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ADVANCED ENGINEERING MATHEMATICS

Ordinary Differential Equations (ODE)

in ODE the unknown(s) are functions

e.g. find $y(x)$ s.t. $\frac{d^2y}{dx^2} = \sin(x)$] an ODE

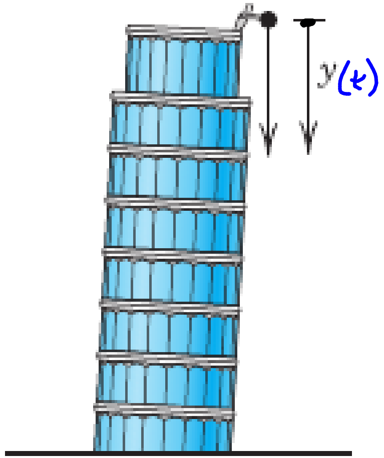
e.g. find $y(x)$ s.t. $(y(x))^2 + \sin(x) = 0$ not an ODE

Equation in which we have a term which contains a derivative of the "unknown" is a differential equation

$F = ma$

$mg = my''$

$\boxed{y'' = g}$

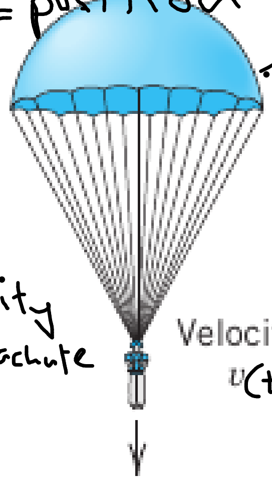


Falling stone

$y'' = g = const.$
(Sec. 1.1)

$y(t)$ = position of time t

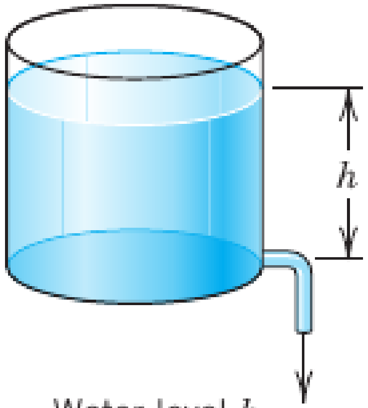
$v(t)$ = velocity of parachute



Parachutist

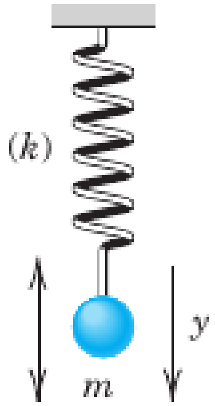
$\boxed{mv' = mg - bv^2}$
(Sec. 1.2)

$my'' = mg - b(y(t))^2$



Outflowing water

$\boxed{h' = -k\sqrt{h}}$
(Sec. 1.3)

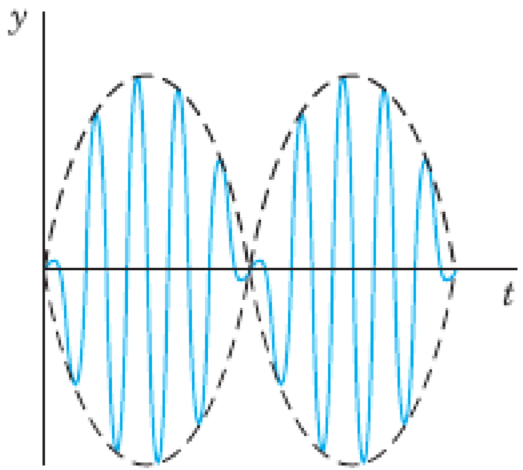


Displacement y

Vibrating mass on a spring

$$my'' + ky = 0$$

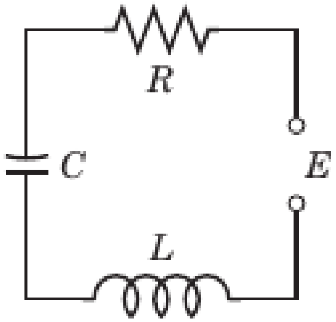
(Secs. 2.4, 2.8)



