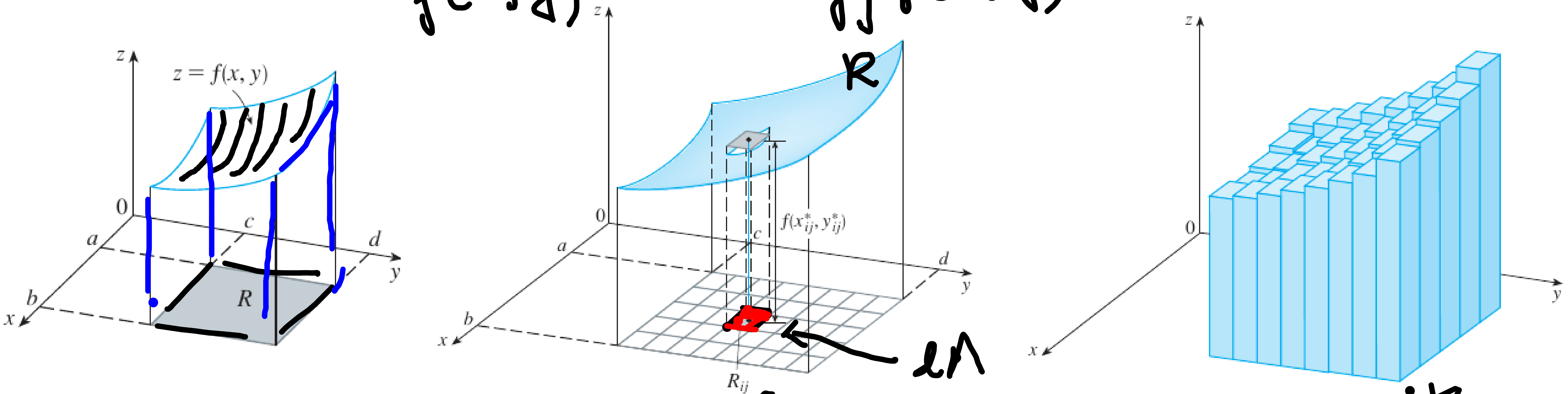
12 MULTIPLE INTEGRALS

$f(x,y)$ $\iint f(x,y)\, dA$

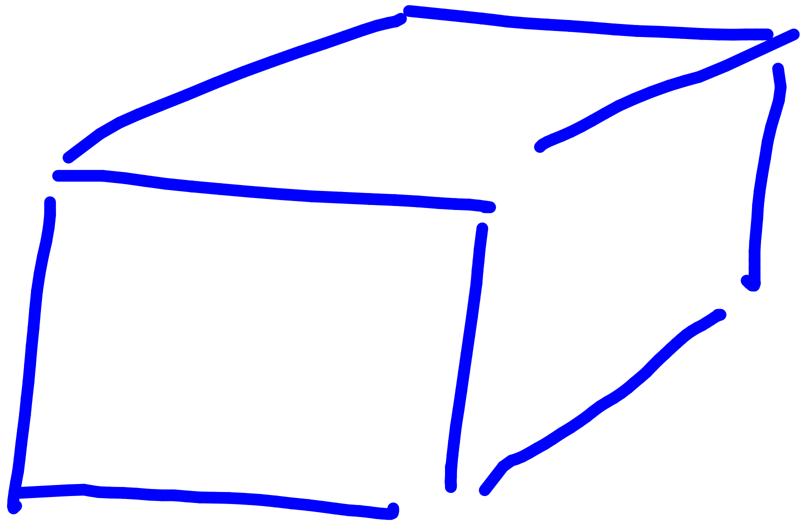


