Single Variable Regression: Introduction

EC420 MSU Spring 2021

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This lecture

Goal:

1. Introduce the problem we'll be working on for a while
2. Define the **Population Regression Function**
3. Intuition of "fitting a line"
4. Define the assumptions for OLS
5. Ordinary Least Squares estimator
6. Computing OLS estimates in the **Sample Regression Function**
7. Descriptive analysis vs. causal
8. Other methods of calculating OLS

The problem at hand...

We have some data on two (or more, later) variables that we think move together in an interesting way.

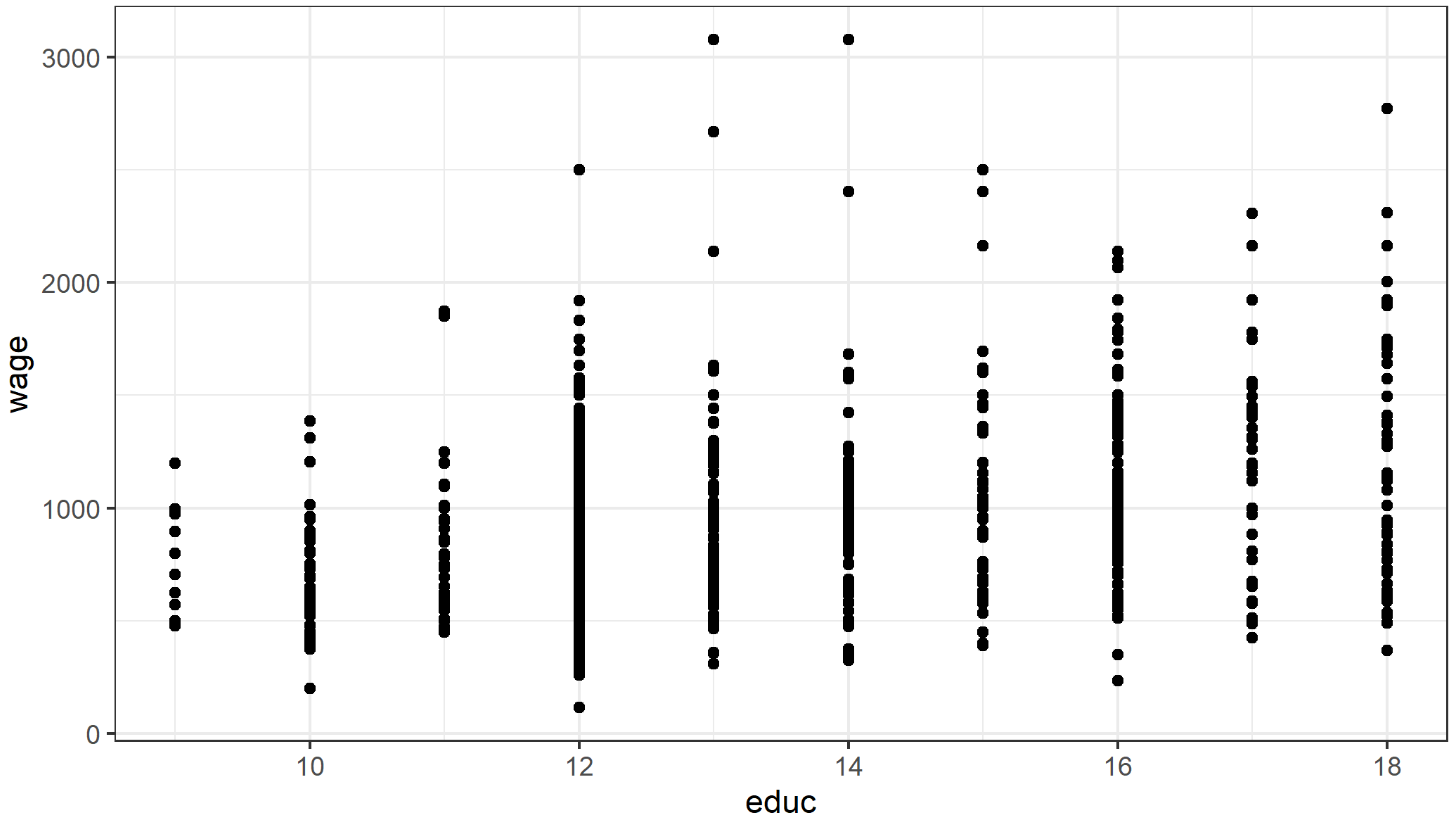
- (Insert one of many examples we've talked about before here)

We want to quantify and test this relationship

- Predict a change
- Test a theory
- Win a bet?

We have a **sample**, but want to predict/test something about the population

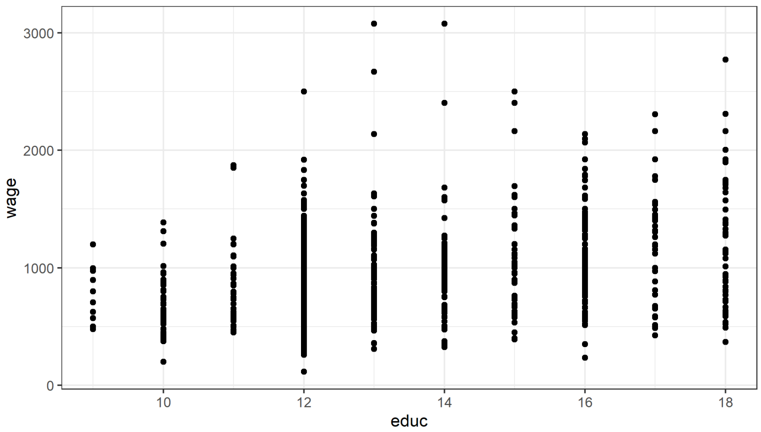
Wage data used in Wooldridge `wage2`



Data from Blackburn and Neumark (1992), "Unobserved Ability, Efficiency Wages, and Interindustry Wage Differentials" *Quarterly Journal of Economics* 107, 1421-1436

The data looks like this:

wage	educ
769	12
808	18
825	14
650	12
562	11
1400	16



- $N = 935$
- $\overline{wage} = 957.95$
- $\overline{educ} = 13.47$

What we'd like to have is a function that tells us how $wage$ and $educ$ move together in the **population**

In a perfect world, we would have some function for $X = educ$ and $Y = wage$:

$$g(x) = y$$

Where we give the function any realization of x, and it spits out exactly y.

