# CHAPTER 15 MULTIPLE INTEGRALS

## 15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy\, dy\, dx = \int_1^2 \left[xy^2\right]_0^4 dx = \int_1^2 16x\, dx = \left[8x^2\right]_1^2 = 24$

2. $\int_0^2 \int_{-1}^1 (x-y)\, dy\, dx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx = \int_0^2 2x\, dx = \left[x^2\right]_0^2 = 4$

3. $\int_{-1}^0 \int_{-1}^1 (x+y+1)\, dx\, dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^1 dy = \int_{-1}^0 (2y+2)\, dy = \left[y^2 + 2y\right]_{-1}^0 = 1$

4. $\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dx\, dy = \int_0^1 \left[x - \frac{x^3}{6} - \frac{xy^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy = \left[\frac{5}{6}y - \frac{y^3}{6}\right]_0^1 = \frac{2}{3}$

5. $\int_0^3 \int_0^2 \left(4 - y^2\right) dy\, dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^3 \frac{16}{3}\, dx = \left[\frac{16}{3}x\right]_0^3 = 16$

6. $\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dy\, dx = \int_0^3 \left[\frac{x^2y^2}{2} - xy^2\right]_{-2}^0 dx = \int_0^3 \left(4x - 2x^2\right) dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$

7. $\int_0^1 \int_0^1 \frac{y}{1+xy}\, dx\, dy = \int_0^1 \left[\ln|1+xy|\right]_0^1 dy = \int_0^1 \ln|1+y| dy = \left[y \ln|1+y| - y + \ln|1+y|\right]_0^1 = 2\ln 2 - 1$

8. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\, dy = \int_1^4 \left[\frac{1}{4}x^2 + x\sqrt{y}\right]_0^4 dy = \int_1^4 (4 + 4y^{1/2}) dy = \left[4y + \frac{8}{3}y^{3/2}\right]_1^4 = \frac{92}{3}$

9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dy\, dx = \int_0^{\ln 2} \left[e^{2x+y}\right]_1^{\ln 5} dx = \int_0^{\ln 2} \left(5e^{2x} - e^{2x+1}\right) dx = \left[\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}\right]_0^{\ln 2} = \frac{3}{2}(5-e)$

10. $\int_0^1 \int_1^2 x\, y\, e^x dy\, dx = \int_0^1 \left[\frac{1}{2}x\, y^2 e^x\right]_1^2 dx = \int_0^1 \frac{3}{2}x\, e^x dx = \left[\frac{3}{2}x\, e^x - \frac{3}{2}e^x\right]_0^1 = \frac{3}{2}$

11. $\int_{-1}^2 \int_0^{\pi/2} y \sin x\, dx\, dy = \int_{-1}^2 \left[-y\cos x\right]_0^{\pi/2} dy = \int_{-1}^2 y\, dy = \left[\frac{1}{2}y^2\right]_{-1}^2 = \frac{3}{2}$

12. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy = \int_\pi^{2\pi} \left[-\cos x + x\cos y\right]_0^\pi dy = \int_\pi^{2\pi} (2 + \pi\cos y)\, dy = \left[2y + \pi\sin y\right]_\pi^{2\pi} = 2\pi$

13. $\displaystyle\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy = \int_1^4 \left(\left[\frac{(\ln x)^2}{2y}\right]_{x=1}^{x=e}\right) dy = \int_1^4 \frac{1}{2y}\, dy = \left.\frac{\ln y}{2}\right]_1^4 = \ln 2$

14. $\int_{-1}^{2}\int_{1}^{2} x\ln y\,dy\,dx = \int_{-1}^{2}\left(x(y\ln y - y)\right]_{y=1}^{y=2}\right)dx = \int_{-1}^{2}(2\ln 2-1)x\,dx = (2\ln 2-1)\frac{x^2}{2}\Big]_{-1}^{2} = 3\ln 2 - \frac{3}{2}$

