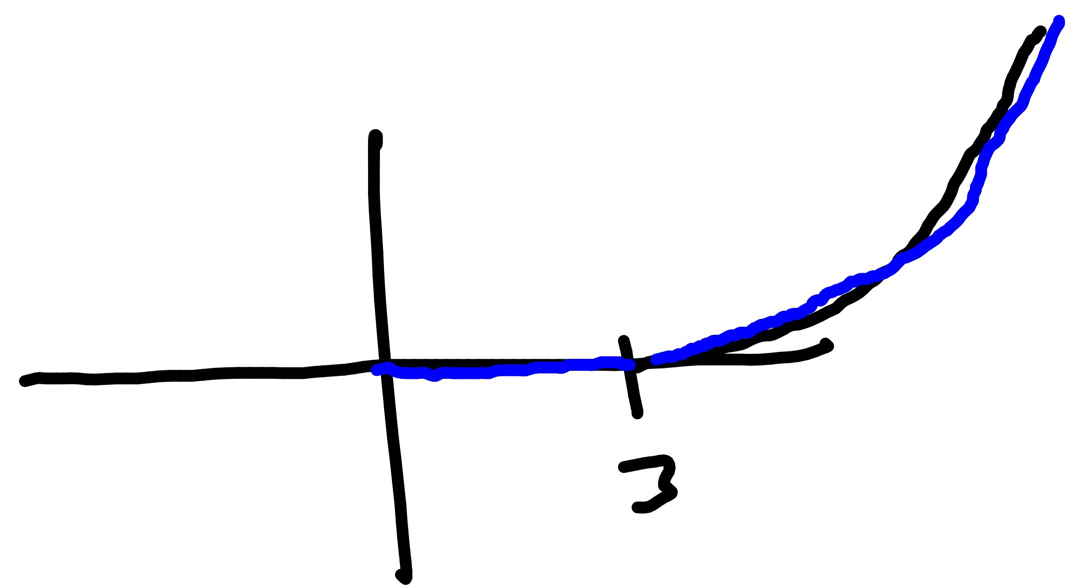
e^{-3s}/s^4

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\,u(t-a)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = ?? = \frac{t^3}{6}$$

$$\mathcal{L}^{-1}\left(e^{-3s}\frac{1}{s^4}\right) = \frac{(t-3)^3}{6}\,u(t-3)$$





INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

$$\mathcal{L}^{-1}\left(e^{-3s}/(s-1)^3\right) = ??$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$$

$$\mathcal{L}(f(t)) = F(s)$$

$$\mathcal{L}(e^t f(t)) = F(s-1)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right) = \frac{e^t t^2}{2}$$

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)u(t-a)$$

$$\mathcal{L}^{-1}\left(e^{-3s}\frac{1}{(s-1)^3}\right) = e^{(t-3)}\frac{(t-3)^2}{2}u(t-3)$$

INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

$6(1 - e^{-\pi s})/(s^2 + 9)$

Q

$$= \frac{6(1-e^{-\pi s})}{s^2+9}$$

$$\Rightarrow 2\sin 3t - 2\cdot\frac{3e^{-\pi s}}{s^2+9}$$

$$\Rightarrow 2\left[\sin 3t - \sin 3(t-\pi)u(t-\pi)\right]$$

Proof

given $\mathcal{L}(f(t)) = F(s)$

why $\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}(f(t-a)u(t-a)) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

change of variable: $u = t-a$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-as} \underbrace{\boxed{\int_0^\infty e^{-su} f(u)\,du}}_{= F(s) \;=\; \int_0^\infty e^{-st} f(t)\,dt} \quad ??$$

Theorem:

if $\mathcal{L}^{-1}\big(F(s)\big) = f(t)$

then $\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau$

$$\iff \frac{1}{s}F(s) = \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$g'(t) = f(t)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$$

$$\mathcal{L}(f) = s\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) - 0$$

Recall

$$\mathcal{L}(f) = F(s)$$

$$F(s) = s\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

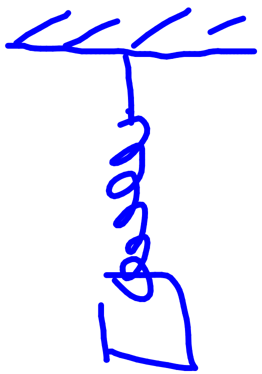
$$\frac{1}{s}F(s) = \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(\tau)\,d\tau$$

