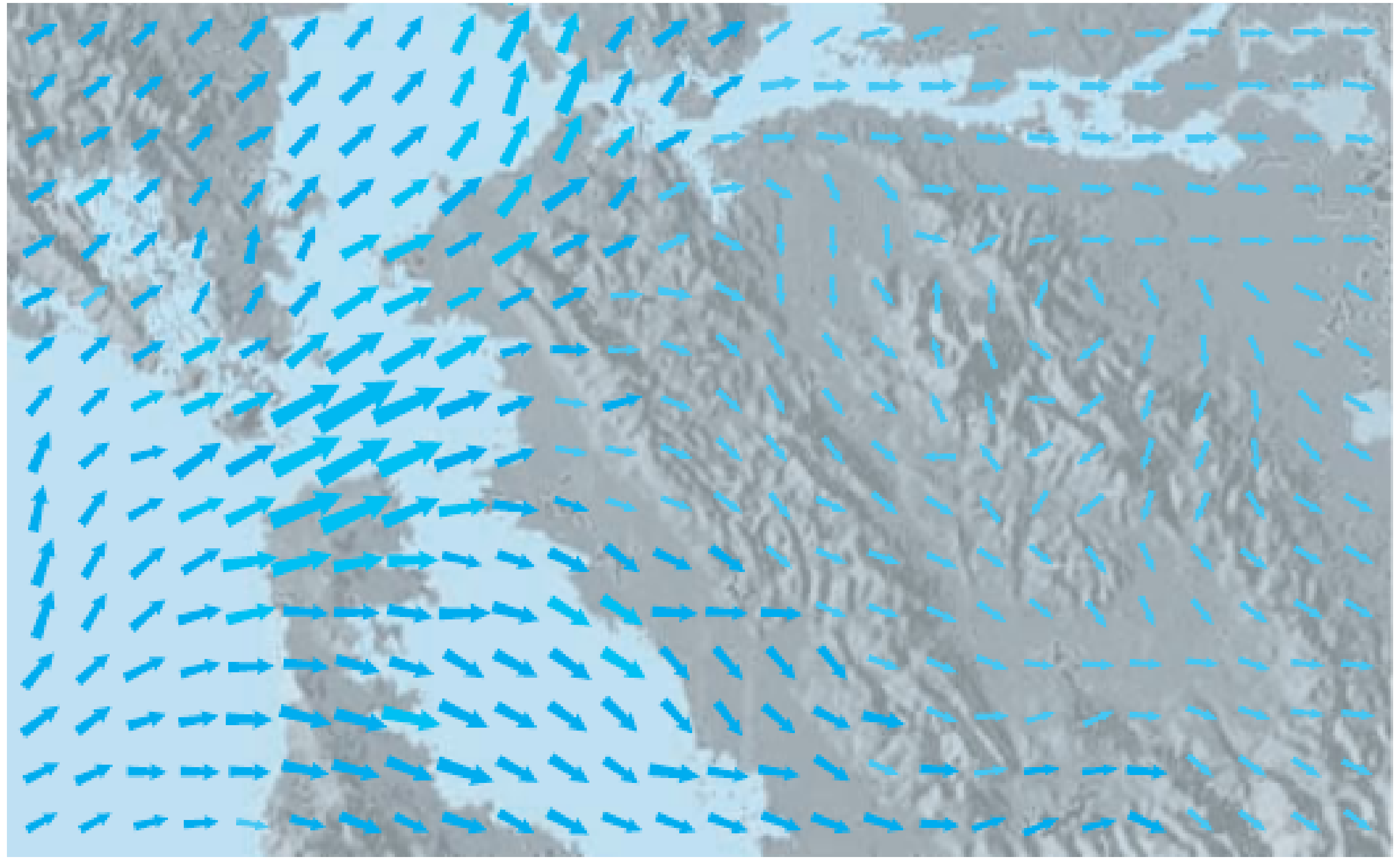
13 VECTOR CALCULUS

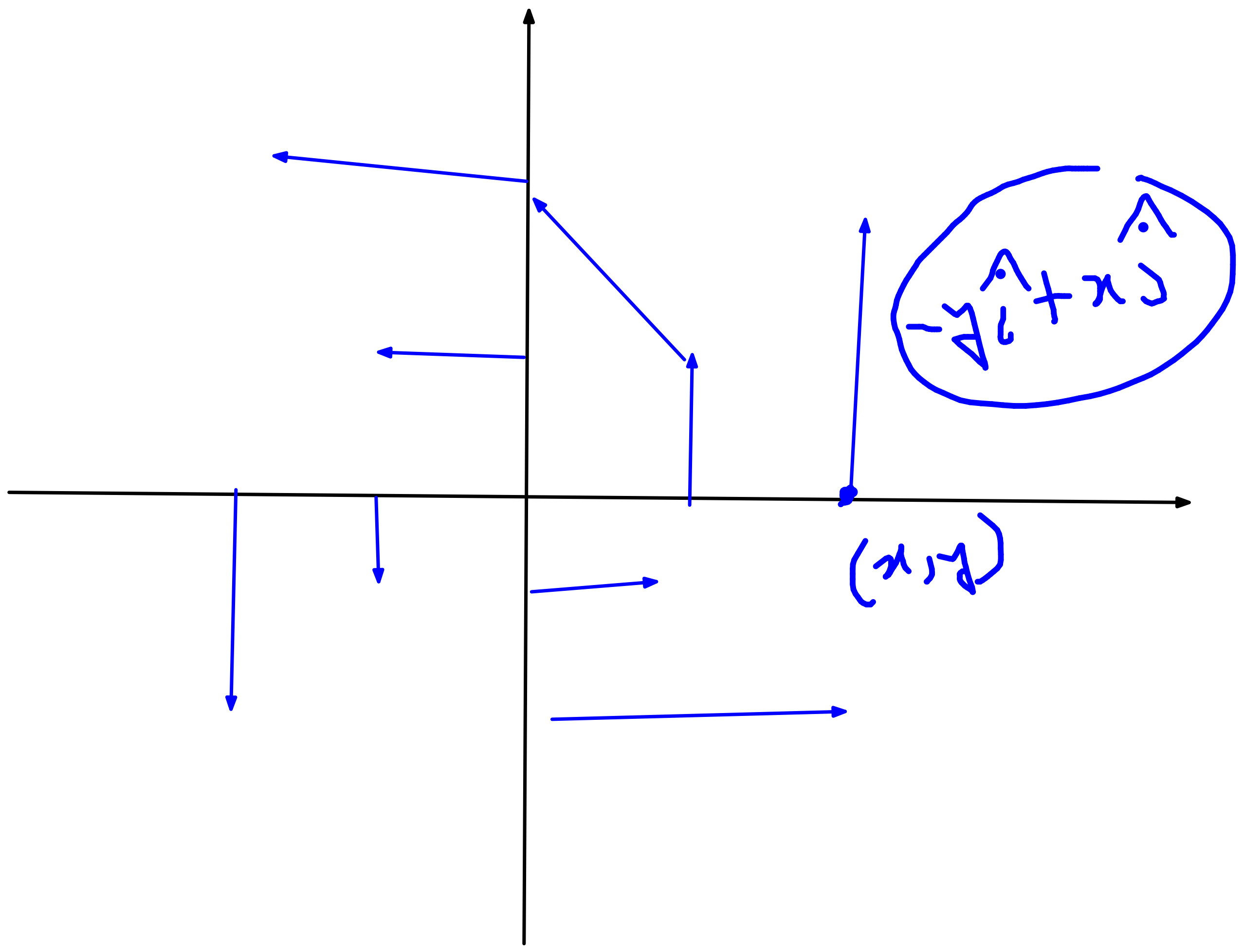
started last time

13.1 VECTOR FIELDS



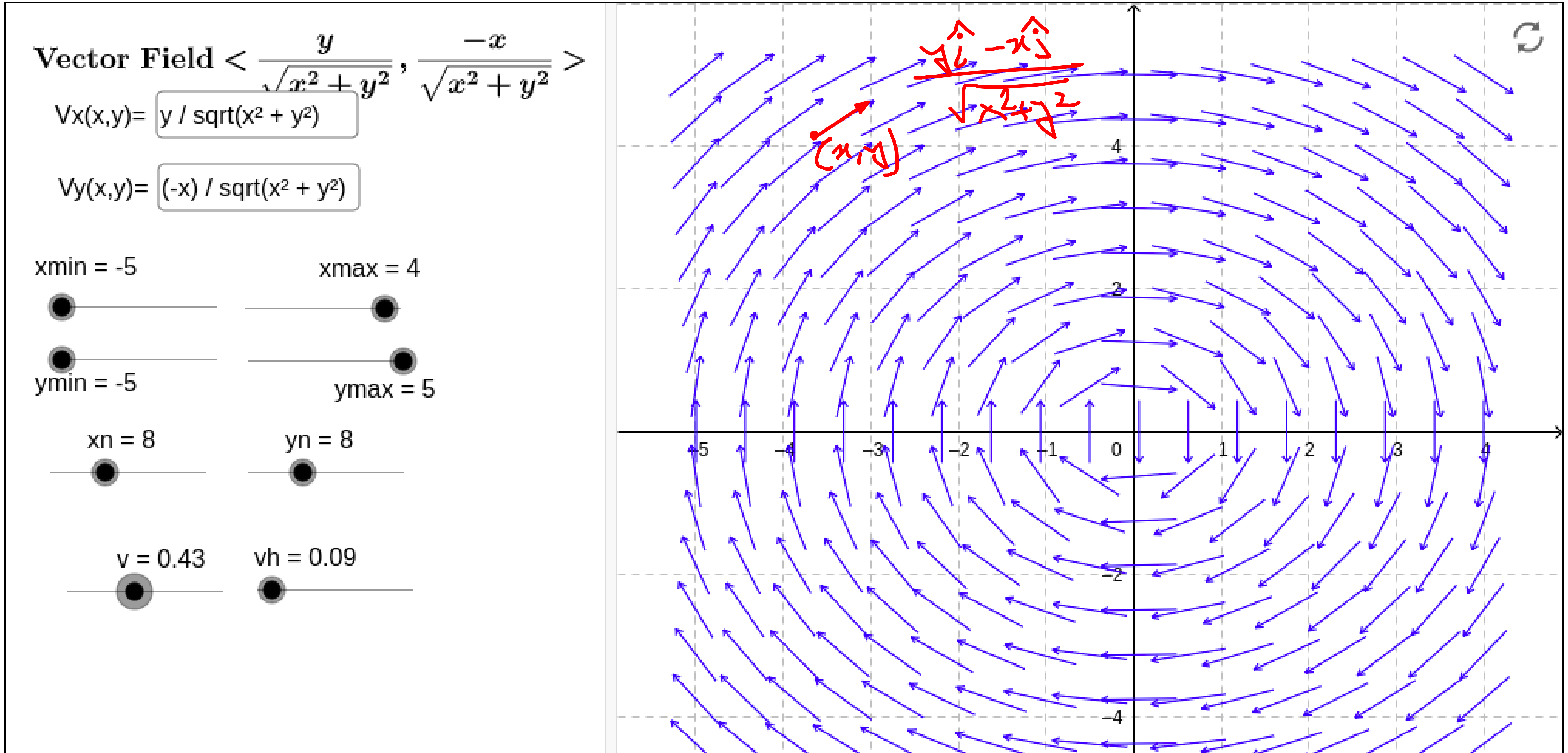
Sketch the vector field **F**

$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}.$$



Sketch the vector field **F** $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

geogebra.org/m/QPE4PaDZ

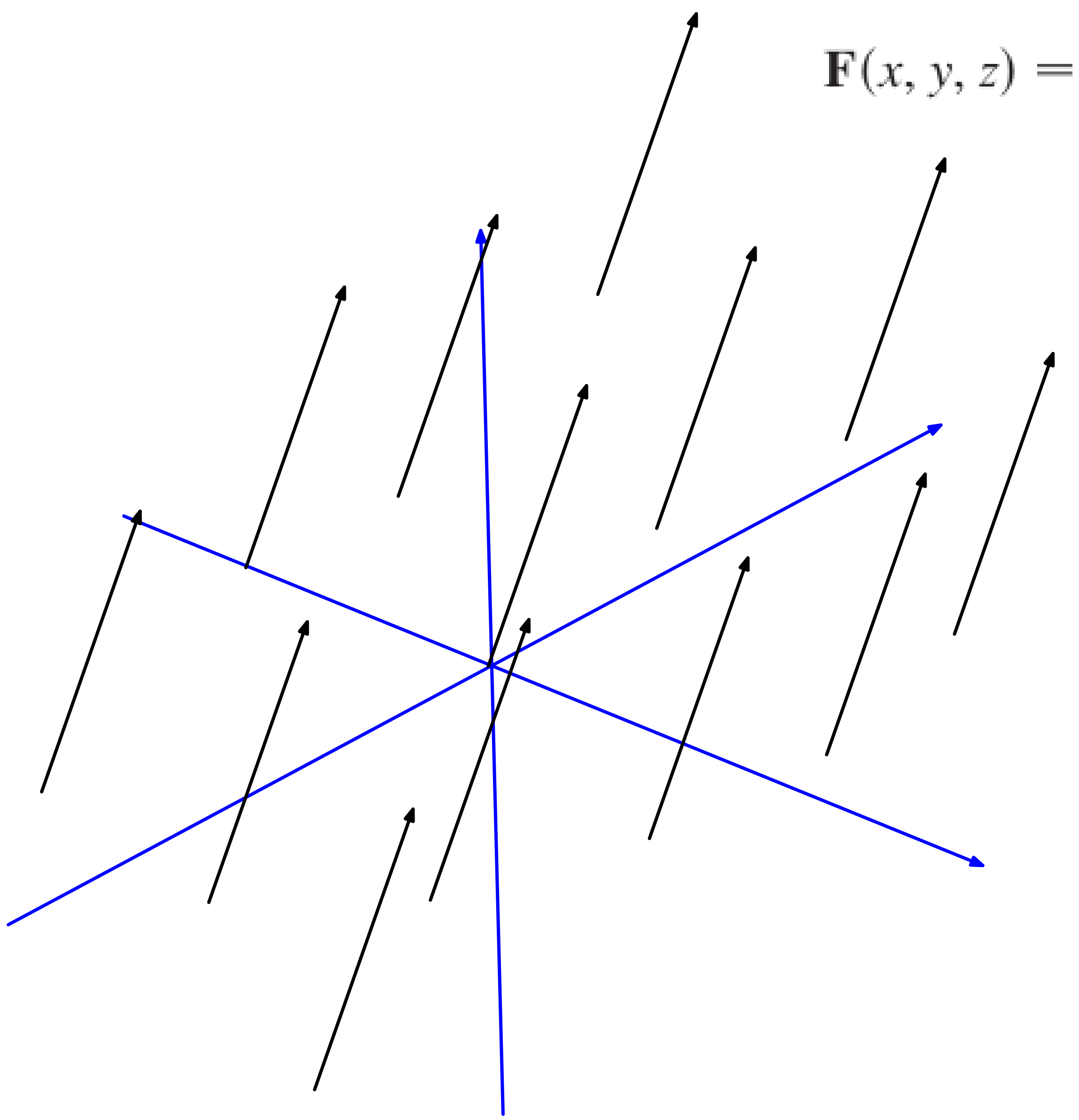


Sketch the vector field **F**

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$$

Sketch the vector field **F**

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$$



Sketch the vector field **F**

$$\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$$

Vector Fields

(point in plane) $\rightarrow$ vector

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x,y) = P\,\hat{i} + Q\,\hat{j}$$

$$= F_1\,\hat{i} + F_2\,\hat{j}$$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\vec{F}(x,y,z) = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$$

$$= F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

GRADIENT FIELDS

| 13.1
later

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

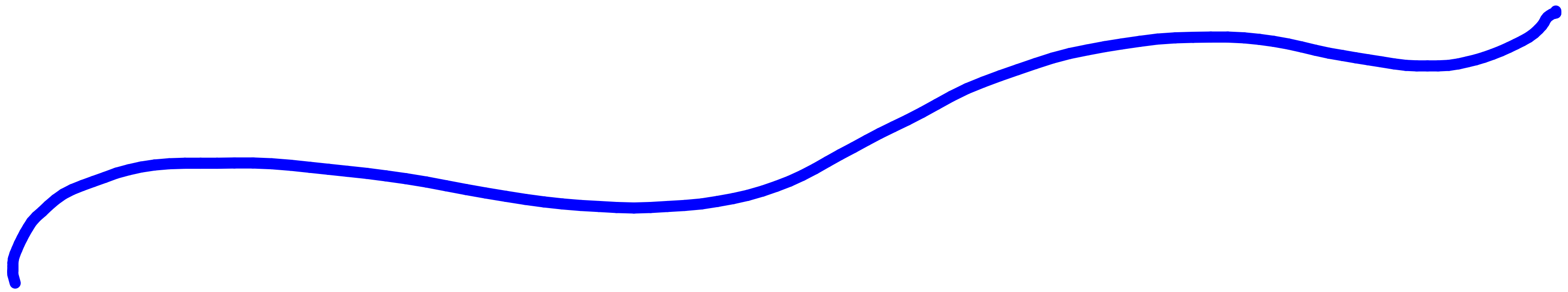
V EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