15. $\iint_R \left(6y^2-2x\right)dA = \int_0^1\int_0^2\left(6y^2-2x\right)dy\,dx = \int_0^1[2y^3-2xy]_0^2\,dx = \int_0^1(16-4x)\,dx = [16x-2x^2]_0^1 = 14$

16. $\iint_R \frac{\sqrt{x}}{y^2}\,dA = \int_0^4\int_1^2\frac{\sqrt{x}}{y^2}dy\,dx = \int_0^4\left[-\frac{\sqrt{x}}{y}\right]_1^2 dx = \int_0^4 \frac{1}{2}x^{1/2}dx = \left[\frac{1}{3}x^{3/2}\right]_0^4 = \frac{8}{3}$

17. $\iint_R xy\cos y\,dA = \int_{-1}^{1}\int_0^{\pi} xy\cos y\,dy\,dx = \int_{-1}^{1}\left[xy\sin y + x\cos y\right]_0^{\pi}\,dx = \int_{-1}^{1}(-2x)\,dx = \left[-x^2\right]_{-1}^{1} = 0$

18. $\iint_R y\sin(x+y)\,dA = \int_{-\pi}^{0}\int_0^{\pi} y\sin(x+y)\,dy\,dx = \int_{-\pi}^{0}\left[-y\cos(x+y)+\sin(x+y)\right]_0^{\pi}dx$

$= \int_{-\pi}^{0}\left(\sin(x+\pi)-\pi\cos(x+\pi)-\sin x\right)dx = \left[-\cos(x+\pi)-\pi\sin(x+\pi)+\cos x\right]_{-\pi}^{0} = 4$

19. $\iint_R e^{x-y}dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}dy\,dx = \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2} dx = \int_0^{\ln 2}\left(-e^{x-\ln 2}+e^x\right)dx = \left[-e^{x-\ln 2}+e^x\right]_0^{\ln 2} = \frac{1}{2}$

20. $\iint_R x\,y\,e^{x\,y^2}\,dA = \int_0^2\int_0^1 x\,y\,e^{x\,y^2}\,dy\,dx = \int_0^2\left[\frac{1}{2}e^{x\,y^2}\right]_0^1 dx = \int_0^2\left(\frac{1}{2}e^x-\frac{1}{2}\right)dx = \left[\frac{1}{2}e^x-\frac{1}{2}x\right]_0^2 = \frac{1}{2}\left(e^2-3\right)$

21. $\iint_R \frac{xy^3}{x^2+1}\,dA = \int_0^1\int_0^2\frac{xy^3}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{xy^4}{4\left(x^2+1\right)}\right]_0^2 dx = \int_0^1\frac{4x}{x^2+1}\,dx = \left[2\ln\left|x^2+1\right|\right]_0^1 = 2\ln 2$

22. $\iint_R \frac{y}{x^2y^2+1}dA = \int_0^1\int_0^1\frac{y}{(xy)^2+1}\,dx\,dy = \int_0^1[\tan^{-1}(xy)]_0^1\,dy = \int_0^1\tan^{-1}y\,dy = \left[y\tan^{-1}y-\frac{1}{2}\ln|1+y^2|\right]_0^1 = \frac{\pi}{4}-\frac{1}{2}\ln 2$

23. $\int_1^2\int_1^2\frac{1}{xy}\,dy\,dx = \int_1^2\frac{1}{x}(\ln 2-\ln 1)\,dx = (\ln 2)\int_1^2\frac{1}{x}\,dx = (\ln 2)^2$

24. $\int_0^1\int_0^{\pi} y\cos xy\,dx\,dy = \int_0^1[\sin xy]_0^{\pi}\,dy = \int_0^1\sin \pi y\,dy = \left[-\frac{1}{\pi}\cos\pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

25. $V = \iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{-1}^{1}\left(x^2+y^2\right)dy\,dx = \int_{-1}^{1}\left[x^2y+\frac{1}{3}y^3\right]_{-1}^{1}dx = \int_{-1}^{1}\left(2x^2+\frac{2}{3}\right)dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_{-1}^{1} = \frac{8}{3}$

