Research Article

The use of fuzzy arithmetic in decision-making problems

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Abstract
The article examines the rules for performing basic arithmetic operations on fuzzy gradations. The tables for performing operations of addition, limited addition, multiplication, as well as operations of minimum and maximum are obtained. A generalization of Markov models of decision making in the case of ambiguity, when there is no information about the probabilities of states, is proposed. In this case, the components of the model are considered as fuzzy and the concept of fuzzy risk is additionally introduced. An example of application of Markov model to economic problem is considered. The models of optimal choice were studied for various decision-making strategies, namely, the threshold criterion method, the worst criterion method, additive convolution, and the distance method for different selection functions. Theoretical studies are accompanied by numerical calculations.

Keywords Fuzzy set · Fuzzy arithmetic · Decision theory · Markov decision-making models

1 Introduction

In decision-making problems, traditionally, two groups of methods are used to describe a subject field: statistical probabilistic methods and methods based on the theory of fuzzy sets (the theory of possibilities). The methods of the first group are applied when the subject domain can be described by probabilistic characteristics (distribution law, distribution moments, etc.). The methods of the second group are used in conditions of ambiguity, uncertainty, incompleteness of data. In the methods of the first group, each alternative is represented by a confidence interval with an indication of the probability characteristics (confidence probability, expectation, variance, etc.). The main advantage of these methods is a fairly complete description of the subject domain. The disadvantage of these methods is the need for a large amount of information, which is often impossible in practice. If the amount of data is insufficient, then this leads to erroneous and unreliable decisions; systematic errors occur that are difficult to control. This group includes Markov models, Bayesian approach, regression analysis methods, cluster analysis, factor analysis, interval analysis, method based on utility function, the method of eigenvalues (hierarchy analysis method) and others. An overview of the methods of the first group can be found in the works [1–19]. In the methods of the second group, each alternative is represented as a measure of possibility (necessity), a fuzzy number, or a linguistic variable. As a measure of possibility (necessity), as a rule, the membership function is used. A fuzzy number is represented by various distributions (exponential, rectangular, triangular, trapezoidal, etc.) with an indication of the membership function of this distribution. The linguistic variable is represented by verbal evaluations with an indication of the membership function of a fuzzy utterance. The main advantage of these methods is their suitability for modeling vague situations with incomplete or fuzzy data. In addition, they give a smoothed description of the subject domain, which allows us to save all the useful information about possible solutions and makes these methods convenient in the tasks of managing technical objects. The main disadvantage of these methods is that the membership functions for individual criteria

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(aims) cannot be summed or multiplied directly. Therefore, the calculations are performed on the basis of the Zadeh generalization principle, which uses idempotent operations max and min. These operations are not valid (adequate) for the operations of summation and multiplication, which belong to the group of archimedean operations. With an increase in the number of terms (multipliers), the distribution expands ("spreads"), the estimates are highly leveled and become indistinguishable within the error, which reduces the reliability of the choice of the preferred solution. In addition, the disadvantage is the need for rationing weights. To eliminate these difficulties, complex procedures are used, for example, solving a linear programming problem, which is associated with machine time and can lead to ambiguous estimates. So, the calculations are cumbersome, the results are not very clear, and they are difficult to interpret [20–22]. Approximation of a fuzzy set of values by different distributions (uniform, exponential, triangular, etc.) also does not allow using the advantages of fuzzy representation [23–25]. It should be borne in mind that in many system tasks of the optimal choice related to the analysis and synthesis of systems, decision making, control and evaluation, the specific numerical content of the quantities does not matter, at least at the stage of algorithmization of the task and solution search, but only the order relation between them. Therefore, it becomes necessary to operate on quantities without being tied to a numerical context. Many methods of this group are a generalization of the methods of the first group for a fuzzy case. These include regression analysis of possibilities, fuzzy cluster analysis, fuzzy arithmetic, and others. In particular, fuzzy arithmetic is an attempt to generalize the concept of a number to a fuzzy case. An overview of the methods of the second group and achievements in this area can be found in the works [20–31]. In [25] an extensive bibliography of works on fuzzy numbers and their applications is given. The author suggests an approach based on the use of fuzzy gradations, in which the numbers are replaced by quantities. In [27–30], the advantages of this approach in solving various problems were shown. The purpose of this article is to generalize the most important methods for applications: Markov models and a multi-criteria utility function using fuzzy gradation techniques applied to decision-making tasks. This approach eliminates the difficulties noted above. It is closest to the theory of interval estimation and models of the linguistic description of the subject domain. The main advantage of this approach is that it allows us to perform arithmetic operations directly without using the membership function and its application does not require the procedure of rationing weights. The advantage of this approach is also a quick analysis of many alternatives with preservation of all useful information. The main disadvantage of the proposed approach is the difficulty of its generalization to fuzzy sets of high order (hierarchical systems).

