Assignment 1

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May 13, 2018

Contents

1 Question 1

The book *Introduction to Machine Learning* by Nils J, Nilsson can be found at http://robotics.stanford.edu/people/nilsson/MLBOOK.pdf and is 1.855 MB.

The book *A first encounter with Machine Learning* by Max Welling can be found at http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.441.6238&rep=rep1&type=pdf and is 416 KB.

2 Question 2

We can represent some learning functions in machine learning as Boolean function. We can define a Boolean function as a function of the form

$$f(x_1, x_2, x_3, ...x_n) \tag{1}$$

Which maps a n-tuple of (0,1) values to 0,1. 0,1 can also be expressed as false,true.

There are 3 basic types of operations can be performed in Boolean functions. Firstly we have the "and" operation which uses the connective "." as in

$$x_1.z_2 \tag{2}$$

Which only returns true if x_1 and x_2 is true. Then there is the "or" operation represented as a "+"

$$x_1 + x_2 \tag{3}$$

Where it returns true if x_1 is true or if x_2 or both are true. Thirdly we have the negation operation indicated by a $^-$ as in

$$\bar{x_1} \tag{4}$$

This operation returns true if x_1 is false and false if x_1 is true.

The . and + operations are commutative (the order of the terms do not matter)

$$\begin{aligned} x_1.x_2 &= x_2.x_1 \\ x_1 + x_2 &= x_2 + x_1 \end{aligned} \tag{5}$$

and associative (the way we group them to evaluate them do not matter)

$$\begin{aligned} x_1.x_2(x_3) &= x_1(x_2.x_3) \\ x_1 + x_2 + (x_3) &= x_1 + (x_2 + x_3) \end{aligned} \tag{6}$$

To commute between . and + we use DeMorgan's laws

$$\begin{aligned} \overline{x_1.x_2} &= \overline{x_1} + \overline{x_2} \\ \overline{x_1 + x_2} &= \overline{x_1}.\overline{x_2} \end{aligned} \tag{7}$$

Boolean functions can be broken down into various sub-classes. The first subclass is called *terms*. We can write these as $k_1k_2k_3...k_n$ where k_i are literals. An example would be the following term of size 4, $x_2.x3.\overline{x_6}.x_{12}$. This is also called a conjunctive literal as in all the terms are separated by the *and* operation.

Secondly we have *clauses* or . A clause is a function where the literals are separated by the or function. As in $k_1 + k_2 + ... + k_n$. An example would be $x_1 + \overline{x_5} + x_8$.

Disjunctive normal functions are functions that can be written as a disjunction of terms. In other words where terms are separated by the + (or) operator. For instance

$$f = x_1x_2 + x_3x_4 \tag{8}$$

where the terms are x_1x_2 and x_3x_4. This has a dual in the Conjunctive Normal Form (CNF) which is a conjunction of clauses. Example

$$f = (x_1 + x_3)(x_2 + x_4) \tag{9}$$

where the clauses x_1+x_3 and x_2+x_4 are conjoined using the "and" operation. We can move between DNF and CNF using De Morgan's laws.

The next class of boolean functions is called decision lists. We can define a decision list as an ordered pair of terms and boolean values. As in

$$\begin{aligned} &(t_q, v_q) \\ &(t_{q-1}, v_{q-1}) \\ &... \\ &(t_i, v_i) \\ &... \\ &(t_2, v_2) \\ &(T, v_1) \end{aligned} \tag{10}$$

where t_i are terms as defined above, T is a term whose value is 1 and v_i are boolean values 0 or 1. The value of v_i, for the first value of t_i that is equal to 1, is the value of the decision list. As an example we have

$$\begin{aligned} &f = \\ &(x_1\overline{x_2}x_3x_5, 0) \\ &(x_1x_3, 1) \\ &(x_1x_2, 0) \\ &(1, 0) \end{aligned} \tag{11}$$

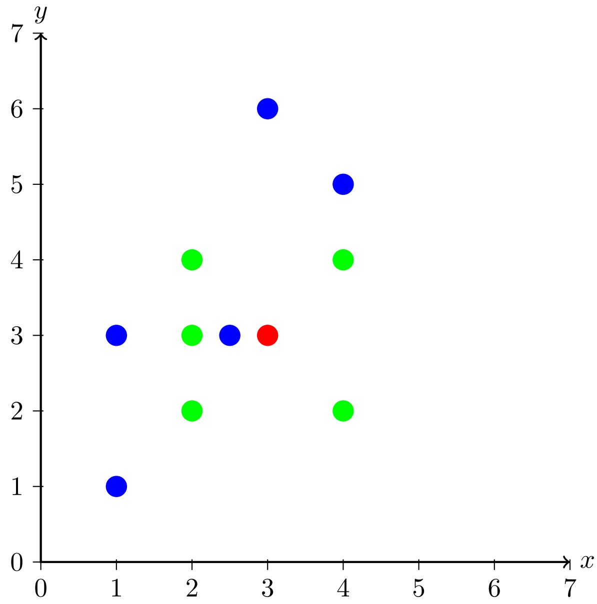
If $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ and $x_5 = 0$, the the decision list will have value 1 because the first term that is equal to 1 is x_1x_3. If $x_1 = x_2 = x_3 = x_5 = 0$ then the first term that will have value 1 is the last term. Then the value of the decision list will be 0. A decision list of k terms is called a k-DL.

A symmetric function is a function whose value does not change(invariant) for permutations of the input variables. For instance, if it takes in 5 variables, and 3 of them are true and 2 are false, then the function will be symmetric if it does not matter which variables are false or true as long as the same number of variables stay true and false. Stated another way. In this example as long as any 3 variables are true and 2are false and the value of the function stays the same, then the function is symmetric. An example would be the parity function

$$f = \overline{x_1}x_2x_3 + x_1\overline{x_2}x_3 + x_1x_2\overline{x_3} + \overline{x_1x_2x_3} \tag{12}$$

A voting function is a subclass of symmetric functions. A voting function has value 1 if a certain number k of inputs or more have value 1. This is called a k-voting function. The function is a majority function if it returns 1 if k is half the number of terms.

3 Question 3: Report on k-Nearest Neighbour Algorithm



Consider the above diagram. Suppose we need to classify the red dot as either a blue dot or a green dot for an arbitrary machine learning task. We could use the k-nearest neighbour algorithm. Where k refers to how many of the red dots neighbours we use to deduce its value.

Let the Euclidean distance be defined as

$$d(x, y) = \sqrt{\sum_i (x_i - y_i)^2} \tag{13}$$

(University of Wasthington, 2012)

We use the euclidean distance to determine the nearest neighbour. If we only use the 1-nearest neighbour we find the nearest point to be $(2.5, 3)$. We can deduce that the dot should be blue. On the other hand, if we use the 6-nearest neighbours we can see that the value should be green, since most of the 6 nearest neighbours are green. (Specifically referring to the points $\{(2.5, 3), (2, 3), (2, 2), (4, 2), (4, 4), (2, 4)\}$)

An odd number of nearest neighbours should be chosen in order to eliminate the possibility of a tie. As in the case with 2 nearest neighbours $\{(2.5, 3), 2, 3)\}$.

This case be generalised to a problem space (as in Mitchell (1997)) where a an instance x is described by the feature vector

$$\langle a_1(x), a_2(x), ..., a_n(x)\rangle \tag{14}$$

where a denotes the a attributes of x. Then the distance will be defined as

$$d(x_i, x_j) \equiv \sqrt{\sum_{r=1}^{n} (a_r(x_i) - a_r(x_j))^2} \tag{15}$$

(Mitchell, 1997)

This algorithm is susceptible to the *curse of dimensionality*. As the number of dimensions (or attributes) increases the contribution of an individual attribute decreases. If the data can only be categorised through a few or even 1 attribute, then i contribution of the important attributes will be "drowned out". (Murphy, 2012) To reduce the *curse of dimensionality*, the attributes could be weighted and additionally weighting attributes, we want to ignore, with 0.(Mitchell, 1997)

One issue with the k-nearest neighbour algorithm is that all computation generally done is do during the querying phase. Unlike decision trees where most of the computation can be done initially and the querying process being quite simple. (Mitchell, 1997)

This algorithm can also be extended to continuous functions as in

$$\hat{f}(x_q) \leftarrow \frac{\sum_{i=1}^{k} f(x_i)}{k} \tag{16}$$

where $\hat{f}(x_q)$ is the resultant function at the point we are querying x_q.

4 Question 4

Firstly we need to covert the positions (x, y) into our hypothesis space $\langle a, b\rangle$. The quantities are related by the equation:

$$h \leftarrow \langle a < \sqrt{x^2 + y^2} < b\rangle \tag{17}$$

Given dataset

index	x	y
1	5	5
2	-6	4
3	-3	-4
4	2	-4
5	-1	2
6	-2	0
7	6	7
8	8	-8

For datapoint 1 we have

$$\begin{aligned} a < \sqrt{x^2 + y^2} < b \\ a < \sqrt{5^2 + 5^2} < b \\ a < \sqrt{50} < b \\ a < 7.071 < b \end{aligned} \tag{18}$$

Given that a a b are integers it is safe to assume they are the next integer i.e. $\langle a, b\rangle = \langle 7, 8\rangle$.

A similar calculation gives us the values of the other data points in the hypothesis space. Note that in cases like $(x, y) = (-3, -4)$

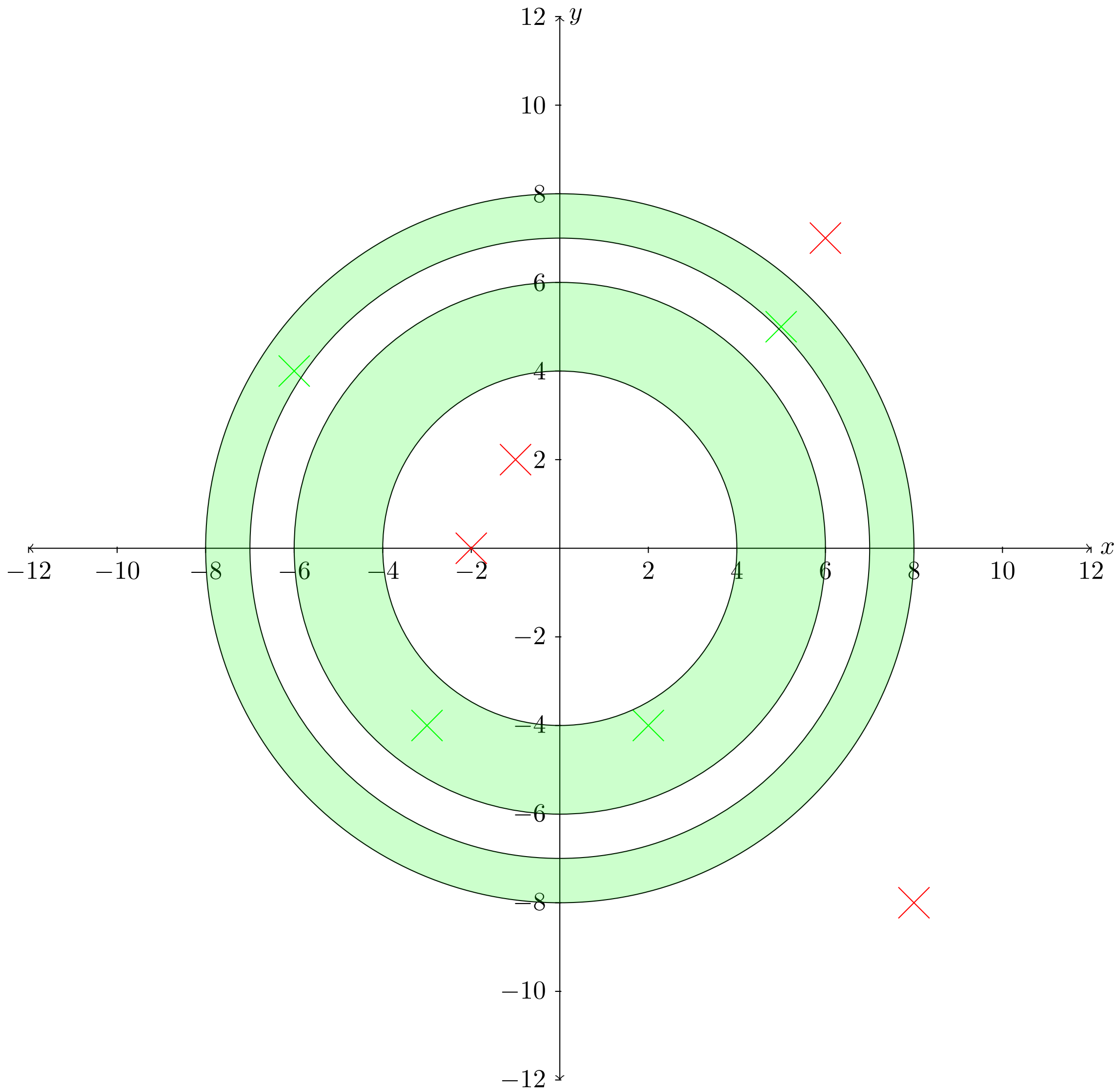
$$
\begin{aligned}
a < \sqrt{x^2 + y^2} < b \\
a < \sqrt{(-3)^2 + (-4)^2} < b \\
a < \sqrt{25} < b \\
a < 5 < b
\end{aligned} \tag{19}
$$

the middle term is a whole integer. Meaning $\langle a, b \rangle = \langle 4, 6 \rangle$
Thus our dataset becomes

index	x	y	a	b
1	5	5	7	8
2	-6	4	7	8
3	-3	-4	4	6
4	2	-4	4	5
5	-1	2	2	3
6	-2	0	1	3
7	6	7	9	10
8	8	-8	11	12

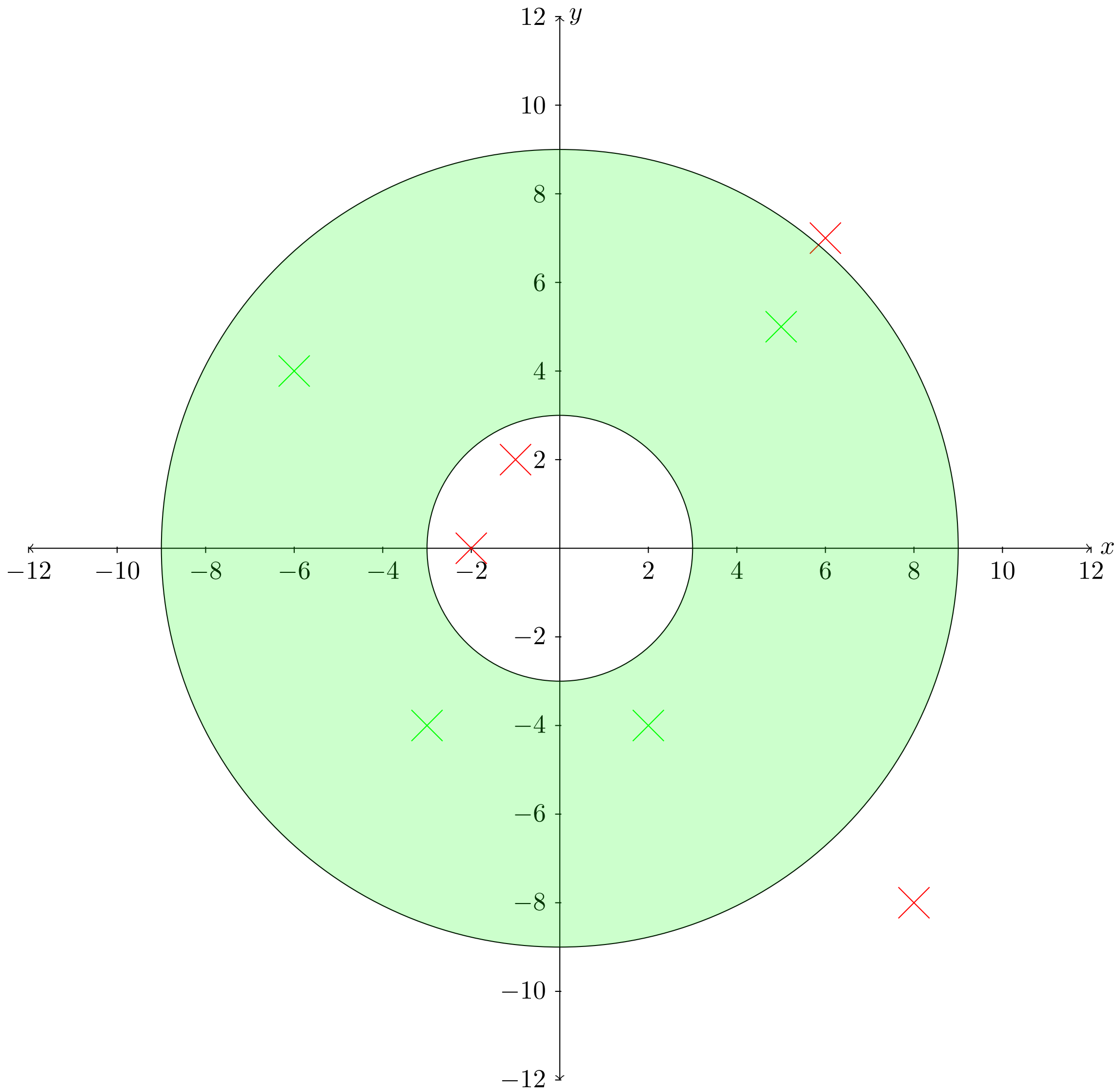
4.1 Question 4a

$$
S = \{\langle 4, 6 \rangle, \langle 7, 8 \rangle\} \tag{20}
$$



4.2 Question 4b

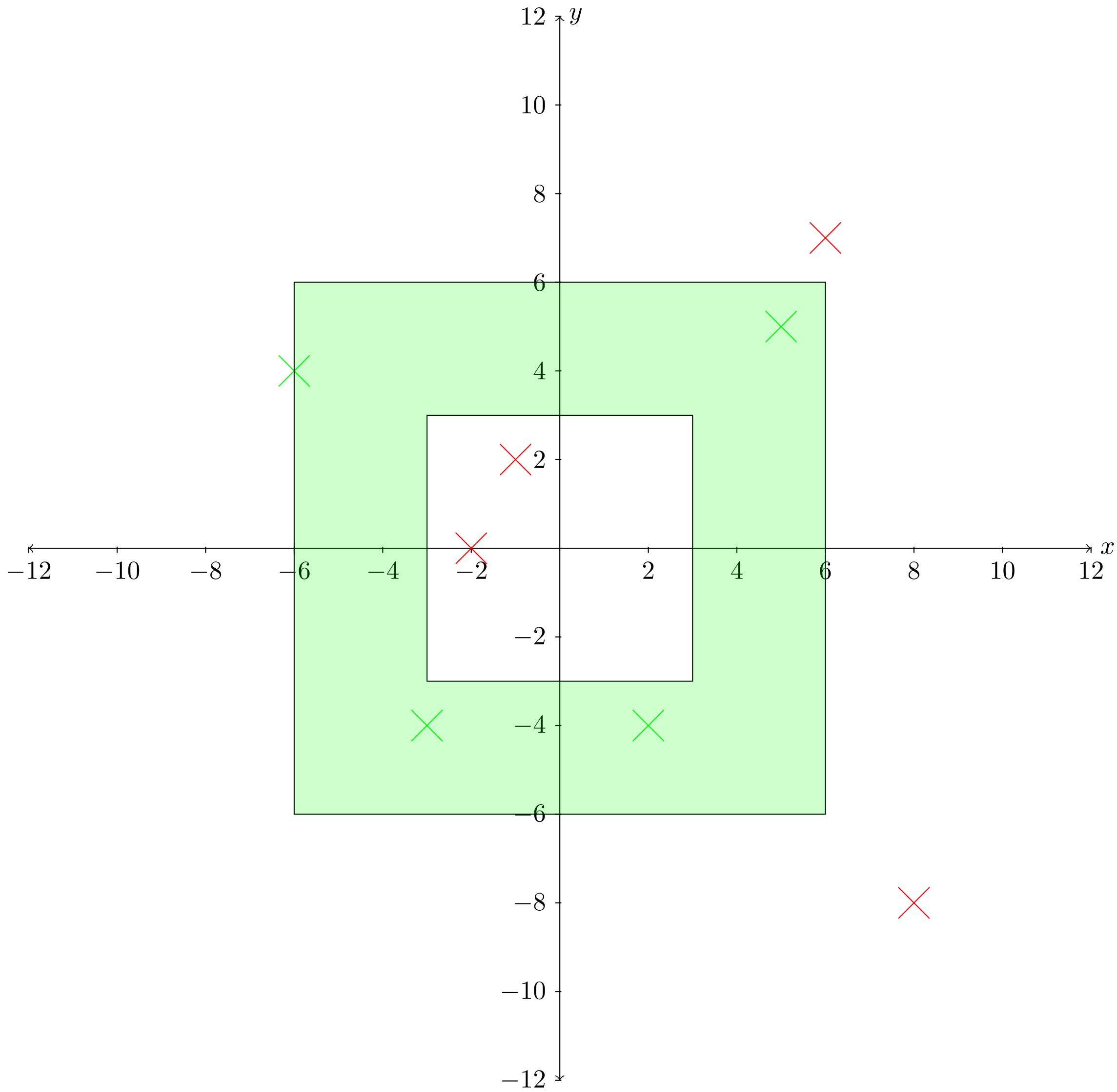
$$S = \{\langle 3, 9 \rangle\} \tag{21}$$



4.3 Question 4c

4.4 Question 4d

We could use a rectangle of the form $\langle (a, b), (c, d) \rangle$ where $a, b, c, d \in Z$. (a,b) is the position of the lower left hand corner of the outer square and (c,d) is the lower left hand corner of the inner square. $h \leftarrow \langle (-6, -6), (-3, -3) \rangle$

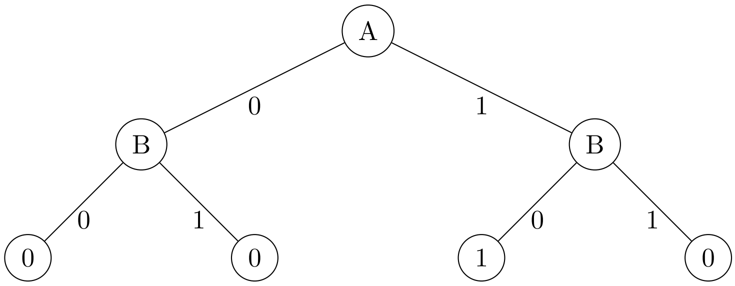


5 Question 5

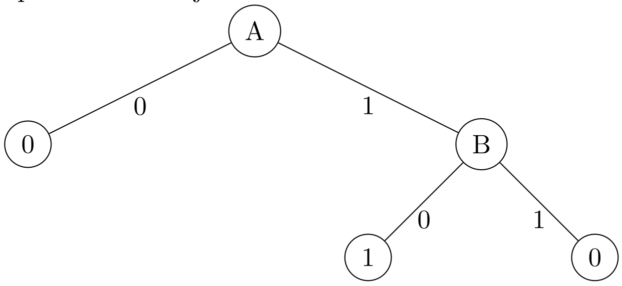
5.1 Question 5.1

For the equation $A \wedge \neg B$:

A	B	$\neg B$	$A \wedge \neg B$
0	0	1	0
0	1	1	0
1	0	1	1
1	1	0	0



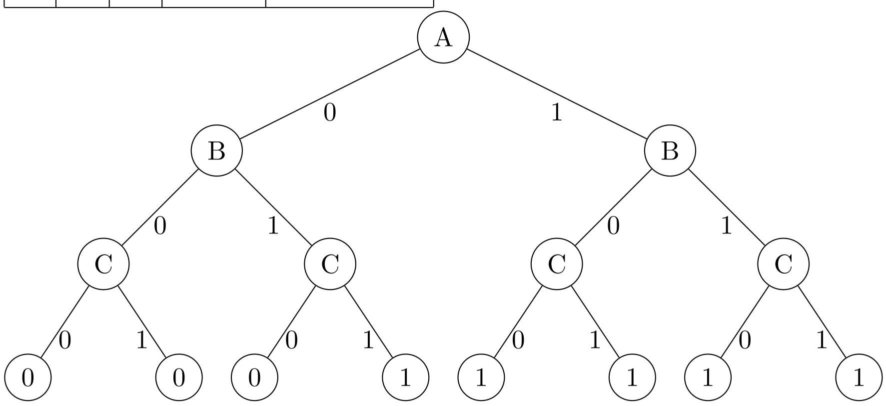
When A=0, B returns 0 whether we have the value 0 or 1 for B. We can collapse this B and just return 0.



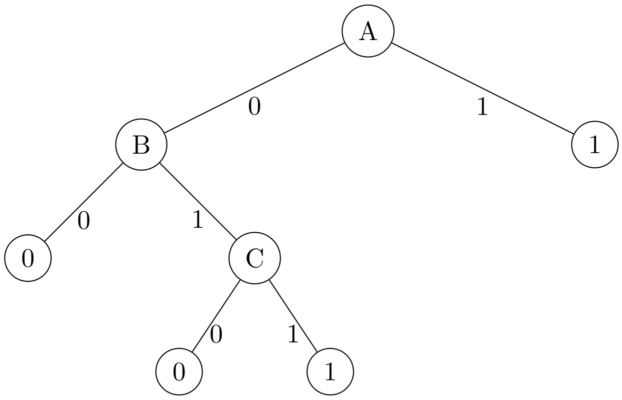
5.2 Question 5.2

For the equation $A \vee [B \wedge C]$:

A	B	C	$B \wedge C$	$A \vee [B \wedge C]$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



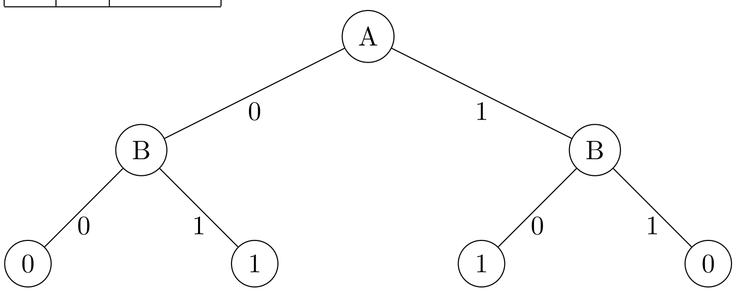
When $A = 0$ and $B = 0$, the result is 0 independent form the value of C. This branch will be reduced. Additionally When $A = 1$ the result will be 1 no matter the value of B or C. This will also be reduced. Our new decision tree will be:



5.3 Question 5.3

For the equation $A \bigoplus B$:

A	B	$A \bigoplus B$
0	0	0
0	1	1
1	0	1
1	1	0

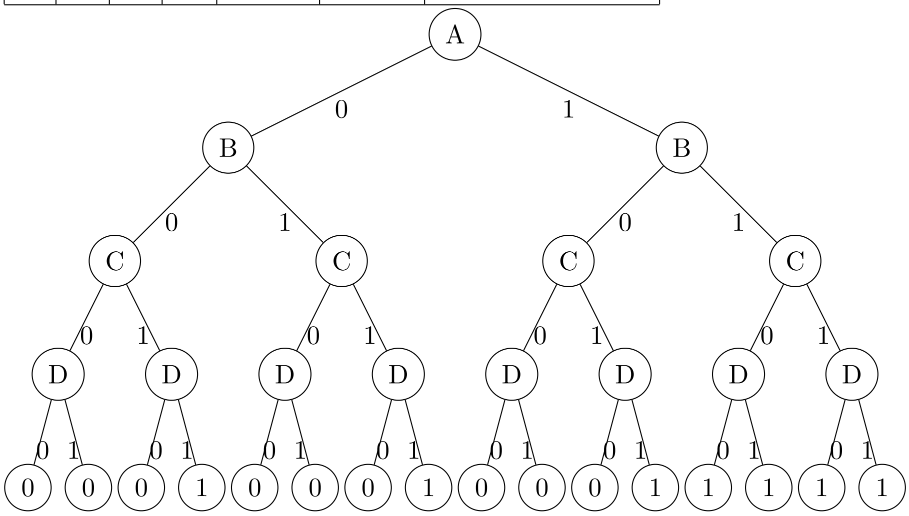


This tree cannot be simplified further

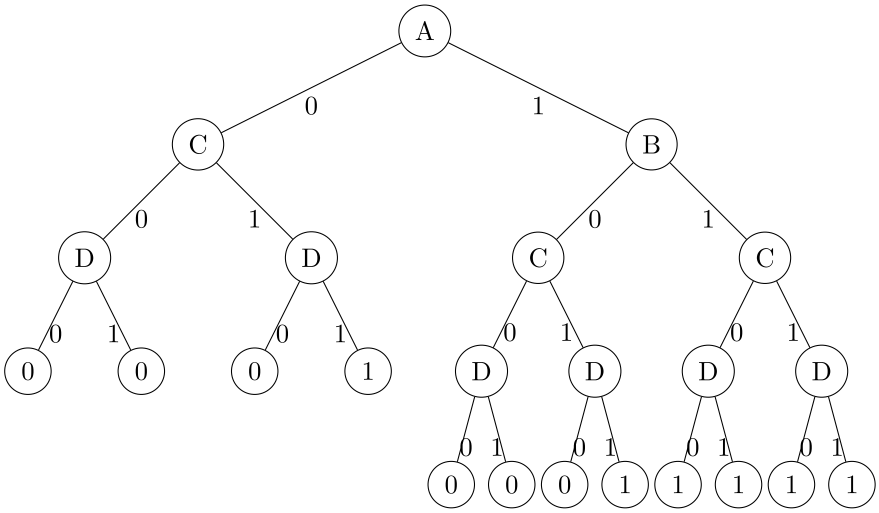
5.4 5.4

For the equation $[A \wedge B] \vee [C \wedge D]$:

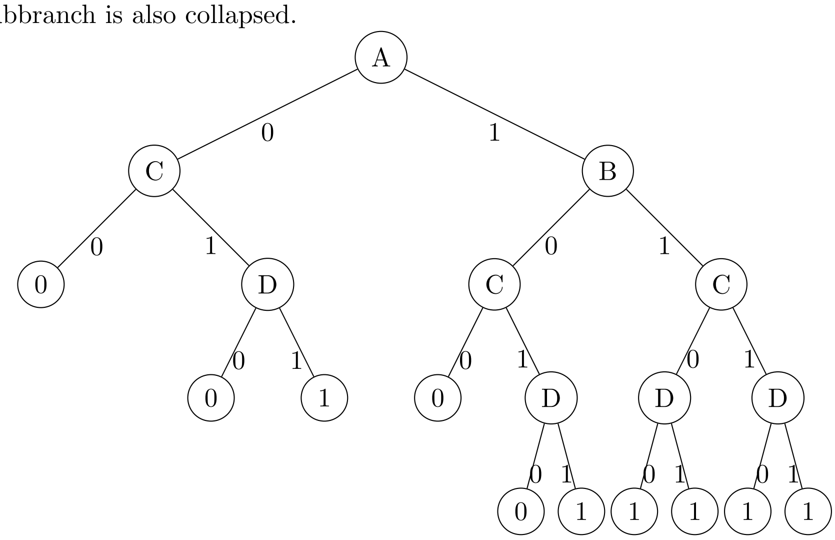
A	B	C	D	$A \wedge B$	$C \wedge D$	$[A \wedge B] \vee [C \wedge D]$
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	0	1	1	0	1	1
0	1	0	0	0	0	0
0	1	0	1	0	0	0
0	1	1	0	0	0	0
0	1	1	1	0	1	1
1	0	0	0	0	0	0
1	0	0	1	0	0	0
1	0	1	0	0	0	0
1	0	1	1	1	1	1
1	1	0	0	1	0	1
1	1	0	1	1	0	1
1	1	1	0	1	0	1
1	1	1	1	1	1	1



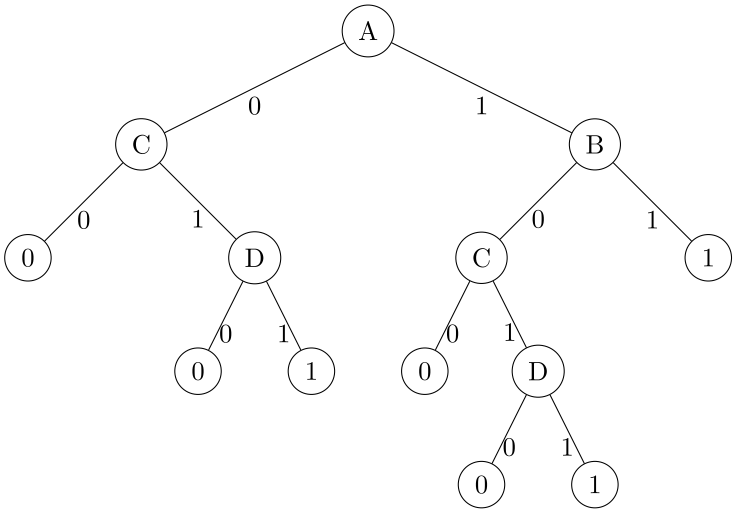
Firstly we can see that when A = 0, the trees under B = 0 and B = 1 are identical so they can be combined.



When A = 0 and C = 0, the result is not dependent on D. D collapses to 0. Additionally when A = 1, B = 0 and C = 0 the answer is always 0. That subbranch is also collapsed.



Finally when A = 1 and B = 1 the aswer will always be 0 and not dependent on C or D. This branch is collapsed.



6 Question 6

6.1 Question 6a

Entropy is used to calculate a decision tree in the ID3 algorithm. For a collection of Samples (S):

$$Entropy(S) = \sum_{i=1}^{c} -p_i log_2 p_i \tag{22}$$

If we assume we only have 2 outcomes, then we let the number of positive outcomes by p and the number of negative outcomes be n. Then we can write Equation 22 as

$$Entropy(p, n) = -\frac{p}{p+n} log_2 \frac{p}{p+n} - \frac{n}{p+n} log_2 \frac{n}{p+n} \tag{23}$$

More specifically we compare the gain of different attributes of the sample set. Then create the tree based on the attributes with the greatest gain

$$Gain(S, A) \equiv Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \tag{24}$$

We need to determine $Entropy(S)$

$$\begin{aligned} Entropy(p, n) &= -\frac{p}{p+n} log_2 \frac{p}{p+n} - \frac{n}{p+n} log_2 \frac{n}{p+n} \\ &= -\frac{3}{3+1} log_2 \frac{3}{3+1} - \frac{1}{3+1} log_2 \frac{1}{3+1} \\ &= 0.811 \end{aligned} \tag{25}$$

Now we need to determine the gain for the 6 attributes to determine which one will be the root node.

For Sky we have sunny $p_1 = 3, n_1 = 0)$ and rainy $p_2 = 0, n_2 = 1)$

$$\begin{aligned}
Gain(S, Sky) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{3}{4} Entropy(S_{Sunny}) - \frac{1}{4} Entropy(S_{Rainy}) \\
&= 0.811 - \frac{3}{4} Entropy(3, 0) - \frac{1}{4} Entropy(0, 1) \\
&= 0.811 - \frac{3}{4}(-\frac{3}{3} log_2 \frac{3}{3} - \frac{0}{3} log_2 \frac{0}{3}) - \frac{1}{4}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\
&= 0.811 - 0 \\
&= 0.811
\end{aligned} \tag{26}$$

For attribute AirTemp we have warm $p_1 = 3, n_1 = 0)$ and cold $p_2 = 0, n_2 = 1)$

$$\begin{aligned}
Gain(S, AirTemp) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{3}{4} Entropy(S_{Warm}) - \frac{1}{4} Entropy(S_{Cold}) \\
&= 0.811 - \frac{3}{4} Entropy(3, 0) - \frac{1}{4} Entropy(0, 1) \\
&= 0.811 - \frac{3}{4}(-\frac{3}{3} log_2 \frac{3}{3} - \frac{0}{3} log_2 \frac{0}{3}) - \frac{1}{4}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\
&= 0.811 - 0 \\
&= 0.811
\end{aligned} \tag{27}$$

For attribute Humidity we have normal $p_1 = 1, n_1 = 0)$ and high $p_1 = 2, n_2 = 1)$

$$\begin{aligned}
Gain(S, Humidity) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{1}{4} Entropy(S_{normal}) - \frac{3}{4} Entropy(S_{high}) \\
&= 0.811 - \frac{1}{4} Entropy(1, 0) - \frac{3}{4} Entropy(2, 1) \\
&= 0.811 - \frac{1}{4}(-\frac{1}{1} log_2 \frac{1}{1} - \frac{0}{1} log_2 \frac{0}{1}) - \frac{3}{4}(-\frac{2}{3} log_2 \frac{2}{3} - \frac{1}{3} log_2 \frac{1}{3}) \\
&= 0.811 - 0.689 \\
&= 0.123
\end{aligned} \tag{28}$$

For attribute Wind we only have strong $p_1 = 3, n_1 = 1$

$$
\begin{aligned}
Gain(S, Wind) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{4}{4} Entropy(S_{strong}) \\
&= 0.811 - \frac{4}{4} Entropy(3, 1) \\
&= 0.811 - \frac{4}{4}(-\frac{4}{4} log_2 \frac{4}{4} - \frac{0}{4} log_2 \frac{0}{4}) \\
&= 0.811 - 0.811 \\
&= 0.0
\end{aligned} \tag{29}
$$

For attribute water we have warm $p_1 = 2, n_1 = 1)$ and cold $p_1 = 1, n_2 = 0)$

$$
\begin{aligned}
Gain(S, Water) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{3}{4} Entropy(S_{warm}) - \frac{1}{4} Entropy(S_{high}) \\
&= 0.811 - \frac{3}{4} Entropy(2, 1) - \frac{1}{4} Entropy(1, 0) \\
&= 0.811 - \frac{3}{4}(-\frac{2}{3} log_2 \frac{2}{3} - \frac{1}{3} log_2 \frac{1}{3}) - \frac{1}{4}(-\frac{1}{1} log_2 \frac{1}{1} - \frac{0}{1} log_2 \frac{0}{1}) \\
&= 0.811 - 0.689 \\
&= 0.123
\end{aligned} \tag{30}
$$

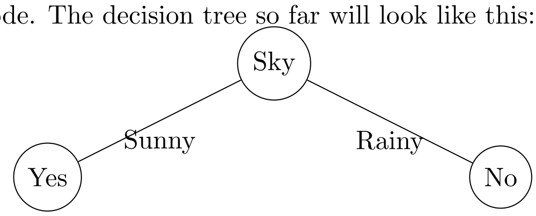
For attribute forecast we have same $p_1 = 2, n_1 = 0)$ and change $p_1 = 1, n_2 = 1)$

$$
\begin{aligned}
Gain(S, Forecast) &= Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{|S|} Entropy(S_v) \\
&= 0.811 - \frac{2}{4} Entropy(S_{same}) - \frac{2}{4} Entropy(S_{change}) \\
&= 0.811 - \frac{2}{4} Entropy(2, 0) - \frac{1}{1} Entropy(1, 0) \\
&= 0.811 - \frac{2}{4}(-\frac{2}{2} log_2 \frac{2}{2} - \frac{0}{2} log_2 \frac{0}{2}) - \frac{2}{4}(-\frac{1}{2} log_2 \frac{1}{2} - \frac{1}{2} log_2 \frac{1}{2}) \\
&= 0.811 - 0.5 \\
&= 0.311
\end{aligned} \tag{31}
$$

To recap:
Gain(Sky) = 0.811 from Equation 26
Gain(AirTemp) = 0.811 from Equation 27
Gain(Humidity) 0.123 from Equation 28
Gain(Wind) = 0 from Equation 29

Gain(Water) = 0.123 from Equation 30
Gain(Forecast) = 0.311 from Equation 31

We find that Sky and Airtemp have the greatest information gain. We could choose either to be the root node. We arbitrarily will choose Sky as the root node. The decision tree so far will look like this:



For the leaf node Sky = Sunny, we need to determine the sub-tree.
For Sky = Sunny:

$$\begin{aligned} Entropy(S_{Sky=Sunny}) &= Entropy(p,n) \\ &= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\ &= -\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3} \\ &= 0 \end{aligned} \tag{32}$$

We need to calculate the information gain for the other attributes when Sky = sunny
For AirTemp we only have warm ($p_1 = 3, n_1 = 0$)

$$\begin{aligned} Gain(S_{Sky=Sunny}, AirTemp) &= 0.0 - \frac{3}{3}Entropy(S_{Sky=Sunny;AirTemp=Warm}) \\ &= 0.0 - \frac{3}{3}Entropy(3,0) \\ &= 0.0 - 1(-\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3}) \\ &= 0.0 \end{aligned} \tag{33}$$

For Humidity normal ($p_1 = 1, n_1 = 0$) and high ($p_2 = 2, n_2 = 0$)

$$\begin{aligned} Gain(S_{Sky=Sunny}, Humidity) &= 0.0 - \frac{1}{3}Entropy(S_{Sky=Sunny;Humidity=Normal}) \\ &- \frac{2}{3}Entropy(S_{Sky=Sunny;Humidity=High}) \\ &= 0.0 - \frac{1}{3}Entropy(1,0) - \frac{2}{3}Entropy(2,0) \\ &= 0.0 - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) \\ &= 0.0 \end{aligned} \tag{34}$$

For Wind we only have strong ($p_1 = 3, n_1 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny}, Wind) &= 0.0 - \frac{3}{3}Entropy(S_{Sky=Sunny;Wind=Strong}) \\
&= 0.0 - \frac{3}{3}Entropy(3,0) \\
&= 0.0 - 1(-\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3}) \\
&= 0.0
\end{aligned} \tag{35}
$$

For Water we have warm($p_1 = 2, n_1 = 0$) and cool($p_2 = 1, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny}, Water) &= 0.0 - \frac{2}{3}Entropy(S_{Sky=Sunny;Water=Warm}) \\
&- \frac{1}{3}Entropy(S_{Sky=Sunny;Water=Cool}) \\
&= 0.0 - \frac{2}{3}Entropy(2,0) - \frac{1}{3}Entropy(1,0) \\
&= 0.0 - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) \\
&= 0.0
\end{aligned} \tag{36}
$$

For Forecast we have same($p_1 = 2, n_1 = 0$) and change($p_2 = 1, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny}, Forecast) &= 0.0 - \frac{2}{3}Entropy(S_{Sky=Sunny;Forecast=Same}) \\
&- \frac{1}{3}Entropy(S_{Sky=Sunny;Forecast=Change}) \\
&= 0.0 - \frac{2}{3}Entropy(2,0) - \frac{1}{3}Entropy(1,0) \\
&= 0.0 - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) \\
&= 0.0
\end{aligned} \tag{37}
$$

This means that the node Sky = Sunny in our tree will be a leaf node, sice at this node the gain gor all the other attributes are 0.

Next we need to calculate the subtree under the node Sky = Rainy

Starting with

$$
\begin{aligned}
Entropy(S_{Sky=Rainy}) &= Entropy(p,n) \\
&= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\
&= -\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1} \\
&= 0
\end{aligned} \tag{38}
$$

For AirTemp we only have cold ($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, AirTemp) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Rainy;AirTemp=Cold}) \\
&= 0.0 - \frac{1}{1}Entropy(0,1) \\
&= 0.0 - 1(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.0
\end{aligned} \tag{39}
$$

For Humidity we only have High ($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, Humidity) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Rainy;Humidity=High}) \\
&= 0.0 - \frac{1}{1}Entropy(0,1) \\
&= 0.0 - 1(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.0
\end{aligned} \tag{40}
$$

For Wind we only have Strong ($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, Wind) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Rainy;Wind=Strong}) \\
&= 0.0 - \frac{1}{1}Entropy(0,1) \\
&= 0.0 - 1(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.0
\end{aligned} \tag{41}
$$

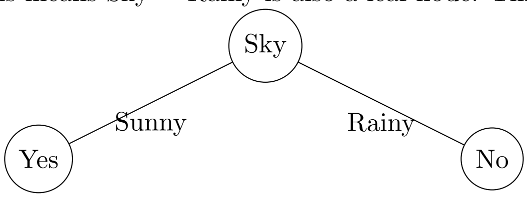
For Water we only have Warm ($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, Water) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Rainy;Water=Warm}) \\
&= 0.0 - \frac{1}{1}Entropy(0,1) \\
&= 0.0 - 1(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.0
\end{aligned} \tag{42}
$$

For Forecast we only have Change ($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, Forecast) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Rainy;ForeCast=Change}) \\
&= 0.0 - \frac{1}{1}Entropy(0,1) \\
&= 0.0 - 1(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.0
\end{aligned} \tag{43}
$$

This means when Sky = Rainy, the gain from all the other attributes are 0. This means Sky = Rainy is also a leaf node. This means the final tree will be:



6.2 Question 6b

Now we add a new row of training data. We will be recalculating the decision tree. Starting by determining the root node of the tree by comparing the information gain each attribute gives us.

First we need to determine the entropy of S

$$\begin{aligned} Entropy(p,n) &= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\ &= -\frac{3}{3+2}log_2\frac{3}{3+2} - \frac{2}{3+2}log_2\frac{2}{3+2} \\ &= 0.971 \end{aligned} \tag{44}$$

For the attribute Sky we have Sunny ($p_1 = 3, n_1 = 1$) and Rainy ($p_2 = 0, n_2 = 1$)

$$\begin{aligned} Gain(S,Sky) &= 0.971 - \frac{4}{5}Entropy(S_{Sunny}) - \frac{1}{5}Entropy(S_{Rainy}) \\ &= 0.971 - \frac{4}{5}Entropy(3,1) - \frac{1}{5}Entropy(0,1) \\ &= 0.971 - \frac{4}{5}(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) - \frac{1}{5}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\ &= 0.971 - 0.649 \\ &= 0.322 \end{aligned} \tag{45}$$

For the attribute AirTemp we have Warm ($p_1 = 3, n_1 = 1$) and Cold ($p_2 = 0, n_2 = 1$)

$$\begin{aligned} Gain(S,AirTemp) &= 0.971 - \frac{4}{5}Entropy(S_{Warm}) - \frac{1}{5}Entropy(S_{Cold}) \\ &= 0.971 - \frac{4}{5}Entropy(3,1) - \frac{1}{5}Entropy(0,1) \\ &= 0.971 - \frac{4}{5}(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) - \frac{1}{5}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\ &= 0.971 - 0.649 \\ &= 0.322 \end{aligned} \tag{46}$$

For the attribute Humidity we have Normal ($p_1 = 1, n_1 = 1$) and High ($p_2 = 2, n_2 = 1$)

$$\begin{aligned}
Gain(S, Humidity) &= 0.971 - \frac{2}{5}Entropy(S_{Normal}) - \frac{3}{5}Entropy(S_{High}) \\
&= 0.971 - \frac{2}{5}Entropy(1,1) - \frac{3}{5}Entropy(2,1) \\
&= 0.971 - \frac{2}{5}(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) - \frac{3}{5}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.971 - 0.951 \\
&= 0.020
\end{aligned} \tag{47}$$

For the attribute Wind we have Strong($p_1 = 3, n_1 = 1$) and Weak ($p_2 = 0, n_2 = 1$)

$$\begin{aligned}
Gain(S, Wind) &= 0.971 - \frac{4}{5}Entropy(S_{Strong}) - \frac{1}{5}Entropy(S_{Weak}) \\
&= 0.971 - \frac{4}{5}Entropy(3,1) - \frac{1}{5}Entropy(0,1) \\
&= 0.971 - \frac{4}{5}(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) - \frac{1}{5}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\
&= 0.971 - 0.649 \\
&= 0.322
\end{aligned} \tag{48}$$

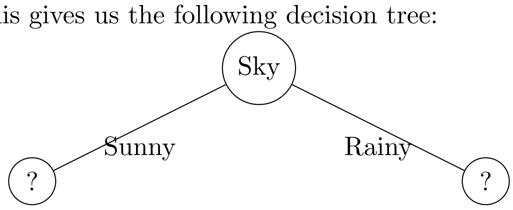
For the attribute Water we have Warm ($p_1 = 3, n_1 = 1$) and Cool ($p_2 = 1, n_2 = 0$)

$$\begin{aligned}
Gain(S, Water) &= 0.971 - \frac{4}{5}Entropy(S_{Warm}) - \frac{1}{5}Entropy(S_{Cool}) \\
&= 0.971 - \frac{4}{5}Entropy(3,1) - \frac{1}{5}Entropy(0,1) \\
&= 0.971 - \frac{4}{5}(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) - \frac{1}{5}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) \\
&= 0.971 - 0.649 \\
&= 0.322
\end{aligned} \tag{49}$$

For the attribute Forecast we have Same ($p_1 = 2, n_1 = 1$) and Change ($p_2 = 1, n_2 = 1$)

$$
\begin{aligned}
Gain(S, Forecast) &= 0.971 - \frac{3}{5}Entropy(S_{Same}) - \frac{2}{5}Entropy(S_{Change}) \\
&= 0.971 - \frac{3}{5}Entropy(2,1) - \frac{2}{5}Entropy(1,1) \\
&= 0.971 - \frac{3}{5}(-\frac{2}{3}log_2\frac{2}{3} - \frac{1}{3}log_2\frac{1}{3}) - \frac{2}{5}(-\frac{1}{2}log_2\frac{1}{2} - \frac{1}{2}log_2\frac{1}{2}) \\
&= 0.971 - 0.951 \\
&= 0.020
\end{aligned}
\tag{50}
$$

We find that the attributes Sky,AirTemp,Wind and Water all have the highest gain of 0.322. This is already very different from the previous question. For consistency we now choose Sky from these 4 attributes to be our root node. This gives us the following decision tree:



Continuing with Sky = Sunny, we need to determine the gain of the other attributes so that we can determine the sub-tree. We know that there are 3 positive data rows and one negative for Sky = Sunny:

$$
\begin{aligned}
Entropy(S_{Sky=Sunny}) &= Entropy(p,n) \\
&= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\
&= -\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4} \\
&= 0.811
\end{aligned}
\tag{51}
$$

For attribute AirTemp we have only the value Warm $(p_1 = 3, n_1 = 1)$

$$
\begin{aligned}
Gain(S_{Sky=Sunny}, AirTemp) &= 0.811 - \frac{3}{3}Entropy(S_{Sky=Sunny;AirTemp=Warm}) \\
&= 0.811 - \frac{4}{4}Entropy(3,1) \\
&= 0.811 - 1(-\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4}) \\
&= 0.811 - 0.811 \\
&= 0
\end{aligned}
\tag{52}
$$

For attribute Humidity we have Normal$(p_1 = 1, n_1 = 1)$ and High $(p_2 = 2, n_2 = 0)$

$$\begin{aligned}
Gain(S_{Sky=Sunny}, Humidity) &= 0.811 - \frac{2}{4}Entropy(S_{Sky=Sunny;Humidity=Normal}) \\
&- \frac{2}{4}Entropy(S_{Sky=Sunny;Humidity=High}) \\
&= 0.811 - \frac{2}{4}Entropy(1,1) - \frac{2}{4}Entropy(2,0) \\
&= 0.811 - \frac{2}{4}(-\frac{1}{2}log_2\frac{1}{2} - \frac{1}{2}log_2\frac{1}{2}) - \frac{2}{4}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2} \\
&= 0.811 - 0.5 \\
&= 0.311
\end{aligned} \tag{53}$$

For attribute Wind we have Strong($p_1 = 3, n_1 = 0$) and Weak ($p_2 = 0, n_2 = 1$)

$$\begin{aligned}
Gain(S_{Sky=Sunny}, Wind) &= 0.811 - \frac{3}{4}Entropy(S_{Sky=Sunny;Wind=Strong}) \\
&- \frac{1}{4}Entropy(S_{Sky=Sunny;Wind=Weak}) \\
&= 0.811 - \frac{3}{4}Entropy(3,0) - \frac{1}{4}Entropy(0,1) \\
&= 0.811 - \frac{3}{4}(-\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3}) - \frac{1}{4}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1} \\
&= 0.811 - 0.0 \\
&= 0.811
\end{aligned} \tag{54}$$

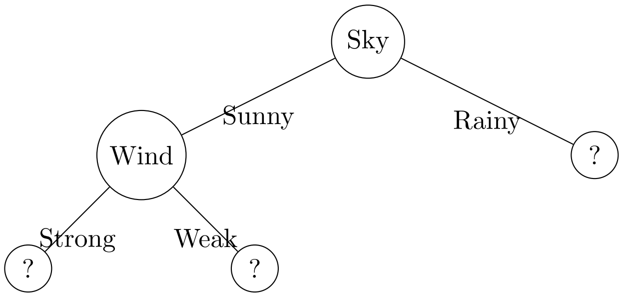
For attribute Water we have Warm($p_1 = 2, n_1 = 1$) and Cool ($p_2 = 1, n_2 = 0$)

$$\begin{aligned}
Gain(S_{Sky=Sunny}, Water) &= 0.811 - \frac{3}{4}Entropy(S_{Sky=Sunny;Water=Warm}) \\
&- \frac{1}{4}Entropy(S_{Sky=Sunny;Water=Cool}) \\
&= 0.811 - \frac{3}{4}Entropy(2,1) - \frac{1}{4}Entropy(1,0) \\
&= 0.811 - \frac{3}{4}(-\frac{2}{3}log_2\frac{2}{3} - \frac{1}{3}log_2\frac{1}{3}) - \frac{1}{4}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1} \\
&= 0.811 - 0.689 \\
&= 0.123
\end{aligned} \tag{55}$$

For attribute Forecast we have Same($p_1 = 2, n_1 = 1$) and Weak ($p_2 = 1, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny}, Forecast) &= 0.811 - \frac{3}{4}Entropy(S_{Sky=Sunny;Forecast=Same}) \\
&- \frac{1}{4}Entropy(S_{Sky=Sunny;Forecast=Change}) \\
&= 0.811 - \frac{3}{4}Entropy(2,1) - \frac{1}{4}Entropy(1,0) \\
&= 0.811 - \frac{3}{4}(-\frac{2}{3}log_2\frac{2}{3} - \frac{1}{3}log_2\frac{1}{3}) - \frac{1}{4}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1} \\
&= 0.811 - 0.689 \\
&= 0.123
\end{aligned} \tag{56}
$$

Therefore the attribute with the greatest gain when Sky=Sunny is Wind. The Wind attribute will now form our new sub-tree at Sky = Sunny



Now we have to calculate the gains for Sky=Sunny and Wind=Strong. Starting with the Entropy(S) calculation. We can see from the data that there is 3 positive data rows and 0 negative data rows for Sky=Sunny and Wind=Strong

$$
\begin{aligned}
Entropy(p,n) &= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\
&= -\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3} \\
&= 0.0
\end{aligned} \tag{57}
$$

For attribute AirTemp we have only the value Warm ($p_1 = 3, n_1 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Strong}, AirTemp) &= 0.811 - \frac{3}{3}Entropy(S_{Sky=Sunny;Wind=Strong;AirTemp=Warm}) \\
&= 0.0 - \frac{3}{3}Entropy(3,0) \\
&= 0.0 - 1(-\frac{3}{3}log_2\frac{3}{3} - \frac{0}{3}log_2\frac{0}{3}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{58}
$$

For attribute Humidity we have the values Normal ($p_1 = 1, n_1 = 0$) and High($p_2 = 2, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Strong}, Humidity) &= 0.0 - \frac{1}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Humidity=Normal}) \\
&- \frac{2}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Humidity=High}) \\
&= 0.0 - \frac{1}{3}Entropy(1,0) - \frac{2}{3}Entropy(2,0) \\
&= 0.0 - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{59}
$$

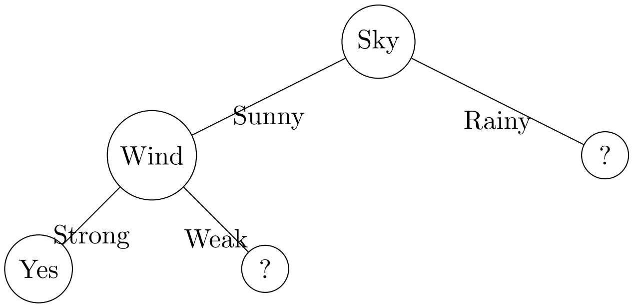
For attribute Water we have values Cool ($p_1 = 1, n_1 = 0$) and Warm($p_2 = 2, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Strong}, Water) &= 0.0 - \frac{1}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Water=Cool}) \\
&- \frac{2}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Water=Warm}) \\
&= 0.0 - \frac{1}{3}Entropy(1,0) - \frac{2}{3}Entropy(2,0) \\
&= 0.0 - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{60}
$$

For attribute Forecast we have values Change ($p_1 = 1, n_1 = 0$) and Same($p_2 = 2, n_2 = 0$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Strong}, Forecast) &= 0.0 - \frac{1}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Forecast=Change}) \\
&- \frac{2}{3}Entropy(S_{Sky=Sunny;Wind=Strong,Forecast=Same}) \\
&= 0.0 - \frac{1}{3}Entropy(1,0) - \frac{2}{3}Entropy(2,0) \\
&= 0.0 - \frac{1}{3}(-\frac{1}{1}log_2\frac{1}{1} - \frac{0}{1}log_2\frac{0}{1}) - \frac{2}{3}(-\frac{2}{2}log_2\frac{2}{2} - \frac{0}{2}log_2\frac{0}{2}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{61}
$$

We can see that the gains for all the other attributes, when Sky = Sunny and Wind = Strong, are 0. This means that this branch of the tree is complete giving:



We can classify the leaf node to be yes because all training rows when Sky = Sunny and Wind = Strong has a yes value for EnjoySport.

Now we need to classify the branch Sky = Sunny and Wind = Weak. There are no positive and one negative data rows for Sky = Sunny and Wind = Weak

$$\begin{aligned} Entropy(S_{Sky=Sunny,Wind=Weak}) &= Entropy(p,n) \\ &= -\frac{p}{p+n}log_2\frac{p}{p+n} - \frac{n}{p+n}log_2\frac{n}{p+n} \\ &= -\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1} \\ &= 0 \end{aligned} \tag{62}$$

For the attribute AirTemp we have Warm($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Sunny;Wind=Weak}, AirTemp) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Sunny;Wind=Weak;Airtemp=Warm}) \\ &= 0.0 - \frac{1}{1}Entropy(0,1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{63}$$

For the attribute Humidity we have Normal($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Sunny;Wind=Weak}, Humidity) &= 0.0 - \frac{1}{1}Entropy(S_{Sky=Sunny;Wind=Weak;Humidity=Normal}) \\ &= 0.0 - \frac{1}{1}Entropy(0,1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1}log_2\frac{0}{1} - \frac{1}{1}log_2\frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{64}$$

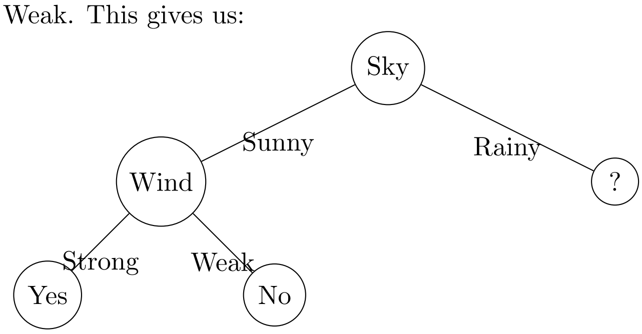
For the attribute Water we have Warm($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Weak}, Water) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Wind=Weak;Water=Warm}) \\
&= 0.0 - \frac{1}{1} Entropy(0,1) \\
&= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{65}
$$

For the attribute Forecast we have Same($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Sunny;Wind=Weak}, Forecast) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Wind=Weak;Forecast=Same}) \\
&= 0.0 - \frac{1}{1} Entropy(0,1) \\
&= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{66}
$$

Now we can see that the gains of all the attributes, when Sky = Sunny and Wind = Weak, are 0. Additionally we can see that this node would return No, because it is the only option according to our data for Sky = Sunny and Wind = Weak. This gives us:



The only branch that still needs classifying is the Sky = Rainy branch. Now we will calculate the gains for the other attributes in this case. For Sky = Rainy, we have only one data row. Which means we have 0 positive and 1 negative data row.

$$
\begin{aligned}
Entropy(S_{Sky=Rainy}) &= Entropy(p,n) \\
&= -\frac{p}{p+n} log_2 \frac{p}{p+n} - \frac{n}{p+n} log_2 \frac{n}{p+n} \\
&= -\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1} \\
&= 0
\end{aligned} \tag{67}
$$

For the attribute AirTemp we have Cold($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Rainy}, AirTemp) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;AirTemp=Cold}) \\ &= 0.0 - \frac{1}{1} Entropy(0, 1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{68}$$

For the attribute Humidity we have High($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Rainy}, Humidity) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Humidity=High}) \\ &= 0.0 - \frac{1}{1} Entropy(0, 1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{69}$$

For the attribute Wind we have Strong($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Rainy}, Wind) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Wind=Strong}) \\ &= 0.0 - \frac{1}{1} Entropy(0, 1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{70}$$

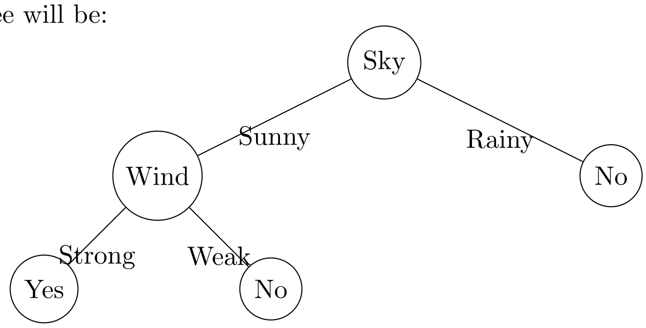
For the attribute Water we have Warm($p_1 = 0, n_1 = 1$)

$$\begin{aligned} Gain(S_{Sky=Rainy}, Water) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Water=Warm}) \\ &= 0.0 - \frac{1}{1} Entropy(0, 1) \\ &= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\ &= 0.0 - 0.0 \\ &= 0 \end{aligned} \tag{71}$$

For the attribute Forecast we have Change($p_1 = 0, n_1 = 1$)

$$
\begin{aligned}
Gain(S_{Sky=Rainy}, Forecast) &= 0.0 - \frac{1}{1} Entropy(S_{Sky=Sunny;Forecast=Change}) \\
&= 0.0 - \frac{1}{1} Entropy(0, 1) \\
&= 0.0 - \frac{1}{1}(-\frac{0}{1} log_2 \frac{0}{1} - \frac{1}{1} log_2 \frac{1}{1}) \\
&= 0.0 - 0.0 \\
&= 0
\end{aligned} \tag{72}
$$

We can can see from the gains for the attributes when Sky = Rainy, that we do not create a new sub-tree but make this a leaf node. From the only applicable data row we can see that this node will return No. Therefore the final decision tree will be:



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