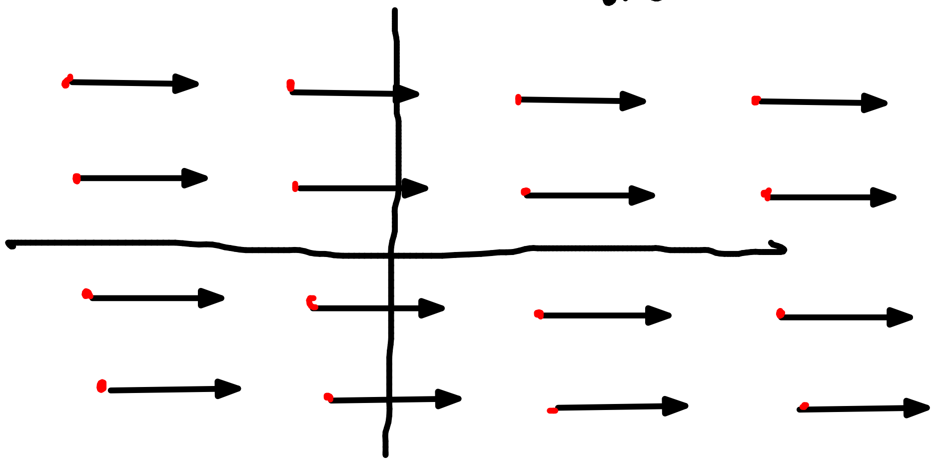
13.1 VECTOR FIELDS

functions whose range set are vector sets

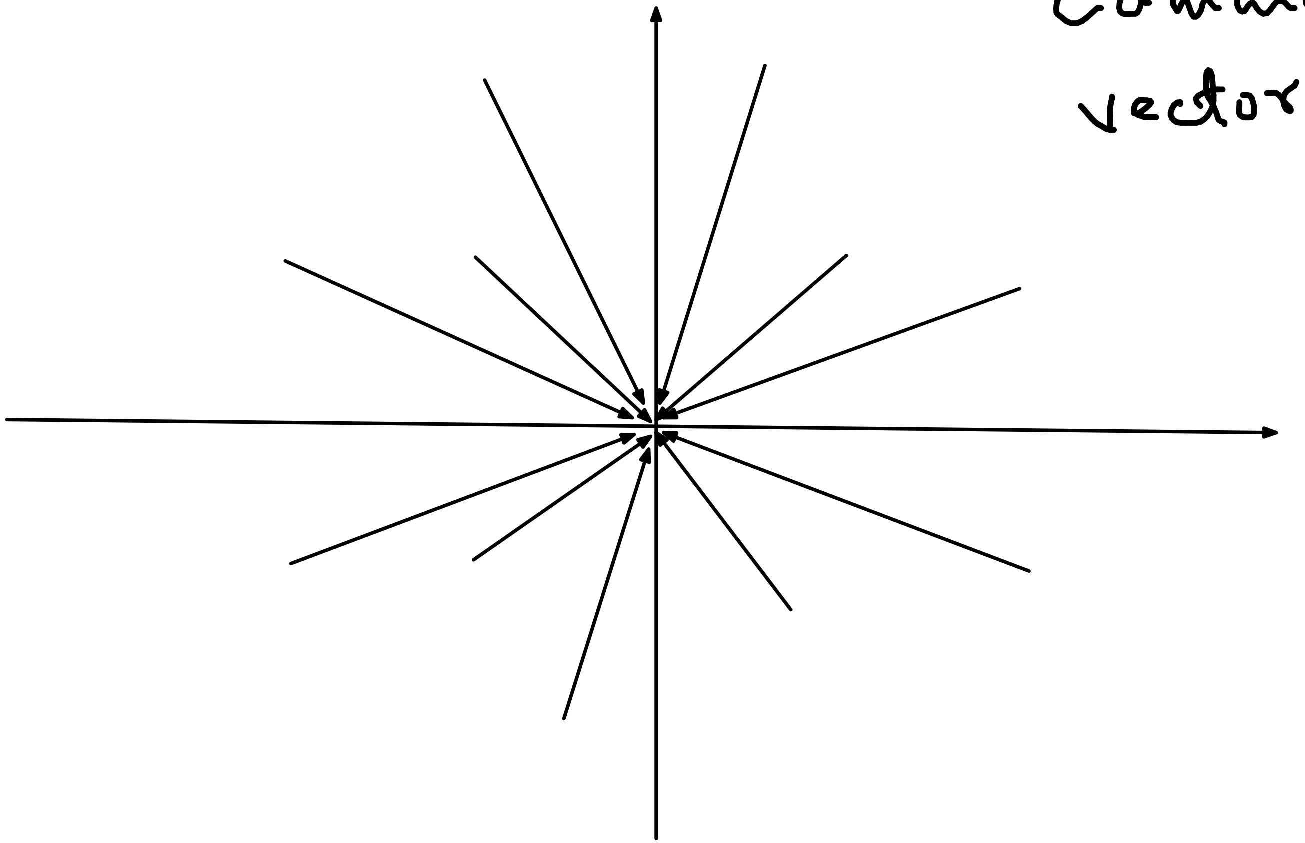
e.g. $\vec{F}(x,y) = \hat{i}$

VF $\mathbb{R}^2 \to \mathbb{R}^2$

