Multivariate Regression: Part V - Interactions and Interpretations

EC420

Justin Kirkpatrick

Last updated February 24, 2021

This lecture

Goal:

1. Answer any questions on last lectures (inference and fixed effects)
2. Fixed Effects with Multiple Groups
   - Interpretation
   - Construction
   - "Within-group variation" and partialling out
   - Time fixed effects
3. Interactions w/dummies
   - Functional form
   - Interpretation
   - Interpretation
   - Interpretation!

Questions from last week?

Fixed Effects

MICHIGAN STATE UNIVERSITY

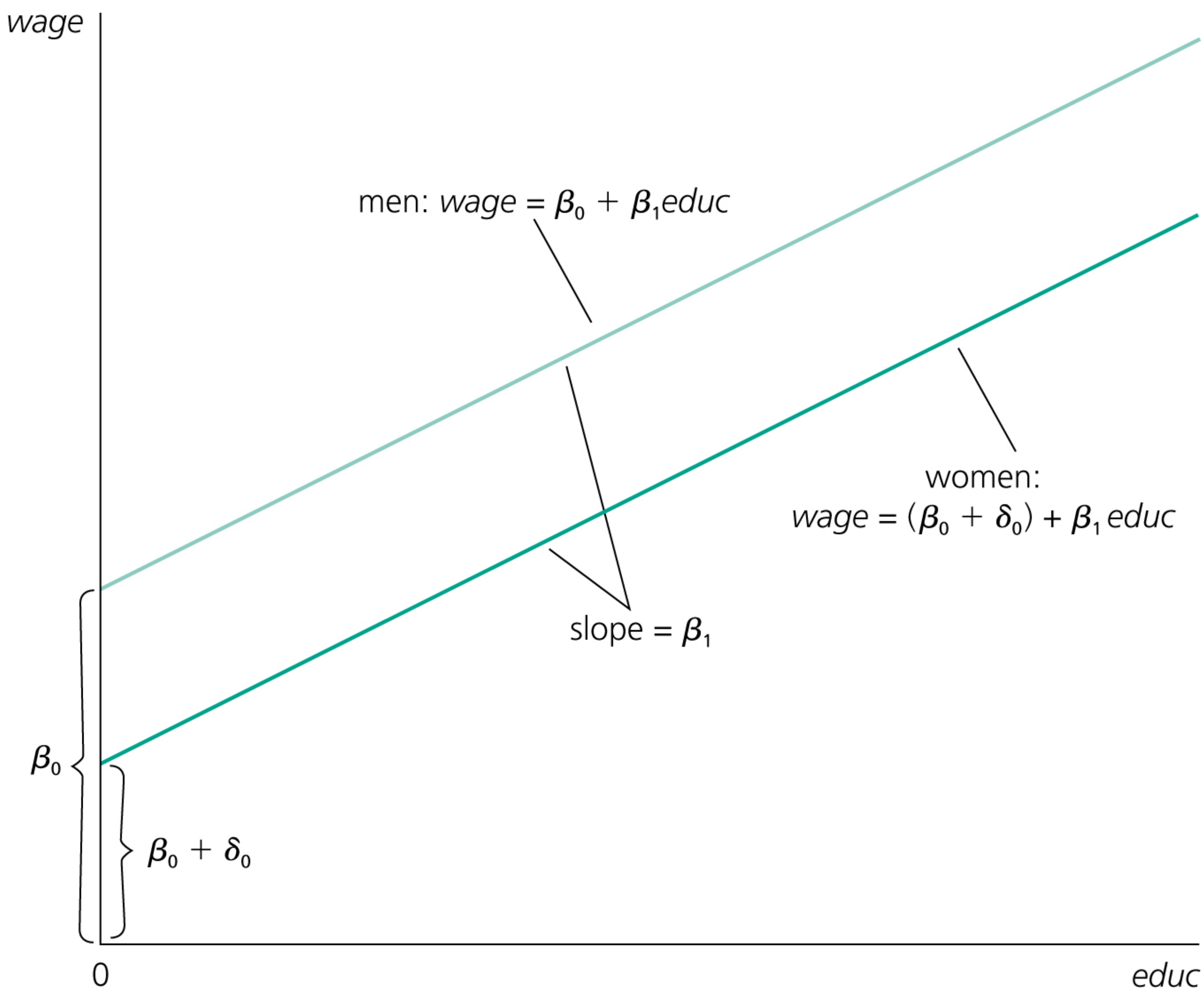
Last class, we covered a binary variable fixed effect

- One variable taking the value of $\{0, 1\}$. For example
  - $1(female)$
  - $1(age > 65)$

Interpretation

- A shift in the intercept **relative to the base group**
  - Base group: $male$
  - Base group: $age < 65$
- The "base group" (or "base level") is represented in the intercept: β_0
- The other group(s) ($female$, $age > 65$) have shifted intercepts:
  - β_0 for the base (male, under 65)
  - $\beta_0 + \beta_1$ for a female under 65
  - $\beta_0 + \beta_1 + \beta_2$ for a female over 65
  - $\beta_0 + \beta_2$ for a male over 65

$$y = \beta_0 + \beta_1 1(female) + \beta_2 1(age > 65) + u$$



wage
men: $wage = \beta_0 + \beta_1 educ$
women:
$wage = (\beta_0 + \delta_0) + \beta_1 educ$
slope = β_1
β_0
$\beta_0 + \delta_0$
0
educ

Ceteris Paribus still applies

Interpretation of the dummy variable fixed effect coefficient is:

> "The change in the expectation of Y from being in the group relative to being in the base group, *ceteris paribus*"

We are using "in the group" here to mean "the observations for which the dummy is true"

The "base level" is very important

- Since the in-group intercept is $E[Y|in-group]=\beta_0+\beta_1$, but the coefficient is β_1, we have to be careful.
- The coefficient is the *difference* between the base level and the in-group.
- The "base" group is sometimes called **the omitted level**

Interpretation of $\{0, 1\}$ dummy variables

In the Wooldridge example

$$wage = \beta_0 + \beta_1 1(female) + \beta_2 educ + u$$

> "Conditional on education, females make on average β_1 more/less than males, ceteris paribus"

More/less depending on whether or not the coefficient is negative

MICHIGAN STATE UNIVERSITY

In the age example:

$$Out - of - pocket = \beta_0 + \beta_1 age + \beta_2 1(age > 65) + u$$

> "Individuals over 65 years of age pay β_2 more/less in out-of-pocket expenses relative to those under 65, controlling for the linear effect of age, ceteris paribus"

Here, we have to be a little more specific since the fixed effect and the continuous variable, age, both refer to age. It would be strange to say "conditional on age, being 65 means paying β_1 more/less".

We can have more than one fixed effect:

$$wage = \beta_0 + \beta_1 1(female) + \beta_2 1(age > 65) + u$$

- $E[wage|male, under65]$ = β_0
- $E[wage|female, under65]$ = ??
- $E[wage|female, over65]$ = ??

Fixed effects with multiple groups

What if we have three groups?

Take *education* as an example - we can "bin" education into:

High School or less		2- or 4-year college degree		Graduate degree
"HS"		"College"		"Graduate"

When this is represented with one variable, it's called a **categorical** variable

Our three groups would work as follows:

wage	experience	educ	education
9000	0	12	HS
20000	5	16	College
60000	12	14	College
27000	2	18	Graduate
32000	10	9	HS

In the US, primary (required) education is 12 years, undergraduate is 4 additional years, and graduate school is 2-5+ additional years.

Base level with categorical variable

- There is still a "base level" (or "omitted level")
- It is *your* choice as to which one is the "base level"
    - Coefficient estimates will still add up the same.
    - Interpretability is easier if you choose wisely
    - We should choose "HS" as the "base level" here, so that estimates are relative to HS
    - This is incorporating *ordinal* information since we think $wage_{grad} > wage_{college} > wage_{HS}$

Numeric representation

- To represent a categorical variable with 3 categories, we need to create **two** more columns
    - If there are K categories, then we need $K-1$ new columns
    - Whichever one we don't create a column for is the "base"
    - It's effect will be found in the β_0 (the intercept)

Using "HS" as the base level:

wage	experience	education	education==College	education==Graduate
9000	0	HS	0	0
20000	5	College	1	0
60000	12	College	1	0
27000	2	Graduate	0	1
32000	10	HS	0	0

If we run this in R (leaving out the "education" column), we would get a coefficient for $education == College$ and $education == Graduate$

In R, categorical variables are a special type of variable called "**factor**"

```
df$education = as.factor(df$education)
```

- R stores the labels separately, but will let you refer to them
- If we use `str(df)`, we can see the factor structure
- I'm going to switch to a dataset that has a categorical in it

```
census = wooldridge::census2000
str(census)
```

```
## 'data.frame':    29501 obs. of  6 variables:
##  $ state   : Factor w/ 51 levels "Alabama","Alaska",..: 41 39 11 29 3 5 38 27 14 19 ...
##  $ puma    : int  100 2502 1800 100 206 1601 1309 100 3301 1600 ...
##  $ educ    : int  13 13 12 13 16 12 13 13 16 16 ...
##  $ lweekinc: num  6.47 6.09 7.03 6.69 7.34 ...
##  $ exper   : int  37 14 21 12 18 15 29 14 22 26 ...
##  $ expersq : int  1369 196 441 144 324 225 841 196 484 676 ...
```

To go from a factor to a character string

```
census$state = as.character(census$state)
head(census)
```

```
##             state puma educ lweekinc exper expersq
## 1 South Carolina  100   13 6.471038    37    1369
## 2   Pennsylvania 2502   13 6.087648    14     196
## 3        Georgia 1800   12 7.034049    21     441
## 4         Nevada  100   13 6.694181    12     144
## 5        Arizona  206   16 7.338538    18     324
## 6     California 1601   12 6.422247    15     225
```

```
str(census)
```

```
## 'data.frame':    29501 obs. of  6 variables:
##  $ state   : chr  "South Carolina" "Pennsylvania" "Georgia" "Nevada" ...
##  $ puma    : int  100 2502 1800 100 206 1601 1309 100 3301 1600 ...
##  $ educ    : int  13 13 12 13 16 12 13 13 16 16 ...
##  $ lweekinc: num  6.47 6.09 7.03 6.69 7.34 ...
##  $ exper   : int  37 14 21 12 18 15 29 14 22 26 ...
##  $ expersq : int  1369 196 441 144 324 225 841 196 484 676 ...
```