26. $V = \iint_R f(x,y)\,dA = \int_0^2\int_0^2\left(16-x^2-y^2\right)dy\,dx = \int_0^2\left[16y-x^2y-\frac{1}{3}y^3\right]_0^2 dx = \int_0^2\left(\frac{88}{3}-2x^2\right)dx = \left[\frac{88}{3}x-\frac{2}{3}x^3\right]_0^2$

$= \frac{160}{3}$

27. $V = \iint_R f(x, y)\, dA = \int_0^1 \int_0^1 (2 - x - y)\, dy\, dx = \int_0^1 \left[ 2y - xy - \frac{1}{2}y^2 \right]_0^1 dx = \int_0^1 \left( \frac{3}{2} - x \right) dx = \left[ \frac{3}{2}x - \frac{1}{2}x^2 \right]_0^1 = 1$

28. $V = \iint_R f(x, y)\, dA = \int_0^4 \int_0^2 \frac{y}{2}\, dy\, dx = \int_0^4 \left[ \frac{y^2}{4} \right]_0^2 dx = \int_0^4 1\, dx = [x]_0^4 = 4$

29. $V = \iint_R f(x, y)\, dA = \int_0^{\pi/2} \int_0^{\pi/4} 2 \sin x \cos y\, dy\, dx = \int_0^{\pi/2} [2 \sin x \sin y]_0^{\pi/4} dx = \int_0^{\pi/2} \left( \sqrt{2} \sin x \right) dx$
$= \left[ -\sqrt{2} \cos x \right]_0^{\pi/2} = \sqrt{2}$

30. $V = \iint_R f(x, y)\, dA = \int_0^1 \int_0^2 \left( 4 - y^2 \right) dy\, dx = \int_0^1 \left[ 4y - \frac{1}{3}y^3 \right]_0^2 dx = \int_0^1 \left( \frac{16}{3} \right) dx = \left[ \frac{16}{3} x \right]_0^1 = \frac{16}{3}$

31. $\int_1^2 \int_0^3 kx^2 y\, dx\, dy = \int_1^2 \left( \left[ \frac{k}{3} x^3 y \right]_{x=0}^{x=3} \right) dx = \int_1^2 9ky\, dy = \left[ \frac{9}{2} ky^2 \right]_1^2 = \frac{27}{2} k$
Thus we choose $k = 2/27$.

32. $\int_0^{\pi/2} \sin\left( \sqrt{y} \right) dy$ is some number, say $a$. Then $\int_{-1}^1 \int_0^{\pi/2} x \sin\left( \sqrt{y} \right) dy\, dx = a \int_{-1}^1 x\, dx = 0$ since the integral of the odd function $x$ over an interval symmetric to 0 is equal to 0.

33. By Fubini's Theorem,
$$\int_0^2 \int_0^1 \frac{x}{1+xy}\, dx\, dy = \int_0^1 \int_0^2 \frac{x}{1+xy}\, dy\, dx$$
$$= \int_0^1 \left( [\ln(1+xy)]_{y=0}^{y=2} \right) dx = \int_0^1 \ln(1+2x)\, dx = \frac{(1+2x)}{2} [\ln(1+2x) - 1] \Big]_0^1 = \frac{3}{2} \ln 3 - 1$$

34. By Fubini's Theorem,
$$\int_0^1 \int_0^3 xe^{xy}\, dx\, dy = \int_0^3 \int_0^1 xe^{xy}\, dy\, dx$$
$$= \int_0^3 \left( \left[ e^{xy} \right]_{y=0}^{y=1} \right) \Big] dx = \int_0^3 \left( e^x - 1 \right) dx = \left( e^x - x \right) \Big]_0^3 = e^3 - 4 \approx 16.086$$

35. (a) MAPLE gives $\int_0^1 \int_0^2 \frac{y-x}{(x+y)^3}\, dx\, dy = \frac{1}{3}$ and $\int_0^2 \int_0^1 \frac{y-x}{(x+y)^3}\, dy\, dx = -\frac{2}{3}$. This does not contradict Fubini's Theorem since the integrand is not continuous on the region $R$: $0 \le x \le 2,\ 0 \le y \le 1$.

36. Since $f$ is continuous on $R$, for fixed $u$ $f(u,v)$ is a continuous function of $v$ and has an antiderivative with respect to $v$ on $R$, call it $g(u,v)$. Then $\int_c^y f(u,v)\,dv = g(u,y) - g(u,c)$ and

$$F(x,y) = \int_a^x \int_c^y f(u,v)\,dv\,du = \int_a^x (g(u,y) - g(u,c))\,du.$$

$$F_x = \frac{\partial}{\partial x}\int_a^x (g(u,y) - g(u,c))\,du = g(x,y) - g(x,c).$$

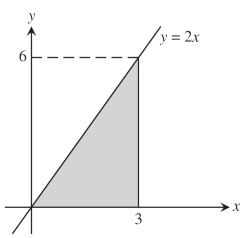
Now taking the derivative with respect to $y$, we get

$$F_{xy} = \frac{\partial}{\partial y}(g(x,y) - g(x,c)) = f(x,y).$$

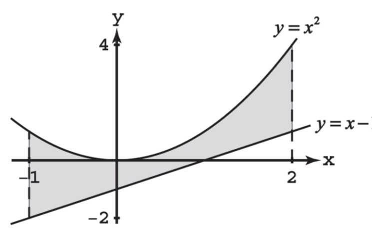
To evaluate $F_{yx}$ we use Fubini's Theorem to rewrite $F(x,y)$ as $\int_c^y \int_a^x f(u,v)\,du\,dv$ and make a similar argument. The result is again $f(x,y)$.

## 15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

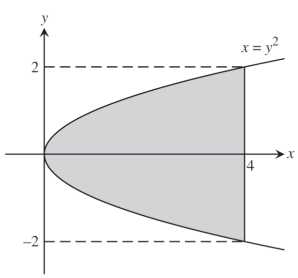
1.



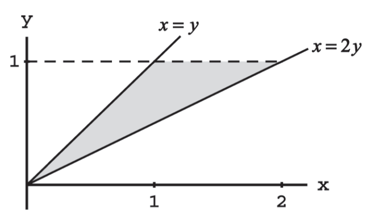

2.



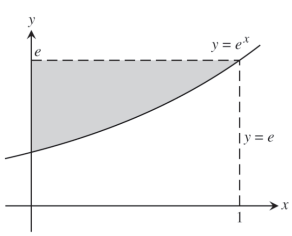

3.



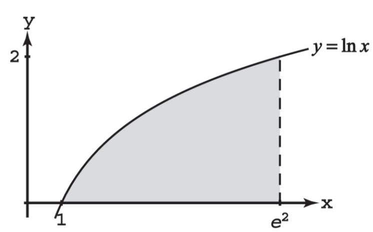

4.



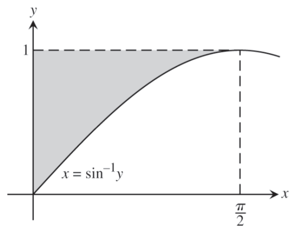

5.



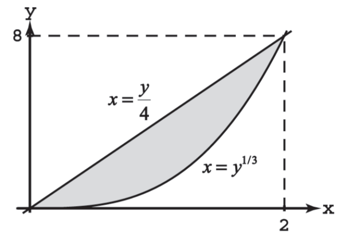

6.



7.




8.




