This notebook goes through what an option is, derives the Black-Scholes model from a symmetric random walk in an intuitive sense, goes through all of the greeks, provides plots to show the greeks sensitivities, goes over option terminology, and explains delta hedging which is what will be simulated after.

What is the Black-Scholes Model?

- A pricing model for finding the fair price of European options which can only be exercised on their expiration date.

What is an option?

An option is a financial contract between two parties. It gives the buyer a right, but not an obligation to either buy or sell an underlying asset at a pre-agreed price K (The strike price) on a specific date T (The expiration date)

- A **call** option gives the buyer the right (but not the obligation) to **buy** the underlying asset at a fixed price K at the expiration date
- A **put** option gives the buyer the right (but not the obligation) to **sell** the underlying at the strike K at time T.

Notation

- T - Expiration date (The date that they agree to transact if the buyer chooses)
- S_t - Price of the underlying asset at time t
- S_T - Price of the underlying at expiration
- K - Strike price (Agreed upon price to potentially transact at)

Option Payoff per unit of the underlying asset

- Call payoff per unit: max(S_T - K, 0)
- Put payoff per unit: max(K-S_T, 0)

Modeling the underlying which will be referred to as the stock moving forward

Given we are trying to model the price of an option on a stock, it naturally follows that we first need to model the stock price itself. The Black-Scholes model assumes that the stock price follows a stochastic process called Geometric Brownian motion, which captures both the expected return and the random fluctuations of the stock

Building to Geometric Brownian Motion (GBM)

Symmetric Random Walk to Brownian motion

Think about standing on the sidewalk and flipping a coin to determine if you should take one step forward or one step backward where your starting position is 0. So we have (the size of your step forward or backward) Δy = 1 step. Lets say your time step (the time it takes you to flip the coin and move forward or backward) Δt = 1 second. Let X_i define if you stepped forward or backward at each time i. So we have

$$X_i = \begin{cases} +1, & \text{with probability } 0.5 \\ -1, & \text{with probability } 0.5 \end{cases}$$

Δt = 1

Δy = 1

Your steps are i.i.d (independently and identically distributed), because coin flips are i.i.d.

Your position at time n seconds P_n = $\sum_{i=1}^{n} X_i * \Delta y$

so $P_n = \Delta y(X_1 + X_2 + ... + X_n)$

since our X_i are i.i.d, we know $E[P_n]$ = $E[\Delta y(X_1 + X_2 + ... + X_n)]$ = $\Delta y E[(X_1 + X_2 + ... + X_n)]$ = $\Delta y \sum_{i=1}^{n} E[X_i]$

We know that $E[X_i] = (1).5 + (-1).5 = 0$, so $E[P_n] = 0$

Now consider $Var(P_n) = Var(\Delta y(X_1 + X_2 + ... + X_n))$ = $\Delta y^2 Var((X_1 + X_2 + ... + X_n))$, = $(\Delta y)^2 n * Var(X_i)$, since variance of i.i.d is additive

$Var(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i^2] - 0 = (1)^2.5 + (-1)^2.5 = 1$

so, $Var(P_n) = (\Delta y)^2 n$

To summarize:

$$E[P_n] = 0$$
$$Var[P_n] = (\Delta y)^2 n$$

See plot below to see what a bunch of simulated random walk looks like

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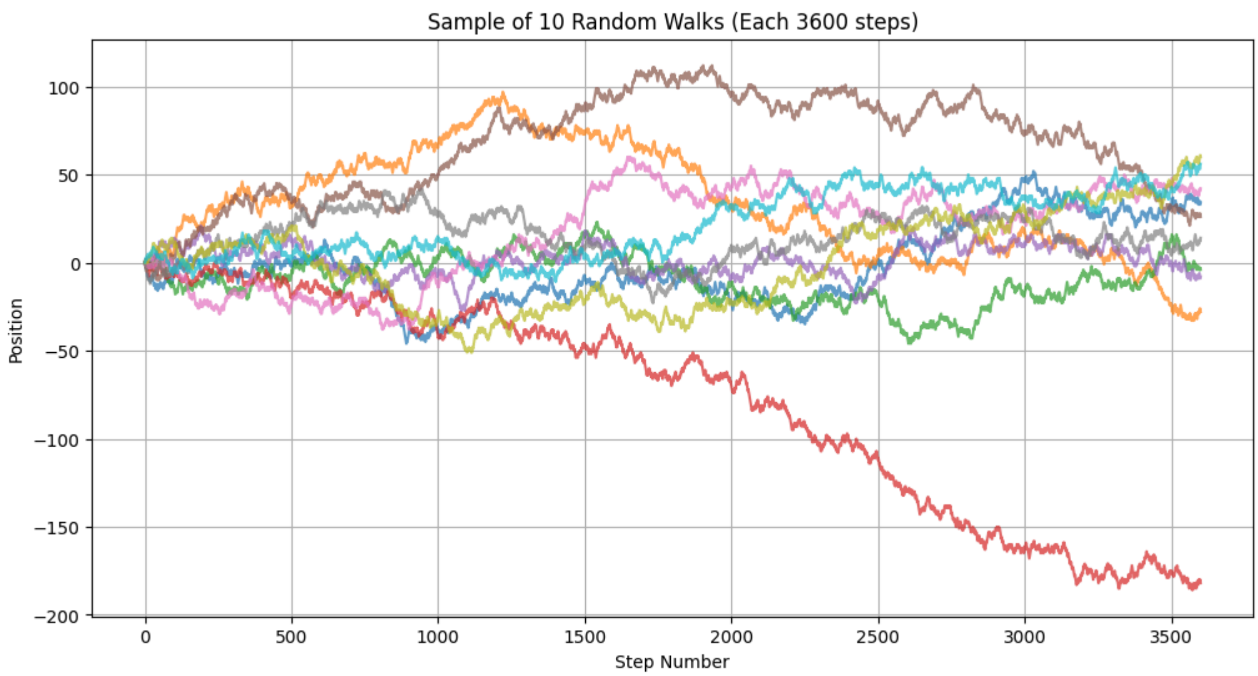


Turning our random walk into brownian motion

Note that n = $\frac{t}{\Delta t}$, so we can rewrite $Var[P_n] = (\Delta y)^2 n$, as $(\Delta y)^2 \frac{t}{\Delta t}$, and our mean stays the same.

If we take the limit as $\Delta t -> 0$, we turn our discrete time random walk into a continuous-time process. In order to have a meaningful limit we must scale our Δy down so our variance does not approach infinity or 0 due to Δt being very small. So, we want to make it such that the variance of our limit is a constant multiple of t, so we can say Δy = $\sigma\sqrt{\Delta t}$, for a constant σ>0

Resulting $Var(P_n)$ and $E[P_n]$ as Δt ->0

$Var(P_n) = \sigma^2 t \quad E[P_n] = 0$

The limit as Δt ->0 of our symmetric random walk is known as Brownian motion denoted W_t

Properties of Brownian Motion

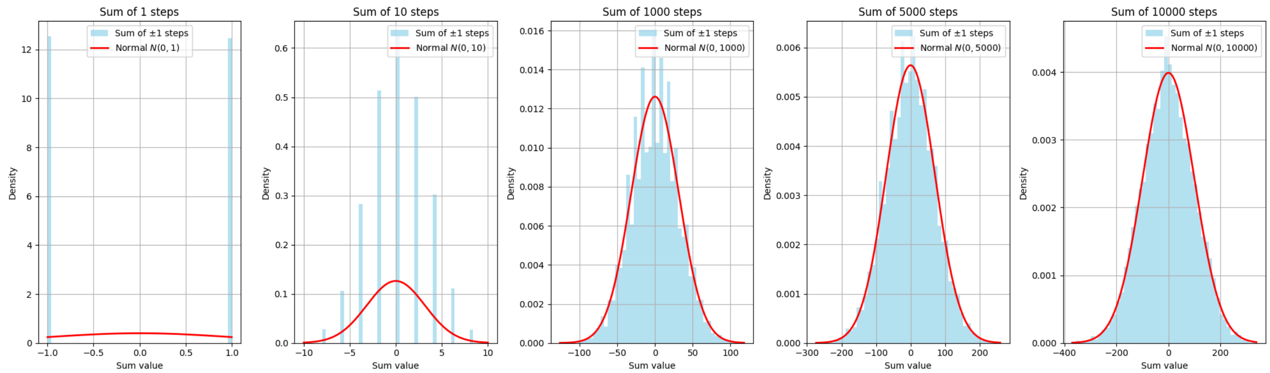
1. $W_0 = 0$ **(Starts at 0)**
2. For any 0 ≤s ≤ t, the increment $W_t - W_s$ is independent of the past values {W_u : u ≤ s} **(Memoryless)**
3. $W_t - W_s \in N(0, \sigma^2(t-s))$**(time increments are normally distributed)**

- We get this property because each increment is the sum of many i.i.d steps with finite variance. By the Central Limit Theorem, the sum converges in distribution to a normal random variable. Because the steps between s and t are i.i.d, W_t - W_s is normally distributed with mean 0 and variance proportional to the elapsed time t - s.

