COS 4807 Assignment 3

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1 Question 1i

Let $\mathscr{I}$ be an arbitrary interpretation such that $v_{\mathscr{I}}(\forall xp(x) \vee \exists xq(x)) = F$. From the truth value of disjunction $v_{\mathscr{I}}(\forall xp(x) = F$ and $v_{\mathscr{I}}\exists xq(x)) = F$. From this using Theorem 7.22 we for all assignments $v_{\sigma\mathscr{I}}(p(x)) = F$ and for some assignments $v_{\sigma\mathscr{I}}(q(x)) = F$. Then by the truth value of disjunction $v_{\sigma\mathscr{I}}(p(x) \vee q(x)) = F$. Then by using Theorem 7.22 $v_I(\forall(p(x) \vee q(x))) = F$. Then if $v_I(\forall xp(x) \vee \exists xq(x)) = F$ then $v_{\mathscr{I}}(\forall x(p(x) \vee q(x) \rightarrow (\forall xp(x) \vee \exists xq(x)) = T))$ by the truth value of implication. And since $\mathscr{I}$ is an arbitrary interpretation, the formula is valid

2 Question 1ii

Let $\mathscr{I}$ ba an arbitrary interpretation such that $v_{\mathscr{I}}(\forall x\neg p(x) \vee \forall x\neg q(x)) = F$. Then from the definition of disjunction $v_{\mathscr{I}}(\forall x\neg p(x)) = F$ and $v_{\mathscr{I}}(\forall x\neg q(x)) = F$. Using the theorem from question 3ii we get for all assignments $v_{\sigma\mathscr{I}}(\neg p(x)) = F$ and for all assignmets $v_{\sigma\mathscr{I}}(\neg q(x)) = F$. Then by the truth values of negation, $v_{\sigma\mathscr{I}}p(x) = T$ and $v_{\sigma\mathscr{I}}q(x) = T$. Then by theorem 7.22 and the definition of conjunction $v_{\mathscr{I}}(\exists xp(x) \wedge q(x)) = T$. Now we have shown that $v_{\mathscr{I}}(\forall x\neg p(x) \vee \forall x\neg q(x)) = F$ and $v_{\mathscr{I}}\exists(xp(x) \wedge q(x)) = T$. Combining these into the original formula we get $v_{\mathscr{I}}(\exists x(p(x) \wedge q(x)) \wedge (\forall x\neg p(x) \vee \forall x\neg q(x))) = F$ by the definition of conjunction. And since $\mathscr{I}$ is an arbitrary interpretation, the formula is unsatifiable.

3 Question 2i

$$\forall x(\neg p(x) \leftrightarrow \exists y(q(a, x, y) \wedge r(x, y)) \tag{1}$$

4 Question 2ii

$\{\mathscr{N}, prime, equals, less\_than\}$

5 Question 3i

6 Question 3ii

7 Question 4i

To prove that

$$\forall x(p(x) \vee q(x)) \rightarrow (\forall xp(x) \vee \exists xq(x)) \tag{2}$$

is valid with sematic tableau we need to show that

$$\neg\forall x(p(x) \vee q(x)) \rightarrow (\forall xp(x) \vee \exists xq(x)) \tag{3}$$

closes.

Applying rule $\neg(A_1 \rightarrow A_2)$ we get

$$\forall x(p(x) \vee q(x)), \neg(\forall xp(x) \vee \exists xq(x))) \tag{4}$$

Applying rule $\neg(A_1 \vee A_2)$ we get

$$\forall x(p(x) \vee q(x)), \neg\forall xp(x), \neg\exists xq(x) \tag{5}$$

Using the dual of the 2 quantifiers

$$\forall x(p(x) \vee q(x)), \exists \neg x p(x), \forall \neg x q(x) \tag{6}$$

Using the γ rule ofr existential qualification

$$\forall x(p(x) \vee q(x)), \neg p(a_1), \forall \neg x q(x) \tag{7}$$

$$\forall x(p(x) \vee q(x)), p(a_1) \vee q(a_1), \neg p(a_1), \forall \neg x q(x), \neg q(a_1) \tag{8}$$

Using $B_1 \vee B_2$the branches split into

$$\forall x(p(x) \vee q(x)), p(a_1), \neg p(a_1), \forall \neg x q(x), \neg q(a_1) \tag{9}$$

and

$$\forall x(p(x) \vee q(x)), q(a_1), \neg p(a_1), \forall \neg x q(x), \neg q(a_1) \tag{10}$$