9. (a) $\int_0^2 \int_{x^3}^8 dy\, dx$ (b) $\int_0^8 \int_0^{y^{1/3}} dx\, dy$

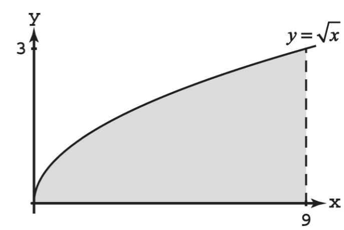
10. (a) $\int_0^3 \int_0^{2x} dy\, dx$ (b) $\int_0^6 \int_{y/2}^3 dx\, dy$

11. (a) $\int_0^3 \int_{x^2}^{3x} dy\, dx$ (b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx\, dy$

12. (a) $\int_0^2 \int_1^{e^x} dy\, dx$ (b) $\int_1^{e^2} \int_{\ln y}^2 dx\, dy$

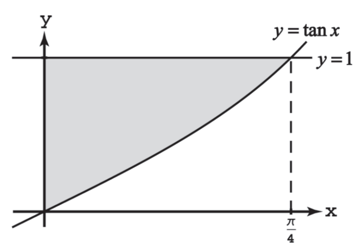
13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$

(b) $\int_0^3 \int_{y^2}^9 dx\, dy$



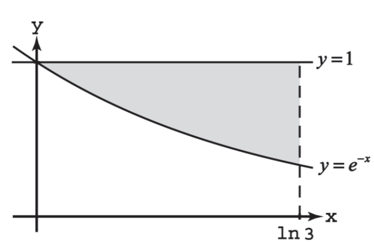

14. (a) $\int_0^{\pi/4} \int_{\tan x}^1 dy\, dx$

(b) $\int_0^1 \int_0^{\tan^{-1} y} dx\, dy$



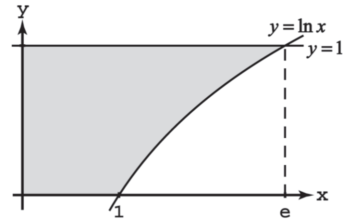

15. (a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy\, dx$

(b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx\, dy$



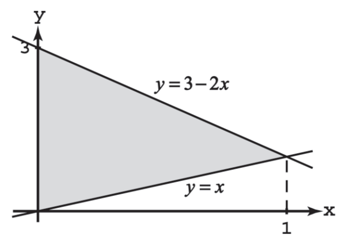
16. (a) $\int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^1 dy\, dx$

(b) $\int_0^1 \int_0^{e^y} dx\, dy$



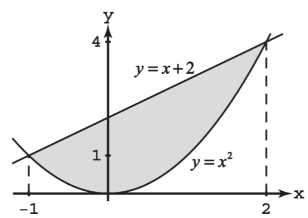

17. (a) $\int_0^1 \int_x^{3-2x} dy\, dx$

(b) $\int_0^1 \int_0^y dx\, dy + \int_1^3 \int_0^{(3-y)/2} dx\, dy$




18. (a) $\int_{-1}^2 \int_{x^2}^{x+2} dy\, dx$

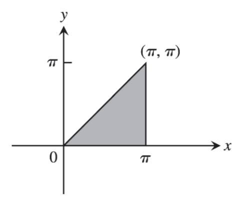
(b) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^3 \int_{y-2}^{\sqrt{y}} dx\, dy$




19. $\int_0^\pi \int_0^\pi (x \sin y)\, dy\, dx = \int_0^\pi \left[-x \cos y\right]_0^x dx$

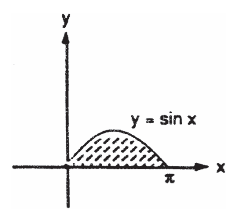
$= \int_0^\pi (x - x\cos x)\, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x)\right]_0^\pi$

$= \frac{\pi^2}{2} + 2$




20. $\int_0^\pi \int_0^{\sin x} y\, dy\, dx = \int_0^\pi \left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^\pi \frac{1}{2} \sin^2 x\, dx$

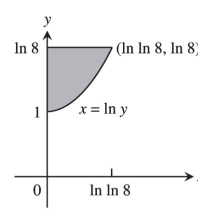
$= \frac{1}{4} \int_0^\pi (1 - \cos 2x)\, dx = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi}{4}$




21. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx\, dy = \int_1^{\ln 8} \left[e^{x+y}\right]_0^{\ln y} dy$

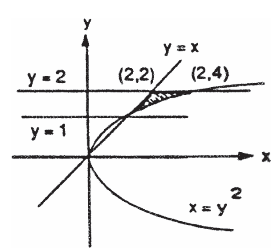
$= \int_1^{\ln 8} \left(ye^y - e^y\right) dy = \left[(y-1)e^y - e^y\right]_1^{\ln 8}$

$= 8(\ln 8 - 1) - 8 + e = 8 \ln 8 - 16 + e$



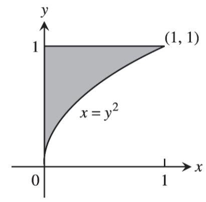

22. $\int_1^2 \int_y^{y^2} dx\, dy = \int_1^2 \left(y^2 - y\right) dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$

$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



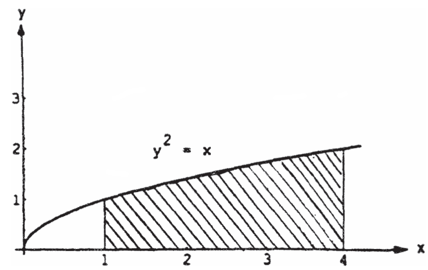
23. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy = \int_0^1 \left[3y^2 e^{xy}\right]_0^{y^2} dy$

$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2\right) dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$




24. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx = \int_1^4 \left[\frac{3}{2}\sqrt{x} e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$

$= \frac{3}{2}(e-1)\int_1^4 \sqrt{x}\, dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$




25. $\int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx = \int_1^2 \left[x \ln y\right]_x^{2x} dx = (\ln 2)\int_1^2 x\, dx = \frac{3}{2} \ln 2$

26. $\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

27. $\int_0^1 \int_0^{1-u} \left(v - \sqrt{u}\right) dv\, du = \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du = \int_0^1 \left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right] du$

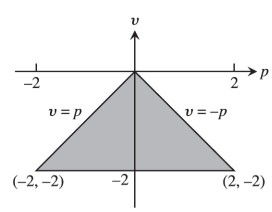
$= \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

28. $\int_1^2 \int_0^{\ln t} e^s \ln t\, ds\, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t)\, dt = \left[\frac{t^2}{2}\ln t - \frac{t^2}{4} - t\ln t + t\right]_1^2$

$= (2\ln 2 - 1 - 2\ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

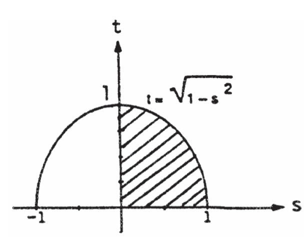
29. $\int_{-2}^0 \int_v^{-v} 2\, dp\, dv = 2\int_{-2}^0 \left[p\right]_v^{-v} dv = 2\int_{-2}^0 -2v\, dv$

$= -2\left[v^2\right]_{-2}^0 = 8$

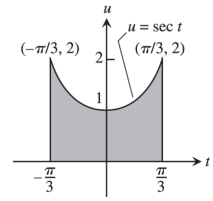



30. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\, dt\, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$

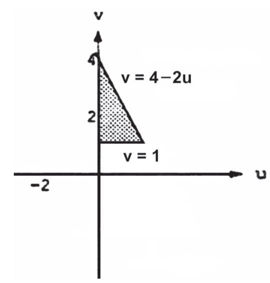
$= \int_0^1 4\left(1-s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



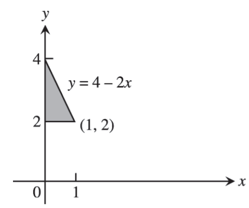
31. $\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t\, du\, dt = \int_{-\pi/3}^{\pi/3}\left[(3\cos t)u\right]_0^{\sec t}$
$= \int_{-\pi/3}^{\pi/3} 3\, dt = 2\pi$



