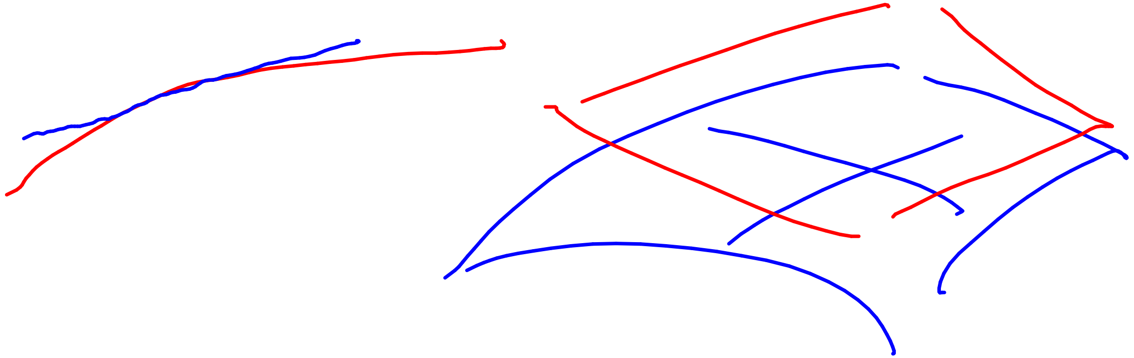
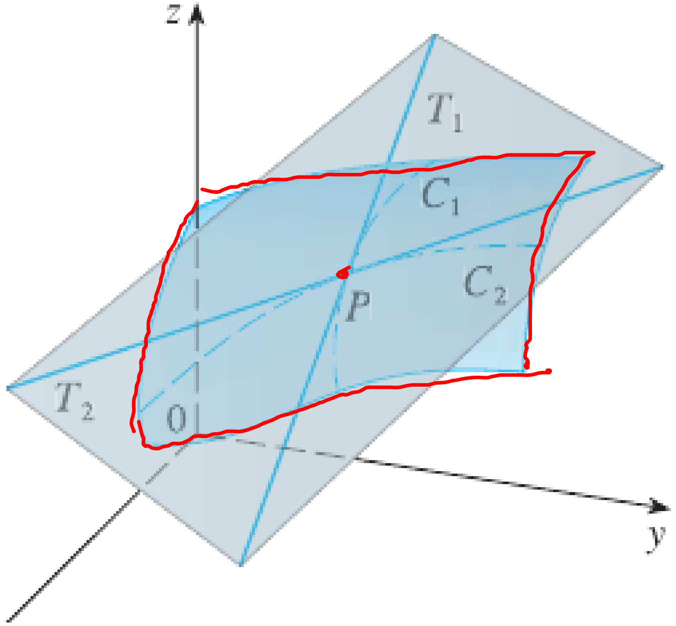
11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

11.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

TANGENT PLANES





2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$z_0 = f(x_0, y_0)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