Recall position function from physics

→ a particle is moving in a path



→ $\vec{r}(t)$ = position of particle at time t

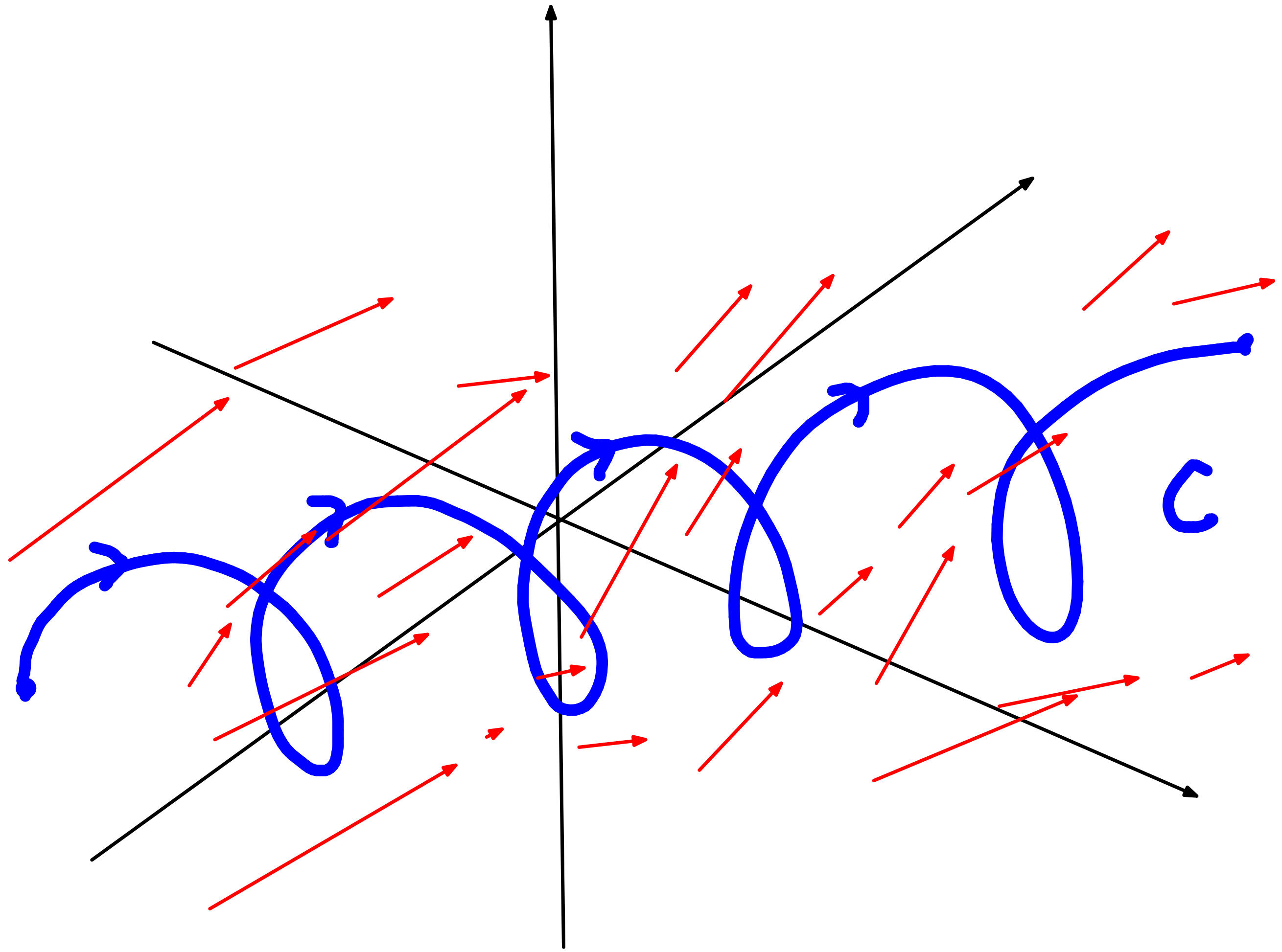
$= (x(t), y(t), z(t))$

→ Here we refer to these $\vec{r}(t)$ as curves.

& t is a parameter for the curve

We will focus for now:

Calculus on curves or paths.



→ Length of the curve?

→ $\int_C f \, d\vec{r}$: integration of scalar function

→ $\int_C \vec{F} \cdot d\vec{r}$: integration of vector functions on curves.

13.2 LINE INTEGRALS

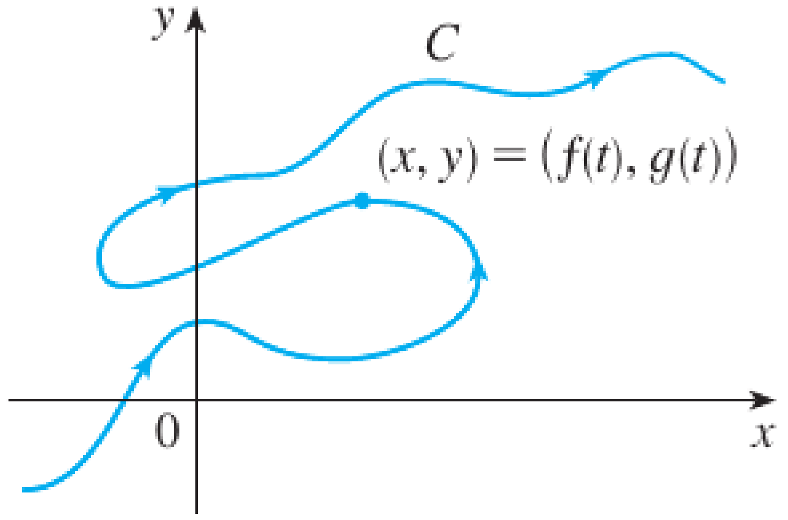
next time

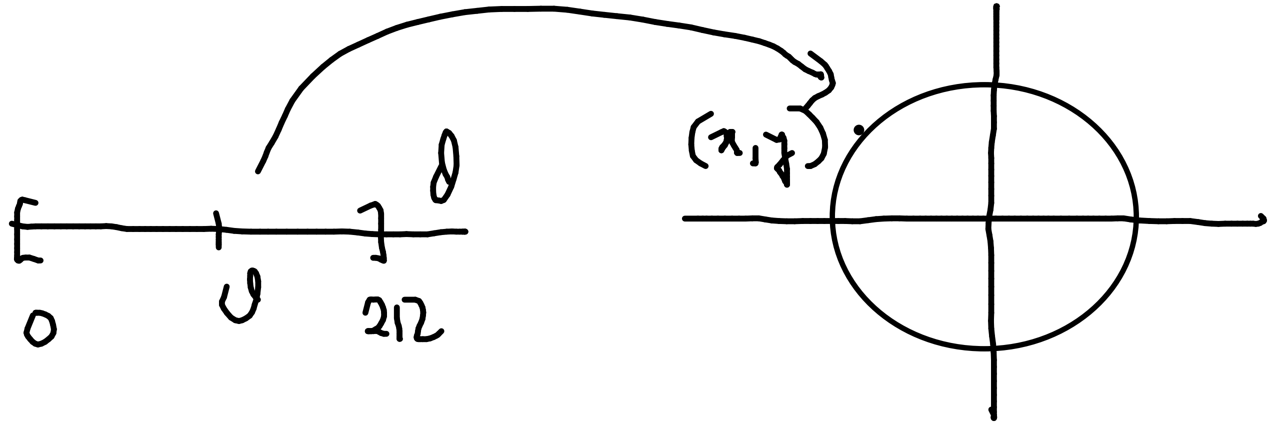
$\vec{r}(t) = (x(t), y(t))$

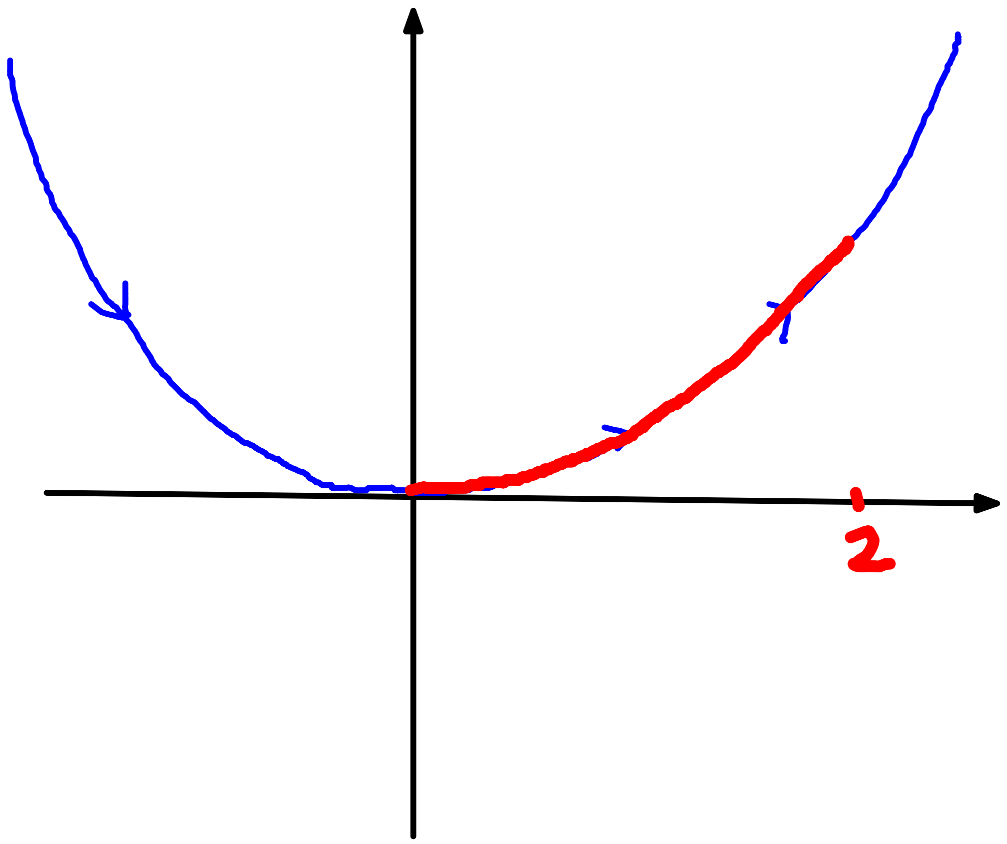
9.1 PARAMETRIC CURVES

:

$$\boxed{\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned} \qquad 0 \le \theta \le 2\pi}$$