But that isn't going to happen

Think about the data we just looked at - $educ = 12$ we observed $wage = 769$ and $wage = 650$. The dream function doesn't exist! There are other things not accounted for besides $educ$.

So we settle for something that tells us about the **expectation** of Y. The Population Regression Function

$$E[Y|X] = \beta_0 + \beta_1 X$$

The *Population Regression Function* (PRF) describes the relationship between X and the **conditional expectation** of Y.

- X and Y are random variables
- β_0 and β_1 are **population parameters**
- We have restricted the $E[Y|X]$ to be a *linear* function of X.
  - It can be drawn as a straight line with an intercept and constant slope
  - We will be estimating β_0 and β_1

Population Regression Function

The PRF:

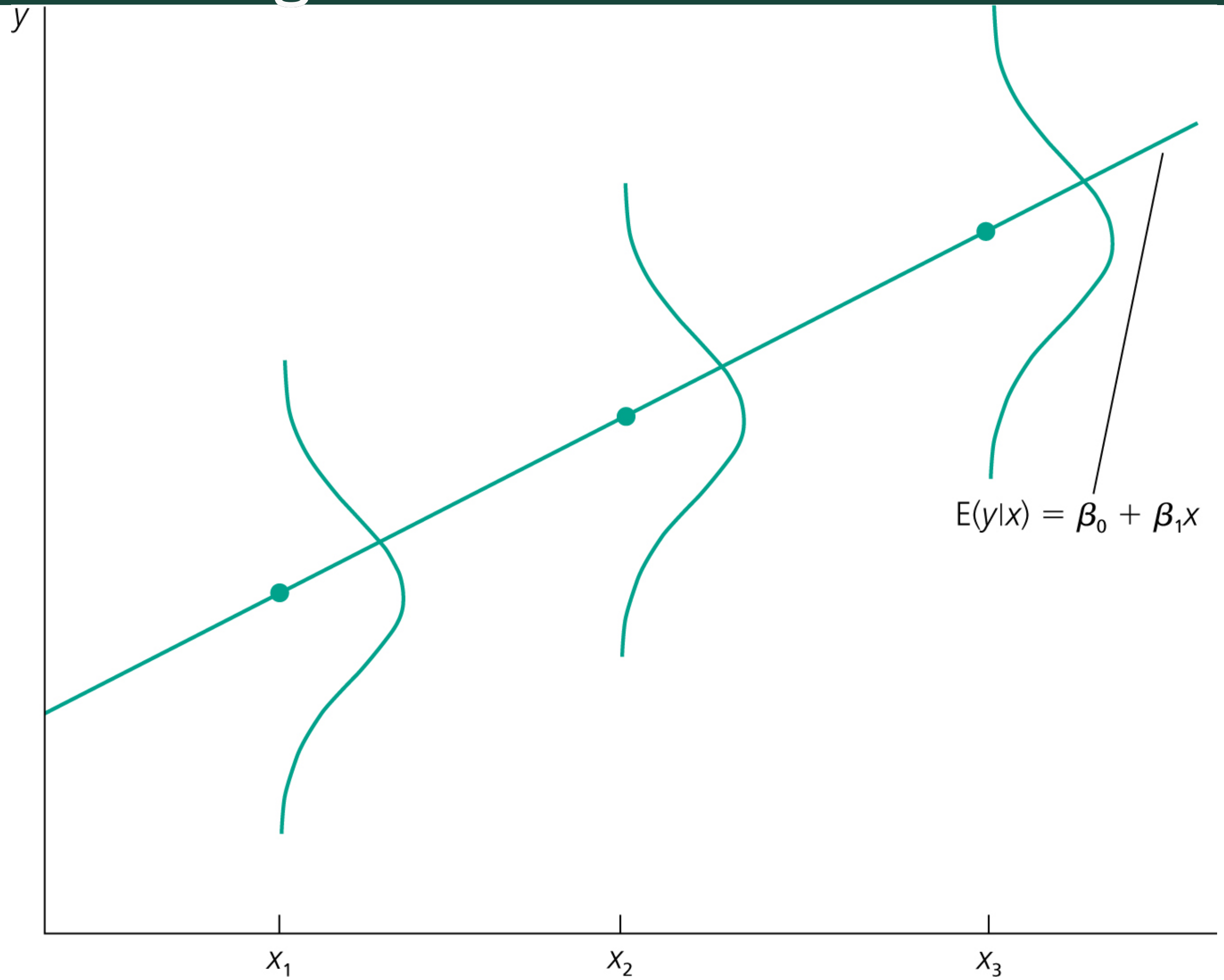
$$E[Y|X] = \beta_0 + \beta_1 X$$

Let $Y = wage$ and $X = educ$

- $E[Y|X = x]$ gives us the expectation of Y (wage) conditional on some realized value of $X = x$ (educ)
- So, if $educ = 16$, then $E[Y|X = 16] = \beta_0 + \beta_1 \times 16$
  - We can plug in any x_i and get the **expected value** of the paired y_i

