12.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

Analog of u subsitution for multivariable integration.

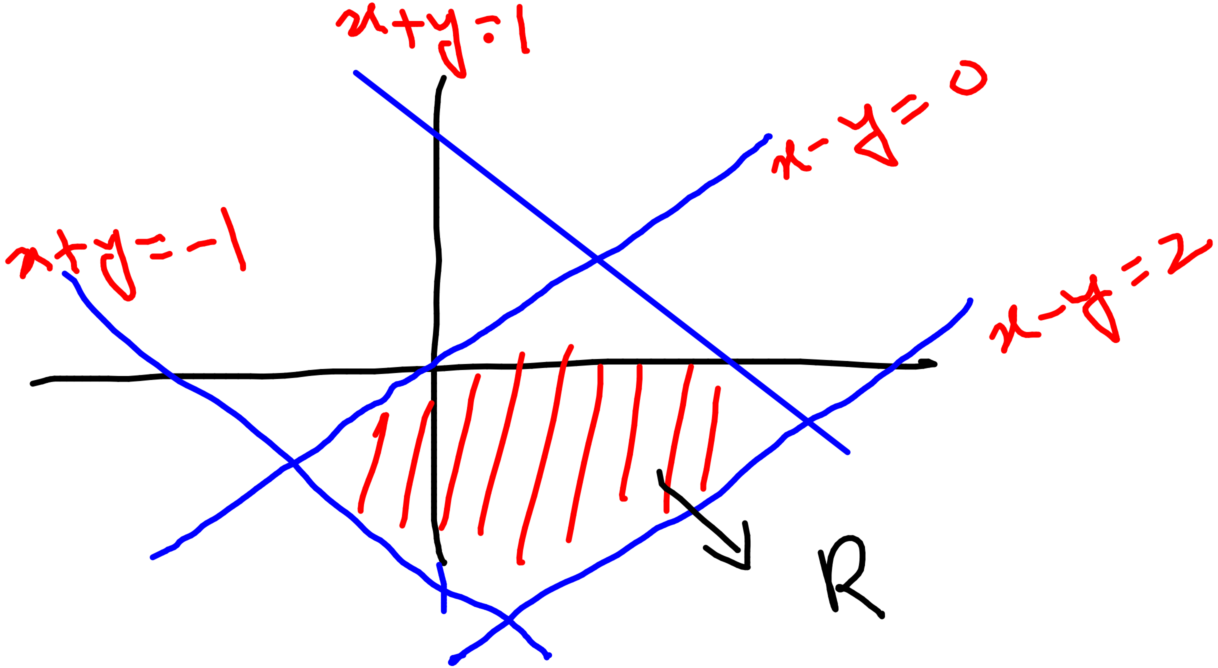
$$dx\,dy = (\ ??? \ )\,du\,dv$$

Jacobian

$$= \frac{\partial(x,y)}{\partial(u,v)} = \text{abs}\left[\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right]$$

Suppose our region of integration is a parallelogram like

$$-1 \le x+y \le 1 \quad , \quad 0 \le x-y \le 2$$



$$u = x+y \qquad x = (u+v)/2$$

$$v = x-y \qquad y = (u-v)/2$$

$$\int_0^2 \int_{-1}^1 ???? \, du \, dv$$

area of region R

$$= \iint_R 1\,dx\,dy = \int_0^2 \int_{-1}^1 (\text{Jacobian})\,du\,dv$$

$$= \int_0^2 \int_{-1}^1 \frac{1}{2}\,du\,dv = 2$$

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \begin{vmatrix} 1/2 & 1/2 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left| -\frac{1}{2} \right| = \frac{1}{2}$$

$$\partial(x,y) = \frac{1}{2}\,\partial(u,v)$$

$$dx\,dy = \frac{1}{2}\,du\,dv$$

7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Similarly (in 3d)

