Comprehensive Fuzzy Turing Machines, An Evolution to the Concept of Finite State Machine Control

Najmeh Ahang, Amin Torabi Jahromi, and Mansour Doostfatemeh

Abstract—The Turing machine is an abstract concept of a computing device which introduced new models for computation. The idea of Fuzzy algorithms defined by Zadeh and Lee [7] was followed by introducing Fuzzy Turing Machine (FTM) to create a platform for a new fuzzy computation model [10]. Then, in his investigations on its computational power, Wiedermann showed that FTM is able to solve undecidable problems [11]. His suggested FTM structure, which highly resembles the original definition was one of the most well-known classical definitions of FTM lately.

To improve some of its weaknesses and vague points which will be discussed extensively in this paper, we will develop a more complete definition for fuzzy Turing machines. Our proposed definition of FTM, which encompasses the conventional definition, is motivated from the definition of General Fuzzy Automata (GFA) introduced by Doostfatemeh and Kremer [3]. As it improved the conventional

Amin Torabi Jahromi is with the Department of Electrical and Computer Engineering, Persian Gulf University, Bushehr, Iran, e-mail: a.torabi@pgu.ac.ir

Najmeh Ahang and Mansour Doostfatemeh are with the Department of Mathematics, Shiraz University, Shiraz, Iran e-mail: {najmeh.ahang,dfatemeh}@shirazu.ac.ir

definition of fuzzy automata, especially the problem of membership assignment and multi-membership resolution, we also improved the same aspects of FTM through the definition of Comprehensive Fuzzy Turing Machine (CFTM). In addition, we address on some possible vaguenesses in FTM was not the subject of focus in fuzzy automata. As example, we investigate the issue of multipath and multi-direction which are possible in case of non-determinism. Finally, we show the simplicity, applicability and computational efficiency of the CFTM through an explanatory example.

Index Terms—General Fuzzy Automata, Comprehensive Fuzzy Turing Machine, Multi-membership Resolution, Multi-direction Resolution, Multi-symbol Resolution

I. INTRODUCTION

Incorporation of Fuzzy sets concepts in various branches of science and technology has led to their applicability and flexibility. Although the computational complexity has increased, the results has become more accurate and closer to the real world

application requirements. In computer science, the combination of fuzzy logic and computational systems has resulted to new more effective and complex computational methods. Fuzzy automata was the result of incorporation of fuzzy logic into automata theory. Another computational concept which was introduced and well developed in past decades was Turing Machine (TM) followed by its fuzzy counterpart, Fuzzy Turing Machine (FTM). Alen Turing introduced the concept of TM with the claim that it is as powerful as the human mind. Years later following the introduction of Fuzzy Turing Machine (FTM) and investigation of its computational power, Wiedermann showed that FTM is much more powerful than classical TM, and claimed that FTM has unique capabilities such as modeling and solving undecidable problems [11]. This fact reaffirms the new capabilities of FTM through which many fuzzy algorithms are implementable and many fuzzy languages are accepted. However, recent investigations introduced some languages which was not possible to be accepted by an FTM with its current form of definition [5]. From there, Gerla concludes that conventional FTM is not eligible to be Universal Fuzzy Turing Machine [11],[5].

Wiedermann claimed that the conventional fuzzy Turing machine to be capable of accepting Recursive Enumerable (R.E.) sets and co-R.E. sets [11], [12]. He also concluded that these machines are able to solve the halting problem. In [2], the

Wiedermann's above statement is investigated by Bedregal and was proved that is not completely correct. He then gave a characterization of the class of R.E. sets in terms of associated fuzzy languages accepted by fuzzy Turing machines leading to the nonexistence of a universal fuzzy Turing machine [4].

The rest of the paper is organized as follows; In the next section, we look into the definition of Wiedermann's Fuzzy Turing Machine (FTM) as the standard classical definition of FTM. We will also study its strength and weaknesses. Then, in section. IV we will develop a more complete formulation for fuzzy Turing machines, cover those vague aspects of the conventional definitions, and propose our own definition of FTM named as Comprehensive Fuzzy Turing Machine (CFTM) which is motivated from the definition of General Fuzzy Automata (GFA). In the light of Generalized Fuzzy Automata (GFA) proposed by [3], we developed a more complete definition for two problems already existed in fuzzy Turing machines which covers those faint faces of the membership assignment and multi-membership resolution problem for the states. As a result, membership values are no longer associated with IDs and they are directly associated with states. Moreover, for each time step, these membership values are calculated based on the current membership values of states and active transitions and are assigned to the successor states. Due to nondeterminism, there is

always a possibility that more than one membership values are assigned to a single state. To resolve the membership assignment problem, we defined a multi-membership resolution function similar to the one existed in the GFA definition.

However, we noticed that in FTMs, the membership assignment is not the only vague issue. Each active transition requires the machine to move its head in a specific direction and also mandates a predefined symbol to be written on the tape. Therefore, at each time step it is usually more than one symbol to be written on the tape and also more than one direction for the machine to move. Hence, in Section .IV we defined two more functions to resolve the above mentioned issues, multi-direction and multi-symbol resolution functions to decide on a single direction and a single head movement based on the weight of the active transitions and the membership values of their predecessor states. It is easy to prove that each conventional fuzzy Turing machine can be modeled in the form of the novel Comprehensive Fuzzy Turing Machine (CFTM). Lastly, some comparison on the volume of calculations on conventional FTM and the novel CFTM is performed.

II. DEFICIENCIES IN CONVENTIONAL FTM DEFINITION

In conventional definition of FTM, there is a key concept called instantaneous description (ID) of FTM \mathbf{T} working on the string w at time $t\geqslant 0$ that it is defined as "a unique description of the

machine's tape content, its state, and the position of the tape head after performing the tth move on the input w". Also, there is a function μ which assigns a weight in [0,1] to each transition $\delta \in \Delta$. In the 5-tuple $\delta = (q_1, a, q_2, b, D), q_1$ and q_2 are current state and next state, respectively. The symbol a is the input symbol just read by the head from the tape. The symbol b is the symbol which will be written on the tape by the active transition, and Dis the direction of head movement. Then, each ID is assigned a membership value which is calculated based on the transition weight. It means each ID has a membership value calculated from the path it is reached from the previous ID. But, how about the states? Are they assigned any membership values during the process as it was conventionally common in fuzzy automata?

In the following example we followed the calculations of a conventional FTM where the calculations are ID-based for few time steps.

Obviously, the amount of calculations are exhaustive and in case of nondeterminism, the possibility of infinite loop is high.