Beats of a vibrating system

$$y'' + \omega_0^2 y = \cos\ \omega t, \quad \omega_0 \approx \omega$$

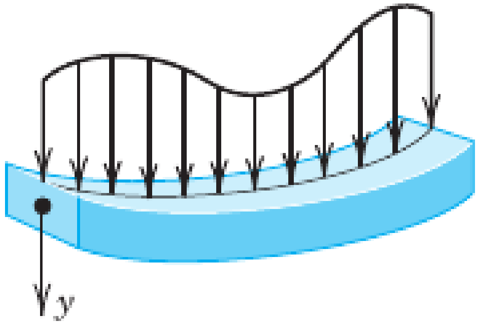
(Sec. 2.8)



Current I in an RLC circuit

$$\boxed{LI'' + RI' + \frac{1}{C} I = E'}$$

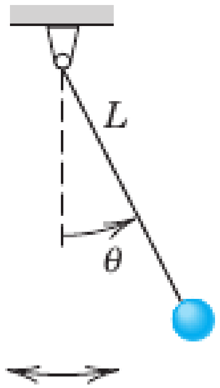
(Sec. 2.9)



Deformation of a beam

$$EIy^{iv} = f(x)$$

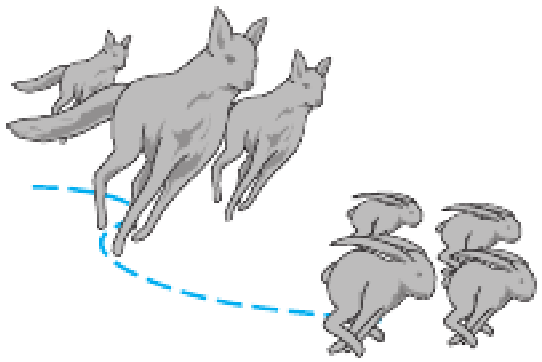
(Sec. 3.3)



Pendulum

$$L\theta'' + g \sin \theta = 0$$

(Sec. 4.5)



Lotka–Volterra predator–prey model

$$y_1' = ay_1 - by_1y_2$$
$$y_2' = ky_1y_2 - ly_2$$

(Sec. 4.5)

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(1) order 1 $$y' = \cos x$$

(2) order 2 $$y'' + 9y = e^{-2x}$$

(3) order 3 $$y'y''' - \tfrac{3}{2}y'^2 = 0$$

$$\cancel{y^2 + \frac{d}{dx}(\sin(x))} = 0$$

not an ODE

Q. $$y' + (y'')^3 + y''' = 5$$ order 3

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**. Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

$$F(x, y, y') = 0 \tag{4}$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

(a) Verify that y is a solution of the ODE. **(b)** Determine from y the particular solution of the IVP. **(c)** Graph the solution of the IVP.

$$y' + 4y = 1.4, \quad y = ce^{-4x} + 0.35, \quad \underline{y(0) = 2}$$

IVP: Initial Value Problem:

a) plug in $y = ce^{-4x} + 0.35$ in $y' + 4y = 1.4$

verify LHS = RHS

$$\begin{aligned} LHS &= y' + 4y \\ &= -4\cancel{ce^{-4x}} + 4(\cancel{ce^{-4x}} + 0.35) \\ &= 1.4 \end{aligned}$$

$y = ce^{-4x} + 0.35$ is a "general solution"

b) Particular Solution: $y(x)$ must satisfy ODE + extra condition given

find $y(x)$ which solves $y' + 4y = 1.4$ & $y(0) = 2$

$$y = ce^{-4x} + 0.35$$

$$y(0) = 2$$

$$ce^{-4(0)} + 0.35 = 2$$

$$c = 1.65$$

Particular Solution

$$y = 1.65e^{-4x} + 0.35$$

(a) Verify that y is a solution of the ODE. **(b)** Determine from y the particular solution of the IVP. **(c)** Graph the solution of the IVP.