More important, how to go from character string to factor

```
census$state = as.factor(census$state)
head(census)
```

```
##             state puma educ lweekinc exper expersq
## 1 South Carolina  100   13 6.471038    37    1369
## 2   Pennsylvania 2502   13 6.087648    14     196
## 3        Georgia 1800   12 7.034049    21     441
## 4         Nevada  100   13 6.694181    12     144
## 5        Arizona  206   16 7.338538    18     324
## 6     California 1601   12 6.422247    15     225
```

```
str(census)
```

```
## 'data.frame':    29501 obs. of  6 variables:
##  $ state   : Factor w/ 51 levels "Alabama","Alaska",..: 41 39 11 29 3 5 38 27 14 19 ...
##  $ puma    : int  100 2502 1800 100 206 1601 1309 100 3301 1600 ...
##  $ educ    : int  13 13 12 13 16 12 13 13 16 16 ...
##  $ lweekinc: num  6.47 6.09 7.03 6.69 7.34 ...
##  $ exper   : int  37 14 21 12 18 15 29 14 22 26 ...
##  $ expersq : int  1369 196 441 144 324 225 841 196 484 676 ...
```

If you use a factor variable in a regression, R will construct the additional columns

```
census.small = census[census$state=='South Carolina'|census$state=='Arizona'|
                      census$state=='Nevada',c('lweekinc','state','educ', 'exper')]
census.small$statefactor = as.factor(census.small$state)
head(census.small, 10)
```

```
##     lweekinc          state educ exper    statefactor
## 1   6.471038 South Carolina   13    37 South Carolina
## 4   6.694181         Nevada   13    12         Nevada
## 5   7.338538        Arizona   16    18        Arizona
## 17  7.129298        Arizona   12    44        Arizona
## 31  6.738426 South Carolina   12     8 South Carolina
## 40  6.827713 South Carolina   12    42 South Carolina
## 58  7.495542         Nevada   13    15         Nevada
## 81  6.357709        Arizona   12    25        Arizona
## 119 6.645391         Nevada   12    32         Nevada
## 152 5.860730        Arizona   13    27        Arizona
```

MICHIGAN STATE UNIVERSITY

How does R convert factors to data columns?

```
head(model.matrix(lweekinc ~ educ + exper + statefactor, data = census.small)) # we won
```

```
##    (Intercept) educ exper statefactorNevada statefactorSouth Carolina
## 1            1   13    37                 0                         1
## 4            1   13    12                 1                         0
## 5            1   16    18                 0                         0
## 17           1   12    44                 0                         0
## 31           1   12     8                 0                         1
## 40           1   12    42                 0                         1
```

Question: What is the base level?

Fixed Effects

```
summary(lm(lweekinc ~ educ + exper + statefactor, data = census.small))
```

```
##
## Call:
## lm(formula = lweekinc ~ educ + exper + statefactor, data = census.small)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.6167 -0.3284  0.0254  0.3749  3.2501
##
## Coefficients:
##                            Estimate Std. Error t value Pr(>|t|)
## (Intercept)                5.529636   0.177034  31.235  < 2e-16 ***
## educ                       0.070970   0.011758   6.036 2.14e-09 ***
## exper                      0.004664   0.001954   2.387   0.0172 *
## statefactorNevada          0.043311   0.054585   0.793   0.4277
## statefactorSouth Carolina -0.059640   0.044963  -1.326   0.1850
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6706 on 1119 degrees of freedom
## Multiple R-squared:  0.03657,    Adjusted R-squared:  0.03312
## F-statistic: 10.62 on 4 and 1119 DF,  p-value: 1.903e-08
```

A "within-group" interpretation

- Group fixed effects explain the *mean* of the y variable within that group
  - E.g. our Cuba/Colombia example on Monday
  - The intercept is just the difference in means (conditional on the other x's)

- The group fixed effect accounts for the averge difference *between* groups
  - And leaves the rest of the x's to explain the variation in y *within* the group

- If we think of "partialling out" the fixed effect, this makes even more sense.

Let's go to our wage/education/experience example. We might think there is a "gender experience gap" where men tend to be more experienced (e.g. due to not giving birth):

$$wage = \beta_0 + \beta_1 1(female) + \beta_2 experience + u$$

Partial out the fixed effect:

$$experience = \delta_0 + \delta_1 1(female) + v$$

$\hat{v}$ is $experience$ that isn't associated with being female. It has had the "gender experience gap" *removed*.

That is, the variation in $\hat{v}$ does not reflect the "male experience gap", so we are *identifying* β_2 off of variation within the group, eliminating variation between $male$ and $female$.

So, a regression using $\hat{v}$ in place of $experience$:

$$wage = \beta_0 + \beta_2 \hat{v}$$

Gives us the correct β_2 (remember our "partialling out" of x_1, x_2) using the "within group" variation in $experience$.

