AN ELEMENTARY INTRODUCTION TO CATEGORIES

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Abstract. A brief on categories in an elementary context. This is by no means original work, and much of the intellectual debt in terms of organization and exercises is owed to [3], where I first came across the material.

Contents

1. Some Linear Algebra

Exercise 1.1.

(a) Show that $\{\mathbf{0}\}$ is the unique (up to isomorphism) vector space U such that for all vector spaces Z, $\hom_{\mathbf{F}}(U, Z)$ is a singleton.

(b) Show that $\{\mathbf{0}\}$ is the unique (up to isomorphism) vector space U such that for all vector spaces Z, $\hom_{\mathbf{F}}(Z, U))$ is a singleton.

(c) Let $f \in \hom_{\mathbf{F}}(U, V)$. Show that f is injective if and only if for all vector spaces Z, and all $g_1, g_2 \in \hom_{\mathbf{F}}(Z, U)$, $f \circ g_1 = f \circ g_2 \iff g_1 = g_2$.

(d) Let $f \in \hom_{\mathbf{F}}(U, V)$. Show that f is surjective if and only if for all vector spaces Z, and all $g_1, g_2 \in \hom_{\mathbf{F}}(V, Z)$, $g_1 \circ f = g_2 \circ f \iff g_1 = g_2$.

(e) Show that $U \oplus V$ (the vector space structure on $U \times V$), together with the maps $u \mapsto (u, 0)$ and $v \mapsto (0, v)$ has the property that for any vector space Z, the map $\hom_{\mathbf{F}}(U \oplus V, Z) \to \hom_{\mathbf{F}}(U, Z) \times \hom_{\mathbf{F}}(V, Z)$ given by restriction is a linear isomorphism.

(f) Suppose that the triple (W, ι_U, ι_V) of a vector space W and maps from U, V to W satisfies the property of (e). Show that there is a unique isomorphism $\phi : W \to U \oplus V$ such that $\phi \circ \iota_U$ is the inclusion of U in $U \oplus V$, and similarly for V.

2. Categories

Informally, a category consists of a collection of objects, collections of maps between any two such objects, and a rule of composition for such maps. More formally, we have the following definition.

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Definition 2.1. A *category* $\mathcal{C}$ consists of:

- A class $\mathrm{ob}(\mathcal{C})$ of *objects*.
- A class $\hom(\mathcal{C})$ of *morphisms* (or *arrows*, or *maps*) between the objects. Each morphism f has a *source object* a and a *target object* b where a and b are in $\mathrm{ob}(\mathcal{C})$. We write $f : a \to b$ and say that 'f is a morphism from a to b.' We then write $\hom_{\mathcal{C}}(a, b)$ (or $\hom(a, b)$) should no confusion about the category referred to arise) to denote the *hom-class* of all morphisms from a to b.
- For every three objects a, b, and c, a binary operation $\hom(a, b)\times\hom(b, c) \to \hom(a, c)$ called the *composition of morphisms*; the composition of $f : a \to b$ and $g : b \to c$ is written $g \circ f$ or gf.

We also require the following axioms to hold:

- (Associativity) if $f : a \to b$, $g : b \to c$, and $h : c \to d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
- (Identity) for every object x, there exists a morphism $\mathbb{1}_x : x \to x$ called the *identity morphism* for x, such that for every morphism $f : a \to x$ and every morphism $g : x \to b$, we have $\mathbb{1}_x \circ f = f$ and $g \circ \mathbb{1}_x = g$.

Note that the identity morphism is unique. Relations among morphisms are often (and conveniently) represented by *commutative diagrams*; to illustrate, an example: $h \circ f = k \circ g$ is represented by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Definition 2.2.

- A category $\mathcal{C}$ is called *small* if both $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ are sets and not *proper classes* (informally, in Zermelo-Fraenkel, a class that is not a set). It is called *large* otherwise.
- A category is *locally small* if for all objects a and b the hom-class $\hom(a, b)$ is a set, called a *homset*. Many important (and common) categories (such as **Set**; see below) are, though not small, at least locally small.

Let us consider some examples.

Examples 2.3.

(a) **Set** is the category of sets. We have $\hom_{\mathbf{Set}}(X, Y) = Y^X$, the set of maps from X to Y, composition is usual composition of maps, and $\mathbb{1}_X$ is the identity map $X \to X$.

(b) **Top** is the category of topological spaces with continuous maps. We have $\hom_{\mathbf{Top}}(X, Y) = C(X, Y)$, the set of continuous maps $X \to Y$.

(c) **Grp** is the category of groups with group homomorphisms, i.e., $\hom_{\mathbf{Grp}}(G, H)$ is the set of group homomorphisms $G \to H$.

(d) **Ab** is the category of abelian groups. Observe that for abelian groups A, B we have $\hom_{\mathbf{Ab}}(A, B) = \hom_{\mathbf{Grp}}(A, B)$, and we say that **Ab** is a *full subcategory* of **Grp**.

(e) $\mathbf{Vect}_{\mathbf{F}}$ is the category of vector spaces over the field $\mathbf{F}$. We have $\hom_{\mathbf{Vect}_{\mathbf{F}}}(U, V) = \hom_{\mathbf{F}}(U, V)$, the space of linear maps $U \to V$.

Observe that in each example (aside from **Set**) the existence of identities and closure under composition is a familiar proposition.

Exercise 2.4. Show that any preordered set $(P, \leq)$ forms a small category (a *preorder* is a reflexive and transitive binary relation).

Exercise 2.5. (Constructing new categories) Convince yourself that any category $\mathcal{C}$ gives rise to another category, denoted $\mathcal{C}^{\mathrm{op}}$ and called the *dual category*, with the same objects and all morphisms reversed. Do the same for the following construction: if $\mathcal{C}$ and $\mathcal{D}$ are categories, we can form the product category $\mathcal{C} \times \mathcal{D}$ with objects and morphisms as pairs (hence we may compose componentwise).

3. Morphisms and Objects

Definition 3.1. Fix a category $\mathcal{C}$, objects $X, Y \in \mathrm{ob}(\mathcal{C})$, and a morphism $f \in \hom(X, Y)$.

- f is a *monomorphism* if for every object Z and every two morphisms $g_1, g_2 \in \hom(Z, X)$ we have $f \circ g_1 = f \circ g_2 \iff g_1 = g_2$.
- f is an *epimorphism* if for every object Z and every two morphisms $g_1, g_2 \in \hom(Y, Z)$ we have $g_1 \circ f = g_2 \circ f \iff g_1 = g_2$.
- f is an *isomorphism* if there is a morphism $f^{-1} \in \hom(Y, X)$ such that $f^{-1} \circ f = \mathbb{1}_X$ and $f \circ f^{-1} = \mathbb{1}_Y$.

Exercise 3.2. Suppose that f is an isomorphism.

(a) Show that f is both a monomorphism and an epimorphism.
(b) Show that f^{-1} is unique, and also an isomorphism.
(c) Show that composition with f gives bijections $\hom(X, Z) \to \hom(Y, Z)$ and $\hom(W, X) \to \hom(W, Y)$ which respect composition (we assume $\mathcal{C}$ is a *locally small category*, and in fact unless mentioned otherwise this assumption holds throughout this brief; recall this means that $\hom(\mathrm{a}, \mathrm{b})$ is a set for all objects a, b).

Part (c) of the above exercise formalizes the notion that isomorphic objects 'are the same.' The following exercise is again a triviality, but serves to introduce some terminology.

Exercise 3.3. We say a morphism $f : a \to b$ is a *retraction* if it has a right inverse, and a *section* if it has a left inverse. Show the following are equivalent:

(a) f is a monomorphism and a retraction;
(b) f is an epimorphism and a section;
(c) f is an isomorphism.

Exercise 3.4. For each category in Examples 2.3, show that

(a) f is a monomorphism if and only if it is set-theoretically injective; and
(b) f is an epimorphism if and only if it set-theoretically surjective.
(c) Find a category for which f being an epimorphism and set-theoretically surjective are not equivalent (in particular, find an example of such a map). Which direction always holds? Can you determine an equivalence condition for the example you came up with? Can you find a category for which f being a monomorphism and set-theoretically injective are not equivalent?

Hint: For the first part, think about conditions one might impose on topological spaces. For the latter part, think of as simple a constructive example as you can.

It follows from (a) and (b) that two sets are isomorphic if and only if they have the same cardinality, and in this vein isomorphism reduces to its more familiar notions in particular categories (as we should hope).

Definition 3.5. We call an object $I \in \mathcal{C}$ *initial* if for every object X, $\hom(I, X)$ is a singleton. We call $F \in \mathcal{C}$ *final* if for every object X, $\hom(X, F)$ is a singleton.

Exercise 3.6. (Uniqueness)

(a) Let I_1, I_2 be initial. Show that there is a unique isomorphism $f \in \hom(I_1, I_2)$.
(b) Let F_1, F_2 be final. Show that there is a unique isomorphism $g \in \hom(F_1, F_2)$.

Exercise 3.7. (Existence)

(a) Show that $\emptyset$ is initial and $\{\emptyset\}$ is final in **Set**. Why is $\{\emptyset\}$ not an initial object?
(b) Show that $\{\mathbf{0}\}$ is both initial and final in $\mathbf{Vect}_{\mathbf{F}}$.
(c) Find the initial and final objects in the categories of groups and abelian groups.