Example II.1. In this example we investigate through calculations the amount of ID-based computations and the complexity of the conventional FTM in [11] and presented in Fig. 2. Each of the rectangles represent an ID in each time step. Due to huge amount of calculation, we stopped it after 4 time steps to save space. Yet,

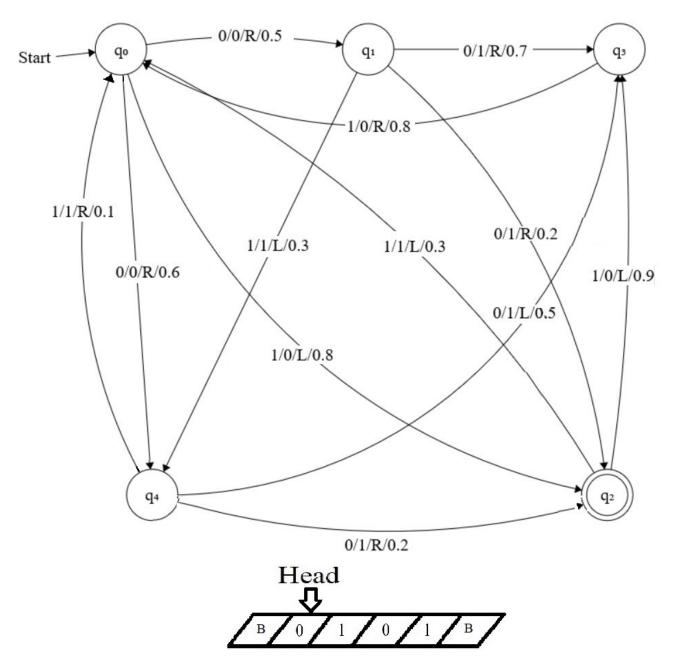


Figure 1. A Nondeterministic Fuzzy Turing Machine

it is worth noting that the calculations wont be $Q=\{q_0,q_1,q_2,q_3,q_4\},\ \Sigma=\{0,1\},\ q_0$: start state, reaching to their end even after 9 time steps.

and q_2 : final state.

The original configuration of the tape and initial At time step t=0, the input symbol is "0" and head position in Fig. 1 is:

	1						
В	0	1	0	1	В		

The FTM **T** has the following details:

the machine starts at state q_0 . Therefore, there are two possible moves for the FTM finite state control; via transitions $(q_0, 0, q_1, 0, R, 0.5)$, and

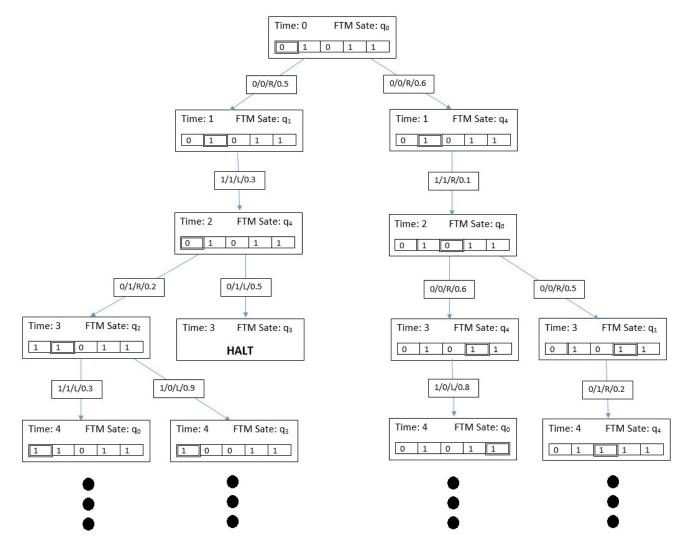


Figure 2. A Nondeterministic Fuzzy Turing Machine calculations using conventional FTM ID-based method. Each rectangle represents an ID.

branches will be:

 Q_1 in which, machine moves to q_1 via a transition with weight 0.5 and tape will be:

В	0	1	0	1	В					
$\overline{Q_2}$	in	wh	ich,	m	achi	ne	moves	to	q_4	via
		1 1				·	0.6	and	tape	is
ъ		1		1	ъ					

The calculations for the next time steps are represented in Fig. 2. It is clear that there are many time

 $(q_0, 0, q_4, 0, L, 0.6)$. Therefore the next IDs in two steps of calculations required for the conventional FTM. By the time all of the branches reach to Halt mode, i.e. there is no more moves possible or the string reaches to its end, the branches stop growing and the tree would be ready for truth degree calculations. By tracing each individual branches and considering the weight of each transition passed through the branch, the weight for that branch tip final ID would be determined. In case of more than one final ID, the maximum of truth degrees will determine the truth degree associated with that input

string.

In conventional definition, it is only the transitions that have weights and assign membership values to the successor IDs all the way to final ID. The method of computation is very similar to transition-based method introduced in [8] and well investigated in [3]. One might address the conventional FTM method for truth degree assignment as "ID-based method". Although the transition-based method can work well for fuzzy Turing machine realized to accept certain types of fuzzy grammars, it has some disadvantages which makes it unsuitable for many applications as its focus is mostly on the acceptor mode of FTMs. In addition to high computational load in simple FTM, as presented in Example. II.1, to study other consequences of transition-based membership, refer to [3].

Wiedermann definition of FTM operation resembles the transition-based method where only transition weights are considered in the assignment of truth degree to the final ID of FTM, and the *mv*'s of states are not considered and discussed.

In order to generalize the definition to an applicational one, we follow the methodology in General Fuzzy Automata (GFA) developed in [3] and incorporate a new function in the definition of conventional FTM considering both the transition weight and the *mv* of the predecessor state to assign membership values to the states rather than final IDs.

Introducing their GFA, ÙŘDoostfatemeh and Kermer devised a method for fuzzy calculations that moved forward the fuzzy automata calculations to become best suited to practical issues [3]. As seen in natural processes, the phenomenon that occurs at a later time (time t+1) is affected by the steps and events that have taken place at the present time (time t). Therefore, it is reasonably expected that the goal to be achieved in the next period in a fuzzy automata would be the product of steps taken up to the present time. Precisely, in GFA, the membership value of the next state not only depends on the weight of the active transition, but also incorporates the mv of the current state as well. Hence, same method might be utilized on the conventional ID-based method in FTMs.

III. STATE MEMBERSHIP ASSIGNMENT IN FTMS

Based on what we discussed in previous section about GFA, to assign a membership value to a next state, both mv of current state and the weight of the active transition have to be effective in mv calculations. Hence, we suggest a function which incorporates these two values to assign a membership value to the next state. There are various options for this function which can be opted based on the application. In the following, we bring some conventions to simplify the presentations.

