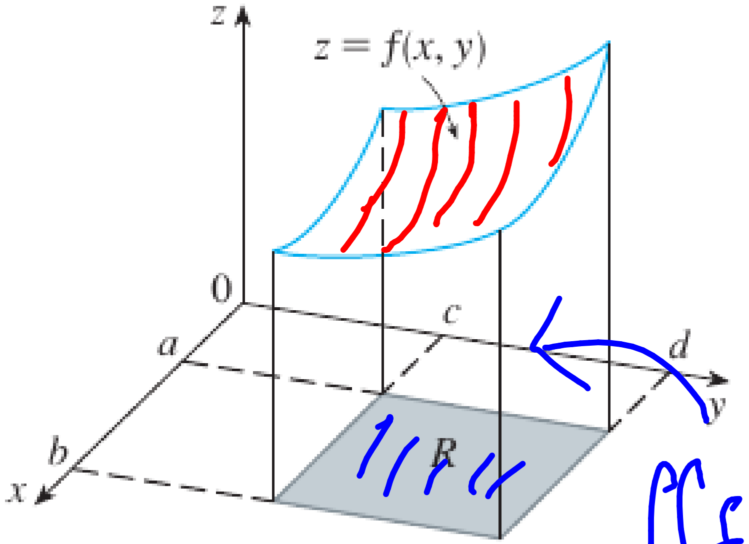
12 MULTIPLE INTEGRALS

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

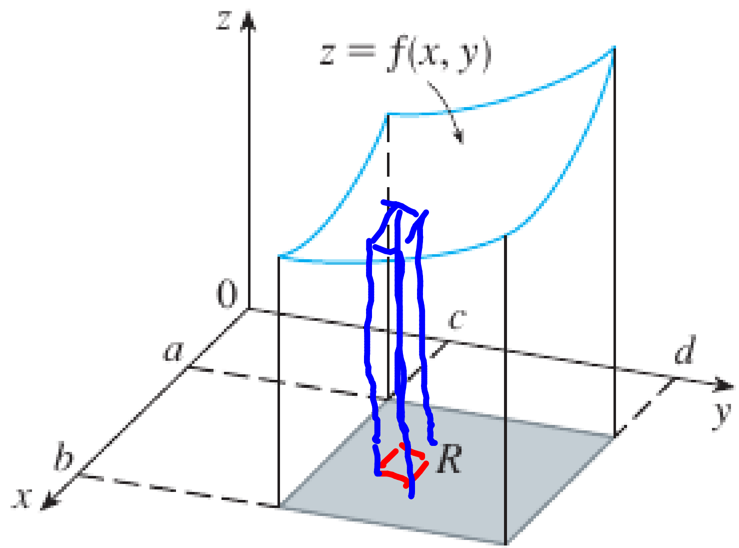


$$\boxed{V = \iint_R f(x, y)\, dA}$$

$$\iint_R f(x,y)\, dA$$

= volume under the graph of $f(x,y)$ on top of region R

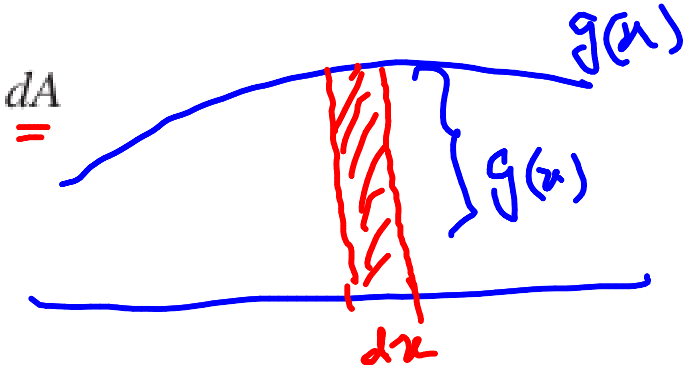
If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is



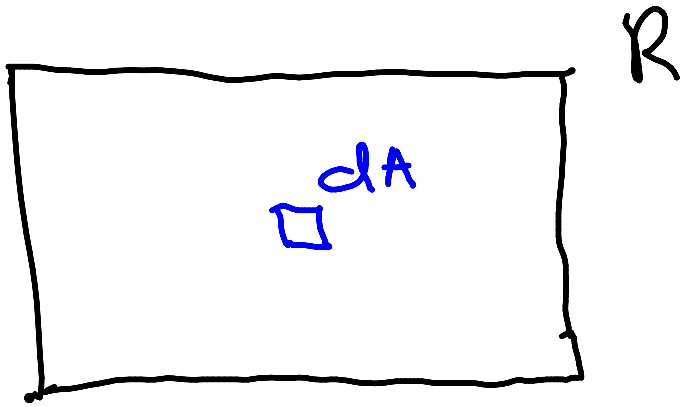
$$V = \iint_R f(x, y)\, dA$$

$$\int_a^b g(x)\, dx$$

why does this matter??



