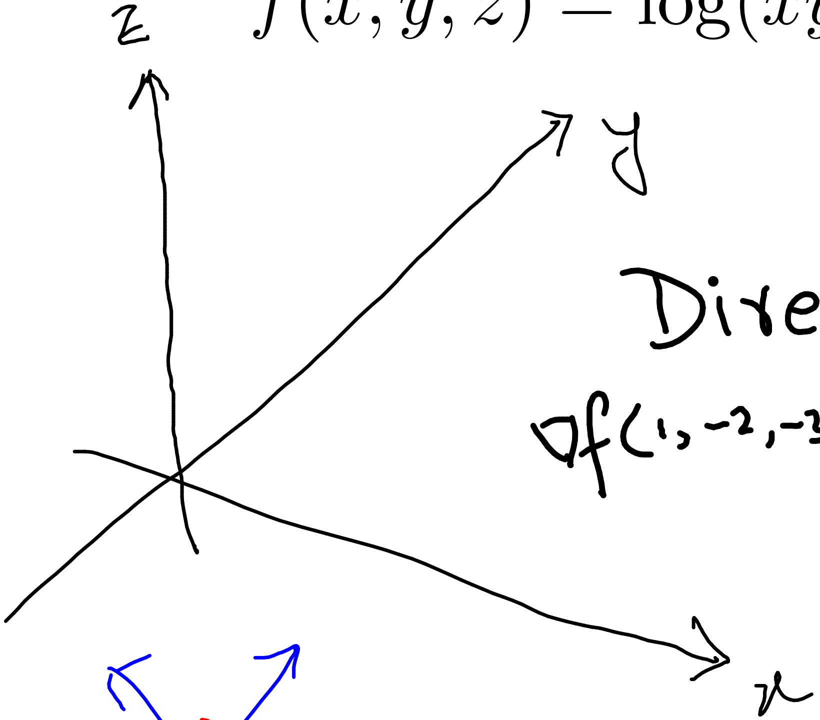
Today’s topic

- Derivatives of vector valued functions (Sec 10.7)

Practice 1: Find the maximum rate of change of f and direction in which it occurs.

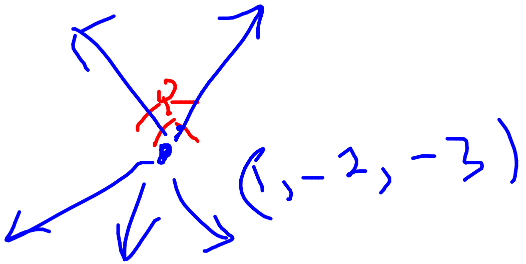
$$f(x,y,z) = \log(xy^2z^2), \qquad \text{at } (1,-2,-3)$$



Direction of fastest ascent

$$\nabla f(1,-2,-3) = \frac{1}{xy^2z^2}\begin{bmatrix} y^2z^2 \\ 2xyz^2 \\ 2xy^2z \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 \\ -36 \\ -24 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -\frac{2}{3} \end{bmatrix}$$

$x=1$
$y=-2$
$z=-3$

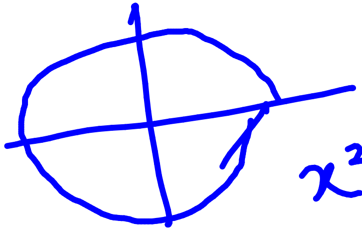


max rate of change $= \left| \hat{i} - \hat{j} - \frac{2}{3}\hat{k} \right| = \sqrt{\frac{22}{9}}$