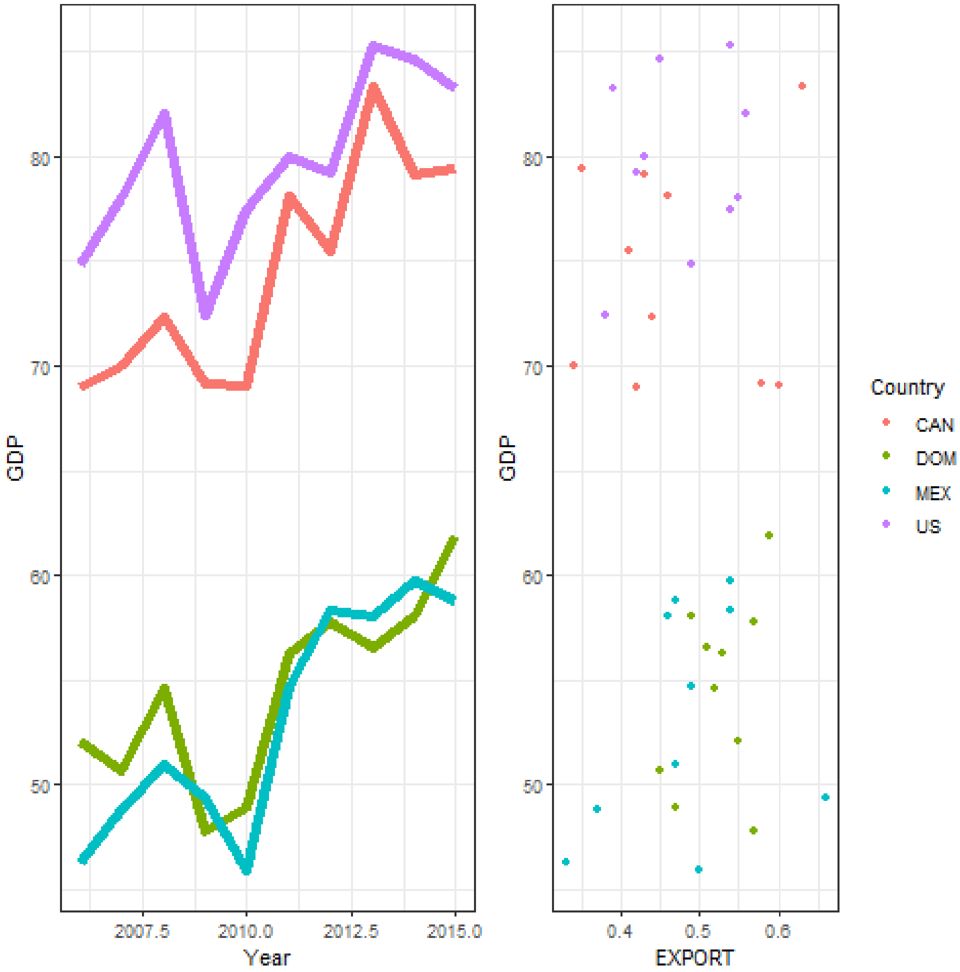
Time fixed effects

What if we have N observations and T time periods (a common type of panel data), but instead of worrying about group-level differences giving us biased estimates, we worried that some time trend or time-specific shock is making one time period different from the others?

Here, let's look at (entirely fake) data on North American GDP and EXPORT (share of GDP from exports).

We want to know if higher EXPORTS are associated with higher GDP.

Fixed Effects

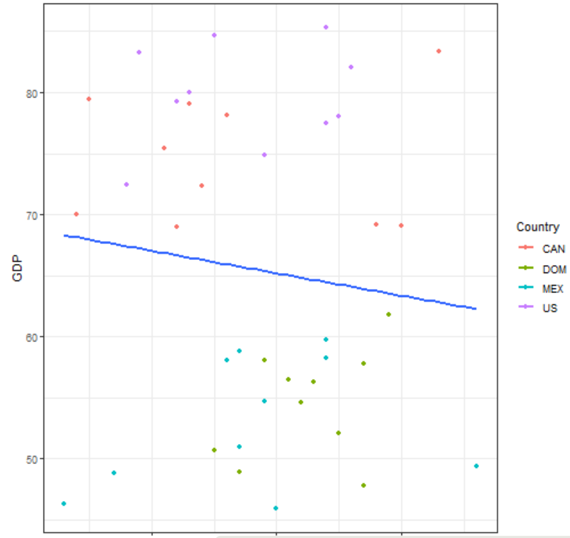


Since this is constructed (fake) data, I know the right coefficient on $EXPORT$, $\beta_{export} = 20$

Fixed Effects

```r
coeftest(lm(GDP ~ EXPORT, df), vcov = vcovHC, type = 'HC1')
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value  Pr(>|t|)
## (Intercept)   74.415     13.843  5.3757 4.085e-06 ***
## EXPORT       -18.417     27.902 -0.6601    0.5132
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



Note I used a simplier call to `coeftest`. Before we had `vcov = vcovHC(OLSobject, 'HC1'))`, but that required two steps: one to create the OLS object, and one to call `coeftest`. This does it all at once.

```r
coeftest(lm(GDP ~ EXPORT + as.factor(Year), df), vcov = vcovHC, type='HC1')
```

```
##
## t test of coefficients:
##
##                     Estimate Std. Error t value  Pr(>|t|)
## (Intercept)         69.69517   16.29177  4.2779 0.0001872 ***
## EXPORT             -20.42546   30.00917 -0.6806 0.5014976
## as.factor(Year)2007  0.91621   10.83901  0.0845 0.9332170
## as.factor(Year)2008  5.43941   10.38025  0.5240 0.6042500
## as.factor(Year)2009  1.16664    9.38704  0.1243 0.9019494
## as.factor(Year)2010  1.41325   11.04306  0.1280 0.8990514
## as.factor(Year)2011  7.32123    9.72125  0.7531 0.4574514
## as.factor(Year)2012  7.90148    8.67860  0.9105 0.3700877
## as.factor(Year)2013 12.03960   11.23519  1.0716 0.2927386
## as.factor(Year)2014 10.44856    9.64676  1.0831 0.2876807
## as.factor(Year)2015 10.34100    9.05938  1.1415 0.2630128
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
# note - you can use "as.factor" in the ~ formula
```

Fixed Effects

```
coeftest(lm(GDP ~ EXPORT + as.factor(Year) + as.factor(Country), df), vcov = vcovHC, typ
```

```
##
## t test of coefficients:
##
##                         Estimate Std. Error  t value  Pr(>|t|)
## (Intercept)             61.51985    1.32211  46.5317 < 2.2e-16 ***
## EXPORT                  18.99747    3.33745   5.6922 5.466e-06 ***
## as.factor(Year)2007      1.70466    0.59633   2.8586 0.0082725 **
## as.factor(Year)2008      3.46827    0.92018   3.7691 0.0008512 ***
## as.factor(Year)2009     -2.77565    0.93890  -2.9563 0.0065428 **
## as.factor(Year)2010     -1.74059    1.30822  -1.3305 0.1949056
## as.factor(Year)2011      6.13854    0.85611   7.1702 1.293e-07 ***
## as.factor(Year)2012      6.42312    0.97667   6.5766 5.659e-07 ***
## as.factor(Year)2013      8.59009    1.07620   7.9819 1.846e-08 ***
## as.factor(Year)2014      9.26587    0.65546  14.1365 1.023e-13 ***
## as.factor(Year)2015     10.24245    0.70164  14.5978 4.860e-14 ***
## as.factor(Country)DOM -21.18461    0.68779 -30.8008 < 2.2e-16 ***
## as.factor(Country)MEX -21.73305    0.58530 -37.1312 < 2.2e-16 ***
## as.factor(Country)US    5.05193    0.66545   7.5917 4.656e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Yes, you can specify more than one set of categorical variables

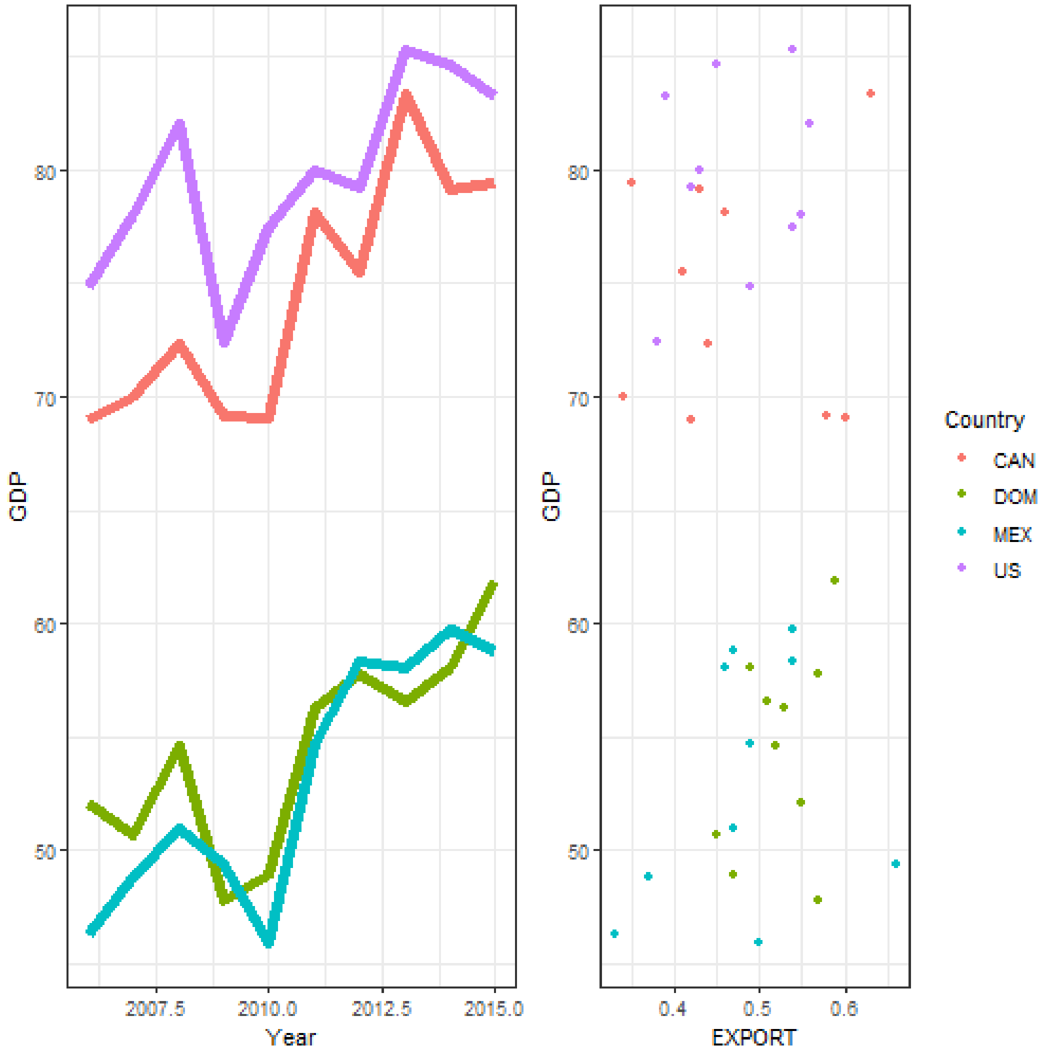
- Just as you can have more than one dummy variable
- The interpretation of each one is still the same: the effect of being in the group/time period relative to the base group/time period, *ceteris paribus*.
- These are called **two-way fixed effects** (TWFE)
  - When used on panel data
  - And when there is one fixed effect for each of the panel data's dimensions
  - N countries and T years here.