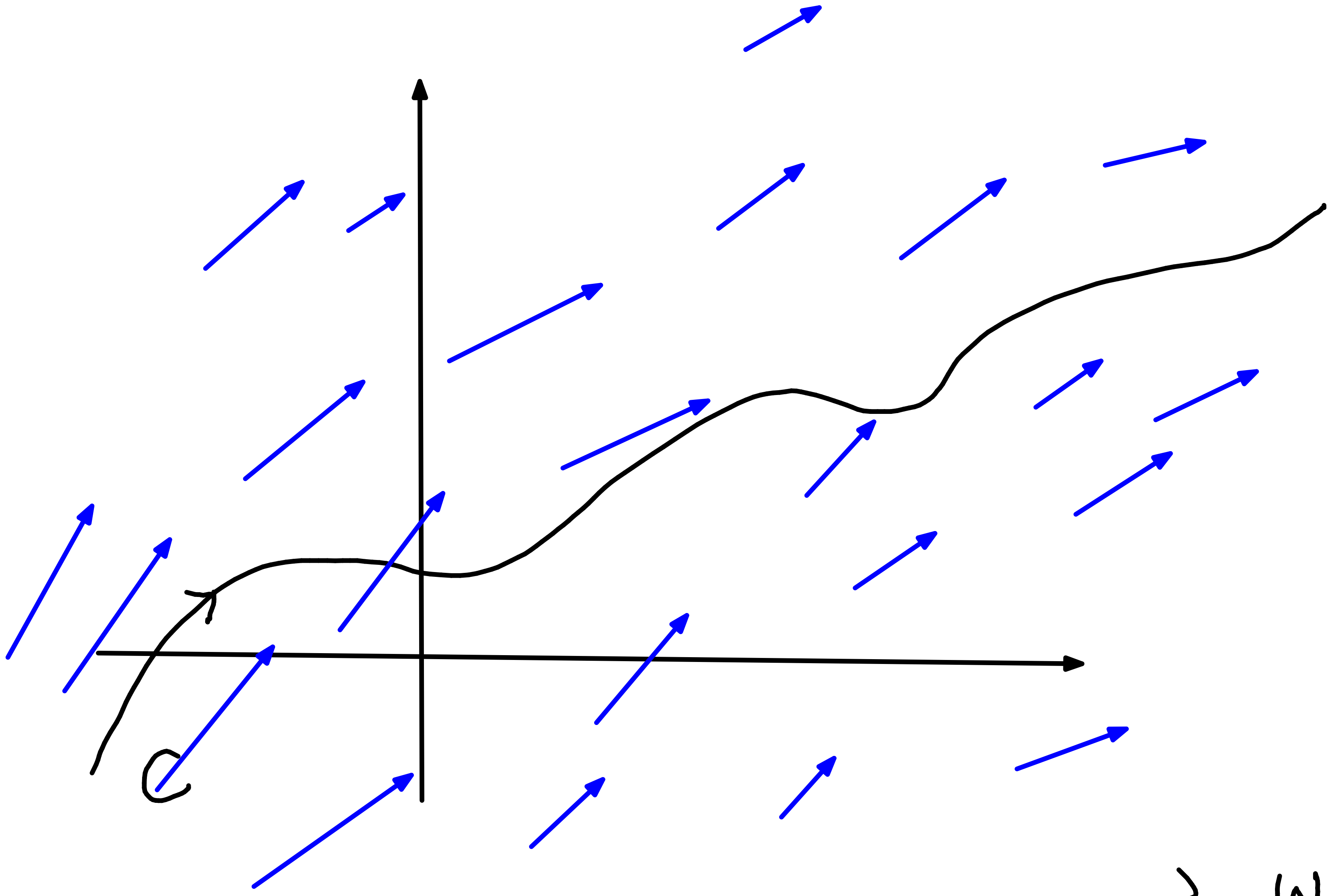


$y = x^2$

$$\left[\begin{array}{l} -\infty \le t \le \infty \\ x = t \\ y = t^2 \end{array}\right]$$

$$\boxed{\begin{array}{l} 0 \le t \le 2 \\ x = t \\ y = t^2 \end{array}} \longrightarrow$$

what curve ??



- Force field $\vec{F}$
- a particle is moving along a curve C

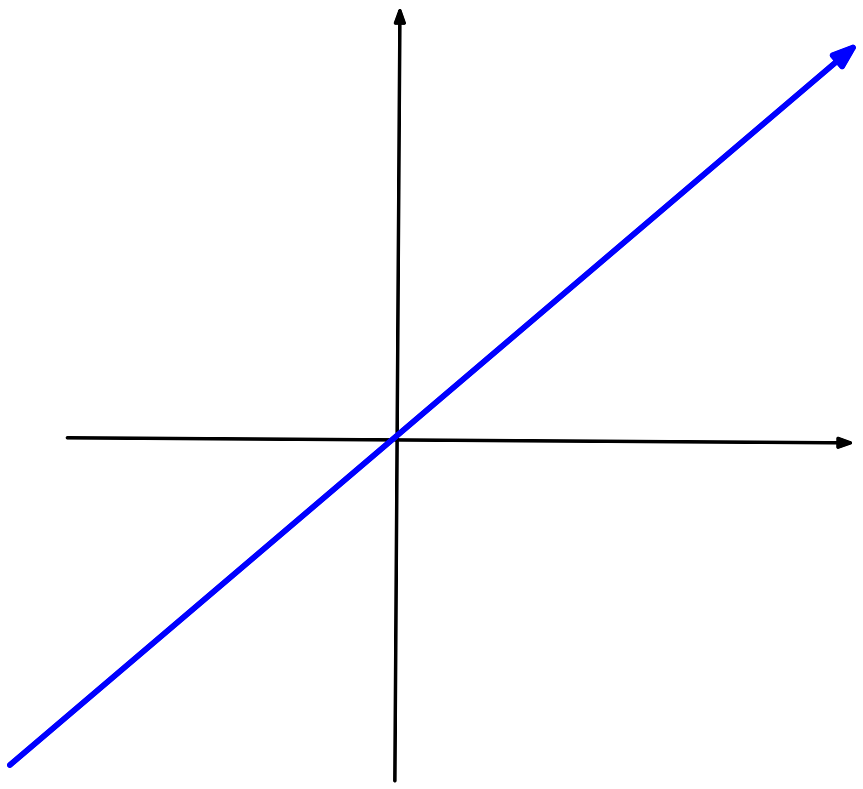
Q: find work done by $\vec{F}$ in moving the particle along the curve ??

→ we need a better precise description of whats a curve or path.