4. The function t to W_t is continuous without any jumps or breaks **(continuous)**
5. The distribution of $W_t - W_s$ depends only on the size of $t - s$ not on t

See the plot below for some intuition on why 3 is valid

Show code



Since we are taking infintely many infintesimally small steps for Brownian motion, we converge to a normal distribution by the Central Limit Theorem.

Standard Brownian Motion is when σ = 1 and μ = 0. Here is the generalized form

$dP = adt + bdW_t$

a: the drift coefficient

b: the diffusion coefficient

W_t: the Wiener Process

dW_t: has $E[dW_t] = 0 \quad and \quad Var(dW_t) = dt$, such that dW_t is independent of dt

Mean and Variance change per time unit

$E[dP] = adt$(see down below), $\quad so \quad E[\frac{dP}{dt}] = a \, Var(dP) = b^2 \, dt$ (see down below) = $Var(\frac{dP}{dt}) = b^2$, since dt is a constant

$E[bdW_t]$ and $Var(bdW_t)$:

$E[dW_t] = 0$, $Var(dW_t) = dt$,

so scaling by b gives $E[bdW_t] = b * 0 = 0$, and $Var(bdW_t) = b^2 \, dt$

E[dP] and Var(dP):

$E[dP] = E[adt + bdW_t] = E[adt] + bE[dW_t] = adt$

$Var(dx) = Var(adt) + Var(bdW_t) = b^2 \, dt$

Note: $Var(adt) = 0$, becasue dt is deterministic, meaning that there is no randomness

Ito Process

An Ito Process is a stochastic process that satisfies the SDE

$dx = a(t, x)dt + b(t, x)dW_t$

Here $a(t, x)$ is a drift function that determines the trend and $b(t, x)$ is the diffusion function that controlls the variance

Geometric Brownian Motion

$dS_t = \mu S_t dt + \sigma S_t dW_t$

S_t is the stock price at time t

μ is the expected rate of return

σ is the volatility

W_t is Standard Brownian Motion

GBM from an Ito Process

$dS_t = a(t, S_t)dt + b(t, S_t)dW_t$

For GBM let $a(t, S_t) = \mu S_t$, and $b(t, S_t) = \sigma dW_t$

If you divide out by S_t, the constants μ and σ, being the mean return and volatility of returns, makes sense using our above calculation of expectation and variance

Itos Lemma

Itos lemma describes how a function F(x, t) where x is a stochastic process evolves over time.

For Itos lemma take x which follows an Ito process:

$dx = a(x, t)dt + b(x, t)dW_t$

Itos Lemma says that the function $F(x, t)$ follows

$dF = (\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{\partial F^2 b^2}{2\partial x^2})dt + \frac{\partial F b dW_t}{\partial x}$, This comes from taylour series expansion and some stochastic calculus rules that are outside the scope of this derivation

Brief intuition

In ordinary calculus we have $dF \approx \frac{\partial F dx}{\partial x} + \frac{\partial F dt}{\partial t}$, but in stochastic calculus you need that extra term to account for the randomness giving: $dF = \frac{\partial F}{\partial t}\,dt + \frac{\partial F}{\partial x}\,dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\,(dx)^2$, which then gets simplified down to what is above after substituting what dx is and using the fact that $(dx)^2 = b^2\,dt$ based on some properties of itos calculus listed below.

1. $(dt)^2 = 0$
2. $dtdW_t = 0$
3. $(dW_t)^2 = dt$

So we assume our stock follows: $dS_t = \mu S_t dt + \sigma S_t dW_t$, and there is a constant risk free rate r.

Now let the value of the option $V = V(t, S_t)$.

We can model how V evolves using Itos Lemma and substitution as follows:

$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial^2 V dS^2}{2\partial S^2}$ where our drift is: μs, and our Diffusion is σS.

Which then simplifies down to: $dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt + \sigma S\frac{\partial V}{\partial S}\,dW_t$

Now lets set our portfolio $\Pi_t = V_t - \Delta_t S_t + C_t$, where Δ is the hedging ratio and C is our cash position.

Now consider $d\Pi_t = dV_t - d(\Delta_t S_t) + dC_t$

$d(\Delta_t S_t) = \Delta_t dS_t + S_t d\Delta_t + d < \Delta, S >_t$ by a stochastic calculus rule.