R

dA

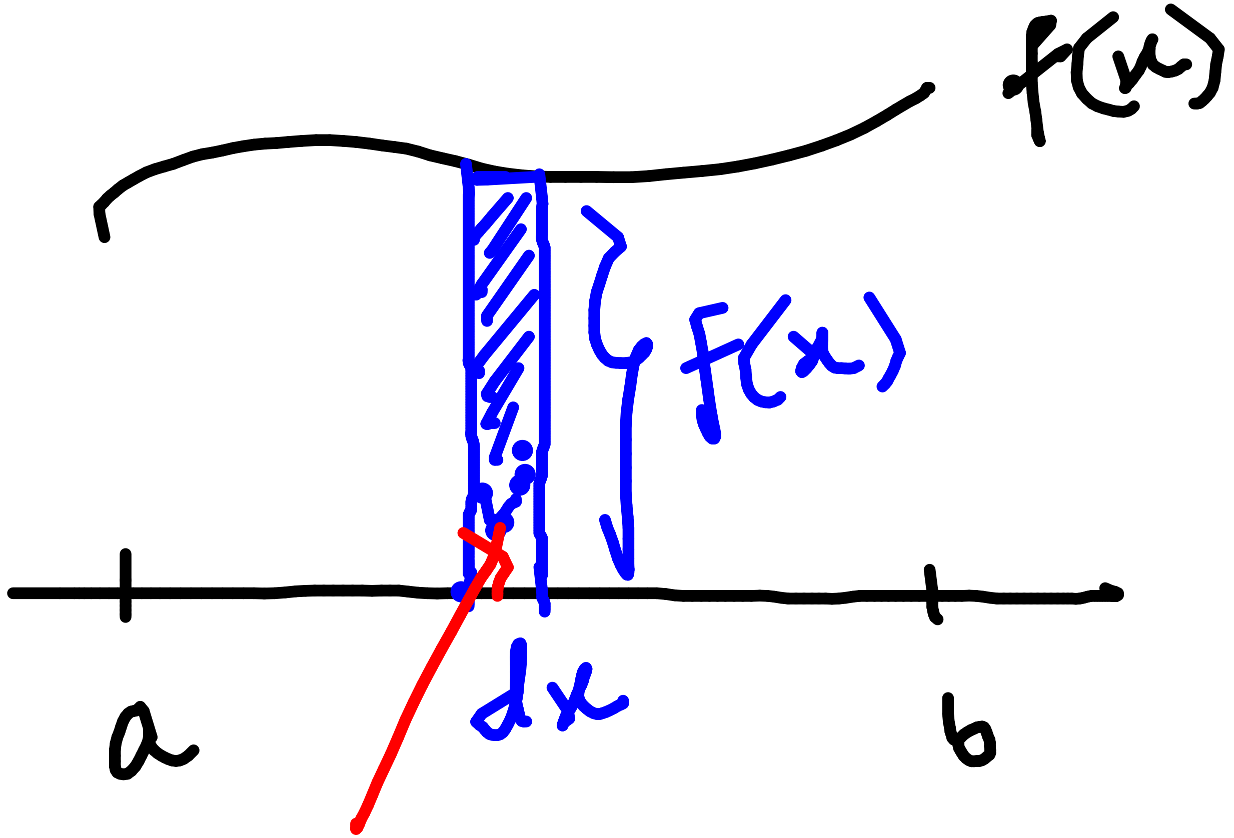
$$\iint_R f(x,y)\, dA = \text{volume with } f(x,y) \text{ on top}$$