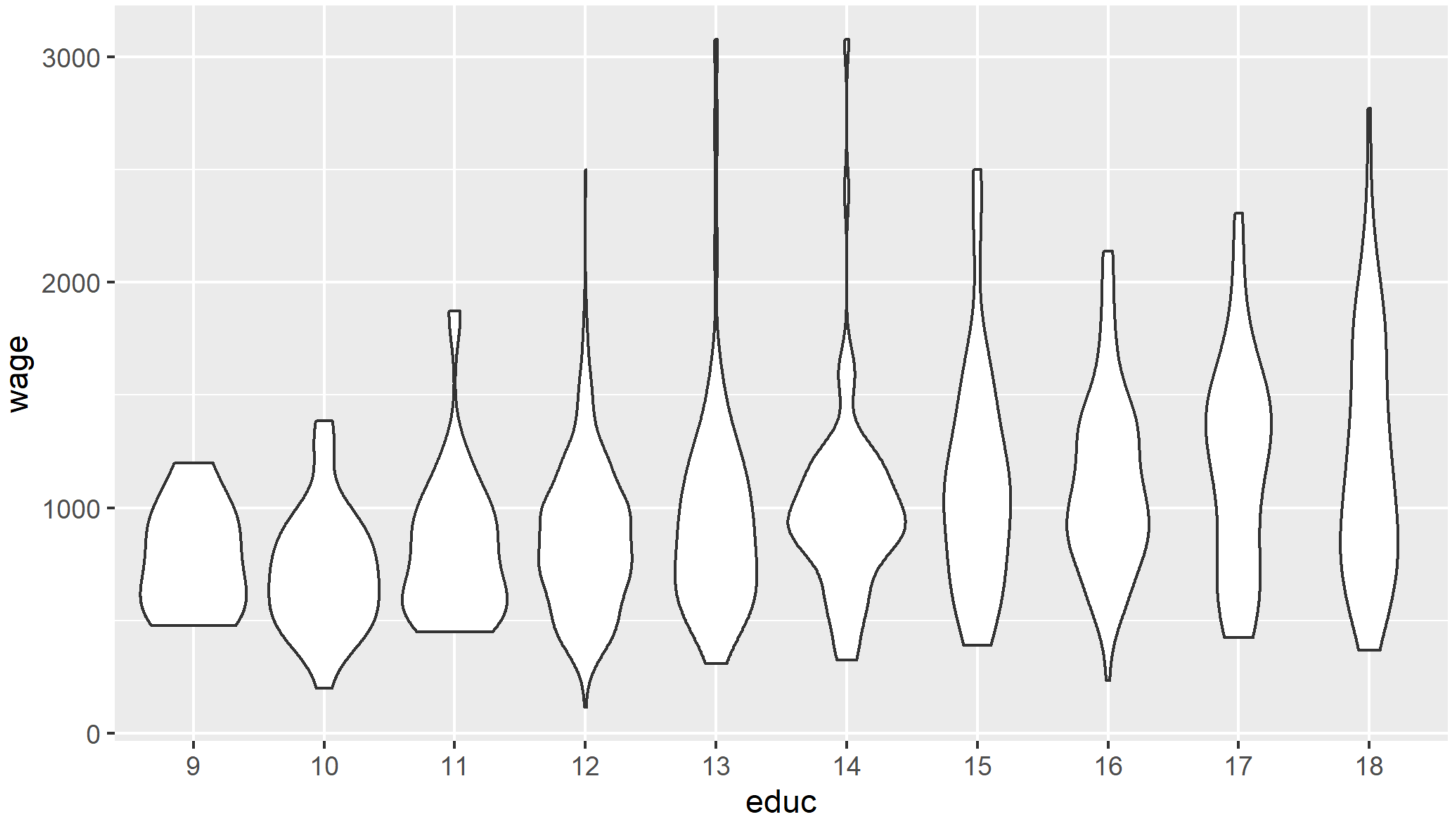
Question: Will the PRF return exactly y_i given a value x_i?