→ Exercise : prove that this is indeed tangent to the surface

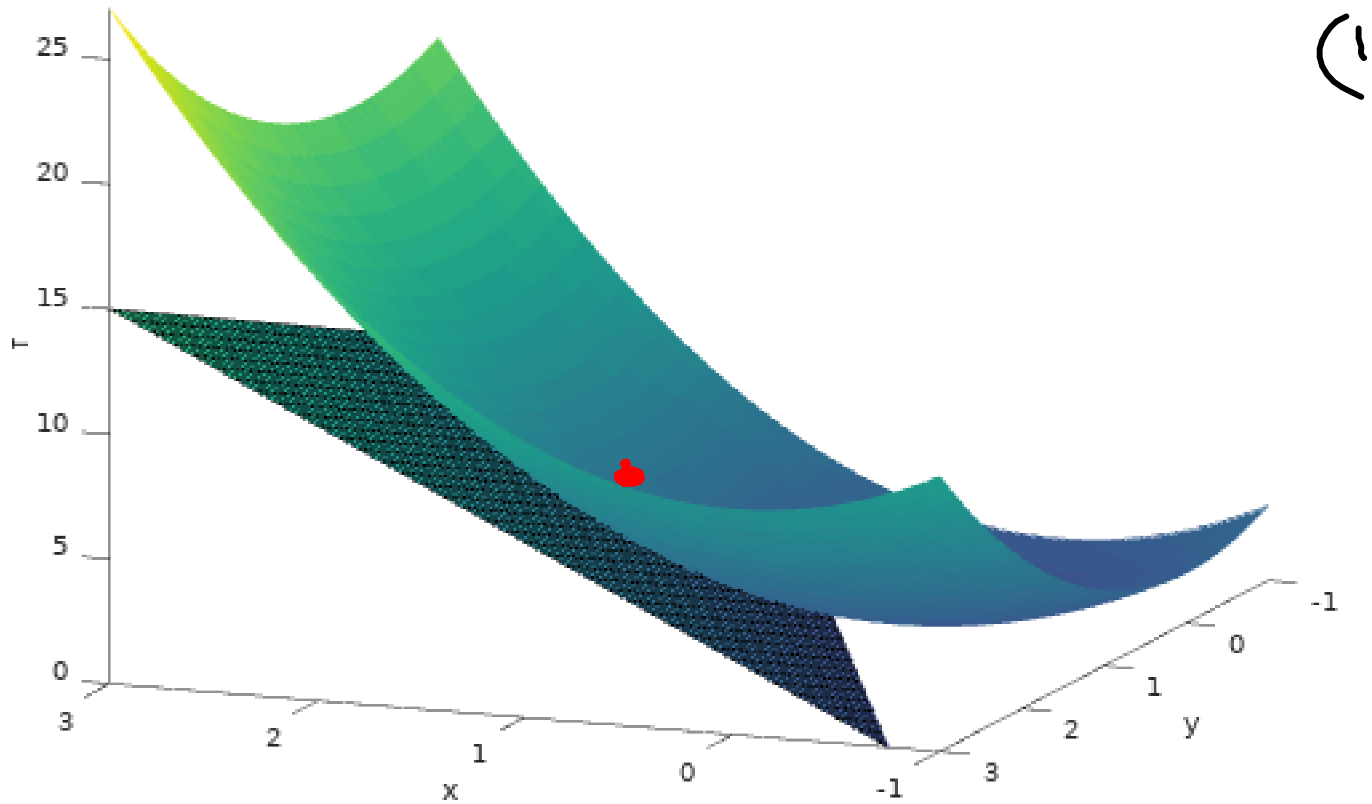
EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$f_x = 4x$ | $f_x(1,1) = 4$

$f_y = 2y$ | $f_y(1,1) = 2$

$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$



(1,1,3)

Find an equation of the tangent plane to the given surface at the specified point.