general sol^n

$$y' = y - y^2, \quad \boxed{y = \frac{1}{1 + ce^{-x}},} \quad y(0) = 0.25$$

a)

$$y' = \frac{ce^{-x}}{(1+ce^{-x})^2} = \text{LHS}$$

$$y - y^2 = \frac{\cancel{1}+ce^{-x} - 1}{(1+ce^{-x})^2} = \frac{ce^{-x}}{(1+ce^{-x})^2} = \text{RHS}$$

Particular Sol^n

$$y(0) = 0.25$$

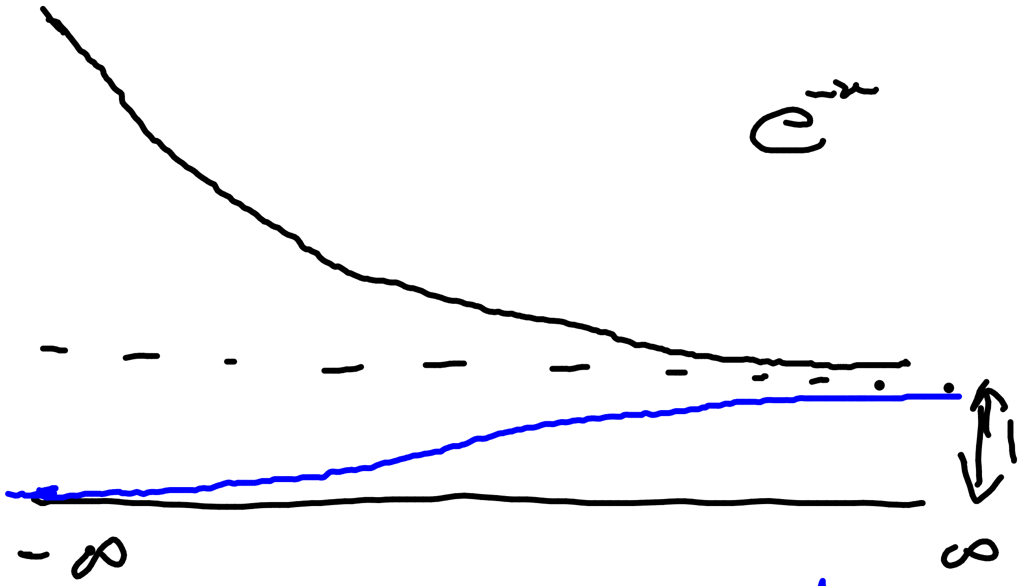
$$\frac{1}{1+c} = 0.25$$

$$c = 3$$

$$y(x) = \frac{1}{1+3e^{-x}}$$

c) Graph of $y(x) = \frac{1}{1+3e^{-x}}$

e^{-x}



$y = \frac{1}{1+3e^{-x}}$

Solve the ODE by integration or by remembering a differentiation formula.

$$\boxed{y' + 2 \sin 2\pi x = 0}$$

$$y' = -2 \sin 2\pi x$$

$$\boxed{y(x) = \frac{\cos(2\pi x)}{\pi} + C}$$

general solution

Solve the ODE by integration or by remembering a differentiation formula.

$$y' = -1.5y$$

try $y(x) = e^{-1.5x}$

general solution : $y(x) = ce^{-1.5x}$

Solve the ODE by integration or by remembering a differentiation formula.

$$y'' = -y$$

$$y(x) = c_1 \sin(x) + c_2 \cos(x)$$

general sol^n :

19. Free fall. In dropping a stone or an iron ball, air resistance is practically negligible. Experiments show that the acceleration of the motion is constant (equal to $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$, called the **acceleration of gravity**). Model this as an ODE for $y(t)$, the distance fallen as a function of time t. If the motion starts at time $t = 0$ from rest (i.e., with velocity $v = y' = 0$), show that you obtain the familiar law of free fall

$$y = \tfrac{1}{2}gt^2.$$

20. Exponential decay. Subsonic flight. The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near 35,000 ft. Find the air pressure $y(x)$ at this height. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure. At 18,000 ft it is half its value $y_0 = y(0)$ at sea level. *Hint.* Remember from calculus that if $y = e^{kx}$, then $y' = ke^{kx} = ky$. Can you see without calculation that the answer should be close to $y_0/4$?