maps a point in $\mathbb{R}^2$ to a 2 dimensional vector



$$\vec{F}(x,y) = -x\hat{i} - y\hat{j}$$



command for plotting vector fields in matlab/octave

"quiver"

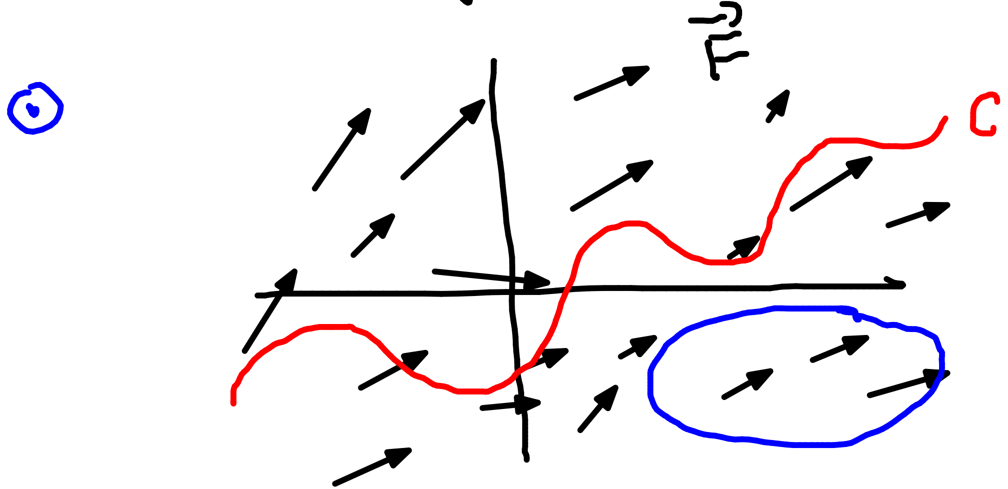
$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$$

