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Fuzzy and Intuitionistic Fuzzy Turing Machines

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Abstract. First we define a new class of fuzzy Turing machines that we call Generalized Fuzzy Turing Machines. Our machines are equipped with rejecting states as well as accepting states. While we use a t-norm for computing degrees of accepting or rejecting paths, we use its dual t-conorm for computing the accepting or rejecting degrees of inputs. We naturally define when a generalized fuzzy Turing machine accepts or decides a fuzzy language. We prove that a fuzzy language L is decidable if and only if L and its complement are acceptable. Moreover, to each r.e. or co-r.e language L, we naturally correspond a fuzzy language which is acceptable by a generalized fuzzy Turing machine. A converse to this result is also proved. We also consider Atanasov's intuitionistic fuzzy languages and introduce a version of fuzzy Turing machine for studying their computability theoretic properties.

Keywords: Theory of Computing, Fuzzy Language, Fuzzy Turing Machine, Generalized Fuzzy Turing Machine, Intuitionistic Fuzzy Turing Machine

1. Introduction

For a fixed set E, a fuzzy subset of E is a function from E to $[0, 1]$ considered as the degree of membership. A fuzzy language is a fuzzy subset of Σ^*, where Σ is a finite set of alphabets and Σ^* is the set of all finite sequences of Σ.

The notion of fuzzy algorithm was introduced by Lotfi Zadeh in [15]. This line was followed by some works like [10], [12], [13] and after that remained untouched for many years. More recently,

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a natural formal computing model for studying computability theoretic properties of fuzzy languages based on Turing machine model was proposed by Wiedermann, see [14]. The model is called classical fuzzy Turing machine. A classical fuzzy Turing machine is defined as the ordinary non-deterministic Turing machine with the difference that a real number is attached to each transition of the machine considered as the degree of membership of the transition in the set of possible transitions of the machine. To give a degree to a sequence of transitions, one may use any t-norm, for example minimization or product. Degree of acceptance of a word σ is the maximum degree of accepting paths (i.e., sequences of transitions starting from the initial state with input σ to an accepting state) of the machine. Now the language of a machine is defined in the obvious way.

Let $0 \leq b < a \leq 1$. Wiedermann proved that a (classical) language L is r.e. if and only if there is a classical fuzzy Turing machine such that if $\sigma \in L$ then the machine accepts σ with degree a and if $\sigma \notin L$ then the machine accepts σ with degree b. See [2] for a clarification of Wiedermann's results. In [7], classical fuzzy Turing machines are generalized to the case of lattice-valued fuzzy Turing machines.

In this paper we first consider a version of fuzzy Turing machines equipped with both accepting and rejecting states. We also use the t-conorm dual to the t-norm used for calculating the degree of each path for calculating the accepting degree of of a word, instead of maximization. We call this version of fuzzy Turing machine Generalized Fuzzy Turing Machine, GFTM. We study basic properties of fuzzy languages and GFTMs. We also consider Atanasov's intuitionistic fuzzy languages and introduce a version of fuzzy Turing machine called Intuitionistic Fuzzy Turing Machine, IFTM, for studying their computability theoretic properties.

The paper is organized as follows. In Section 2, we recall the definition of Classical Fuzzy Turing Machine of Wiedermann and mention some facts concerning it. In Section 3, we introduce GFTM and prove some of its basic properties. In Section 4, we show how certain fuzzy version of a r.e. (classical) language can be decided by a GFTM. In Section 5, we introduce and study IFTM. Section 6 is devoted to the final remarks.

2. Classical Fuzzy Turing Machine

In this section we recall some important facts and definitions concerning fuzzy languages and classical fuzzy Turing machines. The main references are [14] and [2]. [6] discusses variants of fuzzy Turing machines. In [7], a notion of lattice-valued fuzzy Turing machine is introduced. A standard reference book for mathematical fuzzy logic is [5], see also the newer book [11] where fuzzy logic and fuzzy set theory are discussed in a broader sense.

A t-norm is a binary operation $* : [0,1]\times[0,1] \longrightarrow [0,1]$ which is commutative, associative and nondecreasing in both arguments and with the properties $0 * x = 0, 1 * x = x$, for all x. Each t-norm gives a truth function for the binary connective (strong) conjunction in fuzzy logic, see [5].