$-6\pi \leq t \leq 6\pi$

$x = t$

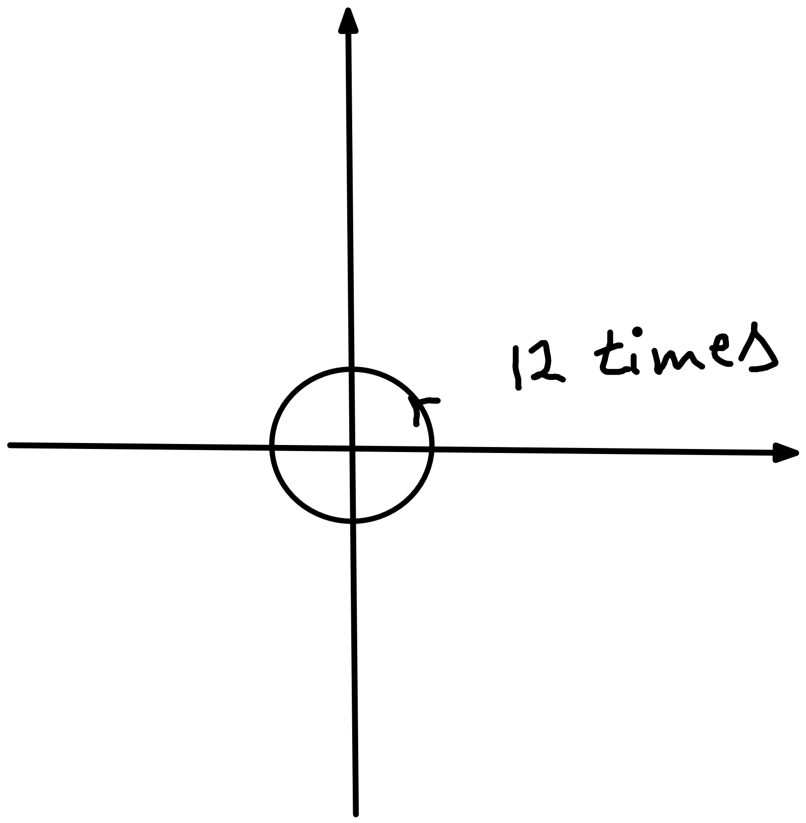
$y = t$



$-6\pi \leq t \leq 6\pi$

$x = 0.1 \cos(2t)$

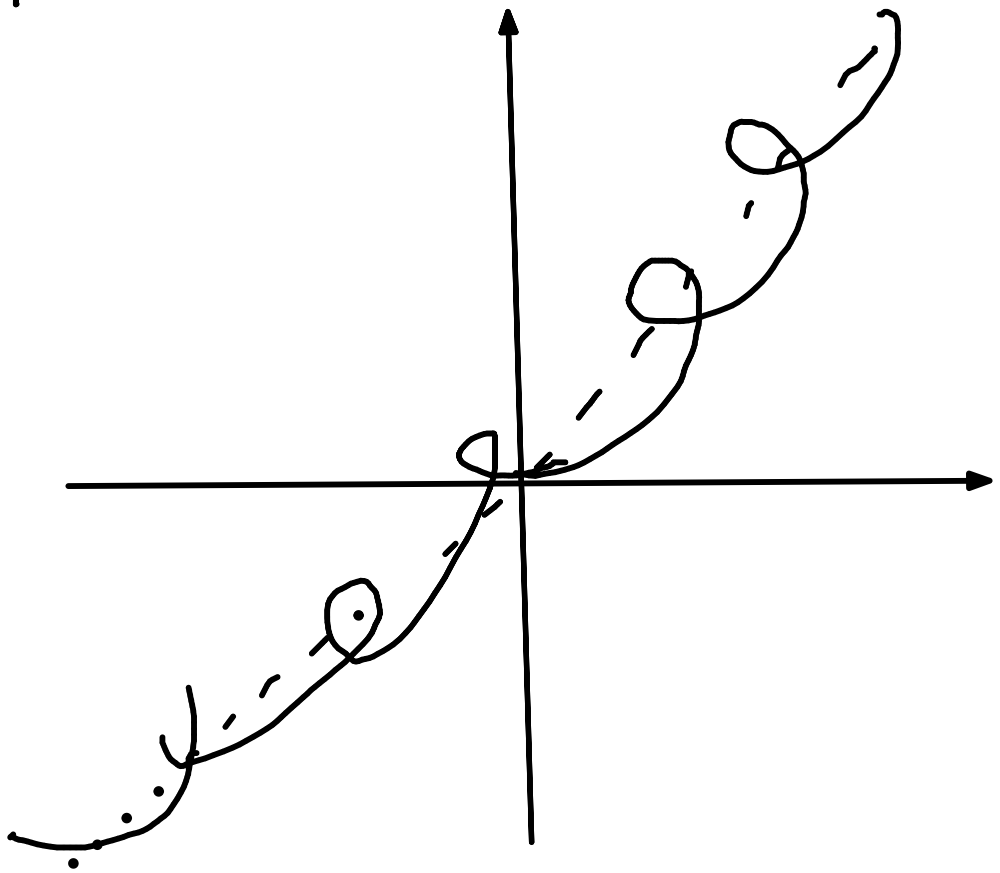
$y = 0.1 \sin(2t)$

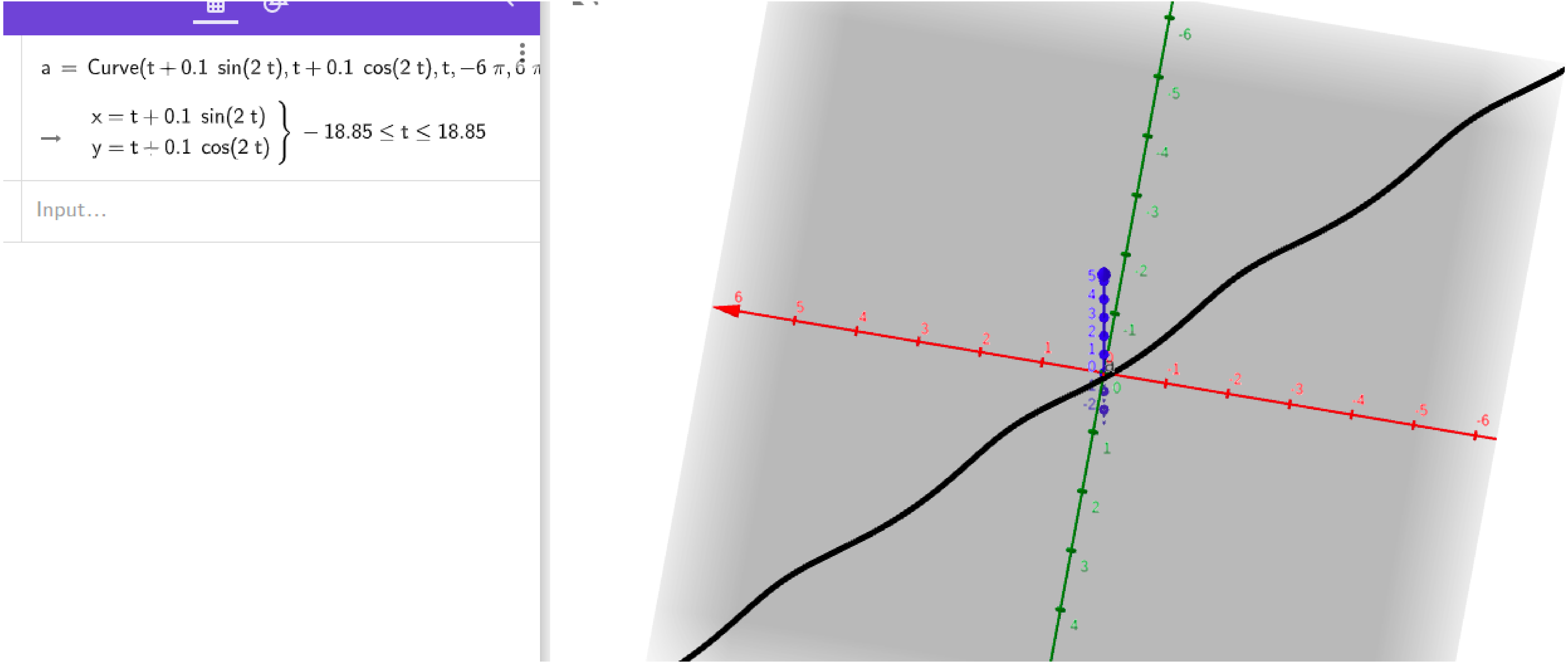


$-6\pi \leq t \leq 6\pi$

$x = t + 0.1 \cos(2t)$

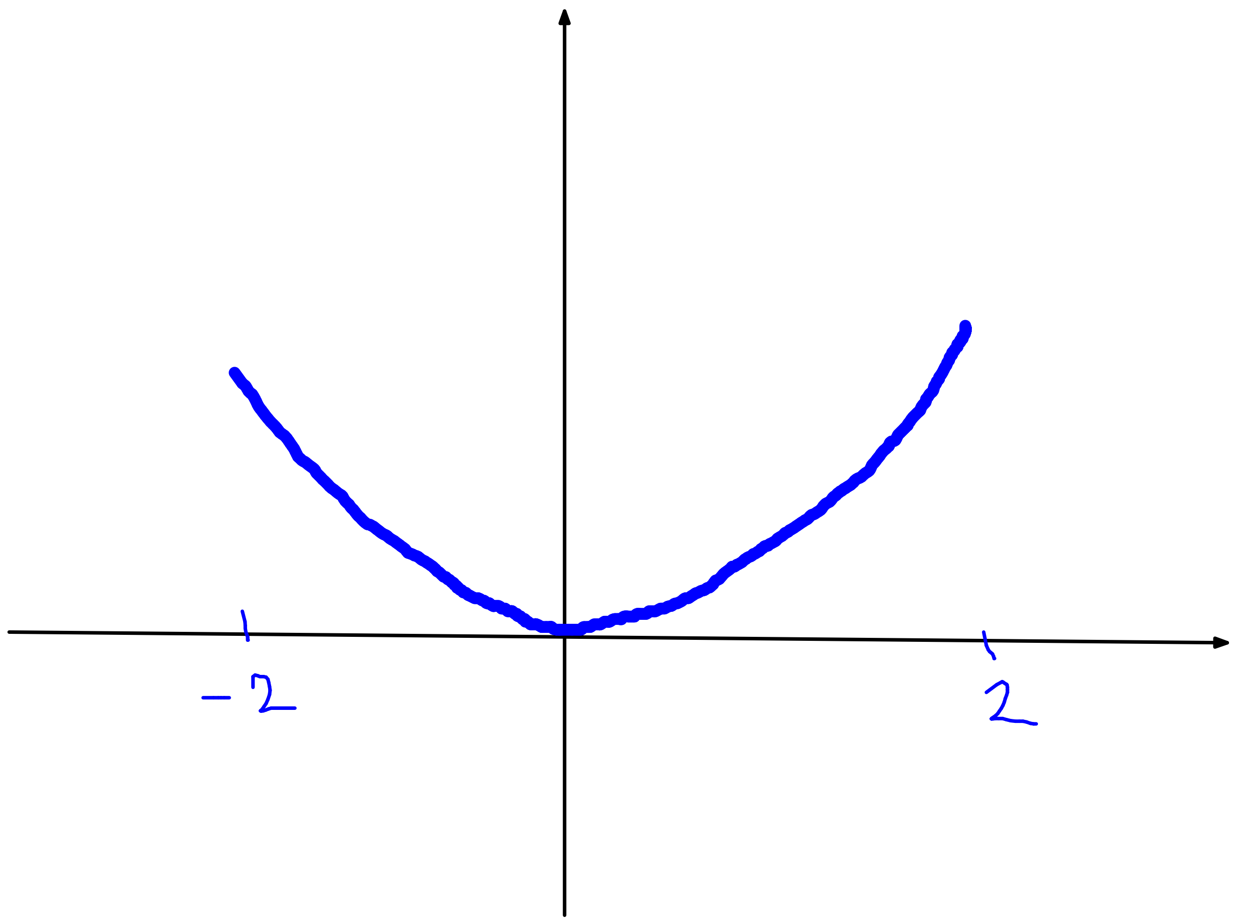
$y = t + 0.1 \sin(2t)$




a = Curve(t + 0.1 sin(2 t), t + 0.1 cos(2 t), t, −6 π, 6 π
x = t + 0.1 sin(2 t)
y = t + 0.1 cos(2 t)
− 18.85 ≤ t ≤ 18.85
Input…