$$z = y\cos(x - y), \quad (2, 2, 2)$$

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

$f_x = -y\sin(x - y)$

$f_y = \cos(x - y) + y\sin(x - y)$

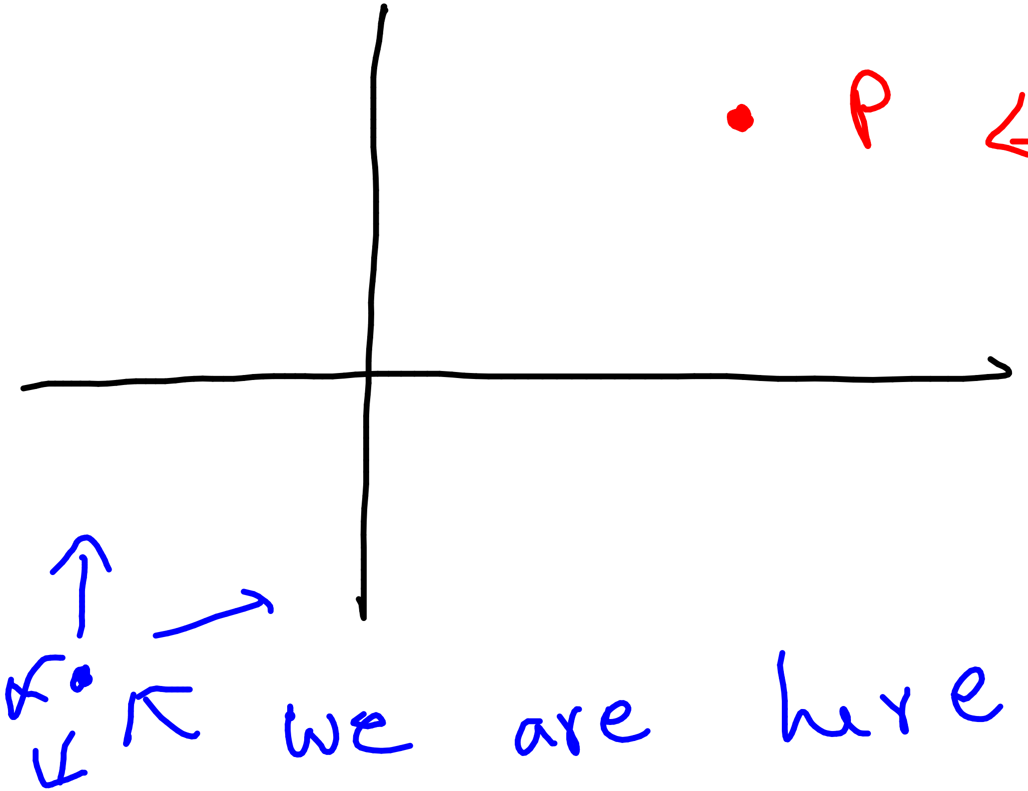
$f_x(2,2) = 0$

$f_y(2,2) = 1$

$z - 2 = y - 2$

Optimization:

we look for minimum point of $f(x,y)$



point of minimum

we always find $-$gradients for fastest descent

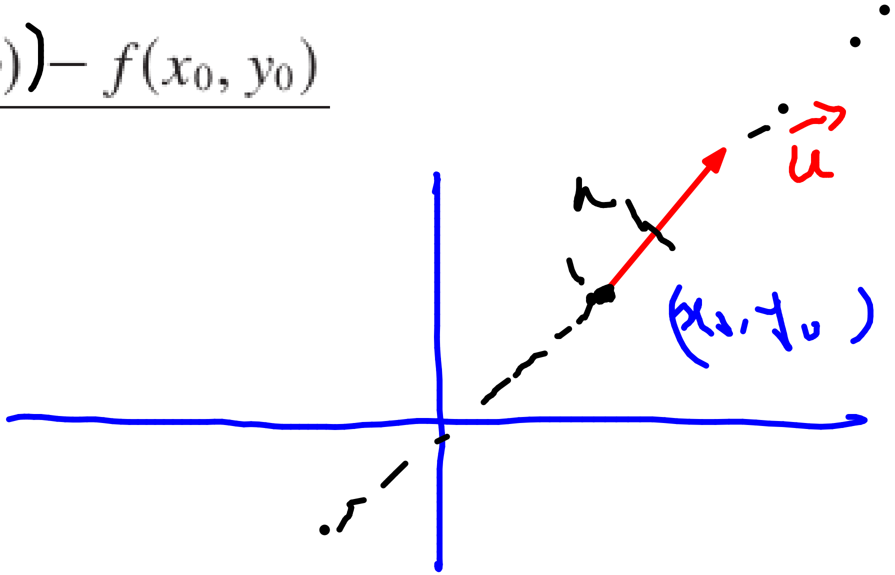
DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$(x_0, y_0) + h(a, b)$

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.



$\dfrac{\partial f}{\partial x}$ $\qquad$ $\dfrac{\partial f}{\partial y}$

3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

$$= \underbrace{(f_x, f_y)} \cdot (a, b)$$

$$=$$

3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Proof: $LHS = D_u f(x,y) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$

$$= \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x, y+hb) + f(x, y+hb) - f(x,y)}{h}$$

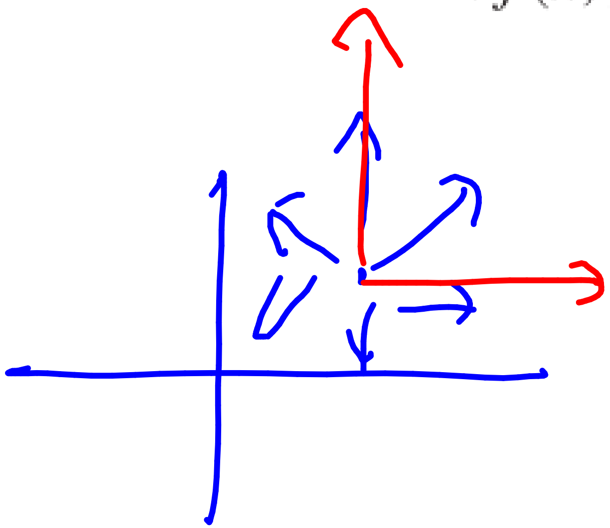
$$= \lim_{h \to 0} \left[\frac{f(x+ha, y+hb) - f(x, y+hb)}{ah} \right] a + \lim_{h \to 0} \left[\frac{f(x, y+hb) - f(x, y)}{bh} \right] b$$

$$= f_x(x, y)\, a + f_x(x, y)\, b$$

THE GRADIENT VECTOR

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$



[Captures the rate of change of f in all direction]

EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\,\mathbf{j}$.