$f(x,y)\,dA$



a metal plate

$\rho(x,y)$ = density (mass/area) at point (x,y)

Q: find mass of this plate

$$dm = \rho(x,y)\, dA$$

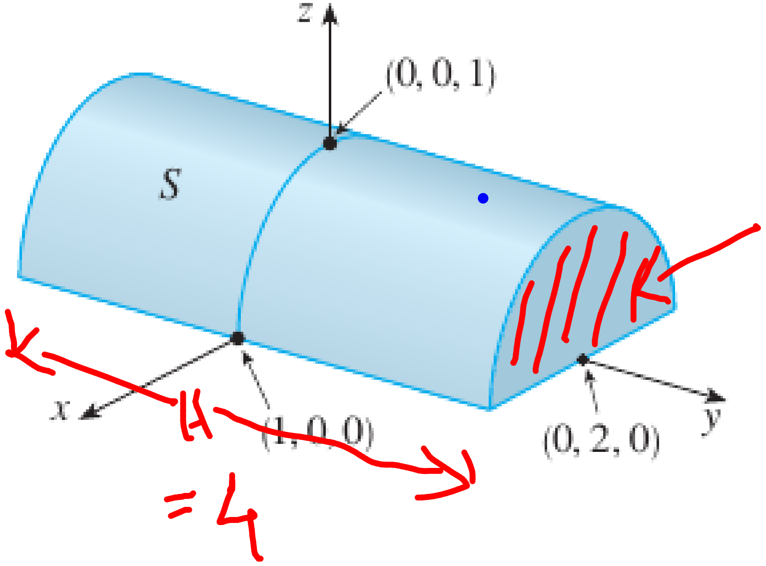
$$\text{total mass} = \iint_R dm = \iint_R \rho(x,y)\, dA$$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

$z = \sqrt{1 - x^2}$

$z^2 + x^2 = 1$

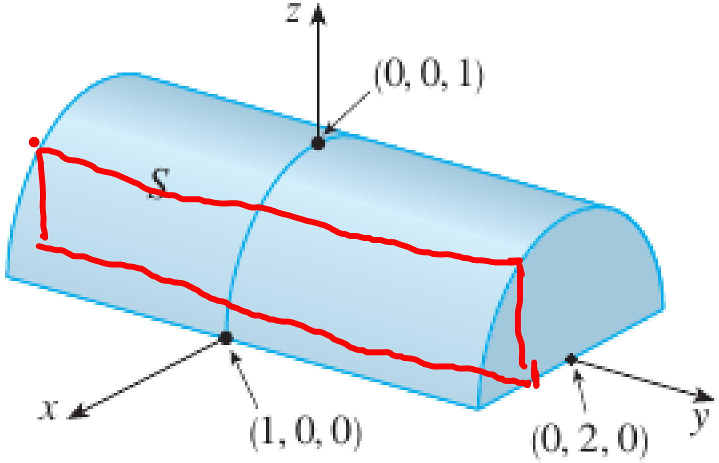


area $= \pi/2$

volume $= 4\,\dfrac{\pi}{2} = 2\pi$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$



