In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
import time
```

Exercise 44

Let

$$D_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad D_2 = D_1 - \sigma\sqrt{T-t}.$$

We show that

$$S_t\varphi(D_1) = Ke^{-r(T-t)}\varphi(D_2),$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$.

Proof: To see that the statement is true, consider

$$\begin{aligned}
\log\left(\frac{\varphi(D_1)}{\varphi(D_2)}\right) &= \log\left(\frac{\frac{e^{-D_1^2/2}}{\sqrt{2\pi}}}{\frac{e^{-D_2^2/2}}{\sqrt{2\pi}}}\right) = \log(\exp(\frac{1}{2}(D_2^2 - D_1^2))) \\
&= \frac{1}{2}(D_2^2 - D_1^2) = \frac{1}{2}(D_2 - D_1)(D_2 + D_1) \\
&= \frac{1}{2}(-\sigma\sqrt{T-t})(2D_1 - \sigma\sqrt{T-t}) \\
&= -D_1\sigma\sqrt{T-t} + \frac{\sigma^2(T-t)}{2} \\
&= -\log\left(\frac{S_t}{K}\right) - (r + \frac{\sigma^2}{2})(T-t) + \frac{\sigma^2(T-t)}{2} \\
&= \log\left(\frac{K}{S_t}\right) - r(T-t).
\end{aligned}$$

Then we take exponentials on both sides of the equation and obtain

$$\frac{\varphi(D_1)}{\varphi(D_2)} = \exp\left(\log\left(\frac{K}{S_t}\right) - r(T-t)\right) = \frac{K}{S_t}\exp(-r(T-t))$$

which is equivalent to

$$S_t\varphi(D_1) = Ke^{-r(T-t)}\varphi(D_2).$$

Exercise 45

In [2]:
```
#Function returning the price of European Call option in Black-Scholes model
def black_scholes_call(S, K, r, tau, sigma):
```

```
    d1 = (np.log(S / K) + (r + 0.5 * sigma**2) * tau)/(sigma * np.sqrt(tau))
    d2 = d1 - sigma * np.sqrt(tau)
    tmp1 = S * norm.cdf(d1, loc=0, scale=1)
    tmp2 = K * np.exp(-r * tau) * norm.cdf(d2, loc=0, scale=1)
    price = tmp1 - tmp2
    return price


myprice = black_scholes_call(S=50, K=50, r=0.05, tau=1.0, sigma=0.3)
print('Price of European call option is: {:.4f}'.format(myprice))
```

```
Price of European call option is: 7.1156
```

In [3]:

```
#Function computing implied volatilities using the Bisection method
def impliedvol_bisection(S, K, r, tau, marketprice, a, b, tolerance):

    f_a = black_scholes_call(S, K, r, tau, a) - marketprice
    f_b = black_scholes_call(S, K, r, tau, b) - marketprice

    if f_a*f_b >= 0:
        print("Choose new interval [a, b]!")
        return

    l_n = a
    r_n = b

    while (np.abs(r_n-l_n) > tolerance):

        f_a = black_scholes_call(S, K, r, tau, l_n) - marketprice
        f_b = black_scholes_call(S, K, r, tau, r_n) - marketprice

        y = (l_n + r_n) / 2
        f_y = black_scholes_call(S, K, r, tau, y) - marketprice

        if f_a * f_y < 0:
            r_n = y
        elif f_b * f_y < 0:
            l_n = y
        elif f_y == 0:
            return y
        else:
            print("Error in bisection method.")
    return ((r_n + l_n) / 2)

iv = impliedvol_bisection(S=50, K=50, r=0.05, tau=1.0, marketprice = 7.11562, a=0.1, b=0.6, t
print(iv)
```