2 Arithmetic operations on fuzzy gradations

To describe the object area we use the fuzzy gradations in the range VL…VH. The range comprising gradations VL, L, M, H, VH (see below) we call the basic scale and with the adding of the intermediate gradations—extended scale [27]. Introduce also two marginal gradations out of range: VVL (lowest value) and VVH (highest value). Depending on condition of the task gradation VVL can be interpreted as zero, lower bound, exact lower bound etc. and gradation VVH—as unit, infinity, upper bound, exact upper bound etc. For any two gradations x and y of the scale we introduce a measure of distance. The relationship of similarity has the form

$$d(x, y) = \alpha(x, y) \tag{1}$$

where $\alpha(x, y)$—the degree of proximity of the two gradations. Then the relationship of difference has the form

$$d(x, y) = \bar{\alpha}(x, y) \tag{2}$$

where $\bar{\alpha}(x, y)$—the opposite value, i.e. if $\alpha(x, y)$=VH, then $\bar{\alpha}(x, y)$=VL, etc. Values $\alpha(x, y)$ are given in Table 1, where due to symmetry $\alpha(x, y) = \alpha(y, x)$. Since the relationship of similarity is symmetric, in Table 1 the values that are below the main diagonal, i.e. in cells (i, k) for $i > k$, they are equal to the corresponding values above the main diagonal, i.e. in cells (i, k) for $i < k$, where i is a row, k is a column. Therefore, the values below the main diagonal in Table 1 are omitted.

For most applications the accuracy of Table 1 is sufficient; if necessary Table 1 can be made in increments of half or a quarter of the gradation, however the simplicity and reliability of the calculations are lost. Consider the arithmetic operations (summation and multiplication) on the set of fuzzy gradations. The results for multiplication of fuzzy gradations defined in the extended range (scale) are given in Table 2 and for summation—in Table 3. When compiling the tables, it was assumed that the entire range of gradations corresponds to a numerical range of 0…1, which was divided into five equal intervals in the number of basic gradations. This establishes a one-to-one correspondence between each fuzzy gradation of the scale and the corresponding numerical interval. It is assumed that the value of the gradation is concentrated in the center of the interval, so modal values of fuzzy gradations VL, L, M, H, VH correspond with the values 0.1; 0.3; 0.5; 0.7; 0.9 respectively. Modal values of intermediate gradations VL–L, L–M, M–H, H–VH correspond to values 0.2; 0.4; 0.6; 0.8 respectively. The value VVL in the Table 2

Table 1 The matrix of correspondences

Fuzzy gradations $\alpha(x,y)$	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VH	H–VH	H	M–H	M	L–M	L	VL–L	VL
VL–L		VH	H–VH	H	M–H	M	L–M	L	VL–L
L			VH	H–VH	H	M–H	M	L–M	L
L–M				VH	H–VH	H	M–H	M	L–M
M					VH	H–VH	H	M–H	M
M–H						VH	H–VH	H	M–H
H							VH	H–VH	H
H–VH								VH	H–VH
VH									VH

VL: very low value, VL–L: between very low and low, L: low, L–M: between low and middle, M: middle (medium), M–H: between middle and high, H: high, H–VH: between high and very high, VH: very high

Table 2 The calculation of the product of two fuzzy gradations

Fuzzy gradations of factors	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VVL	VVL	VVL	VVL	VL	VL	VL	VL	VL
VL–L		VVL	VL	VL	VL	VL	VL	VL–L	VL–L
L			VL	VL	VL–L	VL–L	VL–L	VL–L	L
L–M				VL–L	VL–L	VL–L	L	L	L–M
M					L	L	L–M	L–M	M
M–H						L–M	L–M	M	M
H							M	M–H	M–H
H–VH								M–H	H
VH									H–VH

The values are given with rounding to nearest gradation of extended scale

Table 3 The calculation of sum of two fuzzy gradations

Fuzzy gradations of summands	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VL–L	L	L–M	M	M–H	H	H–VH	VH	VVH
VL–L		L–M	M	M–H	H	H–VH	VH	VVH	VVH
L			M–H	H	H–VH	VH	VVH	VVH	VVH
L–M				H–VH	VH	VVH	VVH	VVH	VVH
M					VVH	VVH	VVH	VVH	VVH
M–H						VVH	VVH	VVH	VVH
H							VVH	VVH	VVH
H–VH								VVH	VVH
VH									VVH

The values are given with rounding to nearest gradation of extended scale

means that the result is outside the left boundary of the range (this value corresponds to 0); the value VVH means that the result is outside the right boundary of the range (this value corresponds to 1). Fuzzy gradations obey the order relation VVL < VL < ⋯ < VH < VVH. As is well known, for this relationship, a permissible transformation is arbitrary monotone transformation that preserves the order between gradations. For example, all gradations can be simultaneously multiplied or divided by the same number, so that the values do not go beyond the range of 0…1 (see Sect. 5). Therefore, any numerical interval can be reduced to a unit interval (0, 1). For this, it suffices to apply a linear transformation of the form $x = (z - z_{\min})/(z - z_{\max}) \pm 0.1$, where the plus sign corresponds to the value of $z_{\min}$, and the minus sign to the value of z_{max}. Here x is a standardized variable from the interval (0, 1), z is a "physical" variable, determined by measurement or expert method, which takes values in the interval $[z_{min}, z_{max}]$. Its values are represented by named numbers or dimensionless estimates. After the tables are compiled, all calculations are performed on the tables. It should be noted that in calculations it makes no sense to introduce small gradation shares, and rounding should be used towards the nearest gradation, since this does not affect the accuracy of the final result (see below). When the number of factors (summands) is greater than two, the result is also determined using tables. The process quickly converges as the number of components (factors or terms) increases, in the sense that, for three to four components, the extreme limits of the range are reached. From the tables, you can determine the results for inverse operations (subtraction and division). Of course, if necessary, fuzzy gradations can be transformed into values of the "physical" variable corresponding to the subject domain in question.

The operation of bounded summation is also used in practical applications; the results for it are given in Table 4.

