2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$

undetermined coefficients

- easier
- range of r(x) is small

variation of parameters

- slightly tedious
- more general

2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x) \qquad (*)$$

↳ step ① solve a corresponding homogeneous eqⁿ

$$y'' + py' + qy = 0$$

→ let solution y_h

↳ step ② find a particular solution y_p

$$y'' + py' + qy = r$$

either by
→ undetermined coefficients
→ variation of parameters

↳ final step : $y = y_h + y_p$ [general solution of Eq ⊛]

Recall the ODE :

→ $$y'' + py' + qy = r$$

→ $$y_h'' + p y_h' + q y_h = 0$$

→ $$y_p'' + p y_p' + q y_p = r$$

$$(y_h + y_p)'' + p\,(y_h + y_p)' + q\,(y_h + y_p) = r$$

End Sem :

May 7

→ mode : similar to mid term

→ syllabus : everything covered this semester

So far we have been solving homogeneous 2nd order ODE

$$y'' + ay' + by = 0$$

now we will solve non-homogenous 2nd order ODE

$$\boxed{y'' + ay' + by = r(x)} \quad \text{———} \quad (*)$$

Step ① find a general solution of the corresponding homogenous sol^n:

y_h : $\qquad y_h'' + a y_h' + b y_h = 0$

Step ② find a particular sol^n : y_p which solves Eq $(*)$

↳ we will see how

Q. Does $y = y_h + y_p$ solve Eq (*)

Yes

$y = y_h + y_p$ is called general solution of Eq (*)

how to find y_p

method of undetermined coefficients

Today

Variations of parameters

Sec 2.10

Method of Undetermined Coefficients

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Q. Solve: $y'' + y = 1$

→ Solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p using undetermined coefficients:

$$r(x) = 1$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

∵ our $r(x)$ is a constant

The table is suggesting that y_p will also be a constant

$$y_p = C$$ [C needs to be found]

∴ we substitute $y_p = c$ in the given ODE & solve for C

$$(c)'' + (c) = 1$$

$$c = 1$$

general sol^n

$$y = C_1 \cos x + C_2 \sin x + 1$$

Q. Solve: $y'' + y = 5$

→ Solve the corresponding homogeneous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p:

$$y_p = C$$

$y_p = 5$ will work

→

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Q. Solve: $y'' + y = e^{-2x}$

→ Solve the corresponding homogenous part.

$$y'' + y = 0$$

$$y_h = C_1 \cos x + C_2 \sin x$$

→ find y_p using undetermined coefficients

$r(x) = e^{-2x}$

Try $y_p = Ce^{-2x}$, need to find C

→ plug in $y_p = Ce^{-2x}$ in $y'' + y = e^{-2x}$

$$4Ce^{-2x} + Ce^{-2x} = e^{-2x}$$

$$5C\cancel{e^{-2x}} = \cancel{e^{-2x}}$$

$$5C = 1$$

$$C = 1/5$$

$\Rightarrow y_p = \frac{1}{5} e^{-2x}$

general solution

$$y = C_1 \cos x + C_2 \sin x + \frac{e^{-2x}}{5}$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$\boxed{ke^{\gamma x}}$	$\boxed{Ce^{\gamma x}}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

EXAMPLE 1

Solve the initial value problem

(5) $$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

$y_h = c_1 \cos x + c_2 \sin x$

What should be y_p ?

