Exercise 42: (Black-Scholes option pricing formula - European put option)

Let $X \sim \mathrm{N}(0,1)$. Then,

$$\begin{aligned}
\mathrm{E}\left[e^{-rT}(K-S_T)^+\right] &= \mathrm{E}\left[e^{-rT}(K - S_0\exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}X\Big))^+\right] \\
&= \int_{-\infty}^{+\infty} e^{-rT}(K - S_0\exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\Big))^+ \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx = (*).
\end{aligned}$$

Now observe that

$$\begin{aligned}
& K \geq S_0\exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\Big) \\
\iff & x \leq -\frac{\log\Big(\frac{S_0}{K}\Big)+\Big(r-\frac{\sigma^2}{2}\Big)T}{\sigma\sqrt{T}} = -(D_1-\sigma\sqrt{T}) = -D_1+\sigma\sqrt{T}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(*) &= \int_{-\infty}^{-D_1+\sigma\sqrt{T}} e^{-rT}(K - S_0\exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\Big))\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx \\
&= e^{-rT}K\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx \\
&\quad - e^{-rT}S_0\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\Big)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx.
\end{aligned}$$

We now look at the two integrals separately. We start with the first one:

$$\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx = \Phi(-D_1+\sigma\sqrt{T}).$$

Next we evaluate the second integral:

$$\begin{aligned}
&\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \exp\Big((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\Big)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)dx \\
&= \exp\Big((r-\frac{\sigma^2}{2})T\Big)\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\left(x^2-2\sigma\sqrt{T}x+\sigma^2T-\sigma^2T\right)\Big)dx \\
&= \exp\Big((r-\frac{\sigma^2}{2})T+\frac{\sigma^2 T}{2}\Big)\int_{-\infty}^{-D_1+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\left(x-\sigma\sqrt{T})\right)^2\Big)dx \\
&= e^{rT}\Phi(-D_1+\sigma\sqrt{T}-\sigma\sqrt{T}) \\
&= e^{rT}\Phi(-D_1).
\end{aligned}$$

Combining these results, gives

$$\mathrm{E}\left[e^{-rT}(K-S_T)^+\right] = Ke^{-rT}\Phi(-D_1+\sigma\sqrt{T}) - S_0\Phi(-D_1).$$

Exercise 43: (Variance reduction techniques for option pricing)

1. A Monte Carlo estimator for the European put option with maturity T and strike K is given by

$$V_0^{\mathrm{MC}}(n) = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\left(K - S_0\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}X_i\right)\right)^+,$$

where $X_1, \dots, X_n$ are i.i.d. from the $\mathcal{N}(0,1)$ distribution.

1. Since for $X \sim \mathcal{N}(0,1)$, $(X, -X)$ is an antithetic pair, an antithetic variates estimator for the time-0 price of a European put option is given by

$$\begin{aligned} V_0^{\mathrm{MC}}(n) = &\frac{1}{2n}\sum_{i=1}^{n} e^{-rT}\left(K - S_0\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}X_i\right)\right)^+ \\ &+ \frac{1}{2n}\sum_{i=1}^{n} e^{-rT}\left(K - S_0\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}(-X_i)\right)\right)^+, \end{aligned}$$

   where $X_1, \dots, X_n$ are i.i.d. from the $\mathcal{N}(0,1)$ distribution.

1. Let X_i be i.i.d. random variables from the standard normal distribution and let $S_i(T) = S_0\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}X_i\right)$ and $Y_i = e^{-rT}(K - S_i)^+$. Then a control variate estimator for the time-0 price of a European put option is given by

$$\overline{Y}_n(b) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(e^{-rT}S_i(T) - S(0))),$$

   where b would often be chosen to be $\hat{b}_n^*$ as defined in the lecture notes.

We will implement the different estimators in Python next.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

rng = np.random.default_rng(2468)
```

Analytical formula and standard Monte Carlo estimator for the European put price

First, we implement the analytical formula for the time-0 price of the European put in the Black-Scholes model and a classical Monte Carlo estimator.

In [2]:
```
def black_scholes_put(S0, K, r, T, sigma):
    d1 = (np.log(S0 / K) + (r + 0.5 * sigma**2) * T)/(sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    tmp1 = S0 * norm.cdf(-d1, loc=0, scale=1)
    tmp2 = K * np.exp(-r * T) * norm.cdf(-d2, loc=0, scale=1)
    price = tmp2 - tmp1
    return price

def terminal_stockprice(rng, S0, T, r, sigma, samplesize):
    mystandardnormalsample = rng.standard_normal(size=samplesize)
    tmp1 = (r - 0.5*(sigma ** 2)) * T
    tmp2 = sigma * np.sqrt(T) * mystandardnormalsample
```

