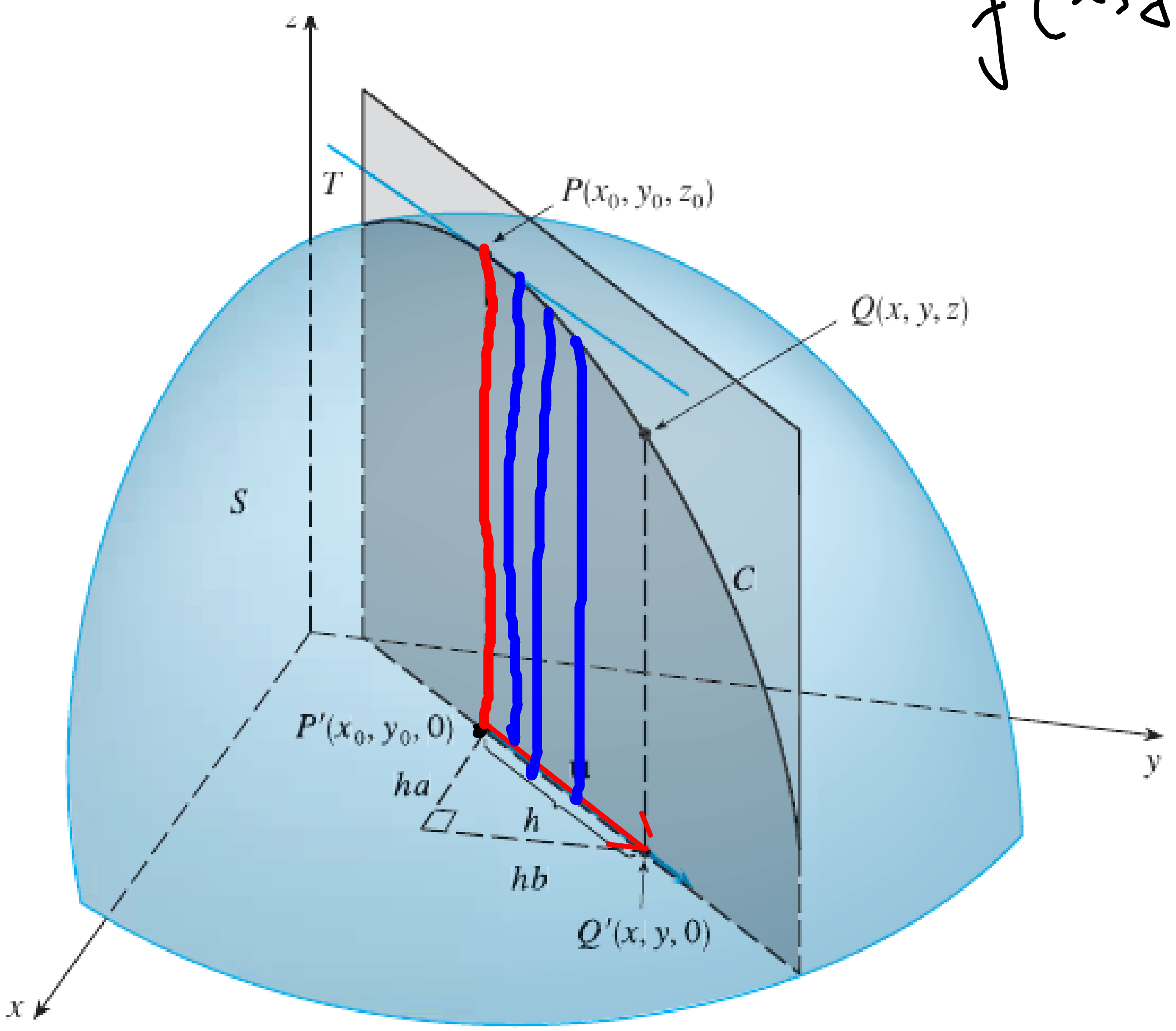
11.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

$f(x,y)$

$\hat{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$
$= u_1\hat{i} + u_2\hat{j}$



