Optimal B/A Pricing Model

Consider the following stylized model for optimal B/A pricing. We denote the cash account by C_{it}, inventories I_t and the underlying price P_{it}. The price of the underlying follows a simple process

$$dP_{it} = P_{it} (\mu_i dt + \sigma_i dW_{it})$$

where μ_i σ_i are constant, and dW_{it} is a Brownian motion.

At time t there is unit mass of agents who can trade with us. The density of sell order arrival if λ_0^s and the density of buy order arrival is λ_0^b. In addition, given our own bid and ask spreads s_{it}^B and s_{it}^A, the probability each agent will trade are

$$\lambda_1^b(s_{it}^A) = e^{-\beta_b s_{it}^A} = \text{Pr agent buys if } s_{it}^A \tag{1}$$
$$\lambda_1^s(s_{it}^B) = e^{-\beta_b s_{it}^B} = \text{Pr agent sells if } s_{it}^B \tag{2}$$

Thus, given that at every t there is unit mass of agents who want to trade with us, we have that the fractions of these agents who accept our trades are

$$\lambda^b(s_{it}^A) = \lambda_0^b \times \lambda_1^b(s_{it}^A) \tag{3}$$
$$\lambda^s(s_{it}^B) = \lambda_0^s \times \lambda_1^s(s_{it}^B) \tag{4}$$

Assume the size of buy and sell order is fixed (for now) at Q_i^b and Q_i^s. We thus have that cash account and inventories follow the processes

$$\begin{aligned} dC_{it} &= P_{it} e^{s_{it}^A} Q_i^b \lambda^b \left(s_{it}^A\right) dt - P_{it} e^{-s_{it}^B} Q_i^s \lambda^s \left(s_{it}^B\right) dt \\ dI_{it} &= \left[-Q_{it}^b \lambda^b \left(s_{it}^A\right) + Q_{it}^s \lambda^s \left(s_{it}^B\right)\right] dt \end{aligned}$$

The value of the position at any time t is

$$\Pi_{it} = C_{it} + I_{it} P_{it}$$

We want to determine the optimal dynamics spread strategy $\{s_{it}^A, s_{it}^B\}_{t=0}^{t=T}$ to maximize the objective function

$$\max_{\{s_{it}^A, s_{it}^B\}} E_0 \left[-e^{-\gamma \Pi_T}\right] \tag{5}$$

Questions

1. The distribution of Π_T is endogenous. However, assuming that Π_T is normal with mean $E[\Pi_T]$ and variance $V[\Pi_T]$, i.e.

$$\Pi_T \sim N\left(E[\Pi_T], V[\Pi_T]\right)$$

   what is the logic of the objective function (5)? How can we interpret γ?

2. Use the Bellman principle to obtain the Hamilton, Jacobi, Bellman equation for the program (5) (It may be useful to define $\pi_t = \Pi_t / P_{it}$)

3. Find the solutions for s_{it}^A and s_{it}^B. Interpret economically your solution.

4. Obtain the final PDE to be solved

5. Illustrate what methods you would use to numerically solve the PDE (or solve for it!)