$$y_p = K_0 + K_1 x + K_2 x^2$$

We need to find K_0, K_1, K_2 by substituting y_p in $y'' + y = 0.001x^2$

$$K_2 x^2 + K_1 x + (K_0 + 2K_2) = 0.001x^2$$

→ matching the x, x^2, & constants in LHS & RHS

$K_2 = 0.001$

$K_1 = 0$

$K_0 = -0.002$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$\boxed{kx^n \ (n = 0, 1, \cdots)}$	$\boxed{K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0}$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

r	y_p
1	K_0
x	$K_0 + K_1 x$
x^2	$K_0 + K_1 x + K_2 x^2$
x^3	$K_0 + K_1 x + K_2 x^2 + K_3 x^3$

general sol^n:

$$y = c_1 \cos x + c_2 \sin x - 0.002 + 0.001x^2$$

remaining work:

find c_1 & c_2 using $y(0) = 0$

$y'(0) = 1.5$

$\downarrow$ $\boxed{\text{work}}$

$\rightarrow$ $c_1 = 0.002$, $c_2 = 1.5$

EXAMPLE 2

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{6}$$

Table 2.1 **Method of Undetermined Coefficients**

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

$\to y_h = ??$

$$\lambda^2 + 3\lambda + 2.25 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{9 - 9}}{2} = -3/2$$

$$y_h = c_1 e^{-3/2 x} + c_2 x e^{-3/2 x}$$

$\to y_p = Ce^{-1.5x}$ should have worked but it's not.

trick: try $y_p = Cxe^{-1.5x}$ ✗

try $y_p = Cx^2e^{-1.5x}$

$$2.25\left(y_p = c\cancel{x^2 e^{-1.5x}} \right)$$

$$+\quad 3\left(y_p' = c\,\cancel{2x e^{-1.5x}} - \cancel{\tfrac{3c}{2} x^2 e^{-1.5x}} \right)$$

$$+\quad \left(y_p'' = c\,2e^{-1.5x} + 2(2x)\cancel{\left(-\tfrac{3}{2}\right)e^{-1.5x}} + cx^2\,\cancel{\tfrac{9}{4}\,e^{-1.5x}} \right)$$

$$-10e^{-1.5x} = 2c\,e^{-1.5x}$$

$$c = -5$$

$\Rightarrow$ general sol^n : $y = c_1 e^{-1.5x} + c_2 x e^{-1.5x} - 5x^2 e^{-1.5x}$

then find c_1 & c_2 using $y(0) = 1$, $y'(0) = 0$

EXAMPLE 3

Solve the initial value problem

(7) $$\boxed{y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x,} \qquad y(0) = 2.78, \qquad y'(0) = -0.43.$$

Table 2.1 **Method of Undetermined Coefficients**

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	

→ step ① $y'' + 2y' + 0.75y = 0$

$$\lambda^2 + 2\lambda + 0.75 = 0$$

$$\lambda = -0.5, \ -1.5$$

$$y_h = c_1 e^{-0.5x} + c_2 e^{-1.5x}$$

→ step ② Find y_p

$$r(x) = 2\cos x - 0.25\sin x + 0.09x$$

$$\boxed{r_1(x) = 2\cos x - 0.25\sin x} \quad y_p \ \Big| \ y_p'' + 2y_p' + 0.75y_p = 2\cos x - 0.25\sin x$$

$$r_2(x) = 0.09x \quad z_p \ \Big| \ z_p'' + 2z_p' + 0.75z_p = 0.09x$$

EXAMPLE 3

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) = \tag{7}$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

2a) find y_p: with $r(x) = 2\cos x - 0.25\sin x$

Try: $0.75\,(y_p = K\cos x + M\sin x)$

$2\,(y_p' = -K\sin x + M\cos x)$

$y_p'' = -K\cos x - M\sin x$

find K & M

$y_p'' + 2y_p' + 0.75\,y_p = 2\cos x - 0.25\sin x$

$(2M - 0.25K)\cos x + (-0.25M - 2K)\sin x = 2\cos x - 0.25\sin x$

$2M - 0.25K = 2$

$-0.25M - 2K = -0.25$

Partial y_p: $\sin x$

mathpapa.com/algebra-calculator.html

MathPapa

ALGEBRA CALCULATOR PRACTICE LESSONS PI

$5 \cdot K = 2, \; -0.25 \cdot M - 2 \cdot K = -0.25$

CALCULATE IT!