Fixed effects and Partialling Out

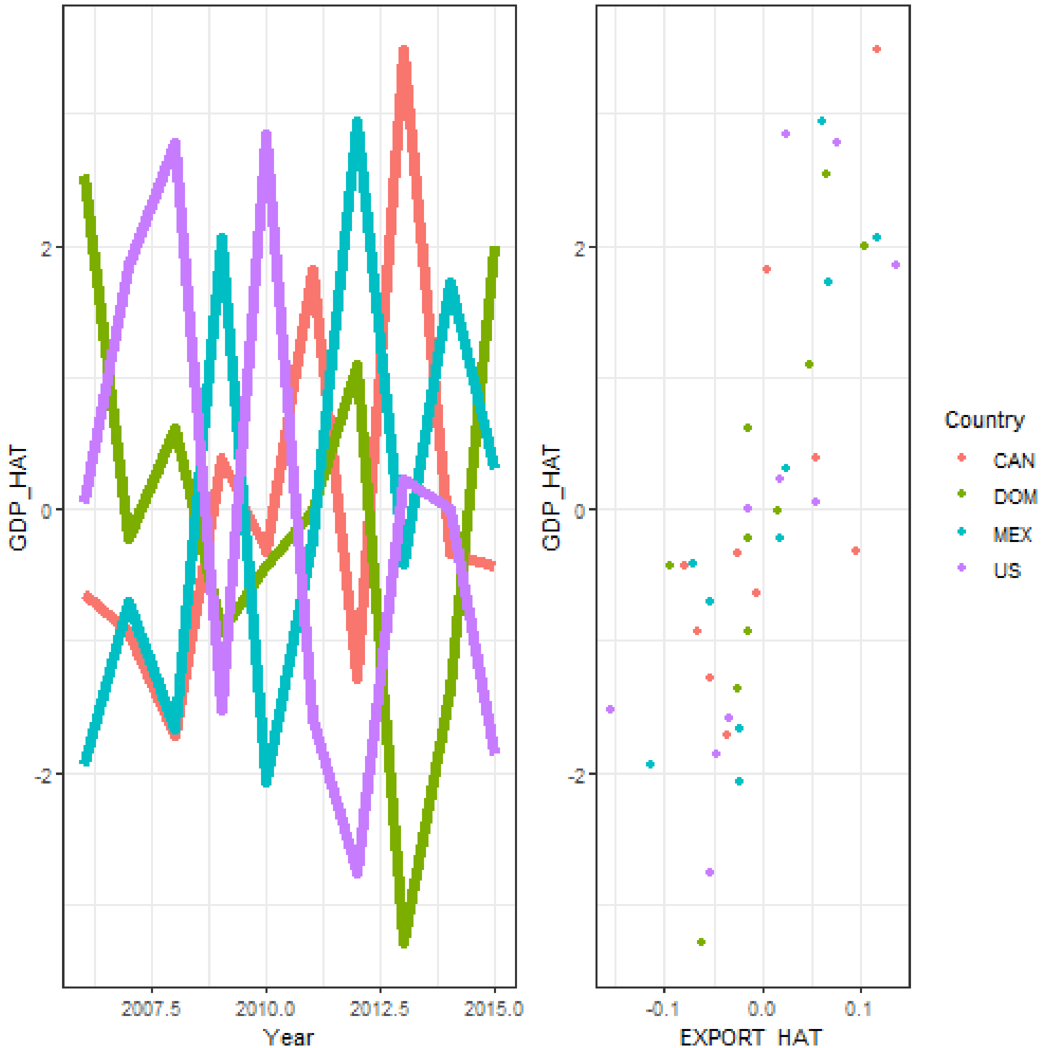
On the next slide, GDP_HAT is the residual from regressing GDP on categorical Year and Country

- Same for EXPORT_HAT

Our original data before partialling out:



After paritalling out left: YEAR and right: YEAR and COUNTRY



MICHIGAN STATE UNIVERSITY

The code I used to parital out and plot the prior slide:

```
df = df %>%
  dplyr::mutate(EXPORT_HAT = resid(lm(EXPORT ~ as.factor(Year) + as.factor(Country), data=df)
                GDP_HAT = resid(lm(GDP ~ as.factor(Year) + as.factor(Country), data=df)))


c = ggplot(df, aes(x = Year, y=GDP_HAT, col=Country)) + geom_line(lwd=2) +
theme_bw() + theme(legend.position='none')

d = ggplot(df, aes(x = EXPORT_HAT, y = GDP_HAT, col=Country)) + geom_point() + theme_bw()

plot_grid(c,d)
```

Interactions with Dummies

Dummy variables shift *the intercepts*

- Very useful when a group (or time) has a different mean
- Covers *"unobserved, time-invariant differences"*

But what if we think that the *slopes* differ

- For instance, maybe each country in our GDP/EXPORT example has *it's own unique relationship* between GDP and $EXPORT$?
- This can be *in addition* to thinking that each country has its own unique intercept
  - In fact, it would be odd to think that they'd have their own unique slope but *not* a unique intercept.

How do we let the slopes vary?