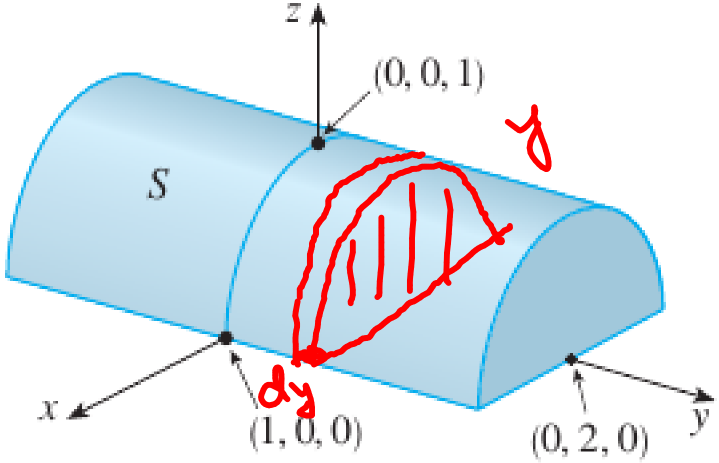
Do it with iterative integration

$$= \int_{-1}^{1} \left(\int_{-2}^{2} \sqrt{1-x^2} \, dy \right) dx$$

$$= \int_{-1}^{1} \left(\sqrt{1-x^2}\right) 4 \, dx = 4 \int_{-1}^{1} \sqrt{1-x^2} \, dx = 4 \frac{\pi}{2}$$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$



Do it with iterative integration

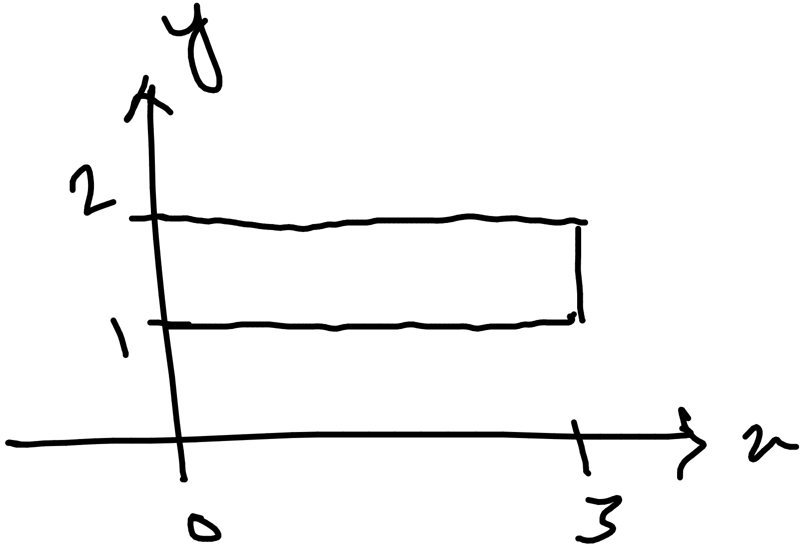
$$= \int_{-2}^{2} \underbrace{\int_{-1}^{1} \sqrt{1-x^2}\, dx}\; dy$$

$$= 2\pi$$

d. Sketch the region of integration

Evaluate the iterated integrals.

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx$$



$$= \int_0^3 \left| x^2 \frac{y^2}{2} \right|_{y=1}^{y=2} dx$$

$$= \int_0^3 \frac{3}{2} x^2 \, dx = \frac{\cancel{3}}{2} \cdot \frac{27}{\cancel{3}}$$

Evaluate the iterated integrals.

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy$$

10 FUBINI'S THEOREM If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

EXAMPLE 6 Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

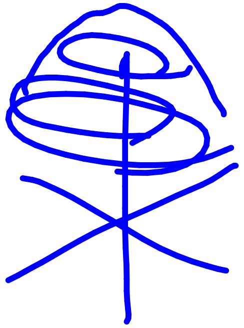
$$\int\left(\int y\sin(xy)\,dy\right)dx \qquad \text{look difficult}$$

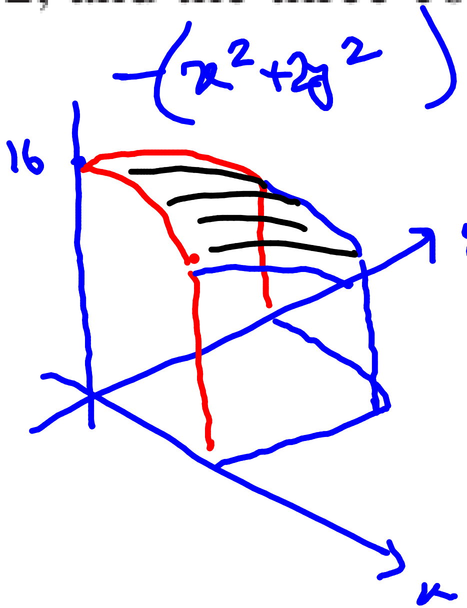
$$\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy = \int_0^{\pi} \cancel{y}\left(-\frac{\cos(xy)}{\cancel{y}}\Big|_{x=1}^{x=2}\right)dy$$

$$= \int_0^{\pi}\left[\cos(y) - \cos(2y)\right]dy$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

$$z = 16 - x^2 - 2y^2$$





→ sketch a nice volume

→ set up the volume formula as a double integration

$$= \int_0^2 \int_0^2 \left(16 - x^2 - 2y^2\right) dy\,dx = 48$$

PROPERTIES OF DOUBLE INTEGRALS

We list here three properties of double integrals that can be proved in the same manner as in Section 5.2. We assume that all of the integrals exist. Properties 12 and 13 are referred to as the *linearity* of the integral.