Solve Lesson

Solve by Substitution

Let's solve your system by substitution.

$$2m - 0.25k = 2; -0.25m - 2k = -0.25$$

Show Step-By-Step

Answer:

$$k = 0 \text{ and } m = 1$$

Solve the initial value problem

(7) $$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \quad y(0) = 2.78, \quad y'(0) = -0.43.$$

find y_p: $\quad y_p'' + 2y_p' + 0.75\,y_p = 0.09x$

Try $\quad y_p = k_1 x + k_0$

find k_1, k_2 by substituting y_p in & comparing constants & x-coeffs in LHS & RHS

$k_1 = 0.12$

$k_0 = -0.32$

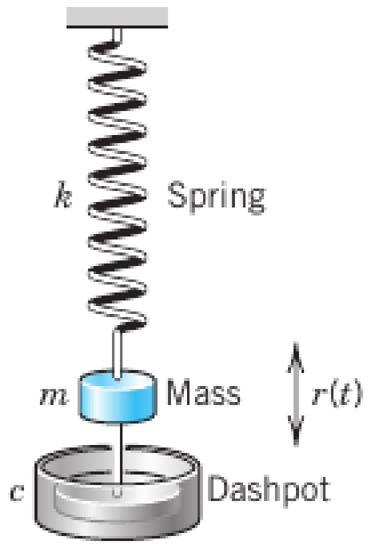
$$y_p = -0.32 + 0.12x$$

Find general solution

$$y = C_1 e^{-0.5x} + C_2 e^{-1.5x} + \sin x \qquad -0.32 + 0.12x$$

→ find C_1, C_2 using initial conditions.

2.8 Modeling: Forced Oscillations. Resonance



$$my'' + cy' + ky = F_0 \cos \omega t.$$

https://www.youtube.com/watch?v=XwIZBJIp1AA

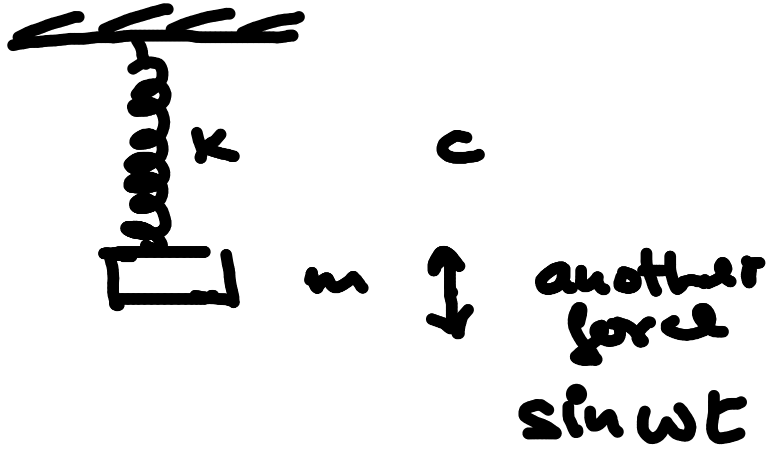
Resonance:

$$my'' + cy' + ky = 0$$

→ assume $c = 0$, $m = 1$, $k = 1$

$$\boxed{y'' + y = \sin \omega t}$$

assume a new force apart from gravity & spring



→ sin ωt : external force

→ frequency determined by ω

Q. How will the general solution look like??

$$y = y_h + y_p$$

$$= c_1 \cos t + c_2 \sin t + y_p$$

y_h: oscillatory sol^n

frequency of y_h is called internal frequency if $\omega \neq 1$ [say $\omega = 2$]

