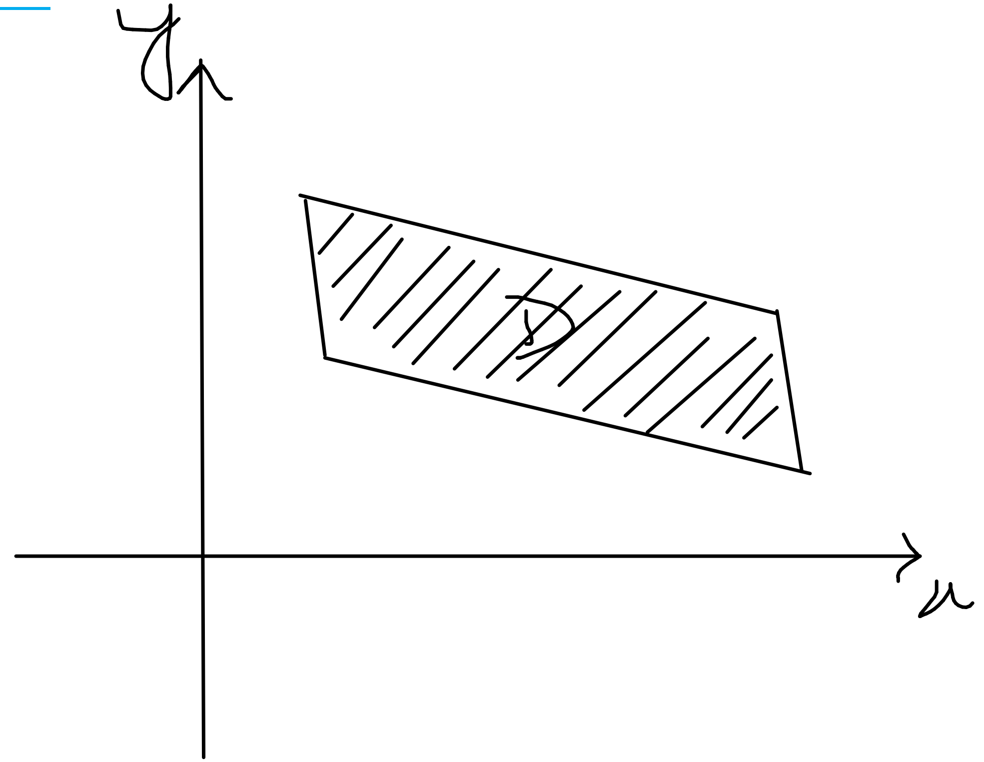
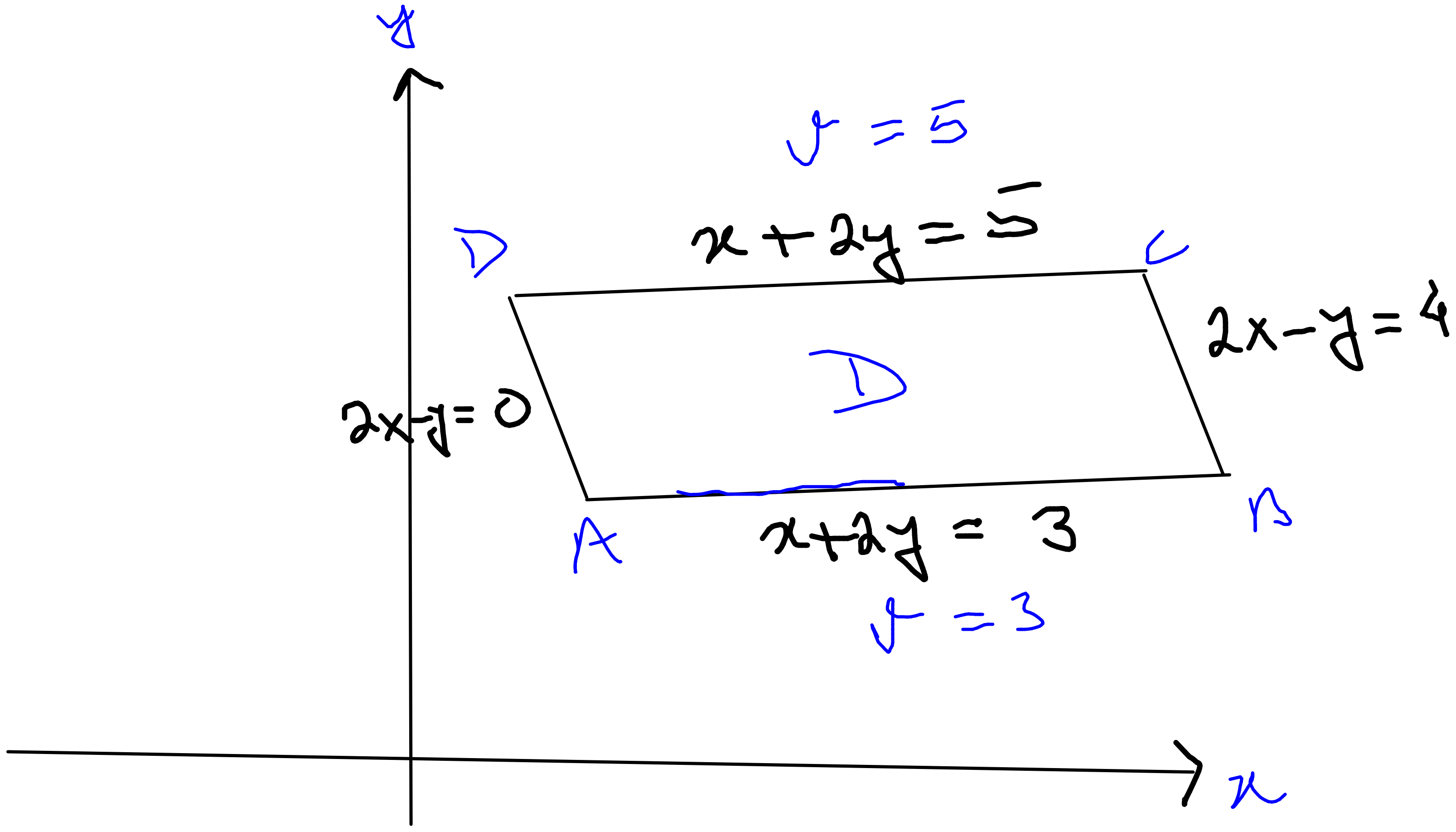
12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

→ $\iint_D f(x,y)\, dA$ — given



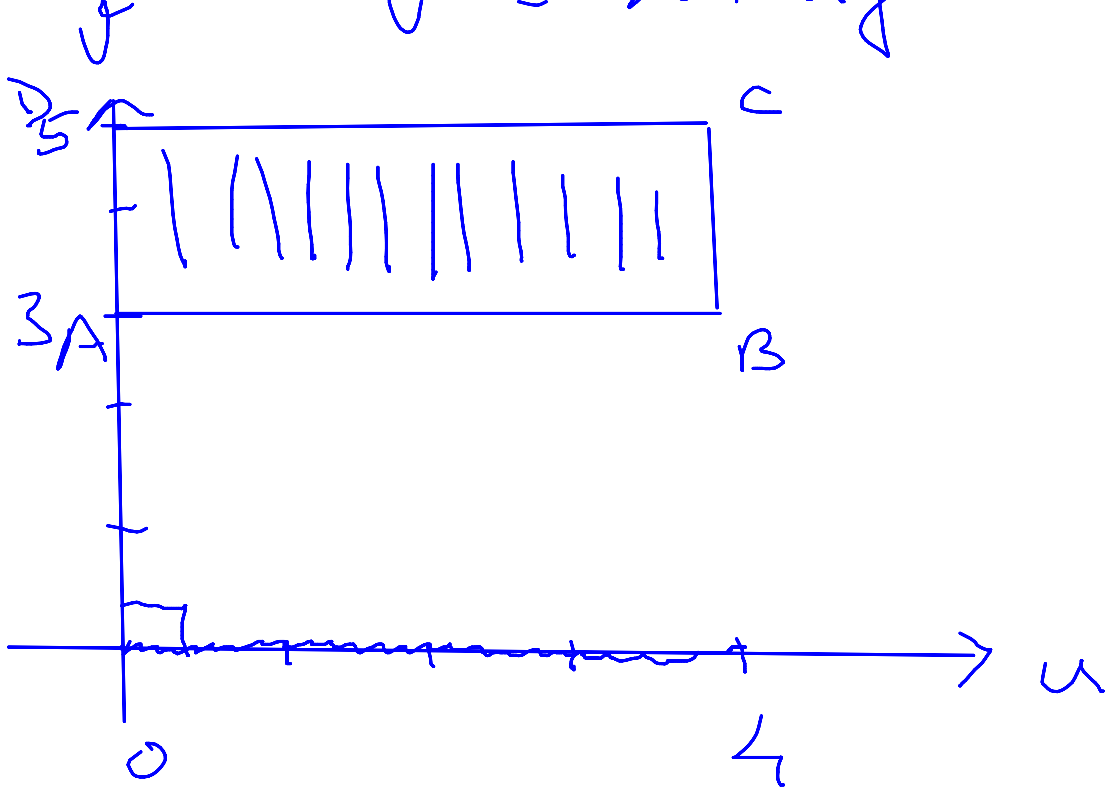
→ D : will be mildly complicated

→ D : will be simplified with change of variables



$u = 2x - y$

$v = x + 2y$



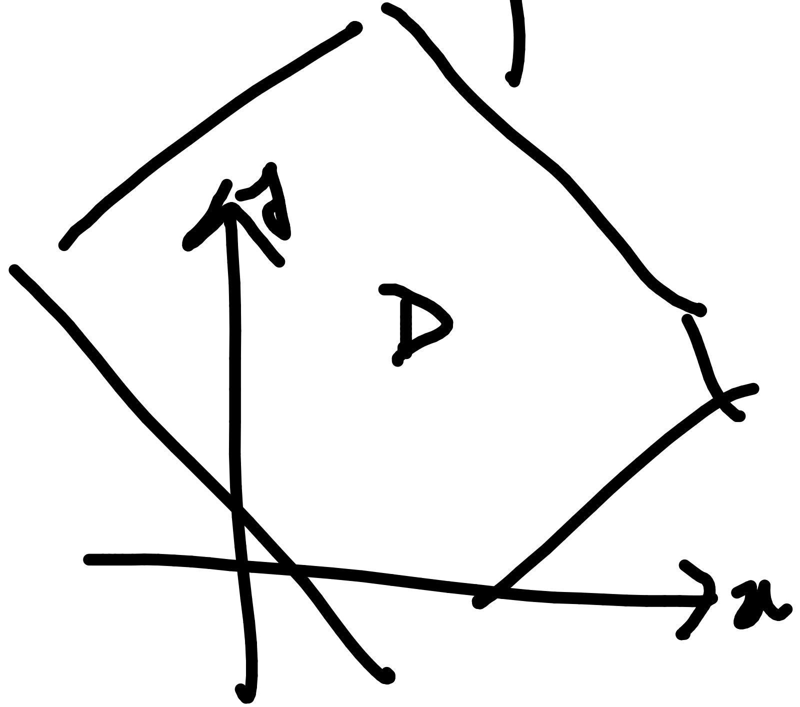
Find the Jacobian of the transformation.

$$x = u + 4v, \quad y = 3u - 2v$$

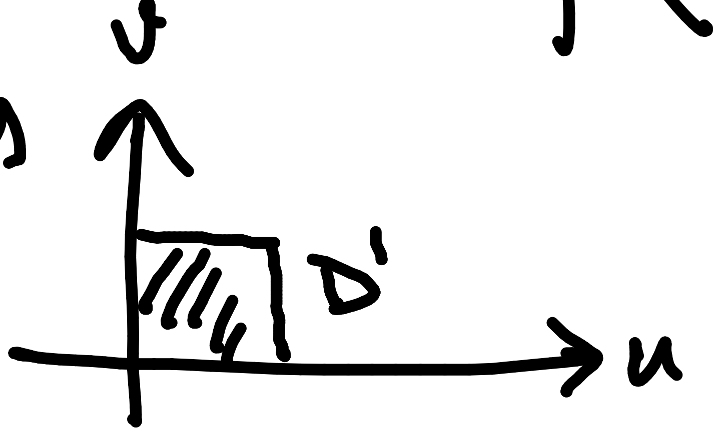
$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \text{abs}\left[\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right]$$

$$= \text{abs}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = |-14| = 14$$

$$\iint_D f(x,y)\,dx\,dy$$



$D' \approx 14$ times smaller than D.



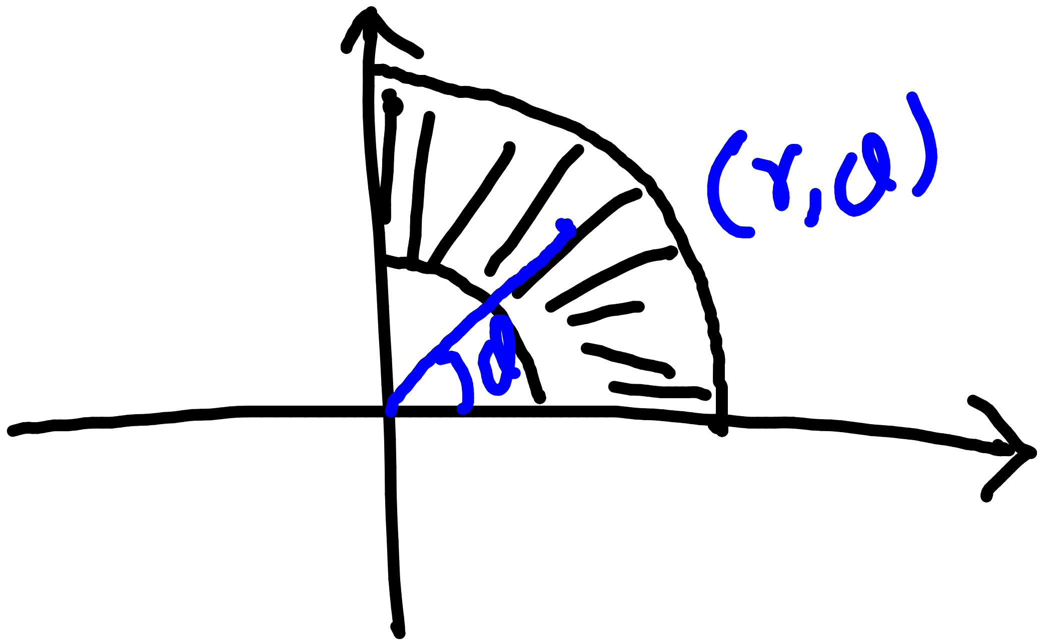
$$\iint$$

Find the Jacobian of the transformation.

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \underline{\underline{r}}$$

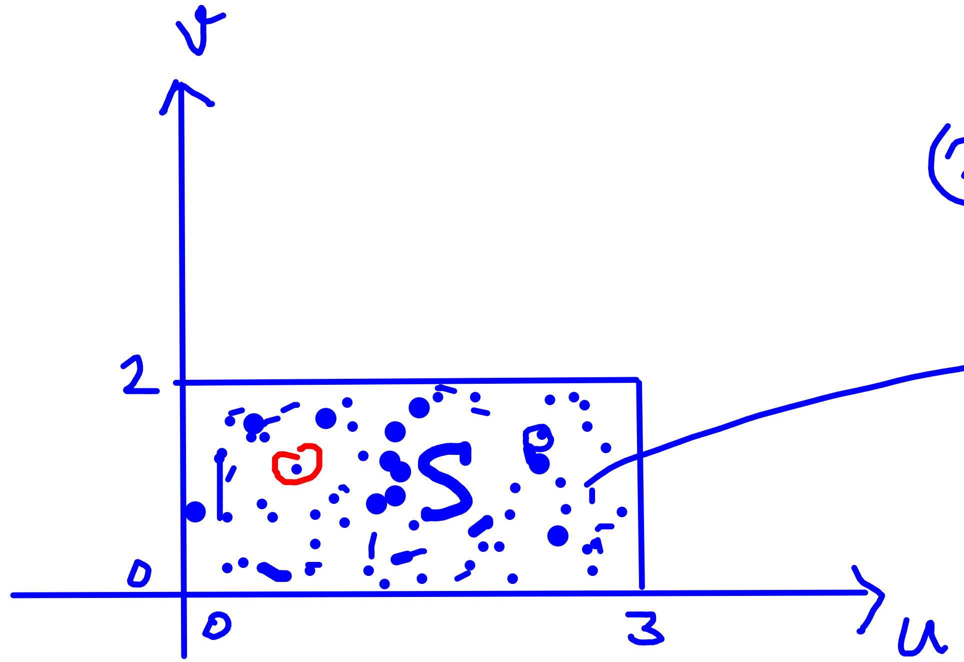


$$dx\,dy = r\,dr\,d\theta$$

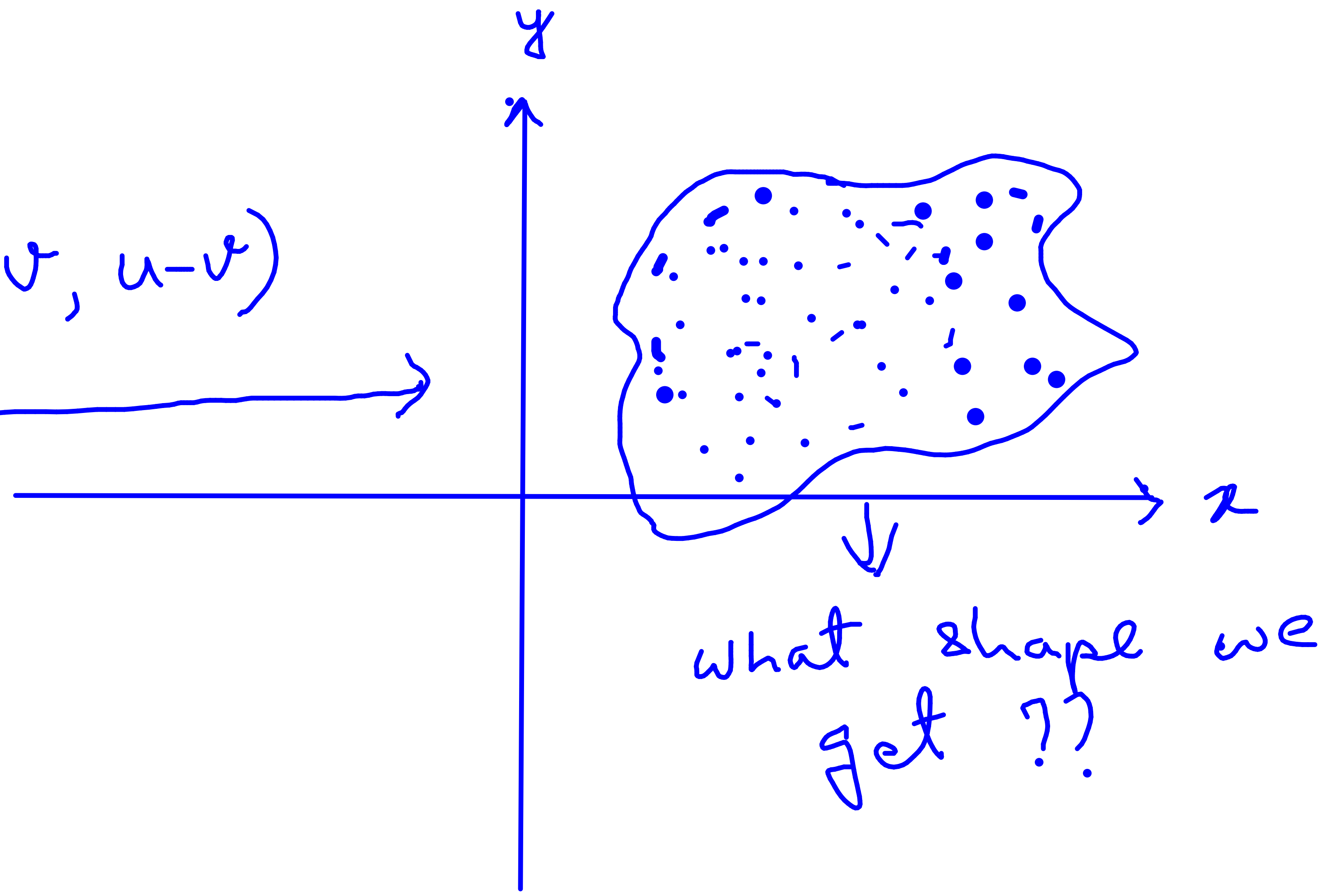
Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$?? shape ??

$x = 2u + 3v,\ y = u - v$



$(x, y) = (2u+3v,\ u-v)$



Find the image of the set S under the given transformation.

$$\boxed{S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};}$$?? shape ??

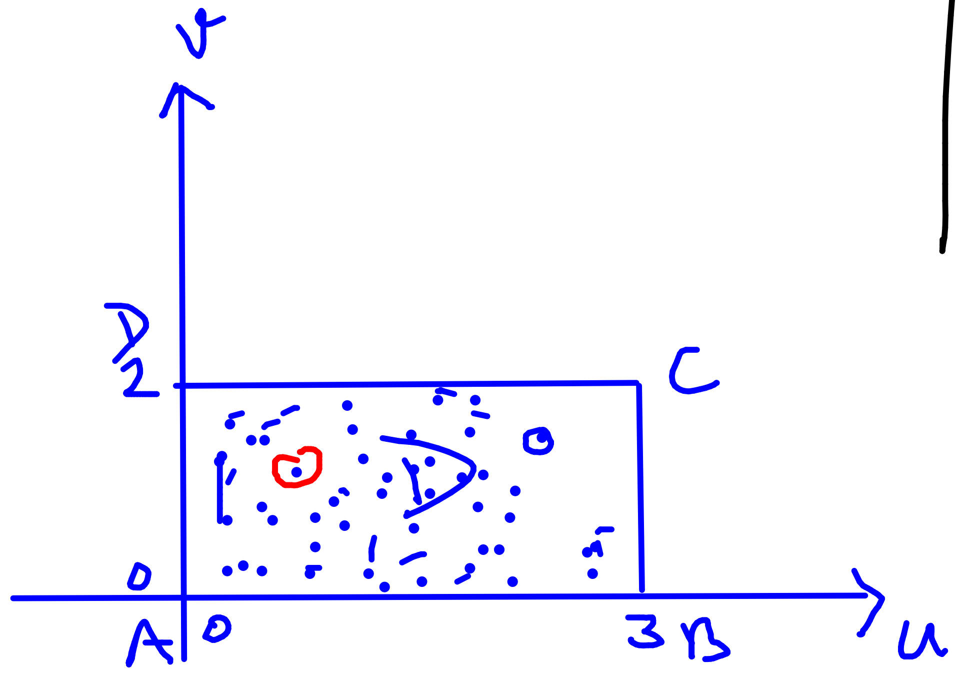
$$x = 2u + 3v, \quad y = u - v$$

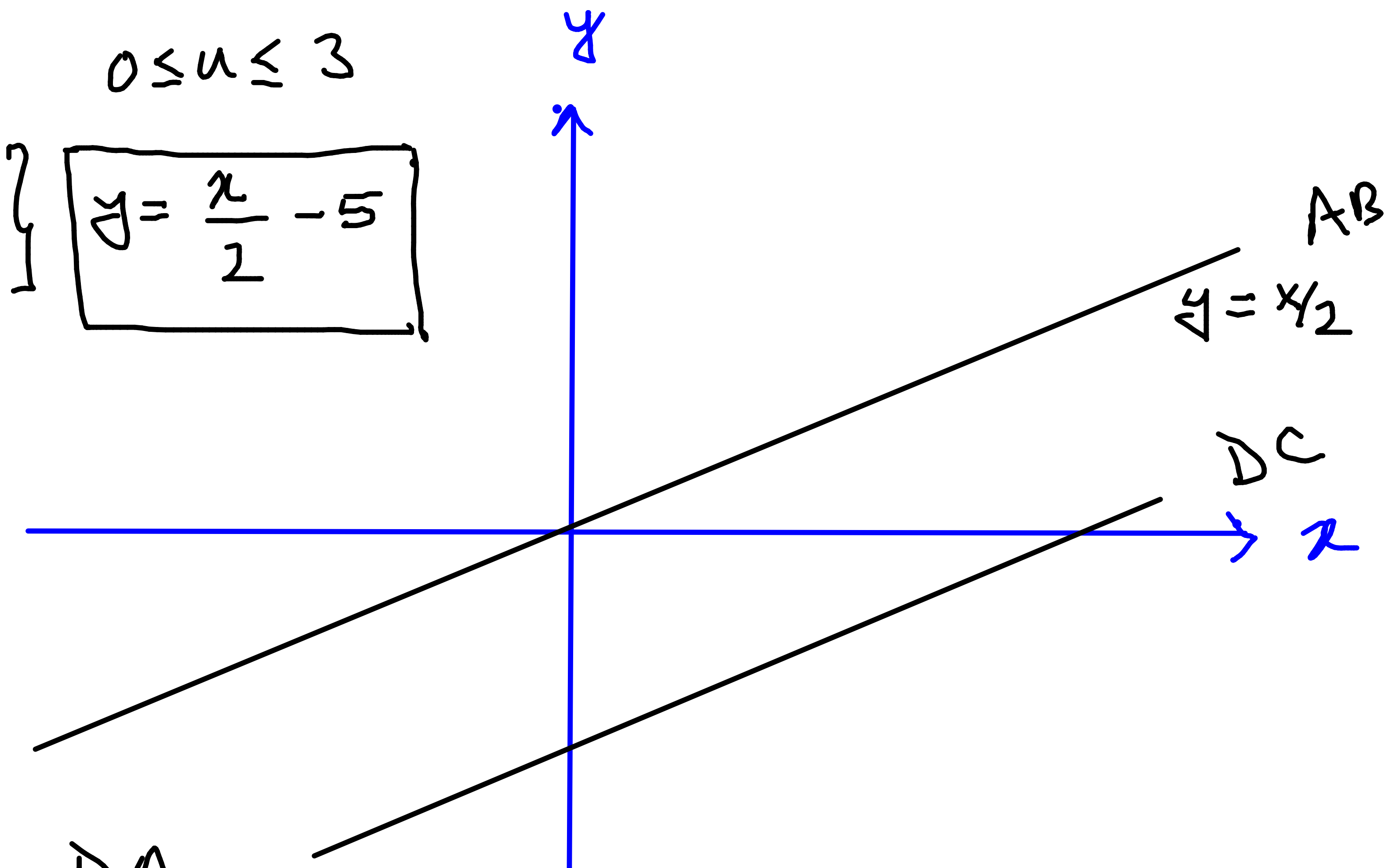
AB , $v = 0$, $0 \le u \le 3$

$\left.\begin{aligned} x &= 2u \\ y &= u \end{aligned}\right\}$ $x = 2y$

DC , $v = 2$ $0 \le u \le 3$

$\left.\begin{aligned} x &= 2u + 6 \\ y &= u - 2 \end{aligned}\right\}$ $\boxed{y = \dfrac{x}{2} - 5}$





strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

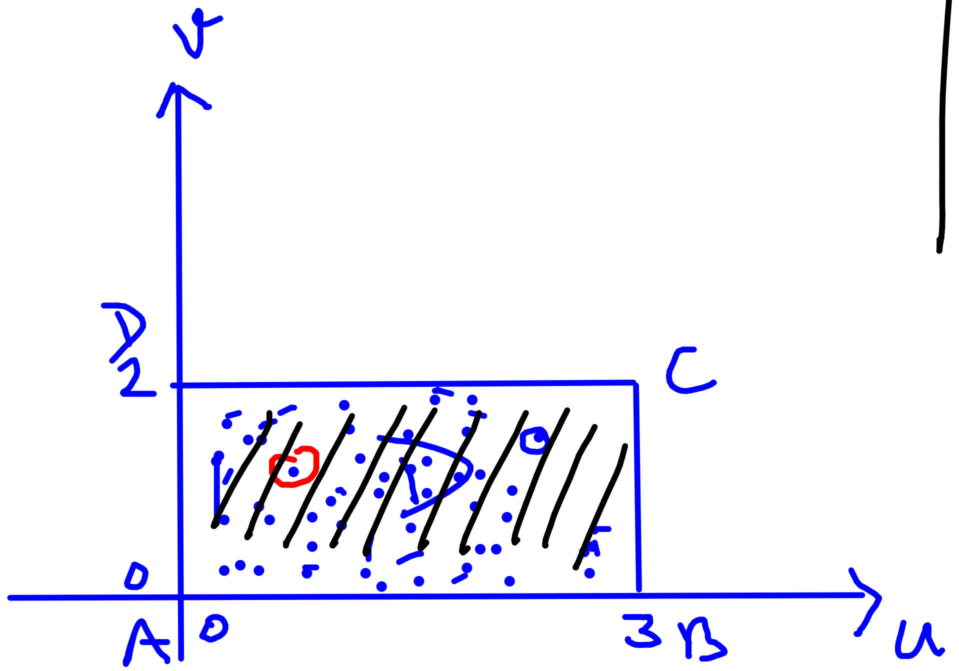
Find the image of the set S under the given transformation.

$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; ?? shape ??

$x = 2u + 3v,\ y = u - v$

AD : $u = 0$, $0 \le v \le 2$

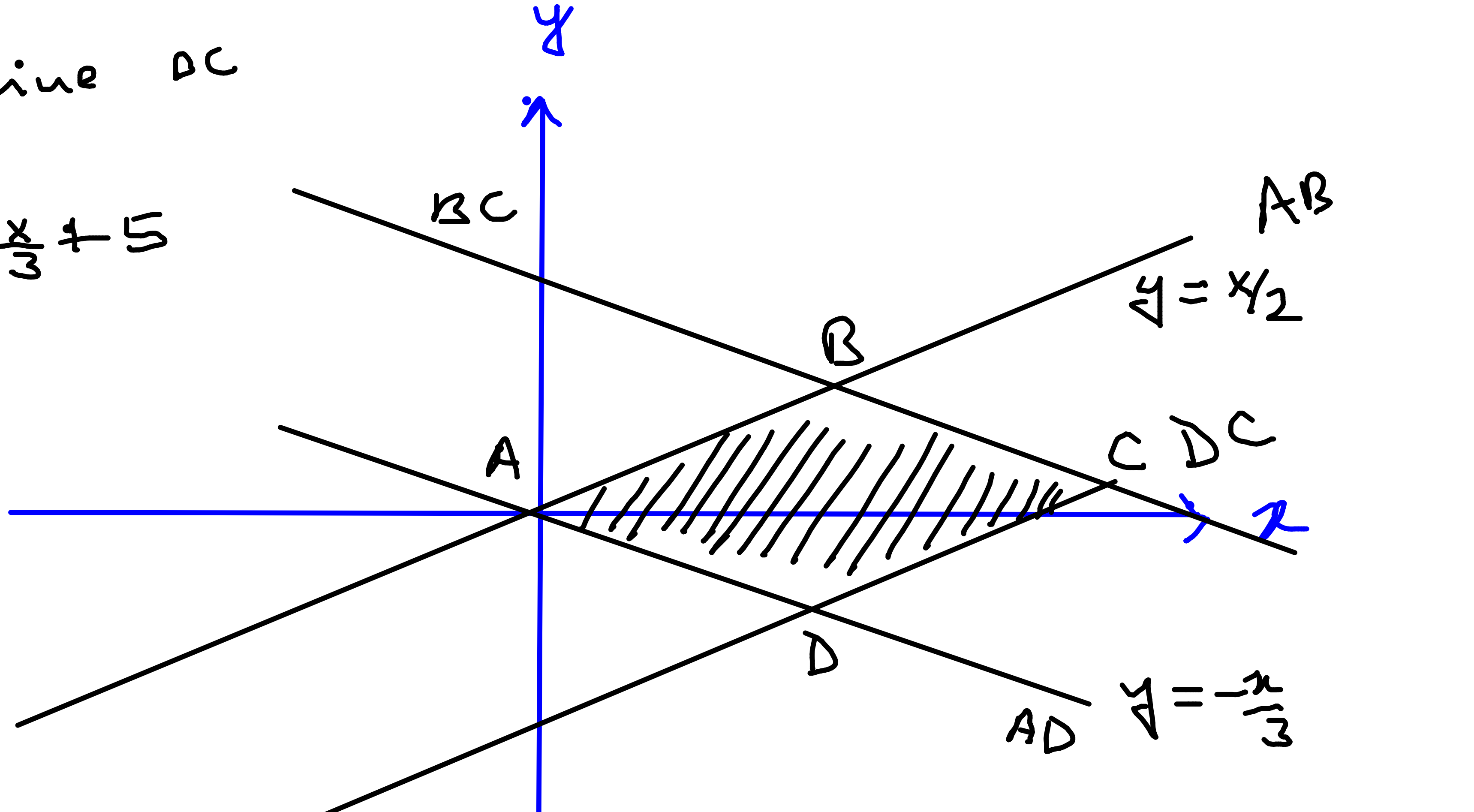
$\left.\begin{matrix} x = 3v \\ y = -v \end{matrix}\right]$ $\begin{matrix} x = -3y \\ y = -\frac{1}{3}x \end{matrix}$



similarly: line DC

$x + 3y = 15$

$y = -\frac{x}{3} + 5$



strategy: for line AB, BC, CD, DA start with eqn in uv variables & convert the eqn from uv to xy

Find the image of the set S under the given

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ y = u - v$$

Find the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

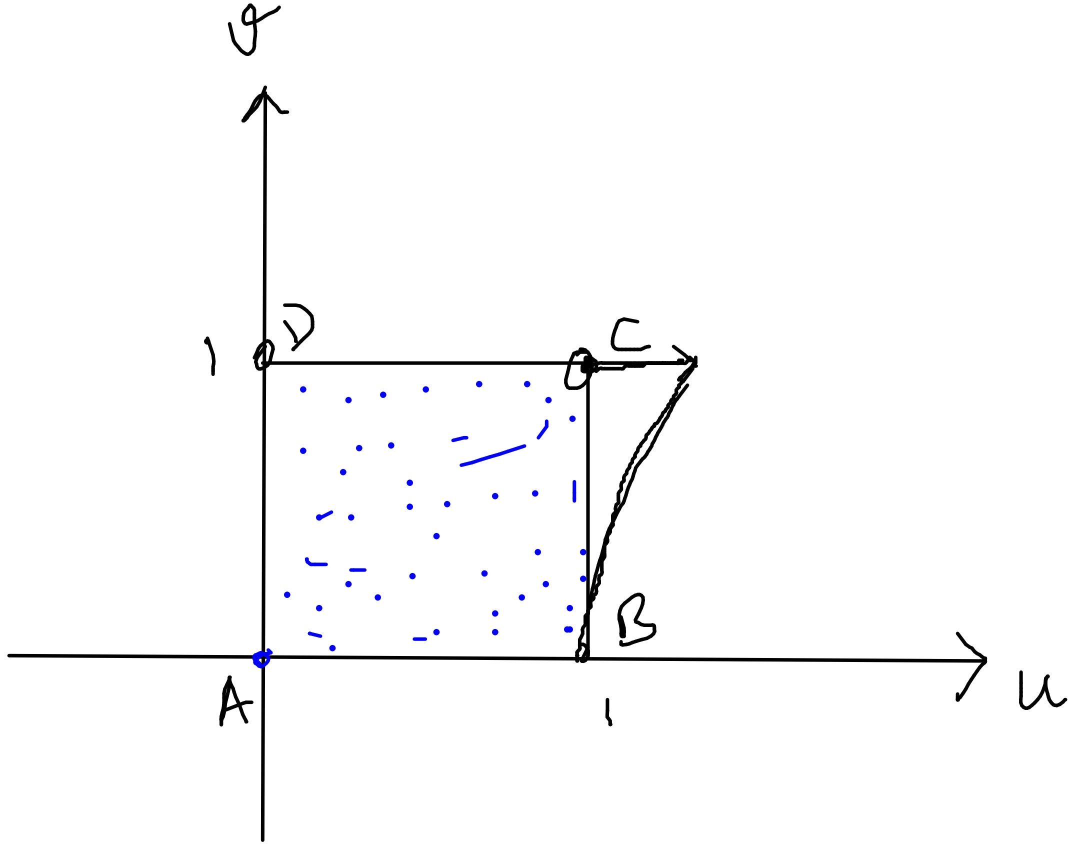
$$dx\,dy = J\,du\,dv$$

$$= \left| \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \right| = |-5|$$

$$= 5$$

Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$



AB

$v = 0$

$\left.\begin{aligned} x &= 0 \\ y &= u \end{aligned}\right\}$ $\begin{aligned} &x = 0 \\ &0 \le y \le 1 \end{aligned}$