Population Regression Function

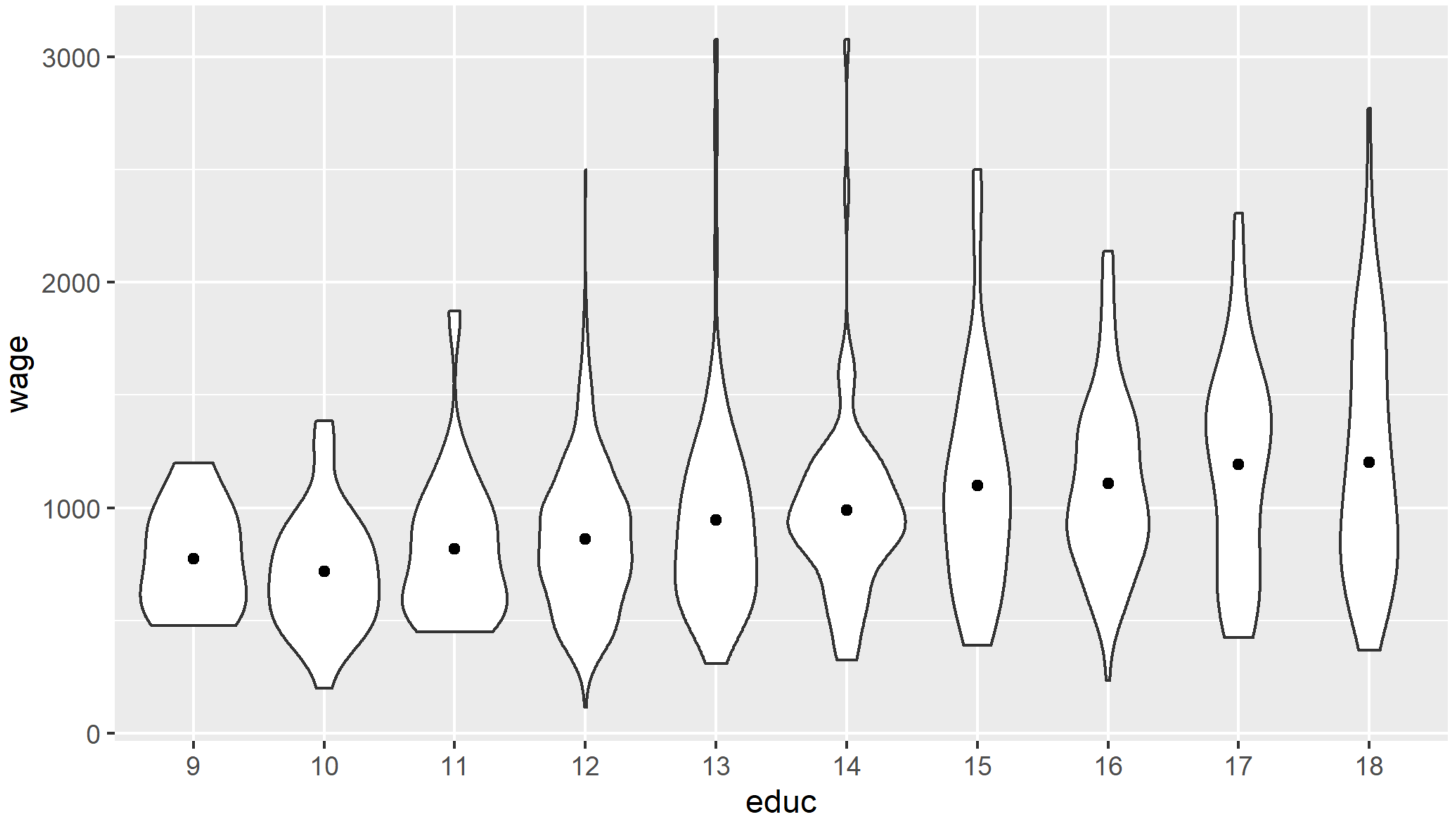


Ch. 2.1 of Wooldridge, example of a conceptual PRF. The line defines the PRF, the expectation of Y conditional on X

Population Regression Function



This is the wage data. Each "blob" is an empirical histogram of the data for that value of $educ$ (they are symmetrical). This is called a *violin plot*. It is the empirical counterpart of the previous plot from Wooldridge



Each point is the sample mean for each value of $educ$.

A (linear) PRF would be the straight line that best connects the points. **Regression fits that line**. A brief look at the line shows that it certainly won't be perfect!

What happens, then, if we want to write Y exactly?

The PRF gives us the *expectation* of Y

- So we add a **stochastic error term**, the difference between $E[Y|X]$ and Y:

$$Y = E[Y|X] + U = \beta_0 + \beta_1 X + U$$

This is the stochastic population regression function

U is also the **population error term**, and is itself a **random variable**.

- It must be that $E[U] = 0$

Now we can write our **simple linear regression model**:

$$y = \beta_0 + \beta_1 x + u$$

This is a statement about the relationship between observed realizations (y_i, x_i) based on the population parameters β_0, β_1

We will call u the **error term** - it is the difference between the conditional expected mean and the observed y_i given a value of x_i.

- It might be different for two identical realizations of x_i

Naturally, we would think that the "right" value of the population parameters, $\beta = \{\beta_0, \beta_1\}^*$, minimizes all of the u_i values in a sample.

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* A parameter vector is just a list of numbers.

The Sample Regression Function

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

The "hats" are important

They mean we have a *sample estimate* of the population parameters.

- β_0, β_1 are the population
- $\hat{\beta}_0, \hat{\beta}_1$ are the sample estimates and will change when the sample changes
  - So they are random variables!

Where did u go?

Since we have a hat on y_i, there is no u, but $\hat{y}_i \neq y_i$.

- Define $\hat{u}_i = \hat{y}_i - y_i$.
- $\hat{u}_i$ is the *residual*.

To summarize:

The PRF is

$$E[Y|X] = \beta_0 + \beta_1 X$$

The simple linear regression model is:

$$y = \beta_0 + \beta_1 x + u$$

The SRF is:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

And if we want to write the sample regression model:

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$$

How do we get those $\hat{\beta}$'s in the SRF?

We make two assumptions:

First, if the expectation of Y equals $\beta_0 + \beta_1 X$, then *in expectation*, $E[U] = 0$. Because:

$$E[Y|X] = \beta_0 + \beta_1 X \quad \textbf{and} \quad Y = \beta_0 + \beta_1 X + U$$

Second, our first assumption should hold no matter what x is. So, it should be true that $E[U|X] = 0$ for **all** possible values of X.

There are very important assumptions as they will define our Sample Regression Function (SRF).

Let's make these assumptions formal:

1. $E[U] = 0$.
   - As long as there is a β_0 (regardless of β_1), this is true. We call this assumption **trivial**.
2. $E[U|X] = E[U]$
   - **Mean independence**. The **mean** of U is the same, regardless of the value of X:

These are **population moments**

- A **moment** is a specific attribute of a distribution
- The mean is the "first moment". Variance is the "second moment".