Since Δ_t is a deterministic function of time with no randomness (We continuously hedge and pick what it is there is no randomness to it. It is based on the realtionship of the option and the stock.) $d < \Delta, S >_t = 0$, so $d(\Delta_t S_t) = \Delta_t dS_t + S_t d\Delta_t$

$dC_t = rC_t dt - S_t d\Delta_t$ This follows from the self financing condition, that all changes in the stock position are financed by deposits and withdrawals from the cash account and the cash earns the risk-free rate r.

$rC_t dt$: is the cash in the bank that growas at the risk free rate r

$-S_t d\Delta_t$: is what happens to your cash if you increase your stock position by $d\Delta_t$

so when we put it together the $-S_t d\Delta_t$ term from dC_t and the $+S_t d\Delta_t$ term from $d\Delta_t S_t$ cancel leaving:

$$d\Pi_t = dV_t - \Delta_t dS_t + rC_t dt = (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW_t - \Delta$$

If we set $\Delta_t = \frac{\partial V}{\partial S}$. We have:

$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rC_t)dt$

Now we will assume Π_t is riskless with no arbitrage. We assume this because we have constructed it so there is no randomness, it is purely deterministic. We assume no arbitrage to force the riskless portfolio to earn the risk free rate. This is reasonable because arbitrage opportunities are fairly rare and they disapear very fast.

By our assumptions of $d\Pi$ being riskless with no arbitrage, we have

$d\Pi_t = r\Pi_t dt$

Now recall $\Pi_t = V_t - \Delta_t S_t + C_t$. We can set $\Delta_t = \frac{\partial V}{\partial S}$, yeilding: $\Pi_t = V_t - S_t \frac{\partial V}{\partial S} + C_t$

Since $d\Pi_t = r\Pi_t dt$, we have:

$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rC_t = r\Big(V_t - S_t \frac{\partial V}{\partial S} + C_t)$. We can then subtract the rC_t terms from both sidesand rearrange the formula by setting it equal to 0 to get the Black-Scholes PDE:

$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$

Solving for put and call values

This section will be expanded later to show the full PDE solution. The key step is a change of variables that reduces the PDE to the heat equation, which has a known Gaussian solution. For now, I focus on Greeks and hedging since those are most relavant to practical trading

After solving using the heat equation we see that the price of a put and call are defined as follows:

$$C = S\,\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \qquad P = Ke^{-r\tau}\Phi(-d_2) - S\,\Phi(-d_1).$$

where: $\tau = T - t, \qquad d_1 = \frac{\ln(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}.$

- S: Current stock price
- K: Strike price of the option
- T: Maturity date
- t: Current time
- τ: Time to maturity ($T - t$)
- r: Constant risk-free rate (T-bill rates)
- σ: Volatility of the stock
- Φ: Standard normal CDF

Reiterating and making assumptions of Black-Scholes Clear:

1. Frictionless markets: No transaction costs, no bid ask spread, no taxes, and perfectly fractional shares
2. No arbitrage opportunities
3. unlimited borrowing and lending at the risk-free rate
4. continuous hedging
5. Assets follow GBM
6. Log normal distribution of stock prices
7. Constant Vol
8. Constant risk-free rate
9. European style excercise
10. No dividends

Moving forward everything discussed is either general or in terms of call options

The Greeks

The Greeks (Note that the Greeks are not unique to the Black-Scholes Model)

- Delta: $\Delta = \frac{\partial V}{\partial S}$: The change in the options price for a $1 change in the stock price
- Gamma: $\Gamma = \frac{\partial^2 V}{\partial S^2}$: The change in delta for a small change in the stock price
- Theta: $\theta = \frac{\partial V}{\partial t}$: How much the option loses each day as expiration approaches. Theta Decay is what happens to the value of an option, it starts out losing a little value per day then as the expiration date gets closer and closer Theta increases
- Vega: $V = \frac{\partial V}{\partial \sigma}$: How much the value of the option changes if implied volatility shifts by 1%
- Rho: $\boldsymbol{\rho} = \frac{\partial V}{\partial r}$: How much the option changes to a 1% change in the risk-free interest rate. Higher rates make calls more valuable because financing the stock is more epxensive, and higher rates make puts less valuable becuuase the present value of the strike decreases over time. $\boldsymbol{\rho}$ is more of a concern for long-dated options also called LEAPS which are 1 year or more.

What is implied volatility?

In practice, we don't know the true volatility of the stock, so we invert the problem to solve for sigma instead of the price since the market has already priced the option. Implied volatility is how volatile the market thinks that the security is going to be for the duration of the option.