Since the multiplication and summation operations are transitive, the values in Tables 2, 3 and 4, located below the main diagonal, i.e. in cells (i, k) for $i>k$, they are equal to the corresponding values above the main diagonal, i.e. in cells (i, k) for $i<k$, where i is a row, k is a column. Therefore, the values below the main diagonal in these tables are omitted. The results of calculations for traditional fuzzy set idempotent operations min and max are given in Tables 5 and 6 respectively.

Since the operations min and max are symmetric, the values in Tables 5 and 6, located below the main diagonal, i.e. in cells (i, k) for $i>k$, they are equal to the corresponding values above the main diagonal, i.e. in cells (i, k) for $i<k$, where i is a row, k is a column. Therefore, the values below the main diagonal in these tables are omitted. It should be noted that for most applications the accuracy of Tables 2, 3, 4, 5 and 6 is sufficient; if necessary, tables can be compiled with a step of half or even a quarter of the gradation, although at the same time the simplicity of computation and reliability are lost. In a certain sense, we can speak of fuzzy arithmetic, since all operations are performed directly on fuzzy gradations. In ordinary arithmetic, the axiom of generating numbers is valid: n (+) $1=n+1$, where $n=1, 2, \ldots$. In fuzzy arithmetic, it should be replaced by the axiom of generating quantities (amounts), since we are dealing with sets: u (+) VL $=u+$VL, where $u=$VL, VL–L,

Table 4 The calculation of restricted sum of two fuzzy gradations: x [+] $y=x+y-xy$

Fuzzy gradations of summands	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VL–L	L	L–M	M	M–H	M–H	H	H–VH	VH
VL–L		VL–L	VL–L	M	M–H	H	H–VH	H–VH	VH
L			M	M–H	H	H	H–VH	VH	VH
L–M				M–H	H	H–VH	H–VH	VH	VH
M					H–VH	H–VH	VH	VH	VVH
M–H						H–VH	VH	VH	VVH
H							VH	VH	VVH
H–VH								VVH	VVH
VH									VVH

The values are given with rounding to nearest gradation of extended scale

Table 5 The calculation of operation min for two fuzzy gradations

Fuzzy gradation of components	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VL	VL	VL	VL	VL	VL	VL	VL	VL
VL–L		VL–L	VL–L	VL–L	VL–L	VL–L	VL–L	VL–L	VL–L
L			L	L	L	L	L	L	L
L–M				L–M	L–M	L–M	L–M	L–M	L–M
M					M	M	M	M	M
M–H						M–H	M–H	M–H	M–H
H							H	H	H
H–VH								H–VH	H–VH
VH									VH

Table 6 The calculation of operation max for two fuzzy gradations

Fuzzy gradation of components	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL–L		VL–L	L	L–M	M	M–H	H	H–VH	VH
L			L	L–M	M	M–H	H	H–VH	VH
L–M				L–M	M	M–H	H	H–VH	VH
M					M	M–H	H	H–VH	VH
M–H						M–H	H	H–VH	VH
H							H	H–VH	VH
H–VH								H–VH	VH
VH									VH

L etc. From the results given in Tables 2, 3, 4, 5 and 6, it follows that for two arbitrary gradations x and y there is a chain of inequalities (in view of rounding)

$$xy < \min(x, y) \le \max(x, y) \le x[+]y \le x + y. \tag{3}$$

Note that operations of multiplication and bounded summation belong to Archimedean operations and the summation operation within the range under consideration belongs to nilpotent operations. This operation can be also defined outside the range, if it is necessary [29]. Fuzzy gradations form an Abelian addition group and an Abelian semigroup for multiplication (with rounding).

3 Generalization of the Markov decision-making model

We apply the results obtained above to generalize the Markov decision models. When modeling the behavior of systems, Markov matrices and Markov models are often used, this is due to their "good" properties [4, 12, 13]. They are applicable in stable situations, but their application in conditions of unexpected changes is very problematic. The weakness of such models is the need to have information about the transition probabilities, which is often impossible. Transitions between states are regarded as random, which in a real situation often does not take place. The second difficulty is that the choice of the best strategy is not determined uniquely by the function of income and can depend on other parameters. We generalize Markov models in two directions. The first generalization is connected with the spread of these models to the case of uncertainty and ambiguity, when there is no information on the transition probabilities. The second generalization is connected with the introduction of additional components in the cortege to eliminate ambiguity of the solution and increase reliability. Let us consider the first generalization. We define a fuzzy Markov chain in the form of a cortege $\langle D, P, W\rangle$ in which the values of P and W are represented by fuzzy gradations in the range VL…VH (see above). Solution D is determined by fuzzy criteria for achieving the goal, the probability P is transformed into an opportunity, income W—into a possible income. The values of P do not necessarily constitute a complete group of events. The second generalization is to take into account another component, namely, the possible risk R. Then the cortege is written in the form $\langle K, P, W, R\rangle$, where R is the set of possible risks. High income, as a rule, is accompanied by a high risk. If transition probabilities are known, then the risk can be estimated based on the probability matrix. But if the external environment is unstable and characterized by uncertainty and ambiguity, then there is a problem of joint accounting of incomes and risks. When determining (assessing) the risk in open systems, both direct and indirect risk should be taken into account: the first relates to the system under consideration and takes into account direct costs, and the second to external systems and takes into account indirect costs (damage to external systems). To jointly account for income and risk, we introduce the safety function as a quantity opposite to risk and define its values on the basis of the difference matrix given in Table 7. It can be seen from the Table 7 that if the risk is very high, then the safety is very low, if high, then low, etc. Therefore, income and safety change the opposite. Using single-purpose models, it can be shown that the best strategy corresponds to a minimum of maximum costs taking into account both income and risk.