& R as base



V

$$\iiint_V f(x, y, z)\, dV$$



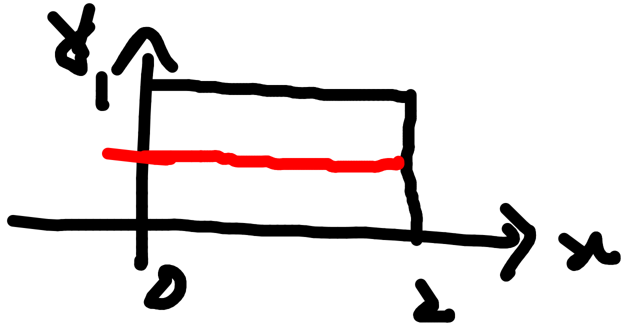
area = $f(x)\,dx$

$$\int_a^b f(x)\,dx$$

$$f(x)\,dx$$

$\int f(x)\,dx$ = infinite sum of $f(x)\,dx$

d. $f(x,y) = x$

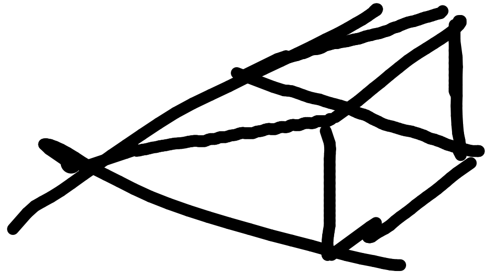


R

$$\iint_R x\, dA = ?? = \int_0^2 \left(\int_0^1 x \, dy \right) dx$$

$$= \int_0^2 \left[xy \Big|_{y=0}^{y=1} \right] dx = \int_0^2 x\, dx$$

$$= 2$$

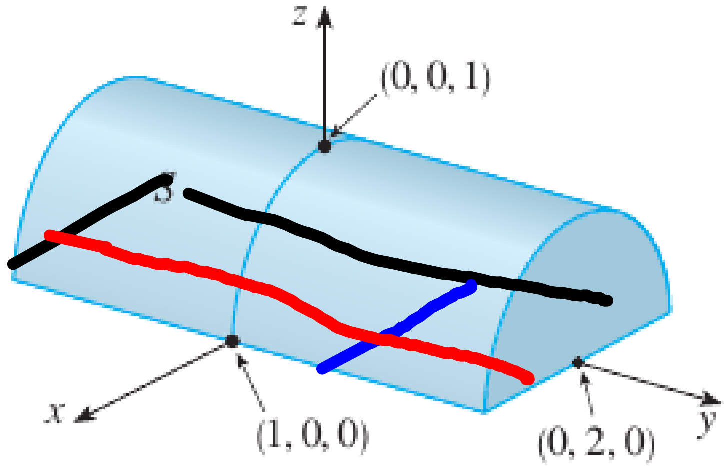


$$= \int_0^1 \left(\int_0^2 x \, dx \right) dy$$

$$= \int_0^1 2\, dy = 2$$

V EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$



$$f(x, y) = \sqrt{1 - x^2}$$

$$z = \sqrt{1 - x^2}$$

$$z^2 + x^2 = 1$$

$$= \int_{-2}^{2}\int_{-1}^{1} \sqrt{1 - x^2}\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-2}^{2} \sqrt{1 - x^2}\, dy\, dx$$

$$= \int_{-1}^{1} 4\sqrt{1 - x^2}\, dx = 2\pi$$

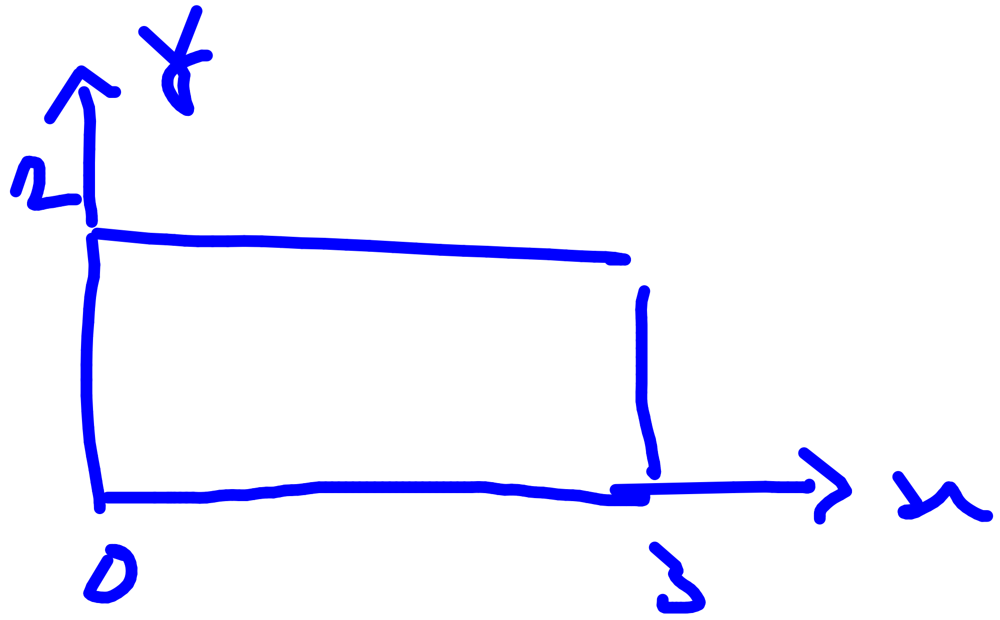
EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