Q. Sketch the path

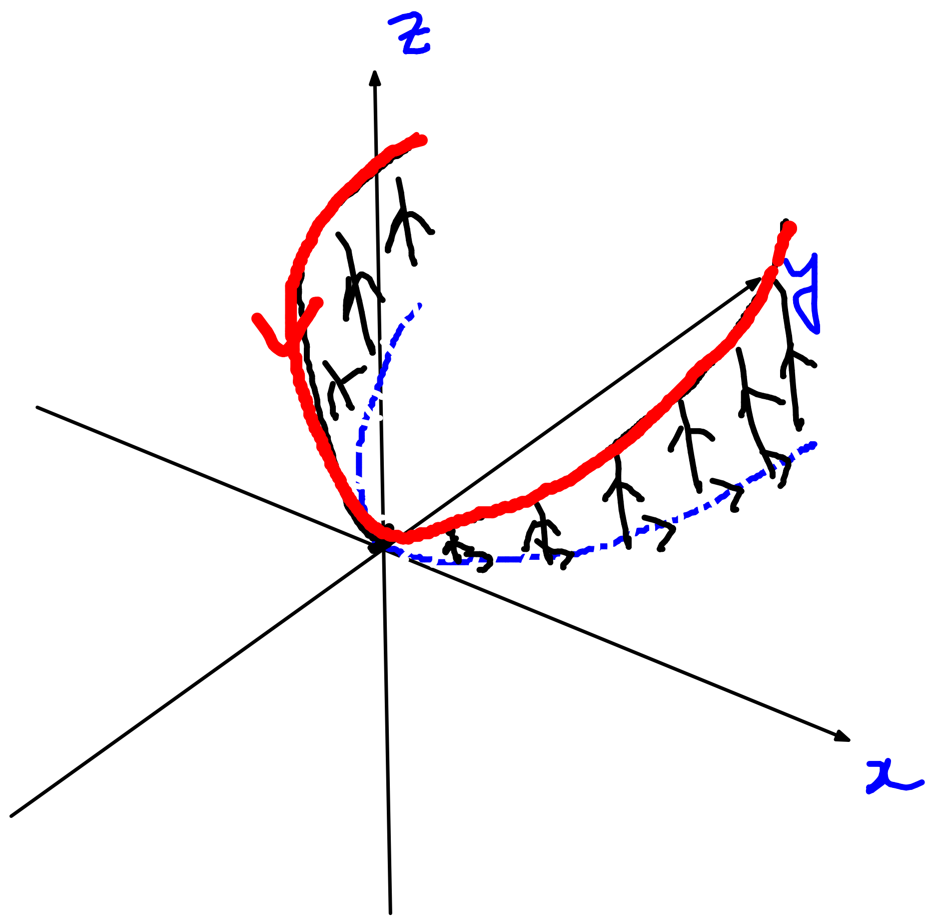
$$x = t \quad , \quad y = t^2 \quad , \quad -2 \le t \le 2$$



