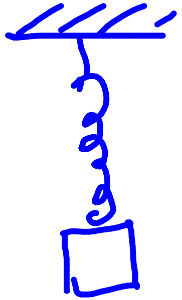
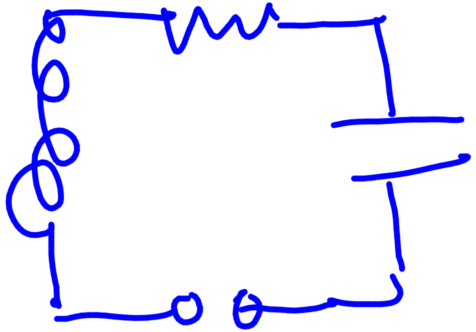
2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x)$$





Method of Undetermined Coefficients

(this section)

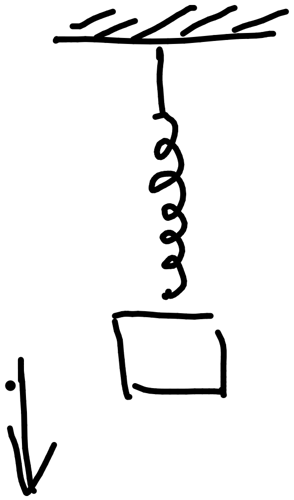
Later: Method of variation of parameters

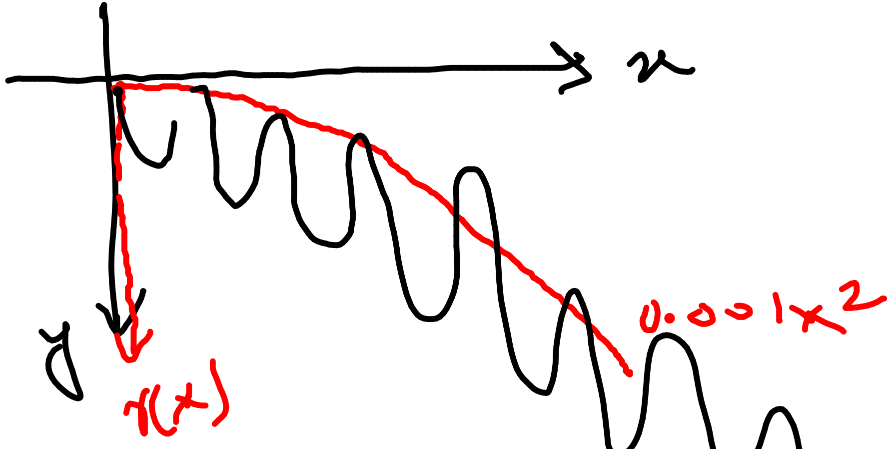
EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

external force

guess the graph of $y(x)$





EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ solve the corresponding homogeneous eqn

$y'' + y = 0$, get $y_h = c_1 y_1 + c_2 y_2$

→ Guess a formula y_p which solves

$y'' + y = 0.001x^2$ [y_p will not have any arbit constants]

→ Final solution: $y = \underbrace{c_1 y_1 + c_2 y_2}_{y_h} + y_p$

EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ solve the corresponding homogeneous eq$_n$

$y'' + y = 0$, get $y_h = c_1 y_1 + c_2 y_2$

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

EXAMPLE 1

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

→ guess a formula y_p which satisfies

$$y'' + y = 0.001x^2$$

→ seems like $y_p = a_0 + a_1 x + a_2 x^2$ will work.

→ find a_0, a_1, a_2 by solving $y_p'' + y_p = 0.001x^2$

& comparing constants, x, x^2 terms in LHS & RHS

$\rightarrow \quad 2a_2 + (a_0 + a_1 x + a_2 x^2) = 0.001 x^2$

$\rightarrow \quad a_2 = 0.001$

$a_1 = 0$

$2a_2 + a_0 = 0$

$a_0 = -0.002$

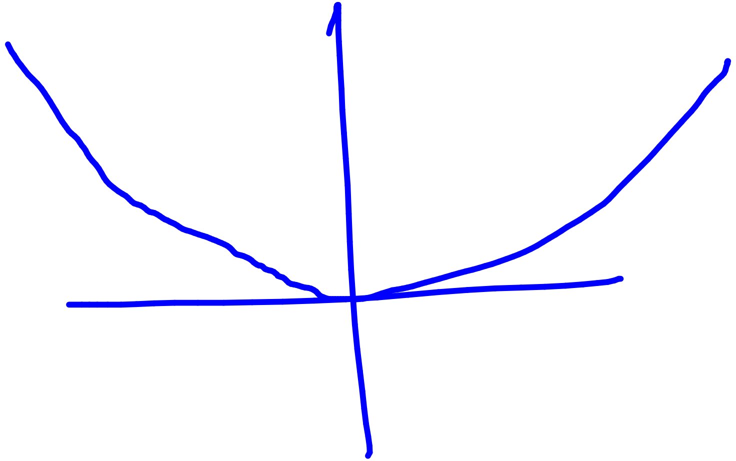
$\rightarrow \quad y_p = -0.002 + 0.001 x^2$

general solution:

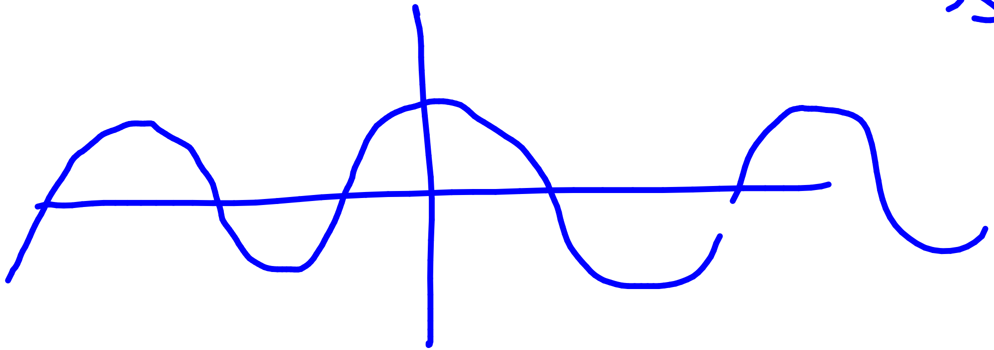
$$y = c_1 \cos x + c_2 \sin x - 0.002 + 0.001 x^2$$