Force field

$$\vec{v}(x,y,z) = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$$

Velocity field

Preview of the chapter



work done by $\vec{F}$ on moving a particle along the given path C

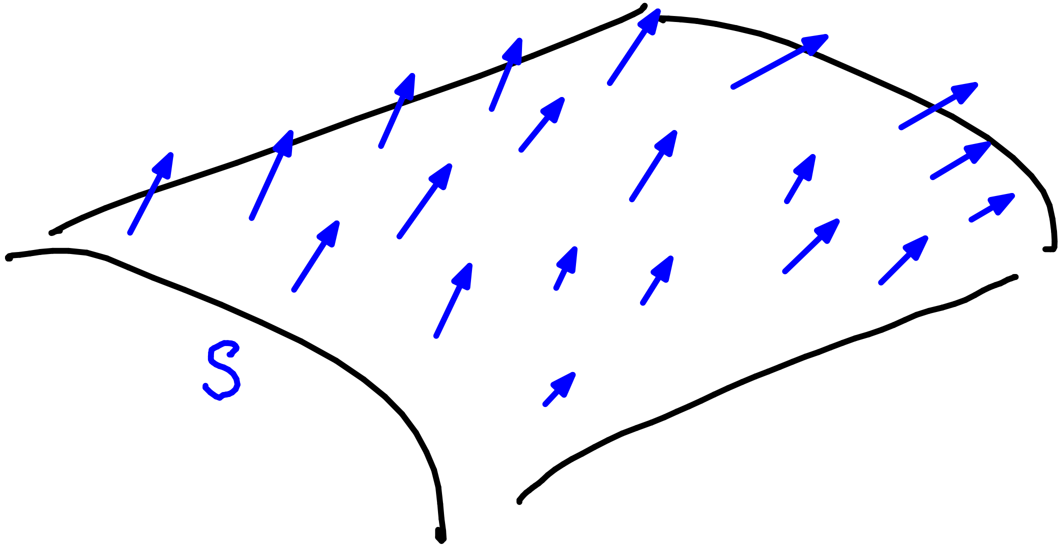
$$\int_C \vec{F}\cdot d\vec{r}$$

- Greens theorem
- Stokes theorem

simplification in $\int_C \vec{F}\cdot d\vec{r}$ if C is a closed loop

- Conservative Vector Fields

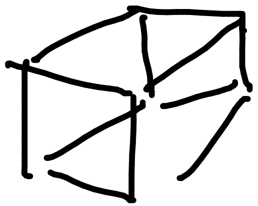
Later half of Chapter 13



$$\iint_S \vec{F} \cdot dA$$

flux of vector fields

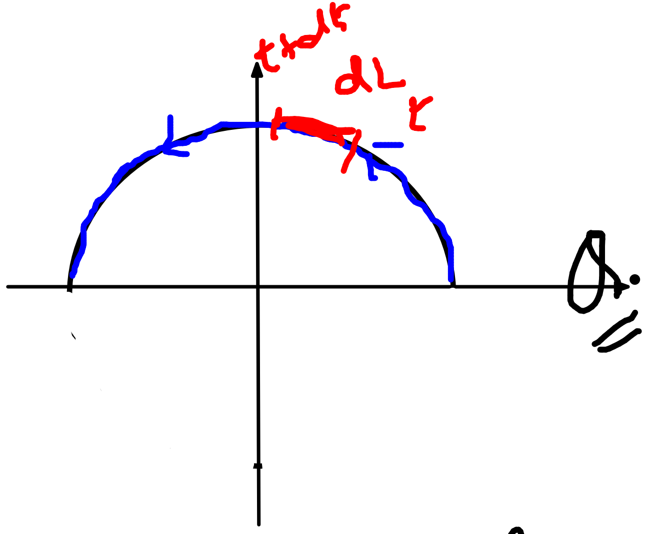
→ Divergence theorem:



13.2 LINE INTEGRALS

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

$f = 2 + x^2 y$: mass per unit length at point (x, y)



$$\int_C (2 + x^2 y)\, ds = \text{mass of } C$$

Can you find length of C ??

$$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$$

$$0 \le t \le \pi$$

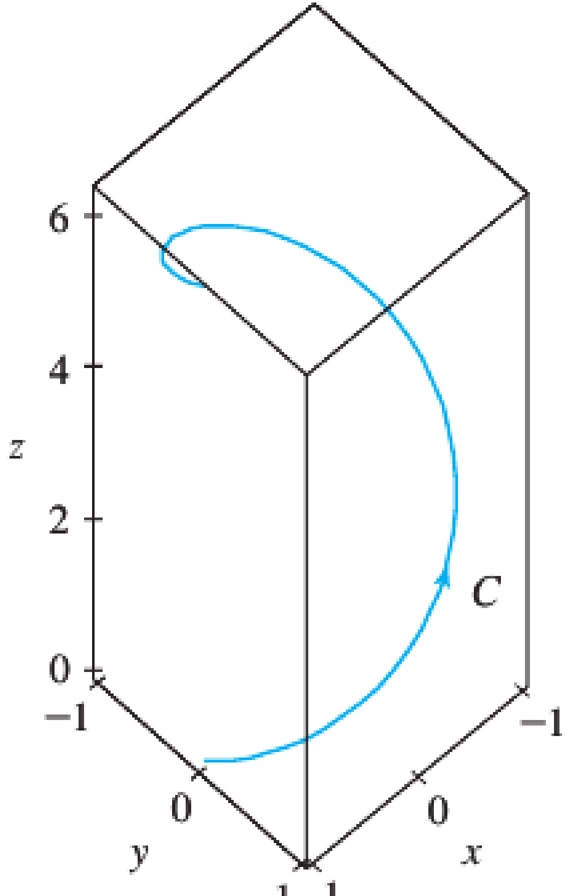
$$dm = (\text{density})\, dL$$

$$= (2+x^2y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Total mass $\quad m = \int_0^{\pi} dm = \int_0^{\pi}(2+x^2y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$= \int_0^{\pi}\left(2+\cos^2 t\,\sin t\right)dt = \text{whatever}$$

EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)



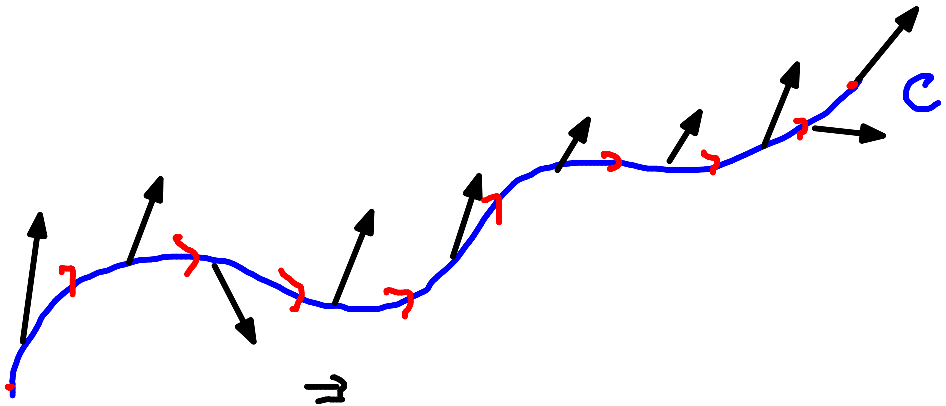
$$\int_C y \sin z \, ds = \int_0^{2\pi} y \sin z \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sin^2(t) \sqrt{2} \, dt = \text{whatever}$$

Aim for Today:

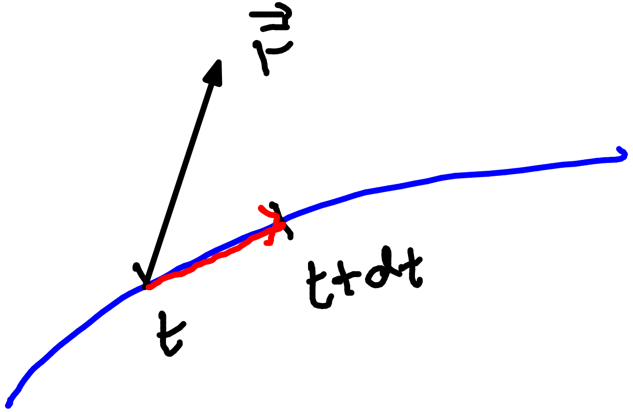
finish 13.3

LINE INTEGRALS OF VECTOR FIELDS



$\vec{F}$ integrate $\vec{F}$ along C

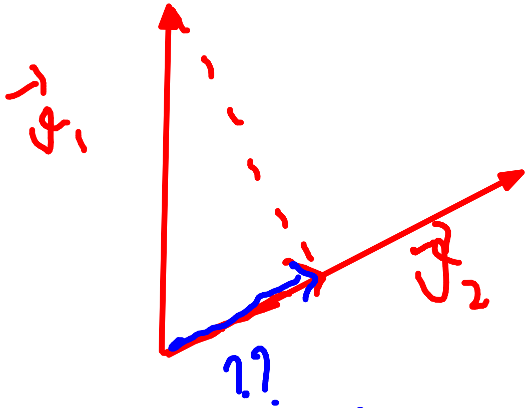
$$\boxed{\int_C \vec{F} \cdot d\vec{r}} = \text{work done by } \vec{F} \text{ in moving a particle along the curve}$$



$W = \vec{F} \cdot \vec{d} = \left(\text{component of } \vec{F} \text{ in the direction of displacement}\right) \times \left(\text{distance travelled}\right)$

$dW = \left(\text{tangential component of } \vec{F}\right) dL$

Tangential Component of $\vec{F} = \vec{F} \cdot \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