not resonance

find y_p: $K\cos 2t + M\sin 2t$

can you find K & M: $y_p'' + y_p = \sin 2t$

$K = 0$, $M = -1/3$

$y_p = -\frac{1}{3}\sin 2t$

general sol^n

$y = c_1\cos t + c_2\sin t - \frac{1}{3}\sin 2t$

Resonance case: $[\omega = 1]$ (internal frequency = frequency of external force

[find y_p: in $y'' + y = \sin t$

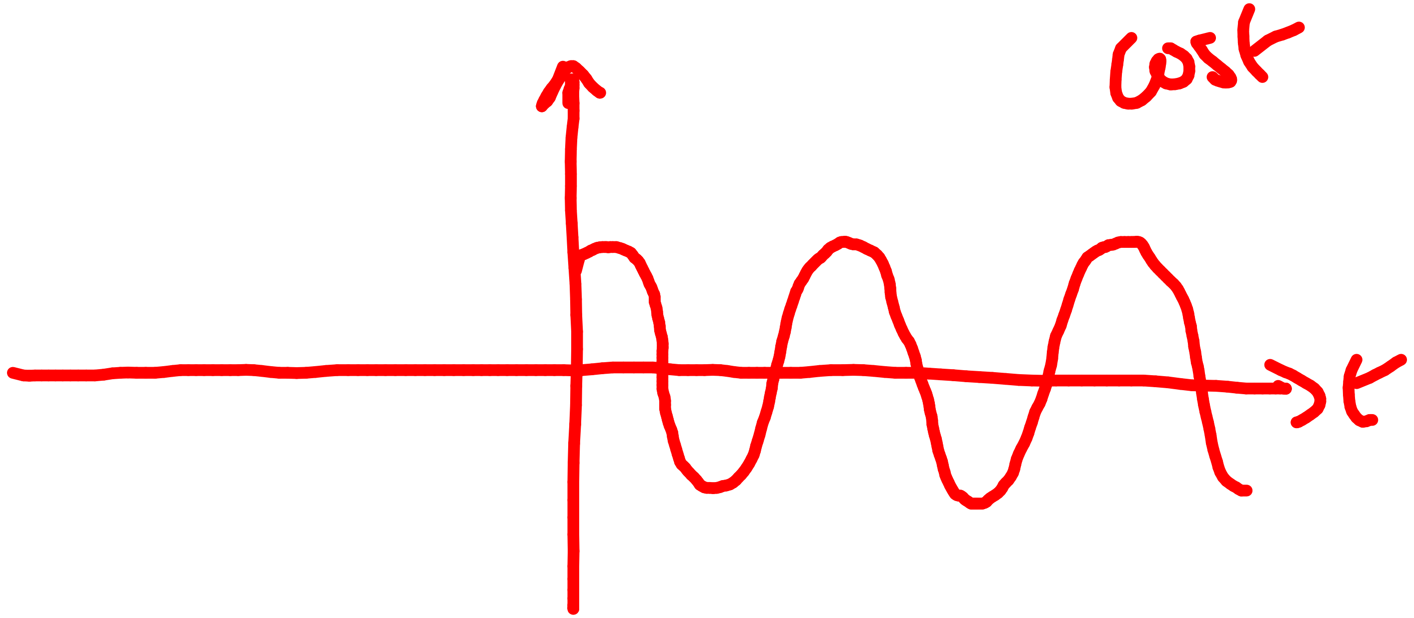
Try $y_p = K\cos t + M\sin t$ won't work