Space Curves: We have discussed earlier. We also call them *parametric curves*

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \qquad a \le t \le b$$

Examples: circle, ellipse, parabola, helix, straight line

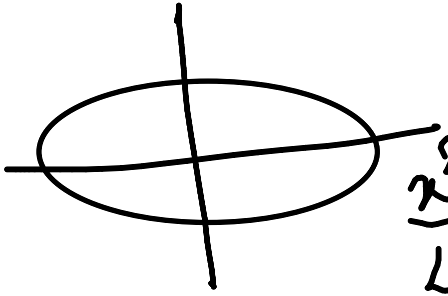


$x^2+y^2=9$

$x = 3\cos(t)$

$y = 3\sin(t)$

$0 \le t \le 2\pi$



$\dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$

$x = 2\cos t$

$y = \sqrt{2}\sin t$

$0 \le t \le 2\pi$

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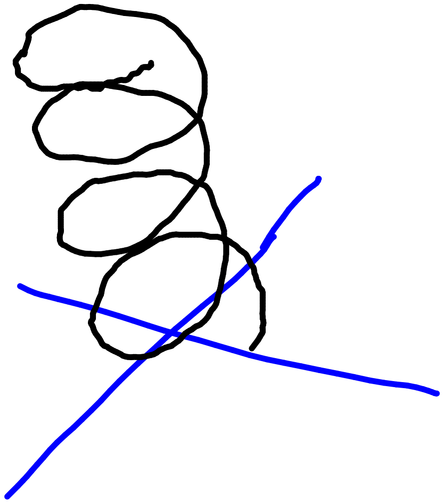


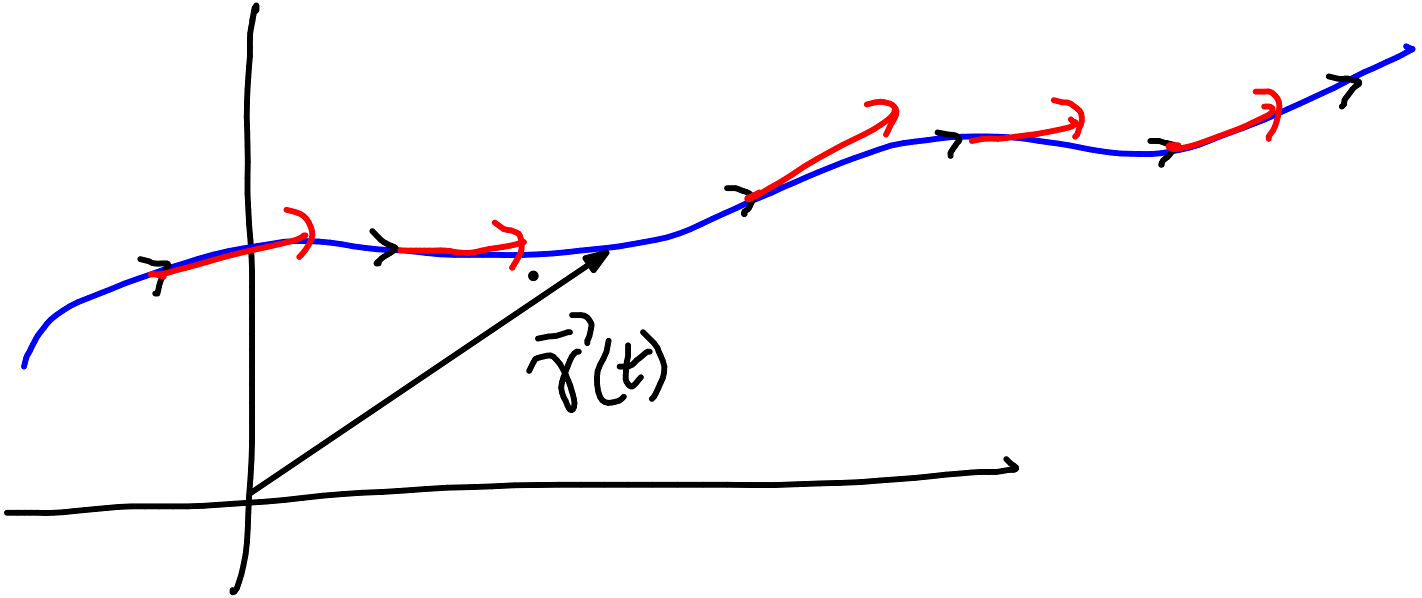
$x = \cos(t)$

$y = \sin(t)$

$z = t$

$0 \leq t \leq 8\pi$

Recall position vectors & velocity vectors in some physics course



DERIVATIVES

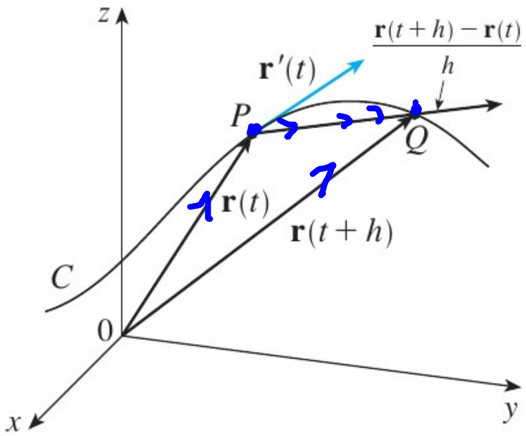
The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

3
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve

unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$



The following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$.

