28th November, 2019

STK-MAT3700

Mandatory assignment 1 of 1

Submission deadline

Thursday 10th OCTOBER 2019, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need a score of at least 50p. All questions have equal weight.

Problem 1. In this problem we review some of the concepts in Lectures 2 and 3.

1. (10p) After how many days will a zero coupon bond purchased for $B(0,1) = 0.85$ produce a 3% return?

2. (10p) How much can you borrow if the annual interest rate is 2%, you can afford to pay NOK10000 each semester and you want to clear the loan in 20 years?

3. (10p) Let C^E, P^E, C^A, and P^A denote prices of a European call option, a European put option, an American call option and an American put option, respectively. All of them with expiry time T and the same strike price K. Let $r \geq 0$ be the continuously compounded interest rate. Show that:

   a) If
   $$C^E - P^E - S(0) + Ke^{-rT} > 0,$$
   then you can make a sure risk-less profit.

   b) If
   $$C^A - P^A - S(0) + K < 0,$$
   then you can make a sure risk-less profit.

4. (10p) A call option with strike price of NOK60 costs NOK2 and one with strike price NOK30 costs NOK6. A put option with strike NOK45 costs NOK4. All options have the same expiry time. Construct a table that shows the profit from a butterfly spread. For what range of stock prices would the butterfly spread lead to a loss? You may assume that the interest rate is zero.

Problem 2. Consider a single-period market consisting of a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, a probability measure $P(\omega) > 0, \omega \in \Omega$, a bank account with $B(0) = 1$, and $B(1) = 1$, and two risky assets, denoted by $S_1 = \{S_1(t)\}_{t=0,1}$ and $S_2 = \{S_2(t)\}_{t=0,1}$

$$S_1(0) = 7, \qquad S_1(1,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 7 & \text{if} \quad \omega = \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3 \end{cases},$$

$$S_2(0) = 3, \qquad S_2(1,\omega) = \begin{cases} 3 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

In this market (so you need to adapt the general definitions given in class to the parameters of this market to answer the following questions):

1. (10p) Define dominant trading strategy and arbitrage opportunity. How are these concepts related?

2. (10p) Define linear pricing measure and risk neutral pricing measure. How are these concepts related?

3. (10p) What is the law of one price? How is it related with the previous concepts of pricing measures?

4. (10p) Does this market contain dominant trading strategies? Does it contain arbitrage opportunities? Check if the following strategies are dominant and/or arbitrage opportunities

   a) $H = (-6, 0, 2)^T$.

   b) $H = (-10, 1, 1)^T$.

Problem 3. Consider a single-period market consisting of a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, a probability measure $P(\omega) > 0, \omega \in \Omega$, a bank account with $B(0) = 1$, and $B(1) = 1 + r$, where $r \geq 0$ is a given interest rate, and one risky asset, denoted by $S_1 = \{S_1(t)\}_{t=0,1}$,

$$S_1(0) = 3, \qquad\qquad S_1(1, \omega) = \begin{cases} 4 & \text{if} \quad \omega = \omega_1 \\ 3 & \text{if} \quad \omega = \omega_2 \\ 2 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

1. (10p) Determine the risk-neutral probability measures. Is the market free of arbitrage? Discuss the result in terms of the possible values of r.

2. (10p) What is the definition of a complete market? Is the market complete? Determine (characterize) the attainable claims. Discuss the result in terms of the possible values of r.

3. (10p) Set $r = 1/6$. Consider the contingent claim $X = (4, 7/2, 4)^T$. Determine the arbitrage-free prices of X.

4. (10p) Set $r = 1/6$. Assume that in the market is introduced a new risky asset S_2 with $S_2(0) = \frac{6}{7}$. Give conditions on $S_2(1) = (S_2(1, \omega_1), S_2(1, \omega_2), S_2(1, \omega_3))^T$ such that the extended market is complete. Check if $S_2(1) = (3/2, 1/3, 1/4)^T$ completes the market and, if this is the case, give the unique risk neutral measure.

Solution Problem 1

1. The price of a zero-coupon bond at time t is given by $B(t,T) = e^{-r(T-t)}$, where r is the implied annual (continuous) compounding rate. Moreover, the return of this bond bond over a period $[s,t] \subset [0,T]$ is given by

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)} = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}}.$$

Here, we have $s = 0, T = 1, B(0,1) = 0.85$ and $R(0,t) = 0.03$. Hence,

$$0.85 = B(0,1) = e^{-r(1-0)} \iff r = -\log(0.85) \simeq 0.1625 = 16.25\%.$$

On the other hand,

$$R(0,t) = \frac{e^{-r(1-t)} - B(0,1)}{B(0,1)},$$

which yields

$$\begin{aligned} t &= 1 + \frac{\log(B(0,1)(1 + R(0,t)))}{r} \\ &= 1 + \frac{\log(0.85(1+0.03))}{0.1625} \simeq 0.1818 \simeq 66.357 \simeq 67 \text{ days.} \end{aligned}$$

2. From the point of view of the lender, the loan can be seen as an annuity with $20 \times 2 = 40$ payments of NOK60000 and with interest rate of $r = 0.02/2 = 0.01$ (The 0.02 rate given is the annual nominal rate, not the annual effective rate. Since it is compounded twice a year we have to divide it by two). The total loan can be computed as the present value of the annuity, given by

$$\begin{aligned} P &= 60000 \times PA(0.01, 40) \\ &= 60000 \times \frac{1 - (1+0.01)^{-40}}{0.01} \\ &\simeq 1.9701 \times 10^6. \end{aligned}$$

3. To answer this question we will build risk-less strategies with positive profit.

   a) Assume that
   $$C^E - P^E - S(0) + Ke^{-rT} > 0.$$
   At time 0
   - Buy one share for $S(0)$.
   - Write and sell one call option for C^E.

- Buy one put option for P^E.
- Invest/borrow $C^E - P^E - S(0)$, depending on the sign, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting (or paying) the amount

$$\left(C^E - P^E - S(0)\right) e^{rT}.$$

- Sell the share for K, either by:
  - exercising the put option if $S(T) \leq K$
  - settling the short position in the call option if $S(T) > K$.

This will give a total profit of

$$\left(C^E - P^E - S(0)\right) e^{rT} + K > 0,$$

which is positive by assumption.

b) Assume that

$$C^A - P^A - S(0) + K < 0,$$

whcih is equivalent to

$$P^A - C^A + S(0) > K.$$

At time $t = 0$

- Sell a put, buy a call and short sell a share, financing the transactions in the money market.

If the American put is exercised at $0 < t \leq T$:

- Borrow K from the money market and buy a share for K. This settles the short position on the American put.
- Return the share to the owner, closing the short position on the share.
- We still have the call option which has a non negative value.
- We close the money market position.

The final balance of this strategy is the value of the call at time t and the amount

$$\left(P^A - C^A + S(0)\right) e^{rt} - K > Ke^{rt} - K > 0.$$

If the American put is not exercised then, at time T, we still need to close the short position on the stock:

- We buy a share of the stock for K, exercising the call option.
- We close the short position on the stock by returning the share to the owner.
- We close the money market position.

The final balance of this strategy gives

$$\left(P^A - C^A + S(0)\right) e^{rT} - K > Ke^{rT} - K > 0.$$

4. Let $0 < K_1 < K_2 < K_3$. To make a butterfly spread you buy a call option with strike K_1 and a call option with strike K_3, then you sell two call options with strike K_2. The profit of the butterfly spread as a function of the final price of the stock S_T is given by

$$\begin{aligned} P(S_T) = {} & (S_T - K_1)^+ + (S_T - K_3)^+ - 2(S_T - K_2)^+ \\ & - C^E(0; K_1) - C^E(0; K_3) + 2C^E(0; K_2). \end{aligned}$$

Note that in this problem $K_1 = 30, K_2 = 45, K_3 = 60$, $C^E(0;30) = 6$, $C^E(0;45) = 3$ and $C^E(0;60) = 2$. Hence,

$$-C^E(0; K_1) - C^E(0; K_3) + 2C^E(0; K_2) = -2 - 6 + 2 \times 3 = -2,$$

and the table of profits is geiven by

S_T	Profit
$S_T < 30$	-2
$30 \le S_T < 45$	$S_T - 32$
$45 \le S_T < 60$	$58 - S_T$
$S_T \ge 60$	-2

If the final price of the stock lies outside the interval $(32, 58)$ the strategy gives a loss.

Solution Problem 2

In the following definitions we take into account the particular instance of single-period market considered in this task. In particular, note that this market has only 2 risky securities. Moreover, $r \neq 0$ and the discounted price processes do not coincide with the price processes.

1. An **arbitrage oportunity** is a trading strategy $H = (H_0, H_1, H_2)^T$ such that its value process

$$V(t) = H_0 B(t) + H_1 S_1(t) + H_2 S_2(t), \qquad t = 0, 1,$$

satisfies that $V(0) = 0$, $V(1,\omega) \geq 0$ for all $\omega \in \{\omega_1, \omega_2, \omega_3\}$ and $\mathbb{E}[V(1)] > 0$.
A trading strategy H is **dominant** if there exists another trading strategy $\hat{H}$ such that their value processes satisfy that $V(0) = \hat{V}(0)$ and $V(1,\omega) > \hat{V}(1,\omega)$ for all $\omega \in \{\omega_1, \omega_2, \omega_3\}$.
If there exist dominant trading strategies then there exist arbitrage opportunities. However, the existence of arbitrage opportunities do not imply, in general, the existence of dominant trading strategies.

2. A **linear pricing measure** is a non-negative vector $\pi = (\pi_1, \pi_2, \pi_3)^T$ such that for every trading strategy H we have that

$$V^*(0) = \sum_{k=1}^{3} \pi_k V^*(1, \omega_k),$$

where $V^* = \left\{V^*(t) = \frac{V^*(t)}{B(t)}\right\}_{t=0,1}$ is the discounted value process associated to H. A probability measure Q on Ω is a **risk neutral pricing measure** if $Q(\omega) > 0, \omega \in \{\omega_1, \omega_2, \omega_3\}$ and

$$\mathbb{E}_Q[S_1^*(1)] = S_1^*(0),$$
$$\mathbb{E}_Q[S_2^*(1)] = S_2^*(0),$$

where $S_i^* = \left\{S_i^* = \frac{S_i(t)}{B(t)}\right\}_{t=0,1}$ for $i = 1, 2$ are the discounted price processes for the two risky assets. A risk neutral pricing measure is always a linear pring measure. However, a linear pricing measure is a risk neutral measure if only if all its components are strictyly positive, i. e., $\pi > 0$.

3. A market model satisfies **the law of one price** if there do not exists two trading strategies H and $\hat{H}$ such that their value processes satisfy that $V(0) > \hat{V}(0)$ and $V(1,\omega) = \hat{V}(1,\omega)$, for all $\omega \in \{\omega_1, \omega_2, \omega_3\}$. In general we can only say that if the law of one price does not hold then the market model does not have linear pricing measures nor risk neutral probability measures.

4. We try to find a risk neutral probability measure or a linear pricing measure for this market, that is, a probability measure Q=$(Q_1, Q_2, Q_3)^T > 0$ such that

$$\begin{aligned}
7 &= S_1^*(0) = \mathbb{E}_Q[S_1^*(1)] = \frac{1}{1+r}\left(\frac{210}{29}Q_1 + 7Q_2 + 4Q_3\right), \\
3 &= S_2^*(0) = \mathbb{E}_Q[S_2^*(1)] = \frac{1}{1+r}(3Q_1 + 6Q_2 + 3Q_3), \\
1 &= Q_1 + Q_2 + Q_3.
\end{aligned}$$

This system of equations has the unique solution

$$Q = \left(\frac{29}{30}, \frac{1}{30}, 0\right)^T.$$

Since $Q_3 = 0$, Q is not a risk neutral probability measure but a linear pricing measure. Hence, by the First Fundamental Theorem of Asset Pricing we can ensure that the market contains arbitrage opportunities. The fact that Q is a linear pricing measure ensure that the market does not contain dominant trading strategies.

a) Let $H = (-6, 0, 2)^T$. By the previous subsection we know that H cannot be a dominant trading strategy. Let's check if H is an arbitrage opportunity. We have that

$$\begin{aligned} V(0) &= H_0 B(0) + H_1 S_1(0) + H_2 S_2(0) \\ &= -6 \times 1 + 0 \times 7 + 2 \times 3 = 0, \end{aligned}$$

but

$$\begin{aligned} V(1, \omega_1) &= H_0 B(1) + H_1 S_1(1, \omega_1) + H_2 S_2(1, \omega_1) \\ &= -6 \times \frac{31}{30} + 0 \times \frac{210}{29} + 2 \times 3 = -\frac{1}{5} < 0. \end{aligned}$$

Therefore, H cannot be and arbitrage strategy.

b) Let $H = (-10, 1, 1)$. By the previous subsection we know that H cannot be a dominant trading strategy. Let's check if H is an arbitrage opportunity. Although

$$\begin{aligned} V(0) &= H_0 B(0) + H_1 S_1(0) + H_2 S_2(0), \\ &= -10 \times 1 + 1 \times 7 + 1 \times 3 = 0, \end{aligned}$$

we have that

$$\begin{aligned} V(1, \omega_3) &= H_0 B(1) + H_1 S_1(1, \omega_3) + H_2 S_2(1, \omega_3) \\ &= -10 \times \frac{31}{30} + 1 \times 4 + 1 \times 3 = -\frac{10}{3} < 0, \end{aligned}$$

and, therefore, H cannot be an arbitrage opportunity.

Solution Problem 3

1. We have that $S_1^*(0) = S_1(0) = 3$ and

$$S_1^*(1,\omega) = \begin{cases} a(1+r)^{-1} & \text{if} \quad \omega = \omega_1 \\ 3(1+r)^{-1} & \text{if} \quad \omega = \omega_2 \\ 2(1+r)^{-1} & \text{if} \quad \omega = \omega_3 \end{cases},$$

$$\Delta S_1^*(\omega) = \begin{cases} (a-3-3r)(1+r)^{-1} & \text{if} \quad \omega = \omega_1 \\ -3r(1+r)^{-1} & \text{if} \quad \omega = \omega_2 \\ -(1+3r)(1+r)^{-1} & \text{if} \quad \omega = \omega_3 \end{cases}.$$

$Q = (Q_1, Q_2, Q_3)^T$ is a risk neutral pricing measure if and only if Q is a probability measure and $\mathbb{E}_Q[\Delta S_1^*] = 0$, that is, the following equations are satisfied

$$\begin{aligned} (a-3-3r)\,Q_1 - 3rQ_2 - (1+3r)\,Q_3 &= 0, \\ Q_1 + Q_2 + Q_3 &= 1, \\ Q_1 > 0, Q_2 > 0, Q_3 &> 0. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} Q_3 &= 1 - Q_1 - Q_2, \\ 1 + 3r &= (a-2)\,Q_1 + Q_2, \\ Q_1 > 0, Q_2 > 0, Q_3 &> 0. \end{aligned}$$

Solving the second equation for Q_2 we get that

$$\begin{aligned} Q_2 &= 1 + 3r - (a-2)\,Q_1, \\ Q_3 &= 1 - Q_1 - 1 - 3r + (a-2)\,Q_1 = -3r + (a-3)\,Q_1. \end{aligned}$$

Combining the previous expressions for Q_2 and Q_3, the constraints $Q_2 \in (0,1)$, $Q_3 \in (0,1)$ and $Q_1 \in (0,1)$, we get that, for $a > 3$ and $r \in [0, \frac{a-3}{3})$ the set of risk neutral probability measures $M(a,r)$ is given by

$$M(a,r) = \left\{ Q_\lambda = (\lambda, 1 + 3r - (a-2)\,\lambda, -3r + (a-3)\,\lambda)^T, \lambda \in \left(\frac{3r}{a-3}, \frac{1+3r}{a-2} \right) \right\}.$$

By the first fundamental theorem of asset pricing the market is arbitrage free iff the set of risk neutral pricing measures is non-empty. Hence, for $a > 3$ fixed:

a) If $r \geq \frac{(a-3)}{3}$, then there are arbitrage opportunities because $M(a,r) = \emptyset$.

b) If $0 \leq r < \frac{a-3}{3}$, the market is arbitrage free because $M(a,r) \neq \emptyset$.

2. A single period market model is complete if any contingent claim X is attainable. That is, if for any contingent claim X there is a trading strategy H such that its value process at time 1 coincides with X, i.e.,

$$X = V(1) = H_0 B(1) + H_1 S_1(1).$$

The set of attainable claims $X = (X_1, X_2, X_3)^T$ is given by the equations

$$\begin{aligned} X_1 &= (1+r) H_0 + a H_1, \\ X_2 &= (1+r) H_0 + 3 H_1, \\ X_3 &= (1+r) H_0 + 2 H_1. \end{aligned}$$

From the first equation we get that $(1+r) H_0 = X_1 - a H_1$ and substituting in the second and third equation this expression for $(1+r) H_0$ we obtain

$$\begin{aligned} X_2 - X_1 &= -(a-3) H_1, \\ X_3 - X_1 &= -(a-2) H_1, \end{aligned}$$

which is equivalent to

$$X_3 - \frac{a-2}{a-3} X_2 + \frac{1}{a-3} X_1 = 0,$$

or

$$X_1 - (a-2) X_2 + (a-3) X_3 = 0. \tag{1}$$

Hence, the attainable claims are characterized by equation (1) which is an hyperplane in $\mathbb{R}^3$. This implies that the market is not complete, regardless of the value of r.
Alternatively, if $r \in [0, \frac{a-3}{3})$ then $M(a,r) \neq \emptyset$ and, as we have shown in the previous section, there are infinitely many risk neutral pricing measures. Then, by the second fundamental theorem of asset pricing (SFTAP), it also follows that the market is incomplete. However, if $r \geq \frac{a-3}{3}$ we cannot use the SFTAP to decide if the market is complete or not, because then $M(a,r) = \emptyset$ and the hypothesis in SFTAP does not hold.

3. If we set $a = 4$ and $r = 1/6$ we have that $B(1) = 7/6$ and

$$M(4, 1/6) = \left\{ Q_\lambda = \left(\lambda, \frac{3}{2} - 2\lambda, \lambda - \frac{1}{2} \right)^T, \lambda \in (1/2, 3/4) \right\}.$$

The contingent claim $X = (4, 7/2, 4)^T$ does not satisfy the equation (1). Therefore, the claim X is not attainable and there is an interval of arbitrage free prices

$[V_-(X), V_+(X)]$, where $V_-(X)$ is the lower hedging price of X and $V_+(X)$ is the upper hedging price of X. Moreover, we know that

$$\begin{aligned}
V_-(X) &= \inf_{Q\in M(1/6)} \left\{ \mathbb{E}_Q \left[\frac{X}{B(1)} \right] \right\} = \frac{6}{7} \inf_{\lambda\in(1/2,3/4)} \{\mathbb{E}_{Q_\lambda}[X]\} \\
&= \frac{6}{7} \inf_{\lambda\in(1/2,3/4)} \left\{ 4\lambda + \frac{7}{2}\left(\frac{3}{2} - 2\lambda\right) + 4\left(\lambda - \frac{1}{2}\right) \right\} \\
&= \frac{6}{7} \inf_{\lambda\in(1/2,3/4)} \left\{ \lambda + \frac{13}{4} \right\} = \frac{6}{7}\left\{ \frac{1}{2} + \frac{13}{4} \right\} \\
&= \frac{45}{14} \simeq 3.2143
\end{aligned}$$

and

$$\begin{aligned}
V_+(X) &= \sup_{Q\in M(1/6)} \left\{ \mathbb{E}_Q \left[\frac{X}{B(1)} \right] \right\} = \frac{6}{7} \sup_{\lambda\in(1/2,3/4)} \{\mathbb{E}_{Q_\lambda}[X]\} \\
&= \frac{6}{7} \sup_{\lambda\in(1/2,3/4)} \left\{ \lambda + \frac{13}{4} \right\} = \frac{6}{7}\left\{ \frac{3}{4} + \frac{13}{4} \right\} \\
&= \frac{24}{7} \simeq 3.4286.
\end{aligned}$$

4. Set $a = 4$ and $r = 1/6$ and let $X = (X_1, X_2, X_3)^T$ be an arbitrary contingent claim. The enlarged payoff matrix is given by

$$S(1,\Omega) = \begin{pmatrix} \frac{7}{6} & 4 & S_2(1,\omega_1) \\ \frac{7}{6} & 3 & S_2(1,\omega_2) \\ \frac{7}{6} & 2 & S_2(1,\omega_3) \end{pmatrix}.$$

The enlarged market is complete iff $S(1,\Omega)H = X$ always have a solution. As this is a system with 3 equations and 3 unknowns, it has a solution iff

$$\det(S(1,\Omega)) \neq 0.$$

We have that

$$\det(S(1,\Omega)) = \begin{vmatrix} \frac{7}{6} & 4 & S_2(1,\omega_1) \\ \frac{7}{6} & 3 & S_2(1,\omega_2) \\ \frac{7}{6} & 2 & S_2(1,\omega_3) \end{vmatrix} = \frac{7}{6}\{-S_2(1,\omega_1) + 2S_2(1,\omega_2) - S_2(1,\omega_3)\}.$$

Therefore, the market is complete iff

$$-S_2(1,\omega_1) + 2S_2(1,\omega_2) - S_2(1,\omega_3) \neq 0. \tag{2}$$

However, this condition may be fulfilled and still have a market with arbitrages. That is, a market with no risk neutral probability measures (actually, if there are risk neutral probability measures there can only be one, by the SFTAP). In order to ensure the existence of risk neutral probability measure for BOTH assets, we will find a subset of $M(4, 1/6)$ such that their elements are still risk neutral probability measures in the extended market. That is, the set of $Q \in M(4, 1/6)$ such that

$$\mathbb{E}_Q[S_2^*(1)] = S_2^*(0) = S_2(0) = 6/7,$$

This translates to the following equation

$$S_2(1, \omega_1)\lambda + S_2(1, \omega_2)\left(\frac{3}{2} - 2\lambda\right) + S_2(1, \omega_3)\left(\lambda - \frac{1}{2}\right) = 1,$$

with $\lambda \in (1/2, 3/4)$. From this equation we get that

$$\lambda = \frac{1 - \frac{3}{2}S_2(1, \omega_2) + \frac{1}{2}S_2(1, \omega_3)}{S_2(1, \omega_1) - 2S_2(1, \omega_2) + S_2(1, \omega_3)}, \tag{3}$$

which combined with $\lambda \in (1/2, 3/4)$ yields the following inequalities that ensure that the set of risk neutral measures in the enlarged market is non-empty

$$2 - S_2(1, \omega_2) - S_2(1, \omega_1) > 0, \tag{4}$$
$$4 - 3S_2(1, \omega_1) - S_2(1, \omega_3) < 0. \tag{5}$$

The conditions given by equations (2), (4), (5) ensure that the enlarged market is arbitrage free and complete. It is straighforward to check that $S_2(1) = (2, 1, 1/2)^T$ satisfies the previous set of equations and equation (3) gives

$$\lambda = \frac{1 - \frac{3}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{4}}{\frac{3}{2} - 2\frac{1}{3} + \frac{1}{4}} = \frac{15}{26},$$

and we can conclude that the unique risk neutral measure in the extended market is given by

$$Q = Q_{\frac{15}{26}} = \left(\frac{15}{26}, \frac{9}{26}, \frac{2}{26}\right)^T.$$

For $S_2(1) = (1, 2, 1)^T$ the market is complete, because equation (2) is satisfied, but there are no risk neutral probability measures, because equations (4) and (5) are not satisfied.