

Arrangements with Crossings

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1 Notation

A finite set of lines L in \mathbb{R}^2 partitions the plane into d -faces where d , signifying dimension, can be zero, one, or two. This notation follows Edelsbrunner [3]. These faces are points for $d = 0$, segments for $d = 1$, and cells for $d = 2$. Here cells are what are commonly referred to as faces. This partition is called an *arrangement*, and is denoted $\mathcal{A}(L)$. If no lines are parallel and no three lines intersect in a common point, then the arrangement is called *simple*. In this case, if there are m lines then there are exactly $\binom{m}{2}$ points of intersection. In this paper all arrangements will be assumed simple.

An *arrangement graph* is a graph $G(L)$ defined from $\mathcal{A}(L)$ with vertex set the $\binom{m}{2}$ intersections in $\mathcal{A}(L)$ and v and w adjacent in $G(L)$ if and only if v and w are the endpoints of a 1-face, i.e. segment, in $\mathcal{A}(L)$. That is, v and w are on a common line l of L , and no other line of L intersects l between v and w . Each line intersects $m - 1$ other lines, so each line contributes $m - 2$ edges to $G(L)$; hence there are $m(m - 2)$ total edges in $G(L)$. Note that $G(L)$ is planar, and that the degree of any vertex is either two, three, or four. Lines in the arrangement graph are the paths whose edges correspond to segments of a common line in $\mathcal{A}(L)$. The set of lines in the arrangement graph will be denoted $l(G)$, and for any line $l \in l(G)$, its corresponding line in $\mathcal{A}(L)$ will be denoted $\rho(l)$. Note that arrangement graphs ignore the half-infinite segments.

We will study sets of nonintersecting lines in \mathbb{R}^3 whose projections to a plane form simple arrangements. We will write elements of \mathbb{R}^3 in the standard basis as triples $\vec{r} = (x, y, z)$, and we will assume that we are projecting downward to the xy -plane. Let L be a set of nonintersecting lines in \mathbb{R}^3 . If l is a line in \mathbb{R}^3 parameterized by $l(t) = (x(t), y(t), z(t))$, then let $\pi(l)$ denote the line $(x(t), y(t), 0)$ in the xy -plane. Suppose additionally that the set $\pi L = \{\pi(l) : l \in L\}$ has $\mathcal{A}(L)$ simple. We form a new structure, which we call an *arrangement with crossings*, denoted by $\mathcal{C}(L)$, as follows. $\mathcal{C}(L)$ is the pair $(c, \mathcal{A}(\pi L))$, where $c : \pi L \times \pi L \rightarrow \{-1, 1\}$ is the *crossing function* defined by $c(\pi(l_1)\pi(l_2)) = 1$ for distinct lines l_1 and l_2 if and only if l_1 passes over l_2 in \mathbb{R}^3 . More precisely, because $\mathcal{A}(\pi L)$ is

simple, $\pi(l_1)$ intersects $\pi(l_2)$ at some point p . Then $c(\pi(l_1), \pi(l_2)) = 1$ if and only if the z -coordinate of the intersection of l_1 with the vertical line passing through p is larger than the z -coordinate of the intersection of l_2 with said line. If $l_1 = l_2$, then $c(\pi(l_1), \pi(l_2))$ is not defined. In any other case, $c(\pi(l_1), \pi(l_2)) = -1$. Note that c is antisymmetric in its arguments. c is called the crossing map, and is only a formal way of describing something familiar: $(c, \mathcal{A}(\pi L))$ is really just a drawing in the plane of lines in \mathbb{R}^3 viewed from above, where the crossings are marked as in a knot diagram FIGURE 2.a.

The *crossing arrangement graph* is the pair $(c_G, G(\pi L))$, where $c_G : l(G) \times l(G) \rightarrow \{-1, 1\}$ where $c_G(l_1, l_2) = c(\rho(l_1), \rho(l_2))$. We will refer to c_G as the crossing function for the graph. That is, the crossing map for the crossing arrangement graph is the same crossing map for the arrangement with crossings, only the domain is a set of paths in the graph rather than lines in the plane. We will translate the crossing arrangement graph into a directed graph $D(L)$ in the following way. The vertex set of $D(L)$ is the vertex set of $G(\pi L)$, plus one vertex $\text{mi}(e)$ per edge e of $G(\pi L)$. The graph will be bipartite and directed. The only edges will be of the form $\text{mi}(e)v$ or $v \text{mi}(e)$ for $e \in E(G(\pi L))$ and $v \in e$. Let l_1 and l_2 in $l(G)$ be lines such that e belongs to l_1 and $v = l_1 \cap l_2$. These lines are uniquely defined, because each point is the intersection of exactly two lines and each edge is in exactly one line. Then $\text{mi}(e)v$ is an edge in $D(L)$ if and only if $c_G(l_1, l_2) = 1$, and $v \text{mi}(e)$ is an edge in $D(L)$ if and only if $c_G(l_1, l_2) = -1$. See FIGURE. $D(L)$ has $|E(G(L))| + |V(G(L))| = m(m-2) + \binom{m}{2}$ vertices and $2m(m-2)$ edges.