Definition 2.1. A *classical fuzzy Turing machine* is a nine-tuple

$(Q, \Sigma, \Gamma, \Delta, q_0, B, F, *, \lambda)$, where
- Q is the set of states,
- Σ is the set of input alphabets,
- Γ is the set of tape alphabets,
- Δ is a fuzzy subset of $Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$,

- $q_0 \in Q$ is the starting state,
- $B \in \Gamma \setminus \Sigma$ is the blank symbol,
- $F \subseteq Q$ is the set of accepting states,
- $*$ is a t-norm,
- and $\lambda : Q \times \Gamma \times Q \times \Gamma \times \{R, L\} \to [0, 1]$ is a function.

In the above definition, for each

$$\delta \in Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$$

$\lambda(\delta)$ gives the truth degree of the proposition $\delta \in \Delta$. Usual notions like *instantaneous description* are defined like ordinary Turing machines, see [14] or [2] for details. [4] is a standard reference for classical theory of computing based on ordinary Turing machine.

If α, α' are two instantaneous descriptions, $\alpha \preceq^r \alpha'$ means that there is a possible move in Δ with degree r leading from α to α' in one step. If $\alpha_0 \preceq^{r_1} \alpha_1 \preceq \cdots, \preceq^{r_t} \alpha_t$ where α_0 is an initial instantaneous description of the machine (with an input ω), then we say that α_t is reachable from α_0 with the degree 1, if $t = 0$, and with the degree $r_1 * \cdots * r_t$, if $t \geq 1$. The path from α_0 to α_f where α_f is an accepting instantaneous description, is defined as an accepting path.

Definition 2.2. With the above conventions, the accepting degree of an input σ, $e(\sigma)$, is defined as follows. If there is at least one accepting path from α_0 on input σ, then $e(\sigma)$ is equal to the Supremum of all degrees of such paths. Otherwise, $e(\sigma) = 0$

Fact 2.3. [W, Theorem 3.1] Let $0 \leq b < 1$ be a real number. If a language L is r.e. then there is a classical fuzzy Turing machine such that for each $\sigma \in \Sigma^*$, if $\sigma \in L$ then $e(\sigma) = 1$ and if $\sigma \notin L$ then $e(\sigma) = b$.

Fact 2.4. [W, Theorem 3.1] Let $0 \leq b < 1$ be a real number. If a language L is co-r.e. then there is a classical fuzzy Turing machine such that for each $\sigma \in \Sigma^*$, if $\sigma \in L$ then $e(\sigma) = b$ and if $\sigma \notin L$ then $e(\sigma) = 1$.

Let us denote the languages accepted by the classical fuzzy Turing machines described in Facts 2.3 and 2.4 by $F_L(1, b)$ and $F_L(b, 1)$, respectively. From the above facts Wiedermann concluded that classical fuzzy Turing machines are more powerful than ordinary non-deterministic Turing machines, since a co-r.e. language can be non-r.e. but here one sees that a fuzzy version of it is acceptable in the fuzzy sense. This claim has been considered debatable by some next authors as different fuzzifications of a fixed (classical) language is considered, see [2].

A computable classical fuzzy Turing machine is a classical fuzzy Turing machine that its λ and $*$ are restricted to rational numbers and are computable, see ([2], Definition 2) for more detailed definition.

Fact 2.5. ([2], Theorem 4) Let $0 \leq b < a \leq 1$ be two rational numbers. A language L is r.e. if and only if there is a computable fuzzy Turing machine such that if $\sigma \in L$ then $e(\sigma) = a$ and if $\sigma \notin L$ then $e(\sigma) = b$.

In [7], similar results obtained in a more general setting of lattice-valued fuzzy Turing machines.

3. Generalized Fuzzy Turing Machine

In this section we introduce the notion of Generalized Fuzzy Turing Machine as a new formal computing model in the context of fuzzy languages. The main difference between our computing model and the classical fuzzy Turing machine is that our model is equipped with rejecting states as well as accepting states. Another difference is that, if we use a t-norm $*$ in the definition of a Turing machine, we use its dual t-conorm $*'$ instead of max for calculating accepting degrees. Although we do not use the duality condition in the most of our results on GFTMs, we think it is a natural condition. Also, it is a base for possible generalization to lattice valued fuzzy Turing machines, see [7].