Convention III.1. $\mu^t(q_m)$ refers to the unique mv of the state q_m at time t.

Convention III.2. $\mu_{q_m}^t$ refers to the set of mv's Function $F_1(\mu, \delta)$ has two arguments as stated associated with the multi-membership state q_m at time t.

Convention III.3. By successor (and predecessor) state q_i , we mean the states which follow q_i (or are followed by q_i) considering a single input symbol read from the tape at the current time.

Convention III.4. In a sample FTM **T**,

- \bullet Q: Set of states.
- Σ Set of tape symbols.
- Δ is the set of all transitions.
- δ : is a function with the following definition: $\delta: Q \times \Sigma \times Q \times \Sigma \times \{-1,0,1\} \rightarrow$ [0, 1]. For example, the weight of the transition (q_i, a, q_j, b, d) is $\delta(q_i, a, q_j, b, d)$. \square

Now, we define a new transition function $\tilde{\delta}$, which is called augmented transition function, as follows:

$$\tilde{\delta}: (Q \times [0,1]) \times \Sigma \times Q \times \Sigma \times D \xrightarrow{F_1(\mu,\delta)} [0,1]$$
 (1)

 δ assigns to the successor state (reached from its predecessor) a value in the interval [0,1] via function $F_1(\mu, \delta)$ defined as follows.

Definition III.1. (Membership assignment function) is a mapping function which is applied via augmented transition function $\tilde{\delta}$ to assign my's to the active states.

$$F_1: [0,1] \times [0,1] \to [0,1]$$
 (2)

above:

- 1) μ : the mv of a predecessor;
- 2) δ : the weight of a transition.

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j, b_k, d)$$

$$= F_1(\mu^t(q_i), \delta(q_i, a_k, q_j, b_k, d))$$
(3)

which means that the mv of the state q_i at time t+1is computed by function F_1 using both the mv of q_i at time t and the weight of the active transition upon input a_k , output b_k , and direction d.

 F_1 should satisfy the following requirements:

Axiom 1. $0 \le F_1(\mu, \delta) \le 1$

Axiom 2.
$$F_1(0,0) = 0$$
 and $F_1(1,1) = 1$.

It is clear that F_1 function is more flexible and applicational compared to the conventional ID-based method. It provides a more suitable platform for generalization of fuzzy computations in our version of FTM. Refer to [3] for more details and discussion on the superiority of F_1 definition.

Example III.1. In Fig. 3, let $F_1(\delta, \mu)$ $\min(\delta, \mu)$. As we know, $\mu^t(q_1) =$ and $\delta(q_1, b, q_2, b, R) = 0.8$ which yields: $\mu^{t+1}(q_2) = \tilde{\delta}((q_1, 0.2), b, q_2, b, R)$ $F_1(\mu^t(q_1), \delta(q_1, b, q_2, b, R)) = \min(0.2, 0.8) = 0.2.\Box$

There are various choices for the function F_1 . However, the best strategy is always determined by the specific application. In the following, we mention just some examples as suggested in [3].

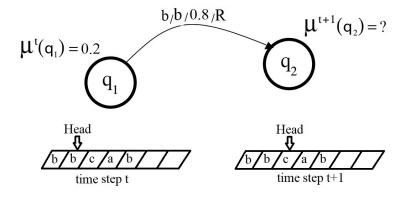


Figure 3. An active transition of a Fuzzy Turing Machine at time step t trying to assign a membership value to a next state

•
$$F_1(\mu, \delta) = Mean(\mu, \delta) = \frac{\mu + \delta}{2}$$

•
$$F_1(\mu, \delta) = GMean(\mu, \delta) = \sqrt{\mu \cdot \delta}$$

•
$$F_1(\mu, \delta) = \begin{cases} \max(\mu, \delta) & t < t_i \\ \min(\mu, \delta) & t \geqslant t_i \end{cases}$$

•
$$F_1(\mu, \delta) = \min \left[1, (\mu^{\omega} + \delta^{\omega})^{1/\omega} \right] \quad \omega > 0$$
(Yager class of t-conorms [6])

It is obvious that ID-based membership assignment to the next configuration can be considered as a special case where $F_1(\mu, \delta) = \delta$. This fact, enables our version of FTM to encompass the conventional versions of FTM.

Example III.2. Let us familiarize ourselves with the FTM fuzzy calculations. In this example, the deterministic FTM includes $Q=\{q_0,q_1,q_2,q_3,q_4,q_5\}$, $\Sigma=\{a,b,c\},\ \Gamma=\{a,b,c,B\},\ q_0=\text{start state,}$ and $F=\{q_5\}$. In this example, to do the states' membership value calculations in Fig. 4 FTM, we apply $F_1(\mu,\delta)=\frac{\mu+\delta}{2}$.

The following table carries the simulation results for a glance.

The performed calculations to fill the above table are as follows:

At time step $t=0,\ input=\epsilon$ (empty input), $\mu^{t_0}(q_0)=1.$

At time step t = 1, input = a, $\begin{bmatrix} B & a & b & c \end{bmatrix}$

$$\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1, x, R))$$
$$= F_1(1, 0.1) = \frac{1 + 0.1}{2} = 0.55]$$

At time step
$$t = 2$$
, $input = b$, $\begin{vmatrix} B & x & b & c & B \end{vmatrix}$

$$[\mu^{t_2}(q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_2, y, R))$$
$$= F_1(0.55, 0.1) = \frac{0.55 + 0.1}{2} = 0.325]$$

At time
$$stept = 3$$
, $input = c$, $\begin{vmatrix} B & x & y & c & B \end{vmatrix}$

$$\mu^{t_3}(q_3) = F_1(\mu^{t_2}(q_2), \delta(q_2, c, q_3, z, L))$$
$$= F_1(0.325, 0.4) = \frac{0.325 + 0.4}{2} = 0.3625]$$

IV. MULTI-MEMBERSHIP, MULTI-SYMBOL, AND MULTI-DIRECTION RESOLUTION

One of the interesting issues which occurs in nondeterministic FTM, similar to its ancestor FFA, is simultaneous transitions to the same state. In previous section, we addressed the membership assignment problem, defining the F_1 function which