Economists spend a lot of time showing mean independence $E[U|X] = E[U]$.

Fitting the line

Two quick reminders before we introduce the Ordinary Least Squares (OLS) estimator for β:

$$Cov(Y, X) = E[YX] - E[Y]E[X]$$

and

$$\text{If} \quad E[U] = 0$$

then

$$Cov(U, X) = E[UX] - E[U]E[X] = E[UX] - 0$$

And note that the simple linear regression model $y = \beta_0 + \beta_1 x + u$ implies that:

$$u = y - \beta_0 - \beta_1 x$$

Since $u = y - \beta_0 - \beta_1 x$:

Let's write Assumption 1 and Assumption 2 using expectations of the regression model from before

- $E[U] = 0 \Rightarrow E[(y - \beta_0 - \beta_1 x)] = 0$
- $E[U|X] = 0 \Rightarrow E[x(y - \beta_0 - \beta_1 x)] = 0$
    - To see this, picture any expected value of x. Now, multiply it by 0.

- How many equations?

- How many unknowns?

Let's solve for β. To the board!

These are *moments*, and this way of deriving β is known as "method of moments".

OLS in 1 variable

What we just derived on the board depends on **population** moments: $Cov(Y, X)$ and $Var(Y, X)$.

But, just as before when we didn't know μ but we could calculate $\bar{y}$ (and we even know something about the distribution of $\bar{y}$)...

...we can calculate sample values for $Cov(y, x)$ and $Var(x)$

OLS in 1 variable

First, let's tackle the *estimate* of β_0.

- We know, from the board, that $\beta_0 = \boldsymbol{E}[\boldsymbol{y}] - \beta_1 \boldsymbol{E}[\boldsymbol{x}]$
- We have a good, unbiased sample estimator for $\boldsymbol{E}[\boldsymbol{y}]$: $\bar{\boldsymbol{y}}$.
- And we have a good, unbiased sample estimator for $\boldsymbol{E}[\boldsymbol{X}]$: $\bar{\boldsymbol{x}}$
  - $\bar{\boldsymbol{y}} = \hat{\beta}_0 + \hat{\beta}_1 \bar{\boldsymbol{x}}$

The hats stand for (sample) estimates! We don't observe β_0, but we can estimate it. This is very common notation.

Of course, we still have to calculate $\hat{\beta}_1$.

We know how to calculate the sample covariance:

- $\widehat{Cov}(Y,X) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$

We know how to calculate the sample variance:

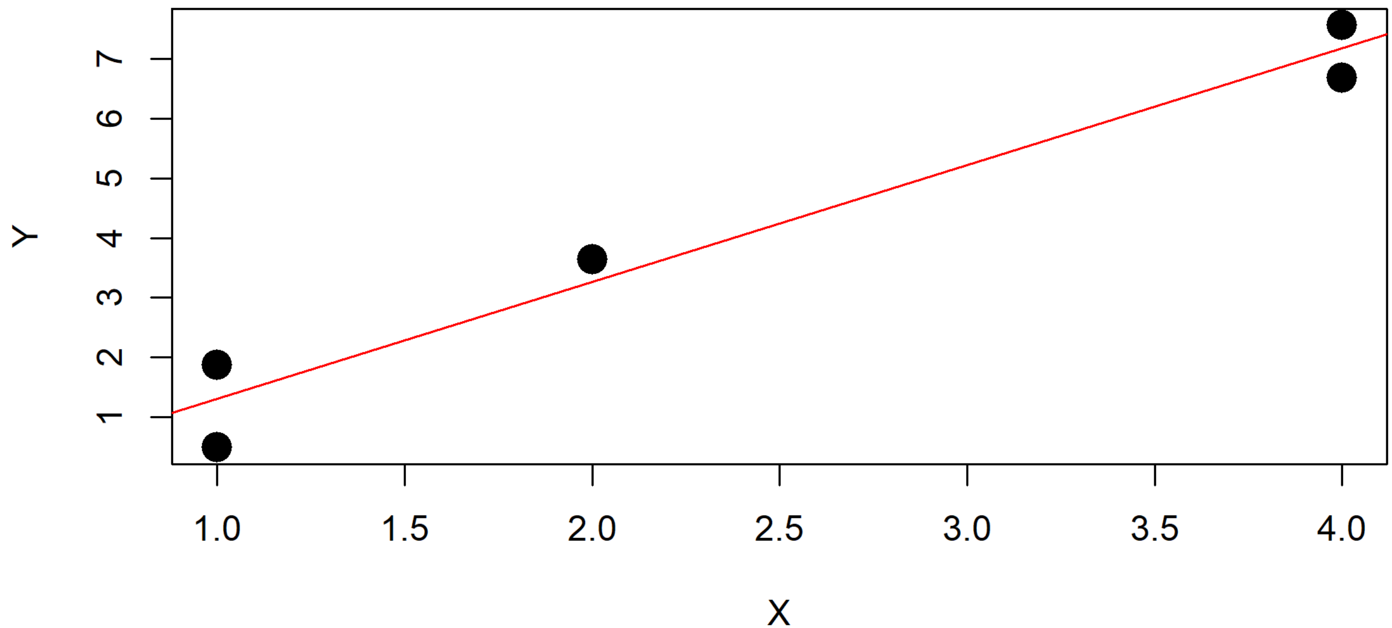
- $\widehat{Var}(X) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$

$$\hat{\beta}_1 = \frac{\widehat{Cov}(Y,X)}{\widehat{Var}(X)} = \frac{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

What is important here is that **these are all observable in the data, and you know how to calculate them**. You know how to calculate $\bar{x}$ and $\bar{y}$, you know how to sum things, and you know x_i and y_i in the data.