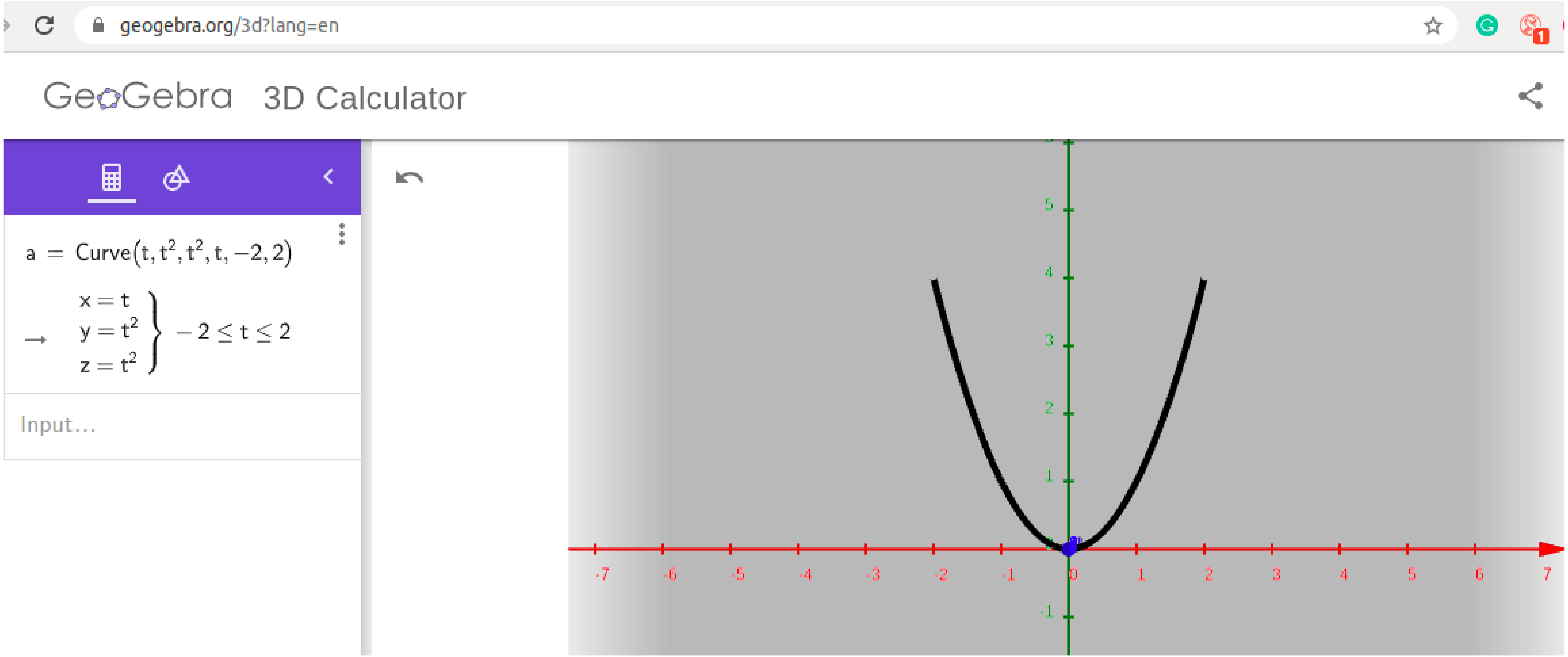
Q. Sketch the path

$$\underbrace{x = t \quad , \quad y = t^2}_{y = x^2} \quad , \quad z = t^2$$

$$-2 \leq t \leq 2$$



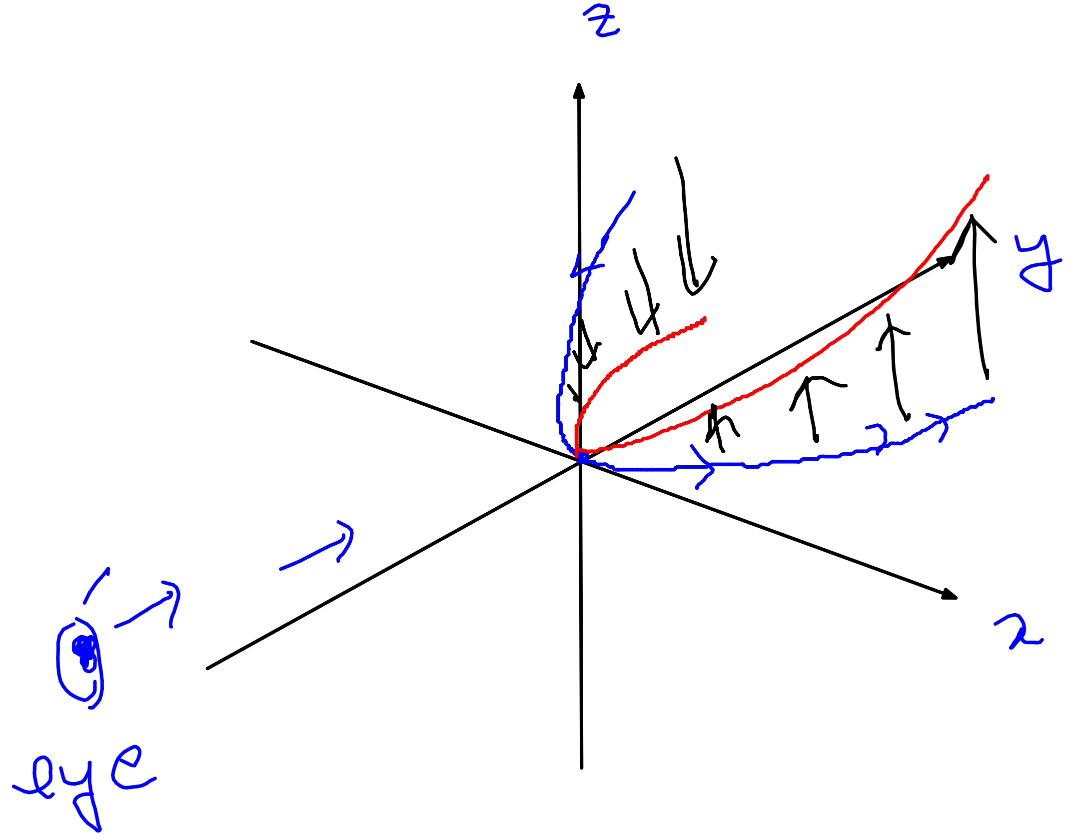
$t = 0 \qquad (x, y, z) = (0, 0, 0)$

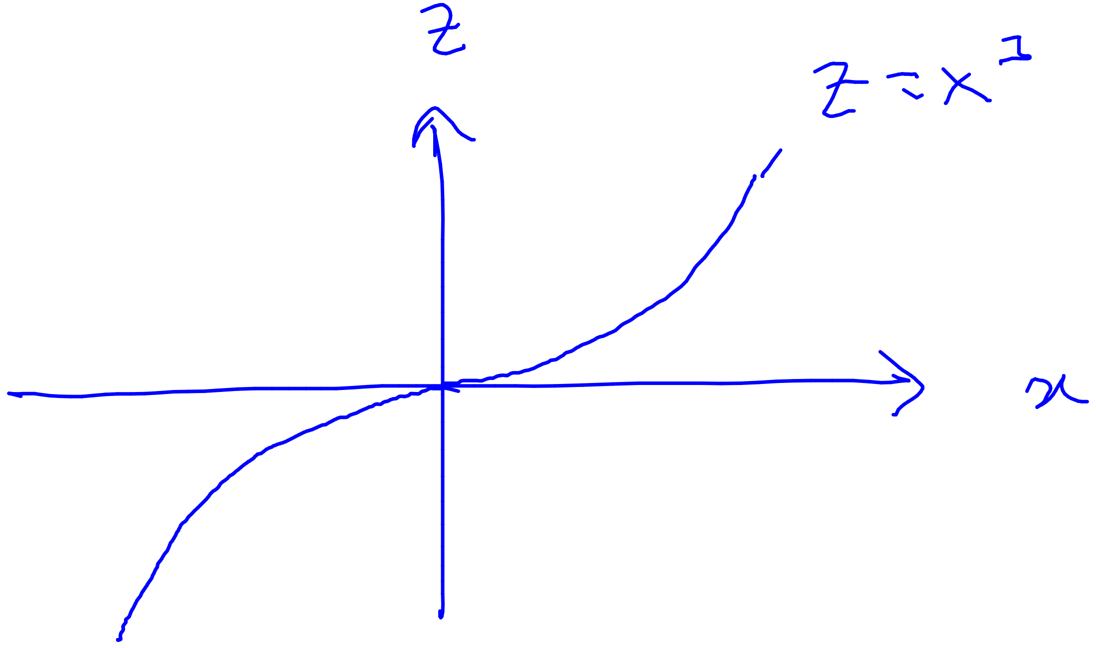

geogebra.org/3d?lang=en
GeoGebra 3D Calculator
a = Curve(t, t², t², t, −2, 2)
x = t
y = t²
z = t²
−2 ≤ t ≤ 2
Input…