4 An example of a study: application of the model

Let us consider an example explaining the application of the generalized model. The broker buys and sells shares (currency) in the market. The situation is characterized by uncertainty, unexpected changes are possible. It is required to determine the strategy of the broker. Suppose the purchase price of one share is c_1, and the sale price is c_2. Let's designate

Table 7 Matrix of difference

Fuzzy gradations of x, y / $d(x,y)$	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL	VL	VL–L	L	L–M	M	M–H	H	H–VH	VH
VL–L		VL	VL–L	L	L–M	M	M–H	H	H–VH
L			VL	VL–L	L	L–M	M	M–H	H
L–M				VL	VL–L	L	L–M	M	M–H
M					VL	VL–L	L	L–M	M
M–H						VL	VL–L	L	L–M
H							VL	VL–L	L
H–VH								VL	VL–L
VH									VL

d(*x*, *y*) is a fuzzy measure of the difference between objects *x* and *y*. Its values are calculated according to (2) and Table 1

$\Delta = c_2 - c_1$, the value Δ can be more, is equal or less than zero. We represent Δ in the form of fuzzy gradations. The possible range of purchase of shares is also determined by fuzzy gradations in the range [VVL, VVH], where VVL—the lowest value and VVH—the highest value (see above). Note that the operations of summation, multiplication, etc. are performed according to the rules of fuzzy arithmetic (see above). We will form the payment matrix; the results are shown in Table 8, where the values on the main diagonal or above the main diagonal are determined by the relation

$$c_{ik} = V_i \cdot \Delta, \tag{4}$$

where $i \leq k$, V_i is the volume of purchase. Values c_{ik} can be either positive or negative, depending on the sign of Δ. If $\Delta \geq 0$, then $c_{ik} \geq 0$, otherwise $c_{ik} < 0$. The values below the main diagonal are determined by the relation

$$c'_{ik} = V_k \cdot c_2 - V_i \cdot c_1 = V_k \cdot \Delta - (V_i - V_k) \cdot c_1, \tag{5}$$

where $i > k$, V_k is the volume of sale, V_i is the volume of purchase. If $\Delta \geq 0$ and it is small in comparison with c_1, then the values below the main diagonal are negative. If Δ is of the order of c_1 or greater than c_1, some of the values can be positive. This explains why sellers are trying to "wind" prices compared to purchase prices, which in turn leads to a spiral of inflation, a decrease in consumption and a drop in the standard of living of the population. We form the loss matrix. The results are shown in Table 9. In Table 9 the values on the main diagonal or above the main diagonal are determined by the relation

$$r_{ik} = (V_k - V_i) \cdot \Delta, \tag{6}$$

where $i \leq k$. The values below the main diagonal are determined by the relation

$$r'_{ik} = (V_i - V_k) \cdot c_1, \tag{7}$$

where $i > k$. Let's determine the expected value of losses. The results are shown in Table 10. The values in Table 10 are determined from the loss matrix.

We use the minimax criterion, so the best strategy corresponds to a minimum of maximum losses. It depends on

Table 8 Payment matrix

Volume of purchase	Volume of sale						
	VVL	VL	L	M	H	VH	VVH
VVL	VVL	VVL	VVL	VVL	VVL	VVL	VVL
VL	$-\text{VL} \cdot c_1$	$\text{VL} \cdot \Delta$	$\text{VL} \cdot \Delta$	$\text{VL} \cdot \Delta$	$\text{VL} \cdot \Delta$	$\text{VL} \cdot \Delta$	$\text{VL} \cdot \Delta$
L	$-\text{L} \cdot c_1$	$-(\text{VL-L}) \cdot c_1 + \text{VL} \cdot \Delta$	$\text{L} \cdot \Delta$	$\text{L} \cdot \Delta$	$\text{L} \cdot \Delta$	$\text{L} \cdot \Delta$	$\text{L} \cdot \Delta$
M	$-\text{M} \cdot c_1$	$-(\text{L-M}) \cdot c_1 + \text{VL} \cdot \Delta$	$-(\text{VL-L}) \cdot c_1 + \text{L} \cdot \Delta$	$\text{M} \cdot \Delta$	$\text{M} \cdot \Delta$	$\text{M} \cdot \Delta$	$\text{M} \cdot \Delta$
H	$-\text{H} \cdot c_1$	$-(\text{M-H}) \cdot c_1 + \text{VL} \cdot \Delta$	$-(\text{L-M}) \cdot c_1 + \text{L} \cdot \Delta$	$-(\text{VL-L}) \cdot c_1 + \text{M} \cdot \Delta$	$\text{H} \cdot \Delta$	$\text{H} \cdot \Delta$	$\text{H} \cdot \Delta$
VH	$-\text{VH} \cdot c_1$	$-(\text{H-VH}) \cdot c_1 + \text{VL} \cdot \Delta$	$-(\text{M-H}) \cdot c_1 + \text{L} \cdot \Delta$	$-(\text{L-M}) \cdot c_1 + \text{M} \cdot \Delta$	$-(\text{VL-L}) \cdot c_1 + \text{H} \cdot \Delta$	$\text{VH} \cdot \Delta$	$\text{VH} \cdot \Delta$
VVH	$-\text{VVH} \cdot c_1$	$-\text{VH} \cdot c_1 + \text{VL} \cdot \Delta$	$-\text{H} \cdot c_1 + \text{L} \cdot \Delta$	$-\text{M} \cdot c_1 + \text{M} \cdot \Delta$	$-\text{L} \cdot c_1 + \text{H} \cdot \Delta$	$-\text{VL} \cdot c_1 + \text{VH} \cdot \Delta$	$\text{VVH} \cdot \Delta$

