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1 Axegine Optimal B/A Pricing Model

1.1 Problem (re) statement and analysis

We are making markets in a single risky asset having a lognormal price process

$$dP_t = P_t \left(\mu \, dt + \sigma \, dW_t \right).$$

Buy and sell orders are modeled as a Poisson counting process with densities λ_0^b and λ_0^s. Our quotes take the form

$$P_t e^{\pm s_t}$$

and the probability the quote is accepted is modeled as

$$\lambda_1^{\cdot}(s_t) = e^{-\beta s_t^{\cdot}}.$$

Note β is a parameter reflecting the fact that we cannot guarantee winning the trade by setting the spread to zero:

$$P(\text{agent buys} \,|\, s \in [0, \delta)) = \int_0^\delta e^{-\beta s} \, ds = 1 - \frac{1}{\beta} e^{-\beta \delta} \approx 1 - \frac{1}{\beta} + \delta.$$

Obviously we must have $\beta \geq 1$, and really only $\beta > 1$ is realistic, as other dealers may be better buyers and sellers (perhaps using a different utility function than us).

Claim: *Accepted trades arrive at rate* $\lambda^{\cdot} = \lambda_0^{\cdot} \times \lambda_1^{\cdot}$.

Trade quantities are denoted by Q. In this setup the change in inventory is goverened by

$$dI_t = \left(Q^s \lambda^s - Q^b \lambda^b \right) \, dt$$

In words, the difference between expected agent sells (our buys) and expected agent buys (our sells) over a small time period.

The change in our cash account is similar, except that we have to reflect the transaction prices:

$$dC_t = P_t \left(Q^b \lambda^b e^{s^A} - Q^s \lambda^s e^{-s^B} \right) \, dt$$

As a mental check, if we take $Q = 1$ and $s = s^A = s^B$ and $\lambda = \lambda^b = \lambda^a$ we have

$$dC_t = Q \lambda P_t \left(e^s - e^{-s} \right) \, dt \approx 2 Q \lambda P_t s \, dt$$

which is twice the ("proportional") spread times the expected quantity traded. (I.e. the spread convention here is a percentage of value, not the more usual absolute amount.)

Our wealth process is simply the sum of the cash account and value of our position (taken at mid):

$$\Pi_t = C_t + I_t P_t,$$

and we are asked to choose a policy $\left\{s_t^A, s_t^B\right\}$ that maximizes exponential utility at some terminal time T:

$$\max \mathbf{E}\left[-e^{-\Pi(T)}\right]$$

1.1.1 Observations

- No time value of money (risk-free rate is zero).
- Can borrow indefinitely, implying no upper limit on long position
- Can short indefinitely
- No penalty for holding inventory; no link between spreads and inventory
- No explicity hedging of position, although P_t could reprsent the price dynamics of a hedged portfolio
- Might re-parameterize to the arrival rate of unit orders - i.e. collapse Q and λ.

1.1.2 Overall solution strategy

- Simulate to check understanding and validate expected sensitivites, such as
- Don't really see any time-dependence here
- Similarly the policy won't depend on position,
- **So perhaps it doesn't depend on wealth, and it's a constant**
- Bid and ask spreads won't be symmetric - the asymmetry should be controlled by μ and the relative arrival rates of buy and sell orders (and relative buy and sell sizes).
- Bid and ask spreads increase with σ
- Would hope that spreads also increase as arrival rates decrease
- Write down the SDE for the wealth process
- Find the infinitessimal generator and pose the optimization as an HJB equation
- Assume a classical solution and "differentiate" to find the optimal policy
- Analyze the PDE more to see what we can say about expected terinal wealth

1.2 Simulation

```
In [1]: import numpy as np
        import matplotlib.pylab as plt
```

```
In [2]: class AxSim(object):
            def __init__(self, P0, T, dt, mu, sig, lam0b, lam0s, beta, ns):
                self.P0 = P0
                self.T = T
                self.dt = dt
                self.mu = mu
                self.sig = sig
                self.lam0b = lam0b
                self.lam0s = lam0s
```