H.W. find c_1, c_2 using other conditions $y(0) = 0$, $y'(0) = 1.5$

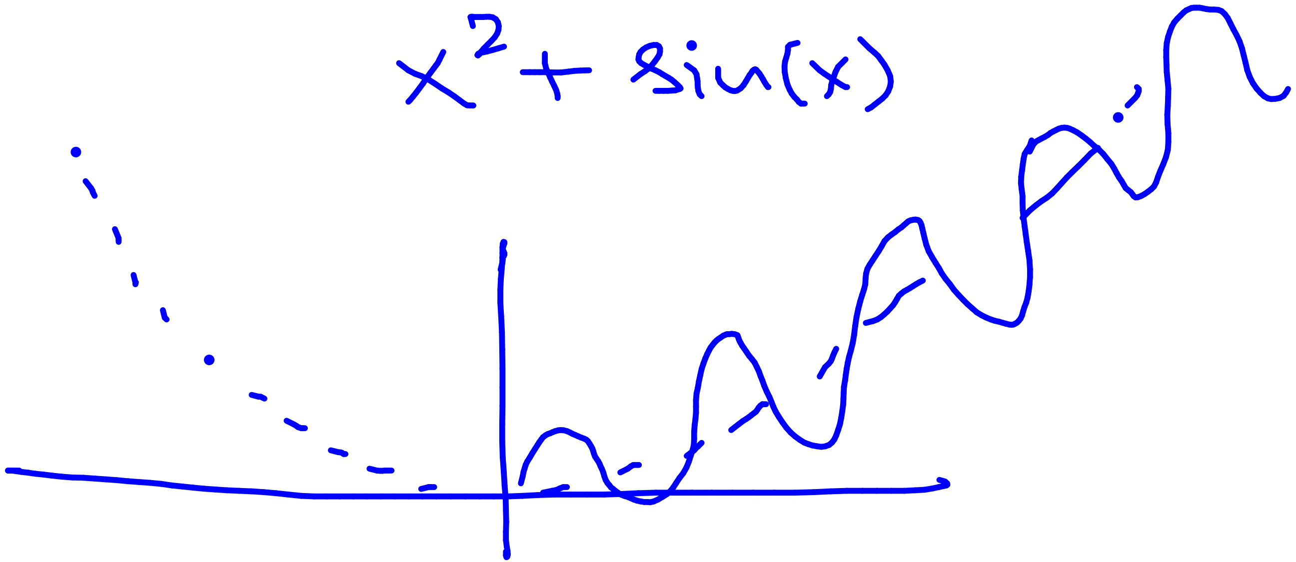
x^2



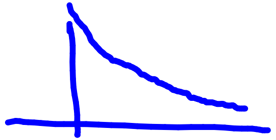
$\sin(x)$



$x^2 + \sin(x)$



EXAMPLE 2



$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

→ Solve the homogeneous part

$y''+3y'+2.25y=0$

$\lambda^2+3\lambda+2.25=0$ $\qquad \lambda = \frac{-3\pm\sqrt{9-9}}{2} = \frac{-3}{2}$

$y_h = e^{-3/2\,x}(c_1x+c_2)$

→ Let's find y_p, try $y_p = ce^{-1.5x}$

find C

will not work (why)

$\rightarrow$ try $y_p = cxe^{-1.5x}$

will not work either (why)

$\rightarrow$ try $y_p = cx^2e^{-1.5x}$

$y_p' = 2cxe^{-1.5x} - 1.5cx^2e^{-1.5x}$

$y_p'' = 2ce^{-1.5x} - 3cxe^{-1.5x}$

$- 3cxe^{-1.5x} + 2.25cx^2e^{-1.5x}$

$= 2ce^{-1.5x} - 6cxe^{-1.5x} + 2.25cx^2e^{-1.5x}$

$$y_p'' + 3y_p' + 2.25\,y_p = -10\,e^{-1.5x}$$

$$2C\,\cancel{e^{-1.5x}} = -10\,\cancel{e^{-1.5x}}$$

$$C = -5$$

→ general solution

$$y = e^{-1.5x}\left(c_1 x + c_2\right) + (-5)x^2 e^{-1.5x}$$

EXAMPLE 3

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) =$$

$y_p = A\cos x + B\sin x + C + Dx$

find y_p by matching cos(x), sin(x), constant & x terms in LHS & RHS.

Today,

→ more problems with undetermined coefficients

→ Resonance

→ LCR circuits equations

Q. Solve this using method of undetermined coefficients

$$y'' + 5y' + 4y = 10e^{-3x}$$

→ homogeneous eqn $y''+5y'+4y=0$

$$y_h = c_1 e^{-4x} + c_2 e^{-x}$$

→ then find y_p which solves

$$y'' + 5y' + 4y = 10e^{-3x}$$

Try $y_p = Ce^{-3x}$

→ find c by solving

$$y_p'' + 5y_p' + 4y_p = 10e^{-3x}$$

$$(9c - 15c + 4c)e^{-3x} = 10e^{-3x}$$

$$-2c\cancel{e^{-3x}} = 10\cancel{e^{-3x}}$$

$$c = -5$$

→ $y_p = -5e^{-3x}$] particular solution

→ Final sol^n $\boxed{y = c_1e^{-4x} + c_2e^{-x} - 5e^{-3x}}$

$$y'' + 3y' + 2y = 12x^2$$

$$2\left(y_p = A + Bx + Cx^2\right)$$

$$3\left(y_p' = B + 2Cx\right)$$

$$y_p'' = 2C$$

$$12x^2 = (2A + 3B + 2C) + (2B + 6C)x + 2Cx^2$$

$$2A + 3B + 2C = 0$$

$$2B + 6C = 0$$

$$2C = 12$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

```
>> M = [2 3 2; 0 2 6; 0 0 2]

M =

     2     3     2
     0     2     6
     0     0     2

>> rhs = [0;0;12]

rhs =

     0
     0
    12

>> ABS = inv(M)*rhs
```

$A = 21$

$B = -18$

$C = 6$

$y_p = 21 - 18x + 6x^2$

general sol^n

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + y_p}$$

8.