4 THEOREM If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

EXAMPLE 8

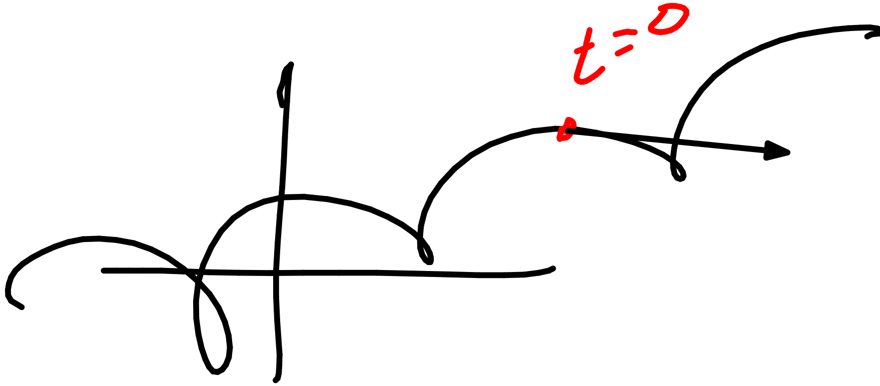
(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

$$\vec{r}\,'(t) = 3t^2\,\hat{i} + (-te^{-t} + e^{-t})\,\hat{j} + 2\cos(2t)\,\hat{k}$$

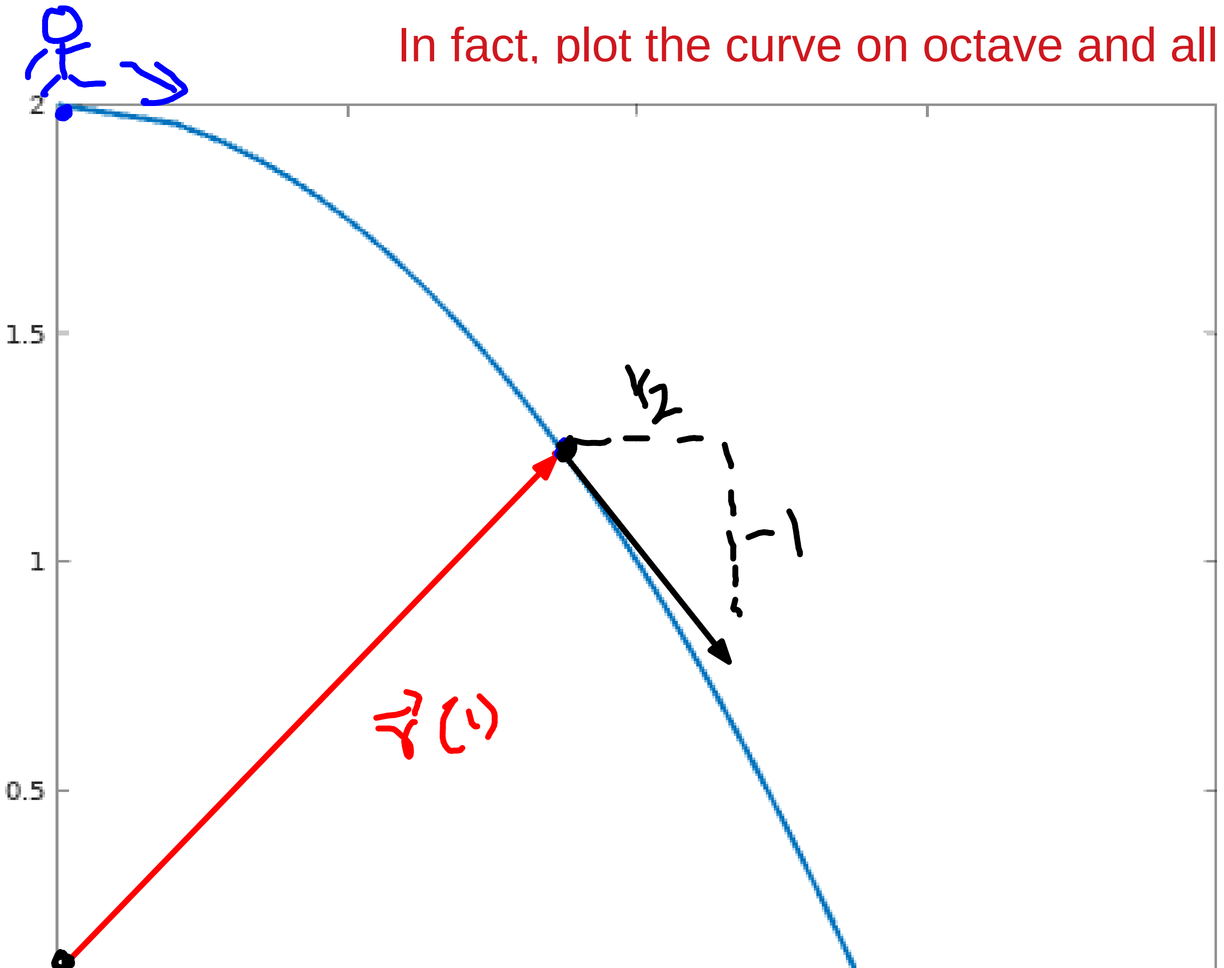
$$\vec{r}\,'(0) = 0\,\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{r}(0) = (1, 0, 0)$$