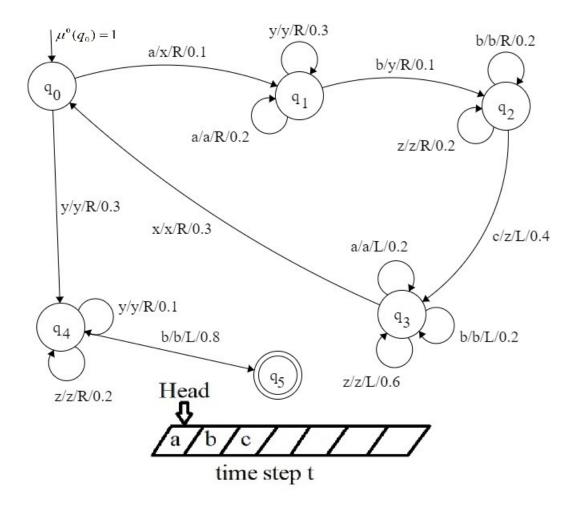


Figure 4. A deterministic FTM which accepts the language $L=\{a^nb^nc^n\}$ for $n\geqslant 1$

time	0	1	2	3
input	ϵ	a	b	c
Q_{act}	q_0	q_1	q_2	q_3
mv	1	0.55	0.325	0.3625
Symbol to Write (output)		X	у	Z
Direction		R	R	L

incorporates *mv* of predecessor state and transition weight to calculate the membership value of the next state. Because of nondeterminism, in some cases we have several membership values to be assigned to a successor state when there are several simultaneous transitions to that state. The question is what will be the actual membership value of the next state?

For example, in Fig. 5, all states have mv's and all transitions have weights. If we consider $F_1(\mu, \delta) = GMean(\mu, \delta) = \sqrt{\mu.\delta}$, then:

At
$$t = t_3$$
, $input = a$, and original state q_1 :

$$[\mu^{t_4}(q_2) = F_1(\mu^{t_3}(q_1), \delta(q_1, a, q_2, a, R))$$
$$= F_1(0.9, 0.4) = \sqrt{0.9 \times 0.4} = 0.6$$

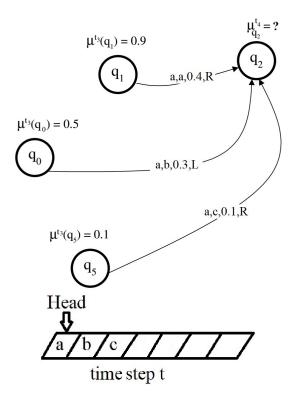


Figure 5. A part of a nondeterministic fuzzy Turing machine which depicts multi-membership

Again, at $t = t_3$, input = a, and original state q_0 :

$$\mu^{t_4}(q_2) = F_1(\mu^{t_3}(q_0), \delta(q_0, a, q_2, b, L))$$
$$= F_1(0.5, 0.3) = \sqrt{0.5 \times 0.3} = 0.387$$

And again, at $t = t_3$, input = a, and original state q_5 :

$$\mu^{t_4}(q_2) = F_1(\mu^{t_3}(q_5), \delta(q_5, a, q_2, c, R))$$
$$= F_1(0.1, 0.1) = \sqrt{0.5 \times 0.3} = 0.1$$

Therefore, q_2 gets activated at t_4 from three different paths with three different mv's $\{0.6, 0.387, 0.1\}$, while only a single mv has to be assigned to q_2 . This issue is called multi-membership problem.

To the best of our knowledge, available literature and research have no solution to characterize the operation of the FTM when it comes to multi-membership problem. Fortunately, there are methodologies in [3] for fuzzy automata to calculate the membership value of the states at time t+1 even in the existence of multi-membership value problem. The idea can be extended to FTM with some minor changes.

Motivated by the method presented in [3], we define some conventions to provide a suitable platform to resolve the multi-membership problem.

In conventional FTM, transition (q_i, a_k, q_j, b_k, d) , includes q_i which represents the current state, a_k that is the incoming symbol (the symbol which is present on the current position of the tape head), q_j which is the next state, b_k is the output symbol which is going to be written on the tape, and d is the direction of the head movement. In our proposed version of

the FTM we utilize the following conventions:

Convention IV.1. Set of all transitions of fuzzy Turing machine \mathbf{F} is denoted by $\Delta_{\mathbf{F}}$.

Definition IV.1. (Successor set): $Q_{Succ}(q_i, a_k)$ is the set of all destination states such as q_j in all transitions with origin q_i like (q_i, a_k, q_j, b_k, d) when the input symbol is a_k .

$$Q_{Succ}(q_i, a_k) = \{q_j | (q_i, a_k, q_j, b_k, d)$$

$$\in \Delta_{\mathbf{F}} \text{ when the input symbol is } a_k \}$$

$$(4)$$

Definition IV.2. (Predecessor set): $Q_{Pred}(q_j, a_k)$ is the set of all states followed by q_j following the input symbol a_k .

$$Q_{Pred}(q_j, a_k) = \{q_i | (q_i, a_k, q_j, b_k, d)$$

$$\in \Delta_{\mathbf{F}} \text{ when the input symbol is } a_k \}$$
(5)

Definition IV.3. (Active state set) After entering input a_k at time t to the FTM, there are some states that have at least one transition directed to them on input symbol a_k . The set of these states along with their membership values is called *active state* set at time t which is denoted as $Q_{Act}(t)$. Note that $Q_{Act}(t)$ is a fuzzy set.

Example IV.1. In Fig. 5 after input 'a' at time t_3 , $Q_{Act}(t_4)$ can be calculated as $\{(q_1, 0.9), (q_0, 0.5), (q_5, 0.1)\}$ which presents clearly a multi-membership problem. \square

In FTM, overlapping of transitions to state is more problematic than fuzzy automata, since it not only

makes the assignment of *mv* to that state ambiguous, but also creates ambiguity to the decision on the direction of head movement and the symbol to be written on the tape and they have to be uniquely determined in a reasonable way.

Referring again to Fig. 5, we notice that in addition to the multi-membership value problem, we have multi-symbol and multi-direction problem to be resolved too. As an example, all three active transitions after incoming symbol 'a', each tries to write its own suggested symbol on the tape. Hence, the problem arises that which member of the set $\{a, b, c\}$ should be written on the tape? Similarly, the movement direction of the head suggested by two of the three transitions is Right while the other tries to move the head to the Left. Again, it will require a proper judgment to be imposed to resolve the multi-direction issue. To the best of our knowledge, these above mentioned issues have never been addressed so far in literature among several available definitions.

To resolve the multi-membership, multi-symbol, and multi-direction problem, we evaluated three options for resolution methods.

