# Mandatory Assignment STK3405 – H19

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① ⓐ $P_1=\{1,2,4\}$; $P_2=\{2,3,5\}$; $P_3=\{1,3,4\}$; $P_4=\{2,4,5\}$; $P_5=\{1,3,5\}$

$K_1=\{1,2\}$; $K_2=\{3,4\}$; $K_3=\{4,5\}$; $K_4=\{1,5\}$; $K_5=\{2,3\}$

for $P_1,\dots,P_5$ := the minimal path set

$K_1,\dots,K_5$ := the minimal cut set

(1b) we use the formula for the minimal path set as following:

$$\phi(\vec{x}) = \coprod_{j=1}^{5} \prod_{i\in P_j} X_i = X_1X_2X_4 \amalg X_2X_3X_5 \amalg X_1X_3X_4 \amalg X_2X_4X_5 \amalg X_1X_3X_5$$

$$= 1-(1-X_1X_2X_4)\cdot(1-X_2X_3X_5)\cdot(1-X_1X_3X_4)\cdot(1-X_2X_4X_5)\cdot(1-X_1X_3X_5) =$$

$$= 1-(1-X_2X_3X_5-X_1X_2X_4+X_1X_2X_3X_4X_5)\cdot(1-X_2X_4X_5-X_1X_3X_4+X_1^2X_2X_3^2X_4X_5)\cdot$$

$$(1-X_1X_3X_5) = 1-\big(1-X_2X_4X_5-X_1X_3X_4+\cancel{X_1X_2X_3X_4X_5}-X_2X_3X_5+X_2X_3X_4X_5$$

$$+\cancel{X_1X_2X_3X_4X_5}-\cancel{X_1X_2X_3X_4X_5}-X_1X_2X_4+X_1X_2X_4X_5+X_1X_2X_3X_4-X_1X_2X_3X_4X_5$$

$$+X_1X_2X_3X_5-\cancel{X_1X_2X_3X_4X_5}-\cancel{X_1X_2X_3X_4X_5}+\cancel{X_1X_2X_3X_4X_5}\big)\cdot(1-X_1X_3X_5)=$$

$$=\cancel{1}-\big(\cancel{1}-X_1X_3X_5-X_2X_4X_5+\cancel{X_1X_2X_3X_4X_5}-X_1X_3X_4+X_1X_3X_4X_5-X_2X_3X_5$$

$$+X_1X_2X_3X_5+X_2X_3X_4X_5-\cancel{X_1X_2X_3X_4X_5}-X_1X_2X_4+\cancel{X_1X_2X_3X_4X_5}+X_1X_2X_4X_5$$

$$-\cancel{X_1X_2X_3X_4X_5}+X_1X_2X_3X_4-X_1X_2X_3X_4X_5-\cancel{X_1X_2X_3X_4X_5}+\cancel{X_1X_2X_3X_4X_5}$$

$$+\cancel{X_1X_2X_3X_5}-\cancel{X_1X_2X_3X_5}\big) = X_1X_2X_4+X_2X_3X_5+X_1X_3X_4+X_2X_4X_5+X_1X_3X_5$$

$$-X_1X_2X_3X_4-X_1X_2X_3X_5-X_1X_2X_4X_5-X_1X_3X_4X_5-X_2X_3X_4X_5$$

$$+X_1X_2X_3X_4X_5 \;//$$

⊛ Note that $x$ can only be 0 or 1, then, for example, $x_1^2 = x_1$ because $0^2=0$ and $1^2=1$.

(1c) We know that $P_i$ are linear functions, because $P_1 = \{x_1, x_2, x_3\}, \cdots, P_5 = \{x_1, x_3, x_5\}$, and $A = \{P_1, \cdots, P_5\}$ is multilinear which all multilinear functions can be written as:

$$\phi(\vec{x}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$$

where $A := \bigcup_{j=1}^{5} P_i$;

$\delta(A) :=$ The coefficient of the term associated to $\prod_{i \in A} X_i$. This is also called the signed domination function of $\phi(\vec{x})$.