The derivative computations are left out, but here are the results under Black-Scholes:

- Gamma: $\Gamma = \frac{\phi(d_1)}{\sigma\sqrt{\tau}}$, where ϕ is the standard normal PDF
- Delta: $\Delta : \Delta_{call} : \Phi(d_1), \Delta_{put} : \Phi(d_1) - 1$
- Theta: $\Theta_{call} = -\frac{S\phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2), \Theta_{put} = -\frac{S\phi(d_1)\sigma}{2\sqrt{\tau}} + rKe^{-r\tau}\Phi(-d_2)$
- Vega: $V = S\phi(d_1)\sqrt{\tau}$
- Rho: $\rho_{call} = K\tau e^{-r\tau}\Phi(d_2), \quad \rho_{put} = -K\tau e^{-r\tau}\Phi(-d_2)$

Quick Vocab

There are 3 states a call option can have

Out of the money (OTM) Stock price < Strike Price At the money (ATM) Stock price ≈ Strike Price In the money (ITM) Stock price > Strik Price Here the stock has intrinsic value if the stock is at 110 and the strike was 100, the option has $10 of intrinsic value per share

A put option has the same 3 states, just flip OTM and ITM to account for the opposite direction

Plot of The options delta vs Stock Price holding all other variables constant

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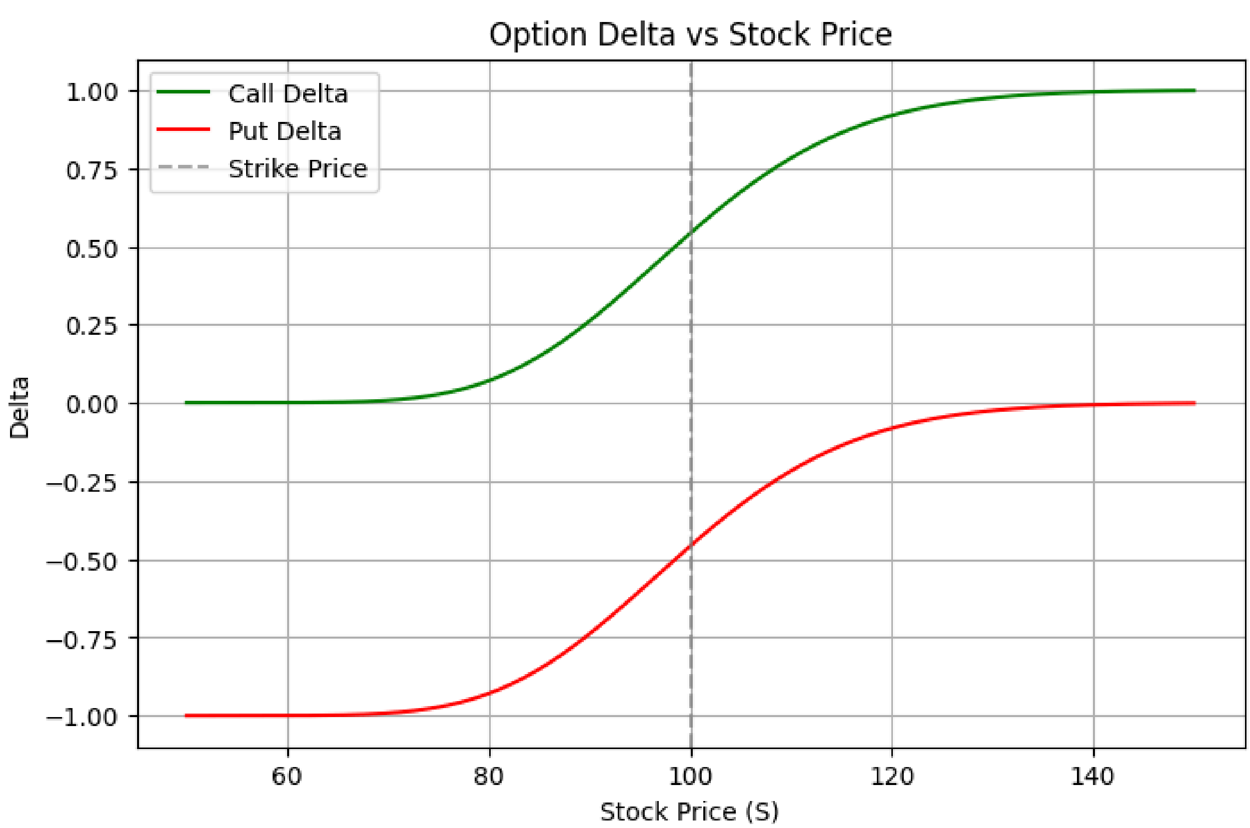

Note that OTM -> ATM -> ITM goes from left to right for the call and left to right for the put. If you look at the absolute value of delta, the absolute value is 1 when the options are very ITM and near 0 when very OTM, which makes since because of the intrinsic value that you have when you get deeper ITM.