Q: Sketch

$$x = t\ ,\quad y = t^2\ ,\quad z = t^3$$

$$-2 \le t \le 2$$





$$z = y^{3/2}$$

between

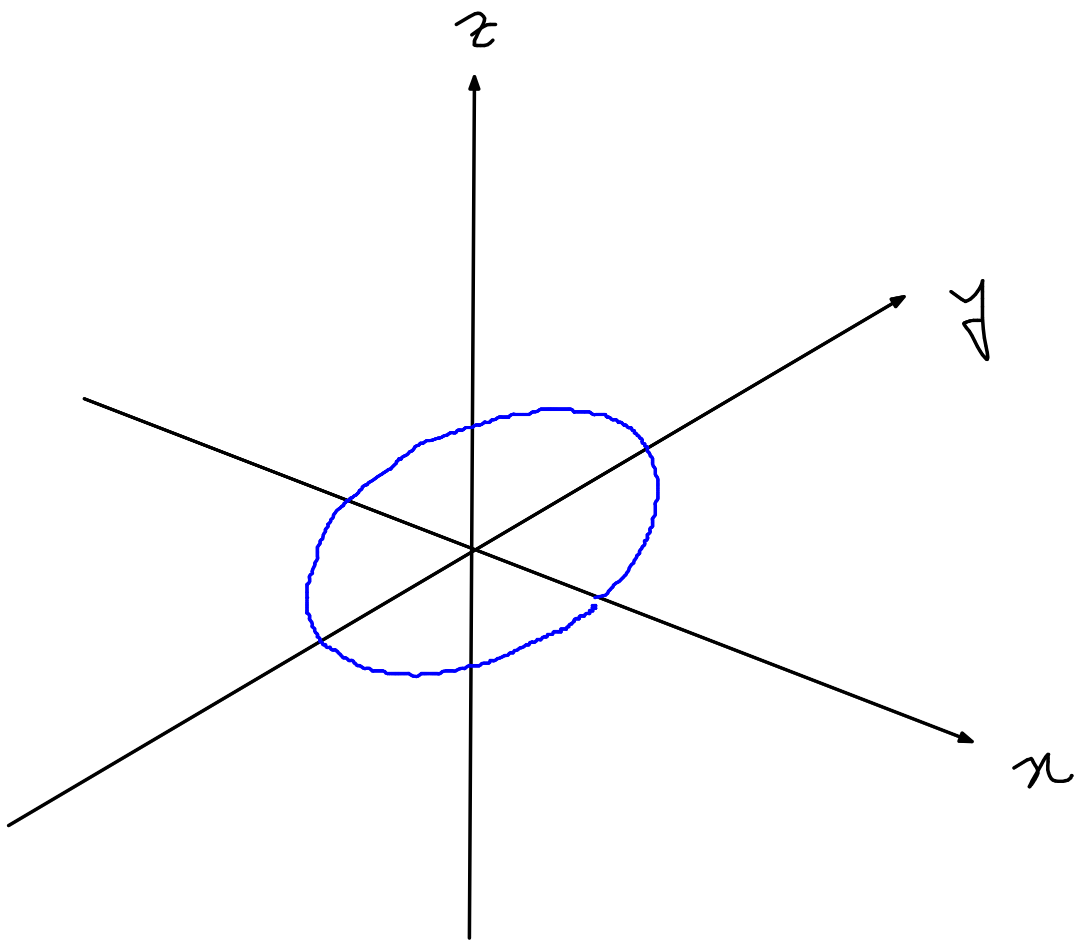
$$z = y$$

$$z = y^2$$

Q. Sketch

$x = \cos(t) \qquad y = \sin(t) \qquad z = 0$

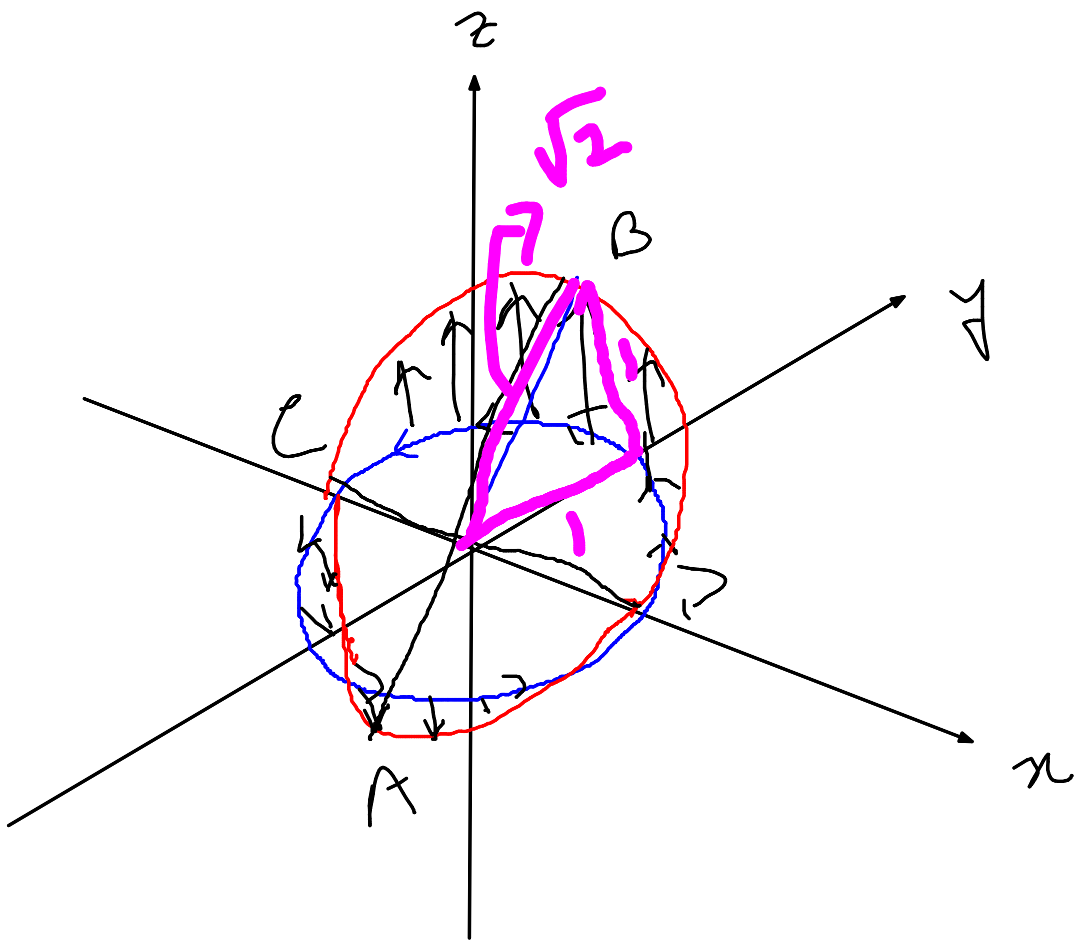
$0 \le t \le 6\pi$



Q. Sketch

$x = \cos(t)$ $y = \sin(t)$ $z = \sin(t)$

$0 \le t \le 6\pi$



$AB \overset{??}{=} CD$

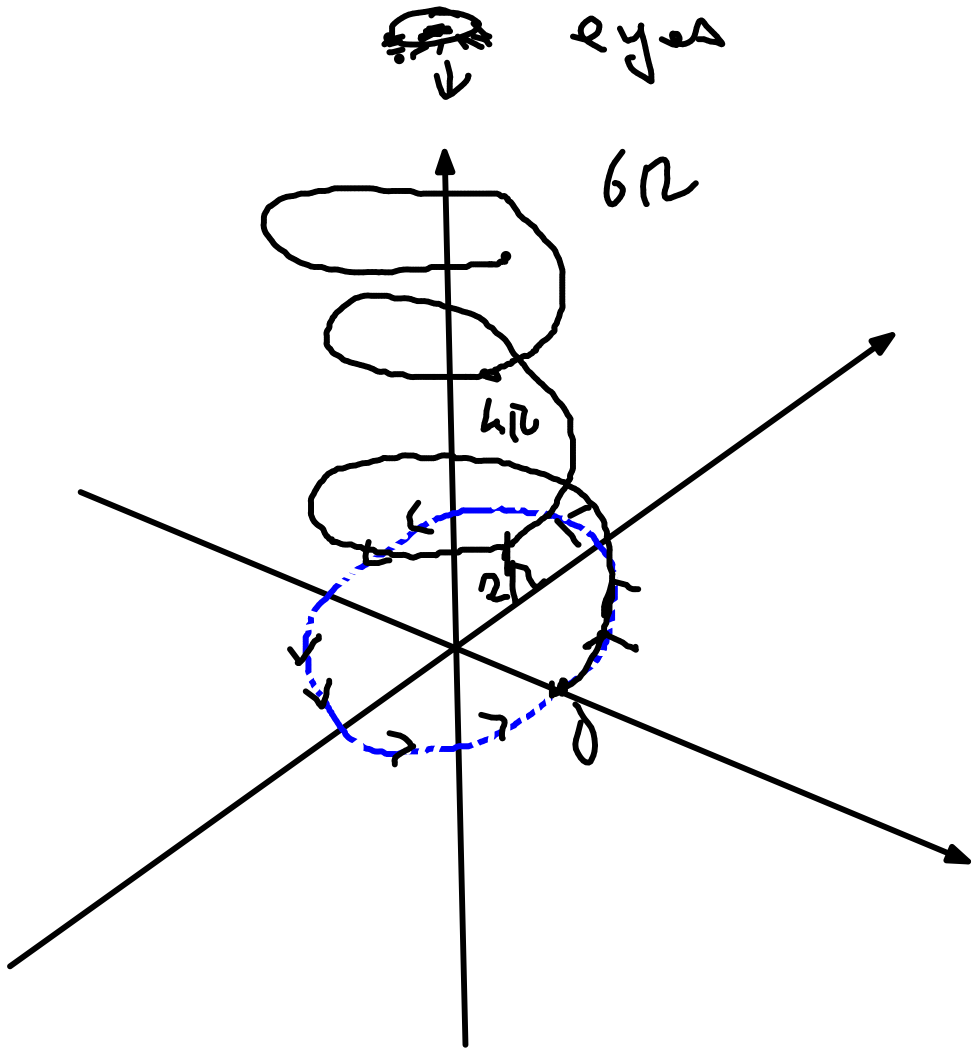
$2\sqrt{2}$ 2

ellipse

Q. Sketch

$x = \cos(t)$ $y = \sin(t)$, $z = t$

$0 \le t \le 6\pi$



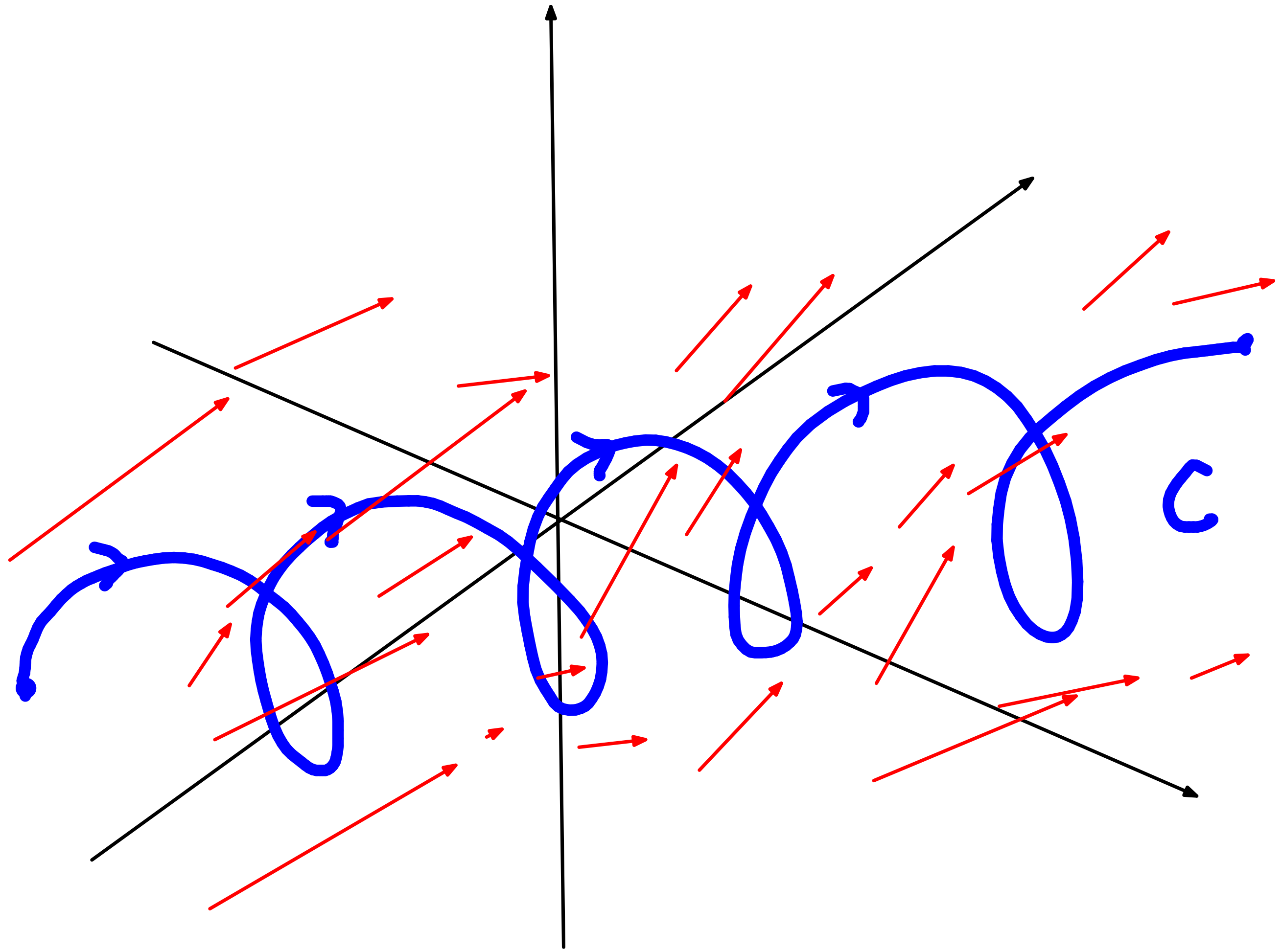
Calculus on curves

— next time

We will focus for now:

next time

Calculus on curves or paths.



→ Length of the curve?

→ $\int_C f \, d\vec{r}$: integration of scalar function