Recall that if $*$ is a t-norm then the t-conorm $*'$ dual to $*$ is obtained from the relation

$$*'(x, y) = 1 - *(1 - x, 1 - y).$$

Actually, t-norms and their dual t-conorms generalize the properties of the $min{-}max$ functions used in set theory as the membership functions of intersection-union or as the truth functions for conjunction-disjunction in logic, respectively.

Definition 3.1. A *Generalized fuzzy Turing machine, GFTM*, is a an eleven-tuple

$(Q, \Sigma, \Gamma, \Delta, q_0, B, F, F', *, *', \lambda)$, where
- Q is the set of states,
- Σ is the set of input alphabets,
- Γ is the set of tape alphabets,
- Δ is a fuzzy subset of $Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$,
- $q_0 \in Q$ is the starting state,
- $B \in \Gamma \setminus \Sigma$ is the blank symbol,
- $F, F' \subseteq Q$ are the sets of accepting and rejecting states,
- $*$ is a t-norm and $*'$ is its dual t-conorm,
- and $\lambda : Q \times \Gamma \times Q \times \Gamma \times \{R, L\} \to [0, 1]$ is a function.

As in classical fuzzy Turing machines, for each

$$\delta \in Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$$

$\lambda(\delta)$ gives the truth degree of the proposition $\delta \in \Delta$.

Usual notions like *instantaneous description* are defined like ordinary Turing machines, see [14] or [2] for details. If α, α' are two instantaneous descriptions, $\alpha \preceq^r \alpha'$ means that there is a possible move in Δ with fuzzy degree r leading from α to α' in one step. If $\alpha_0 \preceq^{r_1} \alpha_1 \preceq \cdots, \preceq^{r_t} \alpha_t$, where α_0 is the initial instantaneous description of the machine (with an input ω), then we say that α_t is reachable from α_0 with the degree 1, if $t = 0$, and the degree $r_1 * \cdots * r_t$, if $t \geq 1$.

The path from α_0 to α_a, where α_a is an accepting instantaneous description, is defined as an accepting path. The path from α_0 to α_r, where α_r is a rejecting instantaneous description, is defined as a rejecting path.

Definition 3.2. Let M be a GFTM as above.

i) The accepting degree of an input σ, $e_M(\sigma)$, is defined as follows. If there is at least one accepting path from α_0 on input σ, then $e_M(\sigma)$ is equal to the Supremum of all $a_1 *' a_2 *' \cdots *' a_n$ where $a_1, a_2, \cdots, a_n$ are degrees of some accepting paths. Otherwise, $e_M(\sigma) = 0$.

ii) The rejecting degree of an input σ, $e'_M(\sigma)$, is defined as follows. If there is at least one rejecting path from α_0 on input σ, then $e'_M(\sigma)$ is equal to the Supremum of all $a_1 *' a_2 *' \cdots *' a_n$ where $a_1, a_2, \cdots, a_n$ are degrees of some rejecting paths. Otherwise, $e'_M(\sigma) = 0$.

Definition 3.3. Let Σ be a finite set (of alphabets) and Σ^* be the set of all finite sequences of Σ. By a fuzzy language, we mean a fuzzy subset L of Σ^*. Each fuzzy language can be considered as a function L from Σ^* to $[0, 1]$.

Let M be a GFTM. We can think of M as the ordered pair (e_M, e'_M) to express that the accepting degree of a word σ by M is $e_M(\sigma)$ and the rejecting degree of a word σ by M is $e'_M(\sigma)$. So we can speak about the fuzzy language accepted by M, $A(M)$ and the fuzzy language rejected by M, $R(M)$. The membership functions of these languages are e_M and e'_M, respectively.

In the sequel we assume that L is a Fuzzy Language.

Definition 3.4.

i) L is acceptable if there is a GFTM M such that $L = A(M)$.

ii) L is decidable if there is a GFTM M such that $L = A(M)$ and $L^c = R(M)$.

The complement of a fuzzy language L, L^c, is defined by $L^c(x) = 1 - L(x)$.

Proposition 3.5. L is decidable if and only if L and L^c are acceptable.

Proof:
$\Rightarrow$ is obvious.