- In a way very similar to letting the intercepts vary
- Let's look at it in an example with only two categories (a single dummy)

$$y = \beta_0 + \beta_1 1(condition) + \beta_2 x_1 + \underbrace{\beta_3 \times x_1 \times 1(condition)}_{\text{The interaction term}} + u$$

A couple things to note:

- x_1 is our variable of interest here
- $condition$ is our group dummy (like $male$ or $age > 65$)
- x_1 appears twice, once with β_2, and *again* in the interaction of $x_1 \times 1(condition)$

$$y = \beta_0 + \beta_1 1(condition) + \beta_2 x_1 + \underbrace{\beta_3 x_1 1(condition)}_{\text{The interaction term}} + u$$

Refreshing our interpretation of the intercept:

- The intercept for the base group is β_0
- The intercept for the in-group defined by $condition$ is $\beta_0 + \beta_1$

Applying the same thought process to the interaction:

- **For the base group**, the marginal change in y from a unit increase in x_1 is β_2
- **For the in-group**, the marginal change in y from a unit increase in x_1 is $\beta_2 + \beta_3$

$$\text{For the base group: } \frac{\Delta y}{\Delta x_1} = \beta_2$$

$$\text{For the in-group: } \frac{\Delta y}{\Delta x_1} = \beta_2 + \beta_3$$

Of course, we can have >2 groups (categorical)

$$
\begin{aligned}
y = \beta_0 &+ \beta_1 1(group == 2) + \beta_3 1(group == 3) + \beta_4 x_1 \\
&+ \beta_5 x_1 1(group == 2) + \beta_6 x_1 1(group == 3) + u
\end{aligned}
$$

What does that look like?

wage	experience	educ	educ = College	educ = Graduate	experience x educ == College	experience x educ == Graduate
9000	0	HS	0	0	0	0
20000	5	College	1	0	5	0
60000	12	College	1	0	12	0
27000	2	Graduate	0	1	0	2
32000	10	HS	0	0	0	0

And in R:

```
lm(wage ~ as.factor(educ) + exper + as.factor(educ)*exper,
data=df)
```

Here, you'll get **intercept** **shift** coefficients on:

- educ = College
- educ = Grad

And you'll get **slope** **shift** coefficients on:

- experience for educ = College
- experience for educ = Graduate

The wage/education/experience regression would be:

$$wage = \beta_0 + \beta_1 1(educ == College) + \beta_2 1(educ == Grad) + \beta_3 exper$$
$$+ \beta_4 x_1 1(educ == Coll) + \beta_5 x_1 1(educ == Grad) + u$$

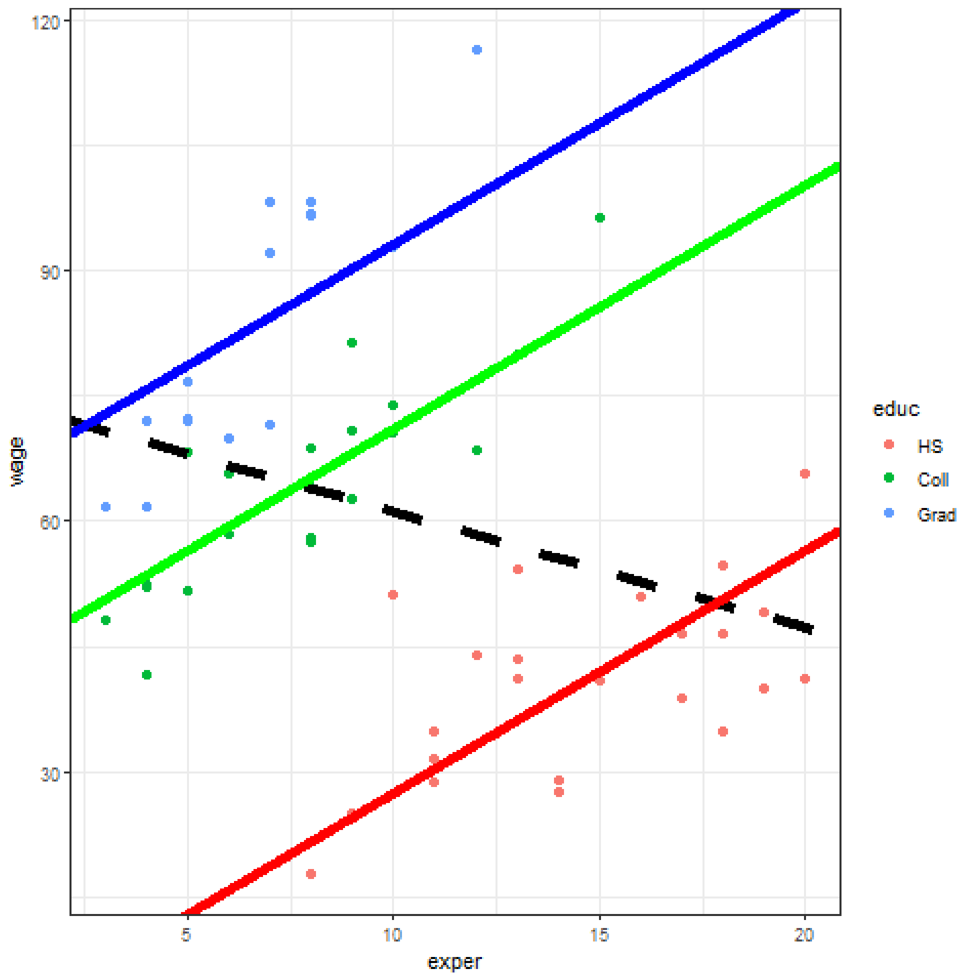
Expected Values conditional on X:

- $E[wage|exper, educ = HS] = \beta_0 + \beta_3 \times exper$
- $E[wage|exper, educ = Coll] = (\beta_0 + \beta_1) + (\beta_3 + \beta_4) \times exper$
- $E[wage|exper, educ = Grad] = (\beta_0 + \beta_2) + (\beta_3 + \beta_5) \times exper$