Table 9 Loss matrix

Volume of purchase	Volume of sale						
	VVL	VL	L	M	H	VH	VVH
VVL	VVL	VL · Δ	L · Δ	M · Δ	H · Δ	VH · Δ	VVH · Δ
VL	VL · c_1	VVL	(VL-L) · Δ	(L-M) · Δ	(M-H) · Δ	(H-VH) · Δ	VH · Δ
L	L · c_1	(VL-L) · c_1	VVL	(VL-L) · Δ	(L-M) · Δ	(M-H) · Δ	H · Δ
M	M · c_1	(L-M) · c_1	(VL-L) · c_1	VVL	(VL-L) · Δ	(L-M) · Δ	M · Δ
H	H · c_1	(M-H) · c_1	(L-M) · c_1	(VL-L) · c_1	VVL	(VL-L) · Δ	L · Δ
VH	VH · c_1	(H-VH) · c_1	(M-H) · c_1	(L-M) · c_1	(VL-L) · c_1	VVL	VL · Δ
VVH	VVH · c_1	VH · c_1	H · c_1	M · c_1	L · c_1	VL · c_1	VVL

Table 10 Expected value of losses (costs)

Volume of purchase	Losses (costs)		Maximum losses
	Due to understated purchase volume	Due to excessive purchase volume	
VVL	VVH · Δ	VVL	VVH · Δ
VL	VH · Δ	VL · c_1	(VH · Δ) ∨ (VL · c_1)
L	H · Δ	L · c_1	(H · Δ) ∨ (L · c_1)
M	M · Δ	M · c_1	(M · Δ) ∨ (M · c_1)
H	L · Δ	H · c_1	(L · Δ) ∨ (H · c_1)
VH	VL · Δ	VH · c_1	(VL · Δ) ∨ (VH · c_1)
VVH	VVL	VVH · c_1	VVH · c_1

∨ logical connection "or". In this case, the largest value is selected from the two values

the ratio of the quantities Δ and c_1. In particular, if $\Delta = \mathrm{VL} \cdot c_1$, then the best strategy corresponds to the volume of purchase VL or VVL (nothing to buy) and the minimum losses is VL · c_1. If $\Delta = c_1$, then the best strategy corresponds to the purchase volume M and the losses are equal M · c_1. If $\Delta = c_1/\mathrm{M}$, then the best strategy corresponds to the volume of purchase H and the losses are equal H · c_1, although the expected losses are increasing. Consequently, the greater the increase (surcharge) to the purchase price, the greater the optimal purchase volume. We now take into account the risk that arises from the uncertainty of the situation. It's clear that risk increases with the purchase volume, and safety decreases. In particular, for V_i = VVL the safety is S_i = VVH, for V_i = VL the safety is S_i = VH etc., for V_i = VVH we have the lowest safety S_i = VVL. Therefore, the best solution will correspond to the maximum safety, i.e. strategy of waiting (do not buy anything, do not invest money). Here we find an explanation for the well-known characteristic of the instability of the situation when it is said that "the future affects the present". A similar problem was solved by the author in [31, pp. 230–232] by the traditional probabilistic method using the "income-loss" model. The calculations were carried out as follows. For different values of the purchase volume and sales volume and different values of the expected probability of demand, the expected income and the expected total losses (costs) were calculated. The result is determined by the minimum total losses. In conditions of uncertainty, the result corresponds to the minimum maximum value of the total losses. The result is consistent with the results of this article, provided that there is no risk. Since the risk criterion is not taken into account, the choice of the best strategy is poorly conditioned and is subjective.

5 Fuzzy models of optimal choice

Now we apply fuzzy arithmetic to the problem of optimal choice. The problem is formulated as follows. There are many possible solutions (alternatives) $X=\{x_1, \ldots, x_m\}$. Each alternative is evaluated by a set of criteria $\{K_1, \ldots, K_n\}$. We also know the weights (importance) of the criteria $\{a_1, \ldots, a_n\}$. The values of the criteria and weights are represented in the form of fuzzy gradations. It is required to determine the fitness of alternatives for purpose and to choose the best solution. To solve the formulated problem, it is expedient to use the method of threshold criteria and the distance method. The first method allows us to obtain a lower estimate, and the second gives an upper estimate and allows us to determine the so-called indirect costs. We also consider two methods occupying an intermediate position: convolution by the worst criterion, at which the risk of a selection error due to the model is the smallest and additive convolution corresponding to the averaging strategy. To apply the threshold criteria method, the threshold values of the criteria should be known. We define them directly from the initial data (see Table 11). Then the value of the general criterion for an arbitrary alternative x is given by the following expression that does not depend on the weight of criteria

$$K(x) = \min_j K_j(x) / \min_x K_j(x), \tag{8}$$

and the best solution is

$$x^* = \arg\max_{x \in X} K(x). \tag{9}$$

Table 11 The initial data for the example

Alternatives, x_i	Criteria, K_j				
	K_1	K_2	K_3	K_4	K_5
x_1	H	M	M	L	VH
x_2	VH	H	M	L	H
x_3	M–H	M	L	M	M–H
x_4	M	M	M	M	M
x_5	M	L	VH	H	M
"Threshold" solution	M	L	L	L	M
"Ideal" solution	VH	H	VH	H	VH