For the $\Leftarrow$ part, let M and M' be two GFTMs such that M accepts L and M' accepts L^c. Let $\widetilde{M}, \widetilde{M'}$ be two GFTMs obtained from M, M' in the following way:

1) Rename the starting states of the machines with some non-starting states,

2) Change the rejecting states coming from M and M' to some new non-rejecting and non-accepting states.

3) Change the accepting states of M' with rejecting states.

Now consider a GFTM $\overline{M}$ with two immediate non-deterministic moves with degree 1 from the starting state to $\widetilde{M}, \widetilde{M'}$. We show that $\overline{M}$ decides the language L.

For each word σ, we have:

$$e_{\overline{M}}(\sigma) = e_M(\sigma) = L(\sigma),$$

$$e'_{\overline{M}}(\sigma) = e_{M'}(\sigma) = 1 - L(\sigma) = L^c(\sigma).$$

So we have $L = A(\overline{M})$ and $L^c = R(\overline{M})$. □

Corollary 3.6. If L is decidable, then L^c is decidable.

Definition 3.7. Let L_1 and L_2 be two fuzzy languages. Let $*$ be a t-norm and $*'$ be its dual t-conorm. Then the languages $L_1 * L_2$ and $L_1 *' L_2$ are defined by

$$(L_1 * L_2)(x) = L_1(x) * L_2(x)$$

and

$$(L_1 *' L_2) = L_1(x) *' L_2(x).$$

Proposition 3.8. Let L_1 and L_2 be two fuzzy languages. Assume that L_1 and L_2 are accepted by GFTMs M_1 and M_2 equipped with the same t-norm $*$. Let $*'$ be the dual t-conorm of $*$. Then $L_1 *' L_2$ is accepted by a machine equipped with the same t-norm.

Proof:
Change the starting state of M_1 and M_2 to two non-starting new states. Consider a GFTM M such that its work starts from the (new) starting state with two immediate transitions with degree 1 to the (modified) machines. The GFTM M obviously accepts the language $L_1 *' L_2$. □

Note that, if the GFTMs M_1 and M_2 in the proof of Proposition 3.8 respectively decides the languages L_1 and L_2, then the defined GFTM M rejects the language $L_1 * L_2$. To see this note that for any input σ, we have

$$\begin{aligned} e'_M(\sigma) &= e'_{M_1}(\sigma) *' e'_{M_2}(\sigma) \\ &= L_1^c(\sigma) *' L_2^c(\sigma) \\ &= (1 - L_1(\sigma)) *' (1 - L_2(\sigma)) \\ &= 1 - [L_1(\sigma) * L_2(\sigma)] \\ &= 1 - [(L_1 * L_2)(\sigma)] \\ &= (L_1 * L_2)^c(\sigma). \end{aligned}$$

Note that that in the above result we used the duality between $*$ and $*'$.

4. Classical Languages and GFTMs

As stated in Section 2, for each r.e. or co-r.e. classical language L, Wiedermann corresponded a fuzzy language which is accepted by a Classical Fuzzy Turing Machine. Actually, he gave a classical description of such a fuzzy language. Below we give a version of this result in the context of Generalized Fuzzy Turing Machines.

Proposition 4.1. If L is a (classical) r.e. language, then there is a GFTM M which (classically describing) behaves in the following way: For each input σ, $\sigma \in L$ if and only if M accepts σ with degree 1 and rejects σ with degree 0.

Proof:
Let M' be the ordinary Turing machine which accepts L (with accepting states only). We can assume that M' behaves in the following way. For each input σ, if $\sigma \in L$ then M' with input σ eventually halts

in an accepting state and if $\sigma \notin L$ then M' with input σ never halts. Construct a GFTM M as follows ($*$ and $*'$ can be any t-norm and its dual t-conorm):

1)Replace the starting state of M' with a fresh arbitrary state,
2) consider degree 1 for each transition of the machine,
3) consider the starting state q_0,
4) make two immediate nondeterministic choices form q_0, one to the new copy of M' with degree 1 and one to a rejecting state with degree 0.

Obviously, M has the desired properties. □

In 4.1, let M' be computable. By a computable GFTM we mean a GFTM which its λ and $*$ are restricted to rational numbers and are computable in the classical sense. In this case the converse of 4.1 also holds. For this we give a simple argument based on Turing's Thesis, a more exact proof can be given like the one in [2], Theorem 4. To show that L is r.e., assume that $\sigma \in SEQ(\Sigma)$. Now run M on the input σ. If $\sigma \in L$ (and just in this case), the machine with input σ will have an accepting path with degree 1 and we can compute this degree. So L is r.e. (note that any word will be rejected by M with degree 0).

Proposition 4.2. If L is a (classical) co-r.e. language, then there is a GFTM M which behaves in the following way: For each input σ, $\sigma \notin L$ if and only if M rejects x with degree 1 and accepts x with degree 0.

Proof:
As L is co-r.e., there is an ordinary Turing machine M' that accepts L^c. Interpret M' as a GFTM M by doing the following ($*$ and $*'$ can be any t-norm and its dual t-conorm):
1)Replace the accepting states of M' with rejecting states,
2) consider degree 1 for each transition of the machine,
3) replace the starting state of M' with a fresh arbitrary state,
4) make two immediate nondeterministic choices form q_0, one to the new copy of M' with degree 1 and one to an accepting state with degree 0.

Obviously, M has the desired properties. □

As before, the converse of 4.2 also holds when we work with computable GFTMs.

Considering efficiency issues in the context of GFTMs, one may replace r.e languages with NP languages and co-r.e. languages with co-NP languages, then the GFTMs described in the two above propositions obviously work in the polynomial time. So one may describe this situation by saying that the languages accepted by them is in Fuzzy-NP or Fuzzy-co-NP, respectively. The converse of these results also hold, if one works with computable GFTMs working in the polynomial time.

5. Intuitionistic Fuzzy Turing Machine

The GFTMs in Section 3 are defined in order to naturally represent computability issues for fuzzy languages. As they are equipped with rejecting states they can explicitly speak about the rejecting degree of an input which can be interpreted as the degree of nonbeing in the language at hand. But the problem here is that in GFTMs there is no connection between accepting degrees and rejecting degrees. Each of them can be any number between 0 and 1 independent of each other. One way to deal with this problem is to consider two $dual$ degrees for each transition, one as the degree of its membership and one as the degree of its nonmembership. This takes us to the world of intuitionistic fuzzy sets. The duality of $* - *'$ play an important role here.

Intuitionistic fuzzy sets were defined by Atanassov, see [1]. For a fixed set E, an intuitionistic fuzzy subset of E is a pair of functions from E to $[0, 1]$ considered as the degrees of membership and nonmembership, respectively. The word "intuitionistic" here should not be confused with the usual usage of this word in intuitionistic logic, see e.g. [3] for an accessible description of intuitionistic logic.

In this paper we are concerned with formal languages in the context of intuitionistic fuzzy sets. An intuitionistic fuzzy language is an intuitionistic fuzzy subset of Σ^* where Σ is a finite set. We suggest a formal computing model for studying intuitionistic fuzzy languages from the viewpoint of the theory of computing. Intuitionistic fuzzy Automata are introduced in [8] in a similar way.

Definition 5.1. An *Intuitionistic Fuzzy Turing Machine, IFTM*, is an eleven-tuple

$(Q, \Sigma, \Gamma, \Delta, q_0, B, F, *, *', \mu, \nu)$, where
- Q is the set of states,
- Σ is the set of input alphabets,
- Γ is the set of tape alphabets,
- Δ is an intuitionistic fuzzy subset of $Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$,
- $q_0 \in Q$ is the starting state,
- $B \in \Gamma \setminus \Sigma$ is the blank symbol,
- $F \subseteq Q$ is the set of accepting states,
- $*$ is a t-norm and $*'$ is its dual t-conorm,
- and $\mu, \nu : Q \times \Gamma \times Q \times \Gamma \times \{R, L\} \to [0, 1]$ are functions where $\mu(\delta) + \nu(\delta) \leq 1$ for each δ.

Note that in the above definition an IFTM is defined like classical fuzzy Turing machines with the difference that each IFTM has a function ν giving degrees of not being in the set of instructions. For each

$$\delta \in Q \times \Gamma \times Q \times \Gamma \times \{R, L\}$$

$\mu(\delta)$ and $\nu(\delta)$ give the truth degree of the proposition $\delta \in \Delta$ and $\delta \notin \Delta$, respectively. We call $(\mu(\delta), \nu(\delta))$ the intuitionistic fuzzy degree of the proposition $\delta \in \Delta$.

Usual notions like *instantaneous description* are defined like ordinary Turing machines naturally. If α, α' are two instantaneous descriptions, $\alpha \preceq^{(r,r')} \alpha'$ means that there is a possible move in Δ with intuitionistic fuzzy degree (r, r') leading from α to α' in one step. If $\alpha_0 \preceq^{(r_1,r'_1)} \alpha_1 \preceq \cdots, \preceq^{(r_t,r'_t)} \alpha_t$ where α_0 is the initial instantaneous description of the machine (with an input ω), then we say that α_t is reachable from α_0 with the degree 1, if $t = 0$, and the degree $r_1 * \cdots * r_t$, if $t \geq 1$. The co-degree of the path is 0, if $t = 0$, and is $r_1 *' \cdots *' r_t$, if $t \geq 1$.

The path from α_0 to α_f, where α_f is an accepting instantaneous description, is defined as an accepting path. Accepting and rejecting degrees of an input are defined as follows.

Definition 5.2. i) The accepting degree of an input σ, $e_M(\sigma)$, is defined as follows. If there is at least one accepting path from α_0 on input σ, then $e_M(\sigma)$ is equal to the Supremum of all $a_1 *' a_2 *' \cdots *' a_n$ where $a_1, a_2, \cdots, a_n$ are degrees of some accepting paths. Otherwise, $e_M(\sigma) = 0$.

ii) The rejecting degree of an input σ, $e'_M(\sigma)$, is defined as follows. If there is at least one accepting path from α_0 on input σ, then $e'_M(\sigma)$ is equal to the Infimum of all $a_1 * a_2 * \cdots * a_n$ where $a_1, a_2, \cdots, a_n$ are co-degrees of some accepting paths. Otherwise, $e'_M(\sigma) = 1$.

The intuition behind the definition of the rejecting degree above is that an input is rejected by an IFTM if it does not lead to any accepting path.

A simple computation shows that, for each input σ for an IFTM M, $e_M(\sigma) + e'_M(\sigma) \leq 1$. To see this, for simplicity, assume that this input leads to two accepting paths with degrees $a * b$ and $c * d$. So,

$$\begin{aligned} e_M(\sigma) &= (a * b) *' (c * d) \\ &= 1 - [(1 - (a * b)) * (1 - (c * d))] \\ &= 1 - [((1 - a) *' (1 - b)) * ((1 - c) *' (1 - d))] \\ &\leq 1 - e'_M(\sigma). \end{aligned}$$

The (last) inequality obtained using the fact that t-norms and t-conorms are nondecreasing.

Definition 5.3. Assume that $L = (\mu, \nu)$ is an IFL.

i) We say that $L = (\mu, \nu)$ is acceptable if there is an IFTM M such that for each $\sigma \in SEQ(\Sigma)$, $e_M(\sigma) = \mu(\sigma)$.

ii) We say that $L = (\mu, \nu)$ is rejectable if there is an IFTM M such that for each σ, $e'_M(\sigma) = \nu(\sigma)$.

iii) We say that $L = (\mu, \nu)$ is decidable if there is an IFTM M such that for each $\sigma \in SEQ(\Sigma)$, $e_M(\sigma) = \mu(\sigma)$ and $e'_M(\sigma) = \nu(\sigma)$.

Definition 5.4. Let $L_1 = (\mu_1, \nu_1)$ and $L_2 = (\mu_2, \nu_2)$ be two IFLs. Then the union and intersection of them are defined respectively as

$$L_1 \cup L_2 = (max(\mu_1, \mu_2), min(\nu_1, \nu_2))$$

and

$$L_1 \cap L_2 = (min(\mu_1, \mu_2), max(\nu_1, \nu_2)).$$

Definition 5.5. Let $L_1 = (\mu_1, \nu_1)$ and $L_2 = (\mu_2, \nu_2)$ be two IFLs. Assume that $*$ is a t-norm and and $*'$ is its t-conorm. Then the generalized union and intersection of L_1, L_2 (w.r.t. $*, *'$) are defined as follows

$$L_1 *' L_2 = (\mu_1 *' \mu_2, \nu_1 * \nu_2)$$

and

$$L_1 * L_2 = (\mu_1 * \mu_2, \nu_1 *' \nu_2).$$

Proposition 5.6. Let $L_1 = (\mu_1, \nu_1)$ and $L_2 = (\mu_2, \nu_2)$ be two IFLs. If L_1 and L_2 are acceptable (rejectable, decidable), then their generalized union $L_1 *' L_2$ is acceptable (rejectable, decidable).