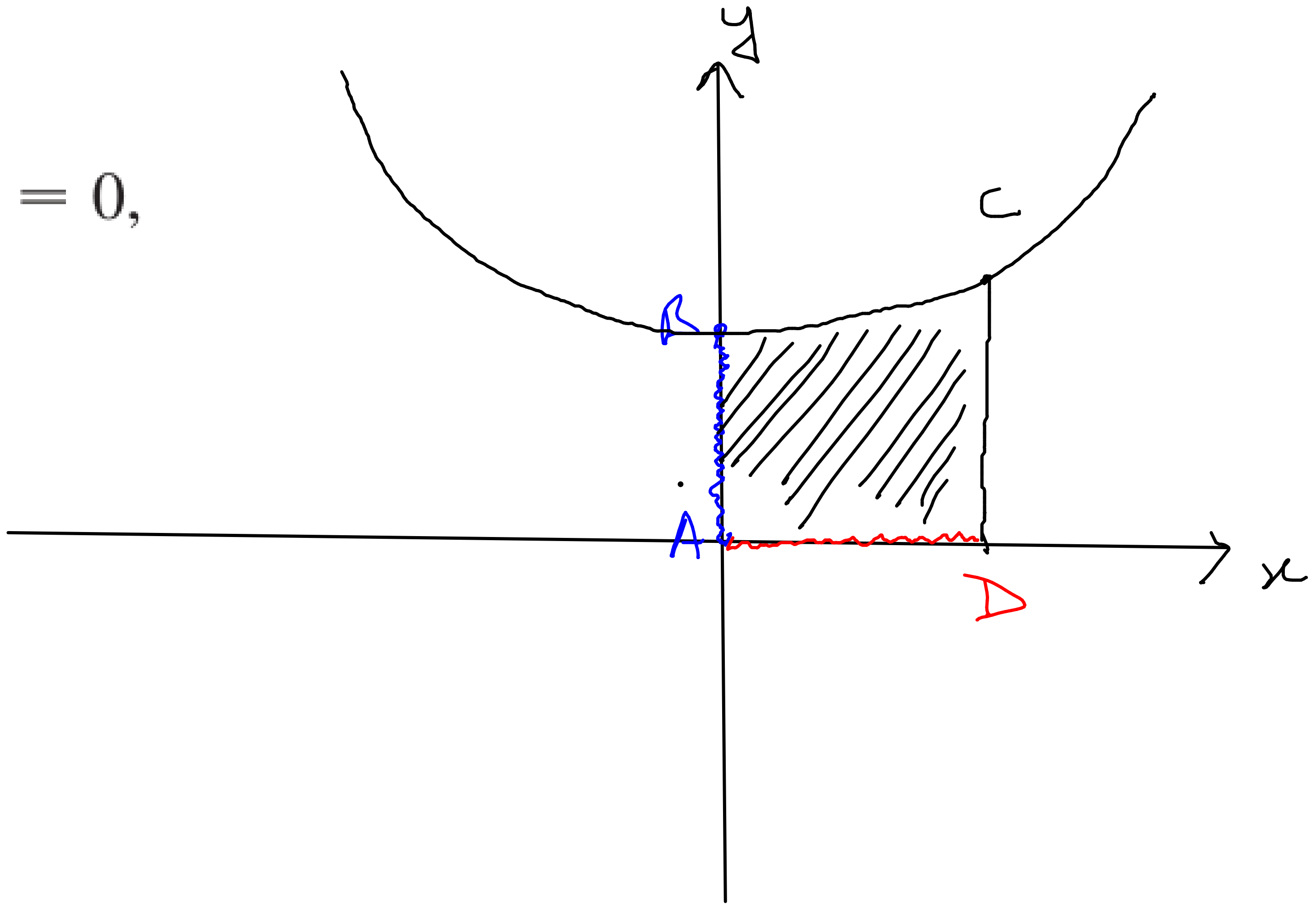
AD $u = 0$, $0 \le v \le 1$

$\left.\begin{aligned} x &= v \\ y &= 0 \end{aligned}\right\}$ $\begin{aligned} &y = 0 \\ &0 \le x \le 1 \end{aligned}$

BC : $u = 1$, $0 \le v \le 1$

$\left.\begin{aligned} x &= v \\ y &= 1 + v^2 \end{aligned}\right\}$ $\begin{aligned} &y = 1 + x^2 \\ &0 \le x \le 1 \end{aligned}$

DC



Find the image of the set S under the given transformation.

Find the Jacobian

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

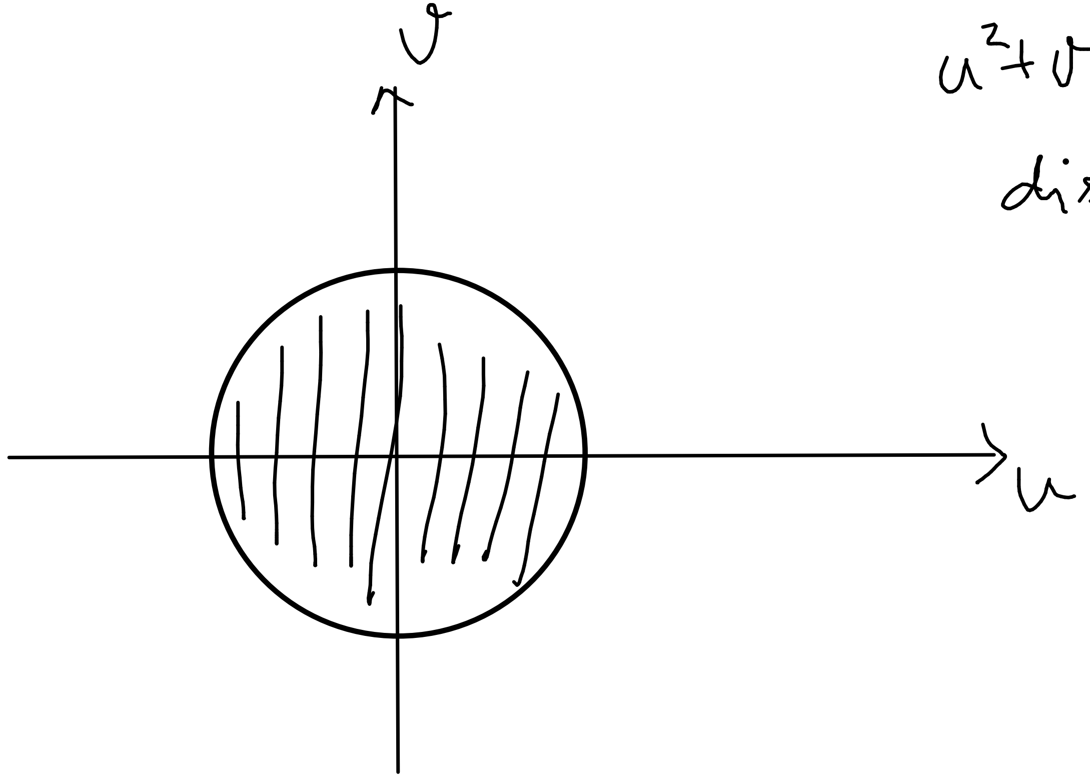
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} =$$

$$= 1+v^2$$

Find the image of the set S under the given transformation.

$$S \text{ is the disk given by } u^2 + v^2 \leqslant 1; \quad x = au, \; y = bv$$

for simplicity, assume $a = 3$, $b = 2$



$u^2+v^2 \leq 1$

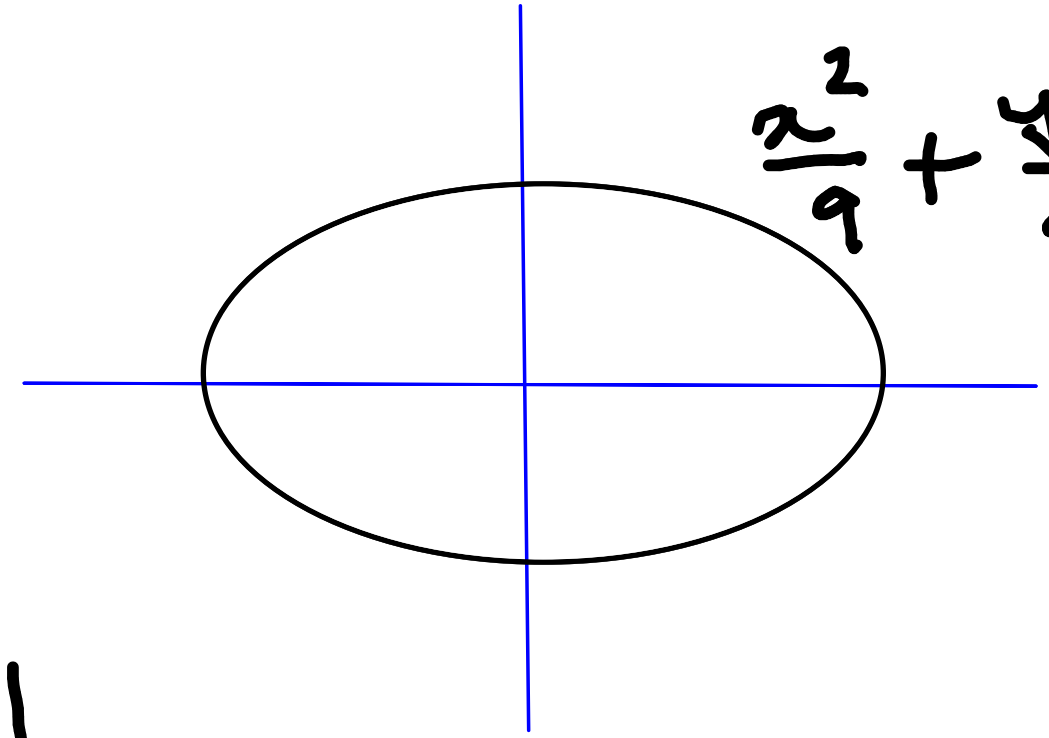
disc

$x = 3u$

$y = 2v$

$u^2+v^2 = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

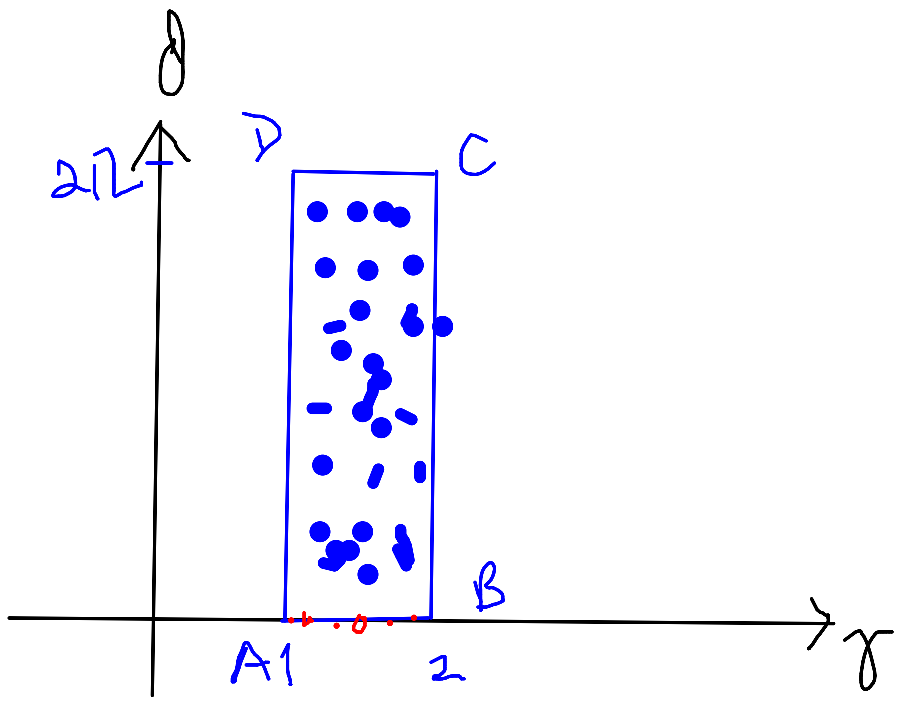


$\frac{x^2}{9} + \frac{y^2}{4} = 1$

Q. Find the image of S under the given transformation.

$$S = \{ (r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 2\pi \}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$



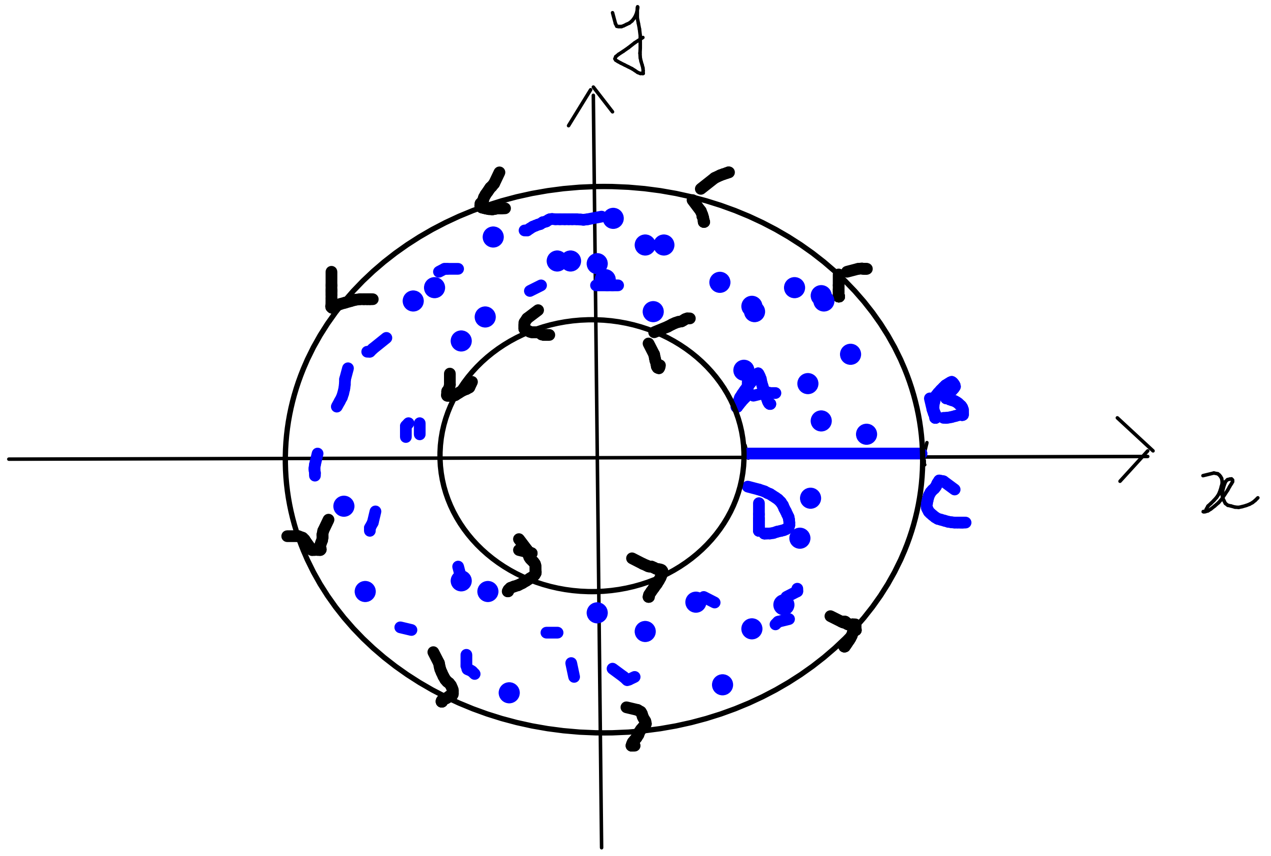
AB

$\theta = 0, \ 1 \le r \le 2$

BC

$r = 2,\ 0 \le \theta \le 2\pi$

$x = 2\cos\theta,\ y = 2\sin\theta$



DC, $\theta = 2\pi$, $1 \le r \le 2$

$x = r\cos 2\pi$, $y = r\sin 2\pi$

$x = r$, $y = 0$

$0 \le x \le 2$

AD:

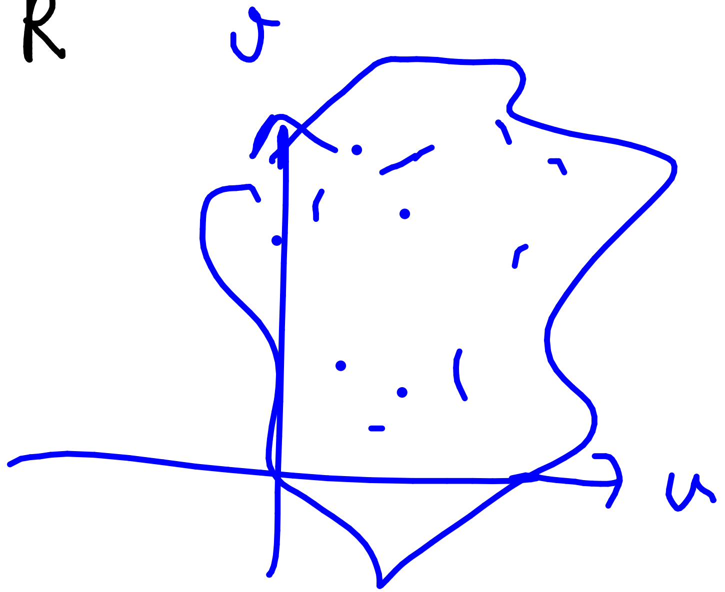
$r = 1$ $\quad 0 \le \theta \le 2\pi$

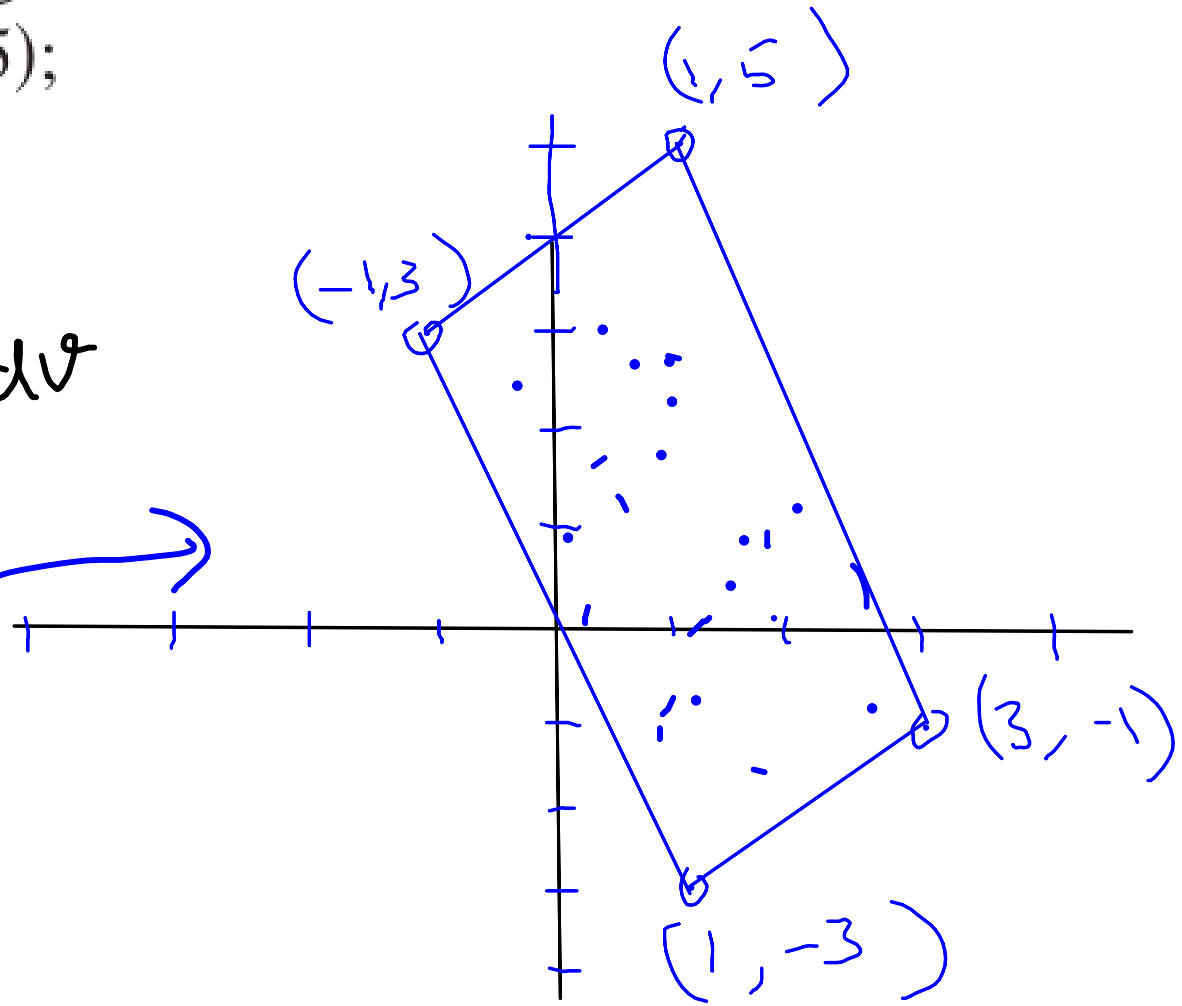
$x = \cos\theta$ $\quad y = \sin\theta$

Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\, dA = \iint_{??}^{??} (??)\, du\, dv$$