$\frac{27}{2}$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

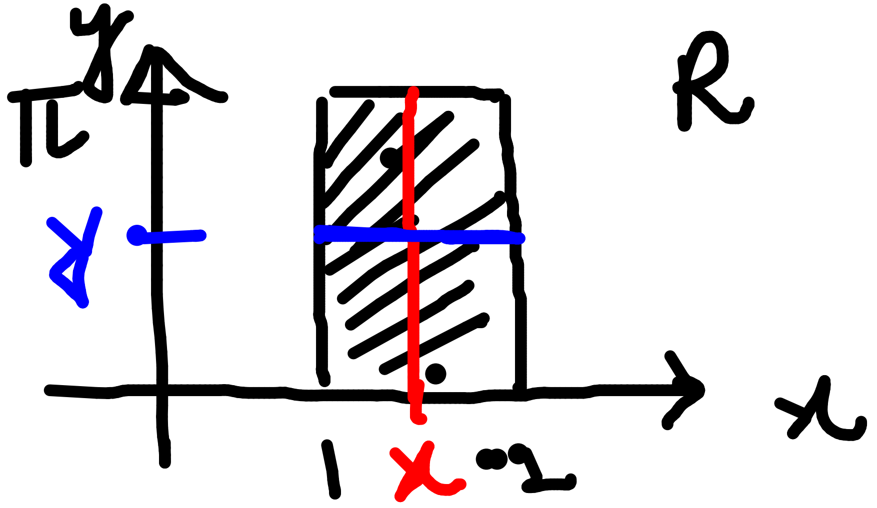
$\frac{27}{2}$



10 FUBINI'S THEOREM If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.



$x \quad y$

$$\iint_R y\sin(xy)\,dA = \int_1^2 \left(\int_0^{\pi} y\sin(xy)\,dy \right) dx$$

$$= \int_0^{\pi} \left(\int_1^2 y\sin(xy)\,dx \right) dy$$

Then, start integrating inside out

$$\int_1^2 y\sin(xy)\,dx = \left[-\cancel{y}\frac{\cos(xy)}{\cancel{y}}\right]_{x=1}^{x=2}$$