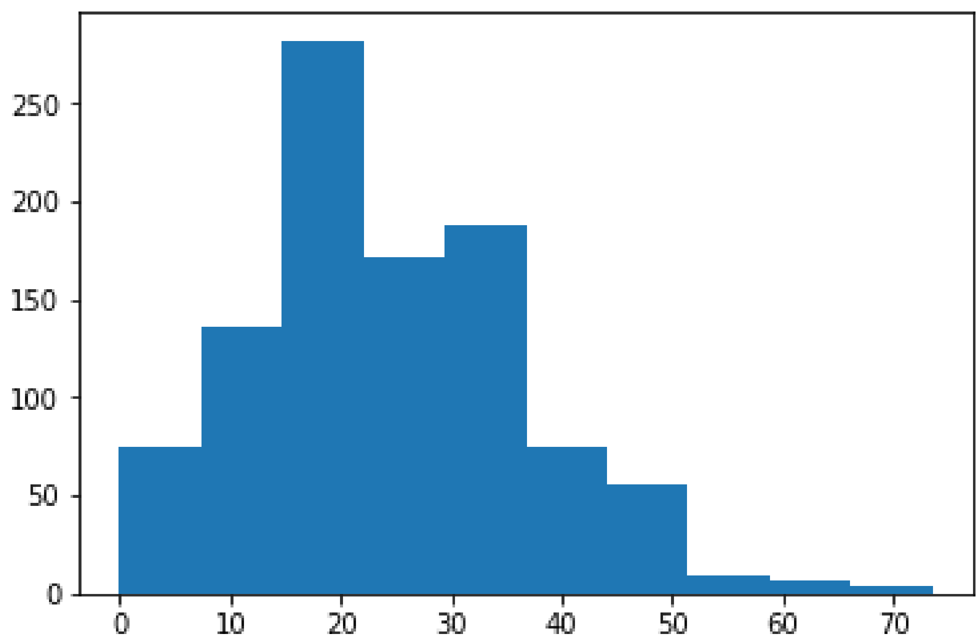
```
        self.beta = beta
        self.ns = ns
        self.Qb = 1.0
        self.Qs = 1.0

    def run_sim(self, sA, sB, prt_out):
        Pt = np.zeros((self.T+1, self.ns))
        Ct = np.zeros((self.T+1, self.ns))
        It = np.zeros((self.T+1, self.ns))
        Vt = np.zeros((self.T+1, self.ns))
        Pt[0,:] = self.P0
        ba_t = np.zeros((self.T+1, self.ns)) # buy arrivals
        sa_t = np.zeros((self.T+1, self.ns))
        bw_t = np.zeros((self.T+1, self.ns)) # buy wins
        sw_t = np.zeros((self.T+1, self.ns))
        for t in range(self.T):
            # price process
            Wt = np.random.normal(scale=np.sqrt(self.dt), size=self.ns)
            X = np.exp((self.mu-0.5*self.sig**2)*self.dt) * np.exp(self.sig*Wt)
            Pt[t+1,:] = Pt[t,:] * X
            # order arrivals
            ba_t[t+1,:] = np.random.poisson(self.lam0b/12, self.ns) # sth wrong?
            sa_t[t+1,:] = np.random.poisson(self.lam0b/12, self.ns)
            # prob our quote wins
            pb, pa = np.exp(-self.beta*sA), np.exp(-self.beta*sB)
            bw_t[t+1,:] = self.Qb*ba_t[t+1,:]*np.random.binomial(1, pb, size=self.ns)
            sw_t[t+1,:] = self.Qs*sa_t[t+1,:]*np.random.binomial(1, pa, size=self.ns)
            It[t+1,:] = It[t,:] + (sw_t[t+1,:] - bw_t[t+1,:]) # cust sells - cust buys
            Ct[t+1,:] = Ct[t,:] + Pt[t,:]*(bw_t[t+1,:]*np.exp(sA)
                                           - sw_t[t+1,:]*np.exp(-sB))
            Vt[t+1,:] = Ct[t+1,:] + It[t+1,:]*Pt[t+1,:] # quote at P(t), mark at P(t+1
        if prt_out:
            print('Bid arrivals per dt=%4.3f'%np.mean(np.mean(ba_t,axis=1)))
            print('Bids won per dt=%4.3f'%np.mean(np.mean(bw_t,axis=1)))
        return Vt, Ct, It
```

```
In [3]: T = 250 # days
        dt = 0.125 # 8 trading hours per day
        Qs = 1.0 # unit customer sell size
        Qb = 1.0 # unit customer buy size
        mu = 0.02/250.0  # 2% per annum
        sig = 0.01/np.sqrt(T) # 1% asset vol (eg high-quality bond)
        lam0s = 0.6033*8.0 # 33% chance per dt via P(N(1/8)=1) = lam/8 exp(-lam/8) = 0.33
        lam0b = lam0s # let's make buys and sells symmetric
        beta = 8.0 # so we are only X% likely to win a trade with a zero spread
        gam = 0.1
        P0 = 100.0
```

```
In [4]: sim1 = AxSim(P0, T, dt, mu, sig, lam0b, lam0s, 75.0, 1000)
```