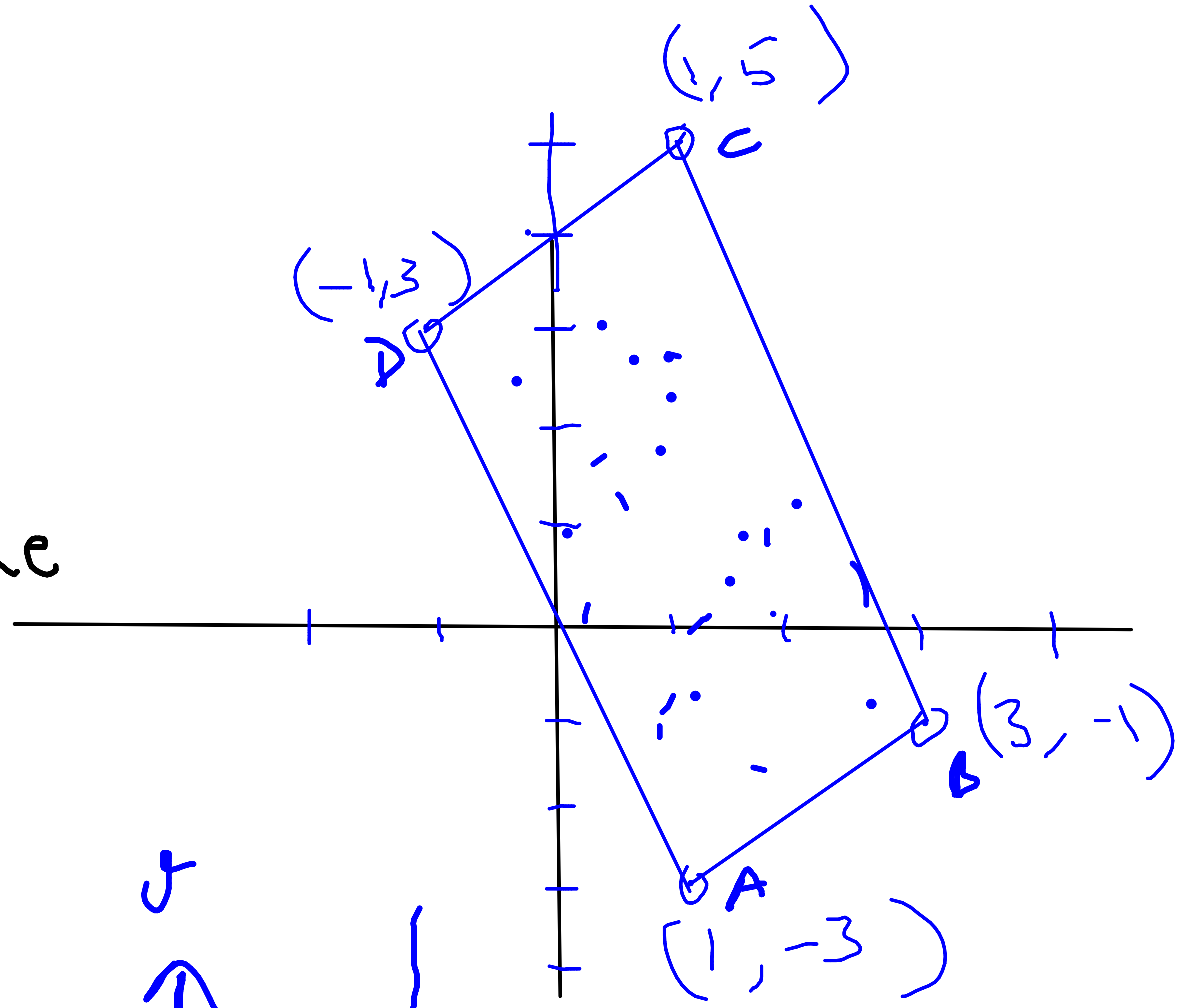




Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

find out the shape of ABCD in the uv-plane



$\rightarrow$ AB: eqn of AB in xy variable.

$$x - y = 4$$

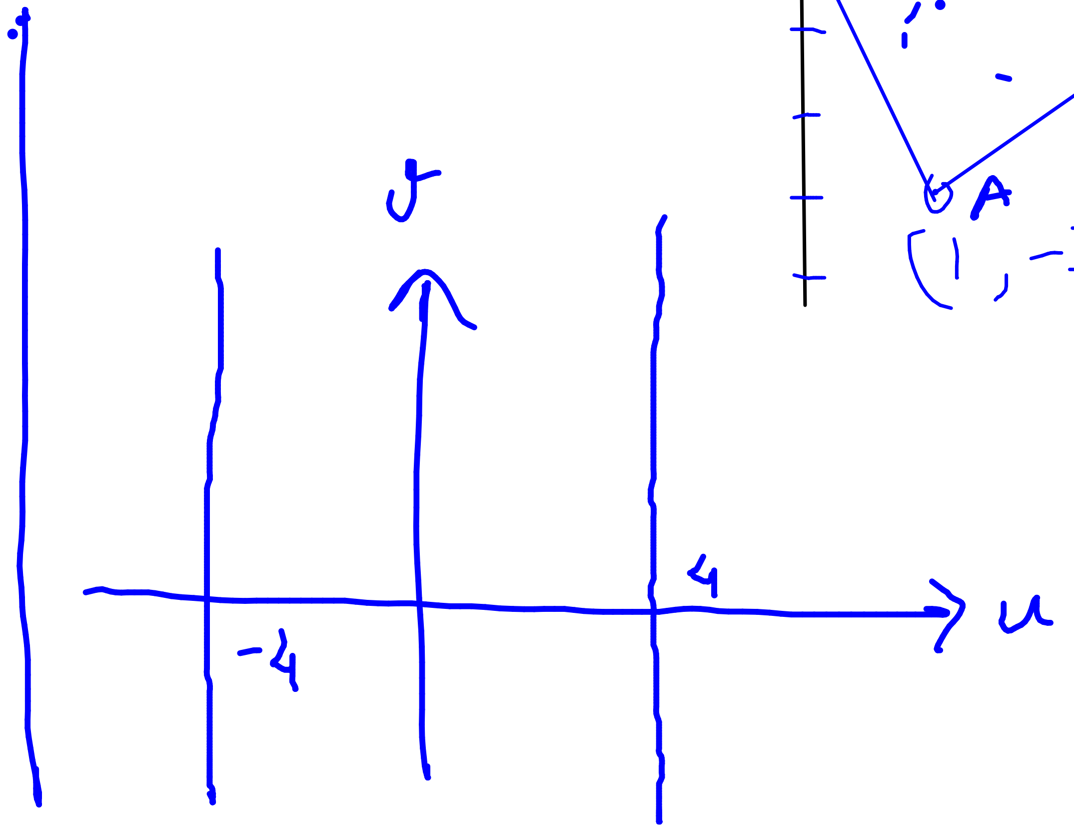
$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = 4$$

$$u = 4$$

$\rightarrow$ DC: $x - y = -4$

$$\frac{1}{4}(u+v) - \frac{1}{4}(v-3u) = -4$$

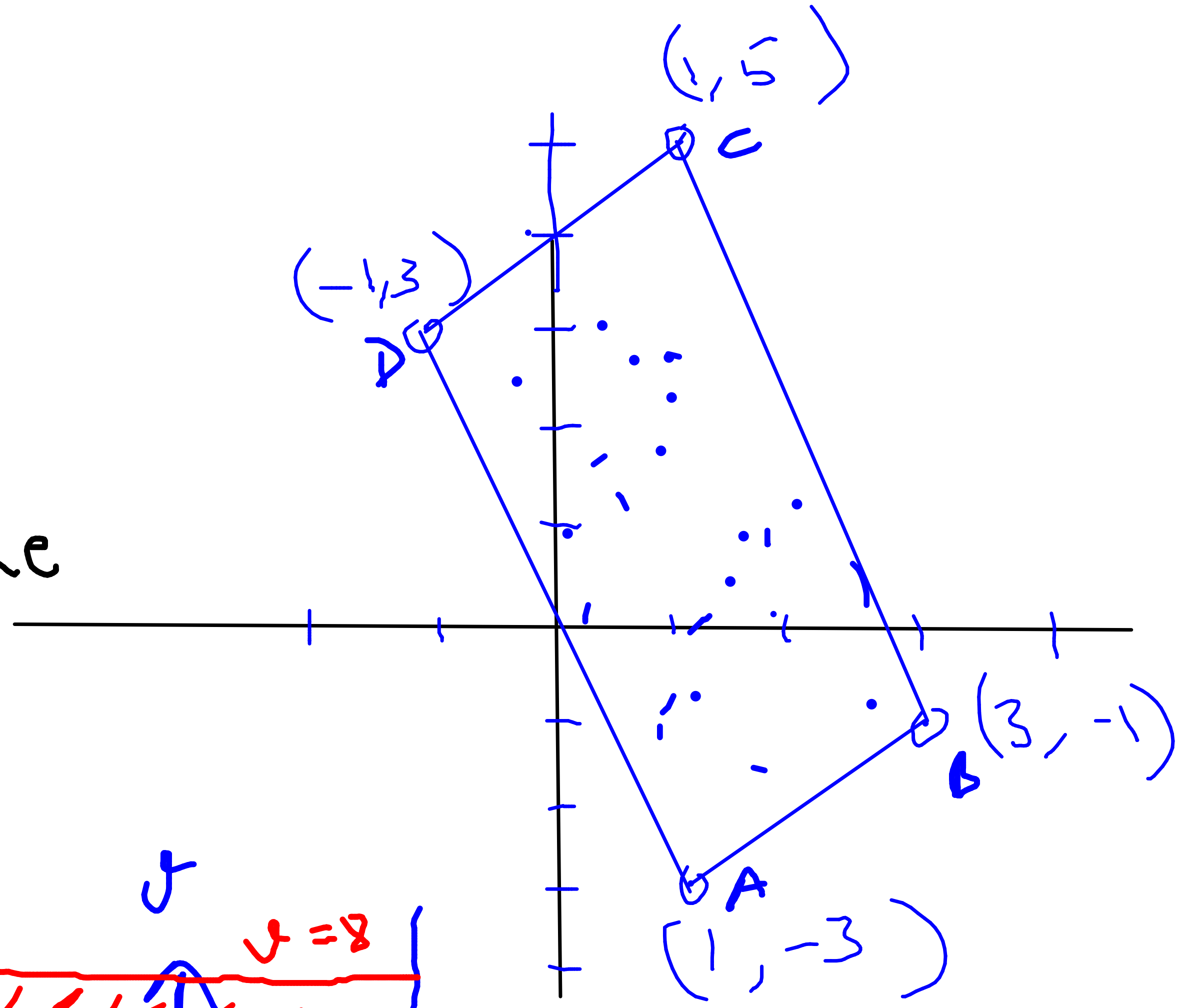
$$u = -4$$



Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

find out the shape of ABCD in the uv-plane



→ AD, $y = -3x$

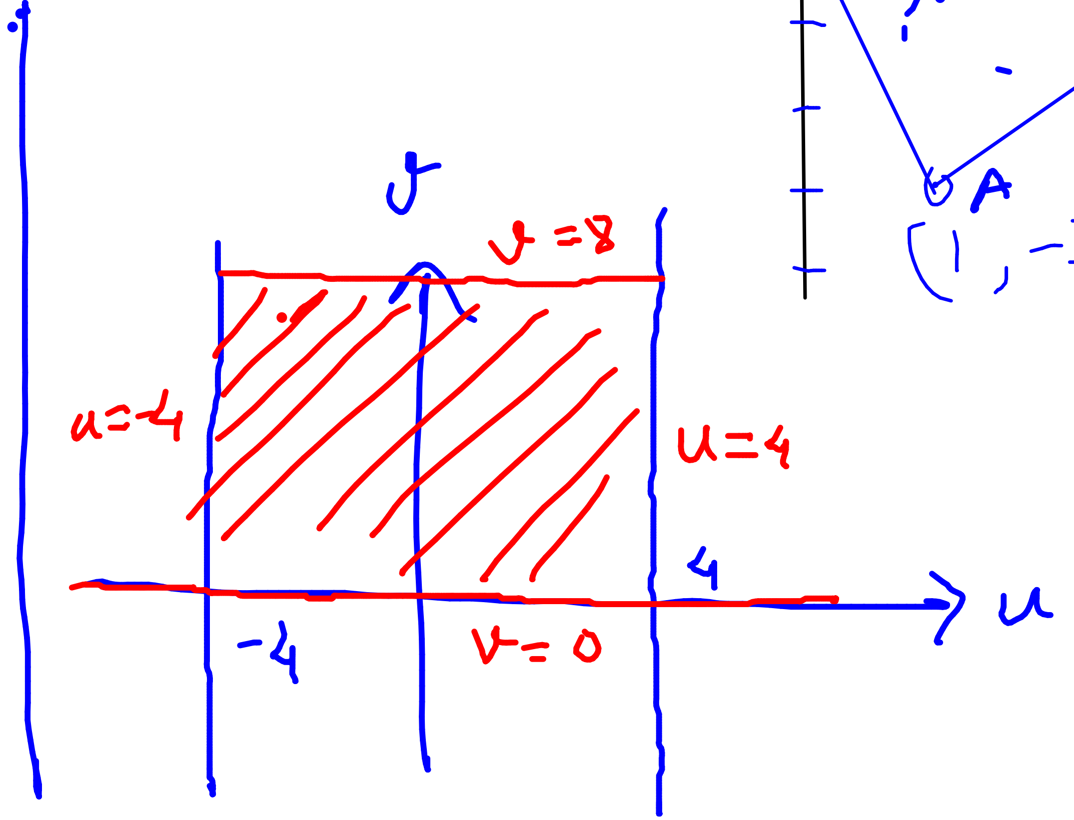
$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)$$

$$v = 0$$

→ BC, $y = -3x+8$

$$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)+8$$

$$v = 8$$



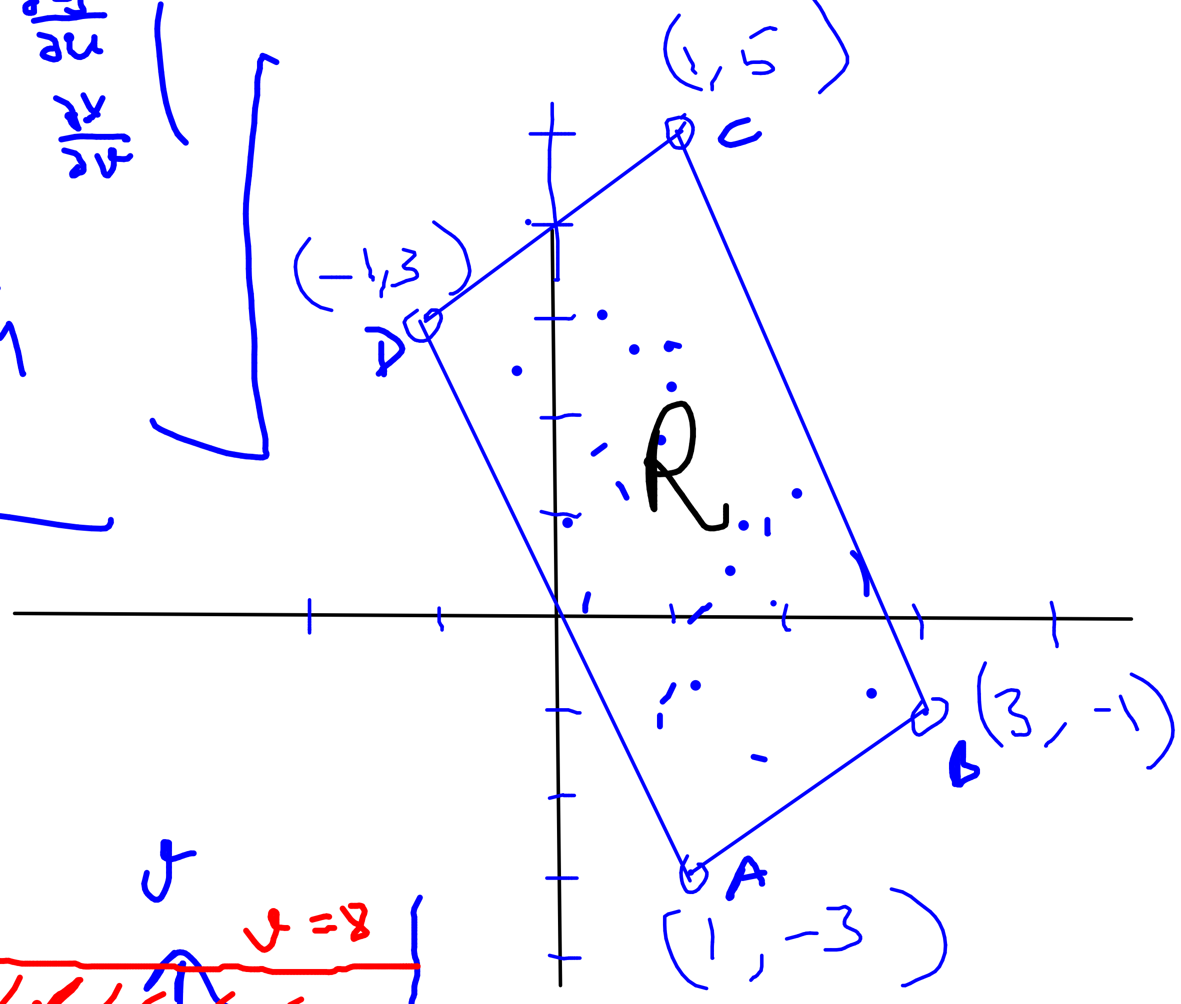
Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