4. Sums and Products

Let us begin with a simple exercise as an easy case of the following.

Exercise 4.1.

(a) For sets X_1, X_2, consider the disjoint union $X_1 \sqcup X_2 = (X_1 \times \{1\}) \cup (X_2 \times \{2\})$ and maps $\iota_j : X_i \to X_1 \sqcup X_2$ defined by $x \mapsto (x, j)$. Show that for any Z, the map $Z^{X_1 \sqcup X_2} \to Z^{X_1} \times Z^{X_2}$ given by restriction $f \mapsto (f \circ \iota_1, f \circ \iota_2)$ is a bijection. If X_1 and X_2 are disjoint, the same result holds for $X_1 \cup X_2$ (maps become simply identity maps).
(b) Suppose that the triple (U, ι_1', ι_2') of a set U and maps $\iota_j' : X_j \to U$ satisfies the property in (a). Show that there is a unique bijection $\phi : U \to X_1 \sqcup X_2$ such that $\phi \circ \iota_j' = \iota_j$.

Definition 4.2. Let $\{X_i\}_{i \in I} \subset \mathcal{C}$ be a collection of objects.

- Their *coproduct* is an object $U \in \mathcal{C}$ together with maps $u_i \in \hom(X_i, U)$ such that for every object Z the map $\hom(U, Z) \to \prod_{i \in I} \hom(X_i, U)$ given by $f \mapsto (f \circ u_i)_{i \in I}$ is a bijection.
- Their *product* is an object $P \in \mathcal{C}$ together with maps $p_i \in \hom(P, X_i)$ such that for every object Z the map $\hom(Z, U) \to \prod_{i \in I} \hom(U, X_i)$ given by $f \mapsto (p_i \circ f)_{i \in I}$ is a bijection.

Exercise 4.3. (Uniqueness)

(a) Show that if U, U' are coproducts then there is a unique isomorphism $\phi \in \hom(U, U')$ such that $\phi \circ u_i = u_i'$.
(b) Show that if P, P' are products then there is a unique isomorphism $\phi \in \hom(P, P')$ such that $p_i' \circ \phi = p_i$.

Exercise 4.4. (Existence)

(a) Consider the category **Set**.

  (i) Show that the disjoint union $\bigsqcup_i X_i \overset{\text{def}}{=} \bigcup_{i\in I}(X_i \times \{i\})$ with maps $u_i(x) = (x, i)$ is a coproduct. In particular, if the X_i are disjoint show that $\bigcup_{i\in I} X_i$ is a coproduct.
  (ii) Show that $\prod_{i\in I} X_i$ with maps $p_j((x_i)_{i\in I}) = x_j$ is a product.

(b) Consider the category **Top**.
  (i) Show that $[0, 2) = [0, 1)\cup[1, 2)$ (with the inclusion maps) is a coproduct in **Set** but not in **Top** (with subspace topologies from $\mathbf{R}$).
  (ii) Show that $\bigsqcup_i X_i$ with the topology $\mathcal{T} = \left\{\bigcup_{i\in I}(A_i \times \{i\}) \mid A_i \subset X_i \text{ open}\right\}$ is a coproduct.
  (iii) Show that the product topology on $\prod_{i\in I} X_i$ is a product.

(c) Consider the category $\mathbf{Vect_F}$.
  (i) Show that $\bigoplus_{i\in I} X_i$ is a coproduct.
  (ii) Show that $\prod_{i\in I} X_i$ is a product.

(d) Consider the category **Grp**.
  (i) Show that the 'coordinatewise' group structure on $\prod_{i\in I} G_i$ is a product. Note that the coproduct exists, and is called the *free product* of the groups G_i, denoted $*_{i\in I}G_i$.

5. A Final Exercise

A category can be thought of as a 'labeled graph': a set of vertices (the objects), a set of directed edges (the morphisms), a composition operator, and an identity morphism. Now suppose you are only given combinatorial data (so the vertices and edges are no longer labeled). The question is: are you able to restore the labels on the objects, and given that, are you able to restore the labels on the morphisms (up to automorphism of each object of course)?

Think about this for each example of categories provided in 1.3.

Hint: Try in the following order: **Set**, **Top**, $\mathbf{Vect_F}$, **Ab**, and then **Grp**.

References

[1] Peter May. `http://math.uchicago.edu/~may/REU2019/TEMPLATE.pdf`
[2] Wikipedia. `https://en.wikipedia.org/wiki/Category_(mathematics)`
[3] Lior Silberman. `https://www.math.ubc.ca/~lior/teaching/1617/412_F16/Notes/412_Categories.pdf`