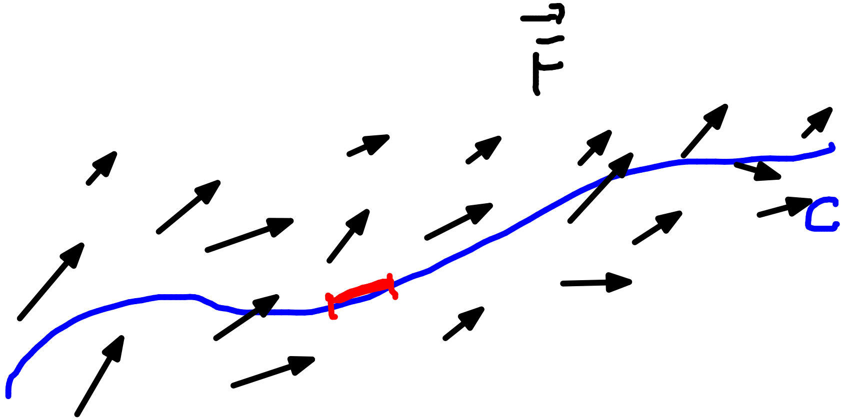


$$= \vec{v}_1 \cdot \frac{\vec{v}_2}{|\vec{v}_2|}$$

$$dW = \vec{F} \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \cdot \cancel{|\vec{r}\,'(t)|}\, dt$$

$$= \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$W = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$$



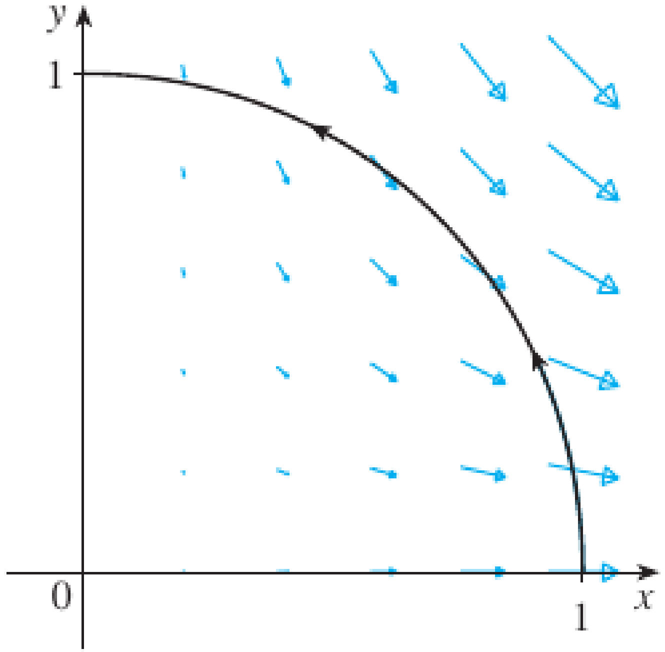
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$a \le t \le b$$

$$\vec{F}(x,y,z) = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C dW = \int_C \underbrace{\vec{F}\,\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}}_{\text{component of } \vec{F} \text{ in the tangential dir}} \underbrace{\cancel{|\vec{r}\,'(t)|}\,dt}_{dL} = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.



$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi/2} \left(\cos^2 t,\ -\cos t \sin t\right) \cdot \left(-\sin t,\ \cos t\right) dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \; dt = -\frac{2}{3}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt = \frac{27}{28}$$

13.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

f: potential f^n

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

whats the theorem??

$f(x,y,z)$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\int_C \vec{F}\cdot d\vec{r}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j}$$

$$= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

for some scalar valued function $f(x,y)$

$$\vec{F} = xy\,\hat{i} + \sin(5+2x)\,\hat{j}$$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

11. $\mathbf{F}(x, y) = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$,

C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

$x^4y^4/4$

find $f(x,y)$

$\nabla f = \vec{F}$

i.e $\frac{\partial f}{\partial x} = x^3y^4$ $\qquad \underbrace{\frac{\partial f}{\partial y} = x^4y^3}$

$\Downarrow$

$f = \frac{x^4y^4}{4} + h(y)$

$\Downarrow$

$\frac{\partial f}{\partial y} = x^4y^3 + h'(y) = x^4y^3$

$h'(y) = 0$

$h =$ constant

$f(x,y) = \frac{x^4y^4}{4}$

or

$f(x,y) = \frac{x^4y^4}{4} + 10$

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{end point of } C) - f(\text{starting point of } C)$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

11. $\mathbf{F}(x, y) = x^3 y^4 \, \mathbf{i} + x^4 y^3 \, \mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t} \, \mathbf{i} + (1 + t^3) \, \mathbf{j}, \quad 0 \leqslant t \leqslant 1$

$\vec{F} = P\hat{i} + Q\hat{j} = \nabla f$

$\Rightarrow \quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$= f(1,2) - f(0,1) = 4 - 0 = 4$$