Proof:
The proof is an obvious modification of the proof of 3.8. Assume that L_1 and L_2 are respectively accepted by the IFTMs M_1 and M_2 equipped with the same t-norm $*$. Let $*'$ be the dual t-conorm of $*$. Change the starting state of the machines to two non-starting new states. Now consider an IFTM M such that its

work starts from the (new) starting state with two immediate transitions with degree 1 and co-degree 0 to the modified M_1 and M_2. The machine M obviously accepts (rejects, decides) the language $L_1 *' L_2$. $\square$

Proposition 5.7. If L_1 is a (classical) r.e. language then the IFL $L = (\mu, \nu)$ is decidable where, for each $x \in L$, $\mu(x) = 1$ and $\nu(x) = 0$, and for each $x \notin L$, $\mu(x) = 0$ and $\nu(x) = 1$.

Proof:
The proof is an obvious modification of the proof of 4.1. Let M' be the machine which accepts L. Construct a IFTM M as follows ($*$ and $*'$ can be any t-norm and its dual t-conorm):

1) Replace the starting state of M' with a fresh arbitrary state,

2) consider degree 1 and co-degree 0 for each transition of the machine,

3) consider the starting state q_0,

4) make two immediate nondeterministic choices form q_0, one to the new copy of M' with degree 1 and co-degree 0 and one to a rejecting state with degree 0 and co-degree 1.

Obviously, M has the desired properties. $\square$

Proposition 5.8. If L is a (classical) co-r.e. language then the IFL language (μ, ν) is decidable where, for each $x \in L$, $\mu(x) = 0$ and $\nu(x) = 1$, and for each $x \notin L$, $\mu(x) = 1$ and $\nu(x) = 0$.

Proof:
Let M' be the machine which accepts L^c. Interpret M' as an IFTM M by doing the following ($*$ and $*'$ can be any t-norm and its dual t-conorm):

1) Consider degree 1 and co-degree 0 for each transition of the machine,

2) replace the starting state of M' with a fresh arbitrary state,

3) make two immediate nondeterministic choices form q_0, one to the new copy of M' with degree 1 and co-degree 0 and one to an accepting state with degree 0 and co-degree 1.

Obviously, M has the desired properties. $\square$

As before one can prove a converse of Proposition 5.7 and Proposition 5.8 when one works with computable intuitionistic fuzzy Turing machines, or better, when one works with intuitionistic fuzzy Turing machines working in the polynomial time.

6. Final Remarks

In this paper we have proposed two new formal computing models, GFTM for fuzzy languages and IFTM for intuitionistic fuzzy languages. In GFTM we have accepting and rejecting states and use a t-norm and its dual t-conorm to compute the degrees of paths and the accepting-rejecting degrees. The condition that the t-conorm be dual of the t-norm can be omitted without substantial change. However this duality has some theoretical advantages. For example, using that we can define generalized union and intersection of the languages in the obvious ways and prove the expected results about them. The existence of rejecting states has the advantage that the rejecting degrees can be explicitly defined what which is implicit in the work of Wiedermann and the authors after him. This claim can be verified by considering two main Wiedermann's results, Facts 2.3 and 2.4. The fuzzy language $F_L(1, b)$ is a language that with our terminology can be accepted with degree 1 and rejected with degree b by a GFTM.

In a GFTM there is no relation between the accepting and rejecting degrees. For example, both of them can be 1. If one sees this as a weak point and wants to work with intuitionistic fuzzy languages one can use IFTMs. In IFTMs we do not have rejecting states but working with t-norms and their dual t-conorms and defining the accepting and rejecting degrees properly enable us to have two dual degrees, sum of them less than or equal 1. For this, the duality between t-norm and t-conorm in a machine is an essential requirement. As far as I know, IFTM is the only version of Turing machine that has been proposed for studying computing properties of intuitionistic fuzzy languages except the one in [9] where the authors defined a version of the machine with accepting and rejecting states as well. We prefer IFTMs since they are simpler and theoretically more convenient to work with.

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