Senior Capstone

The Thue-Morse Sequence

Kenneth Potts

Introduction

The Thue-Morse Sequence is a very specific infinite binary sequence. The Sequence may be written using 0*s* and 1*s* as 0110100110010110... or as a word using *As* and *Bs* as *ABBABAABBAABABBA*....

The sequence was first created by a Norwegian mathematician named Axel Thue. He was interested in sequences with limited to no repetitions(1).

Sequence Definition

The following definitions equivalently construct the Thue-Morse sequence.

Bitwise Negation

We begin with the 0^{th} iteration of the sequence, $T^0 = 0$, then find its bitwise negations, i.e. replace the 0s and 1s with 1s and 0s respectively. We write the negation as $\overline{T^0} = 1$. Next we append $\overline{T^0}$ to T^0 to get $T^1 = T^0\overline{T^0} = 01$. In general we generate the n^{th} iteration of the sequence as

$$T^n = T^{n-1}\overline{T^{n-1}}$$
$$\implies 0 \mapsto 01 \mapsto 0110 \mapsto 01101001$$

Then we can define the full Thue-Morse sequence as(3)

$$\lim_{n\to\infty} T^n = T = 0110100110010110100101100110100 1...$$

Recursive Definition

Let T_i be the i^{th} element of the sequence T for $i \in \mathbb{N}$, element T_i can be defined recursively. Let $T_0 = 0$, then

$$T_{2n} = T_n$$
$$T_{2n+1} = \overline{T_n}$$

Binary Sum Definition

We can also generate the i^{th} element, T_i, as

$$T_i = S_2(i) \bmod 2, i \geq 0$$

where $S_2(i)$ mod 2 is the sum of the binary representation of $i \in \mathbb{N}$, mod2(1).

Prouhet-Tarry-Escott Problem

The Prouhet-Tarry-Escott, or PTE, problem is centered on a specific partitioning of integers. The underlying question is if it is possible to partition the integers $\{0, 1, 2, ..., 2^n - 1\}$ into two disjoint sets A and B such that

$$\sum_{j\in A} j^k = \sum_{j\in B} j^k$$

and if so, for what values of k(1). The trivial value of k, of course, is $k = 0$.

This problem can be generalized to a polynomial f for $0 \leq k \leq n$ as follows

$$\sum_{k=0}^{n-1}\sum_{j\in A} a_k j^k = \sum_{k=0}^{n-1}\sum_{j\in B} a_k j^k$$
$$\implies \sum_{j\in A} f(j) = \sum_{j\in B} f(j)$$

PTE Solution

A partition according to the Thue-Morse sequence is a solution to the PTE problem. If we partition the integers $i < 2^n$, or $\{0, 1, 2, ..., 2^n - 1\}$, into two disjoint sets A and B such that $i \in A$ where $T_i = 0$ and $i \in B$ where $T_i = 1$, then this partition of A and B is a solution to the PTE problem(4).

The Fair Sequence

In games which require participants to take turns, typically the advantage goes to the participant who has the first turn. This assumes that participants will alternate taking turns in an standard *ABABAB*... pattern.

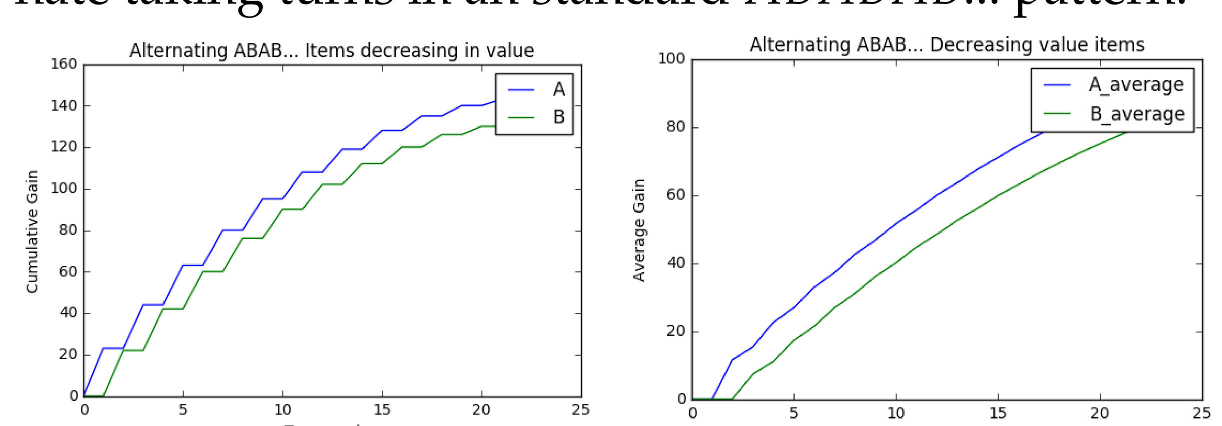

Figure 1: ABAB... Gain

A fair solution is to take turns as dictated by the elements of the Thue-Morse sequence(2). This is true if the payoff of each turn is equivalent in value, decreasing in value, or increasing in value.

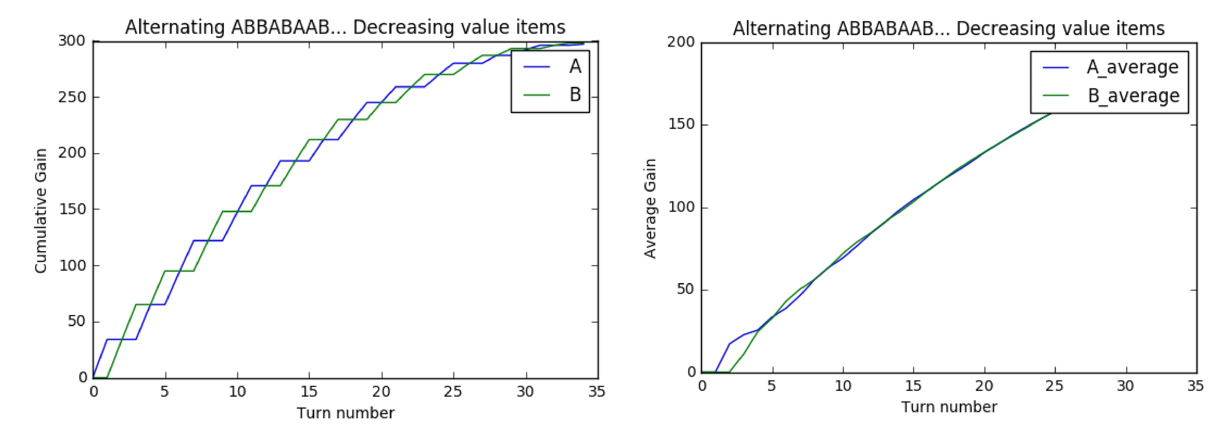

Figure 2: ABBA... (Thue-Morse) Gain

When using the Thue-Morse to take turns, the advantage of any one player disappears. In fact, this is simply another version of the PTE partitioning problem where now the partitions of integers are values that are being evenly spread to participants turn by turn.

Pouring Two Cups of Coffee

Consider a brewed pot of coffee. When brewed, the first drops of coffee are the strongest, with every subsequent drop less potent than the last. If the coffee is poured out into two cups, alternating pours between the cups according to the Thue-Morse sequence, then we can pour two equal strength cups of coffee(5).

Generating Function

The Thue-Morse sequence elements can be used as coefficients to construct a power series over the finite field $\mathbb{GF}(2)$.

$$F(x) = 0 + 1x + 1x^2 + 0x^3 + 1x^4 + 0x^5 + ...$$

The function $F(x)$ is one of two solutions to the quadratic equation

$$(1 + x)F^2 + F = \frac{x}{1 + x^2} \bmod 2$$

the other solution is $\overline{F}$, the conjugate of F.

Base-m and Dimension-d

The Thue-Morse sequence is not limited to base 2 and one dimension. The sequence can be generalized to base m and d dimensions.

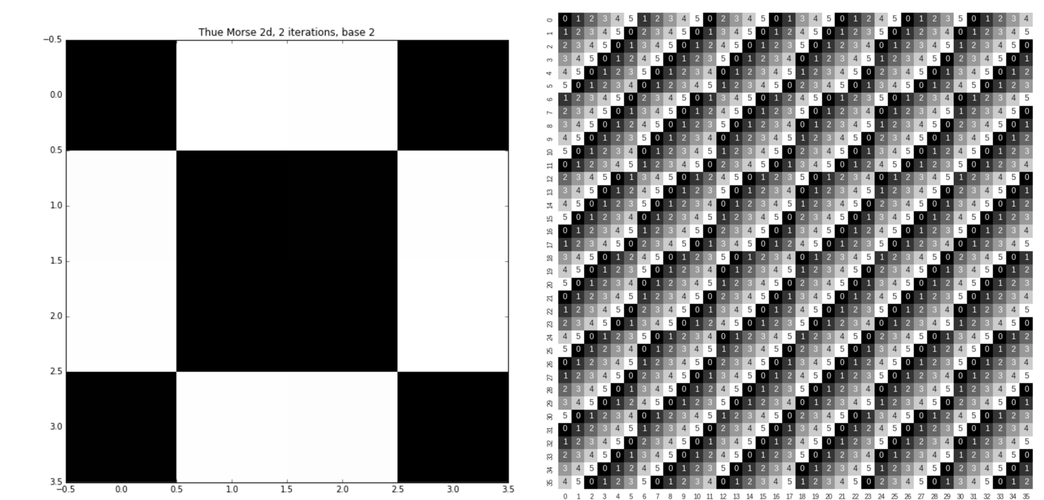

Figure 3: 2 Iterations in 2 dimensions in Bases 2 and 6

Conclusions

The sequence can be used as a fair sequence for taking turns, pouring coffee, and also as a way to partition 2^n integers into disjoint sets which sum to the same value.

References

[1] J.-P. Allouche and J. Shallit, *Sequences and Their Applications: The Ubiquitous Prouhet-Thue-Morse Sequence*, Springer, 1999.

[2] E. D. Bolker, C. Offner, R. Richman, and C. Zara, *The prouhet-tarry-escott problem and generalized thue-morse sequences*, Journal of Combanatorics, 7 (2016), pp. 117–133.

[3] M. Bucci, N. Hindman, S. Puzynin, and L. Q. Zamboni, *On additive properties of sets defined by the thue-morse word*, Journal of Combanatorics Theory, 120 (2013), pp. 1235–1245.

[4] R. Honsberger, *Mathematical Diamonds*, Mathematical Association of America, 2003.

[5] R. M. Richman, *Recursive binary sequences of differences*, Complex Systems, 13 (2001), pp. 381–392.

Acknowledgments

I would like to acknowledge Ted Galanthay and Osman Yurekli as they guided me throughout my capstone project, and also Ryan Bianconi for his previous work on the Thue-Morse Sequence.

Advisor: Ted Galanthay