EXAMPLE 9 For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

In fact, plot the curve on octave and all unit tangent vectors.



$0 \le t \le 2$

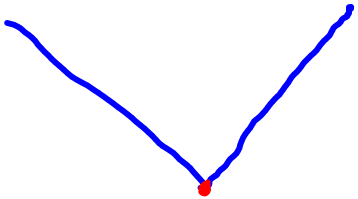
$r(1) = \hat{i} + \hat{j}$

$\vec{r}\,'(t) = \frac{1}{2\sqrt{t}}\hat{i} - \hat{j}$

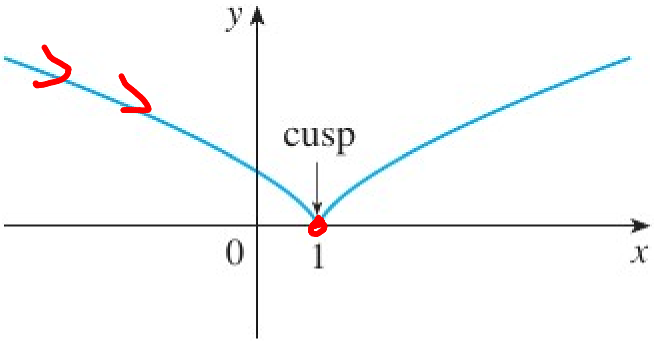
$\vec{r}\,'(1) = \frac{1}{2}\hat{i} - \hat{j}$

A curve given by a vector function $\mathbf{r}(t)$ on an interval I is called **smooth** if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ (except possibly at any endpoints of I). For instance, the helix in Example 10 is smooth because $\mathbf{r}'(t)$ is never $\mathbf{0}$.

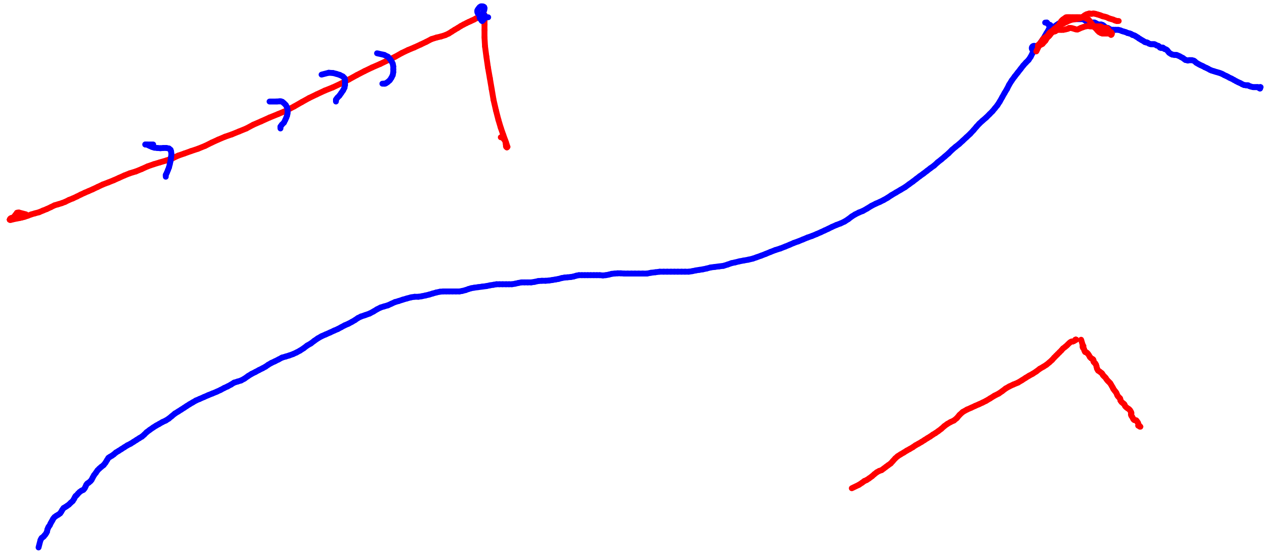
EXAMPLE 11 Determine whether the semicubical parabola $\mathbf{r}(t) = \langle 1 + t^3, t^2 \rangle$ is smooth.