```
0.29999960064888
```

In [4]:

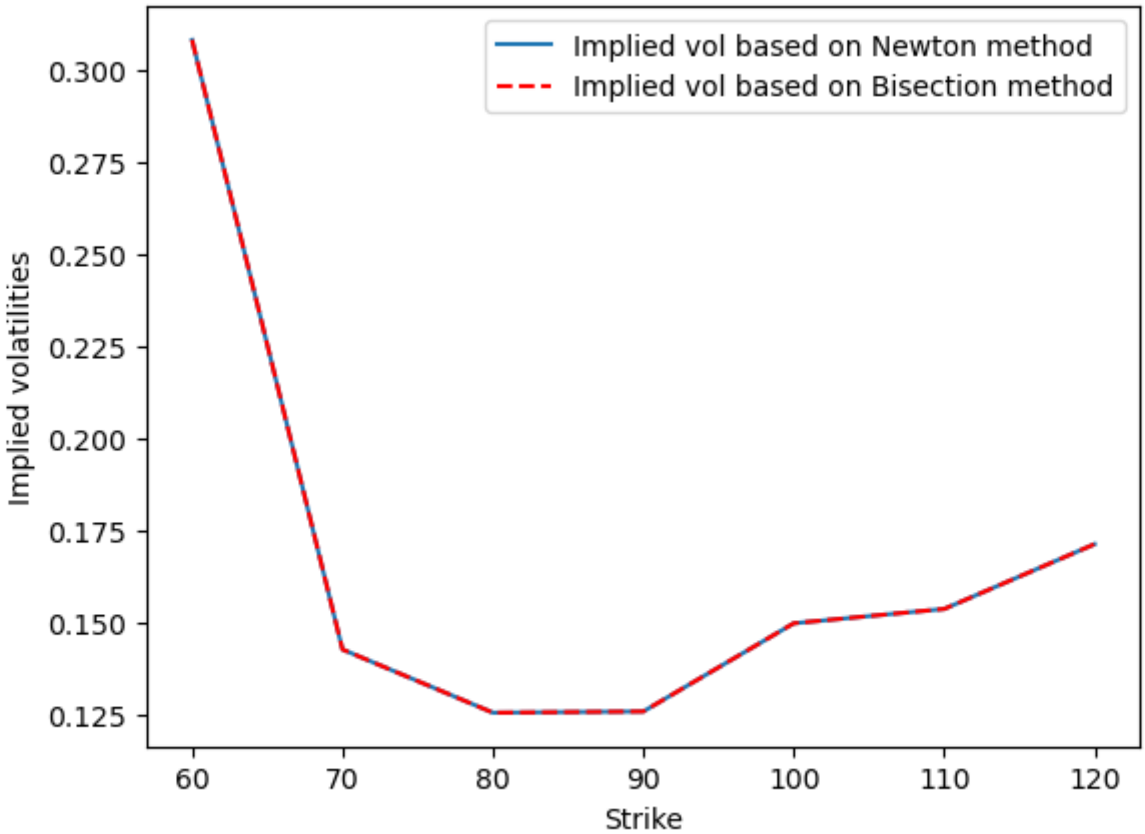
```
# Functions copied from the worksheet for Lecture 12 using the Newton method for comparison
def bs_vega(S, K, r, tau, sigma):
    d1 = (np.log(S / K) + (r + 0.5 * sigma**2) * tau)/(sigma * np.sqrt(tau))
    vega = S * np.sqrt(tau) * norm.pdf(d1, loc=0, scale=1)
    return vega

def find_impliedvol_Newton(S, K, r, tau, marketprice, initialguess):
    sigmaold = initialguess
    maxiteration = 100
    mydiff = 1
    for i in range(maxiteration):
            callprice = black_scholes_call(S, K, r, tau, sigmaold)
            vega = bs_vega(S, K, r, tau, sigmaold)
            if (vega < 0.000000001):
                return "Error in vega"
            else:
                sigmanew =  sigmaold - (callprice - marketprice) / vega
                mydiff = np.abs(sigmaold - sigmanew)
                sigmaold = sigmanew
```

```
            if (mydiff < 0.0000001):
                return sigmaold
        return "Max iteration reached"
```

In [5]:

```
# Example for computing implied volatility

teststrikes = np.array([60, 70, 80, 90, 100, 110, 120])
testprices = np.array([22, 11, 4, 1, 0.4, 0.1, 0.05])
S0=80
testimpliedvols1 = np.zeros(7)
testimpliedvols2 = np.zeros(7)

# Computing implied volatities using Newton's method
start = time.time()
for i in range(teststrikes.size):
    testimpliedvols1[i] = find_impliedvol_Newton(S=S0, K=teststrikes[i], r=0.0, tau=1.0, mark
runtime = time.time()-start
print('Computation time for Newton method is {:.8f}'.format(runtime))

# Computing implied volatities using Bisection method
start = time.time()
for i in range(teststrikes.size):
    testimpliedvols2[i] = impliedvol_bisection(S=S0, K=teststrikes[i], r=0.0, tau=1.0, market
runtime = time.time()-start
print('Computation time for Bisection method is {:.8f}'.format(runtime))

# Plotting implied volatilities
fig, ax = plt.subplots(nrows=1, ncols=1)
ax.plot(teststrikes, testimpliedvols1, label="Implied vol based on Newton method")
ax.plot(teststrikes, testimpliedvols2, color="red", linestyle="dashed", label="Implied vol ba
ax.set_xlabel("Strike")
ax.set_ylabel("Implied volatilities")
ax.legend(loc="upper right");
```

```
Computation time for Newton method is 0.01994276
Computation time for Bisection method is 0.16025805
```



Above we compared the computational time of both methods, and find that the Newton method is faster here, which is an advantage. One possible disadvantage of the Netwon method is that one needs to compute compute the derivative of the function, which is not required for the Bisection method.