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$f(x,y,z)$

13. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

$$\frac{\partial f}{\partial x} = yz \quad \Big| \quad \frac{\partial f}{\partial y} = xz \quad \Big| \quad \frac{\partial f}{\partial z} = xy + 2z$$

$$f = xyz + h(y,z)$$

$$\frac{\partial f}{\partial y} = \cancel{x}z + \frac{\partial h}{\partial y} = \cancel{x}z$$

$$h = h(z)$$

$$\frac{\partial f}{\partial z} = \cancel{x}y + h'(z) = \cancel{x}y + 2z$$

$$h'(z) = 2z$$

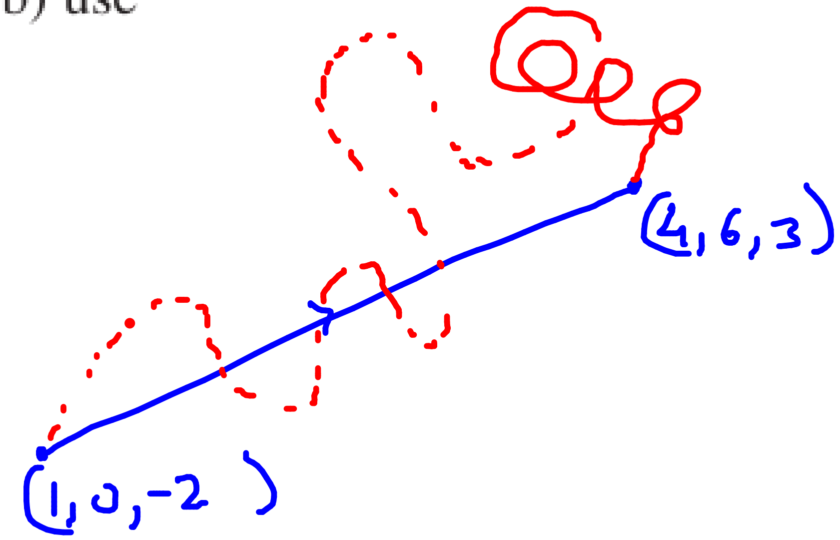
$$h(z) = z^2$$

$$f = xyz + z^2 \Leftarrow$$

will work

11–16 ▪ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

13. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$



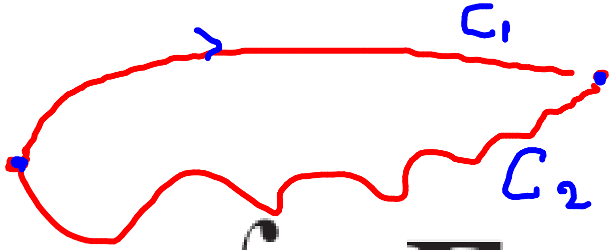
$f = xyz + z^2$

will work

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(4,6,3) - f(1,0,-2)$$

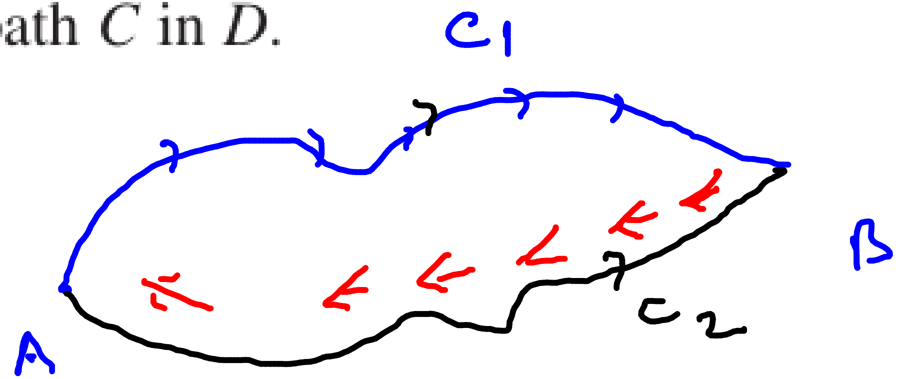
$$= 77$$

INDEPENDENCE OF PATH



$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

3 THEOREM $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.



$$\int_{C_1} \vec{F} \cdot d\vec{r} \overset{??}{=} \int_{C_2} \vec{F} \cdot d\vec{r}$$

given

$$\int_{C_1 \cup (-C_2)} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \quad \int_{C_1} \vec{F} \, d\vec{r} = \int_{C_2} \vec{F} \cdot dr$$