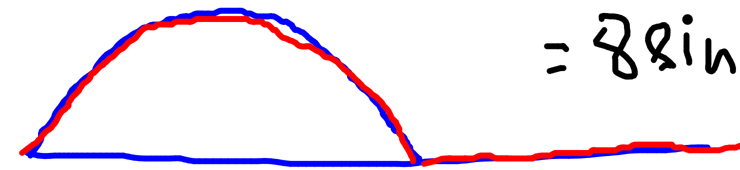
d. $y'' + 9y = \underbrace{8 \sin t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi}_{r(t)}$; $y(0) = 0,\ y'(0) = 4$

$$r(t) = \begin{cases} 8\sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

$$= 8\sin(t)\left[1 - u(t-\pi)\right]$$



external force



$$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0).$$

$$\mathcal{L}(y'' + 9y) = \mathcal{L}(r(t))$$

$$\mathcal{L}(y'') + 9\mathcal{L}(y) = \mathcal{L}\left(8\sin(t)\left[1 - u(t-\pi)\right]\right)$$

$$\boxed{s^2\mathcal{L}(y) - 4 + 9\mathcal{L}(y) = \frac{8}{1+s^2} + \frac{8\,e^{-\pi s}}{s^2+1}}$$

$\Rightarrow$ solve for $\mathcal{L}(y)$ then y.

$$\mathcal{L}\left(8\sin(t)\left[1-u(t-\pi)\right]\right) = \mathcal{L}\left(8\sin(t)\right) - \mathcal{L}\left(8\sin(t)\,u(t-\pi)\right)$$

$$\mathcal{L}\left(8\sin(t)\right) = 8\,\frac{1}{s^2+1}$$

$$\mathcal{L}\left(f(t-\pi)\,u(t-\pi)\right) = e^{-\pi s}F(s)$$

7

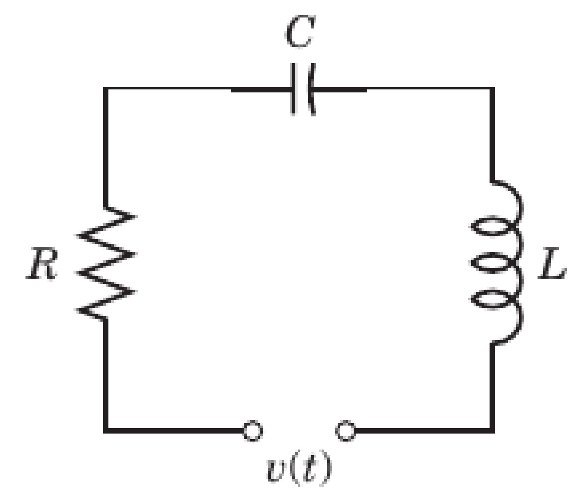
$$\mathcal{L}\left(\sin(t)\,u(t-\pi)\right) = \mathcal{L}\left(-\sin(t-\pi)\,u(t-\pi)\right)$$

$$= -\frac{e^{-\pi s}}{s^2+1}$$

$y'' + y = t$ if $0 < t < 1$ and 0 if $t > 1$; $\quad y(0) = 0, \qquad y'(0) = 0$

do yourself

$R = 2\ \Omega$, $L = 1$ H, $C = 0.5$ F, $v(t) = 1$ kV if $0 < t < 2$ and 0 if $t > 2$



6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

An airplane making a "hard" landing, a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time.

We can model such phenomena and problems by "Dirac's delta function," and solve them very effecively by the Laplace transform.

next time.

see you guys

$\delta(t - a)$ is called the **Dirac delta function**[2] or the **unit impulse function**.

$$\delta(t-a)=\begin{cases}\infty & \text{if } t=a\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\,dt=1,$$

$$\int_0^\infty g(t)\delta(t - a)\, dt = g(a)$$

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}.$$

EXAMPLE 2 **Hammerblow Response of a Mass–Spring System**

$$y'' + 3y' + 2y = \delta(t - 1) \qquad y(0) = 0, \qquad y'(0) = 0.$$

$$y'' + y = \delta(t - \pi) - \delta(t - 2\pi), \qquad y(0) = 0, y'(0) = 1$$