```
    stockprice = S0 * np.exp(tmp1 + tmp2)
    return stockprice

def bs_put_mc(rng, S0, K, T, r, sigma, samplesize, myepsilon):
    # Generate terminal stock prices.
    mystockprice = terminal_stockprice(rng, S0, T, r, sigma, samplesize)
    # Compute payoffs.
    payoffs = np.maximum(K - mystockprice, 0)
    # Discount payoffs
    discountedpayoffs = np.exp(- r * T)*payoffs
    # Compute MC price
    price = np.mean(discountedpayoffs)
    # Compute confidence interval next
    standarddev_rv = np.std(discountedpayoffs, ddof=1)
    standarddev_mcest = standarddev_rv / np.sqrt(samplesize)
    aepsilon = norm.ppf(1.0 - myepsilon * 0.5)
    # Left boundary of CI
    ci_left = price - aepsilon * standarddev_mcest
    # Right boundary of CI
    ci_right = price + aepsilon * standarddev_mcest
    return price, standarddev_mcest, ci_left, ci_right
```

In [3]:
```
# Defining some model parameters
S0 = 50.0
K = 50.0
T = 0.25
r = 0.05
sigma = 0.3
halfsamplesize = 100000
samplesize = 2 * halfsamplesize
myepsilon = 0.05
```

In [4]:
```
print('----------------')
print('The analytical option price of the put is {:.4f}'.format(black_scholes_put(S0, K, r, T
print('-----------------')
MCresults = bs_put_mc(rng, S0, K, T, r, sigma, samplesize, myepsilon)
print('MC put price: {:.4f} and stdev of MC est: {:.4f}'.format(MCresults[0],MCresults[1]))
print('CI based on MC is ("{:.4f}, {:.4f})'.format(MCresults[2], MCresults[3]))
```

```
----------------
The analytical option price of the put is 2.6704
-----------------
MC put price: 2.6776 and stdev of MC est: 0.0085
CI based on MC is ("2.6610, 2.6943)
```

Antithetic variates estimator for the European put price

Second, we implement an antithetic variates estimator for the time-0 price of the European put.

In [5]:
```
def terminal_stockprice_av(rng, S0, T, r, sigma, halfsamplesize):
    """Function computes terminal stock prices based on antithetic pairs. """
    mynormals1 = rng.standard_normal(halfsamplesize)
    mynormals2 = - mynormals1
    tmp1 = (r - 0.5*sigma ** 2) * T
    tmp2 = sigma * np.sqrt(T) * mynormals1
    tmp3 = sigma * np.sqrt(T) * mynormals2
    stockprice1 = S0 * np.exp(tmp1 + tmp2)
    stockprice2 = S0 * np.exp(tmp1 + tmp3)
    allstockprices = np.concatenate((stockprice1, stockprice2))
    return stockprice1, stockprice2, allstockprices


def bs_put_av(rng, S0, K, T, r, sigma, halfsamplesize, myepsilon):
    """ Antithethic variate estimation for European call price in BS model."""
    # Note that 2*halfsamplesize random variables will be used in the AV estimator.
    # Generate terminal stock prices.
    mystockprices = terminal_stockprice_av(rng, S0, T, r, sigma, halfsamplesize)
```

```python
    # Compute payoffs.
    payoffs1 = np.maximum(K - mystockprices[0], 0)
    payoffs2 = np.maximum(K - mystockprices[1], 0)
    # Discount payoffs
    discpayoffs1 = np.exp(- r * T)*payoffs1
    discpayoffs2 = np.exp(- r * T)*payoffs2
    thecov = np.cov(discpayoffs1, discpayoffs2, ddof=1)[0, 1]
    possiblereduction = thecov / (2 * halfsamplesize)
    discpayoffs = np.concatenate((discpayoffs1, discpayoffs2))
    price = np.mean(discpayoffs)
    standarddev_rv = np.std(discpayoffs, ddof=1)
    standarddev_avest = standarddev_rv / np.sqrt(2 * halfsamplesize)
    aepsilon = norm.ppf(1.0 - myepsilon * 0.5)
    ci_left = price - aepsilon * standarddev_avest
    ci_right = price + aepsilon * standarddev_avest
    return price, standarddev_avest, ci_left, ci_right, possiblereduction

print('----------------')
AVresults = bs_put_av(rng, S0, K, T, r, sigma, halfsamplesize, myepsilon)
print('AV price: {:.4f} and stdev of AV est: {:.4f}'.format(AVresults[0], AVresults[1]))
print('CI based on AV is ({:.4f}, {:.4f})'.format(AVresults[2], AVresults[3]))
print('Note that sample covariance/(2n) is {:.8f}'.format(AVresults[4]))
```