Option Gamma as a function of time to maturity and stock price with all else constant

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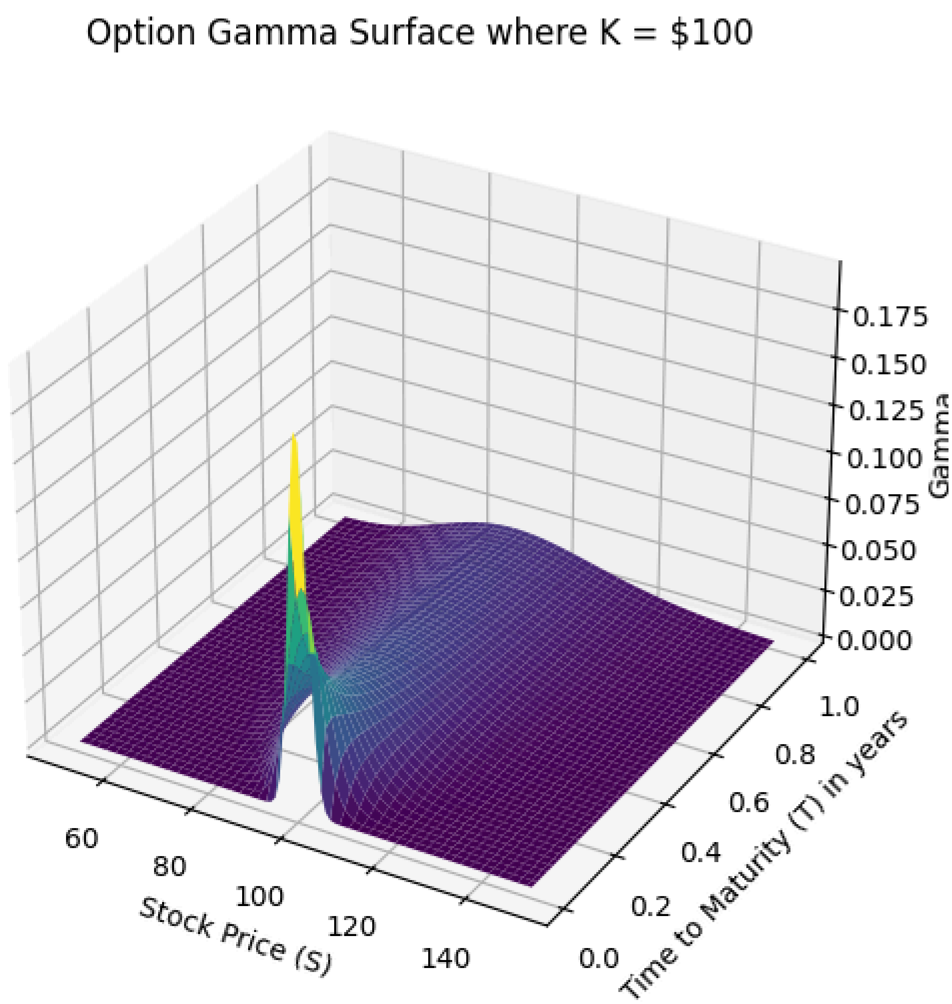

Note that the highest Gamma occurs for ATM options that are close to expiring.