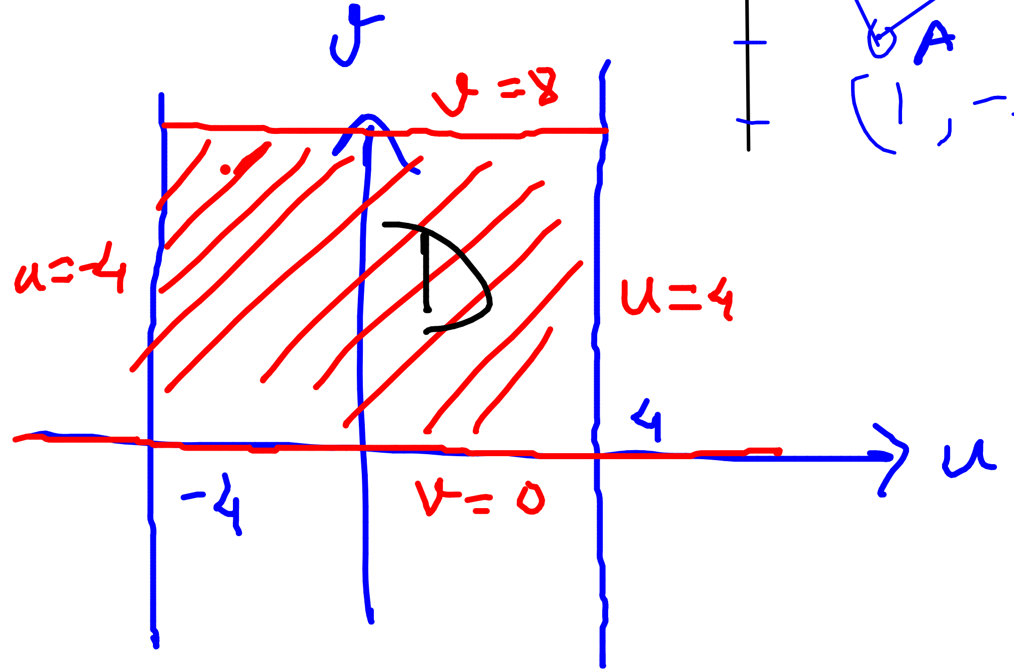
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = 1/4$$

$$4x + 8y = 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u)$$

$$= 3v - 5u$$



$$\iint_R (4x+8y)\, dR = \iint_D (3v-5u)\,(\text{Jacobian})\, dD$$



$$= \int_0^8 \int_{-4}^4 (3v - 5u)\left(\frac{1}{4}\right) du\, dv$$

$$= 192$$

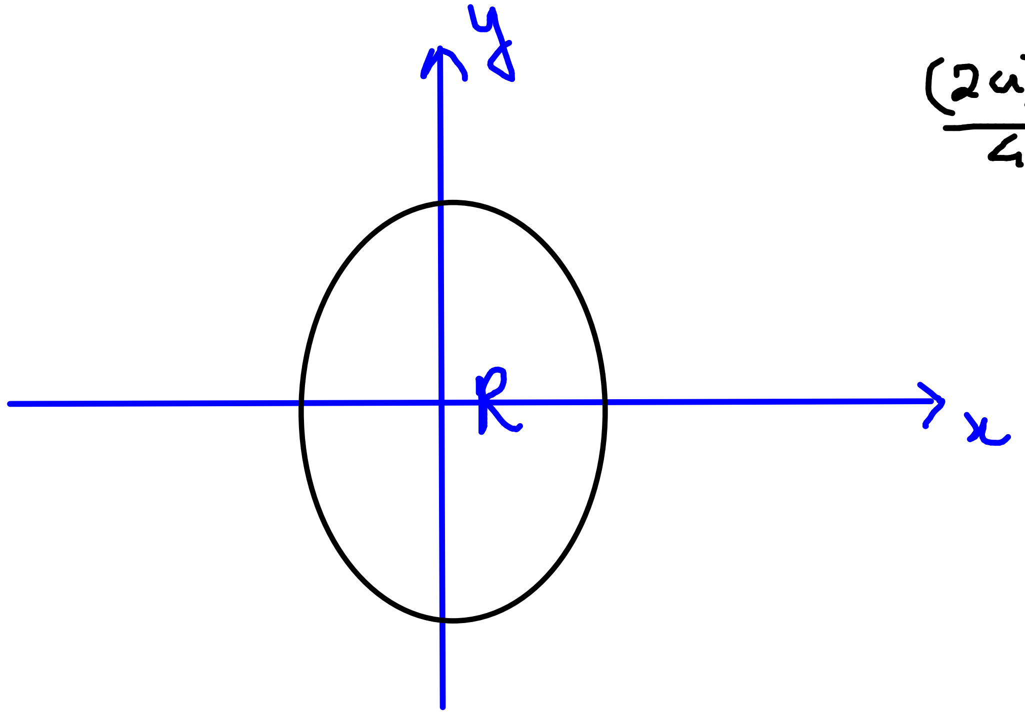
Use the given transformation to evaluate the integral.

$\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $x = 2u$, $y = 3v$

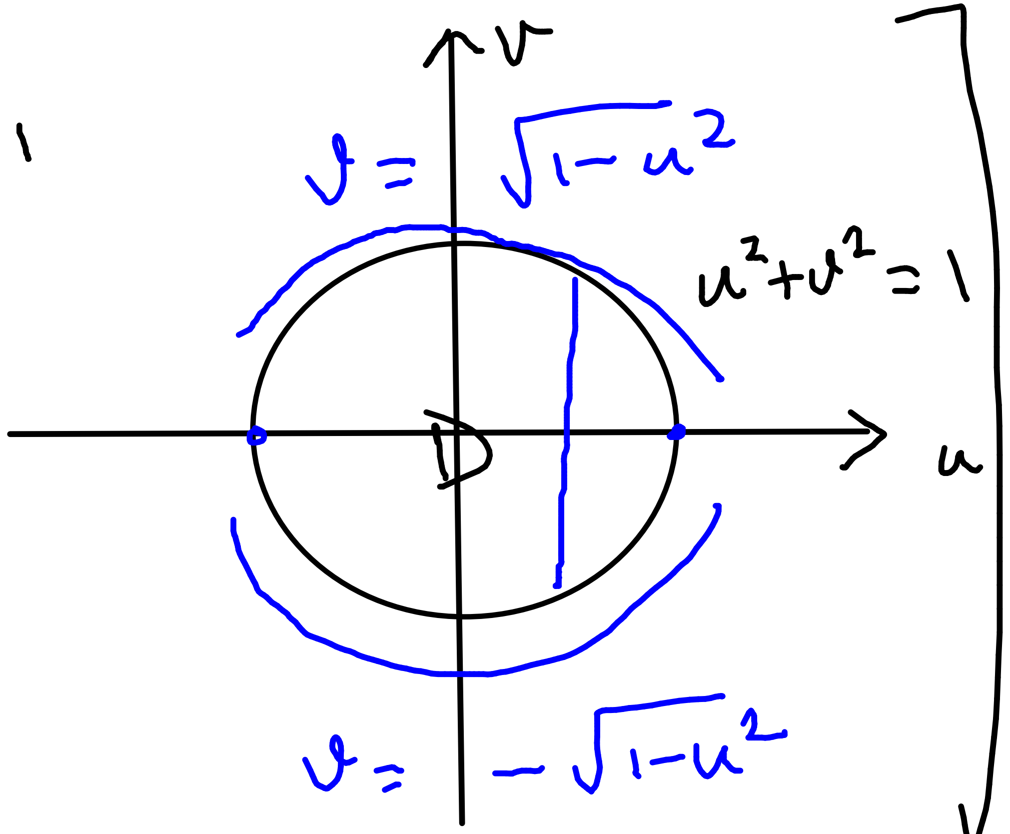
$$\iint_R x^2\, dA = \iint_D (2u)^2 \cdot (\text{Jacobian})\, dD$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{(2u)^2}{4} + \frac{(3v)^2}{9} = 1$$



Jacobian $= 6$



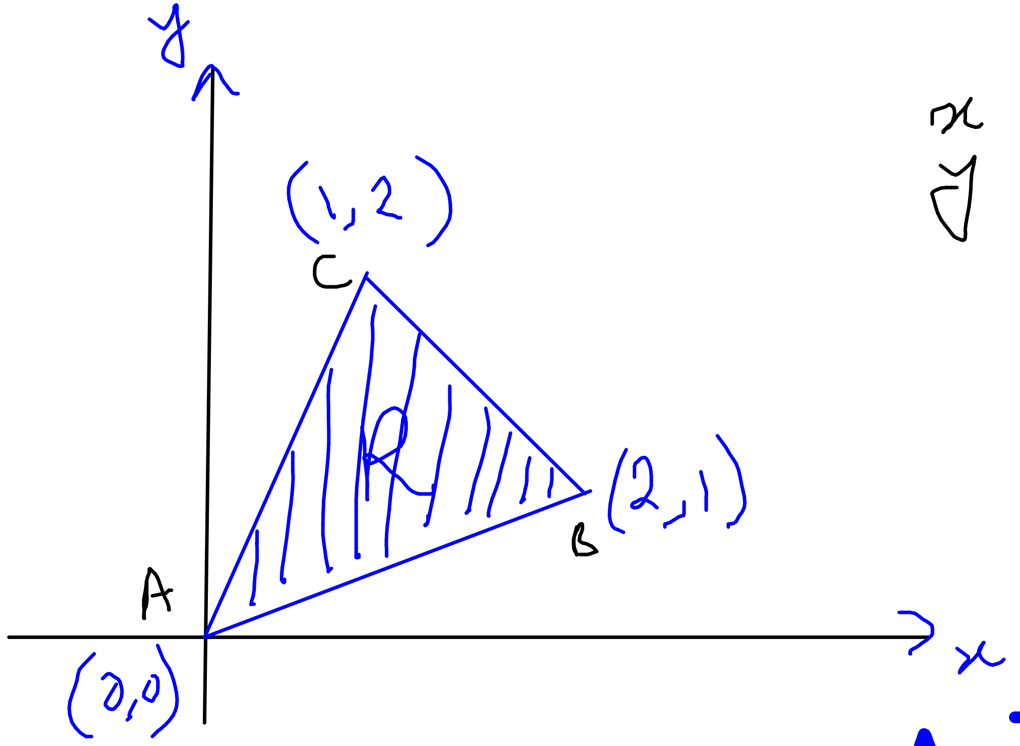
$$= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 4u^2 \cdot 6\, dv\, du$$

$= \_\_\_\_\_\_$

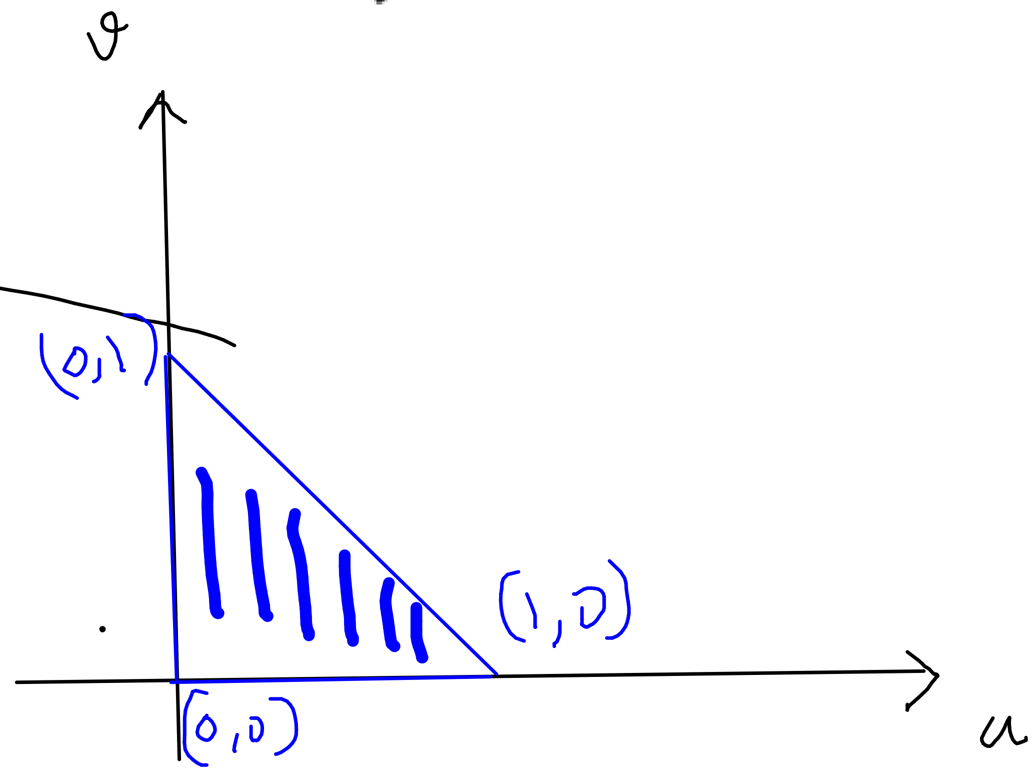
complete it.

Use the given transformation to evaluate the integral.

$\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $x = 2u + v$, $y = u + 2v$



$x = 2u + v$
$y = u + 2v$



AB

$x = 2y$

$2u + v = 2(u + 2v)$

$v = 0$

AC

$y = 2x$

$u + 2v = 2(2u + v)$

$u = 0$

BC

$x + y = 3$

$(2u + v) + (u + 2v) = 3$

$u + v = 1$

$$\iint_R (x - 3y)\, dA = \int_B^A \int_C^D (\;E\;)(\;F\;)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} (-u-5v)\,(3)\,dv\,du$$

Jacobian

$$= -3$$

Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

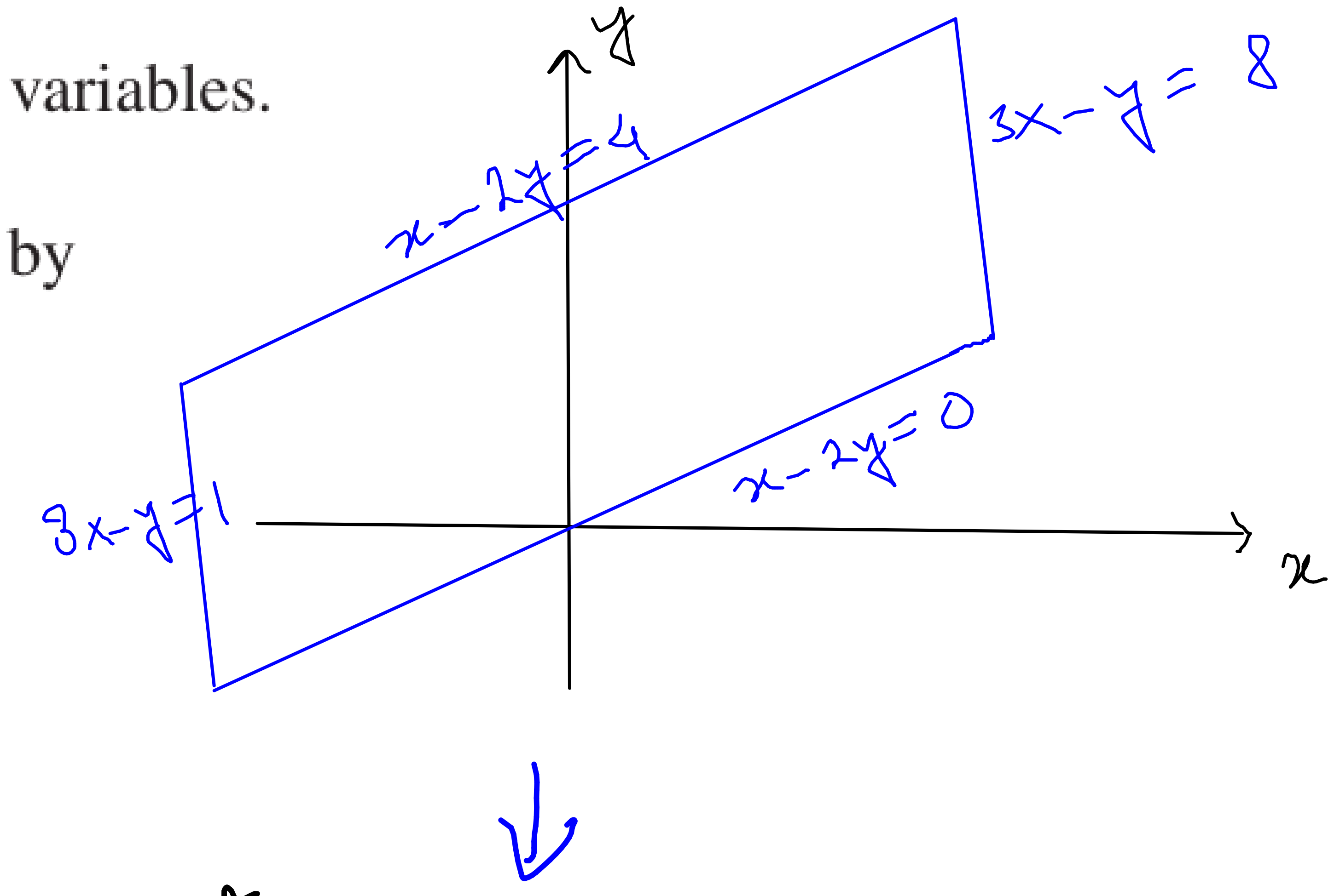
$$x - 3y = (2u+v) - 3(u+2v)$$

Evaluate the integral by making an appropriate change of variables.

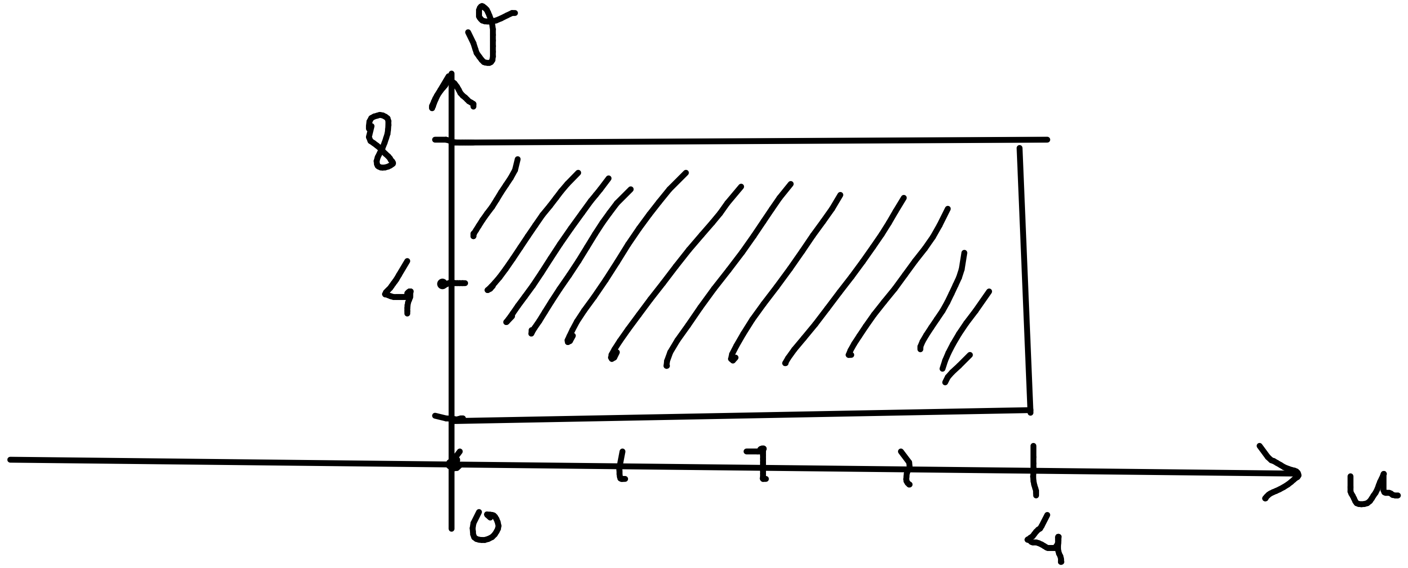
$$\iint_R \frac{x-2y}{3x-y}\,dA,$$ where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$

$u = x-2y$ | $x = (2v-u)/5$

$v = 3x-y$ | $y = (v-3u)/5$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{-3}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}$$

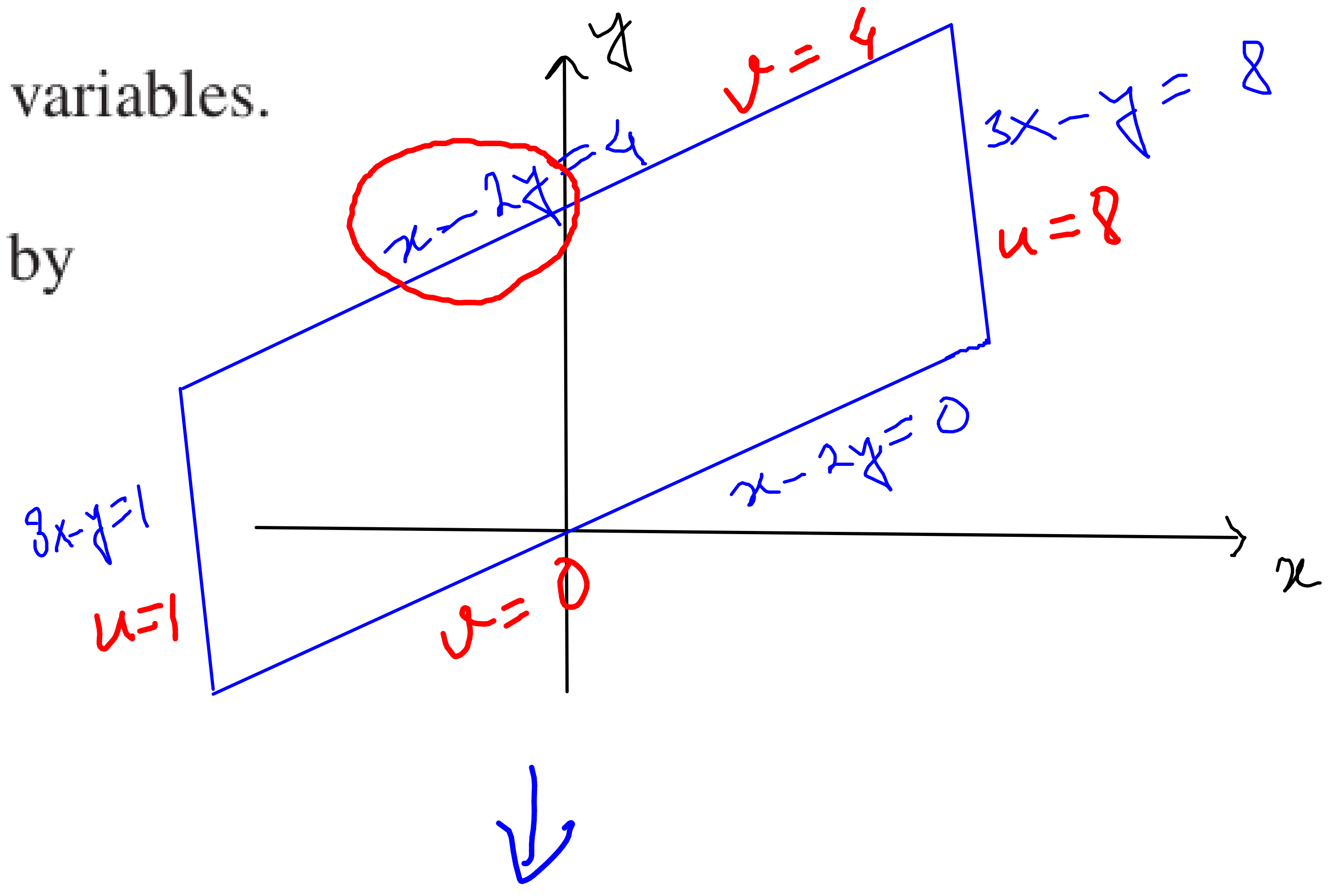


$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4\int_1^8 \frac{u}{v}\left(\frac{1}{5}\right) dv\,du$$

$$= (8\log 8)/5$$

Evaluate the integral by making an appropriate change of variables.

$$\iint_R \frac{x-2y}{3x-y}\,dA,$$ where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$



$$u = 3x - y$$

$$v = x - 2y$$

Evaluate the integral by making an appropriate change of variables.

$\iint_R e^{x+y}\,dA$, where R is given by the inequality $|x| + |y| \leqslant 1$

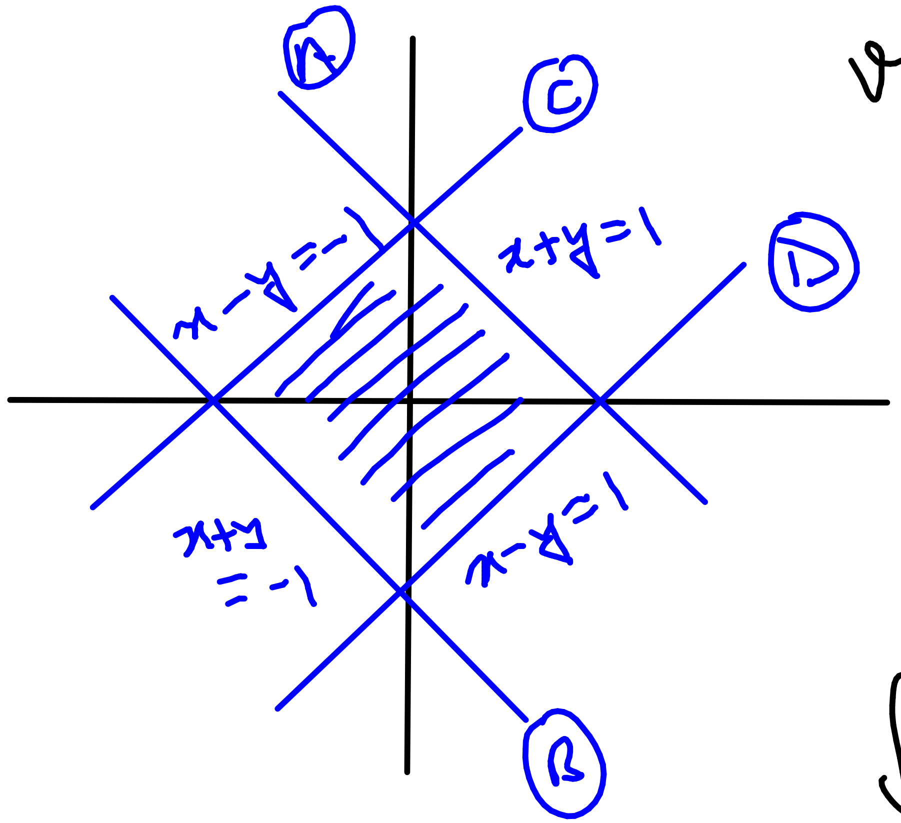
$|x| + |y| \leq 1$

$u = x+y \qquad x = (u+v)/2$

$v = x-y \qquad y = (u-v)/2$

$(\pm x) + (\pm y) \leq 1$

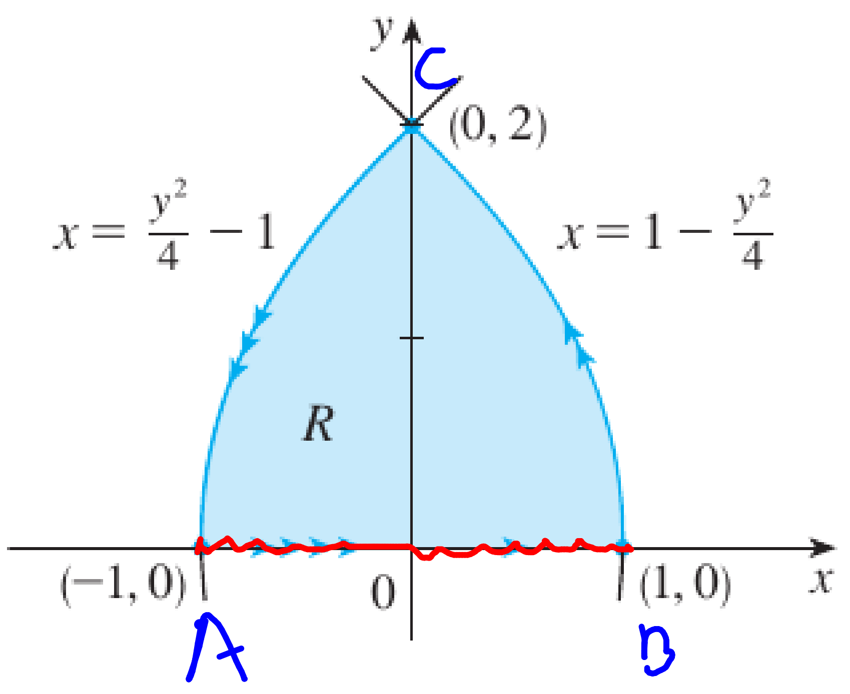
$x+y \leq 1$	$x-y \leq 1$	$-x+y \leq 1$	$-x-y \leq 1$
A	B	C	D



$$\iint_R e^{x+y}\,dA = \int_A^B \int_C^D (E)(F)\,du\,dv = \int_{-1}^{1}\int_{-1}^{1} e^u \left(\tfrac{1}{2}\right) du\,dv$$

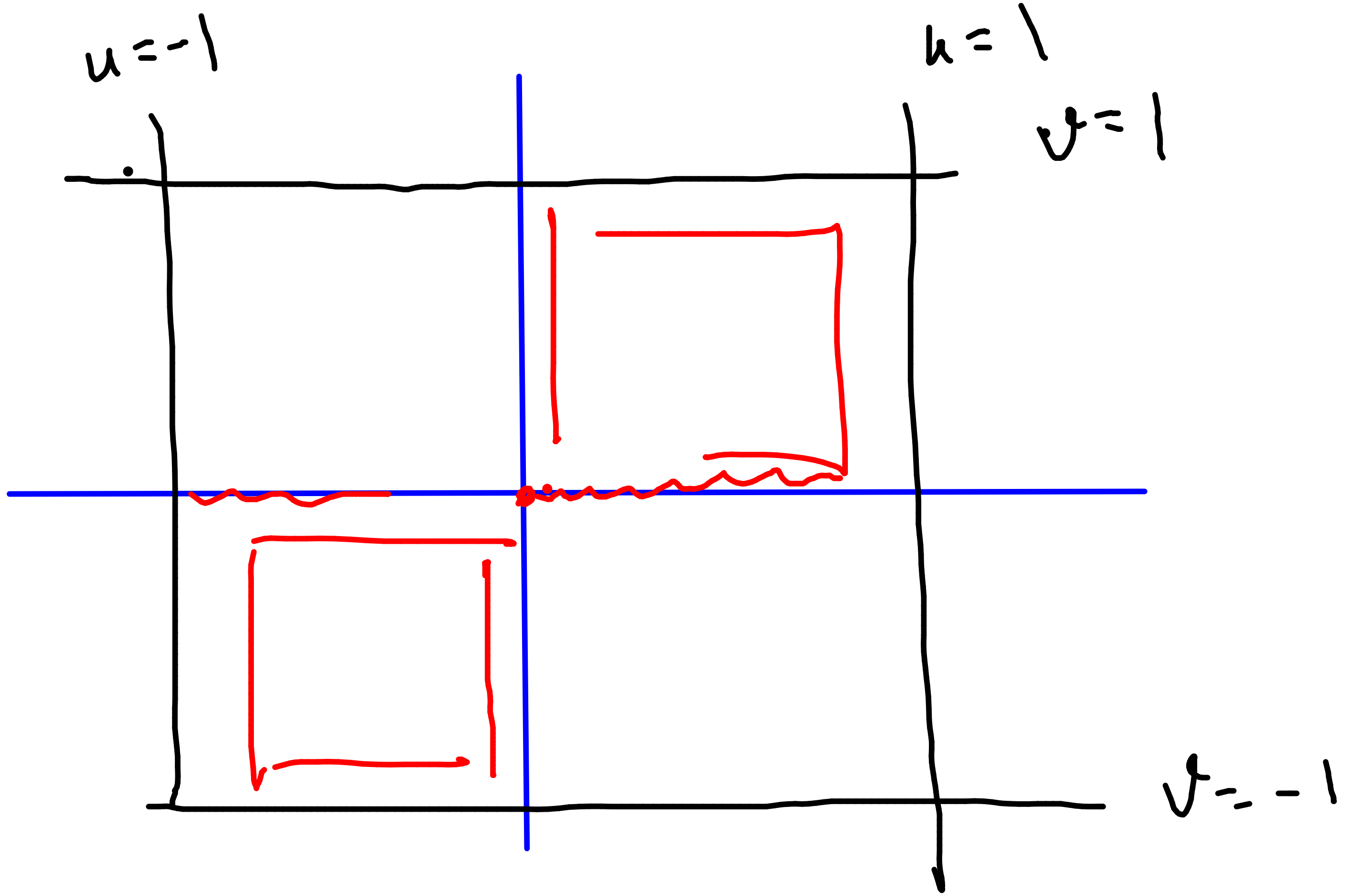
$= e - \frac{1}{e}$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.



BC :

$u = \pm 1$



OB $y = 0$, $x \geq 0$

$u = 0$ or $v = 0$ $\left.\begin{matrix} \\ [u^2 \geq v^2] \end{matrix}\right\} \Rightarrow$

AC

$y^2 = 4 + 4x$

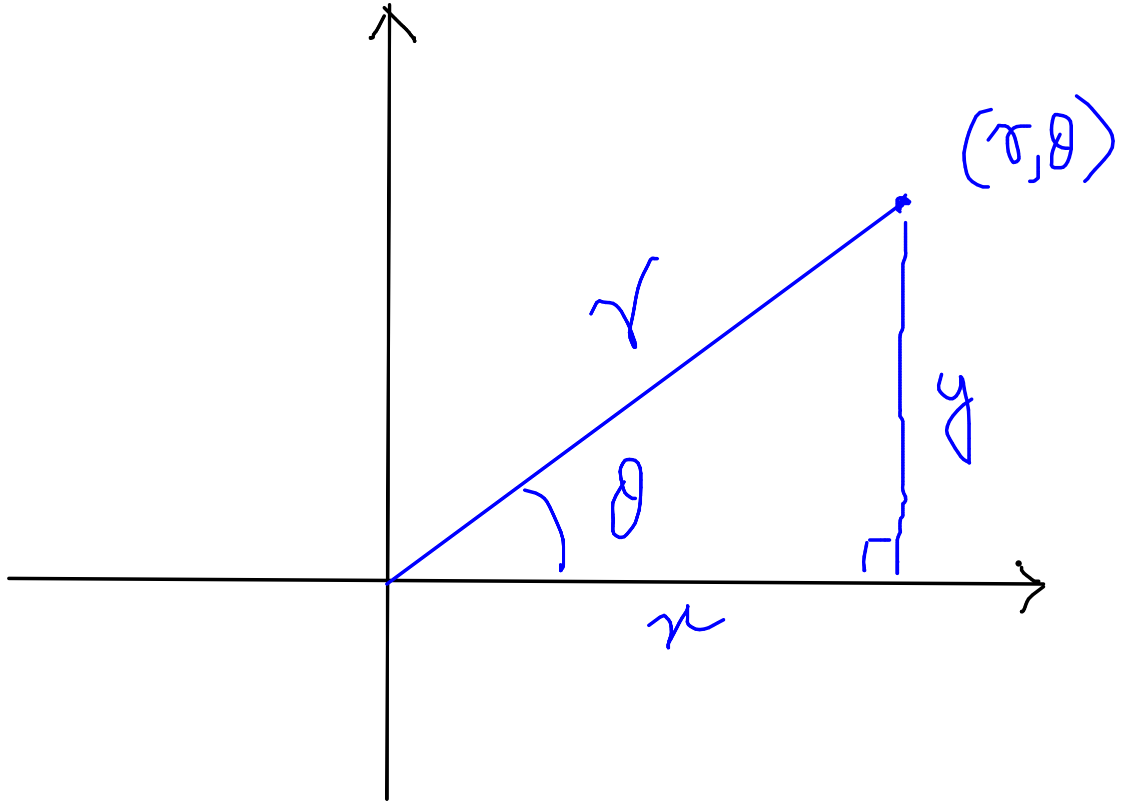
$u^2 v^2 = 1 + u^2 - v^2$

$1 + u^2 - v^2 - u^2 v^2 = 0$

$\cancel{(1+u^2)}(1 - v^2) = 0$

$v = \pm 1$

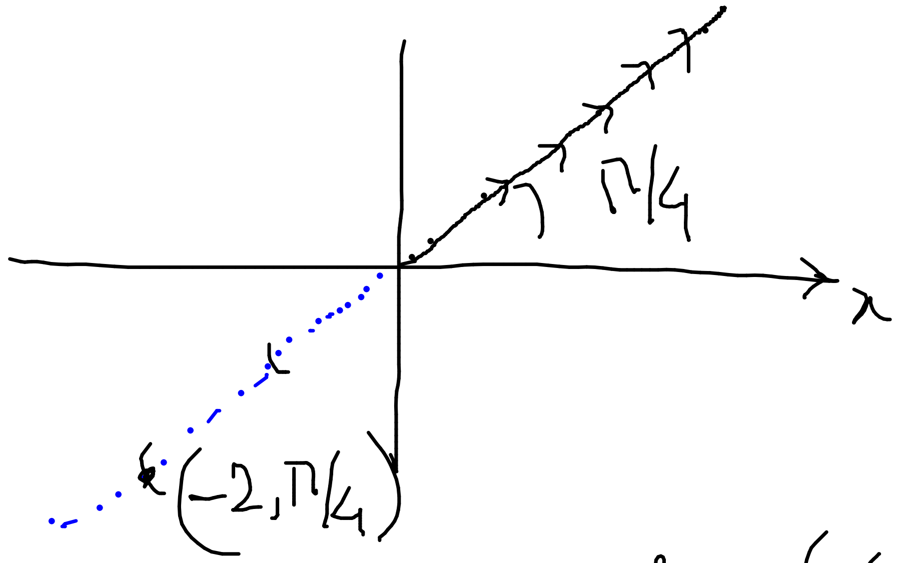
Recall polar coordinates:



$x = r\cos\theta$

$y = r\sin\theta$

$\boxed{\theta = \pi/4}$



mark the point $(r, \theta) = (-2, \pi/4)$

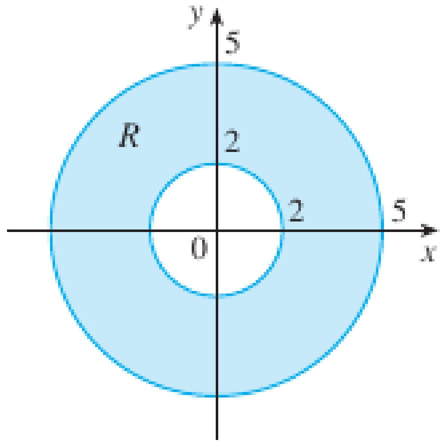
Q. For each region: choose whether it is more convenient to describe the region in xy- or r-θ

①

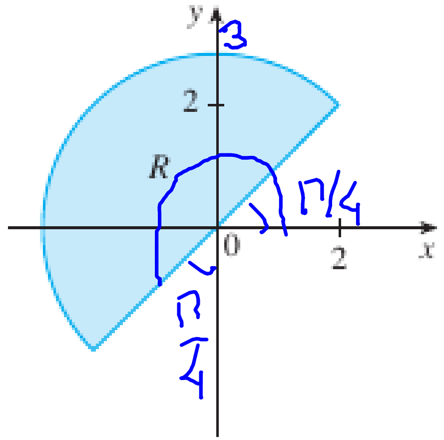
$2 \le r \le 5$

$0 \le \theta \le 2\pi$

1.



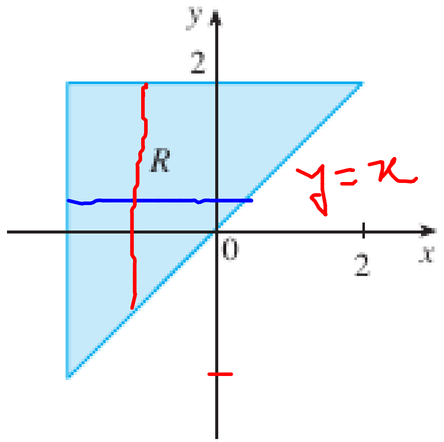
2.



$0 \le r \le 3$

$\frac{\pi}{4} \le \theta \le 5\pi/4$

3.



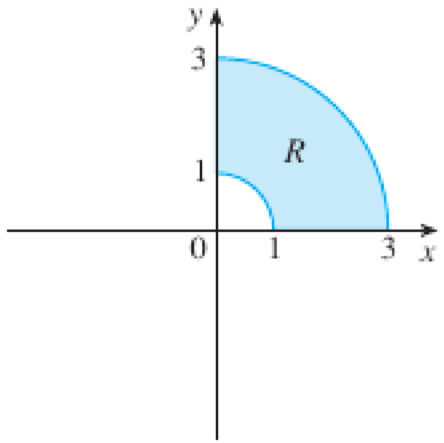
$-2 \le y \le 2$

$-2 \le x \le y$

$-2 \le x \le 2$

$x \le y \le 2$

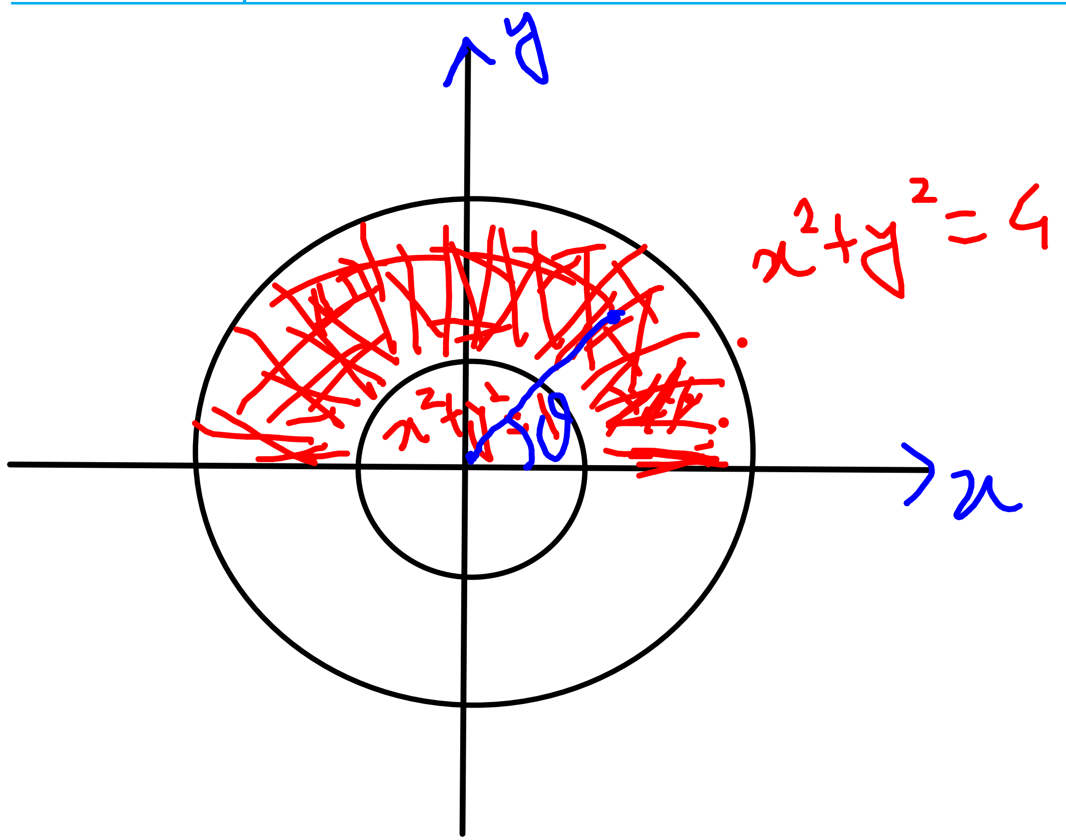
4.



$1 \le r \le 3$

$0 \le \theta \le \pi/2$

12.3 DOUBLE INTEGRALS IN POLAR COORDINATES



range of r & θ for the shaded region

$0 \le \theta \le \pi$

$1 \le r \le 2$

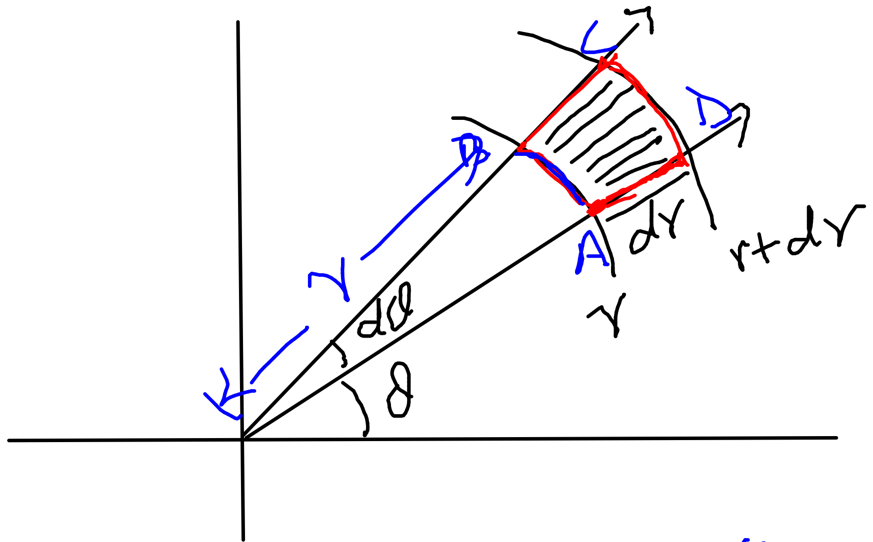
$x = r\cos\theta$

$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$dx\,dy \approx r\,dr\,d\theta$

$\square dy$
dx

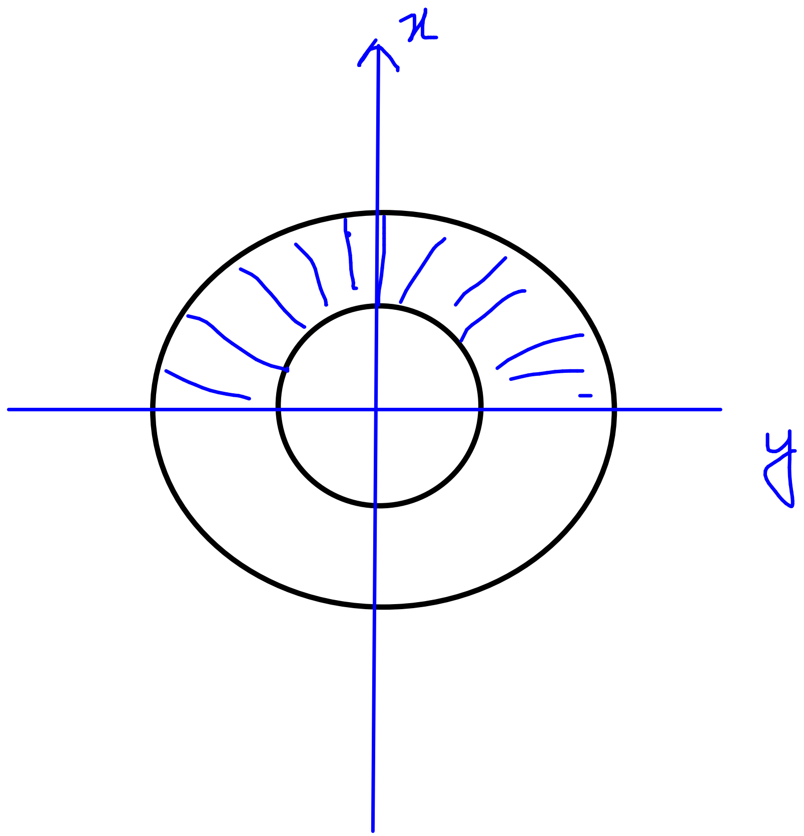


$AB \approx r\,d\theta$

$AD = dr$

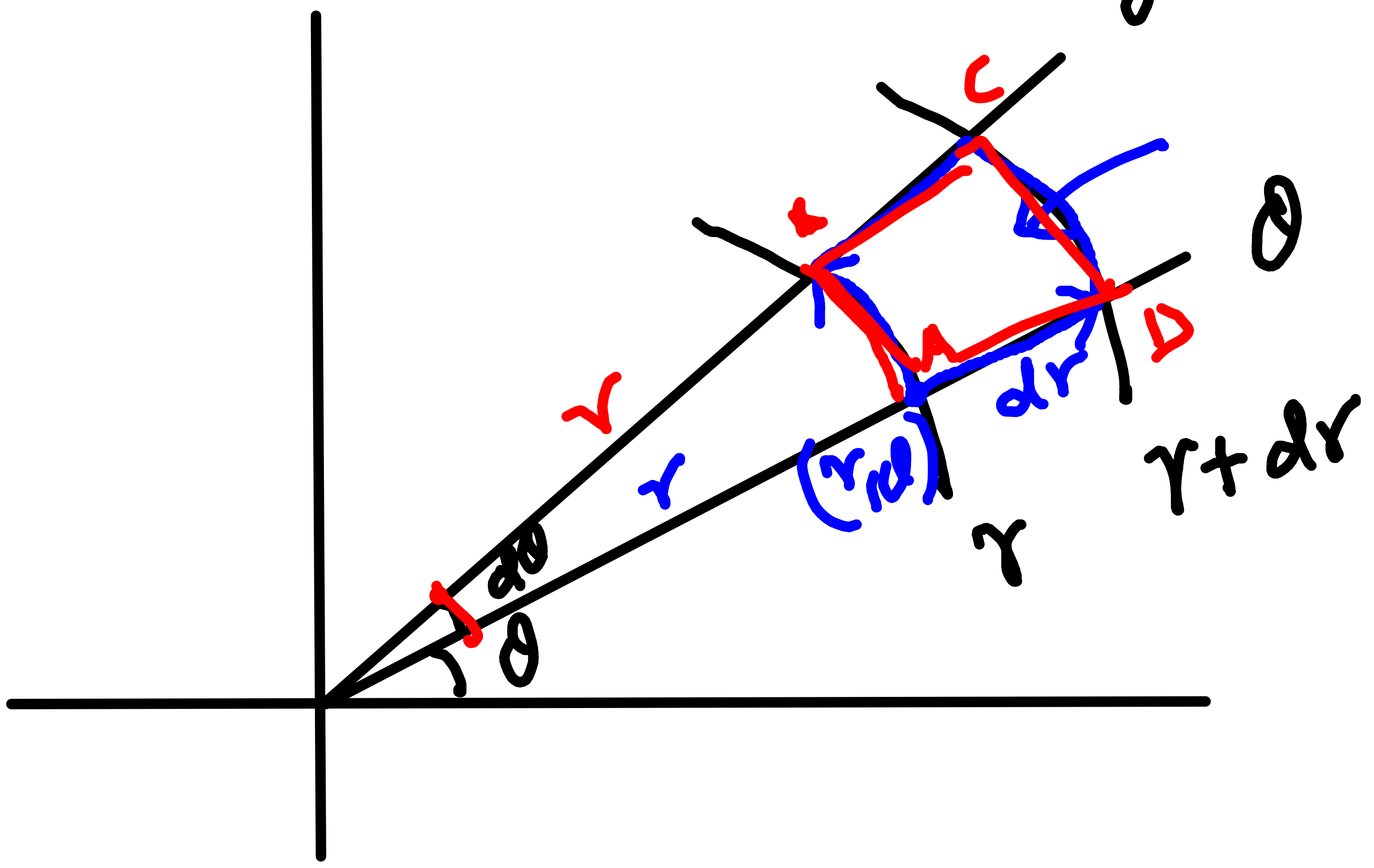
area $(ABCD) = (r\,d\theta)\,dr = r\,dr\,d\theta$

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$\int_1^2 \int_0^{\pi} \left(3r\cos\theta + 4r^2\sin^2\theta\right) r \, d\theta \, dr$$

$$= \frac{15\pi}{2}$$



$\approx r\,dr\,d\theta$

$\overset{\frown}{AB} = r\,d\theta$

$AD = dr$

$\text{area}(ABCD) \approx (r\,d\theta)(dr)$

$= \underline{\underline{r\,dr\,d\theta}}$

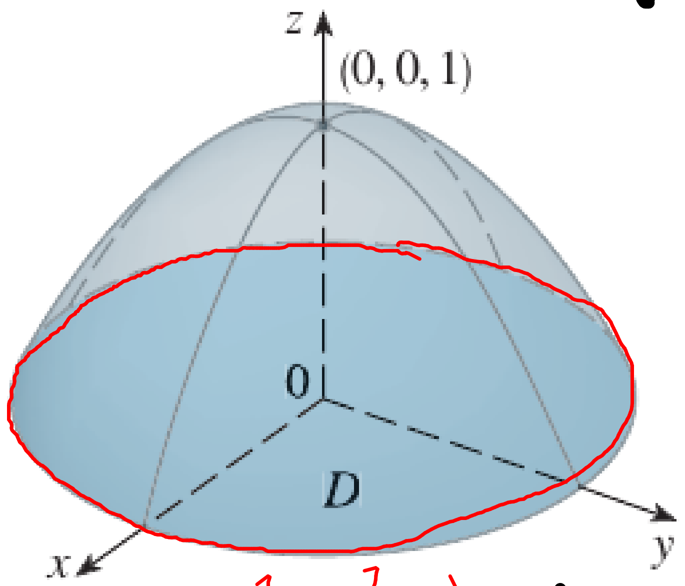
$z = x^2+y^2$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