All values are given in the direct scale. Hereinafter, a short line is used to denote an intermediate gradation, and long line means a subtraction operation. When using the threshold criteria method, calculations according to (8) give $K(x_1)=\min\{$H/M, M/L, M/L, L/L, VL/M$\}=1$ (one unit). Similarly, we have $K(x_2)=1$ (one unit), $K(x_3)=1$(one unit), $K(x_4)=1$ (two units), $K(x_5)=1$ (three units). It follows from (9) that all solutions are equivalent. If we take into account the number of units, i.e. coincidences with the threshold values, then solutions can be distinguished. The most preferable solutions are x_1, x_2, x_3, which correspond to the smallest number of units (one unit). The ranking according to the "degree of admissibility" has the form $\{x_1, x_2, x_3\}$, x_4, x_5. We apply the distance method using the Hamming function, Euclid's function and functions of the greatest and smallest difference. Suppose that the importance of the criteria is the same $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \cdots = \boldsymbol{a}_5 = \mathbf{M}$. In this case, the weight of the criteria may not be taken into account; it can be used to ensure that the result remains within the scale. In the case of the *Hamming function*, calculations from Eq. (10) using Tables 2 and 3 give for alternative x_1: $d(x_1)=$ M·(VH–H) + M·(H–M) + M·(VH–M) + M·(H–L) + M·(VH–VH) = M–H. Similarly for other alternatives, we have $d(x_2)=$M, $d(x_3)=$H–VH, $d(x_4)=$H–VH, $d(x_5)=$M–H. The best solution in accordance with (14) is x_2. The ranking by degree of proximity to the ideal solution has the form x_2, $\{x_1, x_5\}$, $\{x_3, x_4\}$. The solutions in braces are equivalent. In the case of the *Euclid's function*, the results are almost the same, but the solutions are less distinct. We have from (9): $d^2(x_1)=d^2(x_2)=d^2(x_4)=d^2(x_5)=$VL, $d^2(x_3)=$VL–L. The ranking has the form$\{x_1, x_2, x_4, x_5\}$, x_3. In the case of *function of the greatest difference*, calculations from Eq. (12) using Tables 2, 3 and 6 give for alternative x_1: $d(x_1)=\max\{$M·(VH–H); M·(H–M); M·(VH–M); M·(H–L); M·(VH–VH)$\}=$VL–L. Similarly for other alternatives, we obtain $d(x_2)=d(x_4)=d(x_5)=$V–L, $d(x_3)=$L. So, the solutions x_1, x_2, x_4 and x_5 are equivalent. The ranking has the form$\{x_1, x_2, x_4, x_5\}$, x_3. In the case of *function of the smallest difference*, calculations from Eq. (13) using Tables 2, 3 and 5 give for alternative x_1: $d(x_1)=\min\{$M·(VH–H); M·(H–M); M·(VH–M); M·(H–L); M·(VH–VH)$\}=0$ (one zero). Similarly $d(x_2)=d(x_5)=0$ (two zeros), $d(x_3)=d(x_4)=$VL. The best solutions in accordance with (18) are x_2 and x_5. The ranking has the form$\{x_2, x_5\}$, x_1, $\{x_3, x_4\}$. Note that with our initial data, the result for the function of the smallest difference does not depend on the weight of the criteria (see below). In the case of the *convolution by the worst criterion*, calculations from Eq. (15) using Tables 2 and 5 give for alternative x_1: $K(x_1)=\min\{$M·H; M·M; M·M; M·L; M·VH$\}=$VL–L. Similarly $K(x_2)=K(x_3)=K(x_5)=$V–L, $K(x_4)=$L (with rounding). So, the best solution in accordance with (9) is x_4; the solutions x_1, x_2, x_3 and x_5 are equivalent. The ranking has the form x_4, $\{x_1, x_2, x_3, x_5\}$. In the case of the *additive convolution*, calculations from Eq. (16) using Tables 2 and 3 give for alternative x_1: $K(x_1)=$M·H + M·M + M·M + M·L + M·VH = L–M. Similarly $K(x_3)=K(x_4)=K(x_5)=$L–M, $K(x_2)=$M. So, the best solution in accordance with (13) is x_2; the solutions 1, 3, 4 and 5 are equivalent. The ranking has the form x_2, $\{x_1, x_3, x_4, x_5\}$. Let's explore how the result depends on changing the importance of the criteria. Suppose that the importance of criteria increases from K_1 to K_5, and the importance of the criteria K_1 and K_2 is approximately the same. Let $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \mathbf{L}$, $\boldsymbol{a}_3 = \mathbf{L} - \mathbf{M}$, $\boldsymbol{a}_4 = \mathbf{M}$, $\boldsymbol{a}_5 = \mathbf{M} - \mathbf{H}$. In the case of the *Hamming function*, calculations from Eq. (10) using Tables 2 and 3 give for alternative x_1: $d(x_1)=$L·(VH–H) + L·(H–M) + (L–M)·(VH–M) + M·(H–) + (M–H)·(VH–VH) = M. Similarly we have $d(x_2)=$M, $d(x_3)=$H, $d(x_4)=$H, $d(x_5)=$L–M. It follows from (18) that the best solution is x_5. The ranking by degree of proximity to the ideal solution has the form x_5, $\{x_1, x_2\}$, $\{x_3, x_4\}$. For the *Euclid's function*, the results are similar. We have from (15): $d^2(x_1)=d^2(x_2)=d^2(x_3)=d^2(x_4)=d^2(x_5)=$VL, so all solutions are equivalent; the ranking has the form $\{x_1, x_2, x_3, x_4, x_5\}$. In the case of *function of the greatest difference*, calculations from Eq. (12) using Tables 2, 3 and 6 give $d(x_1)=d(x_2)=d(x_3)=d(x_4)=d(x_5)=$VL–L. So, all solutions are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4, x_5\}$. In the case of *function of the smallest difference*, calculations from Eq. (13) using Tables 2, 3 and 5 give the same result as above with an equal weight of the criteria: $d(x_1)=0$(one zero), $d(x_2)=d(x_5)=0$ (two zeros), $d(x_3)=d(x_4)=$VL. The best solutions in accordance with (18) are x_2 and x_5. The ranking has the form$\{x_2, x_5\}$, x_1, $\{x_3, x_4\}$. In the case of the *convolution by the worst criterion*, calculations from Eq. (15) using Tables 2 and 5 give: $K(x_1)=K(x_2)=K(x_3)=K(x_4)=$VL–L, $K(x_5)=$VL (with rounding). So, the solutions x_1, x_2, x_3 and x_4 are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4\}$, x_5. In the case of the *additive convolution*, calculations from Eq. (16) using Tables 2 and 3 give: $K(x_1)=K(x_2)=K(x_5)=$L–M, $K(x_3)=K(x_4)=$L. In the calculation, each term of the sum was multiplied by a gradation L so that the results remained within the scale. The solutions x_1, x_2, x_5 are equivalent. The ranking has the form $\{x_1, x_2, x_5\}$, $\{x_3, x_4\}$. We will change the priorities, assuming that the importance of the criteria decreases from K_1 to K_5, and the importance of the criteria K_4 and K_5 is approximately the same. Assume that $\boldsymbol{a}_1 = \mathbf{M} - \mathbf{H}$, $\boldsymbol{a}_2 = \mathbf{M}$, $\boldsymbol{a}_3 = \mathbf{L} - \mathbf{M}$, $\boldsymbol{a}_4 = \boldsymbol{a}_5 = \mathbf{L}$. In the case of the *Hamming function*, the calculations from (10) using Tables 2 and 3 give $d(x_1)=$(L–H)·(VH–H) + M·(H–M) + (L–M)·(VH–M) + L·(H–L) + L·(VH–VH) = M. Similarly we have for other alternatives $d(x_2)=$L–M, $d(x_3)=d(x_4)=$H, $d(x_5)=$M. The best solution in accordance with (14) is x_2. The ranking has the form x_2, $\{x_1, x_5\}$, $\{x_3, x_4\}$. For *Euclid's function*, we have from (15): $d^2(x_1)=d^2(x_3)=d^2(x_4)=d^2(x_5)=$VL, $d^2(x_2)=$VVL. So, the best solution in accordance with (14) is x_2; the ranking has the form x_2, $\{x_1, x_3, x_4, x_5\}$. In the case of *function of the greatest difference*, we have from Eq. (12): $d(x_1)=d(x_2)=d(x_3)=d(x_4)=d(x_5)=$VL–L. So, all solutions are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4, x_5\}$. In the case of *function of the smallest difference*, the results do not change (see above). In the case of the *convolution by the worst criterion*, we obtain from (15) using Tables 2 and 5: $K(x_1)=K(x_2)=K(x_3)=$VL, $K(x_4)=K(x_5)=$VL–L. The best solutions in accordance with (9) are x_4 and x_5. The ranking has the form $\{x_4, x_5\}$, $\{x_1, x_2, x_3\}$. In the case of the *additive convolution*, we obtain from (16) using Tables 2 and 3: $K(x_1)=$(M–H)·H + M·M + (L–M)·M + L·L + L·VH = L–M. In the calculation, each term of the sum was multiplied by a gradation L so that the results remained within the scale. Similarly for other alternatives, taking into account the same factor L, we obtain $K(x_2)=$L–M, $K(x_3)=$L, $K(x_4)=$L, $K(x_5)=$L–M (the last result is obtained with rounding towards a larger gradation). The best alternatives from (9) are $\{x_1, x_2, x_5\}$. The ranking has the form $\{x_1, x_2, x_5\}$, $\{x_3, x_4\}$. A summary of the results is given in Table 12. So, in our example, the most justified is the application of the distance method (the distance of the smallest difference). The calculations show that the most "intensive" is the variant x_2 (economic components prevail), and the most "gentle"(humane) is x_5 (ergonomic and ecological components prevail)

