COS 4807 Assignment 1

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1 Question 1.1

For Formula

$$(((p \lor q) \lor r) \land (\neg p \land \neg q)) \to r \quad (1)$$

To prove the formula is valid we need to show it is true for all possible interpretations. We will use 0 to represent false and 1 to represent true

p	q	r	$\neg p$	$\neg q$	$p \lor q$	$(p \lor q) \lor r$	$\neg p \land \neg q$	$(((p \lor q) \lor r) \land (\neg p \land \neg q))$	$(((p \lor q) \lor r) \land (\neg p \land \neg q)) \to r$
0	0	0	1	1	0	0	1	0	1
0	0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0	1
0	1	1	1	0	1	1	0	0	1
1	0	0	0	1	1	1	0	0	1
1	0	1	0	1	1	1	0	0	1
1	1	0	0	0	1	1	0	0	1
1	1	1	0	0	1	1	0	0	1

Since this formula is true in all cases (last column), it is valid (tautology).

2 Question1.2

For Formula

$$(p \to (q \to r)) \leftrightarrow ((p \land q) \land \neg r) \quad (2)$$

to show that it is unsatisfiable we need to show that it is false for all interpretations

p	q	r	$q \to r$	$p \to (q \to r)$	$p \land q$	$\neg r$	$(p \land q) \land \neg r$	$(p \to (q \to r)) \leftrightarrow ((p \land q) \land \neg r)$
0	0	0	1	1	0	1	0	0
0	0	1	1	1	0	0	0	0
0	1	0	0	1	0	1	0	0
0	1	1	1	1	0	0	0	0
1	0	0	1	1	0	1	0	0
1	0	1	1	1	0	0	0	0
1	1	0	0	0	1	1	1	0
1	1	1	0	1	1	0	0	0

From this we can see that the Formula is unsatisfiable for all interpretations of p q and r

3 Question1.3

For Formula

$$(p \to (q \to r)) \to ((p \to q) \to r) \quad (3)$$

we need to show that there exists at least 1 interpretation for which this formula is false and at least one interpretation for which this formula is true

p	q	r	$q \to r$	$p \to q$	$p \to (q \to r)$	$(p \to q) \to r$	$(p \to (q \to r)) \to ((p \to q) \to r)$
0	0	0	1	1	1	0	0
0	0	1	1	1	1	1	1
0	1	0	0	1	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	1	1	1
1	0	1	1	0	1	1	1
1	1	0	0	1	0	0	1
1	1	1	1	1	1	1	1

The above trusth table shows that the formula is true for some interpretations and thus the Formula is satisfiable. Also the truth table shows that the formula is false for some interpretations and thus the formula is falsifiable

4 Question2.1

Given Formula

$$F_1 = (((p \lor q) \lor r) \land (\neg p \land \neg q)) \to r \tag{4}$$

We need to show that

$$\nu_I(F_1) = T \tag{5}$$

for any interpretation of p, q and r.
Using proof by contradiction we assume

$$\nu_I(F_1) = F \tag{6}$$

then

$$\nu_I(r) = F \tag{7}$$

and

$$\nu_I(((p \lor q) \lor r) \land (\neg p \land \neg q)) = T \tag{8}$$

The subformula

$$((p \lor q) \lor r) \land (\neg p \land \neg q) \tag{9}$$

is made up of the conjuction between sub formulas

$$(p \lor q) \lor r \tag{10}$$

and

$$\neg p \land \neg q \tag{11}$$

Thus the statement 8 can only be valid if both the statements

$$\nu_I((p \lor q) \lor r) = T \tag{12}$$

and

$$\nu_I(\neg p \land \neg q) = T \tag{13}$$

are true.
But formula 11 is the negation of formula 10 by de Morgans law, taking into account that $\nu_I(r) = F$ and the disjunction of any statement with a false statement has the tuth value of the original statement. Thereore we have a contradiction. Statement 8 cannot be true because the conjunction of a formula and its negation cannot be True for any interpretation. That means our assumption in statement 6 is incorect and therefore statement 5 is true. And since F_1 is true for all interpretations, it is a valid formula.

5 Question 2.2

Given formula

$$(p \to (q \to r)) \leftrightarrow ((p \land q) \land \neg r) \tag{14}$$

We need to prove that

$$\nu_I((p \to (q \to r)) \leftrightarrow ((p \land q) \land \neg r)) = F \tag{15}$$

Using proof by contradiction, we assume

$$\nu_I((p \to (q \to r)) \leftrightarrow ((p \land q) \land \neg r)) = T \tag{16}$$

We know that the equivalence operator only returns true when both sides of the forluma is true. In this case when

$$\nu_I(p \to (q \to r)) = F \tag{17}$$

and

$$\nu_I((p \land q) \land \neg r) = F \tag{18}$$

which we will call Case 1 and

$$\nu_I(p \to (q \to r)) = T \tag{19}$$

and

$$\nu_I((p \land q) \land \neg r) = T \tag{20}$$

which we will call case 2.
For Case 1, formula 17 can only be valid if

$$\nu_I(p) = T \tag{21}$$

and

$$\nu_I(q \to r) = F \tag{22}$$

are valid. In turn, for eaqation 22 to be valid we have to have

$$\nu_I(q) = T \tag{23}$$

and

$$\nu_I(r) = F \tag{24}$$

Thus in case 1 for equation 17 to be valid we have to have the values $\nu_I(p) = T$, $\nu_I(q) = T$ and $\nu_I(r) = F$. Now to prove Case 1 we have to use these values for interpretation 18. But that gives

$$\nu_I((p \land) \land \neg r) = T \tag{25}$$

which is a contradiction to equation 18. Which means formula cannot be true when wh have equations 17 and 18.
Now for case 2:
Equation 20 can only be true if $\nu_I(p) = T$, $\nu_I(q) = T$ and $\nu_I(r) = F$ due to the definition of conjuctions. Equation 19 need to be true for this interpretation. But using these interpretations for p,q and r gives

$$\nu_I(p \to (q \to r)) = F \tag{26}$$

Which is a contradiction with equation 19.
In both cases when formula 14 could have been True, we have a contradiction. The formula has to be false for all interpretations and is thus unsatisfiable.

6 Question 3

This is equivalent to proving the contrapositive namely if some tableau for A is open then A is satisfiable. Further we need the concept of a Hintikka set. Where if U is a set of formulas, U is a Hitikka set when:

1. for all atoms p in a formula of U then either $p \notin U$ or $\neg p \notin U$
2. If $A \in U$ is an α-formula then, $A_1 \in U$ and $A_2 \in U$
3. if $B \in U$ ia a β-formula then, $B_1 \in U$ or $B_2 \in U$

Now we show that the union of all the formulas from an open leaf to the root of a completed tableau is a Hintikka set. Firstly any literal p or $\neg p$ cannot be decomposed futher in semantic tableau. This means that any literal will copied unitl it is iin the leaf node. But since the leaf node is open the first condition for a Hintikka set holds. For the second condition, since the tableau is completed, when some formula A (α-formula) in the set is decomposed, A_1 and A_2 will be in U. Similarly for some formula B (β-formula) when it is decomposed in a completed tableau, either B_1 or b_2 will be in U.

Now a Hintikka set U is satisfiable.

Then if we have an completed open tableau, Then the union of the labels creates a Hintikka set. Since a Hintikka set is satisfiable, then the tableau is satisfiable,

7 Question 4.1

For formula

$$(((p \vee q) \vee r) \wedge (\neg p \wedge \neg q)) \rightarrow r \tag{27}$$

Applying the β formula for implication gives

$$\neg(((p \vee q) \vee r) \wedge (\neg p \wedge \neg q)) \tag{28}$$

and

$$r \tag{29}$$

Where Formula 29 is satisfiable.

Continueing for Formula 28 by applying the β formula for $\neg(B_1 \wedge B_2)$ we get

$$\neg((p \vee q) \vee r) \tag{30}$$

and

$$\neg(\neg p \wedge \neg q) \tag{31}$$

Continueing for Formula 30 and applying α formula $\neg(A_1 \vee A_2)$ we get

$$\neg(p \vee q), \neg r \tag{32}$$

Then applying formula $\neg(A_1 \vee A_2)$ again we get

$$\neg p, \neg q, \neg r \tag{33}$$

Which is satisfiable.

Going back to Formula 31 which can by simplified to

$$p \vee q \tag{34}$$

Then applying β formula $B_1 \vee B_2$ we get formulas

$$p \tag{35}$$

and

$$q \tag{36}$$

Both of which are satifyable. Now looking at all the leaf nodes namely Formalas 29, 33, 35 and 36 we can see that they are all satisfiable. Thus Formula 27 is valid.

8 Question 4.2

$$(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \wedge \neg r) \tag{37}$$

By substituting for double implication operator:

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \wedge \neg r), ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{38}$$

Substituting for the implication operator in the first term gives:

$$\neg(p \rightarrow (q \rightarrow r)), ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{39}$$

and

$$((p \wedge q) \wedge \neg r), ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{40}$$

Equation 39 becomes

$$p, \neg(q \rightarrow r)), ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{41}$$

then

$$p, q, \neg r, ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{42}$$

Substituting for the implication in the above formula gives:

$$p, q, \neg r, \neg((p \wedge q) \wedge \neg r) \tag{43}$$

and

$$p, q, \neg r, (p \rightarrow (q \rightarrow r)) \tag{44}$$

Formula 43 becomes

$$p, q, \neg r, \neg(p \wedge q) \tag{45}$$

and

$$p, q, \neg r, r \tag{46}$$

We can see that Formula 46 is unsatifyable. Then Formula 45 becomes

$$p, q, \neg r, \neg p \tag{47}$$

and

$$p, q, \neg r, \neg q \tag{48}$$

Both of which are unsatifyable. Now having done all the leaves under Formula 43, we continue with the leaves under Equation 44.

Equation 44 becomes

$$p, q, \neg r, \neg p \tag{49}$$

which is unsatifyable and

$$p, q, \neg r, q \rightarrow r \tag{50}$$

which becomes

$$p, q, \neg r, \neg q, r \tag{51}$$

Having finished all the leaves under Formula 39 we continue with Formula 40 which becomes

$$p, q, \neg r, ((p \wedge q) \wedge \neg r) \rightarrow (p \rightarrow (q \rightarrow r)) \tag{52}$$

But Formula 52 is the same as Formula 42. Formula 42 will have the same leaf nodes as Formula 52.

Thus we have all the leaf nodes namely Formulas 46, 47, 48, 49 and 51. All of these Formulas are unsatisfiable, therefore the original Formula 37 is unsatisfiable.

9 Question 4.3

For formula

$$(p \to (q \to r)) \to ((p \to q) \to r) \tag{53}$$

We have to prove that it is satisfiable and falsifiable using semantic tableaux. Applying the β formula fot implication we get

$$\neg(p \to (q \to r)) \tag{54}$$

and

$$((p \to q) \to r) \tag{55}$$

Applying the α formula for negated implication, set 54 becomes

$$p, \neg(q \to r)) \tag{56}$$

Applying the *alpha* formula for negated implication to set 57 we get

$$p, q, \neg r \tag{57}$$

We have found a set that does not contain an atom and its negation. Proving formula 53 is satisfiable.

10 Question 5

1.	$\vdash \neg p$	(axiom)
2.	$\vdash \neg q$	(axiom)
3.	$\vdash \neg r$	(axiom)
4.	$\vdash p, q, r$	(axiom)
5.	$\vdash p \vee q, r$	$\alpha \vee 4$
6.	$\vdash (p \vee q) \vee r$	$\alpha \vee 5$
7.	$\vdash \neg p \wedge \neg q$	$\beta \wedge 1, 2$
8.	$\vdash ((p \vee q) \vee r) \wedge (\neg p \wedge \neg q)$	$\beta \wedge 6, 7$
9.	$\vdash ((p \vee q) \vee r) \wedge (\neg p \wedge \neg q)) \to r$	$\beta \to 3, 8$