Throughout this paper, if G is a graph, $V(G)$ is the vertex set of G and $E(G)$ is the edge set.

2 Geometric to Combinatorial

The goal of this section is to show that $D(L)$ and $D(L')$ isomorphic as graphs if and only if $\mathcal{C}(L) = (c, \mathcal{A}(L))$ and $\mathcal{C}(L') = (c', \mathcal{A}(L'))$ are isomorphic in a certain sense.

First, we say that two arrangements $\mathcal{A}(L)$ and $\mathcal{A}(L')$ are isomorphic if there is a homeomorphism h of \mathbb{R}^2 carrying $\mathcal{A}(L)$ to $\mathcal{A}(L')$ in the sense that for each d -face X of $\mathcal{A}(L)$, we have $h|_X : X \rightarrow Y$ is a homeomorphism of X and Y where Y is a d -face of $\mathcal{A}(L')$. This means that points map to points, segments map to segments homeomorphically, and cells map to cells homeomorphically. In this case we refer to h as an isomorphism of arrangements, and write $\mathcal{A}(L) \cong \mathcal{A}(L')$.

We say that $\mathcal{C}(L) = (c, \mathcal{A}(L))$ and $\mathcal{C}(L') = (c', \mathcal{A}(L'))$ are isomorphic if there is an isomorphism of arrangements $h : \mathcal{A}(\pi L) \rightarrow \mathcal{A}(\pi L')$ such that for any two lines l_1 and l_2 in πL , $c(l_1, l_2) = c'(h(l_1), h(l_2))$, and the lefthand side is defined if and only if the right hand

side is. Similarly, their crossing arrangement graphs are isomorphic if there exists a graph isomorphism $G(\pi L) \rightarrow G(\pi L')$ that preserves the crossing function c_G . This definition only makes sense if $h(l)$ is a line in $\mathcal{A}(\pi L')$ for any $l \in \pi L$, which we will show is the case.

Lemma 1. *Let L, L' be finite sets of lines in the plane. If $h : \mathcal{A}(L) \rightarrow \mathcal{A}(L')$ is an isomorphism of (simple) arrangements, then any line in L is mapped homeomorphically by h to a line of L' .*

Proof. Let e_0 be one of the infinite segments of l . Then $h(e_0)$ is another infinite segment, say f_0 , for this is the only type of face that is homeomorphic to e_0 . Let v_0 be the endpoint of e_0 . Consider the segments e_1, e_2, e_3 that border e_0 at v_0 in clockwise order starting at e_0 . Observe that $e_2 \subset l$. Let l' be the line to which f_0 belongs and let w_0 be the endpoint of f_0 . Again, we consider the segments f_1, f_2, f_3 that border f_0 at w_0 in clockwise order starting at f_0 . $f_2 \subset l'$. We claim that $h(e_2) = f_2$. Because h is a homeomorphism, we know that $h(e_2)$ is a segment bordering f_0 at w_0 . Suppose $h(e_2) = f_1$. Then we must have $\{h(e_1), h(e_3)\} = \{f_2, f_3\}$.

l cuts the plane into two disjoint open half-planes, P_1 containing e_1 and P_3 containing e_3 . $h(l)$ is an unbounded polygonal curve that separates the plane into two disjoint regions R_1 and R_3 . $h(l)$ is unbounded because the half infinite segments belonging to l must map to half infinite segments in $\mathcal{A}(L')$, and it separates the plane into two regions by the Jordan curve theorem applied to the stereographic projection of the plane. Because h is a homeomorphism, we have $h(P_i) = R_i$ without loss of generality. Now note that both f_2 and f_3 are contained together in one of $h(P_1)$ or $h(P_3)$; however, $h(e_1) = f_1 \subset h(P_1)$ and $h(e_3) = f_3 \subset h(P_3)$. This is impossible because $h(P_3) \cap h(P_1) = \emptyset$.

Identical reasoning can be applied to the case $h(e_2) = f_3$, so we conclude $h(e_2) = f_2$. This reasoning can be applied inductively to see that the i^{th} segment of l counting away from e_0 , must map to the i^{th} segment of l' counting away from f_0 . Hence, $h(l) = l'$. \square

Lemma 2. *Two simple arrangements $\mathcal{A}(L)$ and $\mathcal{A}(L')$ are isomorphic if and only if their arrangement graphs $G(L)$ and $G(L')$ are isomorphic as graphs. In particular,*

Proof. Suppose $\phi : G(L) \rightarrow G(L')$ is an isomorphism. Let $\nu : G(L) \rightarrow \mathcal{A}(L)$ be the natural map sending the edges and vertices of $G(L)$ into the corresponding points and segments of $\mathcal{A}(L)$, which is an embedding of $G(L)$ as a plane graph. Note that the vertices of degree at most three in $G(L)$ are exactly those on the unbounded face in this embedding (these are the endpoints of the unbounded segments). $\nu' : G(L') \rightarrow \mathcal{A}(L')$ is an embedding of $G(L')$ with the same property. Then $\nu' \circ \phi$ is an embedding of $G(L)$, the image of which is identical to the embedding of $G(L')$. Following [1], we define a new graph $G(L)^*$ that is an identical copy of $G(L)$ plus a new vertex, x , that is attached to all vertices of degree at most three. Because both embeddings, ν and $\nu' \circ \phi$, of $G(L)$ have all vertices of degree two

and three on the unbounded face, there exist embeddings μ and μ' of $G(L)^*$ extending ν and $\nu' \circ \phi$, respectively. Namely, in both embeddings, x goes in the unbounded face and connects to all vertices on the unbounded face. From [1], $G(L)^*$ is three-connected. By a theorem originally due to Whitney [2], all embeddings of $G(L)^*$ with x on the unbounded face are equivalent, meaning there is a homeomorphism $h : \mu(G(L)^*) \rightarrow \mu'(G(L)^*)$ such that h extends the set map $\mu' \circ \mu^{-1}$ on the vertices and edges. As $\mu' \circ \mu^{-1}$ maps x to x , we must have $h : \nu(G(L)) \rightarrow \nu' \circ \phi(G(L))$. Furthermore, h must also extend the set map $\nu' \circ \phi \circ \nu^{-1}$.

If $h : \mathcal{A}(L) \rightarrow \mathcal{A}(L')$ is an isomorphism of arrangements, then one can easily see that the map induced by h on the vertices of $G(L)$ provides an isomorphism with $G(L')$. \square

Lemma 3. *Two arrangements with crossings $\mathcal{C}(L)$ and $\mathcal{C}(L')$ are isomorphic if their crossing arrangement graphs $(c_G, G(\pi L))$ and $(c'_G, G(\pi L'))$ are isomorphic.*

Proof. Let $\phi : (c_G, G(\pi L)) \rightarrow (c'_G, G(\pi L'))$ be an isomorphism. ϕ is also an isomorphism of graphs $G(\pi L) \rightarrow G(\pi L')$. From the previous lemma we know $\mathcal{A}(L)$ and $\mathcal{A}(L')$ are isomorphic under an isomorphism of arrangements ψ extending $\nu' \circ \phi \circ \nu^{-1}$ on the vertices. We have defined for lines $l_1, l_2 \subset G(\pi L)$, $c_G(l_1, l_2) = c(\nu(l_1), \nu(l_2))$; likewise for c' . Then

$$c'_G(\phi(l_1), \phi(l_2)) = c'(\nu' \phi(l_1), \nu' \phi(l_2)) = c'(\psi \circ \nu(l_1), \psi \circ \nu(l_2)).$$

But ϕ is an isomorphism of crossing arrangement graphs, so

$$c'_G(\phi(l_1), \phi(l_2)) = c_G(l_1, l_2) = c(\nu(l_1), \nu(l_2)).$$

In particular, $c'(\psi \circ \nu(l_1), \psi \circ \nu(l_2)) = c(\nu(l_1), \nu(l_2))$. As ν is a bijection, we see that ψ is an isomorphism of arrangements with crossings. \square

Lemma 4. *If $|L| = m > 3$, $D(L) \cong D(L') \implies (c_G, G(\pi L)) \cong (c'_G, G(\pi L'))$.*