1) The first resolution method is based on the conventional definition for FTM, where transition weights are involved to assign membership value to the successor IDs -very similar to transition-based membership assignment method. The main concern in this method is the final "accepting"

ID membership value and not only other ID's or states. Therefore, the path to the accepting ID is considered to evaluate the degree of acceptance, and a final decision is made in cases there are more than one path to the accepting ID. Hence, the multidirection issues were never faced as there is only one possible path considered and a tree is formed from the machine possible movements, refer to example 2. The same problem holds for the symbol to be written on the tape, (multi-symbol problem). Another aspect of the conventional FTM calculation is the volume of calculation needed to trace each possible path from the initial to the final ID. Due to possible nondeterminism, at each branch in the automata (at least two active transitions from one state), another new truth degree calculation branch is initiated and its respective truth degree is considered as a possible candidate for the final truth degree of the input string. In this method, the truth degree assignment is performed only after each and every new path is finalized. In some cases, it takes many or even infinite calculations for a simple FTM to determine a truth degree for a string.

2) Core idea of the second resolution method for aforementioned issues is extracted from ambiguity removal idea discussed in [9] by Omlin. In his suggested method, when an overlapping problem is observed for a state, a new state is generated for each of the conflicting transitions, and this process is continued until there will be no two conflicting

transitions directed to one single state. In practice, this resolution method causes two major problems:

- Generation of many new states that change the original finite control (FC) to a much more complex one. The new FC is no longer identical to the simple initial one and the original form cannot be distinguished among the numerous newly defined states. This issue is addressed well in [3].
- Due to considerable number of new states created by this method, it increases considerably
 the volume of fuzzy computations, which may
 lead to impracticality for large fuzzy Turing
 machines.

It is quite obvious that, following the above idea for FTM, a set of new tapes have to be created once a multi-symbol problem is faced. There are several issues with this method as described below:

• Each new tape has to be identical to original tape, but they will differ at the place that the head points to at time t. From that moment on, since FTM possess new tapes to handle, the transitions of its FC have to be modified accordingly. For example, suppose there are k number of tapes available at time t, which mandates the FTM transitions to have k input symbols. Also, suppose for a multi-symbol problem before time step t, i new tapes are generated. For the new FTM to manage these tapes, each transition needs to have k + i

input symbols. This implies that the FC of the FTM have to be thoroughly modified, which considerably complicates the calculations and FTM management problem. To illustrate the issue, for any instantaneous description (ID) containing a nondeterminism, one might generate a new tape so that the symbol suggested by each active transition be written on the respective new tape and the tape moves along the suggested direction for the next step. As the number of multi-symbol and multi-direction issues in FTM computations increases, it leads to numerous new tapes which may again lead to FTM blow up. Therefore, generating these tapes, managing their computations, dealing with ever-changing FC are the consequences of this solution which makes it almost impractical to implement and compute.

- In practice, having many new tapes generated with identical content, and moving their heads to a specified location is troublesome.
- From that moment onward, each tape will follow its own direction based on active transition. It simply manifolds the complexity of the multi-symbol problem.
- 3) In the third method, which is our novel approach, we consider a set of active transition(s) in fuzzy Turing machine at each time step. These 5-tuple transitions (q_i, a_k, q_j, b_k, d) are composed of three parts; current and next state, symbol read and to be

written on the tape, and head movement direction. Suppose that in the above mentioned set, there are more than one active transitions directed to a next state q_i , each requires to:

- 1- assign a membership value,
- 2- determine the direction of head movement, and
- 3- write its own symbol on the tape.

A. Multi-membership Resolution

We suggest a solution to first problem using another function that we call F_2 or (multi-membership resolution function):

Definition IV.4. In FTM, the multi-membership resolution function is a function which combines mv's of an active state, and produces a unique membership value for a state to be used in the next time step. $F_2: [0,1]^* \rightarrow [0,1]$.

Similar to what we suggested for F_1 , there are some requirements that F_2 has to meet:

Axiom 3.
$$0 \overset{n}{F_{2}}(\nu_{i})1$$

$$\nu_{i} = F_{1}(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d))$$

Axiom 4. $F2(\emptyset) = 0$.

Axiom 5.
$$F_2^n(\nu_i) = a$$
, if $\forall i, \nu_i = a$.

There can be several options for F_2 , where the best choice have to be determined by the application under consideration. Some possible candidates are as follows [3]:

• Maximum multi-membership resolution:

$$\mu^{t+1}(q_j) = \max_{i=1}^{n} \left[\tilde{\delta} \left((q_i, \mu^t(q_i)), a_k, q_j, b_k, d \right) \right]$$

$$= \max_{i=1}^{n} \left[F_1 \left(\mu^t(q_i), \delta(q_i, a_k, q_j, b_k, d) \right) \right]$$
(6)

• Arithmetic mean multi-membership resolution:

$$\mu^{t+1}(q_j) = \left[\sum_{i=1}^n \tilde{\delta}\left((q_i, \mu^t(q_i)), a_k, q_j, b_k, d\right)\right] / n \text{ nonzero.}$$

$$= \left[\sum_{i=1}^n F_1\left(\mu^t(q_i), \delta(q_i, a_k, q_j, b_k, d)\right)\right] / n$$
active training ac

• Geometric mean multi-membership resolution:

$$\mu^{t+1}(q_j) = \sqrt[n]{\prod_{i=1}^n \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j, b_k, d)}$$

$$= \sqrt[n]{\prod_{i=1}^n F_1(\mu^t(q_i), \delta(q_i, a_k, q_j, b_k, d))}$$
(8)

where n is the number of simultaneous transitions from q_i 's to q_m at time t+1, and $q_i \in Q_{pred}(q_m, a_k)$.

Example IV.2. For the membership value calculations of Fig. 5, the results are gathered in a set like $\{0.6, 0.387, 0.1\}$ which illustrates a simple case of multi-membership problem. In order to resolve this issue, one can utilize an F_2 function like Arithmetic mean. Therefore, the actual membership value which will be assigned to state q_2 is calculated as: $\mu^{t+1}(q_2) = (0.6 + 0.387 + 0.1)/3 = 0.362$

B. Multi-symbol Resolution

To resolve the multi-symbol and multi-direction problem we have to consider some new conventions:

Convention IV.2. Suppose a_k is an input tape symbol at time t. The *active transitions* are those with the form (q_i, a_k, q_j, b_k, d) whose $\mu^t(q_i)$'s are nonzero.

Definition IV.5. (Set of pairs including current active transitions and their weights)

$$\Delta_{Act}^{t}(a_{k}) = \{ [(q_{i}, a_{k}, q_{j}, b_{k}, d), F_{1}(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)) | | (q_{i}, a_{k}, q_{j}, b_{k}, d) \}$$

$$\in \Delta_{FTM}, \mu^{t}(q_{i}) \neq 0,$$

and current input symbol from the tape is a_k (9)

i.e. the set of all active transitions at time step t with regards to the input a_k .