12 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

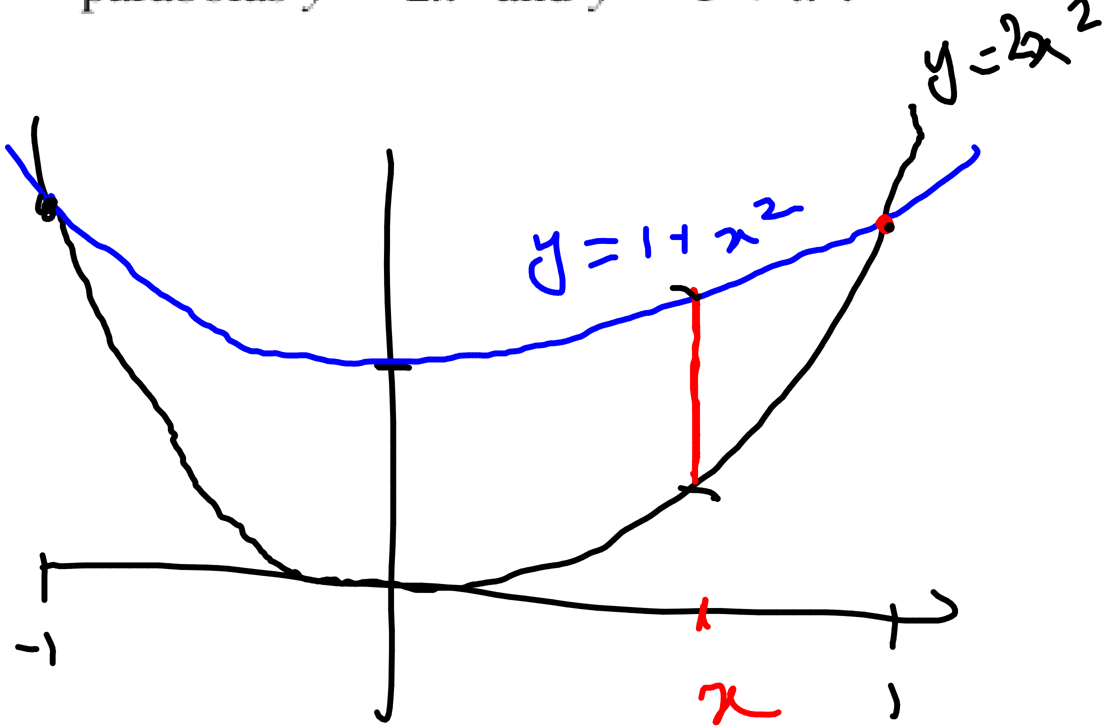
13 $$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