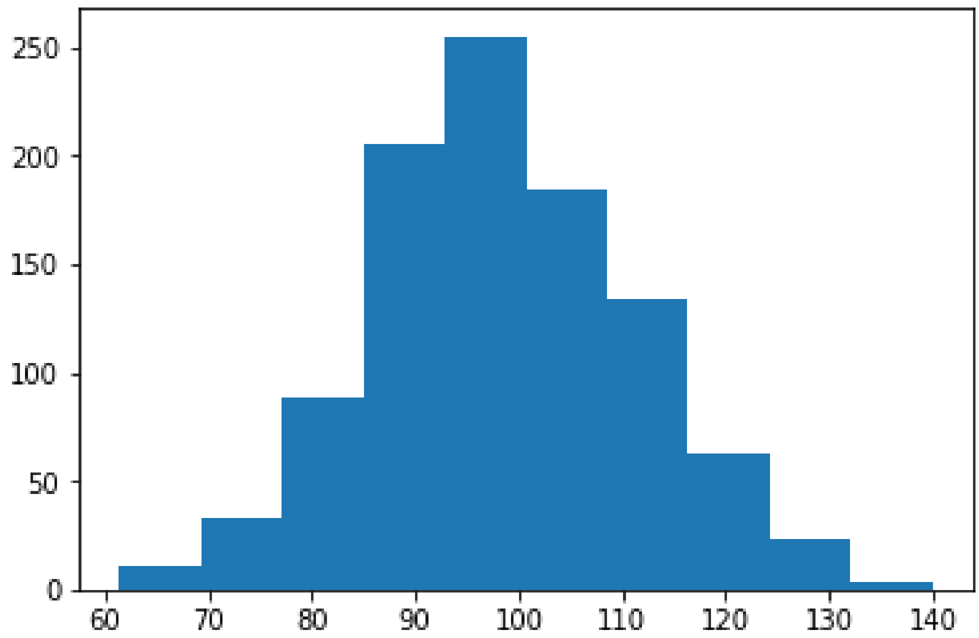
In [5]:
```
np.random.seed(123)
Vt, Ct, It = sim1.run_sim(0.05, 0.05, True)
UT = -np.exp(-gam*Vt[T,])
print(np.mean(Vt[T,]), np.std(Vt[T,]), np.mean(UT))
plt.hist(Vt[T,])
plt.show()
```

```
Bid arrivals per dt=0.402
Bids won per dt=0.010
24.1616420068 12.5644752121 -0.167976113687
```