Example IV.3. In the FTM depicted in Fig. 5, suppose that F_1 function be the algebraic product t-norm. The $\Delta^t_{Act}(a)$ set will simply be:

$$\Delta_{Act}^{t}(a) = \begin{cases} [(a, a, 0.4, R), F_{1}(0.9, 0.4)], \\ [(a, b, 0.3, L), F_{1}(0.5, 0.3)], \\ [(a, c, 0.1, R), F_{1}(0.1, 0.1)] \end{cases}$$

$$= \begin{cases} [(a, a, 0.4, R), (0.9 \times 0.4)], \\ [(a, b, 0.3, L), (0.5 \times 0.3)], \\ [(a, c, 0.1, R), (0.1 \times 0.1)] \end{cases}$$

Each of the active transitions which are members of $\Delta^t_{Act}(a_k)$ suggests a symbol to be written on the tape. To resolve any confusion about these symbols and to agree upon a single symbol which will be written on the tape, we define a function F_3 as following:

Definition IV.6. (Multi-symbol resolution function)

$$F_3: \Delta^t_{Act}(a_k) \to \Sigma$$
 (10)

As is clear, set of pairs of active transitions (q_i, a_k, q_j, b_k, d) , their weights $\delta(q_i, a_k, q_j, b_k, d)$, and the membership value of their origin $\mu^t(q_i)$ at time t when the input symbol a_k is read from the tape are required for calculations of the symbol to be written

on the tape. The transition details are required because they include the symbol to be written on the tape and the δ and $\mu^t(q_i)$ are needed by F_3 to determine the strength of that transition. For the sake of simplicity, let us limit the criterion for choosing the symbol and the direction to be only based on F_1 of each transition. But, the method is open for further modifications in cases when F_3 needs to be independent of F_1 .

There can be several options for F_3 , where the best choice have to be determined by the application. Some possible candidates might be as follows:

• The symbol in the active transition with maximum weight represented in Eq.11.

$$F_{3}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} b_{k} | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ F_{1}\left(\mu^{t}(q_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right) < F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right) \end{cases}$$
(11)

$$F_{3}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} b_{k} | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \sum F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right) < \sum F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right) \end{cases}$$
(12)

• The symbol in the active transition set with maximum cardinal (number of the transitions

that suggest the specific symbol) represented in Eq.13.

$$F_{3}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} b_{k} | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \sum \left[F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] < \sum \left[F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \end{cases}$$
(13)

Some possible candidates are as follows:

maximum weight represented in Eq.15.

• The direction in the active transition with

$$F_{4}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} d | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right) < F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right) \end{cases}$$
(15)

In case of equal weights, one might select the suggestion of the transition with maximum weight of its respective predecesor.

• The direction in the active transition set with

maximum scalar cardinality - sigma-count - of membership values of transitions (summation of the weight of transitions that suggest that specific direction) represented in Eq.16.

$$F_{4}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} d | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \sum F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right) < \sum F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right) \end{cases}$$
(16)

in Eq.17.

$$F_{4}\left(\Delta_{Act}^{t}(a_{k})\right) = \begin{cases} d | \left[\left(q_{i}, a_{k}, q_{j}, b_{k}, d\right), F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \forall \left[\left(q'_{i}, a_{k}, q'_{j}, b'_{k}, d'\right), F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] \in \Delta_{Act}^{t}(a_{k}), \\ \sum \left[F_{1}\left(\mu^{t}(q'_{i}), \delta(q'_{i}, a_{k}, q'_{j}, b'_{k}, d')\right)\right] < \sum \left[F_{1}\left(\mu^{t}(q_{i}), \delta(q_{i}, a_{k}, q_{j}, b_{k}, d)\right)\right] \end{cases}$$
(17)

V. COMPREHENSIVE FUZZY TURING MACHINE

Based on the discussions of the issues of the conventional FTM and the presented solutions for each issue, it is the time to define complete version of CFTM:

Definition V.1. (Comprehensive Fuzzy Turing Machine, CFTM)

A Comprehensive Fuzzy Turing Machine is a single tape 4-tuple fuzzy Turing machine \mathbf{M} denoted as $\mathbf{M} = \left(\mathbf{T}, F, \tilde{\delta}, \mu\right)$ which are defined as follows:

- T is the conventional fuzzy Turing machine which includes:
 - Q is the finite set of states.
 - Σ is the finite set of tape symbols to be printed on the tape that has a leftmost cell, but it is unbounded to the right.
 - D is the set of possible head movement directions.
 - I is the set of input symbols; $I \subset \Sigma$.
 - Δ_{CFTM} is is the next-move relation which is a subset of $Q \times \Sigma \times Q \times \Sigma \times D$. For each possible move of \mathbf{F} there is an element $\delta \in \Delta$ with $\delta = (q_1; a_1; q_2; a_2; d)$. That is, if the current state is q_1 and the tape symbol scanned by the machineâ $\check{\mathbf{A}}$ 2s head is a_1 ; \mathbf{F} will enter the new state q_2 , the new tape symbol a_2 will rewrite the previous symbol a_1 , and the tape head will move in direction d.

- $B \in Q I$ is the blank symbol.
- \tilde{R} is the set of start states.
- Q_f is the set of final states.
- **F** is the set of functions which includes:
 - $F_1: [0,1] \times [0,1] \rightarrow [0,1]$ is the mapping function which is applied via $\tilde{\delta}$ to assign mvs to the active states, thus called mem-bership assignment function.
 - $\tilde{\delta}: (Q \times [0,1]) \times \Sigma \times Q \times \Sigma \times D \xrightarrow{F_1(\mu,\delta)}$ [0,1] is the augmented transition function.Please refer to section III for more details.
 - F₂: [0,1]* → [0,1] is a multi-membership resolution function which resolves multi-membership active states and assigns a single mv to them, thus called multi-membership resolution function.
 - F_3 : $\Delta^t_{Act}(a_k) \to \Sigma$ is multi-symbol resolution function of the tape symbols to be printed on the tape during the FTM computations. F_3 assigns a single selected symbol to be printed on the tape at time t. Δ_{Act} is the set of current active transitions.
 - $F_4: \Delta^t_{Act}(a_k) \to D$ is multi-direction resolution function of the head movements during the FTM computations. F_4 determines a single direction for the head of the FTM tape to move at time t.
- $\delta: \Delta_{CFTM} \to [0,1]$ is a function that assigns transition weight in [0,1] to each transition.
- μ is the array of states membership values.