As long as your assumptions hold, you have an estimate of the PRF.

OLS in 1 variable



The red line is the *sample regression function*, or *SRF*.

Why is it the "sample" regression function?

A couple important terms:

- The **fitted value**, $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$
- The **residual**, $\hat{u}_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$

And note that:

- $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$
  - The $\hat{u}_i$ "trues up" the fitted value.

Note that the residual is not the same as the error term.

- The residual is an empirical estimate from the sample
- The error term, u_i, is different

What's inside the error term?

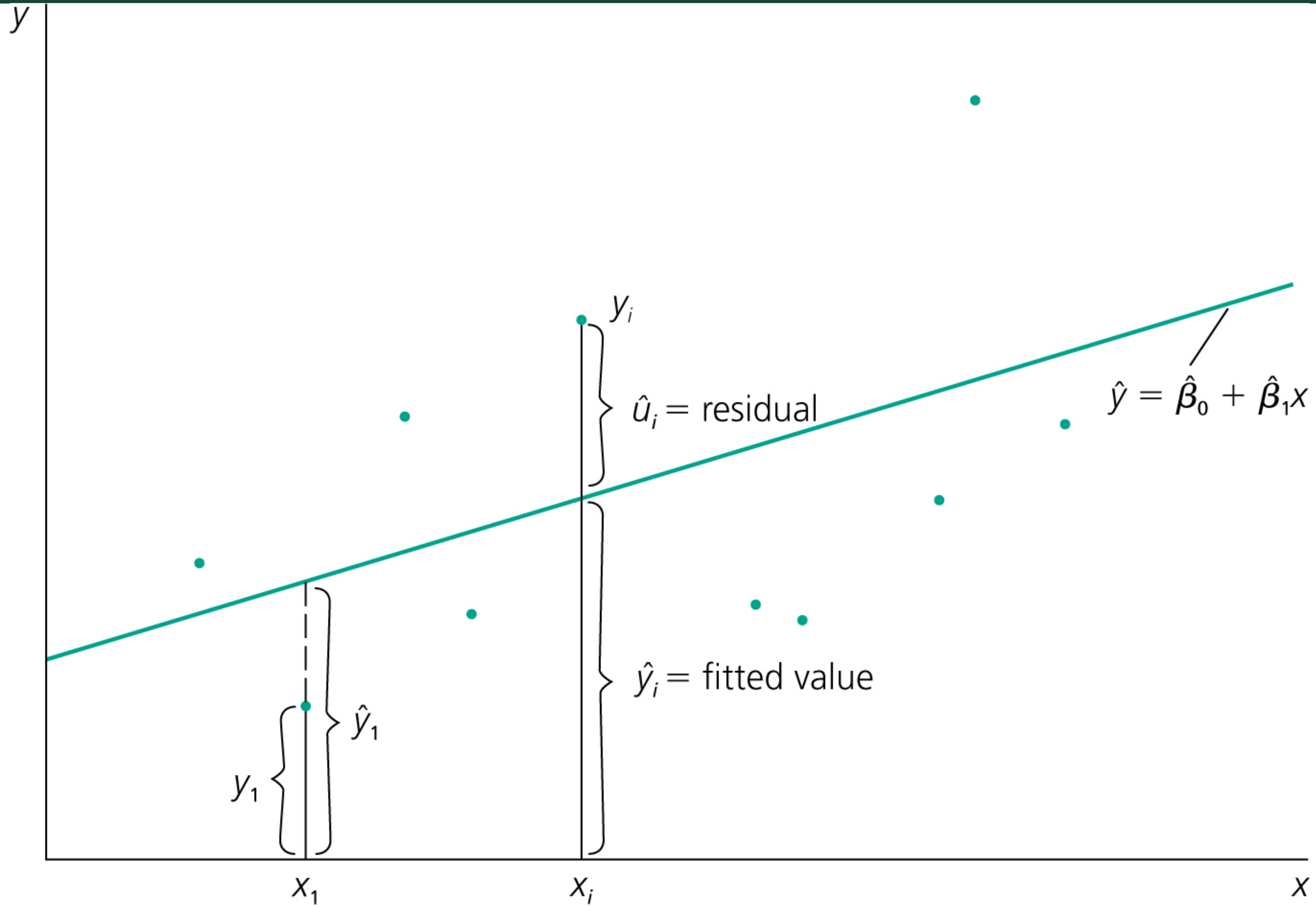
In u_i

- Omitted variables
  - There might be another covariate, x_2, that is missing.
- Measurement error
  - That x might not be correctly measured.
- Non-linearities
  - Maybe there are some non-linear effects included in there.

These are all in u_i.

