Stability Analysis of Rotating Pendulum:

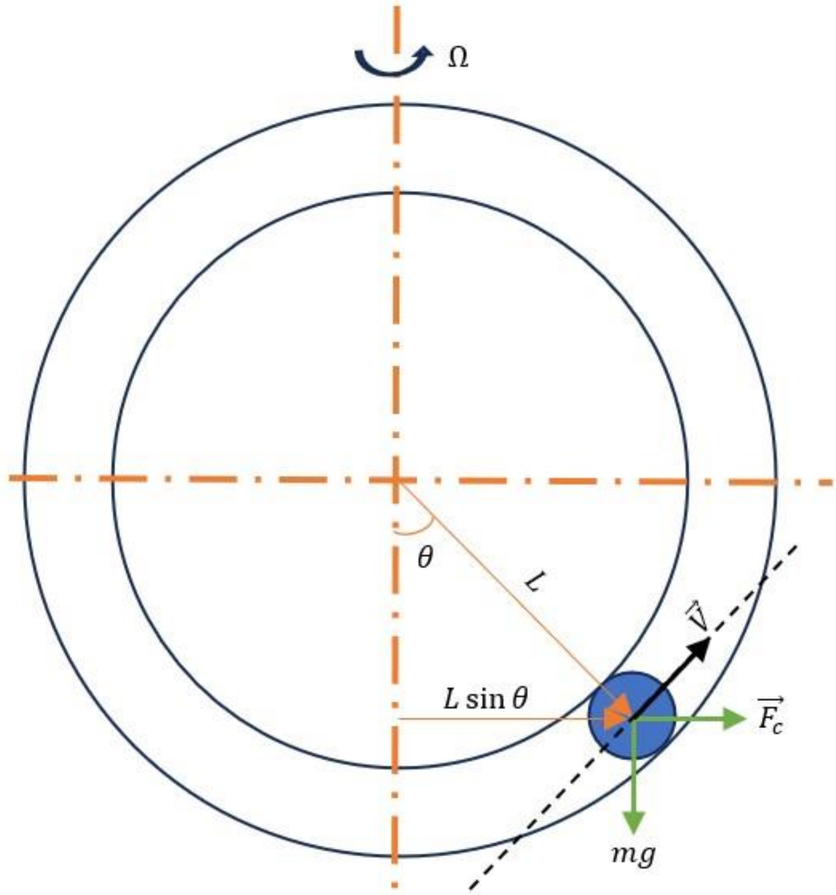

Figure 1. Schematic of Rotating Pendulum

Consider a Rotating Pendulum consists of two vertical concentric circular tracks of mean radius, L and a small spherical ball of mass, m placed in a space between two circular tracks as shown in fig. 1, which guide the motion of ball in circular track. Apart from this whole assemble is rotated about vertical axis by angular velocity, Ω.

<u>Experiment Observation:</u> (Ball Initially displaced from $\theta = 0$)

- When $\Omega = 0$, ball oscillates about $\theta = 0$ and become steady after sometime.
- When Ω is small it shows similar behavior as that of $\Omega = 0$, but as Ω crosses critical value of the system, it doesn't come back to $\theta = 0$, but oscillate about some other $\theta \neq 0$, and get settled after sometime.

Solution:

Step 1: Define Governing Equation of System

To find system equation, we need to balance force acting on ball along instantaneous line of motion as shown in fig. 2.

Various Force acting on the ball are:

- Centrifugal Force (F_c), due to angular velocity along vertical axes, given by:

$$F_c = \mathrm{m}\Omega^2 L \sin\theta \quad \text{......... (A)}$$

- Weight (mg), along instantaneous line of motion represented by black dash line.
- Friction Force (χV), act opposite to the motion of the ball. Where χ depend on dissipation effect experience by ball.

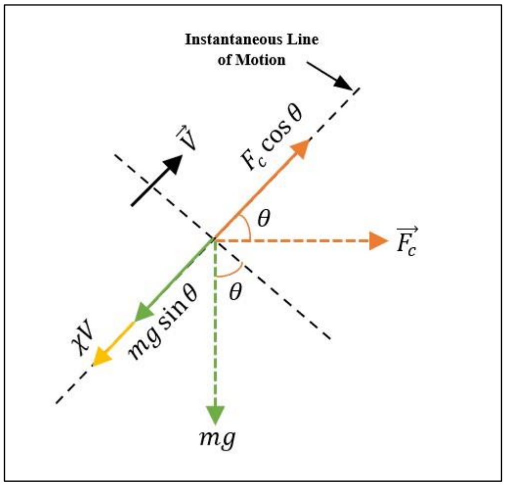

Figure 2. Free Body Diagram (FBD)

Now, Applying Newton's 2nd Law along direction of motion ($\vec{V}$), (i.e. tangential direction)

$$\Sigma F = ma$$

Where, a is the acceleration of ball along line of action

$$F_c - \mathrm{mg}\sin\theta - \chi V = ma$$

Substituting value of F_c, we get

$$(\mathrm{m}\Omega^2 L \sin\theta)\cos\theta - \mathrm{mg}\sin\theta - \chi V = ma$$

We know that, acceleration of ball can be written as $a = \frac{dv}{dt}$

Substituting value of acceleration, we get

$$m\frac{dV}{dt} = (\mathrm{m}\Omega^2 L \sin\theta)\cos\theta - \mathrm{mg}\sin\theta - \chi V$$

Dividing throughout by m,

$$\frac{dV}{dt} = \Omega^2 L \sin\theta\cos\theta - \mathrm{g}\sin\theta - \left(\frac{\chi}{m}\right)V \quad \text{......... (B)}$$

Also, we know that tangential velocity in circular motion can be written as

$$V = L\frac{d\theta}{dt}$$

Substituting above value to equation (B), we get

Governing Equation of system,

$$L\frac{d^2\theta}{dt^2} = \Omega^2 L\sin\theta\cos\theta - \mathrm{g}\sin\theta - \left(\frac{\chi}{m}\right)V \qquad \text{......... (C)}$$

The above equation is "**Second Order Non-Linear Differential equation**"

Step 2: Obtain dimensionless equation of system

For obtaining dimensionless equation, we need to define reference Length scale & Time scale of system. Let,

$L_{ref} = L$ .... Reference Length Scale.

$T_{ref} = \sqrt{\frac{L}{g}}$ .... Reference Time Scale.

Using these reference scale, we need to find non-dimensional variable or parameters. Let the non-dimensional variable be represented with symbol "~" on top.

$\tilde{t} = \frac{t}{T_{ref}}$; $\tilde{L} = \frac{L}{L_{ref}}$ & $\tilde{\theta} = \theta$

Velocity, $V \propto \frac{x}{t}$, $\therefore \tilde{V} = \frac{x/L_{ref}}{t/T_{ref}} = \left(\frac{T_{ref}}{L_{ref}}\right)V$.

Therefore, we get

$t = T_{ref}\,\tilde{t}$, $L = L_{ref}\,\tilde{L}$, $\tilde{\theta} = \theta$ & $V = \left(\frac{L_{ref}}{T_{ref}}\right)\tilde{V}$

Now substituting above value to equation (C),

$$L\frac{d^2\theta}{dt^2} = \Omega^2 L\sin\theta\cos\theta - \mathrm{g}\sin\theta - \left(\frac{\chi}{m}\right)V$$

Dividing throughout by L,

$$\frac{d^2\theta}{dt^2} = \Omega^2 \sin\theta\cos\theta - \frac{\mathrm{g}}{L}\sin\theta - \left(\frac{\chi}{m}\right)\frac{V}{L}$$

$$\frac{1}{(T_{ref})^2}\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = \Omega^2 \sin\tilde{\theta}\cos\tilde{\theta} - \frac{\mathrm{g}}{L}\sin\tilde{\theta} - \left(\frac{\chi}{m}\right)\frac{\tilde{V}}{L}\left(\frac{L_{ref}}{T_{ref}}\right)$$

Multiplying throughout by $(T_{ref})^2$ & substituting $L_{ref} = L$

$$\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = \Omega^2 (T_{ref})^2 \sin\tilde{\theta}\cos\tilde{\theta} - \frac{g}{L}(T_{ref})^2 \sin\tilde{\theta} - \left(\frac{\chi}{m}\right)\frac{\tilde{V}}{\cancel{L}}\left(\frac{\cancel{L}}{\cancel{T_{ref}}}\right)(T_{ref})^{\cancel{2}}$$

Substituting $T_{ref} = \sqrt{\frac{L}{g}}$, we get

$$\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = \frac{\Omega^2 L}{g}\sin\tilde{\theta}\cos\tilde{\theta} - \sin\tilde{\theta} - \left(\frac{\chi}{m}\right)\sqrt{\frac{L}{g}}\,\tilde{V}$$

$$\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = \frac{\Omega^2 L}{g}\sin\tilde{\theta}\cos\tilde{\theta} - \sin\tilde{\theta} - \left(\frac{\chi}{m}\sqrt{\frac{L}{g}}\right)\tilde{V} \qquad \text{......... (D)}$$

Here, we obtained **"non-dimensional equation of system"**

Where, $r = \frac{\Omega^2 L}{g}$ & $p = \left(\frac{\chi}{m}\sqrt{\frac{L}{g}}\right)$ are the two non-dimensional quantity.

Therefore, we can write equation (D) as,

$$\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = r\sin\tilde{\theta}\cos\tilde{\theta} - \sin\tilde{\theta} - p\tilde{V} \qquad \text{......... (E)}$$

The above equation is "non-linear second order non-dimensional differential equation". We can divide a second order differential equation into two first order differential equation.

We know, tangential velocity, $V = L\frac{d\theta}{dt}$ we can transform this to non-dimensional as,

$$\left(\frac{\cancel{L}_{ref}}{\cancel{T}_{ref}}\right)\tilde{V} = \frac{\cancel{L}}{\cancel{T}_{ref}}\frac{d\tilde{\theta}}{d\tilde{t}}$$

$$\therefore \tilde{V} = \frac{d\tilde{\theta}}{d\tilde{t}}$$

Substituting above value to left side of equation (E), we get

$$\frac{d\tilde{V}}{d\tilde{t}} = r\sin\tilde{\theta}\cos\tilde{\theta} - \sin\tilde{\theta} - p\tilde{V} \qquad \text{......... (1)}$$

$$\frac{d\tilde{\theta}}{d\tilde{t}} = \tilde{V} \qquad \text{......... (2)}$$

Equation (1) & Equation (2) are the two **"Non-Linear First order differential equation"**