$$r'(t) = 3t^2\,\hat{i} + 2t\,\hat{j}$$



$$r'(0) = \vec{0}$$



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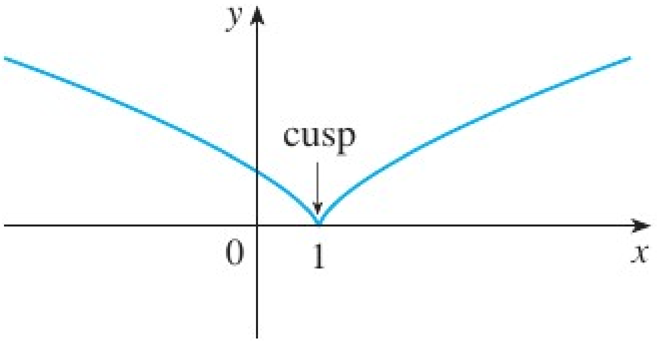
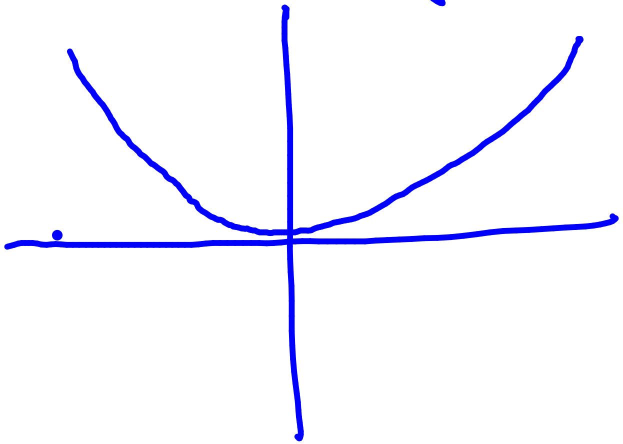




$y = x^2$

(t, t^2)

$$\boxed{\begin{aligned}\vec{\gamma}(t) &= (t^3, t^6) \\ \vec{\gamma}\,'(0) &= \vec{0}\end{aligned}}$$