2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

$$D_{\hat{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

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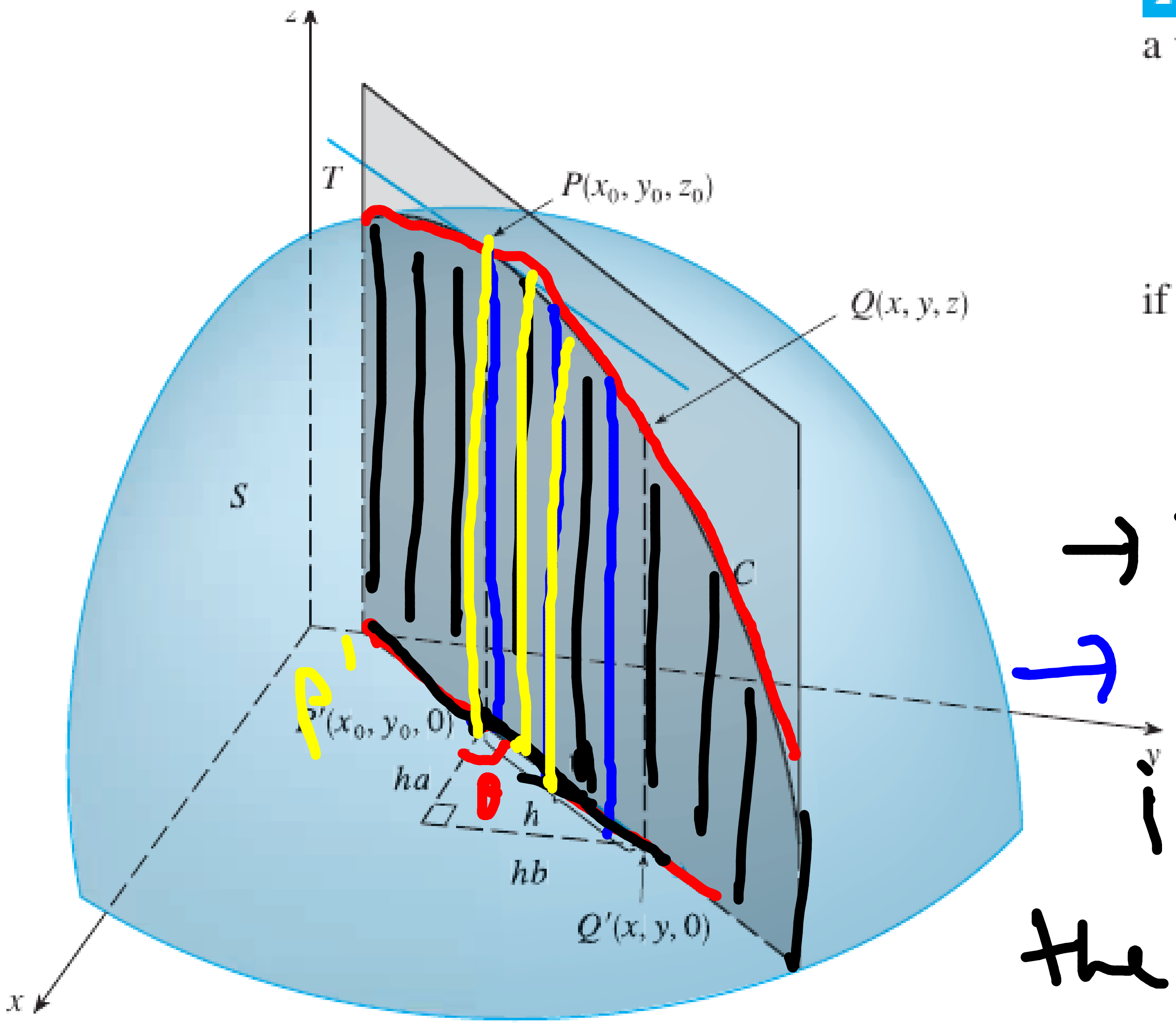

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if this limit exists.

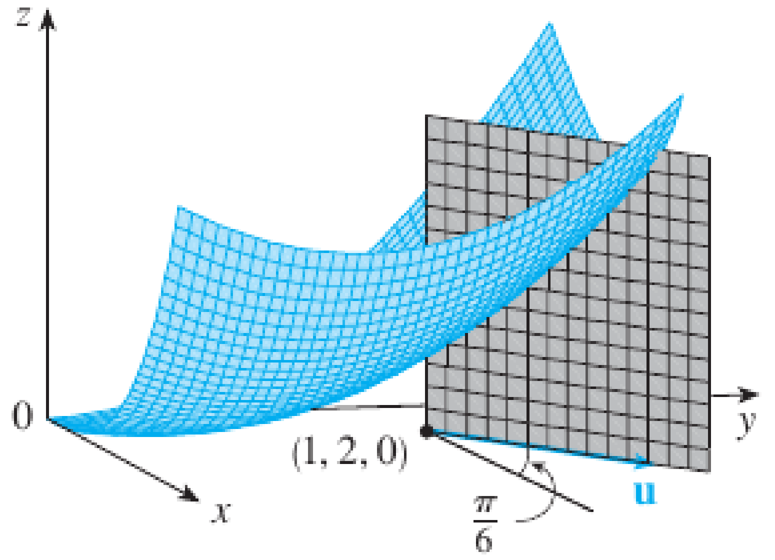
graph of f(x,y)

