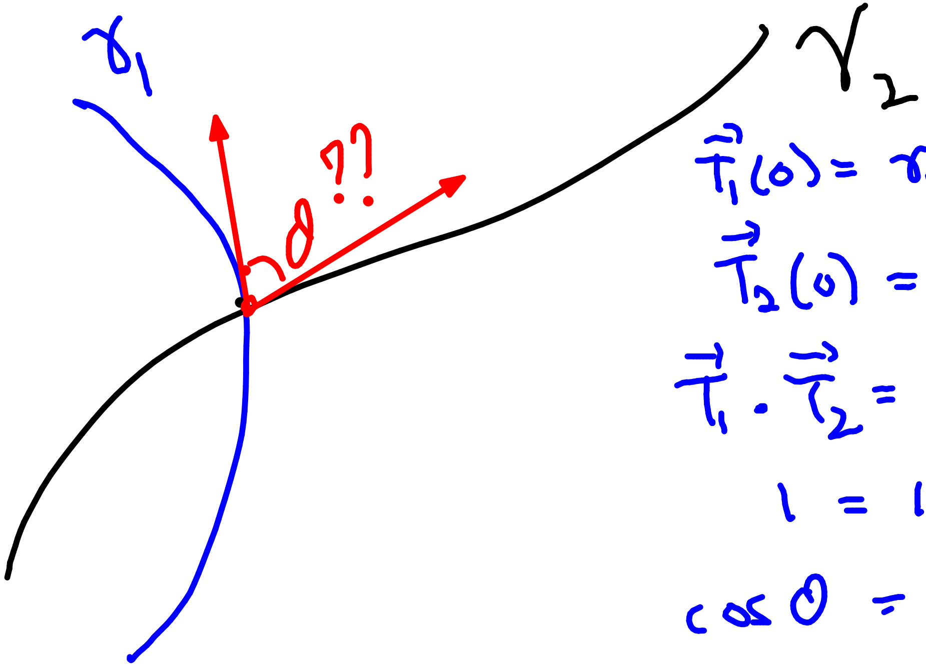
Today’s topic

- Derivatives of vector valued functions (Sec 10.7)
    - Some more problems

- Chain rule (Sec 11.5)

55. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.



$\vec{T}_1(0) = \gamma_1'(0) = \hat{i}$

$\vec{T}_2(0) = \gamma_2'(0) = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{T}_1 \cdot \vec{T}_2 = |\vec{T}_1|\,|\vec{T}_2| \cos\theta$

$1 = 1\sqrt{6}\cos\theta$

$\cos\theta = \frac{1}{\sqrt{6}}$

$\theta \approx 65.9^\circ$

a. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

$$\vec{r}(t) = (t^2 + c_1)\,\hat{i} + (t^3 + c_2)\,\hat{j} + \left(\tfrac{2}{3}t^{3/2} + c_3\right)\hat{k}$$

use $r(1) = \hat{i} + \hat{j}$ to find c_1, c_2, c_3

$c_1 = 0$	$1 + c_2 = 1$	$\frac{2}{3} + c_3 = 0$
	$c_2 = 0$	$c_3 = -2/3$

If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time*. Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geqslant 0$. Do the particles collide?

for collision, there must exist a t s.t. $\vec{r}_1(t) = \vec{r}_2(t)$

i.e. $t^2 = 4t - 3$, | $7t - 12 = t^2$ | $t^2 = 5t - 6$

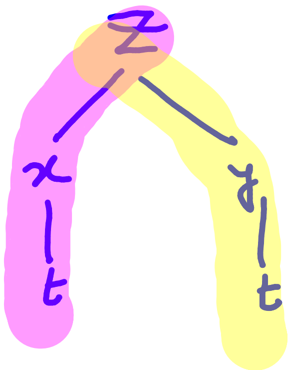
$t = 3, 1$ | $t = 3$ | $9 = 9$ | $t = 3$ | $9 = 9$

collission indud occurs at $t = 3$

11.5 THE CHAIN RULE

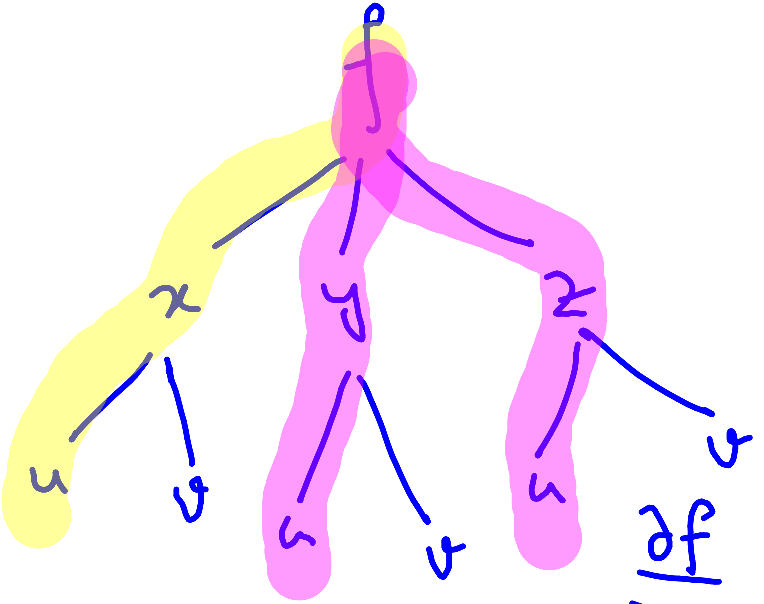
EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