4 THEOREM Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

- Previously: if $\vec{F} = \nabla f$ then the work done is path independent
- Theorem is other way round:
  if the work done is independent of path then $\vec{F}$ must necessarily be ∇f for some f, provided $\vec{F}$ is continuous in the domain

5 THEOREM If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Q: if $\vec{F}$ is conservative, then why $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

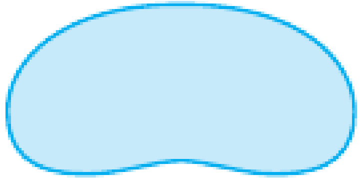
$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial^2 f}{\partial y\,\partial x} \\ \frac{\partial Q}{\partial x} &= \frac{\partial^2 f}{\partial x\,\partial y} \end{aligned}\right\} \text{same ??}$$

Recall Clairut's thm

6 THEOREM Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

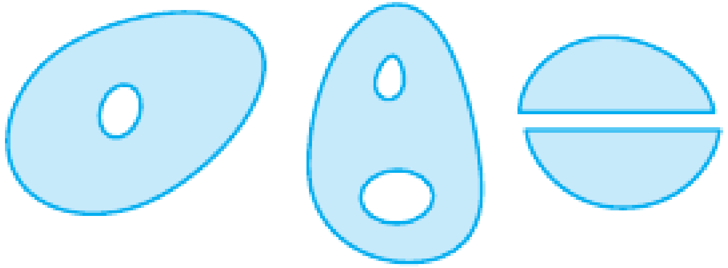
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.



simply-connected region

note: converse of previous theorem

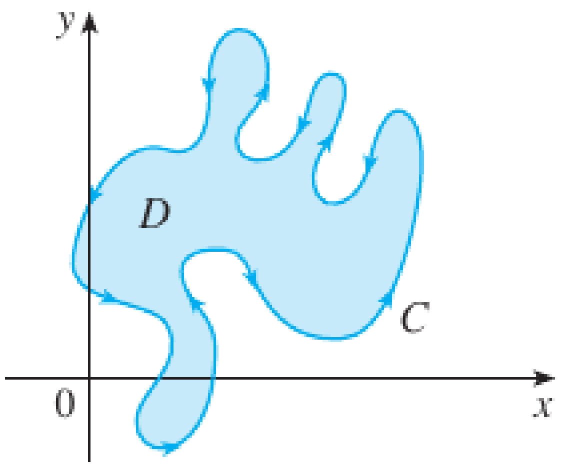


regions that are not simply-connected

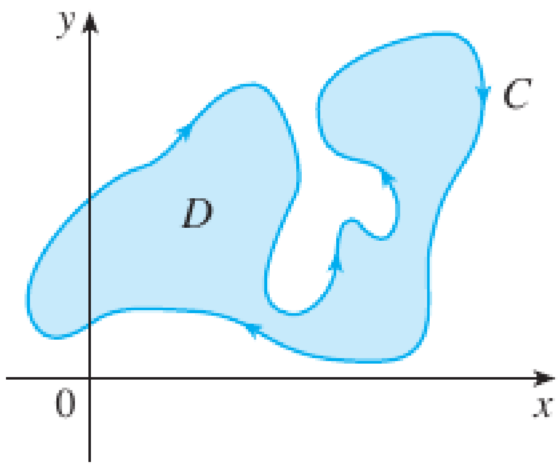
application : next time

13.4 **GREEN'S THEOREM** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

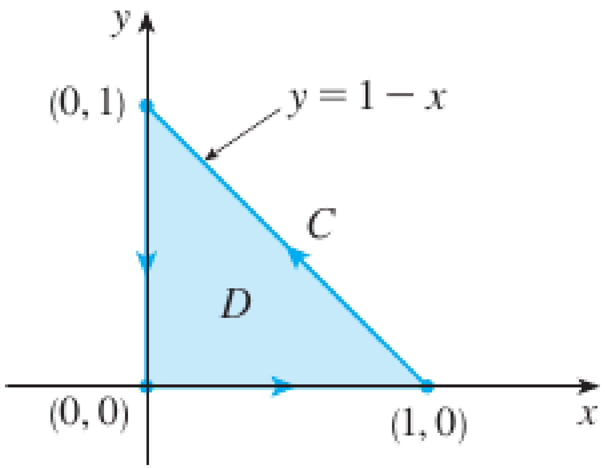


(a) Positive orientation



(b) Negative orientation

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).



V EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right)dy$, where C is the circle $x^2 + y^2 = 9$.

V EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.