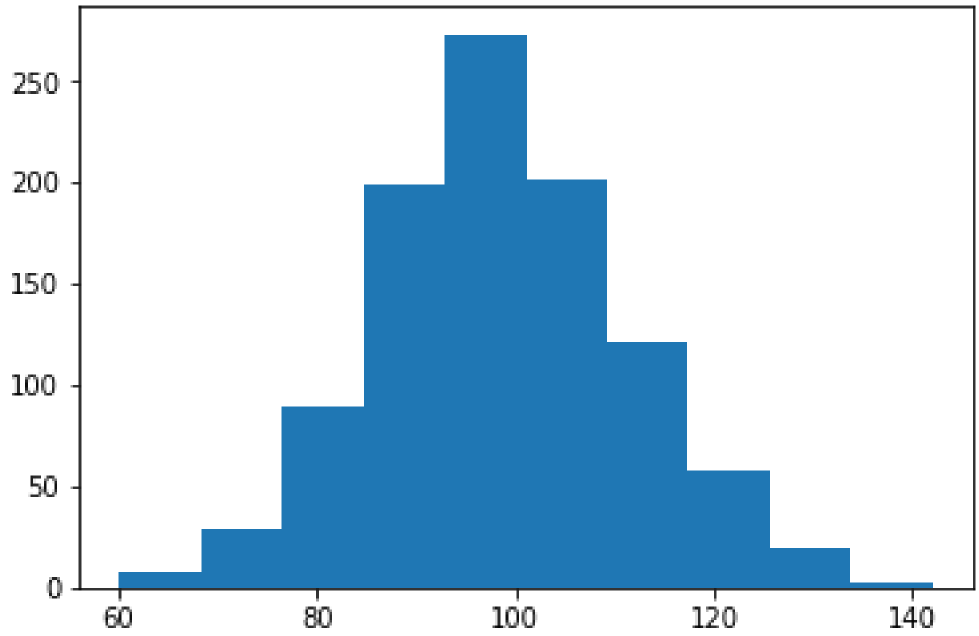
In [6]:
```
np.random.seed(123)
Vt, Ct, It = sim1.run_sim(0.0125, 0.0125, True)
UT = -np.exp(-gam*Vt[T,])
print(np.mean(Vt[T,]), np.std(Vt[T,]), np.mean(UT))
plt.hist(Vt[T,])
plt.show()
```

```
Bid arrivals per dt=0.402
Bids won per dt=0.158
98.5286318419 12.7886723267 -0.000114551699257
```