Table 12 The summary of results

Method (model)	The best solution	Ranking	Weight (importance) of criteria
Threshold criteria method	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}, x_4, x_5$	Arbitrary
Hamming function	x_2	$x_2, \{x_1, x_5\}, \{x_3, x_4\}$	Equal
	x_5	$x_5, \{x_1, x_2\}, \{x_3, x_4\}$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	x_2	$x_2, \{x_1, x_5\}, \{x_3, x_4\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$
Euclid's function	$\{x_1, x_2, x_4, x_5\}$	$\{x_1, x_2, x_4, x_5\}, x_3$	Equal
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	x_2	$x_2, \{x_1, x_3, x_4, x_5\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$
Function of the greatest difference	$\{x_1, x_2, x_4, x_5\}$	$\{x_1, x_2, x_4, x_5\}, x_3$	Equal
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$
Function of the smallest difference	$\{x_2, x_5\}$	$\{x_2, x_5\}, x_1, \{x_3, x_4\}$	Equal
	$\{x_2, x_5\}$	$\{x_2, x_5\}, x_1, \{x_3, x_4\}$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	$\{x_2, x_5\}$	$\{x_2, x_5\}, x_1, \{x_3, x_4\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$
Convolution by the worst criterion	x_4	$x_4, \{x_1, x_2, x_3, x_5\}$	Equal
	$\{x_1, x_2, x_3, x_4\}$	$\{x_1, x_2, x_3, x_4\}, x_5$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	$\{x_4, x_5\}$	$\{x_4, x_5\}, \{x_1, x_2, x_3\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$
Additive convolution	x_2	$x_2, \{x_1, x_3, x_4, x_5\}$	Equal
	$\{x_1, x_2, x_5\}$	$\{x_1, x_2, x_5\}, \{x_3, x_4\}$	$a_1 = a_2 = L, a_3 = L–M, a_4 = M, a_5 = M–H$
	$\{x_1, x_2, x_5\}$	$\{x_1, x_2, x_5\}, \{x_3, x_4\}$	$a_1 = M–H, a_2 = M, a_3 = L–M, a_4 = a_5 = L$

For the application of the distance method, an "ideal" solution should be known. We define it, as above, directly from the experimental data (see Table 11). As a measure of distance, we use the Hamming function and Euclid's function, corresponding to the strategy of the mean and the mean square, respectively, as well as the functions of the greatest and smallest difference, corresponding to the limiting strategies. In the case of the Hamming function, we have for the distance of an arbitrary alternative x to an ideal solution

$$d(x) = \sum_j a_j \left| K_j(x) - \max_{x\in X} K_j(x) \right| \tag{10}$$

In the case of Euclid's function, the analogous expression has the form

$$d(x) = \left(\sum_j a_j^2 \left| K_j(x) - \max_{x\in X} K_j(x) \right|^2 \right)^{1/2} \tag{11}$$

In the case of a function of the greatest difference, we have the expression

$$d(x) = \max_j a_j \left| K_j(x) - \max_{x\in X} K_j(x) \right| \tag{12}$$

In the case of a function of the smallest difference, we can write

$$d(x) = \min_j a_j \left| K_j(x) - \max_{x\in X} K_j(x) \right| \tag{13}$$

The best solution in all cases is defined as the closest to the ideal

$$x^* = \arg\min_{x\in X} d(x). \tag{14}$$

In the case of the convolution by the worst criterion the value of the general criterion for an arbitrary alternative x is given by the following expression

$$K(x) = \min_j a_j K_j(x) \tag{15}$$

In the case of the additive convolution we have the relation

$$K(x) = \sum_j a_j K_j(x) \tag{16}$$

The best solution in both cases is defined from (9). Note that when using a representation in the form of fuzzy gradations, it is not necessary to fulfill the normalization condition for the weights of the criteria, it is only necessary that the results of the calculations do not go beyond the scale, which is ensured by multiplying by a small gradation (see below). It should be borne in mind that the estimates obtained are not absolute, but relative, since they are satisfied in the scale of order and allow any monotonic transformation at which the result remains within the scale. Let's consider a concrete example. Let X—the set of objects, for example, projects of technical system, consisting of five variants of solutions (alternatives), each of which is evaluated according to five criteria, where K_1 is a functional criterion, K_2 is economic, K_3 is ergonomic, K_4 is ecological, and K_5 is social. Generalized criteria were used to simplify the calculations. The initial data are given in Table 11. The degree of certainty (reliability) of the data (β) is assumed to be equal to VH, so that the certainty condition (β > H) is satisfied [29]. It is easy to see that alternatives form the Pareto set, so none of them can be excluded.

Thus, our study shows that the selection of a preferred solution depends on the importance of criteria (priorities) defined by external purpose, as well as, the type of the model, which is determined by the initial data and preferences of the person making the decisions. A similar problem was solved by the author in [31, pp. 212–221] by the traditional method of eigenvalues (hierarchy analysis method). This method is an expert-statistical and allows to obtain fairly accurate (reliable) estimates, as well as take into account the individual preferences of the expert [16]. The method includes the following steps: preliminary ranking of criteria and alternatives in importance using a nine-point scale; determination of the normalized weight (priority) of each criterion and evaluation of the suitability (utility) of alternatives for each criterion; determination of a general criterion for each alternative, characterizing its suitability for all criteria (aims), using different strategies (additive convolution, convolution by the worst criterion, etc.). The calculations were carried out on a computer and the results are consistent with those obtained in the article, but they significantly depend on errors in determining the weight of the criteria. To obtain reliable results, tedious (time-consuming) calculations and high qualification of experts are required. The approach proposed in the article is largely free from these shortcomings.

6 Conclusion

The representation of data in the form of fuzzy gradations makes it possible to take advantage of the fuzzy approach to decision-making and at the same time preserve the clarity and unambiguity of the conclusions. This representation eliminates the problem of standardization of criteria and their importance arising in decision-making on many criteria in connection with the transition from the physical scale to the scale of order for each criterion. It also facilitates calculations and makes them independent of specific numerical values of the quantities. Approximation of calculations and estimates using the rules of fuzzy arithmetic allows us to smooth or eliminate inconsistency and errors of the initial data, which increases the reliability of decision making and allows us to reduce the risk associated with the inadequacy of the optimization model. The proposed generalization of the Markov model allows making meaningful decisions in situations characterized by uncertainty and the possibility of unexpected changes.

Compliance with ethical standards

Conflict of interest No conflict of interest.

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