Both of which close. THus the original formula is valid

8 Question 4ii

We need to prove that

$$\exists x(p(x) \wedge q(x)) \wedge (\forall x \neg p(x) \vee \forall x \neg q(x)) \tag{11}$$

is unsatisfiable. Thus the tableau for this formula has to close.
Applying rule $A_1 \wedge A_2$

$$\exists x(p(x) \wedge q(x)), \forall x \neg p(x) \vee \forall x \neg q(x) \tag{12}$$

Applying rule $B_1 \vee B_2$ we get

$$\exists x(p(x) \wedge q(x)), \forall x \neg p(x) \tag{13}$$

$$\exists x(p(x) \wedge q(x)), \forall x \neg q(x) \tag{14}$$

For branch of Equation 13

$$p(a_1) \wedge q(a_1), \forall x \neg p(x) \tag{15}$$

$$p(a_1) \wedge q(a_1), \forall x \neg p(x), \neg p(a_1) \tag{16}$$

using $A_1 \wedge A_2$

$$p(a_1), q(a_1), \forall x \neg p(x), \neg p(a_1) \tag{17}$$

which closes.
Now for the branch of Equation 14

$$p(a_1) \wedge q(a_1), \forall x \neg q(x) \tag{18}$$

$$p(a_1) \wedge q(a_1), \forall x \neg q(x), \neg q(a_1) \tag{19}$$

using $A_1 \wedge A_2$

$$p(a_1), q(a_1), \forall x \neg q(x), \neg q(a_1) \tag{20}$$

which closes.
Thus the original formula is unsatisfiable

9 Question 5i

We need to show that the tableau for this formula has an open branch

$$(\forall x(p(x) \vee q(x))) \rightarrow (\forall x p(x) \vee \forall x q(x)) \tag{21}$$

Using rule $B_1 \rightarrow B_2$ we get

$$\neg \forall x(p(x) \vee q(x)) \tag{22}$$

and

$$\forall x p(x) \vee \forall x q(x) \tag{23}$$

We will continue with Equation 22

$$\exists \neg x(p(x) \vee q(x)) \tag{24}$$

$$\neg(p(a_1) \vee q(a_1)) \tag{25}$$

Using rule $\neg(A_1 \vee A_2)$

$$\neg p(a_1), \neg q(a_1) \tag{26}$$

Which is an open branch. So the original formula is satisfiable and there is no need to continue with the other branch on Equation 23

10 Question 5ii

We need to prove that the negation of the formula is satisfiable i.e. has an open branch

$$\neg((\forall x(p(x) \vee q(x))) \rightarrow (\forall x p(x) \vee \forall x q(x))) \tag{27}$$

Using rule $\neg(A_1 \rightarrow A_2)$

$$\forall x(p(x) \vee q(x)), \neg(\forall x p(x) \vee \forall x q(x)) \tag{28}$$

Using rule $\neg(A_1 \vee A_2)$

$$\forall x(p(x) \vee q(x)), \neg \forall x p(x), \neg \forall x q(x) \tag{29}$$

$$\forall x(p(x) \vee q(x)), \exists \neg x p(x), \exists \neg x q(x) \tag{30}$$

$$\forall x(p(x) \vee q(x)), \neg p(a_1), \exists \neg x q(x) \tag{31}$$

$$\forall x(p(x) \vee q(x)), p(a_1) \vee q(a_1), \neg p(a_1), \exists \neg x q(x) \tag{32}$$

$$\forall x(p(x) \vee q(x)), p(a_1) \vee q(a_1), \neg p(a_1), \neg q(a_2) \tag{33}$$

$$\forall x(p(x) \vee q(x)), p(a_1) \vee q(a_1), \neg p(a_1), \neg q(a_2), p(a_2) \vee q(a_2) \tag{34}$$

Using rule $B_1 \vee B_2$ on the term $p(a_1) \vee q(a_1)$ we get 2 branches.

$$\forall x(p(x) \vee q(x)), p(a_1), \neg p(a_1), \neg q(a_2), p(a_2) \vee q(a_2) \tag{35}$$

and

$$\forall x(p(x) \vee q(x)), q(a_1), \neg p(a_1), \neg q(a_2), p(a_2) \vee q(a_2) \tag{36}$$

Equation 35 closes so we continue with Equation 36. Using rule $B_1 \vee B_2$ gives

$$\forall x(p(x) \vee q(x)), q(a_1), \neg p(a_1), \neg q(a_2), p(a_2) \tag{37}$$

and

$$\forall x(p(x) \vee q(x)), q(a_1), \neg p(a_1), \neg q(a_2), q(a_2) \tag{38}$$

Equation 38 closes but Equation 37 is an open branch. Therefor the original equation is falsifiable.