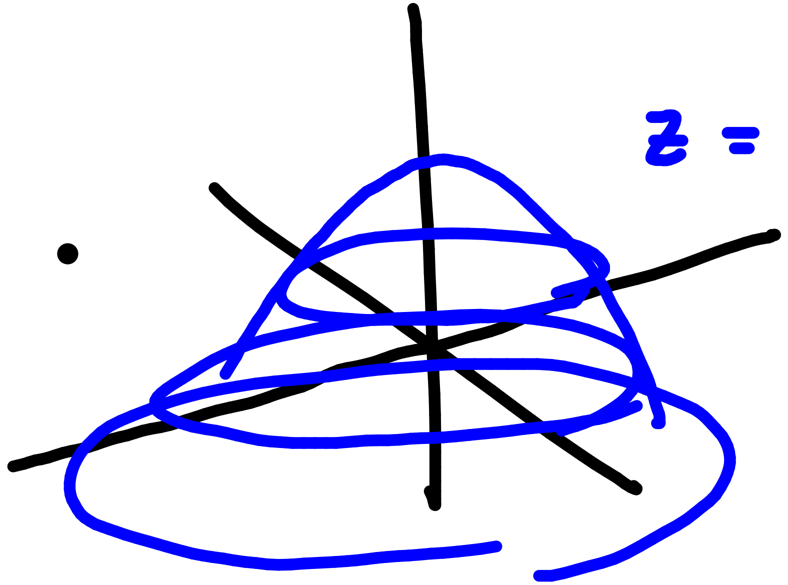
$z = -x^2-y^2$

$z = 1-x^2-y^2$

$$V = \iint_D (1-x^2-y^2)\, dA$$



$x^2+y^2=1$



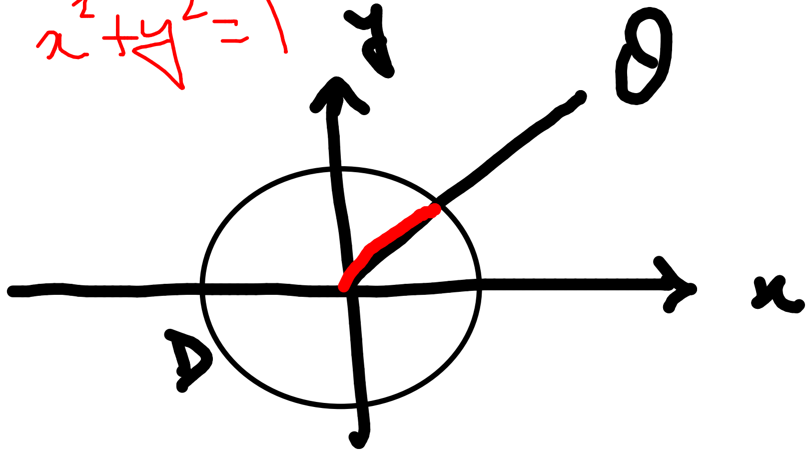
(in polar coordinates

$$= \int_A^B \int_C^D (E)(F)\, dr\, d\theta$$

$1-x^2-y^2 = z = 0$

$1-x^2-y^2 = 0$

$x^2+y^2 = 1$



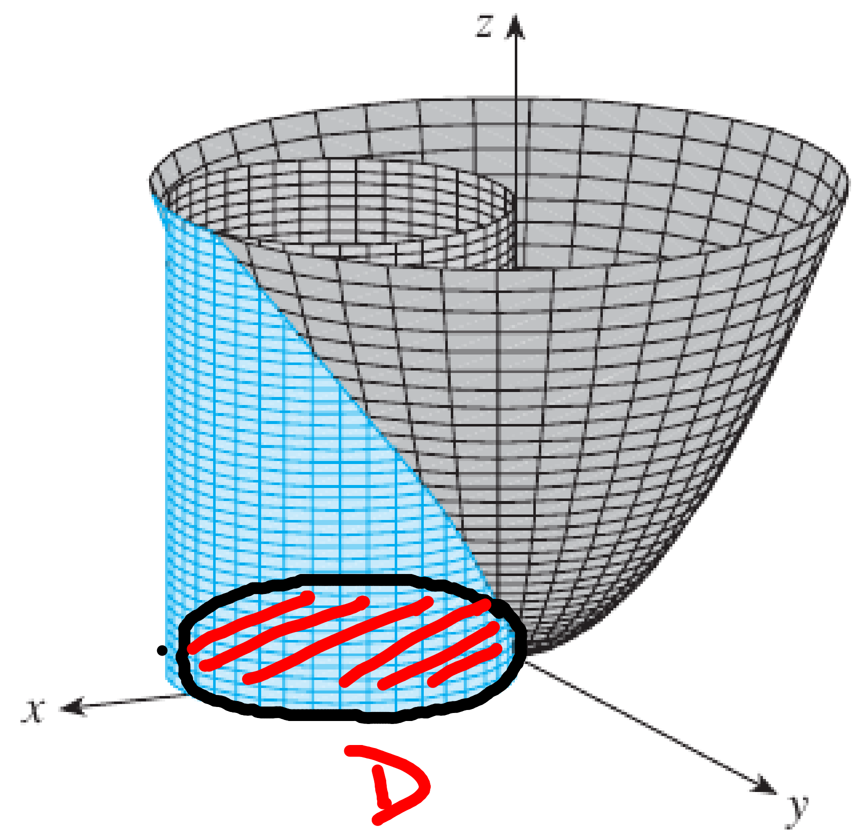
$$= \int_0^{2\pi} \int_0^{1} (1-r^2)\, r \; dr\, d\theta$$

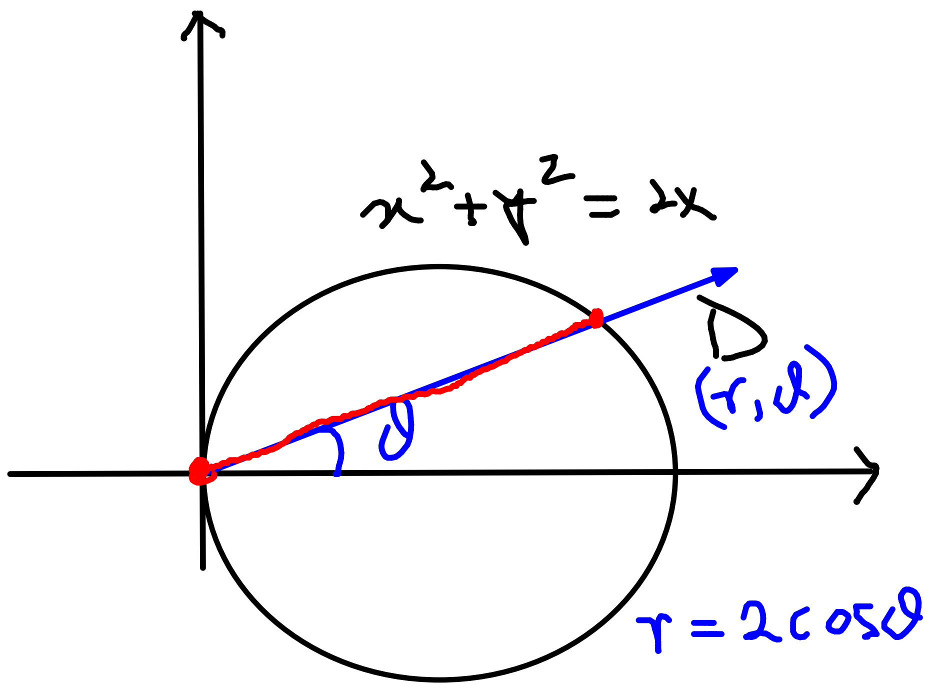
$$= \pi/2$$

$$1-x^2-y^2 = 1-(r\cos\theta)^2-(r\sin\theta)^2$$

$$= 1-r^2(\cos^2\theta+\sin^2\theta) = 1-r^2$$

EXAMPLE 3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.





$$x^2+y^2 = 2x$$

$$(x-1)^2+y^2 = 1$$

circle in xy plane

but cylinder in xyz space

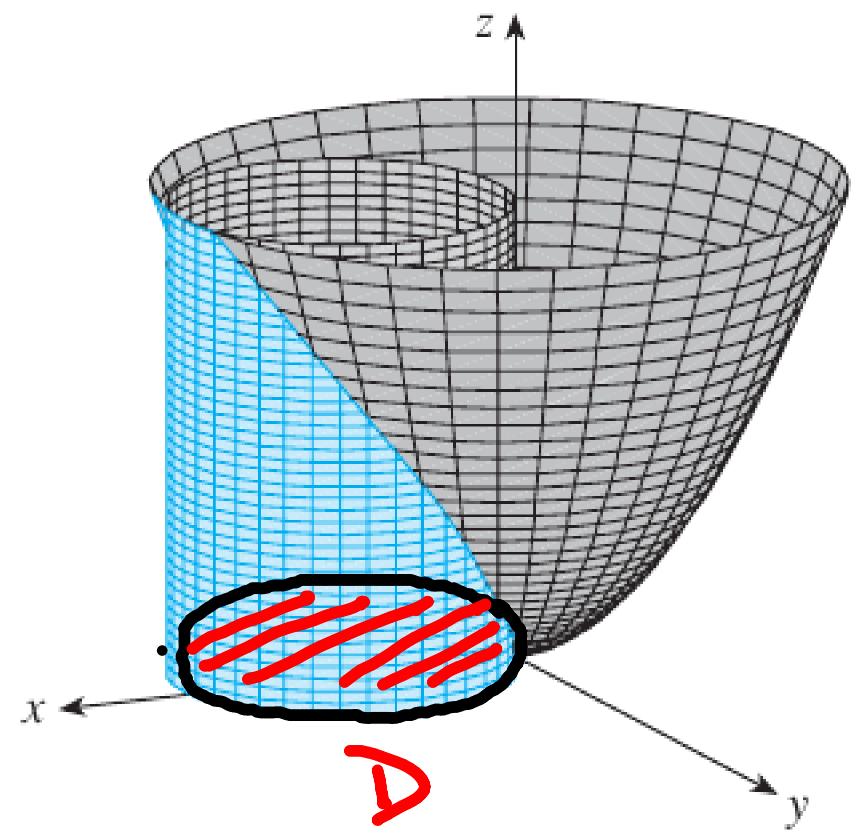
$$x^2+y^2 = 2x$$

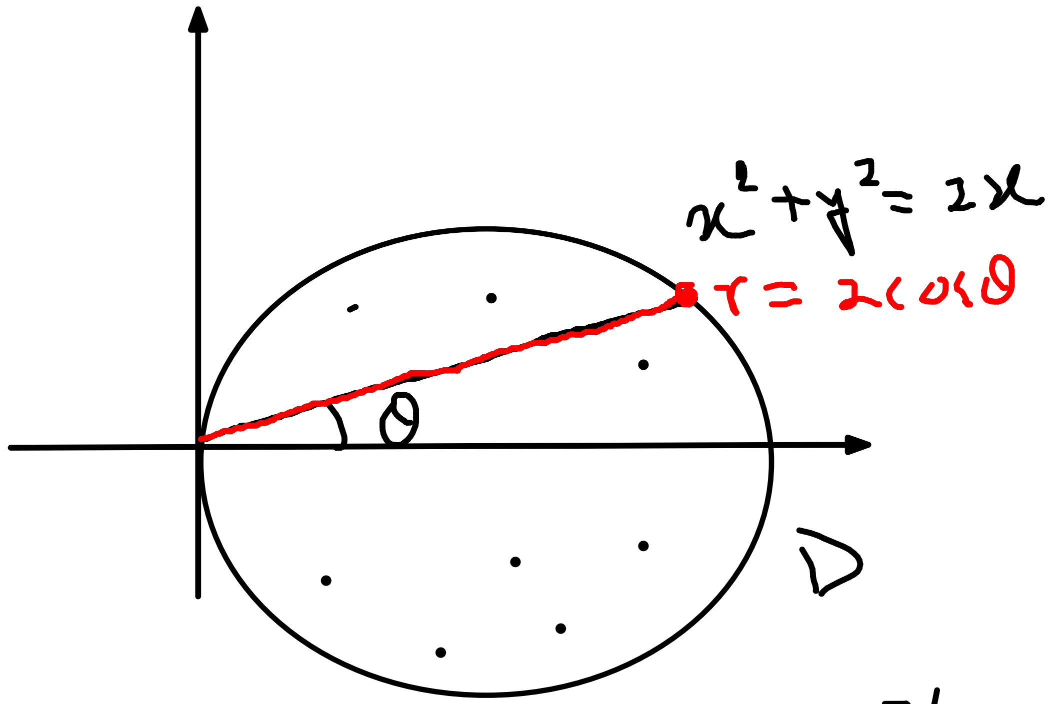
$$r^2 = 2r\cos\theta$$

$$\boxed{r = 2\cos\theta}$$

$$V = \iint_D x^2+y^2 \, dA \xrightarrow[\text{polar coordinates}]{\text{in}} \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \; r\,dr\,d\theta$$

EXAMPLE 3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

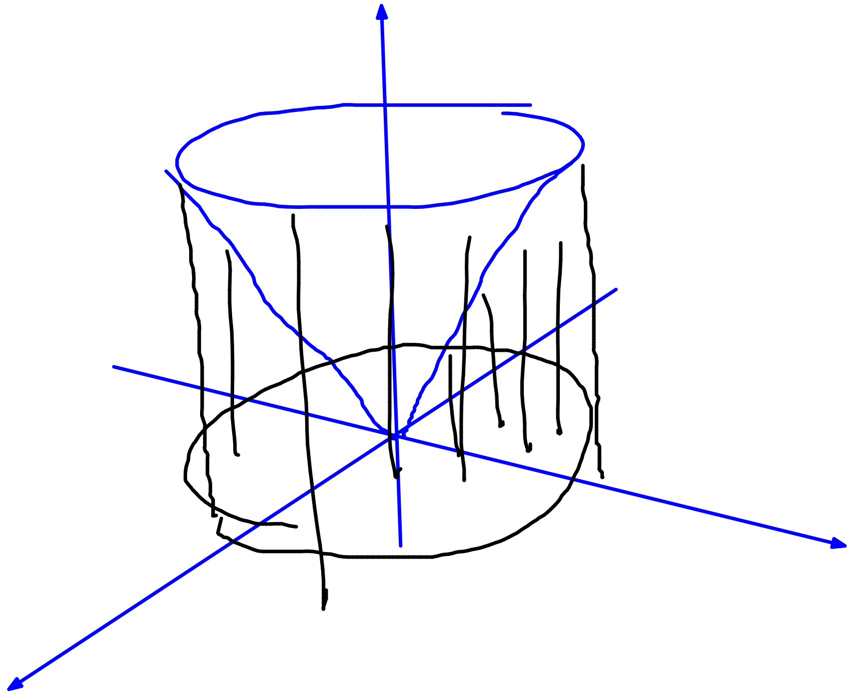




$$V = \iint_D (x^2+y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \; r\, dr\, d\theta$$

Use polar coordinates to find the volume of the given solid.

Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leqslant 4$



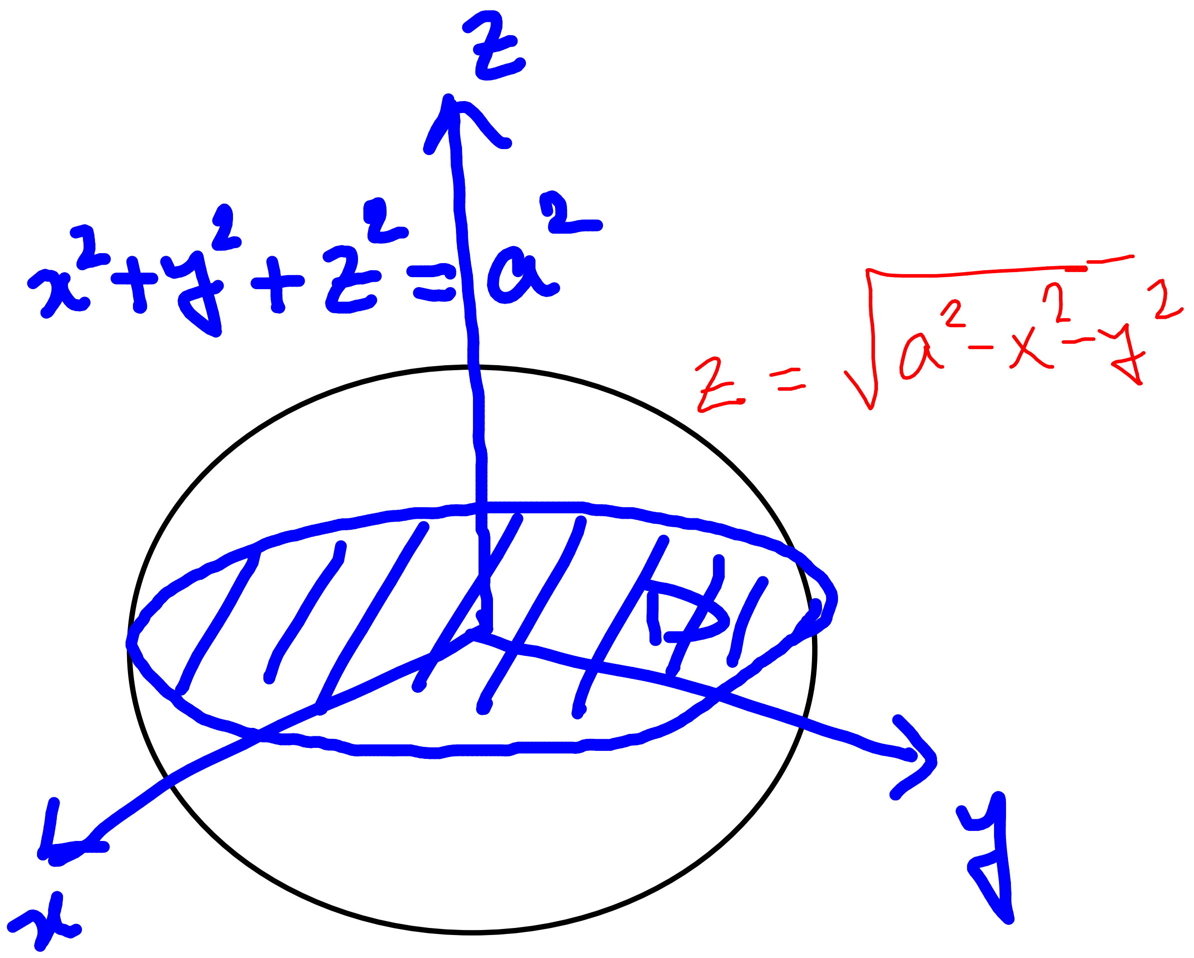
$$\int_0^{2\pi}\int_0^{2} r \; r\,dr\,d\theta = \frac{16}{3}\pi$$

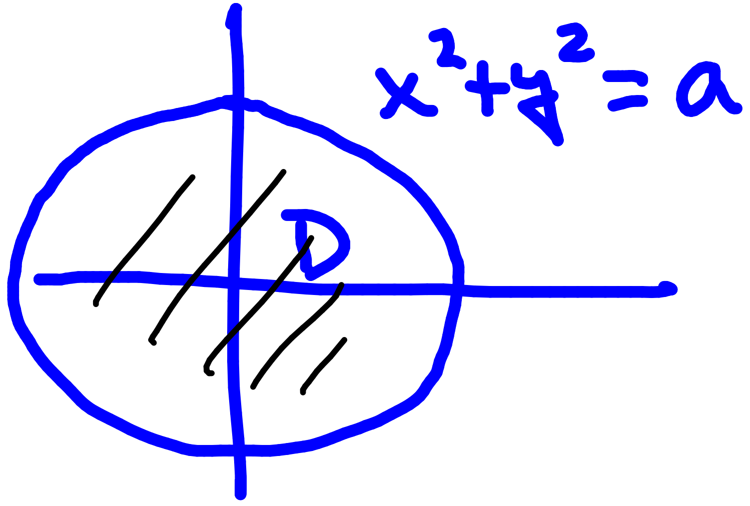
Use polar coordinates to find the volume of the given solid.

A sphere of radius a

Recall sphere volume

$= \frac{4}{3}\pi r^3$





total sphere volume

$= 2$ (volume under the sphere & above the xy plane)

$$= 2 \int_0^{2\pi} \int_0^{a} \sqrt{a^2 - r^2} \, r \, dr \, d\theta \overset{??}{=} 2 \cdot \frac{2}{3} \pi a^3$$

29. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy \, dy \, dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy \, dy \, dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy \, dy \, dx$$

into one double integral. Then evaluate the double integral.

Do yourself.

30. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{a \to \infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

Do yourself

30. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx$$
$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$$

Do yourself

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Do yourself

(d) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

Do yourself