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

we need unit vector

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}$$

$$f_x = 2xy^3 \qquad f_x(2,-1) = -4$$

$$f_y = 3x^2y^2 - 4 \qquad f_y(2,-1) = 8$$

$$D_u f(2,-1) = -4 \cdot \frac{2}{\sqrt{29}} + 8 \cdot \frac{5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

EXAMPLE 2 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

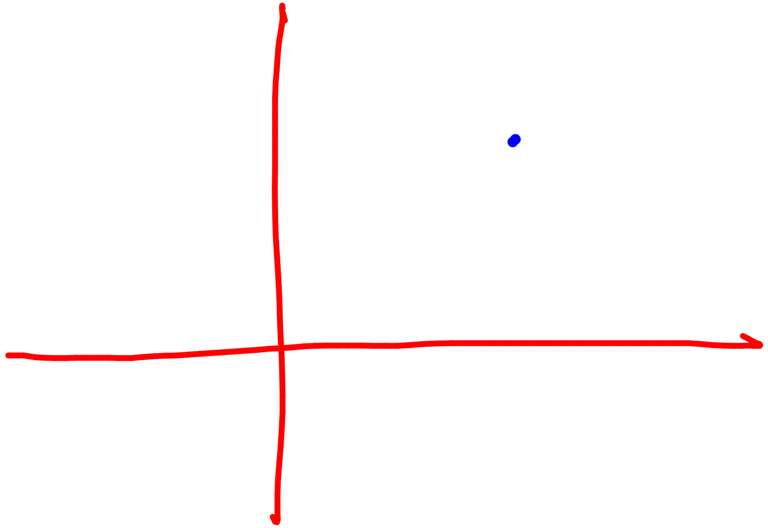
MAXIMIZING THE DIRECTIONAL DERIVATIVE

$\nabla f = f_x \hat{i} + f_y \hat{j}$

15 THEOREM Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

$\hat{u}$

$f(x,y)$



$D_u f = \nabla f \cdot \hat{u}$

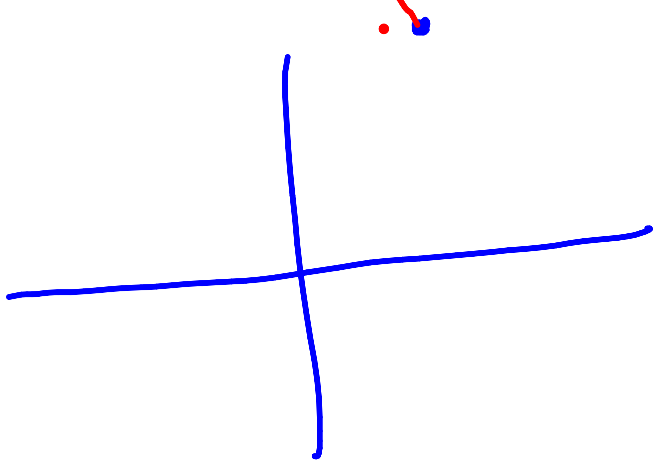
if $\hat{u} \parallel \nabla f \Rightarrow |D_u f| = |\nabla f|$

15–18 ■ Find the maximum rate of change of f at the given point and the direction in which it occurs.

15. $f(x, y) = y^2/x$, $(2, 4)$

$-4\hat{i}+4\hat{j}$

$=$



$$f_x = \frac{-y^2}{x^2} \qquad f_y = \frac{2y}{x}$$

$$f_x(2,4) = -4 \qquad f_y(2,4) = 4$$

$$\nabla f(2,4) = -4\hat{i}+4\hat{j}$$

$$|\nabla f| = \sqrt{32}$$

15–18 ▪ Find the maximum rate of change of f at the given point and the direction in which it occurs.

17. $f(x, y, z) = \ln(xy^2z^3)$, $(1, -2, -3)$