→ $\int_C \vec{F} \cdot d\vec{r}$: integration of vector functions on curves.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

$$x = t + 2\sin 2t$$
$$y = t + 2\cos 5t$$

Evaluate the line integral, where C is the given curve.

$$\int_C y\,ds, \quad C: x = t^2,\ y = t,\ 0 \leqslant t \leqslant 2$$

V EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

Evaluate the line integral, where C is the given curve.

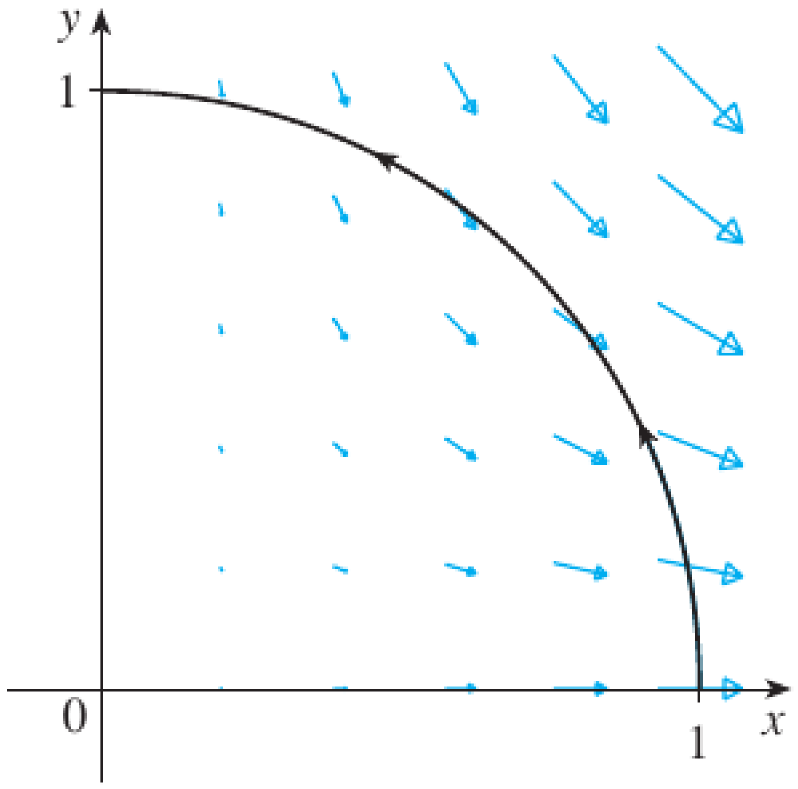
$$\int_C xy^3\, ds,$$
$$C\colon x = 4\sin t,\ \ y = 4\cos t,\ \ z = 3t,\ \ 0 \leqslant t \leqslant \pi/2$$

LINE INTEGRALS OF VECTOR FIELDS

13 DEFINITION Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leq t \leq \pi/2$.



EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$