```
----------------
AV price: 2.6594 and stdev of AV est: 0.0085
CI based on AV is (2.6428, 2.6760)
Note that sample covariance/(2n) is -0.00003536
```

Control variates estimator for the European put price

Third, we implement a control variate estimator for the time-0 price of the European put in the Black-Scholes model.

In [6]:

```python
def bs_put_cv(rng, S0, K, T, r, sigma, samplesize, myepsilon):
    """ Control variate estimation for European put price in BS model."""
    # Generate terminal stock prices.
    mystockprice = terminal_stockprice(rng, S0, T, r, sigma, samplesize)
    # Compute payoffs.
    payoffs = np.maximum(K - mystockprice, 0)
    # Discount payoffs
    discountedpayoffs = np.exp(- r * T)*payoffs
    # Use discounted stock as control
    xs = np.exp(- r * T) * mystockprice
    # Compute sample version of b*
    bstar = np.cov(xs, discountedpayoffs, ddof=1)[0, 1] / np.var(xs, ddof=1)
    # print("In cv bstar=", bstar)
    # Define z= Y(bstar)
    z = discountedpayoffs - bstar * (xs - S0)
    # Compute MC price
    price = np.mean(z)
    # Compute confidence interval next
    standarddev_rv = np.std(z, ddof=1)
    standarddev_cvest = standarddev_rv / np.sqrt(samplesize)
    aepsilon = norm.ppf(1.0 - myepsilon * 0.5)
    # Left boundary of CI
    ci_left = price - aepsilon * standarddev_cvest
    # Right boundary of CI
    ci_right = price + aepsilon * standarddev_cvest
    # Compute the sqared correation rhosquared
    tmpcov = np.cov(xs, discountedpayoffs, ddof=1)[0, 1]
    tmpvarx = np.var(xs, ddof=1)
    tmpvary = np.var(discountedpayoffs, ddof=1)
    rhosquared = (tmpcov ** 2) / (tmpvarx * tmpvary)
    return price, standarddev_cvest, ci_left, ci_right, rhosquared

print('----------------')
CVresults = bs_put_cv(rng, S0, K, T, r, sigma, samplesize, myepsilon)
```

```
print('CV price: {:.4f} and stdev of CV est: {:.4f}'.format(CVresults[0], CVresults[1]))
print('CI based on CV is ("{:.4f}, {:.4f})'.format(CVresults[2], CVresults[3]))
print('Note that rhosquared is {:.4f}'.format(CVresults[4]))
```

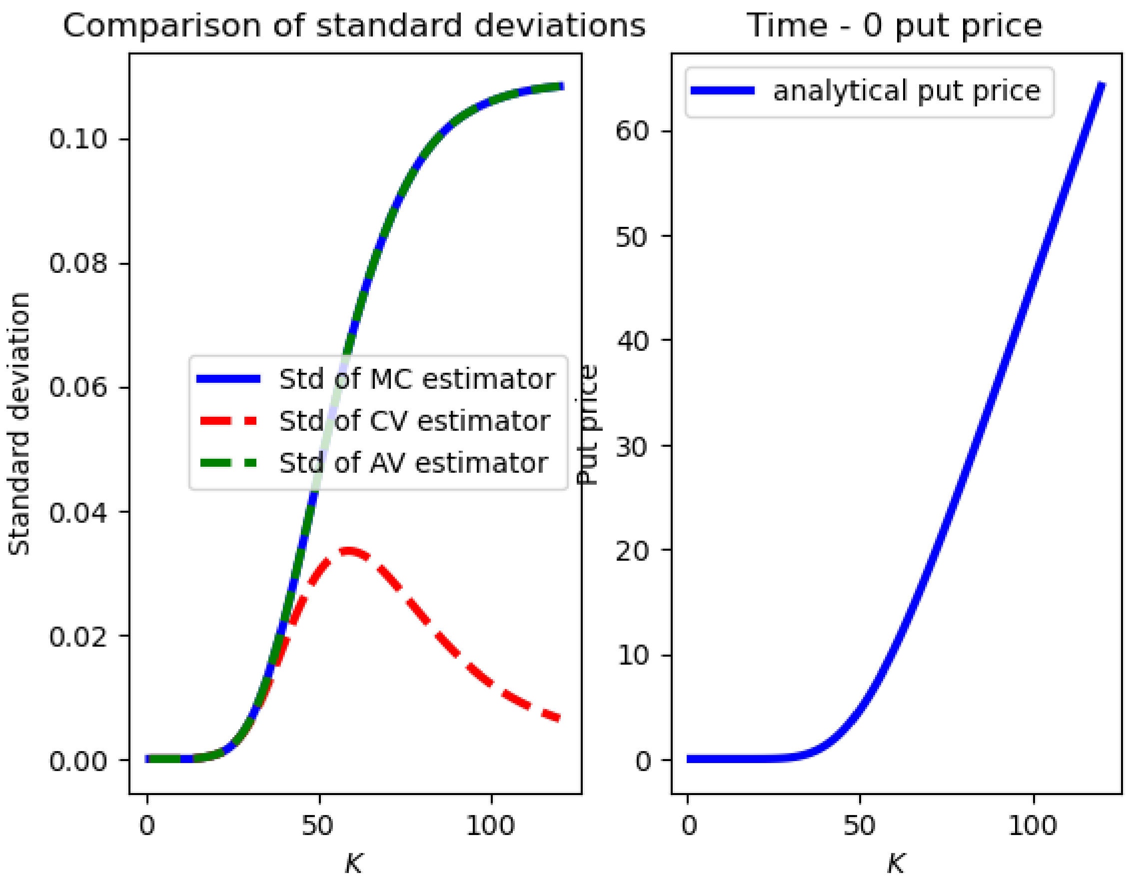
```
----------------
CV price: 2.6669 and stdev of CV est: 0.0050
CI based on CV is ("2.6572, 2.6766)
Note that rhosquared is 0.6570
```

Sensitivity of the variances of the estimators of the time-0 put option price with respect to the strike price

In [7]:

```
# Next we compare the standarddeviation of the MC, AV and the CV estimator
# Defining some model parameters
S0 = 50.0
sigma = 0.3
T = 1.0
r = 0.05
sigma = 0.3
halfsamplesize = 10000
samplesize = 2 * halfsamplesize
myepsilon = 0.05

numberofK = 100
Ks = np.linspace(start=1, stop=120, num=numberofK)

stdMC = np.zeros(numberofK)
stdCV = np.zeros(numberofK)
stdAV = np.zeros(numberofK)
analyticalprice = np.zeros(numberofK)

# Generate the terminal stock prices
mystockprice = terminal_stockprice(rng, S0, T, r, sigma, samplesize)
myavstockprices = terminal_stockprice_av(rng, S0, T, r, sigma, halfsamplesize)
myavstockprice = myavstockprices[2]

for i in range(numberofK):
    K = Ks[i]
    # Compute discounted payoffs
    discountedpayoffs = np.exp(- r * T) * np.maximum(K-mystockprice, 0)
    # Compute standard dev of MC estimator
    stdMC[i] = np.std(discountedpayoffs, ddof=1) / np.sqrt(samplesize)

    # Use discounted stock as control
    xs = np.exp(- r * T) * mystockprice
    # Compute sample version of b*
    bstar = np.cov(xs, discountedpayoffs, ddof=1)[0, 1] / np.var(xs, ddof=1)
    # Define z= Y(bstar)
    z = discountedpayoffs - bstar * (xs - S0)
    # Compute standard dev of CV estimator
    stdCV[i] = np.std(z, ddof=1) / np.sqrt(samplesize)

    #AV:
    mystockprice = myavstockprice
    discountedpayoffs = np.exp(- r * T) *  np.maximum(K-mystockprice, 0)
    # Compute standard dev of AV estimator
    stdAV[i] = np.std(discountedpayoffs, ddof=1) / np.sqrt(samplesize)

    # Compute the analytical put price for different strikes as well
    analyticalprice[i] = black_scholes_put(S0, K, r, T, sigma)


fig, (ax1, ax2) = plt.subplots(nrows=1, ncols=2)
ax1.plot(Ks, stdMC, lw=3, c="blue", label="Std of MC estimator")
ax1.plot(Ks, stdCV, lw=3, c="red", label="Std of CV estimator", linestyle = "dashed")
ax1.plot(Ks, stdAV, lw=3, c="green", label="Std of AV estimator", linestyle = "dashed")
ax1.set_ylabel("Standard deviation")
ax1.set_xlabel(r"$K$")
```

```python
ax1.set_title("Comparison of standard deviations")
ax1.legend(loc = "best")

ax2.plot(Ks, analyticalprice, lw=3, c="blue", label="analytical put price")
ax2.set_ylabel("Put price")
ax2.set_xlabel(r"$K$")
ax2.set_title("Time - 0 put price")
ax2.legend(loc = "best")
plt.show()
```



In [ ]:

In [ ]: