COS 4892 Assignment 2

Adriaan Louw (53031377)

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1 Question 1

We will define

$$Q = ((Q_1 \wedge B) \vee (Q_2 \wedge \neg B)) \tag{1}$$

Now we calculate from Equation 1

$$\begin{aligned} Q \wedge B &= ((Q_1 \wedge B) \vee (Q_2 \wedge B)) \wedge B \\ &= (Q_1 \wedge B \wedge B) \vee (Q_2 \wedge \neg B \wedge B) \\ &= Q_1 \wedge B \end{aligned} \tag{2}$$

and

$$\begin{aligned} Q \wedge B &= ((Q_1 \wedge B) \vee (Q_2 \wedge B)) \wedge B \\ &= (Q_1 \wedge B \wedge \neg B) \vee (Q_2 \wedge \neg B \wedge \neg B) \\ &= Q_2 \wedge \neg B \end{aligned} \tag{3}$$

2 Question 2.1

$$\{?\}x := 3.2z\{wy - 2w^2 < z\} \tag{4}$$

$$\{wy - 2w^2 < z\}x := 3.2z\{wy - 2w^2 < z\} \tag{5}$$

The post condition is not dependant on x. Thus there is notihing to replace by using the assignment Axiom

3 Question 2.2

$$\{?\}x := x - 1; y := y - 1\{z - 1 \leq y < x \leq w\} \tag{6}$$

$$\{z - 1 \leq y - 1 < x - 1 \leq w\}x := x - 1; y := y - 1\{z - 1 \leq y < x \leq w\} \tag{7}$$

$$\{z \leq y < x \leq w + 1\}x := x - 1; y := y - 1\{z - 1 \leq y < x \leq w\} \tag{8}$$

4 Question 2.3

$$\{?\}if even(x) \rightarrow x := x - 1 else odd(x) \rightarrow z := z + yx\{x \geq 0 \wedge z + yx = ab\} \tag{9}$$

Using the retrogressive theorem for the if statement:
Taking the case as of $S1 = even(x) \rightarrow x := x - 1$

$$\{?\}x : x - 1\{x \geq 0 \wedge z + yx = ab\} \tag{10}$$

$$\{x \geq 0 \wedge z + y(x - 1) = ab\}x : x - 1\{x \geq 0 \wedge z + yx = ab\} \tag{11}$$

$$\{x \geq 0 \wedge z + yx - y = ab\}x : x - 1\{x \geq 0 \wedge z + yx = ab\} \tag{12}$$

Taking the case as of $S2 = odd(x) \rightarrow z := z + yx$

$$\{?\}z : z + yx\{x \geq 0 \wedge z + yx = ab\} \tag{13}$$

$$\{x \geq 0 \wedge (z + yx) + yx = ab\}z : z + yx\{x \geq 0 \wedge z + yx = ab\} \tag{14}$$

$$\{x \geq 0 \wedge z + 2yx = ab\}z : z + yx\{x \geq 0 \wedge z + yx = ab\} \tag{15}$$

Now applying the retrogressive theorem for the if statement

$$\begin{aligned}\{(x \geq 0 \wedge z + yx - y = ab \wedge even(x)) \vee (x \geq 0 \wedge z + 2yx = ab \wedge odd(x))\}\\ if even(x) \rightarrow x := x - 1 else odd(x) \rightarrow z := z + yx\\ \{x \geq 0 \wedge z + yx = ab\}\end{aligned} \tag{16}$$

5 Question 2.4

$$\{?\}while 0 \geq c \geq -2 do x := x - 1 endwhile\{x = -3\} \tag{17}$$

$$B = 0 \geq c \geq -2 \tag{18}$$

$$S = x := x - 1 \tag{19}$$

Starting with

$$\begin{aligned}\{Z_0\}S^0\{\neg B\}\\ \{B \wedge C_0\}x := x - 1\{x = -3\}\\ \{0 \geq x \geq -2 \wedge x - 1 = -3\}x := x - 1\{x = -3\}\\ \{0 \geq x \geq -2 \wedge x = -2\}x := x - 1\{x = -3\}\end{aligned} \tag{20}$$

$$\begin{aligned}\{Z_1\}S\{Z_0\}\\ \{0 \geq x \geq -2 \wedge x - 1 = -2\}x := x - 1\{0 \geq c \geq -2 \wedge x = -2\}\\ \{0 \geq x \geq -2 \wedge x = -1\}x := x - 1\{0 \geq c \geq -2 \wedge x = -2\}\end{aligned} \tag{21}$$

$$\begin{aligned}\{Z_2\}S\{Z_1\}\\ \{0 \geq x \geq -2 \wedge x - 1 = -1\}x := x - 1\{1 \geq x \geq -1 \wedge x = -1\}\\ \{0 \geq x \geq -2 \wedge x = 0\}x := x - 1\{1 \geq x \geq -1 \wedge x = -1\}\end{aligned} \tag{22}$$

$$\begin{aligned}\{Z_3\}S\{Z_2\}\\ \{0 \geq x \geq -2 \wedge x - 1 = 0\}x := x - 1\{2 \geq x \geq 0 \wedge x = 0\}\\ \{0 \geq x \geq -2 \wedge x = 1\}x := x - 1\{2 \geq x \geq 0 \wedge x = 0\}\end{aligned} \tag{23}$$

Z_3 is false thus the preconditions of S are

$$\begin{aligned}&\{Z_0 \vee Z_1 \vee Z_2\}\\ &\{(0 \geq x \geq -2 \wedge x = -2) \vee (0 \geq x \geq -2 \wedge x = -1) \vee (0 \geq x \geq -2 \wedge x = 0)\}\end{aligned} \tag{24}$$

6 Question 3

A Hoare triple ,{P}S{Q}, is valid if before execution statement s, P is valid and directly after executing S, Q is valid. Therefor for each possible value of P, statement S has to be exercuted and determined whether the result causes Q to be valid or not.

7 Question 4

In this example P can have value of x = 3, x= -3, x = 4 or x=-4. In the the first case when x = 3, after applying the statement we get y = 3. For the case of x = -3 then y = 3. When x = 4 the results in y = 4 and for x = -4, after apllying the if statement, y = 4. In each of the 4 cases the value of y is within the range defined by the postcondition. Thus the statement is valid.

8 Question 5.1

The programmer specified the loop invariant as:

$$n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge i = 1^{k-1} A(i) \neq X \tag{25}$$

n and k must be integers because they were integers in the precondition and their type cannot be changed. We know that tyhe invariant should contain

$$< \sum k : 1 \leq k \leq n \wedge A(i) \neq X : 1 > \wedge < \sum k : k = n+1 : 1 > \tag{26}$$

because the loops starts with k = 1 up to n where A(i) is not equal to X and k is larger than n for the end condition.

thus

$$\begin{aligned} < \sum k : 1 \leq k \leq n \wedge A(i) \neq X : 1 > \wedge < \sum k : k = n+1 : 1 > = \\ < \sum k : 1 \leq k \leq n \wedge A(i) \neq X \vee k = n+1 : 1 > = \\ < \sum k : 1 \leq k \leq n+1 \wedge i = 1^{k-1} A(i) \neq X : 1 > \end{aligned} \tag{27}$$

9 Question 5.2

We need to verify the Hoare tripple

$$\begin{aligned} \{n \in Z \wedge 0 \leq n\} \\ k := 1 \\ \{ninZ \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge i = 1^{k-1} A(i) \neq X\} \end{aligned} \tag{28}$$

Using the assignment statement we need to show that

$$\begin{aligned} n \in Z \wedge 0 \leq n \\ \rightarrow n \in Z \wedge 1 \in Z \wedge 1 \leq 1 \leq n+1 \wedge i = 1^{1-1} A(i) \neq x \\ \rightarrow n \in Z \wedge true \wedge 0 \leq n \wedge true \\ \rightarrow n \in Z \wedge 0 \leq n \end{aligned} \tag{29}$$

where 1 is an element of Z by definition

10 Question 5.3

We need to prove that

$$n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \wedge k \leq n \wedge A(k) \neq X \rightarrow n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \tag{30}$$

For the first part of the implication

$$n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \wedge k \leq n \wedge A(k) \neq X \tag{31}$$

$k \leq n$ is a subset of $k \in Z \wedge 1 \leq k \leq n+1$ and $A(k) \neq X$ is just the kase when k = 1 in the term $1^{k-1} A(i) \neq X$. Thus Equation 30 becomes

$$n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \rightarrow n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \tag{32}$$

Which is true

11 Question 5.4

we need to prove that $[inv \wedge done \rightarrow Q]$

$$\neg done = k \leq n \wedge A(k) \neq X \tag{33}$$

Thus by de Morgans Law

$$done = k > n \vee A(k) = X \tag{34}$$

Thus

$$\begin{gathered} n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \wedge (k > n \vee A(k) = X) \rightarrow \\ n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge 1^{k-1} A(i) \neq X \wedge (k = n+1 \vee A(k) = X) \end{gathered} \tag{35}$$

Since k = n+1 is a subset of k ¿ n

12 Question 5.5

Using bounding function

$$bf = n - k + 1 \tag{36}$$

We know that to gaurentee temination the following tripple has to be valid

$$\{inv \wedge \neg done]wedgebf > C\}T\{inv \wedge (bf < C \vee done)\} \tag{37}$$

By filling in the post conditions we will apply the assignment statement to reach the precondition.

$$\begin{gathered} \{inv \wedge (bf < C \vee done)\} = \\ n \in Z \wedge k \in Z \wedge 1 \leq k \leq n+1 \wedge i = i^{k-1} A(i) \neq X \wedge (n - k + 1 < 1 \vee k > n \vee A(k) = X) \end{gathered} \tag{38}$$

Applying the assiggnments statement

$$\begin{gathered} n \in Z \wedge k+1 \in Z \wedge 1 \leq k+1 \leq n+1 \wedge i = i^{k+1-1} A(i) \neq X \wedge (n - k + 1 + 1 < 1 \vee k+1 > n \vee A(k+1) = X) = \\ n \in Z \wedge k+1 \in Z \wedge 0 \leq k \leq n \wedge i = i^{k} A(i) \neq X \wedge (n < k - 1 < 1 \vee n < k+1 \vee A(k+1) = X) = (n \in Z \wedge \end{gathered} \tag{39}$$