$$z = f(x,y) = x^2y + 3xy^4$$
$$x = x(t) = \sin(2t)$$
$$y = y(t) = \cos(t)$$



$$t = 0$$
$$x = 0$$
$$y = 1$$

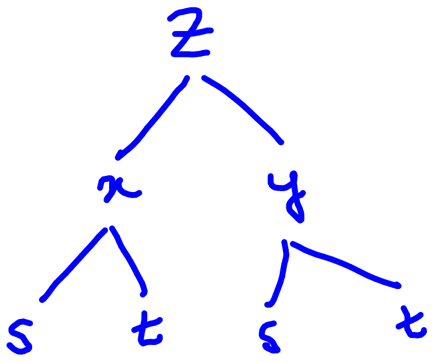
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin(t))$$
$$= 3\cdot 2 + 0 = 6$$



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

find $\frac{\partial f}{\partial v} = \; ??$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.



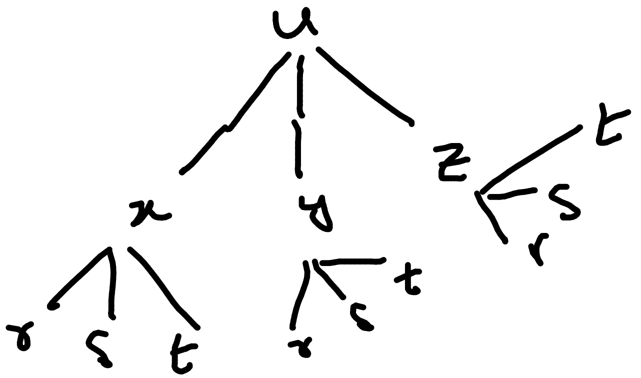
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x \sin y)\, t^2 + (e^x \cos y)(2st)$$

$$\frac{\partial z}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

$x = 2 \qquad y = 2 \qquad z = 0$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)\,2rse^{-t} + (3y^2z^2)(r^2\sin t)$$

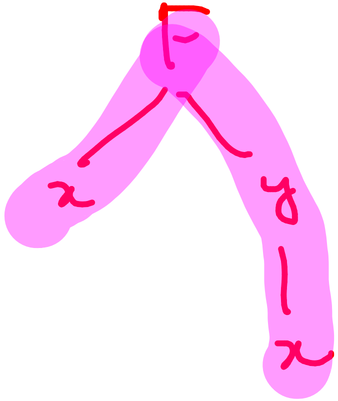
$$= 64 \cdot 2 + (16 + 0)4 + 0$$

$$= 128 + 64 = 192$$

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

$y(x)$ is defined implicity by this eq^n

$$\underbrace{x^3+y^3-6xy}_{F(x,y)} = 0$$



$$\frac{dF}{dx} = 0$$

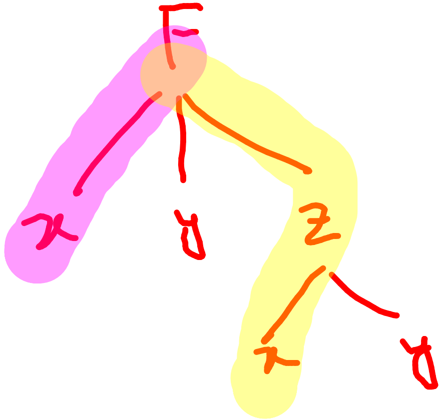
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \underline{\frac{dy}{dx}} = 0$$

$$(3x^2 - 6y) + (3y^2 - 6x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(3x^2 - 6y)}{(3y^2 - 6x)}$$

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

$$\underbrace{x^3+y^3+z^3+6xyz}_{F(x,y,z)}=1$$



find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

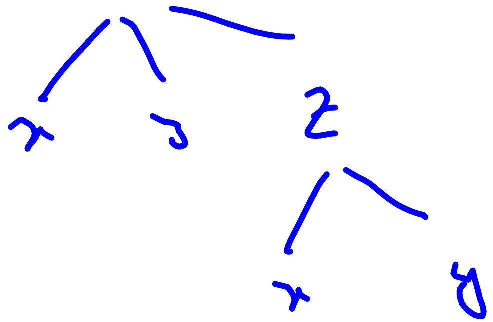
$$(3x^2+6yz)+(3z^2+6xy)\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x} = -\frac{(3x^2 + 6yz)}{3z^2 + 6xy}$$

Similarly

$$\frac{\partial z}{\partial y} = -(3y^2 + 6xz)/(3z^2 + 6xy)$$

$$F(x, y, z(x, y))$$



29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

34. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega/\text{s}$.

37. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$