$$= \cos(y) - \cos(2y)$$

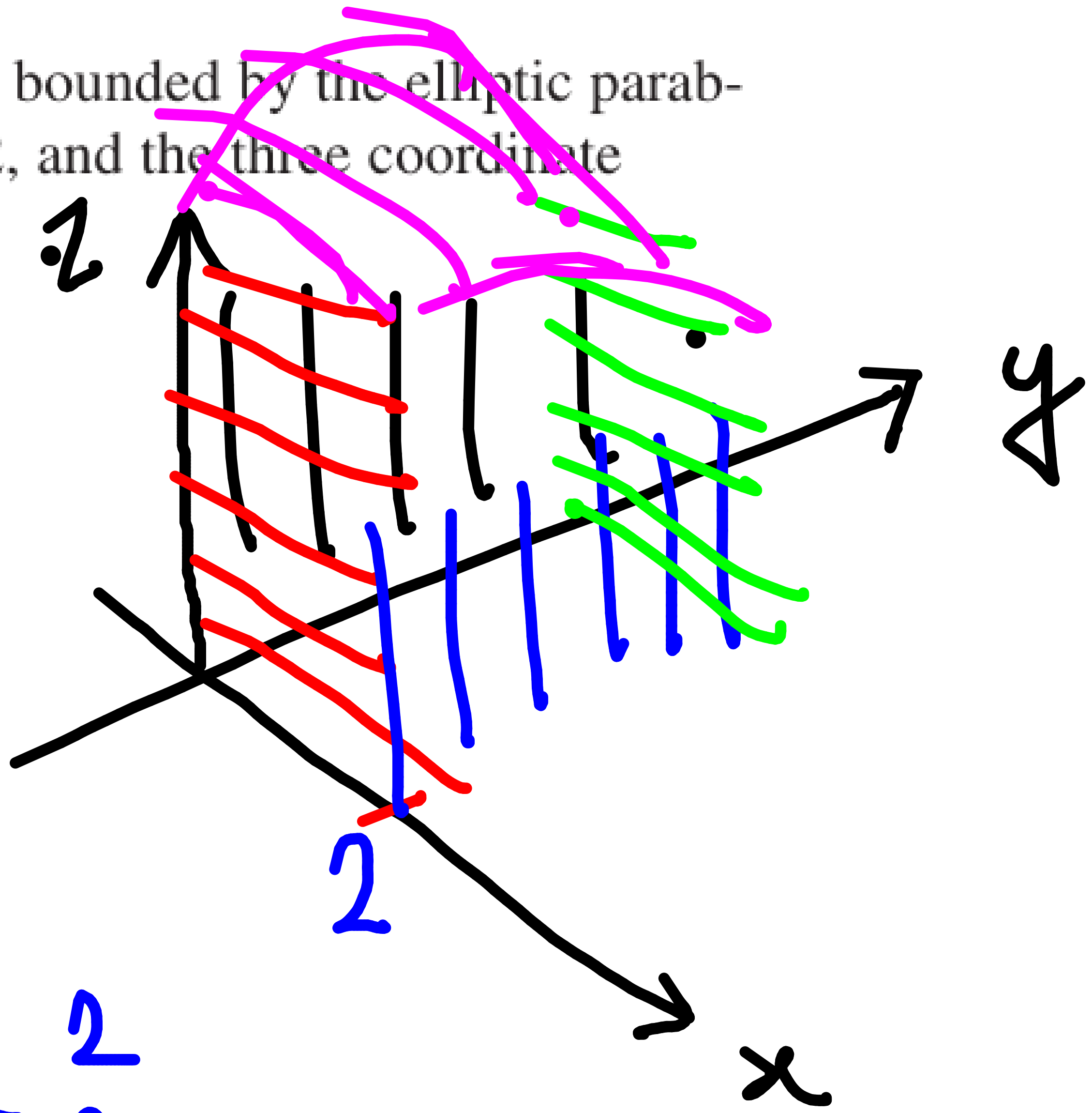
$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\left[\cos(y) - \cos(2y)\right]dy$$

$$= 0$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

$$x^2 + 2y^2 + z = 16$$

$$z = 16 - x^2 - 2y^2$$



$$V = \iint_{[0,2]\times[0,2]} 16 - x^2 - 2y^2 \, dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dy\, dx$$

$$\int_0^2 (16 - x^2 - 2y^2)\,dy = \left[16y - x^2 y - \frac{2y^3}{3}\right]_{y=0}^{y=2}$$

$$= 32 - \frac{16}{3} - 2x^2$$

$$\int_0^2 \left(32 - \frac{16}{3} - 2x^2\right) dx = 48$$

PROPERTIES OF DOUBLE INTEGRALS

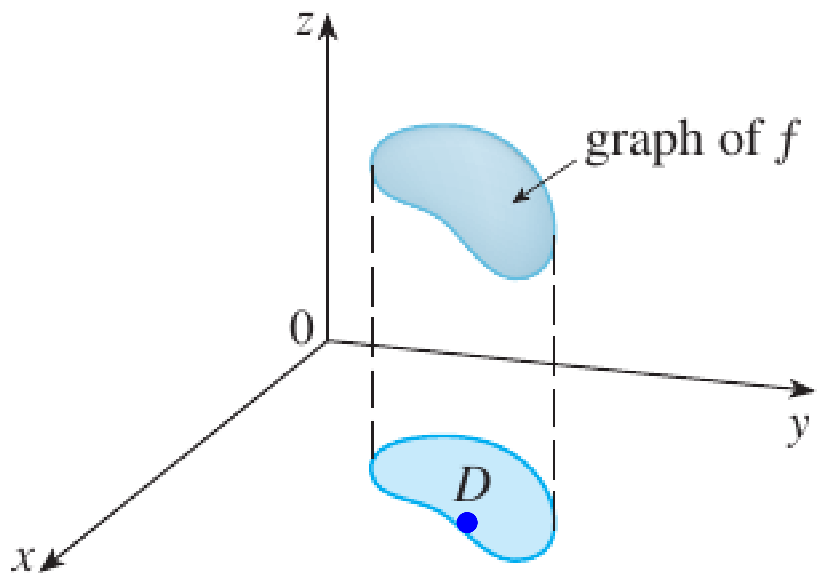
12 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