Step 3: Find Basic solution

We choose steady state solution as basic solution for this problem. Steady state solution obtained where $\frac{d\tilde{\theta}}{d\tilde{t}} = 0$ & $\frac{d\tilde{V}}{d\tilde{t}} = 0$ hence we get steady state solution as,

$$\tilde{V} = 0$$

$$r \sin\tilde{\theta} \cos\tilde{\theta} - \sin\tilde{\theta} - p\tilde{V} = 0$$

Substituting value of $\tilde{V}$, we get

$$r \sin\tilde{\theta} \cos\tilde{\theta} - \sin\tilde{\theta} = 0$$

$$\sin\tilde{\theta}\,(r\cos\tilde{\theta} - 1) = 0$$

$$\sin\tilde{\theta} = 0 \quad or \quad \left(r\cos\tilde{\theta} - 1\right) = 0$$

$$\therefore \ \sin\tilde{\theta} = 0 \qquad\qquad or \qquad\qquad \cos\tilde{\theta} = 1/r$$

$$\tilde{\theta} = 0 \qquad\qquad or \qquad\qquad \tilde{\theta} = \pm\cos^{-1}(1/r)$$

$$\tilde{\theta} = \pi$$

Therefor we got four Basis State $\Phi = [\bar{\theta}\ ,\ \bar{V}]$, as follows

$\Phi_1 = [0\ ,0]$

$\Phi_2 = [\pi\ ,0]$

$\Phi_3 = [\cos^{-1}(1/r)\ \ ,0]$

$\Phi_4 = [-\cos^{-1}(1/r)\ \ ,0]$

Step 4: Find Evolution Equation for Perturbation or disturbance, $\phi' = [\theta', V']$ in as system

To check stability of system at given basic state we need to know how does perturbation behave in system. For that we need to find evolution equation of perturbation.

Let, $\phi' = [\theta', V']$ be the perturbation or disturbance added to the basis state, $\Phi = [\bar{\theta}\ ,\ \bar{V}]$. So, we get solution $\phi = [\tilde{\theta}, \tilde{V}]$ of system as,

$$\phi = \Phi + \phi' = [\bar{\theta} + \theta'\ ;\ \bar{V} + V'\ ]$$

$$\therefore\ \ \tilde{\theta} = \bar{\theta} + \theta' \quad \& \quad \tilde{V} = \bar{V} + V'$$

Adding above value to equation (2) we get

$$\frac{d(\bar{\theta}+\theta')}{d\tilde{t}} = \bar{V} + V'$$

$$\frac{d\theta'}{d\tilde{t}} + \frac{d\bar{\theta}}{d\tilde{t}} = \bar{V} + V'$$

Since basis state also satisfy equation (1) & (2) we get $\frac{d\bar{\theta}}{d\tilde{t}} = \bar{V}$

$$\therefore \ \frac{d\theta'}{d\tilde{t}} + \cancel{\bar{V}} = \cancel{\bar{V}} + V'$$

$$\therefore \ \frac{d\theta'}{d\tilde{t}} = V' \qquad \text{.........} \ (F)$$

Similarly, we can solve for equation (1)

$$\frac{d(\bar{V}+V')}{d\tilde{t}} = r\sin(\bar{\theta}+\theta')\cos(\bar{\theta}+\theta') - \sin(\bar{\theta}+\theta') - p(\bar{V}+V')$$

$$\frac{dV'}{d\tilde{t}} + \frac{d\bar{V}}{d\tilde{t}} = r\sin(\bar{\theta}+\theta')\cos(\bar{\theta}+\theta') - \sin(\bar{\theta}+\theta') - p\bar{V} - pV'$$

We know, $\frac{d\bar{V}}{d\tilde{t}} = r\sin(\bar{\theta})\cos(\bar{\theta}) - \sin(\bar{\theta}) - p\bar{V}$

$$\therefore \ \frac{dV'}{d\tilde{t}} + r\sin(\bar{\theta})\cos(\bar{\theta}) - \sin(\bar{\theta}) - p\bar{V} = r\sin(\bar{\theta}+\theta')\cos(\bar{\theta}+\theta') - \sin(\bar{\theta}+\theta') - p\bar{V} - pV'$$

By some rearrangement we get

$$\frac{dV'}{d\tilde{t}} = r[\sin(\bar{\theta}+\theta')\cos(\bar{\theta}+\theta') - \sin(\bar{\theta})\cos(\bar{\theta})] - [\sin(\bar{\theta}+\theta') - \sin(\bar{\theta})] - pV'$$

.... (G)

Form equation (F) & (G), we get **"Non-linear Evolution Equation for perturbation"** as,

$$\frac{d\theta'}{d\tilde{t}} = V'$$

$$\frac{dV'}{d\tilde{t}} = r[\sin(\bar{\theta}+\theta')\cos(\bar{\theta}+\theta') - \sin(\bar{\theta})\cos(\bar{\theta})] - [\sin(\bar{\theta}+\theta') - \sin(\bar{\theta})] - pV'$$

.... (3 & 4)

Step 5: Linearize the evolution equation for perturbation.

To find Linear stability of system we need to first linearize the evolution equation for perturbation given by equation (3) & (4). Evolution Equation for perturbation can be treated as linear equation when the value of perturbation is very small i.e. $\phi' \ll 1$ ($\theta' \ll 1$ & $V' \ll 1$).

Consider equation (3) & (4)

$$\frac{d\theta'}{d\tilde{t}} = V'$$

$$\frac{dV'}{d\tilde{t}} = r[\sin(\bar{\theta} + \theta')\cos(\bar{\theta} + \theta') - \sin(\bar{\theta})\cos(\bar{\theta})] - [\sin(\bar{\theta} + \theta') - \sin(\bar{\theta})] - pV'$$

Also,

$\sin(\bar{\theta} + \theta') = \sin\bar{\theta}\cos\theta' + \cos\bar{\theta}\sin\theta'$

$\cos(\bar{\theta} + \theta') = \cos\bar{\theta}\cos\theta' - \sin\bar{\theta}\sin\theta'$

For very small value of θ', $\cos\theta' \approx 1$ & $\sin\theta' \approx \theta'$

$\therefore \sin(\bar{\theta} + \theta') \approx \sin\bar{\theta} + (\cos\bar{\theta})\theta'$

$\therefore \cos(\bar{\theta} + \theta') \approx \cos\bar{\theta} - (\sin\bar{\theta})\theta'$

Now,

$\sin(\bar{\theta} + \theta')\cos(\bar{\theta} + \theta') \approx (\sin\bar{\theta} + (\cos\bar{\theta})\theta')(\cos\bar{\theta} - (\sin\bar{\theta})\theta')$

$\therefore \sin(\bar{\theta} + \theta')\cos(\bar{\theta} + \theta') \approx \sin\bar{\theta}\cos\bar{\theta} - (\sin\bar{\theta})^2\theta' + (\cos\bar{\theta})^2\theta' - (\sin\bar{\theta}\cos\bar{\theta})(\theta')^2$

Since, $\theta' \ll 1$, we can neglect higher order term.

Therefore, we get

$\sin(\bar{\theta} + \theta')\cos(\bar{\theta} + \theta') \approx \sin\bar{\theta}\cos\bar{\theta} + (\cos^2\bar{\theta} - \sin^2\bar{\theta})\theta'$

& $\sin(\bar{\theta} + \theta') \approx \sin\bar{\theta} + (\cos\bar{\theta})\theta'$

Substituting above value to equation (4)

we get **"Linearized Evolution Equation for Perturbation"** as

$$\frac{d\theta'}{d\tilde{t}} = V'$$

$$\frac{dV'}{d\tilde{t}} = [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - \cos(\bar{\theta})]\theta' - pV' \qquad \text{......... (5 \& 6)}$$

Above Equation can be written in matrix form as follows,

$$\frac{d\phi'}{d\tilde{t}} = \begin{bmatrix} \frac{d\theta'}{d\tilde{t}} \\ \frac{dV'}{d\tilde{t}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] & -p \end{bmatrix} \begin{bmatrix} \theta' \\ V' \end{bmatrix}$$

Let,

$L_r(\Phi) = \begin{bmatrix} 0 & 1 \\ [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] & -p \end{bmatrix}$ = Coefficient Matrix or Jacobian Matrix

$\phi' = \begin{bmatrix} \theta' \\ V' \end{bmatrix}$

$$\therefore \frac{d\phi'}{d\tilde{t}} = L_r(\Phi)\phi' \qquad \text{.........} (7)$$

Step 6: Stability Analysis:

To Solve equation (7), one of the simplest methods is to find fundamental solution. Since coefficient of linear equation is not a function of time (t) we can assume one of the fundamental solutions as

$\phi' = \tilde{\Phi}e^{st}$

Where, $\tilde{\Phi}$ = some coefficient not a function of time, may be complex ($\tilde{\Phi} = \tilde{\Phi} + i\tilde{\Phi}$)

s = Characteristic root of system, may be complex ($s = s_r + is_i$)

If value of real part of the characteristic root is positive means perturbation will grow in time and if it is negative perturbation will decay with time. Hence, value of characteristic root tells us about stability of system.

Let substitute assumed fundamental solution to equation (7),

$$\tilde{\Phi}\frac{de^{st}}{d\tilde{t}} = L_r(\Phi)\tilde{\Phi}e^{st}$$

$$\tilde{\Phi}se^{st} = L_r(\Phi)\tilde{\Phi}e^{st}$$

$$\tilde{\Phi}sI = L_r(\Phi)\tilde{\Phi}$$

$$\therefore \left(sI - L_r(\Phi)\right)\tilde{\Phi} = 0$$

Where, *I* is Identity Matrix

To get non-trivial solution, determinant of $|sI - L_r(\Phi)| = 0$, which implies that eigen value of coefficient matrix of a system is nothing but the characteristic root of system.

Let consider,

$$|sI - L_r(\Phi)| = 0$$

$$\begin{vmatrix} s & -1 \\ -[r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] & s+p \end{vmatrix} = 0$$

$$s(s+p) - [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] = 0$$

Therefore, **"Characteristic Equation of System"** is given as,

$$\boxed{s^2 + ps - [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] = 0} \quad \text{……… (8)}$$

<u>Exercise 1: To Check Linear Stability of all Basic State for $p = 0$</u>

For p=0, the characteristic equation can be written as

$$s^2 - [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] = 0$$

Let check Stability of all four Base State $\Phi = [\bar{\theta}\ ,\ \bar{V}]$

CASE-I: $\boldsymbol{\Phi_1 = [0\ ,0]}$

$\bar{\theta} = 0$ & $\bar{V} = 0$

Substituting above value of base state to characteristic equation we get,

$$s^2 - (r-1) = 0$$

$$\therefore s^2 = (r-1)$$

$$s = \pm\sqrt{r-1}$$

Here, we get two characteristic values since evolution equation is second order differential equation. Hence, we get solution for perturbation as linear combination of two fundamental solution ($\widetilde{\Phi}_1 e^{s_1 t}\ ,\ \widetilde{\Phi}_2 e^{s_2 t}$) corresponding to two characteristic values ($s_1\ , s_2$).

$$\therefore\ \phi' = A_1\widetilde{\Phi}_1 e^{s_1 t} + A_2\widetilde{\Phi}_2 e^{s_2 t}$$

Where, A_1 & A_2 are some constant.

If the real part of any characteristic value becomes positive it means corresponding fundamental solution grows in a time and hence perturbation also grows in time, which implies that the system is linearly unstable.

Therefore, to check linear stability of a system we need to analyze its characteristics value.

For a given Basic State, two characteristic values of system are:

$$\boxed{s_1 = \sqrt{r-1} \;\; ; \;\; s_2 = -\sqrt{r-1}}$$

For $r < 1$, the value inside square root become negative & hence the characteristics value are imaginary, which means real part of both characteristic value is zero hence perturbation do not grow as well as decay, remain steady in time such system is called as "Lyapunov Stable System".

For $r = 1$, both real & imaginary part of characteristic value becomes zero, result in some stationary pattern of perturbation in a system. We can't say anything about stability at this point.

For $r > 1$, the characteristics value s_1 become positive makes corresponding fundamental solution to grow in time and hence system become "Linearly Unstable" for given basic state.

Here, $r = 1$, is boundary between stable and unstable system, hence it is called as "critical value $(r_c = 1)$"

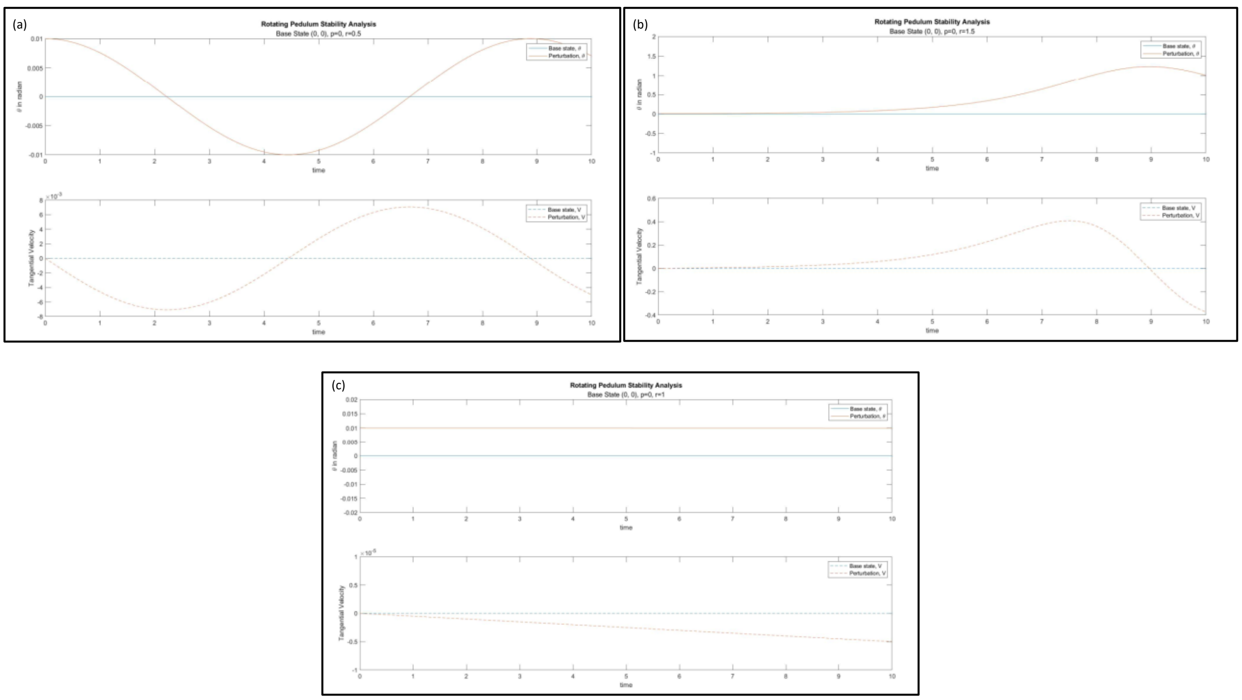

Figure 3. Perturbation Evolution with time (a) $r < 1$, (b) $r > 1$, (c) $r = 1$

In figure we can see that perturbation which is initially, $\phi' = [0.01, 0]$ oscillate about base state with but didn't grow in time for $r < 1$. On the other hand, for $r > 1$ perturbation grows rapidly.

CASE-II: $\boldsymbol{\Phi_2 = [\pi\,, 0]}$

$$\bar{\theta} = \pi \quad \& \; \bar{V} = 0$$

Substituting above value of base state to characteristic equation we get,

$$s^2 - (r+1) = 0$$

$$\therefore s^2 = (r+1)$$

$$s = \pm\sqrt{r+1}$$

For a given Basic State, two characteristic values of system are:

$$\boxed{s_1 = \sqrt{r+1}\ ; \quad s_2 = -\sqrt{r+1}}$$

For

We know that r is always positive value. Hence, for all value of r, s_1 is positive which makes corresponding fundamental solution unstable. Hence System at given basic state is "Linearly Unstable" for all value of r.

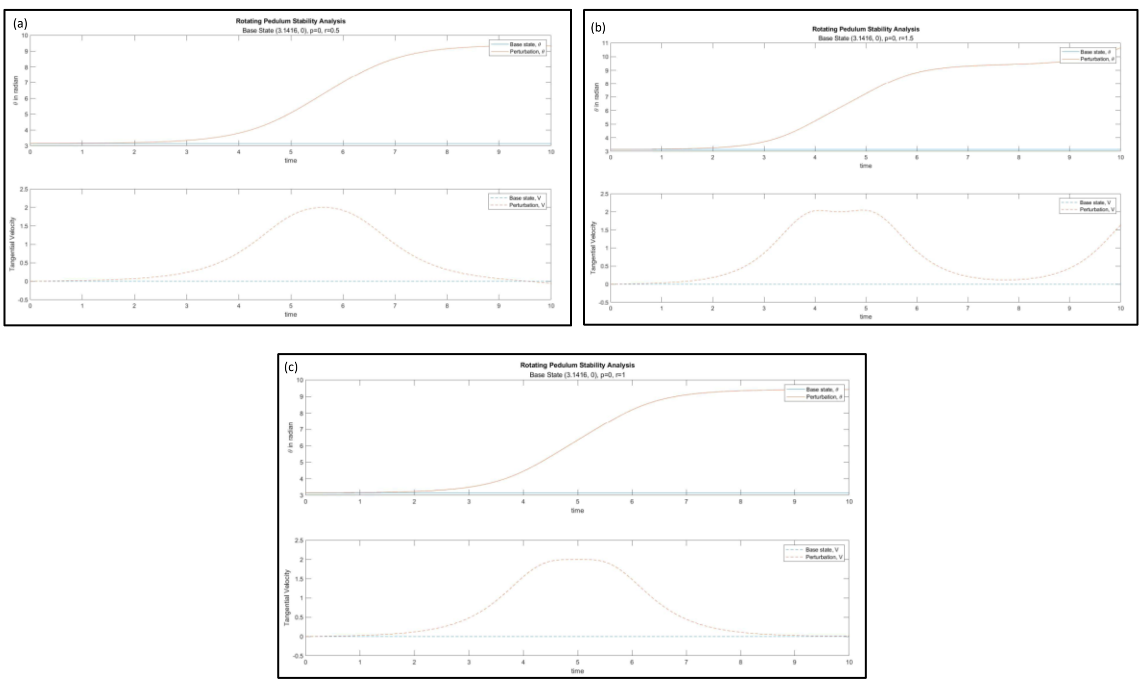

Figure 4. Perturbation Evolution with time (a) $r < 1$, (b) $r > 1$, (c) $r = 1$

Initial perturbation, $\phi' = [0.01, 0]$

In the figure we can see for all values of r, perturbation grows in time and make system unstable at given base state.

CASE-III: $\boldsymbol{\Phi_3 = [\cos^{-1}(1/r)\ , 0]}$

$$\bar{\theta} = \cos^{-1}(1/r) \quad \& \ \bar{V} = 0$$

Substituting above value of base state to characteristic equation we get,

$$s^2 - \left[r\left(\cos^2(\cos^{-1}(1/r)) - \sin^2(\cos^{-1}(1/r))\right) - \cos(\cos^{-1}(1/r))\right] = 0$$

We know that,

$$\cos(\cos^{-1}(1/r)) = \frac{1}{r}$$

$$\cos^2(\cos^{-1}(1/r)) = \left(\frac{1}{r}\right)^2 = \frac{1}{r^2}$$

$$\sin^2(\cos^{-1}(1/r)) = \left(\frac{\sqrt{r^2-1}}{r}\right)^2 = \frac{r^2-1}{r^2}$$

By Substituting above value, we get

$$\therefore s^2 - \left(\frac{1-r^2}{r}\right) = 0$$

$$\therefore s^2 = \left(\frac{1-r^2}{r}\right)$$

$$\therefore s = \pm\sqrt{\frac{1-r^2}{r}}$$

For a given Basic State, two characteristic values of system are:

$$\boxed{s_1 = \sqrt{\frac{1-r^2}{r}} \quad ; \quad s_2 = -\sqrt{\frac{1-r^2}{r}}}$$

For $r > 1$, both characteristic value s_1 and s_2 are imaginary which make real part zero. Hence System is "Lyapunov Stable" for given basic state.

For $r = 1$, Both real & imaginary part of characteristics value become zero result in some stationary pattern of perturbation in a system. We can't say anything about stability at this point.

For $r < 1$, $\cos^{-1}(1/r)$ doesn't exist since value of $\cos^{-1}$ exist for domain [-1 1].

Here, $r = 1$, is the "critical value"

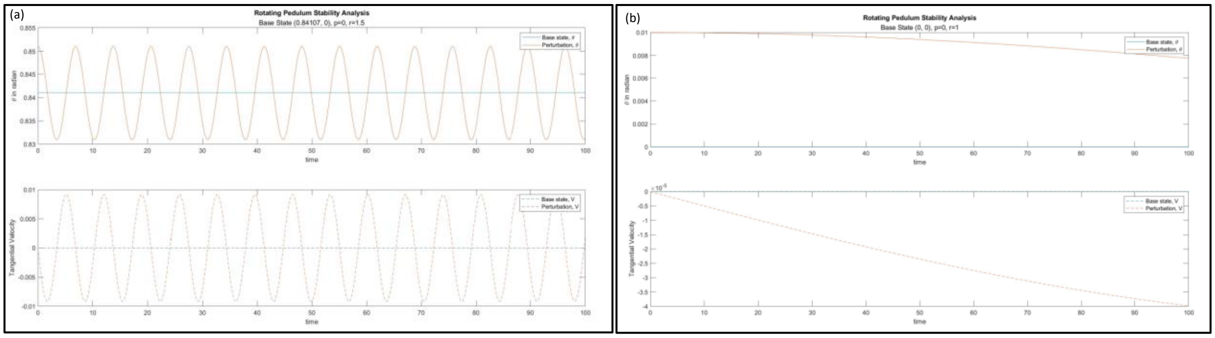

Figure 5. Perturbation Evolution with time (a) $r > 1$, (b) $r = 1$

Initial perturbation, $\phi' = [0.01, 0]$

In the figure we can see that, for $r > 1$, perturbation oscillate about base state but do not grow in time hence system is said to be stable. Solution for $r < 1$, do not exists.

CASE-IV: $\boldsymbol{\Phi_4 = [-\cos^{-1}(1/r)\ , 0]}$

$\bar{\theta} = -\cos^{-1}(1/r) \quad \& \ \bar{V} = 0$

Substituting above value of base state to characteristic equation we get,

$$s^2 - \left[r\left(\cos^2(-\cos^{-1}(1/r)) - \sin^2(-\cos^{-1}(1/r))\right) - \cos(-\cos^{-1}(1/r))\right] = 0$$

We know that,

$$\cos(-\cos^{-1}(1/r)) = \frac{1}{r}$$

$$\cos^2(-\cos^{-1}(1/r)) = \left(\frac{1}{r}\right)^2 = \frac{1}{r^2}$$

$$\sin^2(-\cos^{-1}(1/r)) = \left(-\frac{\sqrt{r^2-1}}{r}\right)^2 = \frac{r^2-1}{r^2}$$

By Substituting above value, we get

$$\therefore s^2 - \left(\frac{1-r^2}{r}\right) = 0$$

$$\therefore s^2 = \left(\frac{1-r^2}{r}\right)$$

$$\therefore s = \pm\sqrt{\frac{1-r^2}{r}}$$

For a given Basic State, two characteristics value of system are:

$$\boxed{s_1 = \sqrt{\frac{1-r^2}{r}} \quad ; \quad s_2 = -\sqrt{\frac{1-r^2}{r}}}$$

For this basic state we get characteristic values same as that for pervious case.

Hence

For $r > 1$, System is "Lyapunov Stable"

For $r = 1$, We can't say anything about stability at this point.

And, $r = 1$, is the "critical value"

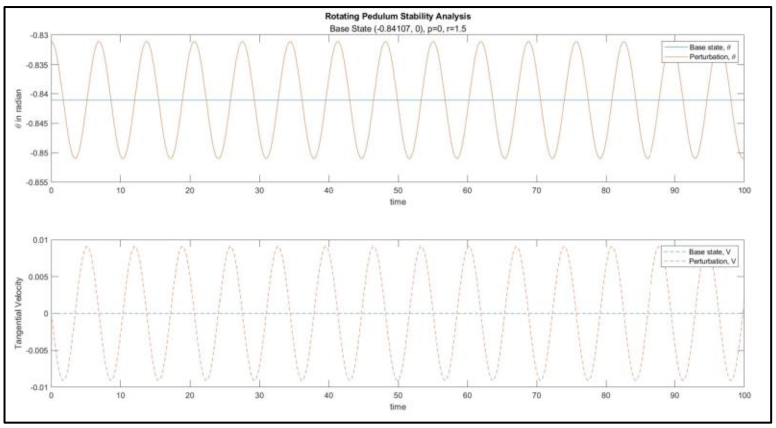

Figure 6. Perturbation Evolution with time, $r > 1$

Exercise 2: To Check Linear Stability of Basic State $\Phi_1 = [0\,,0]$ for $p \neq 0$

The characteristic equation can be written as

$$s^2 + ps - [r(\cos^2\bar{\theta} - \sin^2\bar{\theta}) - cos\bar{\theta}] = 0$$

The Base State $\Phi = [0\,,\ 0]$

$$\bar{\theta} = 0 \ \ \& \ \bar{V} = 0$$

Substituting above value of base state to characteristic equation we get,

$$s^2 + ps - (r - 1) = 0$$

We know that, root of quadratic equation can be written as,

$$s = \frac{-p \pm \sqrt{p^2 + 4(r-1)}}{2}$$

Therefore, the two characteristics value of system are:

$$\boxed{s_1 = \frac{-p + \sqrt{p^2 + 4(r-1)}}{2}\ ;\ s_2 = \frac{-p - \sqrt{p^2 + 4(r-1)}}{2}}$$

For system to be linearly stable, s_1, s_2 should be negative, i.e. $s_1, s_2 < 0$
Let,

$$s_1 < 0$$

$$\therefore -p + \sqrt{p^2 + 4(r-1)} < 0$$

$$\therefore \sqrt{p^2 + 4(r-1)} < p$$

Taking square on both sides, we get

$$p^2 + 4(r-1) < p^2$$

$$\therefore (r-1) < 0$$

$$\therefore r < 1$$

Therefore, s_1 is negative for $r < 1$. If r is greater than 1 the value of s_1 become positive which make corresponding fundamental solution unstable and hence system become linearly unstable.