→ Directional derivative of
→ f(x,y), at P'(x₀,y₀)
in the direction û, is
the rate of change you observe

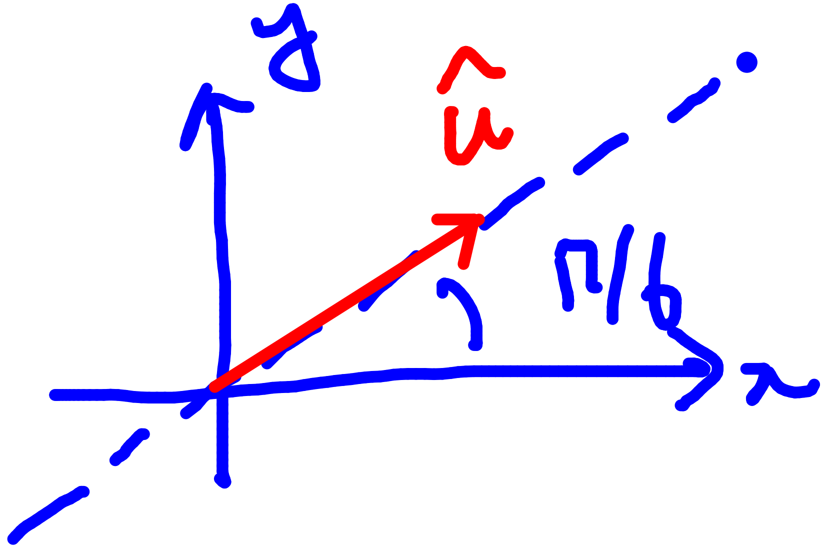
3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

EXAMPLE 1 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?



$$f(x,y) = x^3 - 3xy + 4y^2$$



$$\hat{u} = \cos\frac{\pi}{6}\,\hat{i} + \sin\frac{\pi}{6}\,\hat{j}$$

$$D_{\hat{u}} f(1,2) = \frac{\partial f}{\partial x}(1,2)\cos\frac{\pi}{6} + \frac{\partial f}{\partial y}(1,2)\sin\pi/6$$

$$= \left(\frac{13}{2} - \frac{3\sqrt{3}}{2}\right)$$

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Δ delta

∇ gradient

$f(x,y)$

gradient $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

Q: $f(x,y) = x^3 - 3xy + 4y^2$

$$\nabla f(1,2) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = -3\hat{i} + 13\hat{j}$$

note:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$\hat{u} = u_1\hat{i} + u_2\hat{j}$$

$$D_{\hat{u}} f = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 = \nabla f \cdot \hat{u}$$

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\,\mathbf{j}$.

the direction vector must be a unit vector

$$\hat{v} = \frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j} = \frac{\vec{v}}{|v|}$$ ← length

$$\nabla f(2,-1) = \frac{\partial f}{\partial x}(2,-1)\,\hat{i} + \frac{\partial f}{\partial y}(2,-1)\,\hat{j}$$

$$= -4\hat{i} + 8\hat{j}$$

$$D_{\hat{v}} f = \left(-4\hat{i}+8\hat{j}\right)\cdot\left(\frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}\right) = \frac{32}{\sqrt{29}}$$

EXAMPLE 4 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \sin yz\,\hat{i} + xz\cos(yz)\,\hat{j} + xy\cos(yz)\,\hat{k}$$

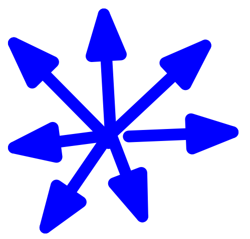
$$= \left(0\hat{i} + 0\hat{j} + 3\hat{k}\right)$$

$$\vec{v} \rightarrow \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}\right)$$