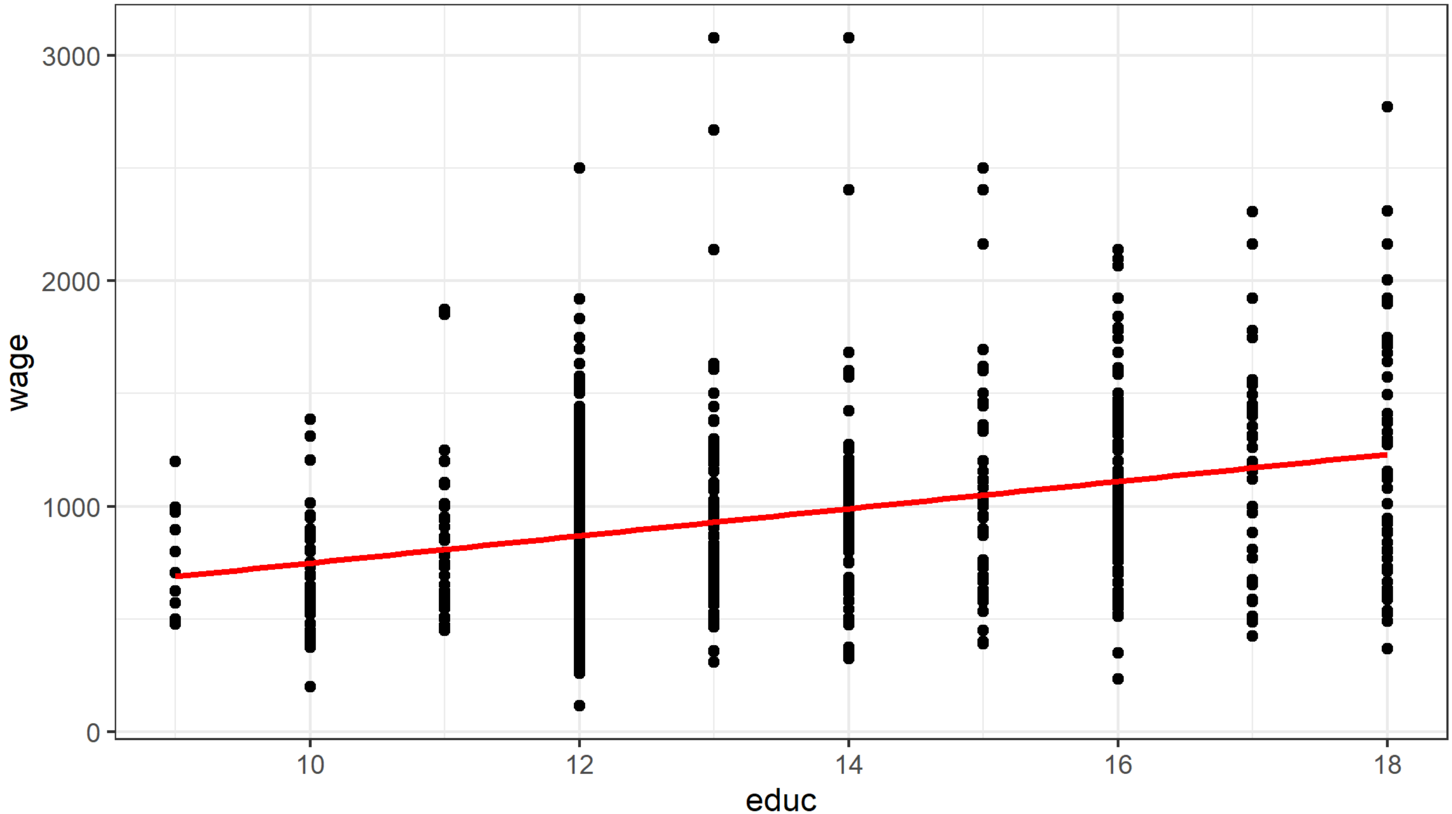
$$y_i = \beta_0 + \beta_1 x_1 + \underbrace{\beta_1(x_1^* - x_1) + \beta_2 x_{omitted} + f(nonlinears) + \tilde{u}_i}_{\text{other things, u}}$$

Our estimator, $\hat{\beta}$ assumes allllll these things are 0 in expectation, no matter the value of x



Wooldridge Fig. 2.4

```
## `geom_smooth()` using formula 'y ~ x'
```



Regression line for wage2 data

It will always be the case that, for any estimates β from a sample:

- $\sum_{i=1}^{N}(\hat{u}_i) = 0$
- $\sum_{i=1}^{N}(x_i \hat{u}_i) = 0$
- The point $(\bar{y}, \bar{x})$ is always on the regression line

In a mathematical sense, we can always calculate a β such that $\bar{u} = 0$ for all values of x.

But what might throw us off is if there is something else unobserved, w, that is "in the error term".

- What is "in the error term?"
  - Everything in the world that isn't $educ$

For our example, let's think about $a = ability$.

- Since $ability$ is unobserved, it is "in the error term"
- $ability$ is also pretty correlated with $educ$ (high ability people go to college)
- In a way, we're attributing the causal effect of $ability$ to $educ$
- So, $E[u|educ = high] > 0$
  - The unobserved error term, u, is higher due to the $ability$ part

For now, we will work as if our assumption $E[U|X] = 0$ is true.

Putting the "Least Squares" in OLS

The "squares" part refers to the squaring of the error term.

The "least" part refers to a minimzation of the (squared) error term.

Let's define the **sum of squared residuals** as:

$$SSR = \sum_{i=1}^{N} \hat{u}_i^2 = \sum_{i=1}^{N} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

And β is the "Least Squares" estimate if it minimizes SSR. How?

Take the derivative and set it equal to zero:

$$\frac{\partial SSR}{\partial \hat{\beta}_0} = 2 \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

and

$$\frac{\partial SSR}{\partial \hat{\beta}_1} = 2 \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0$$

SSR, SSE, and SST

We know that β_{OLS} minimizes the sum of squares. How do we measure how good of a fit we get?

Define two more in addition to SSR:

- Sum of Squares Total: $SST = \sum_{i=1}^{N} (y_i - \bar{y})^2$
    - SST is a total sum of squares (notice no hats).
- Sum of Squares Explained: $SSE = \sum_{i=1}^{N} (\hat{y}_i - \bar{y})^2$
    - SSE can be thought of as how much is *explained* by $\hat{y}_i$, *...relative to just guessing the obvious:* $\bar{y}$

Goodness of fit

$$SST = SSR + SSE$$

The total variance is the sum of the variance of the residuals (what isn't explained by your model) and the SSE (the variance that is explained).