$$y'' - 9y = 18\cos\pi x$$

$$-9\left(y_p = A\cos\pi x + B\sin\pi x\right)$$

$$y_p' = -A\pi\sin\pi x + B\pi\cos\pi x$$

$$\left(y_p'' = -A\pi^2\cos\pi x - B\pi^2\sin\pi x\right)$$

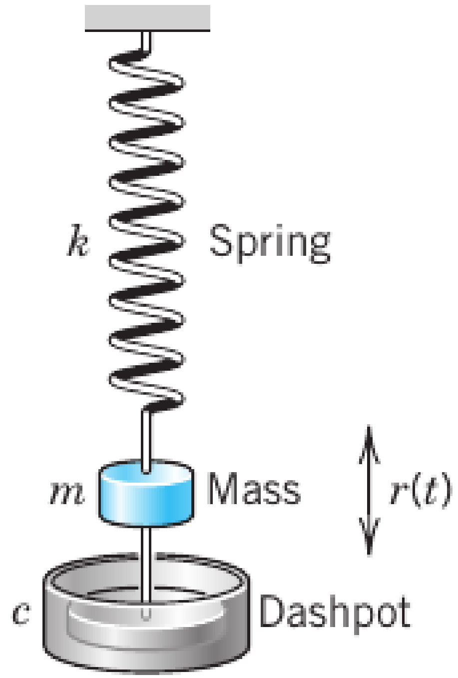
$$18\cos\pi x = -A(9+\pi^2)\cos\pi x - B(9+\pi^2)\sin\pi x$$

$$A = \frac{-18}{9+\pi^2}$$

$$B = 0$$

By comparing sin(πx) & cos(πx) term in LHS & RHS

2.8 Modeling: Forced Oscillations



$$my'' + cy' + ky = r(t).$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

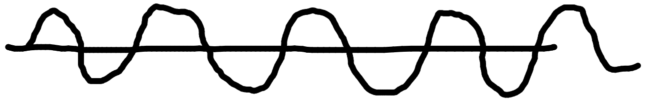
Then we have the nonhomogeneous ODE

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2}$$

Its solution will reveal facts that are fundamental in engineering mathematics and allow us to model resonance.

Q.

$y'' + y = 0$



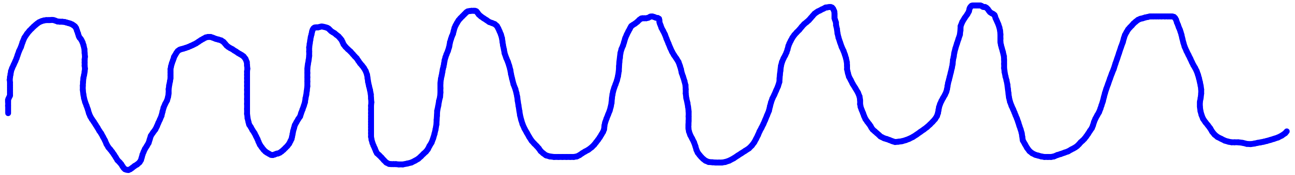
solution : $y = c_1 \cos x + c_2 \sin x$

$= A \cos(x + B)$

now solve

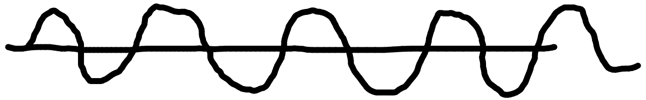
$y'' + y = F_0 \cos(\omega x)$

$$y = c_1 \cos x + c_2 \sin x + \frac{F_0}{1 - \omega^2} \cos(\omega x)$$



Q.

$$y'' + y = 0$$



solution : $y = c_1 \cos x + c_2 \sin x$

$= A \cos(x + B)$

now solve

$$y'' + y = F_0 \cos(\omega x)$$

$$y = c_1 \cos x + c_2 \sin x + \frac{F_0}{1-\omega^2} \cos(\omega x)$$

what if $\omega = 1$??