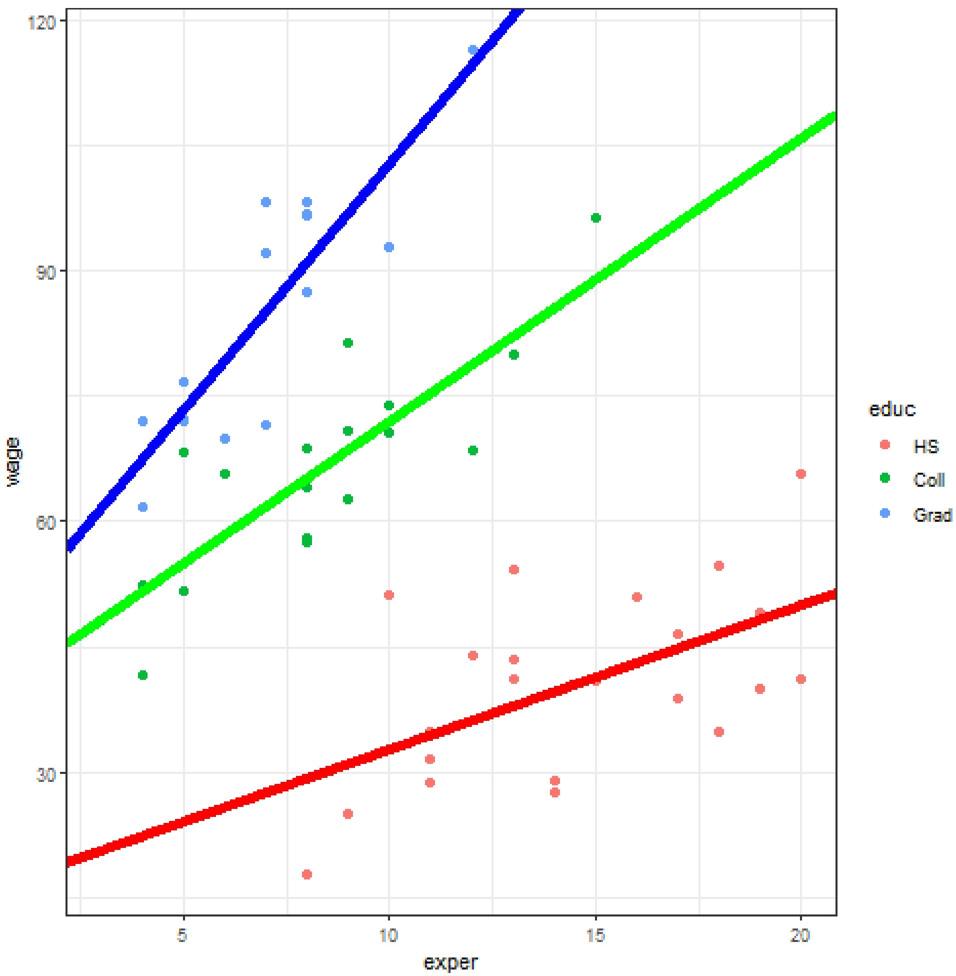
Just as we do with the intercepts, we add to the base level

- Note that when we have three categories $\{HS, Coll, Grad\}$ and we want the $E[wage|exper, educ == Grad]$, we do **not** add in the intercept-shift or slope-shift for $educ == Coll$.

The naive pooled (black) and the intercept-shift only:



And letting *intercept* and *slope* vary:



Interactions

```
##
## t test of coefficients:
##
##                 Estimate Std. Error t value  Pr(>|t|)
## (Intercept)    15.52086    9.33517  1.6626  0.102183
## educColl       22.58345   10.14406  2.2263  0.030189 *
## educGrad       28.32628    9.97941  2.8385  0.006375 **
## exper           1.72848    0.61230  2.8229  0.006649 **
## educColl:exper  1.66493    0.77769  2.1409  0.036816 *
## educGrad:exper  4.17432    0.81558  5.1182 4.211e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The true slopes (since this is fictional data) are:

educ	**Slope**
HS	1
Coll	4
Grad	5

How would we say this?

> β_4 is the college-specific increase in the relationship between per-year-of-experience and wages relative to HS graduates

- We can also just think of it in terms of slope: a positive β_4 means the slope is steeper (more up) than HS

Significance

- The statistical test that is output in these regressions refers to whether or not that coefficient is zero
- For a intercept-shift (β_1 or β_2), the test tells us whether or not the *intercept* (or *mean*) outcome of the in-group is different from the base level.
- For a slope-shift (interaction, e.g. β_3 or β_4), the test tells us whether or not the *slope* is different of the in-group is different from the base level.
    - That is, it asks: "does this group have a *different relationship between exper and wage* than the base group?"

Two-dummy interactions:

$$\begin{aligned} Out - of - pocket = \beta_0 + &\beta_1 1(single) + \beta_2 1(age > 65) \\ &+ \beta_3 1(single) 1(age > 65) + u \end{aligned}$$

We have the same interpretation for β_0 thru β_2

- **But** β_3 tells us the $E[Out - of - pocket|\text{both things true}]$

This means a single person over 65 adds *four* beta's together:

- $E[O - o - p|\text{married, 64 years old}] = \beta_0$
- $E[O - o - p|\text{single, 64 years old}] = \beta_0 + \beta_1$
- $E[O - o - p|\text{married, 66 years old}] = \beta_0 + \beta_2$
- $E[O - o - p|\text{single, 66 years old}] = \beta_0 + \beta_1 + \beta_2 + \beta_3$

This is because a single person over 65 is all four things at once. β_3 is interpreted as the additional effect of being *both* >65 and single.