Proof. Let $\phi : D(L) \rightarrow D(L')$ be an isomorphism of directed graphs. First we claim that the set $M_L = \{\text{mi}(e) : e \in G(\pi L)\} \subset V(D(L))$ is mapped to the corresponding set $M_{L'} \subset V(D(L'))$ by ϕ . Note that the undirected distance $d(\text{mi}(e), v)$ for $\text{mi}(e) \in M_L$ and $v \in V(D(L) \setminus M_L)$ is always odd, and $d(\text{mi}(e_1), \text{mi}(e_2))$ is always even. This is because the $\text{mi}(e)$ are midpoints inserted into edges of the graph $G(L)$. The same distance condition holds for $M_{L'}$ and $V(D(L')) \setminus M_{L'}$. Hence, we must have either $\phi M_L = M_{L'}$ or $\phi M_L = V(D(L')) \setminus M_{L'}$, because isomorphisms preserve distance. Let $m' = |L'|$. Recall that $|V(D(L'))| = m'(m' - 2) + \binom{m'}{2}$, which is monotone increasing for $m' > 2$, so we must have $m' = m$ because $|V(D(L'))| = |V(D(L))| = m(m - 2) + \binom{m}{2}$. Then $|M_L| = |M_{L'}| = m(m - 2)$, and $V(D(L')) \setminus M_{L'} = \binom{m}{2}$. According to Maple, $m(m - 2) > \binom{m}{2}$ for $m > 3$, so we must have $\phi M_L = M_{L'}$ as desired. Hence we also have $\phi V(D(L)) \setminus M_L = V(D(L')) \setminus M_{L'}$, which indicates that the subset of $V(D(L))$ corresponding to vertices in $V(G(L))$ are mapped to those corresponding to vertices in $V(G(\pi L'))$.

Viewing $V(G(\pi L)), V(G(\pi L'))$ as subsets of $V(D(L))$ and $V(D(L'))$, respectively, we claim that $\psi = \phi|_{V(G(\pi L))} : V(G(\pi L)) \rightarrow V(G(\pi L'))$ is an isomorphism. We have just shown that the codomain of ψ is within (and hence all of, by injectivity) $V(G(\pi L'))$, so it remains only to show that it is an isomorphism of arrangement graphs with crossings. First, it is a graph isomorphism. A fortiori, ϕ is an undirected graph isomorphism $D(L) \rightarrow D(L')$. If $vw \in E(G(\pi L))$, then $v \text{ mi}(vw)w$ is an undirected 2-path in $D(L)$. $\phi(v)\phi(\text{mi}(vw))\phi(w)$ remains an undirected 2-path in $D(L')$. As we showed in the previous paragraph, $\phi(v) = \psi(v)$ and $\phi(w) = \psi(w)$ are in $V(G(\pi L'))$. In $D(L')$ they are only connected to vertices of the form $\text{mi}(e)$ for $e \in E(G(\pi L))$, and each such $\text{mi}(e)$ has only two neighbors. As $\psi(v)$ and $\psi(w)$ have a common neighbor in $D(L')$, we can deduce that it is $\text{mi}(\psi(v)\psi(w))$, and hence that $\psi(v)\psi(w) \in E(G(\pi L'))$. In particular, $\phi(\text{mi}(vw)) = \text{mi}(\psi(v)\psi(w))$. The same argument applied to ϕ^{-1} shows if $vw \notin E(G(\pi L))$, then $\psi(v)\psi(w) \notin E(G(\pi L'))$.

Now we show that $c'_G(\psi(l_1), \psi(l_2)) = c_G(l_1, l_2)$ for distinct lines $l_1, l_2 \subset G(\pi L)$. Suppose l_1 and l_2 cross at v , and that w_1 and w_2 are vertices on l_1 and l_2 , respectively, that are adjacent to v . Assume $c_G(l_1, l_2) = 1$. Then we have directed edges $\text{mi}(vw_1)v$ and $v \text{ mi}(vw_2) \in E(D(L))$. As ϕ is an isomorphism of directed graphs, $\phi(\text{mi}(vw_1))\psi(v)\psi(v) = \text{mi}(\psi(v)\psi(w_1))\psi(v)$ and $\psi(v) \text{ mi}(\psi(v)\psi(w_2))$ are directed edges of $D(L')$. We know $\psi(l_i)$ is the line containing $\psi(v)$ and $\psi(w_i)$, for $i = 1, 2$. By definition of $D(L')$, we have $c'_G(\psi(l_1), \psi(l_2)) = 1$. The -1 case follows by antisymmetry (we can swap l_1 and l_2). Hence, ψ is an isomorphism of arrangement graphs with crossings. \square

Theorem 1. *For L, L' finite sets of nonintersecting lines in \mathbb{R}^3 with $|L| > 3$, the following are equivalent:*

1. $D(L) \cong D(L')$.
2. $(c_G, G(\pi L)) \cong (c'_G, G(\pi L'))$.
3. $\mathcal{C}(L) \cong \mathcal{C}(L')$.

Proof. A moments' thought shows that $3 \implies 1$. Lemma 4 gives $1 \implies 2$. Lemma 3 gives $2 \implies 3$. \square

References

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