- Conversely, it is possible to use:

$$\phi(B) = \phi(\vec{1}^B, \vec{0}^B) \quad \text{for } \forall B \subseteq C$$

and

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A| - |B|} \phi(B) \quad \text{for } \forall A \subseteq C$$

where $\phi(B) :=$ the state of the system given that all components in $B$ are functioning, while all components in $B^c$ are failed;

$|A| :=$ # of elements in set $A$;

$|B| :=$ # of elements in set $B$.

(1d) We assume that all components are stochastically independent and have reliability $p$.

$$h(\vec{p}) = E[\phi\,\vec{x}] = E[X_1X_2X_3] + E[X_2X_3X_5] + \dots + E[X_1X_2X_3X_4X_5] =$$

$$= E[X_1]\cdot E[X_2]\cdot E[X_3] + \dots + E[X_1]\cdot E[X_2]\cdot E[X_3]\cdot E[X_4]\cdot E[X_5] =$$

$$= p^3 + p^3 + p^3 + p^3 + p^3 - p^4 - p^4 - p^4 - p^4 - p^4 + p^5$$

$\boxed{\forall\, E[x_i] = p}$

$$\therefore\ h(\vec{p}) = 5p^3 - 5p^4 + p^5$$

■

(1e) $$h(0{,}75) = 5\cdot(0{,}75)^3 - 5(0{,}75)^4 + (0{,}75)^5$$

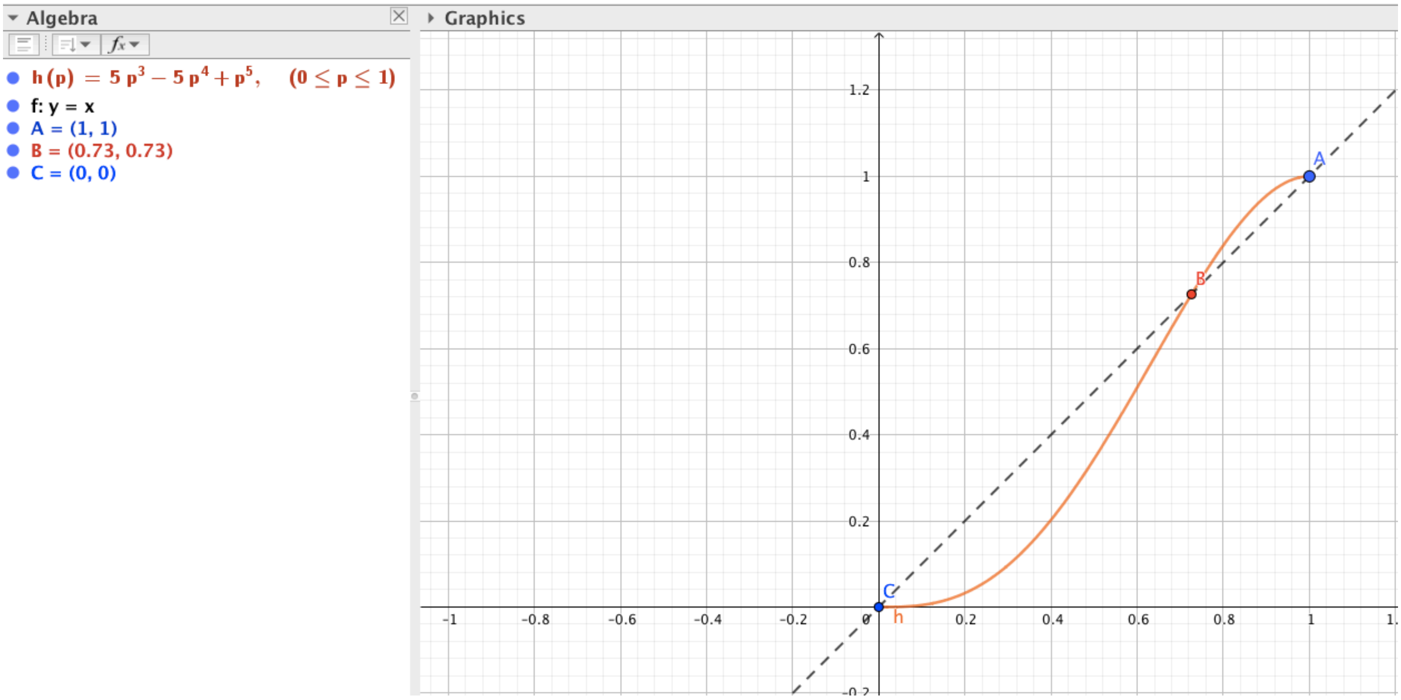
$$= 2{,}11 - 1{,}58 + 0{,}237$$

$$\cong 0{,}7647$$

■

(1f)
- The figure in the next page denotes the function $h(\vec{p}) = 5p^3 - 5p^4 + p^5$ which is in orange for the interval $0 \leq p \leq 1$.
- We define another function $f(x) = x$, denoted as a black dashed line, which intersect $h(p)$ at the point $B \cong \{0{,}73, 0{,}73\}$, that is, it is where $h(p)$ starts to be larger than $p$.
- Hence $h > p$ in around $0{,}73 \leq p \leq 1$

■


Algebra
Graphics
h(p) = 5 p³ − 5 p⁴ + p⁵, (0 ≤ p ≤ 1)
f: y = x
A = (1, 1)
B = (0.73, 0.73)
C = (0, 0)
A
B
C
h
1.2
1
0.8
0.6
0.4
0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

(1g) • $P(X_i=1)$ is the reliability of a component $i$. Since the state variable $(X_i)$ is binary, we have for $\forall\, i \in C$:

$$E[X_i] = 0 \cdot P(X_i=0) + 1 \cdot P(X_i=1) = P(X_i=1) = \alpha \cdot \Theta \quad , \; i=1,\dots,5.$$

• For $Y_1,\dots,Y_5$ and $Z$, we have $P(Y_i=1)=\alpha$ and $P(Z=1)=\Theta$, then:

$$P(X_i=1) = \underbrace{P(Z=1)}_{\alpha} \cdot \underbrace{P(Y_i=1 : Z=1)}_{\Theta} = P(Z=1 \cap Y_i=1)$$

The rule of multiplication in probability

• This means that the probability of a functioning component $i$ is equal to the probability of the electrical power has not failed $P(Z=1)=\alpha$ and (times) the probability of the component $i$ has not burned $P(Y_i=1)=\Theta$, given that $\alpha$ has occurred.

(1h)

$$\mathrm{cov}(X_i, X_j) = E\Big[\big(x - E[X_i]\big)\big(x - E[X_j]\big)\Big]$$

$$= E\Big[(x - p_i) \cdot (x - \alpha\Theta)\Big]$$

$$= E\Big[\cancel{x^2}^{\,x} - x\alpha\Theta - x p_i + p_i \alpha\Theta\Big]$$

$$= E\Big[x \cdot (1 - \alpha\Theta - p_i) + p_i \alpha\Theta\Big]$$

$$= \;?$$

(1i) $h(\vec{p}) = E[\phi(\vec{x})] = 5p^3 - 5p^4 + 5p^5$

since $P(x_i = 1) = p = P(z=1) \cdot P(z=1 \cap Y_i = 1) = \alpha\theta$

$\therefore h(\alpha\theta) = 5(\alpha\theta)^3 - 5(\alpha\theta)^4 + 5(\alpha\theta)$ ∎

## Problem 2:

• Some notes before exercises:

↳ $f(x) = k \cdot \mu \quad f(x) = e^k \quad f(x) = e^\mu$

$f'(x) = k \cdot \mu' \quad f'(x) = e^k \quad f(x) = \mu' \cdot e^\mu$

↳ $P\left(\begin{array}{l}\text{there are no holes in} \\ \text{a given pipeline}\end{array}\right) = e^{-\lambda l}$ where $\lambda > 0 :=$ given #
$l > 0 :=$ the length of the pipeline

(2a) • $\Delta l =$ positive # near to 0

• Taylor series expansion: $e^{-\lambda \Delta l} = 1 - \lambda\Delta l + \frac{(-\lambda)^2(\Delta l)^2}{2} + \frac{(-\lambda)^3(\Delta l)^3}{6} + \dots$

• We see that the taylor series expansion for $e^{-\lambda l}$ is an approximation of $1 - P(\Delta l) \simeq \lambda\Delta l \simeq 1 - e^{-\lambda \Delta l} \simeq \lambda\Delta l - \underbrace{\frac{(-\lambda)^2(\Delta l)^2}{2} - \frac{(-\lambda)^3(\Delta l)^3}{6} - \dots}_{\boxed{\text{very small}}}$

Hence this is an approximation to the probability that there are holes in a given change of the length of a pipeline.

↳ $1 - P\left(\begin{array}{l}\text{there are no holes in a given} \\ \text{change of the length } (\Delta l) \text{ of} \\ \text{a pipeline.}\end{array}\right) = P\left(\begin{array}{l}\text{There are holes in a given} \\ \text{change of the length } (\Delta l) \text{ of} \\ \text{a pipeline}\end{array}\right)$

which is $\simeq \lambda\Delta l$, Then $\lambda \simeq \frac{1 - P(\Delta l)}{\Delta l}$ where we can say that $\lambda$ is a parameter that measures the quality or the probability of a given change in the length of a pipeline per the change of the length of the pipeline. ∎

(2b) • $l_i$ := the length of the $i^{th}$ part, $i = 1, \dots, m$
$\hookrightarrow P(l_i = 1) = e^{-\lambda l_i}$

• $L = \sum_{i=1}^{m} l_i$ := total length of the pipeline

• for $i = 1, \dots, m$: $X_i = \begin{cases} 1 & \text{if there are no holes in the } i^{th} \text{ part} \\ 0 & \text{otherwise} \end{cases}$

and $\phi = \begin{cases} 1 & \text{if there are no holes in the pipeline} \\ 0 & \text{otherwise} \end{cases}$

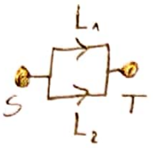
• $\phi(\vec{x}) = \prod_{i=1}^{m} X_i$ := This is the structure function of a binary system that contains $m$ parts connected in series. $X_i$ is the state variable of the $i^{th}$ part in the system.

• $h(\vec{p}) = 0 \cdot P(\phi = 0) + 1 \cdot P(\phi = 1) = P(\phi = 1) = \prod_{i=1}^{m} P(X_i = 1)$ := This is the reliability of the binary system above which represents the probability of the system to not have holes.

• $P(X_i = 1) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = E[X_i]$ := This is the reliability of a part $i^{th}$ to not have holes. ■

Scanned by CamScanner

(2c)




- Yes, this is a reasonable assumption because both pipelines are connected in parallel in the system, which means that the system doesn't need both parts, functioning at the same time, to be functioning, neither one part of the pipeline has influence to the other.

- We assume both pipelines are functioning independently in a binary monotone system $(C, \phi)$ of order 2. They are connected in parallel, then $\phi(\vec{x}) = x_1 \amalg x_2$. Here we assume that the system is coherent, that is both components are relevant. To calculate the relaibility we do the following:

$$\hookrightarrow \quad h = E[\phi(\vec{x})] = E[x_1 \amalg x_2] = E[x_1] \amalg E[x_2]$$

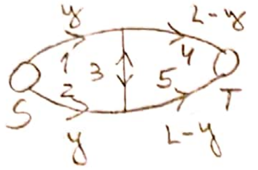
↳ from exercise text we know that $P(L) = e^{-\lambda L}$, that is:

$$E[x_i] = 0 \cdot P(x_i = 0) + 1 \cdot P(x_i = 1) = P(x_i = 1) = p_i = e^{-\lambda L}$$

$$\therefore h(\vec{p}) = e^{-\lambda L} \amalg e^{-\lambda L} = e^{-\lambda L} + e^{-\lambda L} - e^{-\lambda L} \cdot e^{-\lambda L} = 2e^{-\lambda L} - e^{-2\lambda L}$$

$$\hookrightarrow = e^{-\lambda L}\left(2 - e^{-\lambda L}\right) \quad \blacksquare$$

2d




$$\begin{cases} E[x_1] = E[x_2] = a = e^{-\lambda L} \\ E[x_4] = E[x_5] = b = e^{-\lambda(L-y)} \\ E[x_3] = q \end{cases}$$

- We assume that the bridge and pipeline parts are independently of each other and the bridge doesn't change the reliability of the pipelines.
- We use the pivoting technique to find $\phi(\vec{1}_3, x)$, $\phi(0_3, x)$ and its $h_{+3}$ and $h_{-3}$ respectively, as following:

$\hookrightarrow \phi(\vec{1}_3, x) = (x_1 \amalg x_2)\cdot(x_4 \amalg x_5) = (x_1+x_2-x_1x_2)\cdot(x_4+x_5-x_4x_5)$
$= x_1x_4 + x_1x_5 - x_1x_4x_5 + x_2x_4 + x_2x_5 - x_2x_4x_5 - x_1x_2x_4$
$- x_1x_2x_5 + x_1x_2x_4x_5$. //

$\hookrightarrow h_{+3} = E\left[\phi(\vec{1}_3, x)\right] = E[x_1]\cdot E[x_4] + \dots + E[x_1]E[x_2]E[x_4]E[x_5] =$
$= ab + ab - ab^2 + ab + ab - ab^2 - a^2b - a^2b + a^2b^2 =$
$= 4ab - 2ab^2 - 2a^2b + (ab)^2 = 4\cancel{e^{-\lambda y}}e^{-\lambda L}\cancel{e^{+\lambda y}} - 2\cancel{e^{-\lambda y}}e^{-\lambda L}\cancel{e^{+\lambda y}}e^{-\lambda L}e^{+\lambda y}$
$-2\cdot e^{-\lambda y}\cancel{e^{-\lambda y}}e^{-\lambda L}\cancel{e^{+\lambda y}} + \cancel{e^{-\lambda y}}\cancel{e^{-\lambda y}}e^{-\lambda L}\cancel{e^{+\lambda y}}e^{-\lambda L}\cancel{e^{+\lambda y}} = 4e^{-\lambda L} - 2e^{-\lambda L}e^{-\lambda L}e^{+\lambda y}$
$-2e^{-\lambda y}\cdot e^{-\lambda L} + e^{-\lambda L}e^{-\lambda L} = e^{-\lambda L}\left(4 - 2e^{-\lambda L}e^{+\lambda y} - 2e^{-\lambda y} + e^{-\lambda L}\right) =$
$= e^{-\lambda L}\cdot\left(2 - e^{-\lambda y}\right)\cdot\left(2 - e^{-\lambda(L-y)}\right)$ //

$\hookrightarrow \phi(\vec{0}_3, x) = (x_1\cdot x_4) \amalg (x_2x_5) = 1-(1-x_1x_4)(1-x_2x_5) =$
$= 1 - (1 - x_2x_5 - x_1x_4 + x_1x_2x_4x_5) = x_2x_5 + x_1x_4 - x_1x_2x_4x_5$ //

$\hookrightarrow h_{-3} = E\left[\phi(\vec{0}_3, x)\right] = E[x_2]\cdot E[x_5] + E[x_1]\cdot E[x_4] - E[x_1]E[x_2]E[x_4]E[x_5] =$
$= ab + ab - a^2b^2 = 2ab - (ab)^2 = 2\cancel{e^{-\lambda y}}\cdot e^{-\lambda L}\cdot\cancel{e^{\lambda y}} - \cancel{e^{-\lambda y}}\cancel{e^{-\lambda y}}\cdot e^{-\lambda L}\cancel{e^{\lambda y}}e^{-\lambda L}\cdot\cancel{e^{\lambda y}} =$
$= 2e^{-\lambda L} - e^{-\lambda y}\cdot e^{-\lambda L} = e^{-\lambda L}\cdot\left(2 - e^{-\lambda L}\right)$ //

2d $\boxed{\text{continuation}}$

• From the pivoting technique we know that the structure and reliable formulas are the following:

$$\phi(\vec{x}) = x_3 \cdot \phi(\vec{1}_3, x) + (1 - x_3)\,\phi(\vec{0}_3, x)$$

and

$$h = E[\phi(\vec{x})] = E[x_3] \cdot E[\phi(\vec{1}_3, x)] + (1 - E[x_3]) \cdot E[\phi(\vec{0}_3, x)] =$$

$$= q \cdot h_{+3} + (1-q) \cdot h_{-3}$$

$\boxed{\text{from calculations before}}$

$$\therefore\ h_1(y, q) = q\, e^{-\lambda L}\left(2 - e^{-\lambda y}\right)\left(2 - e^{-\lambda(L-y)}\right) + (1-q) \cdot e^{-\lambda L}\left(2 - e^{-\lambda L}\right) \quad \blacksquare$$

(2e) • We get the derivative of $h_1$ with respect to $y$ and equate the equation to zero to determine the value of $y$ that maximizes the system reliability:

$$\frac{d}{dy}\Big[q\cdot e^{-\lambda L}(2-e^{-\lambda y})(2-e^{-\lambda(L-y)}) + \underbrace{\cancel{(1-q)e^{-\lambda L}(2-e^{-\lambda L})}}_{\text{constant}}\Big] =$$

$$= q\,e^{-\lambda L}\Big[\big(\lambda e^{-\lambda y}\cdot(2-e^{-\lambda L}e^{\lambda y})\big)+\big((2-e^{-\lambda y})\cdot(-\lambda e^{\lambda y}e^{-\lambda L})\big)\Big] =$$

$$= q\,e^{-\lambda L}\Big[\big(2\lambda e^{-\lambda y}-\cancel{\lambda e^{-\lambda L}}\big)+\big(-2\lambda e^{\lambda y}e^{-\lambda L}+\cancel{\lambda e^{-\lambda L}}\big)\Big] =$$

$$= q\,e^{-\lambda L}\Big[2\lambda e^{-\lambda y}-2\lambda e^{\lambda y}e^{-\lambda L}\Big] = q e^{-\lambda L}2\lambda e^{-\lambda y} - q e^{-\lambda L}2\lambda e^{\lambda y}e^{-\lambda L} = 0$$

$$\cancel{q}\cancel{e^{-\lambda L}}\cancel{2}\cancel{\lambda}e^{-\lambda y} = \cancel{q}\cancel{e^{-\lambda L}}\cancel{2}\cancel{\lambda}e^{\lambda y}e^{-\lambda L}$$

$$e^{-\lambda y} = e^{\lambda y}e^{-\lambda L}$$

$$-\cancel{\lambda}y = \cancel{\lambda}y - \cancel{\lambda}L$$

$$-2y = -L$$

$$\boxed{\therefore\ y = \frac{L}{2}}$$

• Now we set $y = L/2$ in the equation $h_1(y,q)$ to calculate $h_1$ with this particular value of $y$:

$$h_1\left(\tfrac{L}{2},q\right) = q\cdot e^{-\lambda L}\left(2-e^{-\frac{\lambda L}{2}}\right)\left(2-e^{-\frac{\lambda L}{2}}\right) + (1-q)\cdot e^{-\lambda L}\left(2-e^{-\lambda L}\right)$$

$$'' \quad = q\cdot e^{-\lambda L}\left(4-2e^{-\frac{\lambda L}{2}}-2e^{-\frac{\lambda L}{2}}+e^{-\lambda L}\right)+(1-q)\cdot\left(2e^{-\lambda L}-e^{-2\lambda L}\right)$$

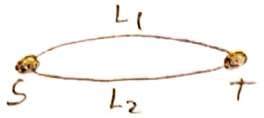
$$'' \quad = \cancel{4qe^{-\lambda L}} - 4qe^{-\frac{3\lambda L}{2}} + \cancel{qe^{-2\lambda L}} + 2e^{-\lambda L} - e^{-2\lambda L} - \cancel{2qe^{-\lambda L}} + \cancel{qe^{-2\lambda L}}$$

$$'' \quad = 2qe^{-\lambda L} - 4qe^{-\frac{3\lambda L}{2}} + 2qe^{-2\lambda L} + e^{-\lambda L}\cdot(2-e^{-\lambda L})$$

or

$$h_1\left(\tfrac{L}{2},q\right) = 2qe^{-\lambda L}\left(1-2e^{-\frac{\lambda L}{2}}+e^{-\lambda L}\right) + e^{-\lambda L}\cdot\left(2-e^{-\lambda L}\right)$$

(2f) • We have the structure represented as:




• The same as exercise (2c), then:

$$\phi(\vec{x}) = x_1 \amalg x_2 = x_1 + x_2 - x_1 x_2$$

• But now the probability of $i^{th}$ parts are:

$$P(x_i = 1) = e^{-\frac{\lambda L}{2}}$$

• Then the reliability of $\phi(\vec{x})$ is expressed as following:

$$h_2 = E[\phi(\vec{x})] = E[x_1] + E[x_2] - E[x_1] \cdot E[x_2]$$

$$h_2 = e^{-\frac{\lambda L}{2}} + e^{-\frac{\lambda L}{2}} - e^{-\frac{\lambda L}{2}} \cdot e^{-\frac{\lambda L}{2}}$$

$$h_2 = 2 e^{-\frac{\lambda L}{2}} - e^{-\lambda L}$$

$$h_2 = e^{-\frac{\lambda L}{2}} \left(2 - e^{-\frac{\lambda L}{2}}\right)$$

# Problem 2g

October 2, 2019

## 1 Problem 2g - oblig STK3405 - H19

- We assume that q=1, L=1000m and lambda=0.001.
- Then we can compare the results for h_1 (bridge) and h_2 (improved pipeline quality), which we import from exercise 2e and 2f respectively.

[7]:
```
from math import exp

q = 1
L = 1000
v_lambda = 0.001

h_1 = q*exp(-v_lambda*L)*(2-exp(-v_lambda*L/2))*(2-exp(-v_lambda*L/
 ↪2))+(1-q)*exp(-v_lambda)*(2-exp(-v_lambda*L))
h_2 = exp((-v_lambda/2)*L)*(2-exp((-v_lambda/2)*L))

print('Reliability for h_1:', h_1, '\n\nReliability for h_2:', h_2)
```

```
Reliability for h_1: 0.7143324073286628

Reliability for h_2: 0.8451818782538245
```

### 1.1 Conclusion:

As we can see, with the same conditions and variables, the reliability for h_1 is smaller than h_2, then if the price to build a bridge is the same as the price of improving the pipeline quality, I would choose to improve the pipeline quality because its reliability is larger!