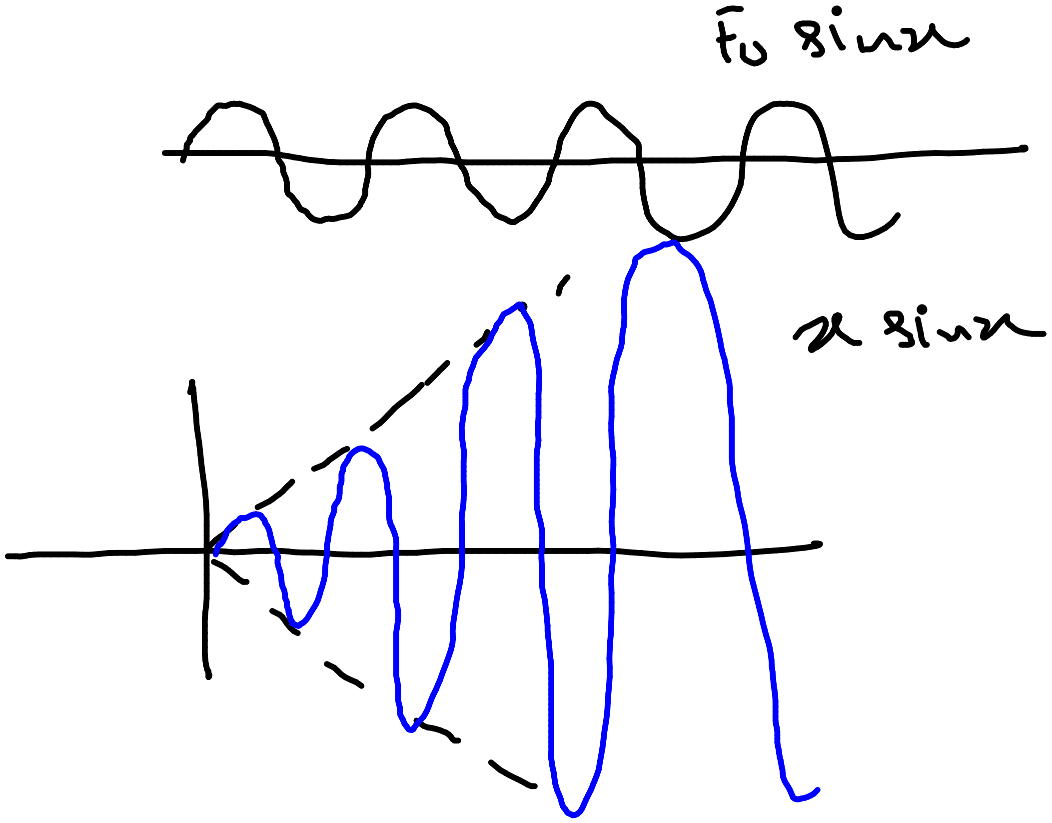
if $\omega = 1$, $y_p = Ax \cancel{\cos x} + Bx \cancel{\sin x}$

$$y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$

$$y_p'' = -2A\sin x - A\cancel{x\cos x} + 2B\cos x - B\cancel{x\sin x}$$

$$F_0\cos x = -2A\sin x + 2B\cos x$$

$$\Rightarrow \quad A = 0, \quad B = \frac{F_0}{2} \qquad \Big| \qquad y_p = \frac{F_0}{2}x\sin x$$



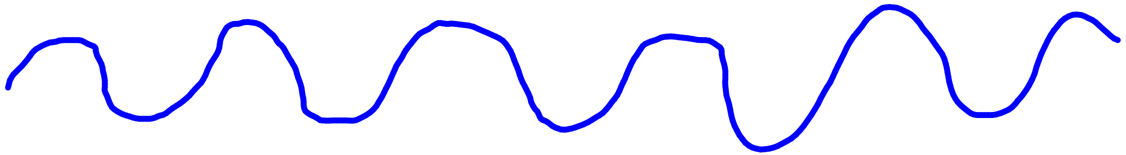
$\boxed{\text{Resonance}}$

System's & input force frequency match

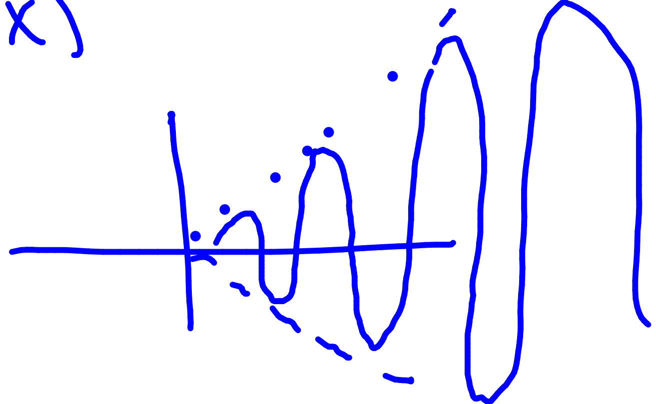
Q. $y'' + \Omega^2 y = F_0 \cos(\omega x)$

draw a rough sketch of the general solution.