14 $$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

12.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



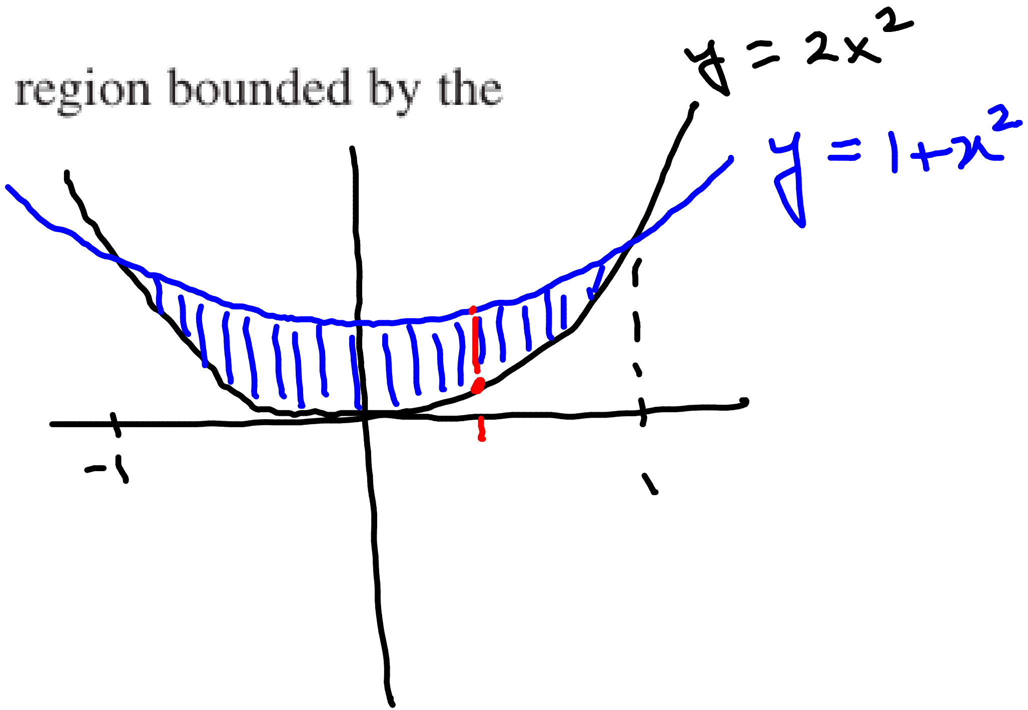
$$= \iint_D (x+2y)\, dA$$

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left. xy + y^2 \right|_{y=2x^2}^{y=1+x^2} dx .$$

$$= \int_{-1}^{1} x(1-x^2) + (1+x^2)^2 - (2x^2)^2 \, dx = \frac{32}{15}$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



$$\int_{?}^{?}\int_{?}^{?} \quad dx\, dy$$

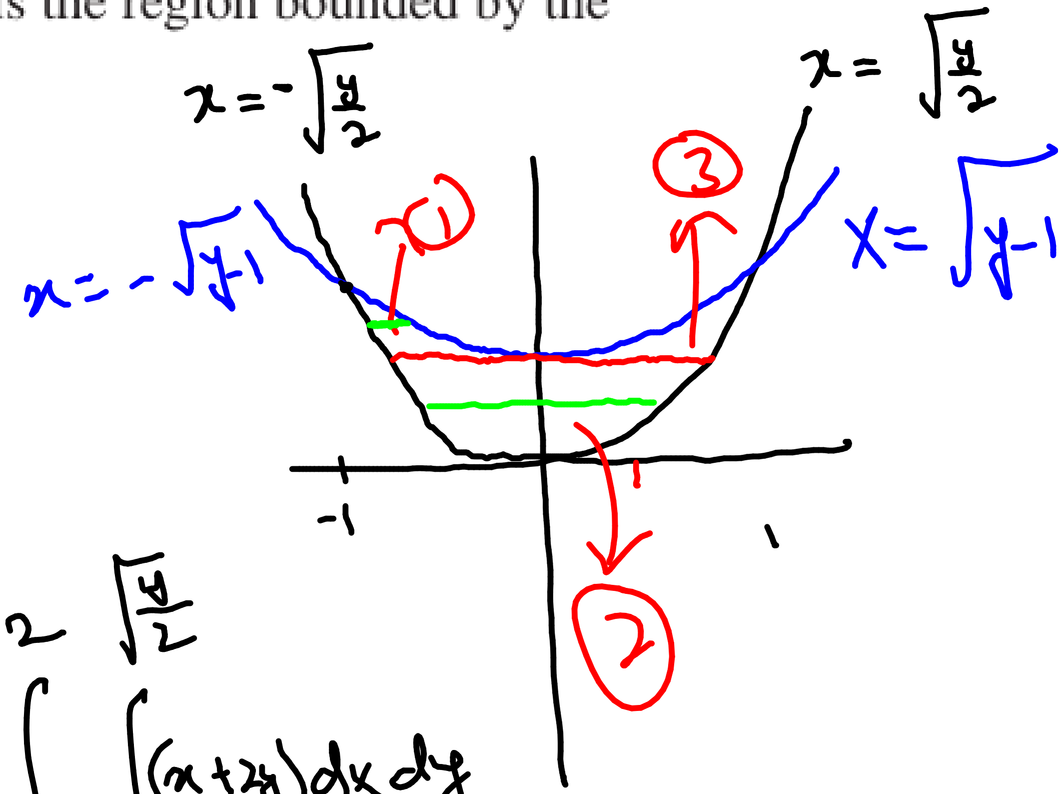
$$= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2x) \quad dy\, dx$$