Let consider second characteristics value,

$$s_2 < 0$$

$$\therefore -p - \sqrt{p^2 + 4(r-1)} < 0$$

we know that, $\sqrt{p^2 + 4(r-1)}$ be either positive or imaginary for all value of r, which make s_2 negative hence corresponding fundamental solution is always negative for all value of r.

Hence the system is “linear stable” when $r < 1$ for the given basic state.

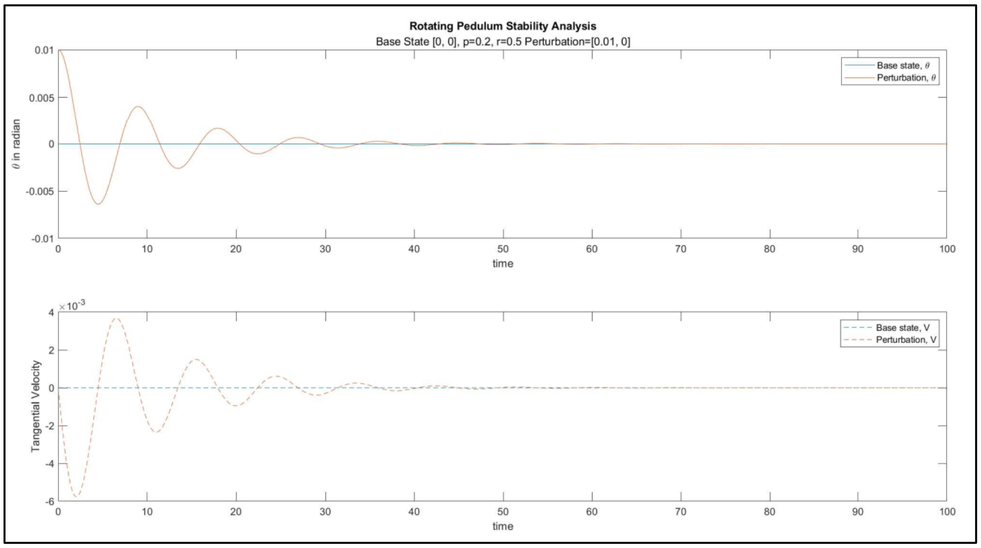

Figure 7. Perturbation Evolution with time, $r < 1$

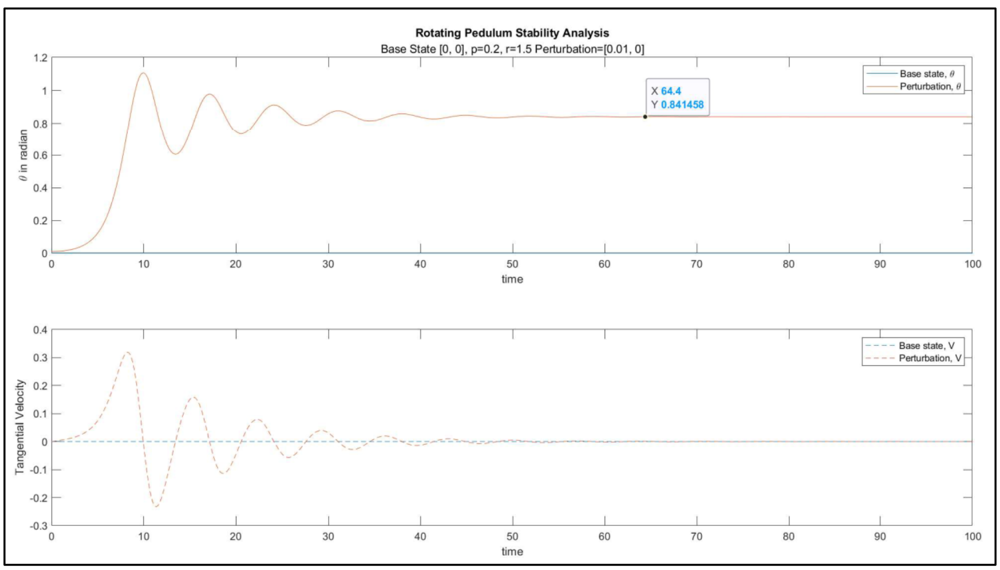

Figure 8. Perturbation Evolution with time, $r > 1$

Initial perturbation, $\phi' = [0.01, 0$

Form fig. 7, we can see that for $r < 1$ perturbation decay in a time and system comes to original the base state. Hence system is “Linearly Stable”. On the other hand, for $r > 1$, perturbation grow in time and after some time it oscillate about some other state and then decay to that sate. Since system attend new stable state for given r. Hence, system is linearly unstable for a given base state.

The new stable state for given r, is $\theta = \cos^{-1}(1/r)$ & V=0