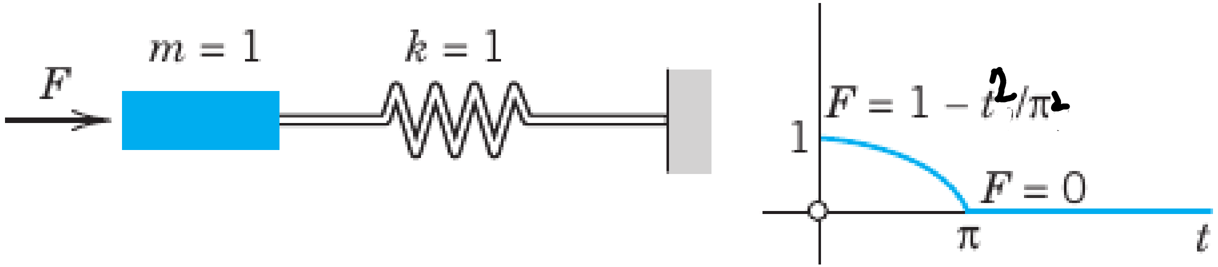
case① $\omega \neq \Omega$



case② $\omega \neq \Omega$ $y_p = C\, x \cos(\Omega x)$



24. Gun barrel. Solve $y'' + y = 1 - t^2/\pi^2$ if $0 \leqq t \leqq \pi$ and 0 if $t \to \infty$; here, $y(0) = 0$, $y'(0) = 0$. This models an undamped system on which a force F acts during some interval of time (see Fig. 59), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). *Hint:* At π both y and y' must be continuous.



$$y'' + y = \begin{cases} 1 - \dfrac{t^2}{\pi^2}, & 0 \le t \le \pi \\ 0, & t \ge \pi \end{cases}$$

→ Step (1) Solve

$$y'' + y = 1 - \frac{t^2}{\pi^2}$$

$y(0) = 0$

$y'(0) = 0$

→ let the solution of step ① is $y_1(t)$.

is valid for $0 \le t \le \pi$

→ then solve

$$y'' + y = 0 \qquad \text{for } \pi \le t$$

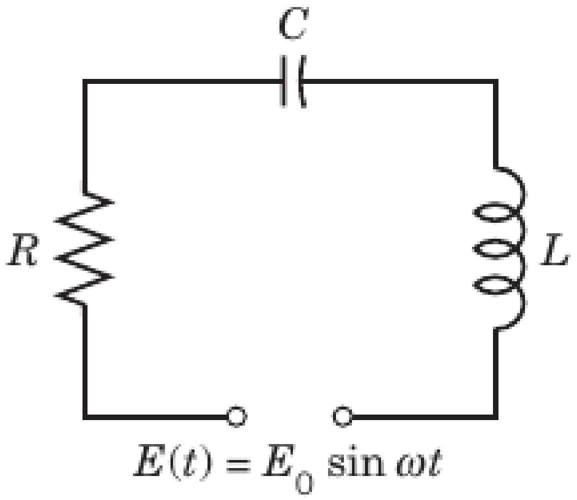
with the condition that

$$y(\pi) = y_1(\pi)$$

$$y'(\pi) = y_1'(\pi)$$

2.9 Modeling: Electric Circuits

next time



Name	Symbol	Notation		Unit	Voltage Drop
Ohm's Resistor		R	Ohm's Resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L\,\dfrac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

EXAMPLE 1 ***RLC*-Circuit**

Find the current $I(t)$ in an RLC-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$ F (farad), which is connected to a source of EMF $E(t) = 110 \sin (60 \cdot 2\pi t) = 110 \sin 377\, t$ (hence 60 Hz = 60 cycles/sec, the usual in the U.S. and Canada; in Europe it would be 220 V and 50 Hz). Assume that current and capacitor charge are 0 when $t = 0$.