V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$\int_?^? \int_?^? dx\, dy$$

$$= \iint_{①} + \iint_{②} + \iint_{③}$$

$$= \int_1^2 \int_{-\sqrt{y/2}}^{-\sqrt{y-1}} (x+2y)\, dx\, dy + \int_0^1 \int_{-\sqrt{y/2}}^{\sqrt{y/2}} (x+2y)\, dy + \int_1^2 \int_{\sqrt{y-1}}^{\sqrt{y/2}} (x+2y)\, dx\, dy$$

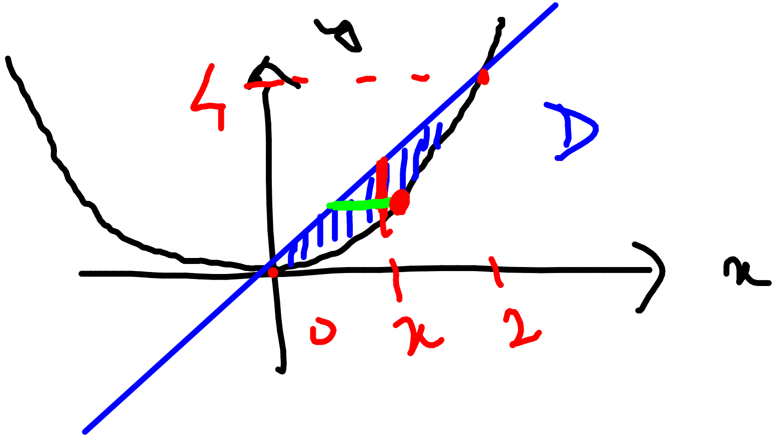


EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

→ sketch D

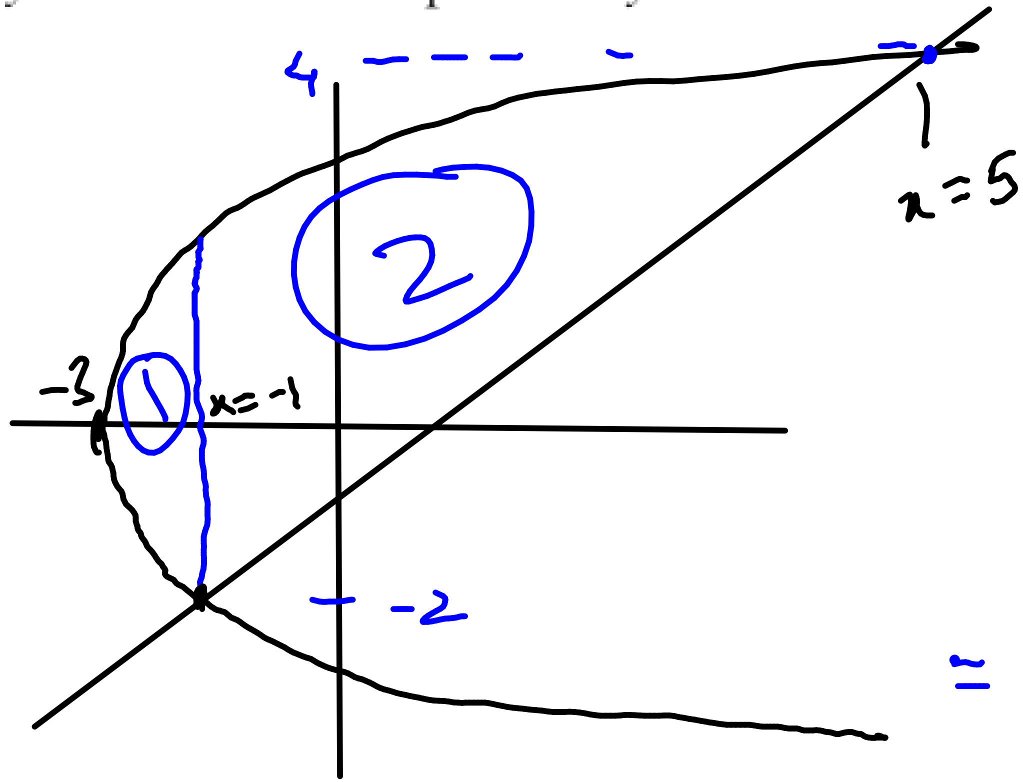
→ graph z

→ sketch the volume



$$\text{Volume} = \iint_D (x^2+y^2)\, dA = \int_0^2 \int_{x^2}^{2x} x^2+y^2 \, dy\, dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2) \, dx\, dy$$

V EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



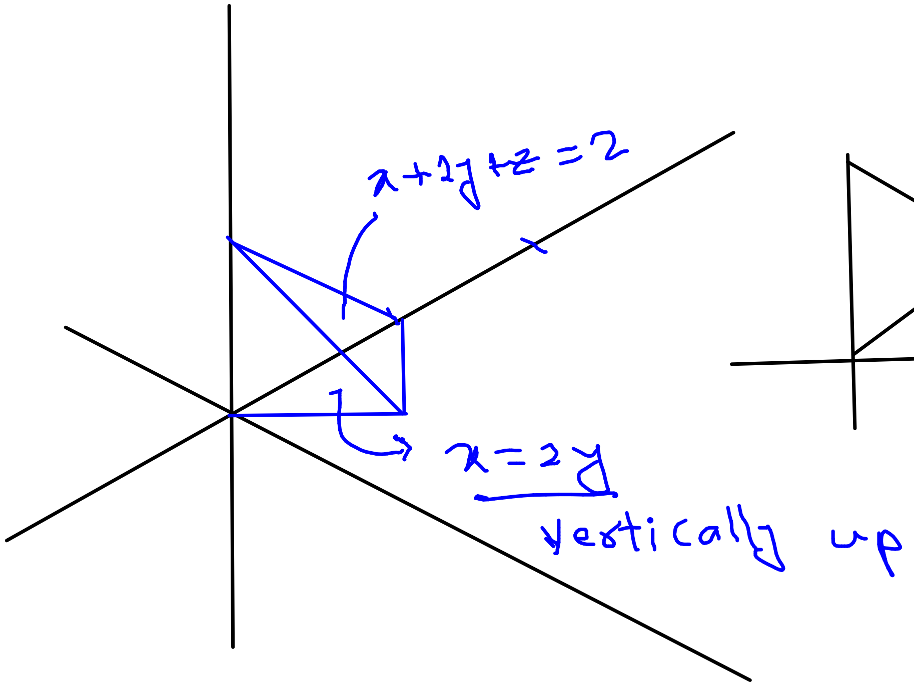
points of intersection

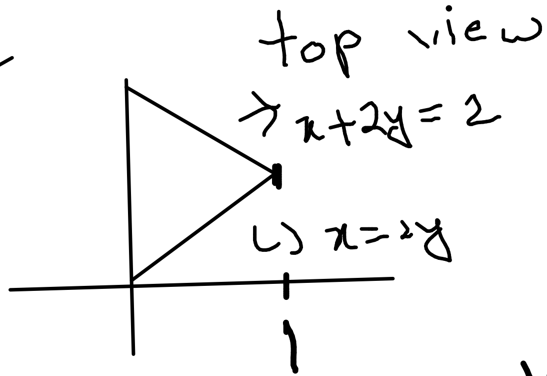
$$y+1 = \frac{y^2-6}{2} \Rightarrow y = -2, 4$$

$$= \int_{-2}^{4}\int_{\frac{(y^2-6)}{2}}^{y-1} xy \; dx\,dy$$

$$= \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \; dy\,dx + \int_{-1}^{5}\int_{x+1}^{\sqrt{2x+6}} xy \; dy\,dx$$

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

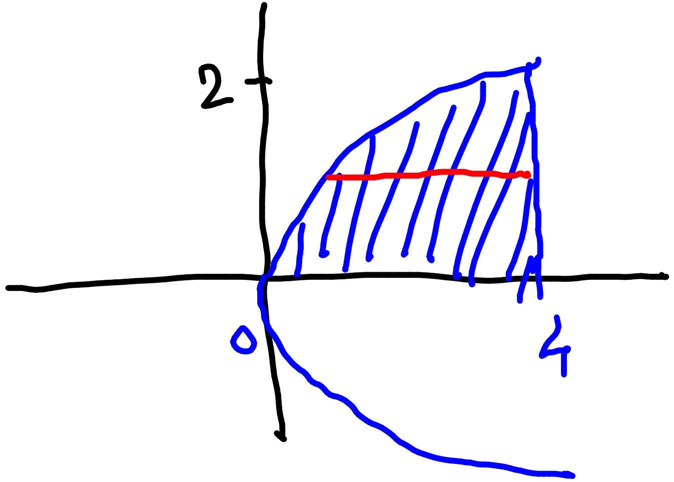




$$\text{Volume} = \int_0^1 \int_{x/2}^{(2-x)/2} (2-x-2y)\,dy\,dx$$

Sketch the region of integration and change the order of integration.