13 $$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

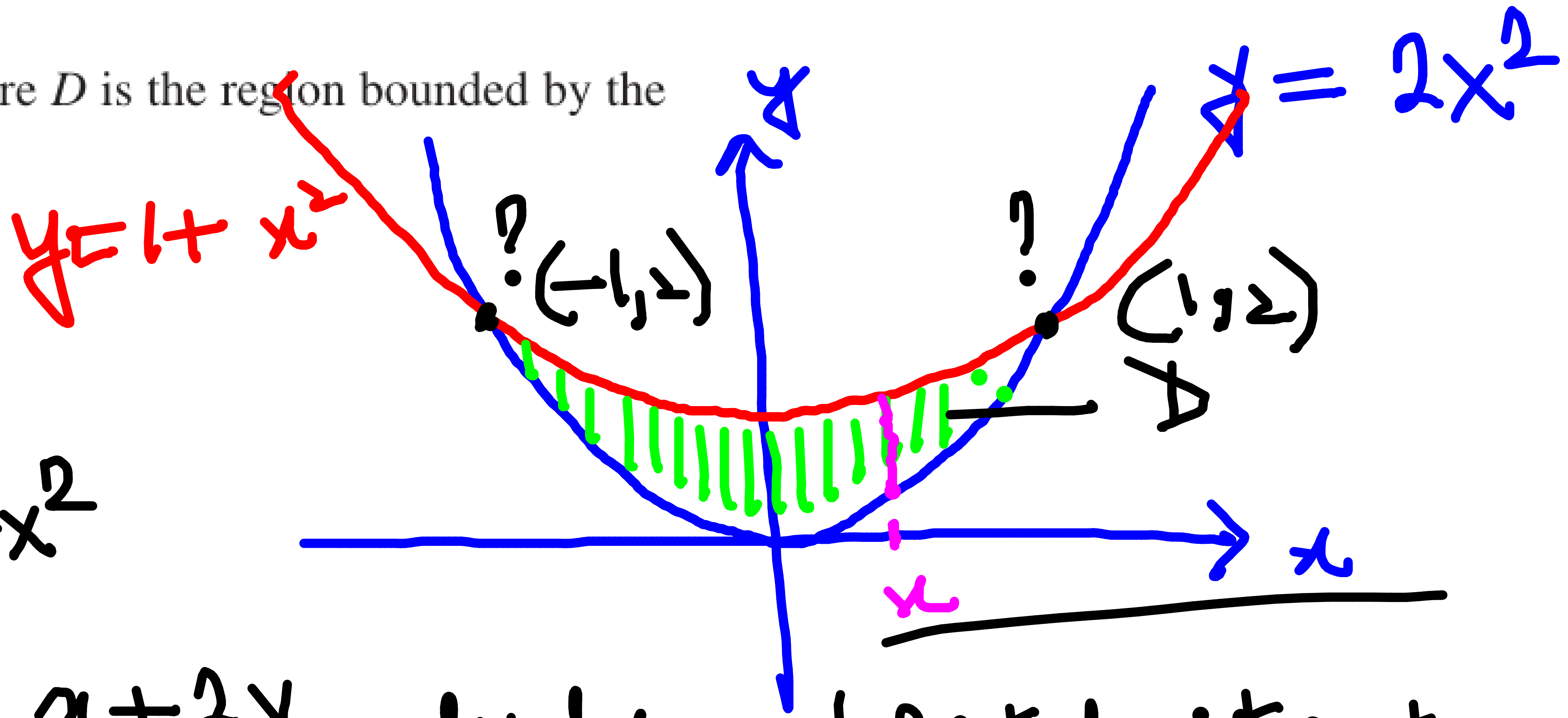
14 $$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

12.2 DOUBLE INTEGRALS OVER GENERAL REGIONS



EXAMPLE 1 Evaluate $\iint_D (x + 2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$z = x + 2y$