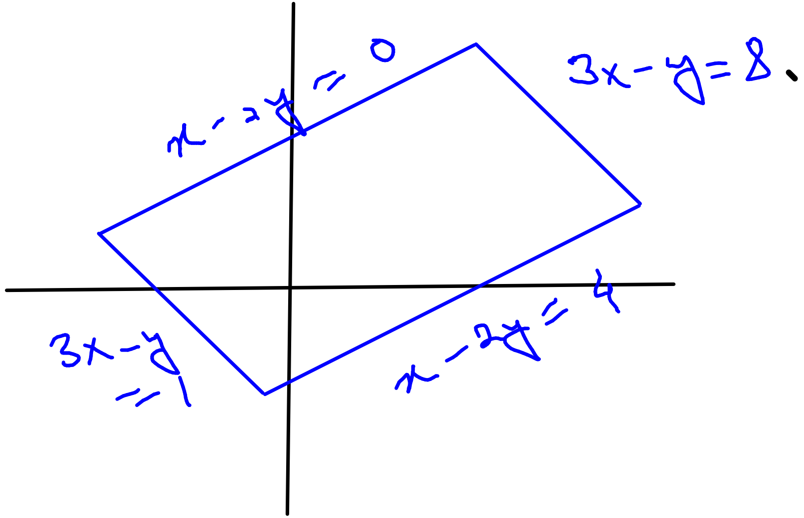
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

9 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \underbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}_{\text{Jacobian}} du\, dv$$

19. $\displaystyle\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$

$$\int_1^8 \int_0^4 \square \; du\,dv$$



choose new variables

$u = x - 2y \qquad v = 3x - y$

$$dA = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= \frac{1}{5}\,du\,dv$$

$x = (2v-u)/5$

$y = (v-3u)/5$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & 2/5 \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix}$$

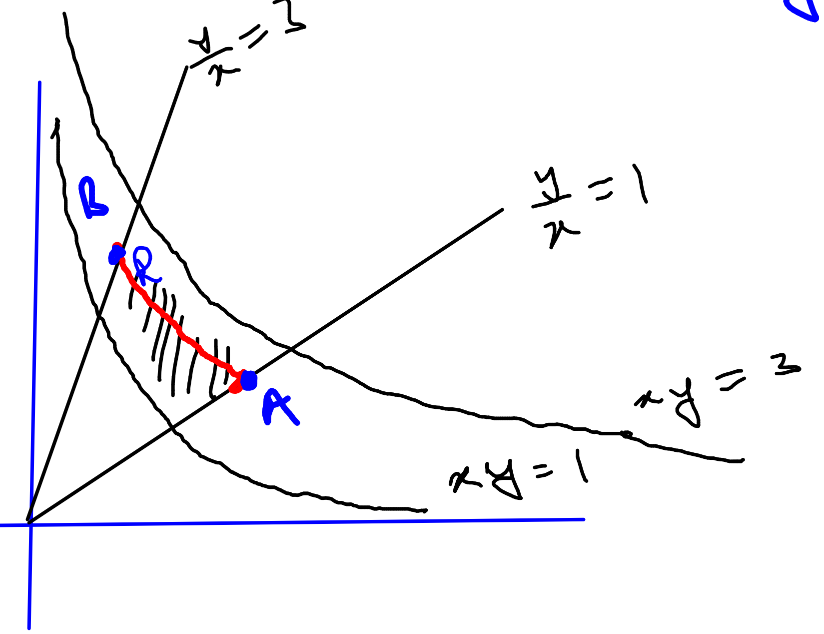
$$= \left|\frac{-1}{25} + \frac{6}{25}\right| = \frac{1}{5}$$

$$\iint_R \frac{x-2y}{3x-y}\, dA = \int_1^8 \int_0^4 \frac{u}{v}\,\frac{1}{5}\, du\, dv$$

15. $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $\underbrace{x = u/v,\ y = v}_{u = xy}$

$$\iint_R xy\, dA = \int_?^? \int_?^? ??\; du\, dv$$

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$



$$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \,\frac{1}{v}\, dv\, du = \text{whatever}$$

at point A: $v =$ what f^n of u

$\frac{y}{x} = 1$ & $xy = u$

$\frac{v}{x} = 1$ $\quad xv = u$

$v^2 = u$

$v = \sqrt{u}$

Similarly for B:

$\frac{y}{x} = 3$ $\quad xy = u$

$\frac{v}{x} = 3$ $\quad xv = u$

$\frac{v^2}{3} = u$

$v = \sqrt{3u}$