$$\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$$



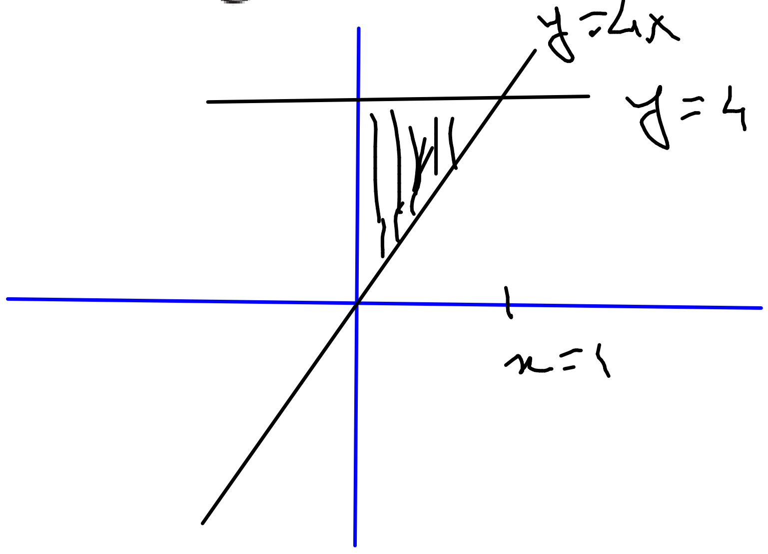
$y = \sqrt{x}$

$x = y^2$

$$= \int_0^2 \int_{y^2}^4 f(x,y)\, dx\, dy$$

Sketch the region of integration and change the order of integration.

$$\int_0^1 \int_{y=4x}^{y=4} f(x,y)\,dy\,dx$$



$$= \int_0^4 \int_0^{y/4} f(x,y)\,dx\,dy$$

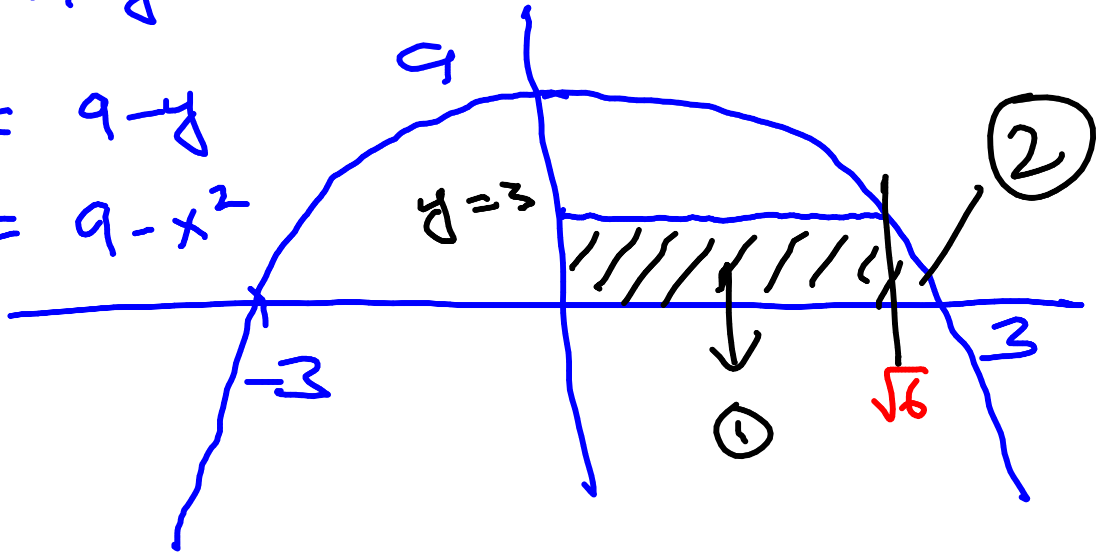
Sketch the region of integration and change the order of integration.

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y)\, dx\, dy$$

$$x = \sqrt{9-y}$$

$$x^2 = 9 - y$$

$$y = 9 - x^2$$



$$= \int_0^{\sqrt{6}} \int_0^{3} f(x,y)\, dx + \int_{\sqrt{6}}^{3} \int_0^{9-x^2} f(x,y)\, dx$$