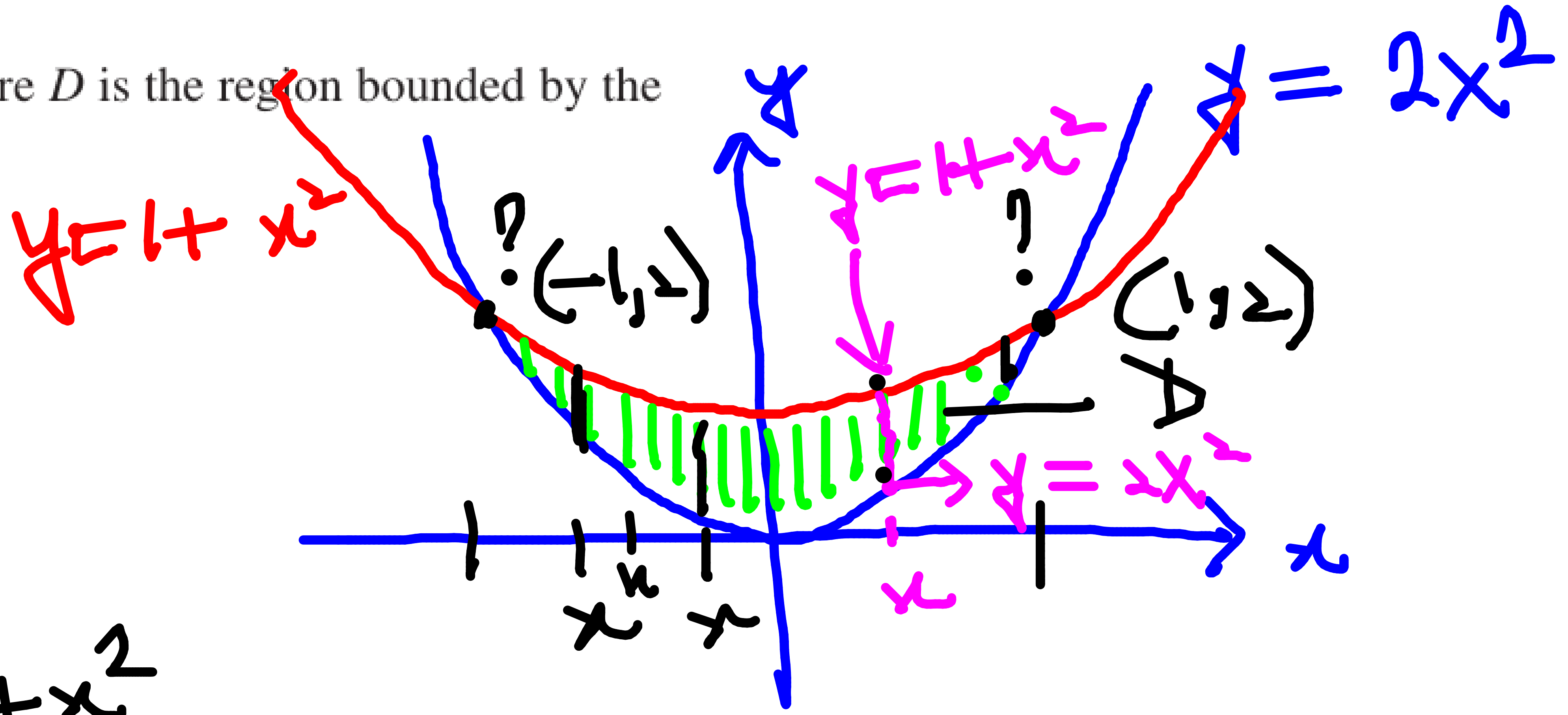
$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y \; dy\,dx$$

Point of intersection

$2x^2 = 1+x^2$

$x = \pm 1$

V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



$$z = x + 2y$$

$$\iint_D (x+2y)\, dA$$

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \; dy \; dx$$

$$\int_{2x^2}^{1+x^2} (x+2y)\,dy = \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2}$$

$$= \underbrace{x(1-x^2)}_{} + (1+x^2)^2 - (2x^2)^2$$

$$\iint_D (x+2y)\,dy = \int_{-1}^{1} x(1-x^2) + (1+x^2)^2 - (2x^2)^2 \, dx = \text{whatever} = \frac{32}{15}$$

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.