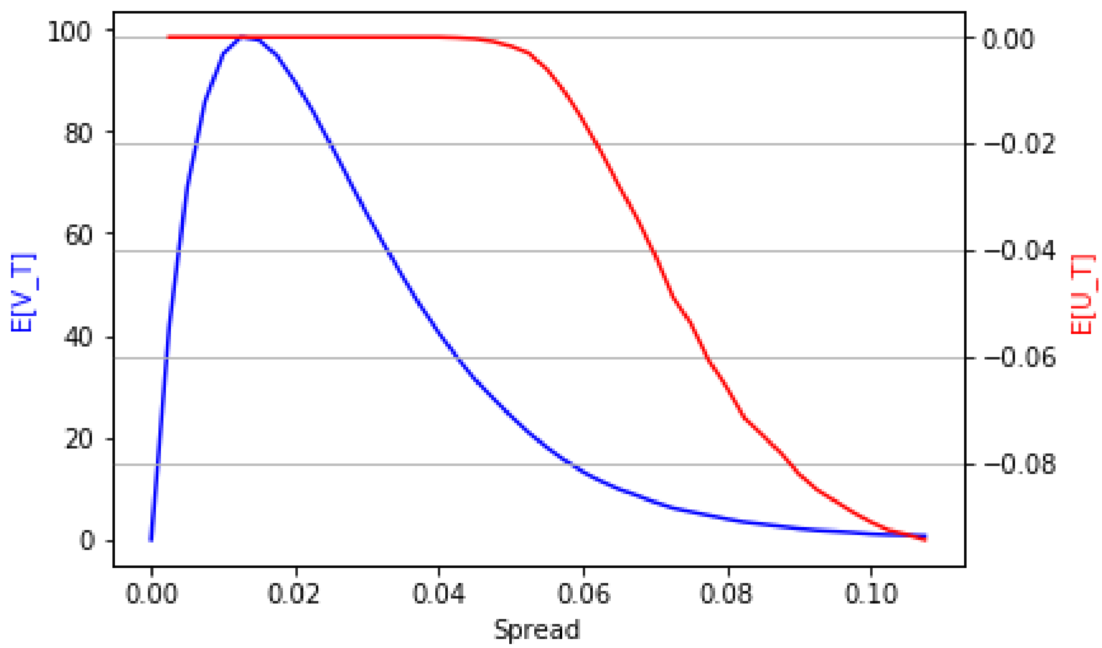


In [7]:
```
np.random.seed(123)
Vt, Ct, It = sim1.run_sim(0.0125, 0.012, True)
UT = -np.exp(-gam*Vt[T,])
print(np.mean(Vt[T,]), np.std(Vt[T,]), np.mean(UT))
plt.hist(Vt[T,])
plt.show()
```

```
Bid arrivals per dt=0.402
Bids won per dt=0.158
98.6382406974 12.6392397031 -0.00011162473986
```




```
In [8]: sj = np.arange(0.0, 0.11, 0.0025)
        mv = np.zeros(np.shape(sj))
        mu = np.zeros(np.shape(sj))
        for j, s_j in enumerate(sj):
            np.random.seed(123)
            Vt, Ct, It = sim1.run_sim(s_j, s_j, False)
            mv[j] = np.mean(Vt[T,:])
            mu[j] = np.mean(-gam*np.exp(-Vt[T,:]))

        fig, ax1 = plt.subplots()
        ax2 = ax1.twinx()

        ax1.plot(sj, mv, 'b');
        ax2.plot(sj[1:], mu[1:], 'r');
        plt.grid(True);
        ax1.set_xlabel('Spread');
        ax1.set_ylabel('E[V_T]', color='b');
        ax2.set_ylabel('E[U_T]', color='r');
        plt.show();

        print('Optimal spread=%f'%sj[np.argmax(mv)])
```




```
Optimal spread=0.012500
```

1.2.1 Observations

- We do observe something basic - spreads should't be too small or too big - more of a code-correctness check
- Utility looks pretty flat where terminal wealth is maximized (at least for these parameters)
- With simulation it's pretty easy to proxy the probability of ruin, say sample $\mathbf{E}(\Pi(T)|\Pi(T) < L)$
- Also [6] and [7] show that by making the spread asymmetic in the expected way, we can get better terminal wealth.
- I.e. because of the positive drift, we should be better bid than offered...

1.3 Towards an SDE for the Wealth Process

Per the hint, let $\pi_t = \Pi_t / P_t$ so that

$$\pi_t = \frac{C_t}{P_t} + I_t$$

Ideally we can write π_t as an Ito process. Let $q_1(s_t^A, s_t^B) = Q^s \lambda^s - Q^b \lambda^b$ and $q_2(s_t^A, s_t^B) = Q^b \lambda^b e^{s^A} - Q^s \lambda^s e^{-s^B}$. Then we have

$$dI_t = q_1 \, dt \quad \text{and} \quad dC_t = P_t q_2 \, dt.$$

We'll need to calculate the differential of π_t and the tricky term is $d\left[\frac{C_t}{P_t}\right]$ which must be done with the Itô rule:

$$d\left[\frac{C_t}{P_t}\right] = \frac{dC_t}{P_t} - \frac{C_t}{P_t}\frac{dP_t}{P_t} + \frac{C_t}{P_t}\frac{d^2P_t}{P_t^2}$$

Using the fact $\frac{C_t}{P_t} = \pi_t - I_t$ we have

$$d\pi_t = q_2\,dt - (\pi_t - I_t)(\mu\,dt + \sigma\,dW_t) + (\pi_t - I_t)\frac{dt}{P_t^2} + q_1\,dt.$$

Of course P_t has a known solution of $P_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)$, but this seems to leave us with an $e^{-2\sigma W_t}$ term in the drift.

1.4 Comment on the utility function

It's not hard to find references on maximizing the exponential utility of terminal wealth - for example Browne 1994. There you find connections to minimizing the probability of ruin (perhaps - not sure I saw necessary and sufficient conditions) and having a constant absolute aversion to risk, i.e. $u''/u' = c$ (which doesn't particularly resonate with any of my intuition beyond the litteral meaning of the words). (The worked example in the paper was also a good review for me generally.)

However, before I did any searching my sense was this * When $\Pi(T) << 0$ the utility is really negative - clearly a reasonable aversion (i.e. staying in business). * When I see $e^{-F(x)}$ I think of something probabilistic - e.g. a likelihood. For example this is how ML objective functions can be brought into a Bayesian framework: anything you want to minmize, I'll exponentiate to get a probability. (Admittedly one should have $F \geq 0$, but perhaps some idea can go through by restricting the domain of F.)

And also if one assume normality $X \sim N(\mu_X, \sigma_X)$ then

$$\mathbf{E}e^X = \int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma}}\,dx.$$

One might recognize this as a value of the moment generating function

$$\mathbf{E}e^{tX} = \exp\left(\mu_X t + \tfrac{1}{2}\sigma_X^2 t^2\right)$$