Option theta as a function of stock price with all else constant

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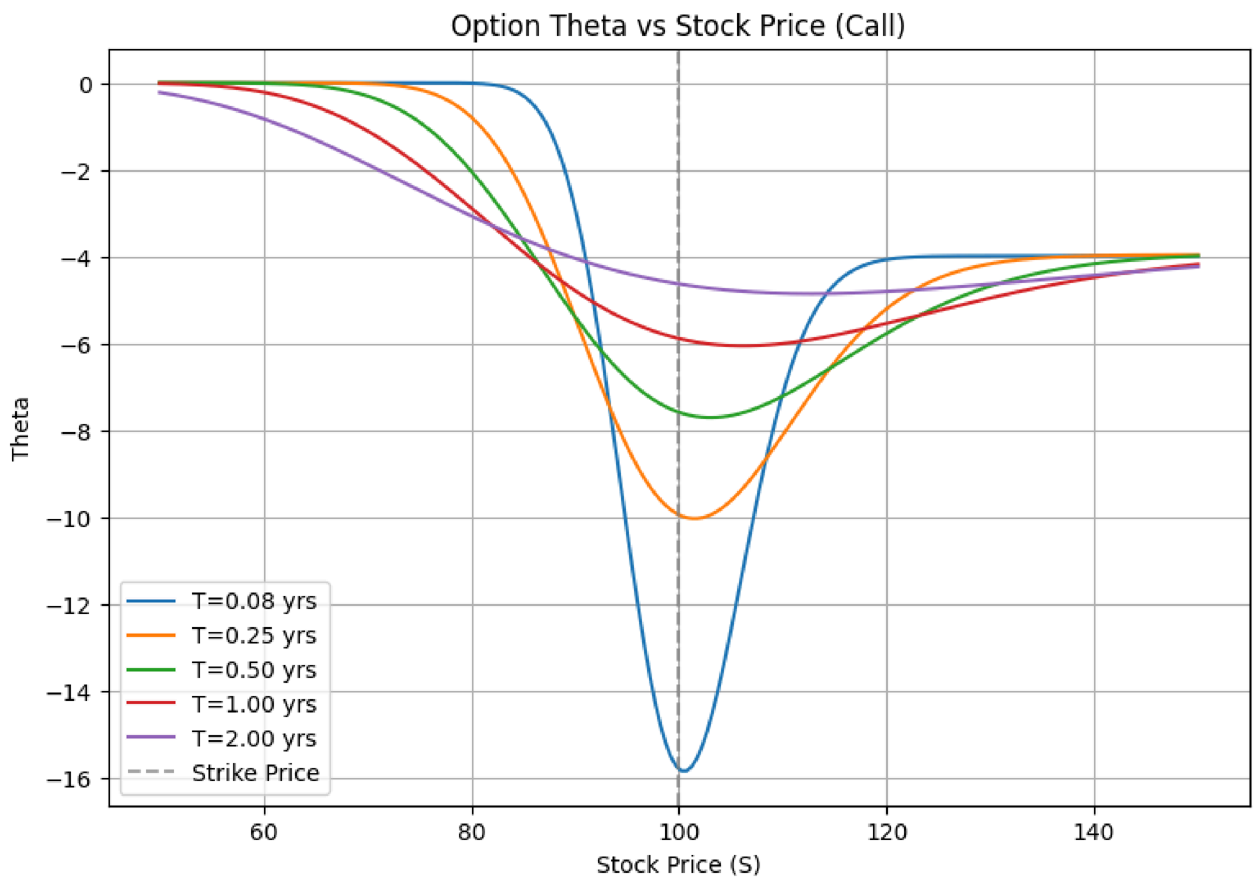

We can see that when the call option is OTM its Theta is nearly 0, as it gets to ATM, the theta spikes, then once it goes to ITM the theta levels off.