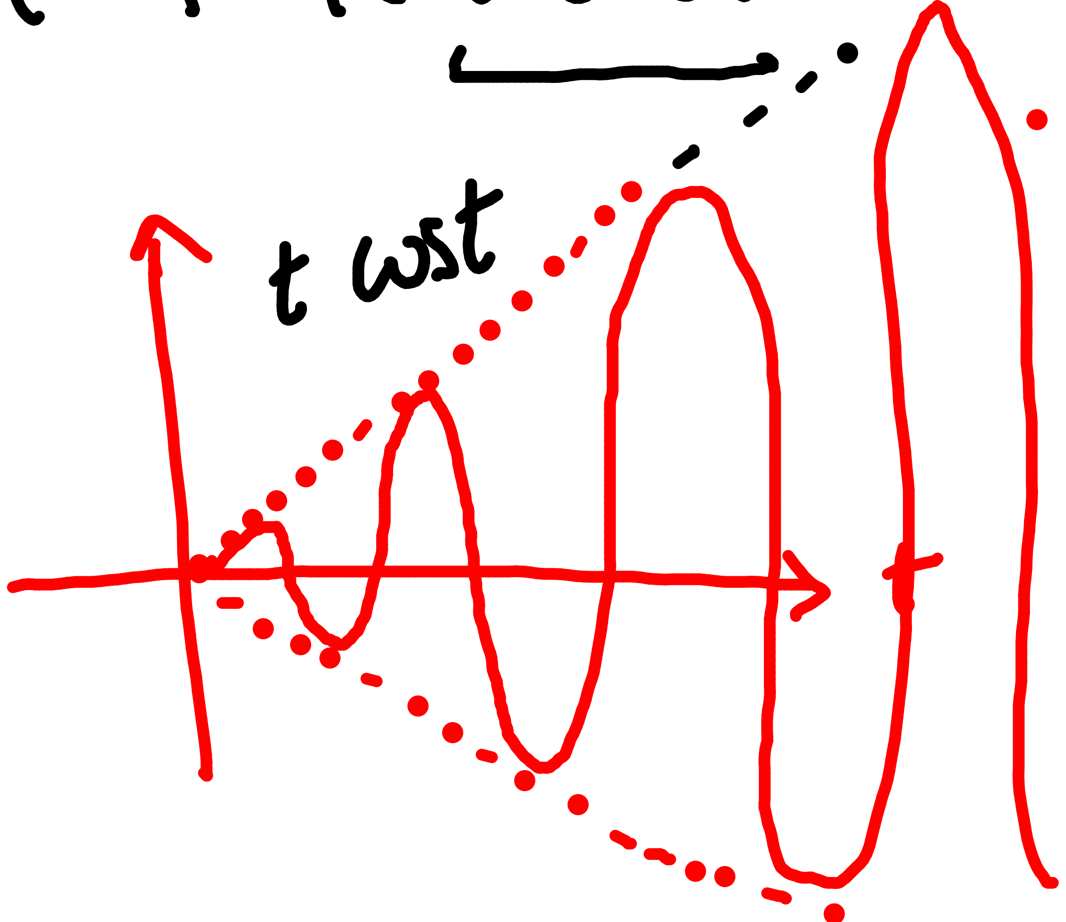
Try $y_p = tK\cos t + tM\sin t$

Lets assume $y_p = k\, t\cos t + M\, t\sin t$ works

general sol^n:

$$y = c_1\cos t + c_2\sin t + k\,\underbrace{t\cos t} + M\,\underbrace{t\sin t}$$





graph ??

2.9 Modeling: Electric Circuits

next time

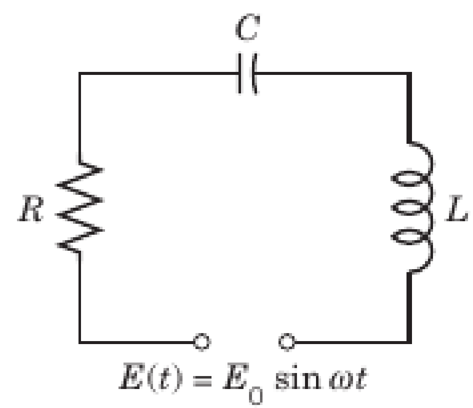


Fig. 61. *RLC*-circuit

Name	Symbol	Notation		Unit	Voltage Drop
Ohm's Resistor		R	Ohm's Resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L \dfrac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

RLC-Circuit

Find the current $I(t)$ in an *RLC*-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$ F (farad), which is connected to a source of EMF $E(t) = 110 \sin(60 \cdot 2\pi t) = 110 \sin 377\, t$ (hence 60 Hz = 60 cycles/sec, the usual in the U.S. and Canada; in Europe it would be 220 V and 50 Hz). Assume that current and capacitor charge are 0 when $t = 0$.