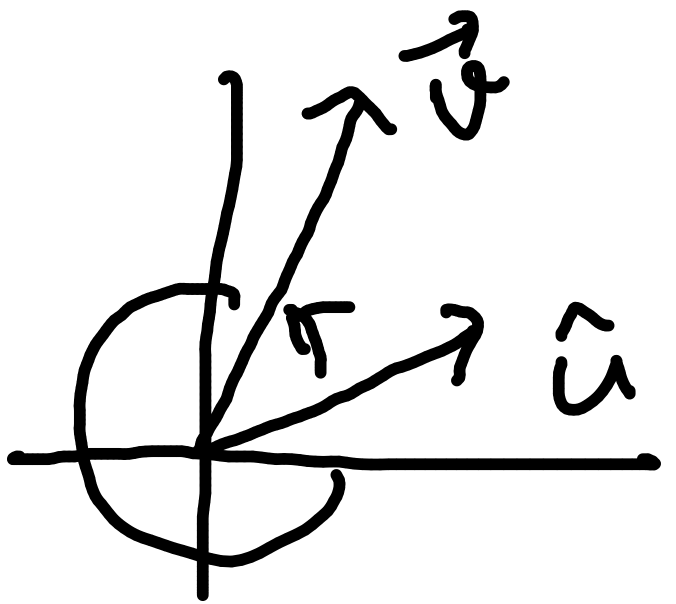
$$D_{\hat{v}}f = \nabla f \cdot \hat{v} = \frac{-3}{\sqrt{6}}$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$



Q: what direction $\hat{u}$ we should choose s.t. $D_{\hat{u}} f$ is maximum among all possible $\hat{u}$.



$\hat{u}$: free to revolve

$\vec{v} \cdot \hat{u} = |\vec{v}| \cos\theta$ is max when $\theta = 0$

in conclusion:

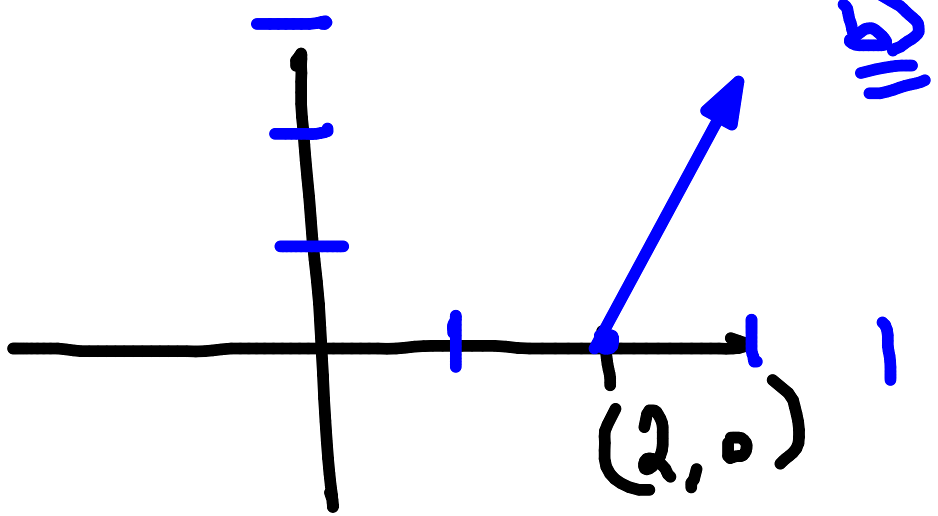
$D_{\hat{u}} f = \nabla f \cdot \hat{u}$ is maximum

if $\hat{u} = \dfrac{\nabla f}{|\nabla f|}$ = parallel to ∇f

EXAMPLE 5

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?



b) the direction of steepest ascent is

$$\frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$$

$$\nabla f = e^y\hat{i} + xe^y\hat{j}$$

$$\nabla f(2,0) = 1\hat{i} + 2\hat{j}$$

rate of change

$$(\nabla f)\cdot\frac{(\nabla f)}{|\nabla f|} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

TANGENT PLANES TO LEVEL SURFACES

Find equations of (a) the tangent plane and (b) the nor mal line to the given surface at the specified point.

$$x^2 - 2y^2 + z^2 + yz = 2, \quad (2, 1, -1)$$

Find equations of (a) the tangent plane and (b) the nor mal line to the given surface at the specified point.

$$yz = \ln(x + z), \quad (0, 0, 1)$$