non-smooth parametrization

5 THEOREM Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

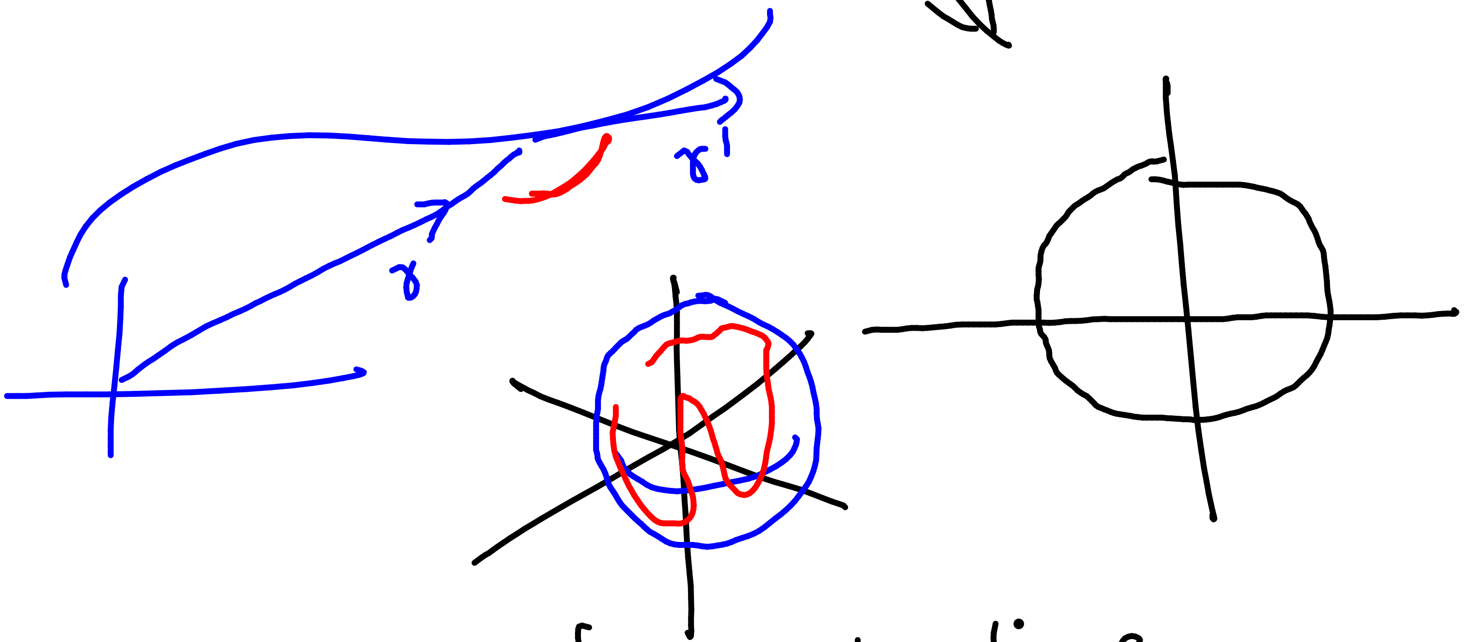
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

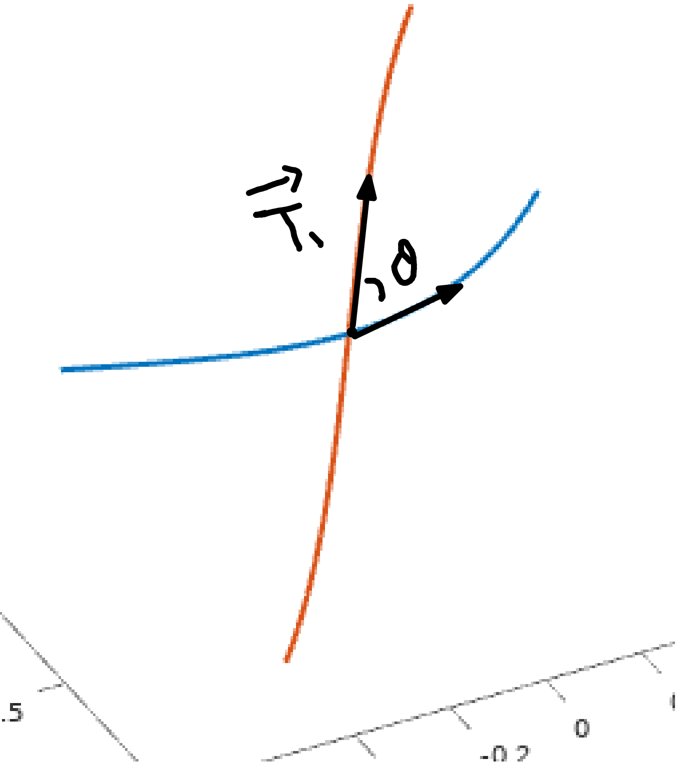
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

EXAMPLE 12 Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.



valid proof next time

55. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.



$$\vec{T}_1 = \vec{r}_1'(0)$$

$$\vec{T}_2 = \vec{r}_2(0)$$

$$\vec{T}_1 \cdot \vec{T}_2 = |\vec{T}_1|\,|\vec{T}_2| \underbrace{\cos(\theta)}_{\text{solve for this}}$$

Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time.* Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geqslant 0$. Do the particles collide?