Same thing for puts

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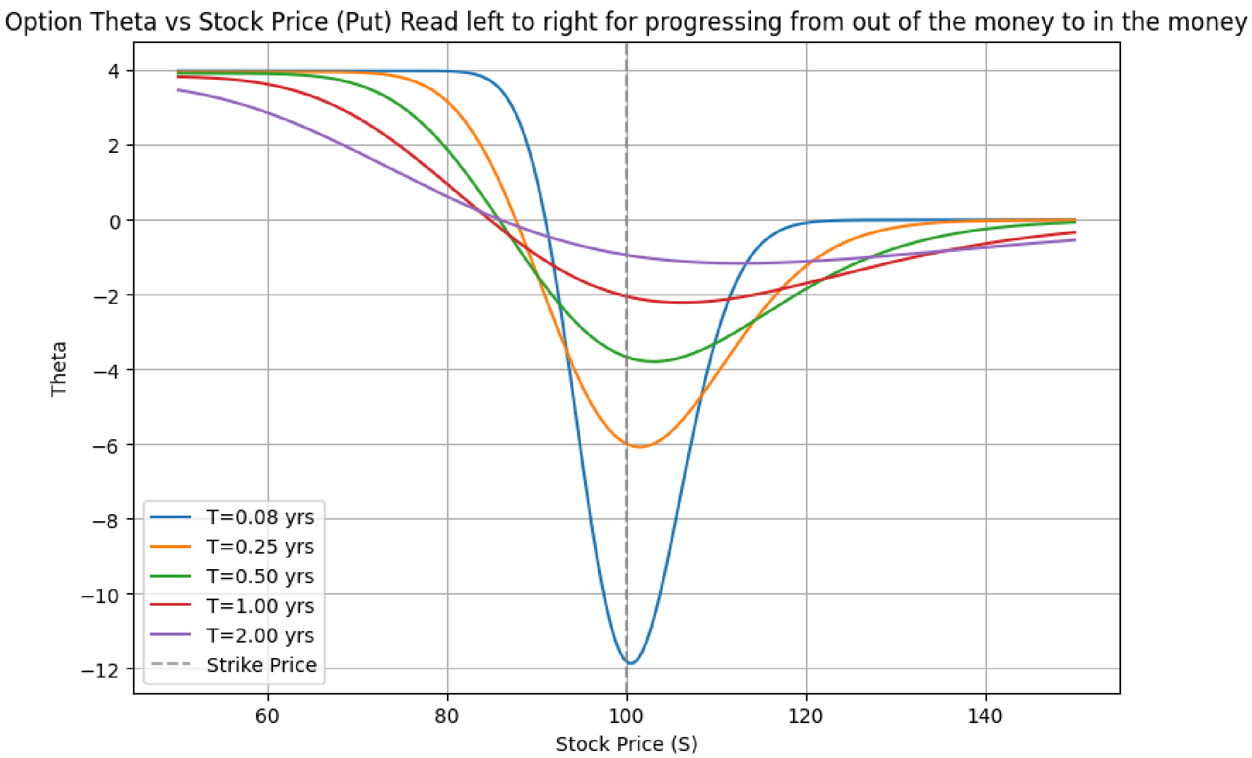

For puts, we read the chart right to left for OTM -> ATM -> ITM and see that theta starts around 0 then spikes negative ATM, then can be positive for very ITM puts. The reason why ITM puts can have positive theta comes from the put-call parity where the second term that is positive, which relies on interest rates can become larger and dominate the negative term at the front.

Intuitively, a positive Θ for deep ITM puts makes sense when interest rates are high enough, because a deep ITM put has almost all of its value locked in as intrinsic value, so there is very little time value left to decay and the part of time value that comes from interest rates can overtake that of decay.

Market sense: A deep ITM put is like shorting a stock and putting cash in t bills. daily t bills earned can be greater than loss of option value for a day because the intrinsic value is so high that it is pretty much just shorting the stock.

Market Makers & Delta hedging

Recall earlier when we derived the Black-Scholes model and said $\Delta = \frac{\partial V}{\partial S}$.

If we are long a call option, we are exposed to the direction of the Stock's moves by $\frac{\partial V}{\partial S}$. If we do not want to be directionally exposed, we can short $\frac{\partial V}{\partial S}$ shares of the stock continuously to have 0 directional exposure at anytime. This is called being Delta-neutral.

Delta Hedging is used by market makers to collect the bid ask spread:

In practice, however, hedging is only possible at discrete intervals. Between re-hedges, the option's delta changes as the stock price moves. This residual exposure is known as gamma risk (the risk that your hedge becomes inaccurate becasue delta itself shifts witht he stock price).

Another big risk that market makers face is known as Vega risk. This is because market makers are typically short vol, so when implied volatility jumps they can lose money

Hedging gamma and vega risk are outside of the scope of the simulation that is to come, but briefly: Gamma risk can be hedged by taking offsetting options positions to balance the Gamma, Vega risk can be hedged by taking offseting exposures across strikes and maturities on the volatility surface (see plot below) using spreads (calendars/diagonals (an example of a calendar spread is where you sell a short dated option and buy a long dated option at the same strike, an example of a diagonal spread is the same thing but you buy at a strike + some change)), and adjusting their quotes to attract trades that rebalance their vol exposure.

You will see Theoretical delta hedging from the perspective of a market maker on a single call option in the simulation that follows. The important thing to know is that the market maker is making a market for an underlying that is following a centi-second discretized GBM under all Black-Scholes assumption except for continuous hedging. The market maker rehedges at 1 second intervals. Since we model the underlying via Geometric Brownian motion, the simulation almost perfectly collects the spread.

Note: It is very important to understand that theoretical delta hedging is not real and that market makers face the risks mentioned above as well as risks associated with the stock having jumps and not following GBM, stocks having dividends, not being European options, transaction costs, liquidity constraints, competition, and adverse selection (if everyone wants to take the other side of my bet should I be doing it?) among others. So market makers must continuously manage risk rather than relying on a theoretical perfect hedge.

› popular IV surface chart

Show code