This is a *decomposition* of variance.

The R^2

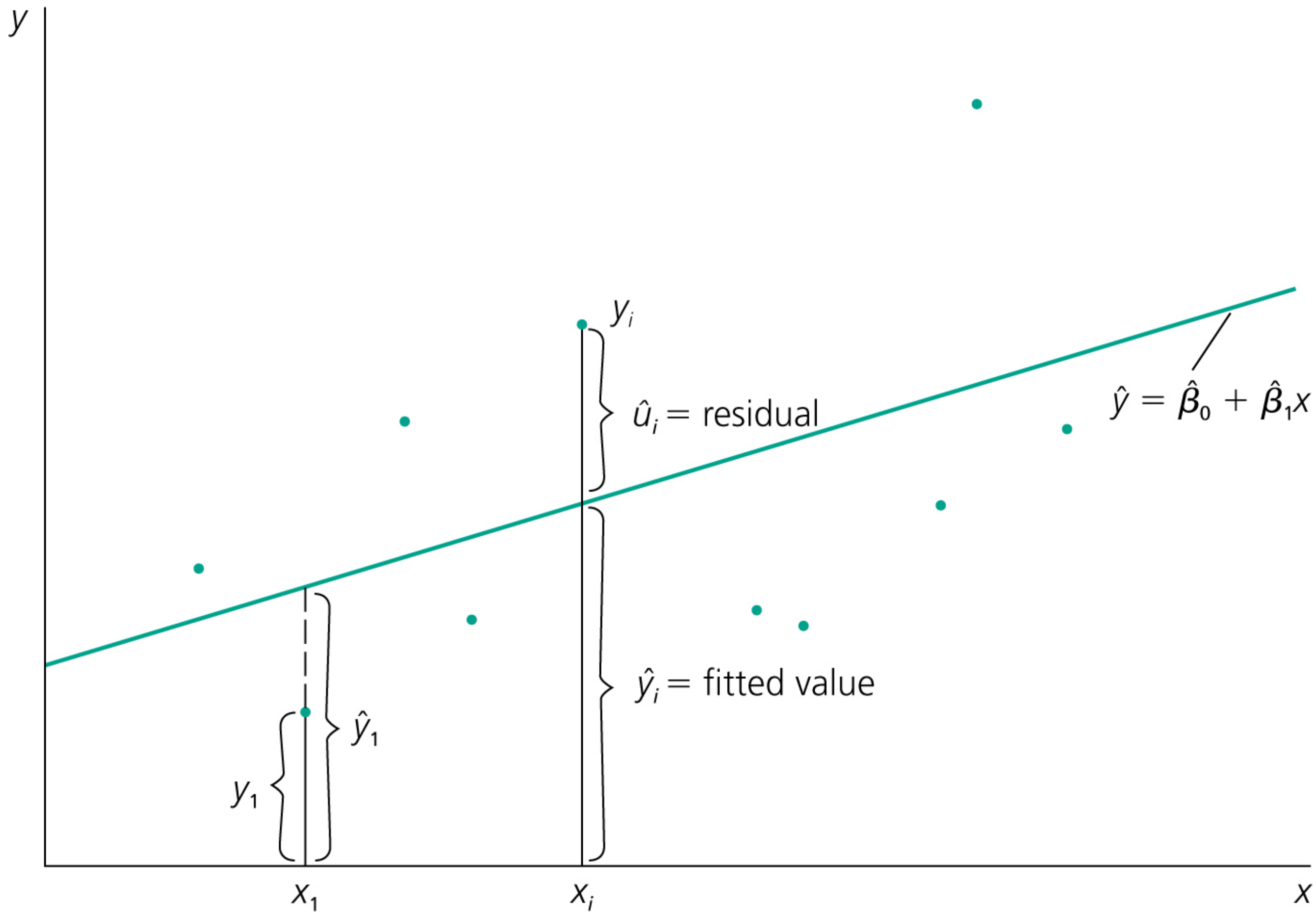
R^2 ('r-square') is the comparison of SSE to SST. Since $SSE < SST$ always, and both are always positive, $0 < R^2 \leq 1$

$$R^2 = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$$

The R^2 is often interpreted as the "fraction of variance explained by the model"

- Your regression, the SRF, is a model
- The variance being explained is the variance in the outcome, y.

From earlier:



Wooldridge Fig. 2.4

Goodness of fit

```r
# Ynum is the column name for the outcome variable
# X is the column name for the independent variable and ex is the name of the
summary(lm(Ynum ~ X, data=ex))
```

```
##
## Call:
## lm(formula = Ynum ~ X, data = ex)
##
## Residuals:
##       1       2       3       4       5
## -0.5046  0.5688 -0.8195  0.3760  0.3793
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -0.6530     0.6501  -1.004  0.38918
## X             1.9591     0.2358   8.308  0.00365 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7153 on 3 degrees of freedom
## Multiple R-squared:  0.9583,    Adjusted R-squared:  0.9445
## F-statistic: 69.02 on 1 and 3 DF,  p-value: 0.003654
```

Terminology

$$y = \beta_0 + \beta_1 x + u$$

y is called

- The dependent variable (DV)
- The "left hand side" (LHS)
- The outcome variable
- The response variable

x is called

- The independent variable
- The "right hand side" (RHS)
- The explanatory variable
- The control variable
- A covariate or a regressor

u is called

- The residual (when $\hat{u}$)
- The error term (when u)

On to transformations and functional forms!