Conventionally, each FTM comes with its view on the concept of instantaneous description of the machine. In the definition below, the ID for our novel machine is presented:

Definition V.2. Instantaneous Description (ID) of Comprehensive Fuzzy Turing Machine (CFTM) M working on the string w at time $t \ge 0$ represented as Q_t is defined as a unique description of the machine's tape, a vector of membership values of all CFTM states, and the position of the machine's head after performing the t-th move on the input w.

Definition V.3. (Acceptance)

A string is said to be accepted by a CFTM if and only if the membership value of at least one final state is not zero after the machine halts. Otherwise, the string is a member of a language which is not supported by the Turing machine. The membership value of the final state is considered as "Truth Degree" or "Acceptance Degree" of that string processed by the CFTM.

Notice: In cases that there are more than one final state with nonzero membership values, the multimembership resolution is required again to determine the acceptance degree. The same conditions and definitions for F_2 is required or one might simply use the same F_2 in CFTM.

Example V.1. An explanatory example of the computations of the CFTM of Fig. 6 comes here. The CFTM **M** includes:

```
Data:
The FTM Information
The Tape Information
Result: The Membership Vales of All States
        after Entering the Input String.
Initialization;
while Not reached to the End of the Tape do
   InputSymbol = Read Tape Symbol;
   for All Transitions in FTM do
       if Transition = Active then
           Calculate F1 of that Transition;
           Add [Transition, F_1] pair to
            \Delta_{Act}^t(InputSymbol) set;
       end
   end
   for All States in Automata of FTM do
       if Single membership value exists for a
        state then
           Determine the membership value of
            the successor state at time step
            (t+1);
       else if Multi-Membership then
           Do MultiMembership Resolution
            via calculation of F_2 for the
            successor state at time step (t+1);
   end
   for All members of \Delta_{Act}^t(InputSymbol)
       Calculate the F_3 for Multi-Symbol
        Resolution to determine the Next
        Symbol to be written on the tape;
       Calculate the F_4 for Multi-Direction
        Resolution to determine the Next
        Direction;
```

 $Q=\{q_0,q_1,q_2,q_3,q_4,q_5\},\ \Sigma=\{0,1\},$ $\Gamma=\{0,1,B\},\ q_0=\text{start state},\ \tilde{R}=\{q_2,q_4\}.$ Also, suppose that the FTM starts with the state q_0 with membership value 1. $F_1=(\mu+\delta)/2,$ $F_2=\sqrt[n]{\tilde{\delta}_1\times\ldots\times\tilde{\delta}_n},$ and F_3 and F_4 are symbol and

Algorithm 1: Pseudocode for CFTM Calculations

end

end

direction with maximum cardinalities, respectively.

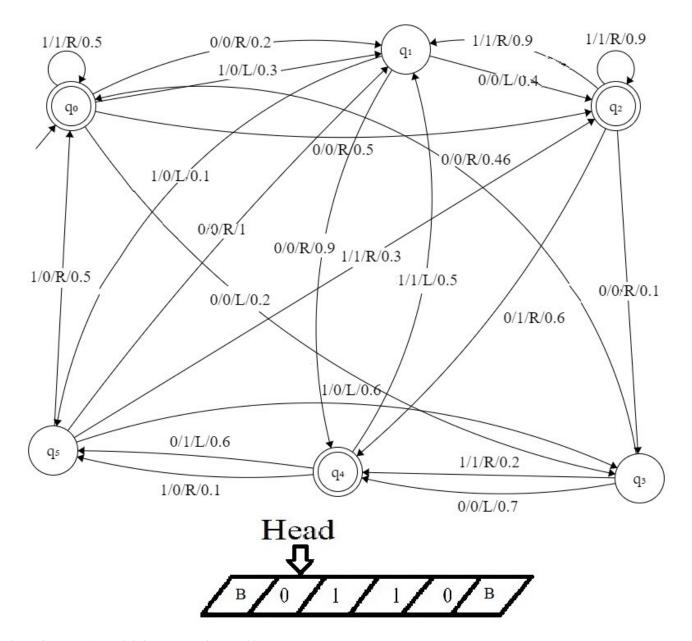


Figure 6. A Nondeterministic Fuzzy Turing Machine

At time t = 0 the ID of the CFTM, Q_0 , is:

Head position is at cell 1 -hypothetically the tape cells are numbered from 0.

The symbol read from the tape is "0".

$$\Delta_{Act}^{0}(0) = \left\{ \begin{bmatrix} (q_0, 0, q_1, 0, R), 0.6], \\ [(q_0, 0, q_3, 0, L), 0.6], \\ [(q_0, 0, q_2, 0, R), 0.75] \end{bmatrix} \right\}$$

Next head movement direction: R, and the symbol written on the tape: 0

No state requires multi-membership resolution.

At time t = 1 the ID of the CFTM, Q_1 , is:

 $\mu(states) = [0.6, 0.75, 0.6, 0, 0, 0],$ tape state:

Head position is at cell 2, and the

symbol read from the tape is "1". $\Delta^1_{Act}(1) =$

$$[(q_1, 1, q_5, 0, R), 0.55], [(q_2, 1, q_2, 1, R), 0.825],$$

$$[(q_2, 1, q_1, 1, R), 0.825], [(q_3, 0, q_0, 0, R), 0.53],$$

$$[(q_3, 0, q_4, 0, L), 0.65]$$

Next head movement direction: R, and the symbol

written on the tape: 1

No state requires multi-membership resolution.

At time t=2 the ID of the CFTM, Q_2 , is:

$$\mu(states) = [0.53, 0.825, 0.825, 0, 0.65, 0.55], \text{ tape}$$

Head position is at cell 3, and the symbol read from the tape is "1".

$$\Delta_{Act}^{2}(1) = \begin{cases} [(q_{5}, 1, q_{2}, 1, R), 0.425], [(q_{5}, 1, q_{3}, 0, L), 0.575], \\ [(q_{2}, 1, q_{2}, 1, R), 0.862], [(q_{3}, 1, q_{4}, 1, R), 0.425], \\ [(q_{0}, 1, q_{0}, 1, R), 0.515], [(q_{1}, 1, q_{5}, 0, L), 0.462], \\ [(q_{5}, 1, q_{0}, 0, R), 0.775], [(q_{2}, 1, q_{1}, 1, R), 0.862], \\ [(q_{0}, 1, q_{1}, 0, L), 0.415] \end{cases}$$

Next head movement direction: R, and the symbol

written on the tape: 1

There are some states that require multi-membership

resolution:

For q_0 , there are two membership value candidates:

$$\{0.515, 0.775\}$$

$$\sqrt{0.515 \times 0.775} = 0.637$$

For q_1 , there are two membership value candidates:

$$\sqrt{0.415 \times 0.862} = 0.598$$

For q_2 , there are two membership value candidates:

$$\{0.425, 0.862\}$$

$$\sqrt{0.425 \times 0.862} = 0.605$$

At time t = 3 the ID of the CFTM, Q_3 , is:

$$\mu(states) = [0.637, 0.598, 0.605, 0.575, 0.425, 0.462],$$

Head position is at cell 4 and the symbol read from the tape is "0".

$$\Delta^3_{Act}(0) =$$

$$\begin{cases}
[(q_3, 0, q_0, 0, R), 0.517], [(q_2, 0, q_4, 1, R), 0.602], \\
[(q_1, 0, q_2, 0, L), 0.499], [(q_5, 0, q_1, 0, R), 0.731], \\
[(q_0, 0, q_2, 0, R), 0.565], [(q_1, 0, q_2, 0, L), 0.499], \\
[(q_4, 0, q_5, 1, L), 0.512], [(q_3, 0, q_4, 0, L), 0.637], \\
[(q_2, 0, q_3, 0, R), 0.352], [(q_1, 0, q_4, 0, R), 0.749], \\
[(q_0, 0, q_1, 0, R), 0.415], [(q_0, 0, q_3, 0, L), 0.415]
\end{cases}$$

Next head movement direction: R, and the symbol

written on the tape: 0

There are some states that require multi-membership

resolution:

For q_1 , there are two membership value candidates:

 $\{0.731, 0.415\}$

$$\sqrt{0.731 \times 0.415} = 0.550$$

For q_2 , there are three membership value candidates:

$$\{0.499, 0.565, 0.499\}$$

$$\sqrt[3]{0.499 \times 0.499 \times 0.565}$$

For q_3 , there are two membership value candidates:

$$\{0.352, 0.415\}$$

$$\sqrt{0.352 \times 0.415} = 0.382$$

For q_4 , there are three membership value candidates:

$$\{0.602, 0.637, 0.749\}$$

$$\sqrt[3]{0.602 \times 0.637 \times 0.749} = 0.659$$

0

For the next time step:

 $1 \mid 1$

 $\mathbf{B} \mid \mathbf{0}$

$$\mu(states) = [0.517, 0.550, 0.605, 0.382, 0.659, 0.512],$$
 and tape state for the next time step:

В

As we reached the end of string here, the CFTM enters the halt mode. It means the machine no longer works and the above configuration, Q_3 is actually the *Final ID*. To determine whether the string "0110" is accepted, we refer to the membership value of the final states, i.e. q_0 , q_2 , and q_4 in the final ID, Q_3 . The calculated membership values for these states at final ID are $\{0.517, 0.605, 0.659\}$. To assign a truth degree to the input string, again we face the multi-membership problem. Referring to the definition V.3, we might utilize the same definition for F_2 used throughout the calculations to resolve multi-membership problem for the acceptance degree. Therefore, the calculated "Acceptance

Degree" of the string "0110" in CFTM **M** will be $= \sqrt[3]{0.517 \times 0.605 \times 0.659} = 0.590.$

Our suggested algorithm of CFTM computations is as presented in Algorithm. 1. Also, we made the source code for computing CFTM is available in Python which can be found in [1].

VI. CONCLUSION

In this paper, we instigated the conventional definition of FTM for their benefits and weaknesses. We noticed that the membership assignment is performed ID-based. In the light of General Fuzzy Automata (GFA) proposed by [3], we developed a more complete definition for two problems already existed in fuzzy Turing machines which covers those vague aspects of the membership assignment and multi-membership resolution issue. we noticed that in FTMs, the membership assignment is not the only vague issue. Each active transition requires the machine to move its head in a specific direction and also mandates a predefined symbol to be written on the tape. Therefore, at each time step it is usually more than one symbol to be written on the tape and also more than one direction for the machine to move. Hence we defined two more functions to resolve the above mentioned issues, multi-direction and multi-symbol resolution functions to decide on a single direction and a single head movement based on the weight of the active transitions and the membership values of their predecessor states. It is easy to prove that each conventional fuzzy Turing machine can be modeled in the form of the novel Comprehensive Fuzzy Turing Machine (CFTM). Lastly, using an example, some comparison on the volume of calculations on conventional FTM and the novel CFTM is performed. It is clear that the CFTM significantly reduces the amount of computations required for fuzzy Turing machine.

REFERENCES

- Najmeh Ahang. Fuzzy Turing Machine and Its Capabilities. Shiraz University Publications, 2018.
- [2] Benjamín Callejas Bedregal and Santiago Figueira. On the computing power of fuzzy turing machines. Fuzzy Sets and Systems, 159(9):1072–1083, 2008.
- [3] Mansoor Doostfatemeh and Stefan C. Kremer. New directions in fuzzy automata. *International Journal of Approximate Reasoning*, 38(2):175–214, February 2005.
- [4] Hadi Farahani. Meta-type fuzzy computations and fuzzy complexity. *Journal of Intelligent & Fuzzy Systems*, 34(1):81–92, 2018.
- [5] Giangiacomo Gerla. Fuzzy turing machines: Normal form and limitative theorems. *Fuzzy Sets and Systems*, 333:87–105, 2018.
- [6] George Klir and Bo Yuan. Fuzzy sets and fuzzy logic, volume 4.Prentice hall New Jersey, 1995.
- [7] Edward T Lee and Lotfi A Zadeh. Note on fuzzy languages. In Fuzzy Sets, Fuzzy Logic, And Fuzzy Systems: Selected Papers by Lotfi A Zadeh, pages 69–82. World Scientific, 1996.
- [8] John N Mordeson and Davender S Malik. Fuzzy automata and languages: theory and applications. Chapman and Hall/CRC, 2002.
- [9] Christian W Omlin, C Lee Giles, and Karvel K Thornber. Equivalence in knowledge representation: automata, recurrent neural networks, and dynamical fuzzy systems. *Proceedings of the IEEE*, 87(9):1623–1640, 1999.
- [10] Eugene S. Santos. Fuzzy algorithms. *Information and Control*, 17(4):326 339, 1970.

- [11] JiřílA Wiedermann. Characterizing the super-turing computing power and efficiency of classical fuzzy turing machines.

 Theoretical Computer Science, 317(1-3):61–69, 2004.
- [12] JiÅŹÃ■ Wiedermann. Fuzzy Turing machines revised. *Computing and Informatics*, 21(3):251–263, 2002.