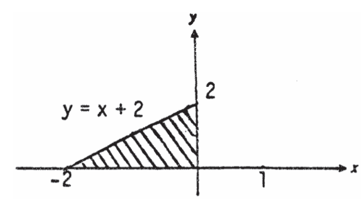

32. $\int_{0}^{3/2}\int_{1}^{4-2u} \frac{4-2u}{v^2}\, dv\, du = \int_{0}^{3/2}\left[\frac{2u-4}{v}\right]_1^{4-2u} du$
$= \int_{0}^{3/2}(3-2u)\, du = \left[3u - u^2\right]_0^{3/2} = \frac{9}{2}$



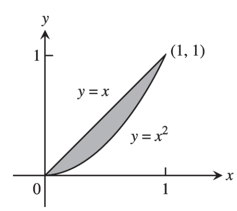

33. $\int_{2}^{4}\int_{0}^{(4-y)/2} dx\, dy$



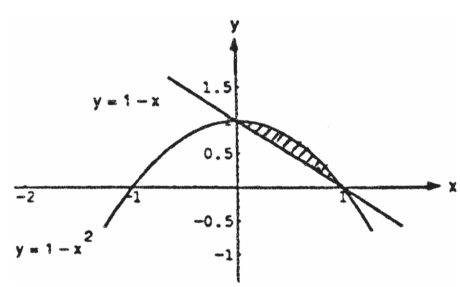

34. $\int_{-2}^{0}\int_{0}^{x+2} dy\, dx$



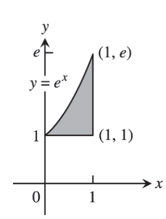

35. $\int_{0}^{1}\int_{x^2}^{x} dy\, dx$



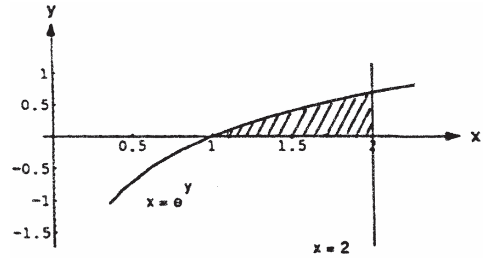

36. $\int_{0}^{1}\int_{1-y}^{\sqrt{1-y}} dx\, dy$



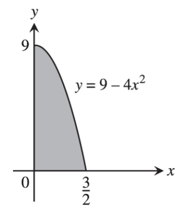

37. $\int_{1}^{e}\int_{\ln y}^{1} dx\, dy$



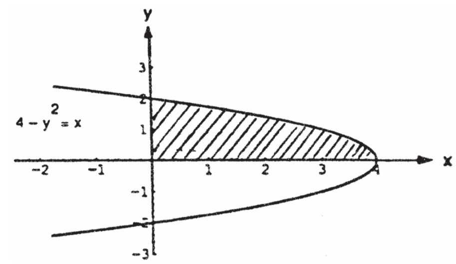
38. $\int_1^2 \int_0^{\ln x} dy\, dx$



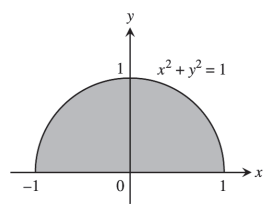

39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\, dx\, dy$



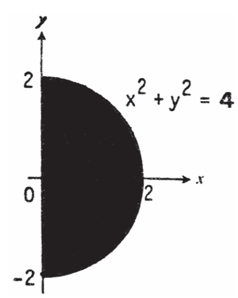

40. $\int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx$



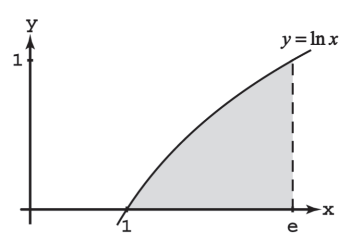

41. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\, dy\, dx$



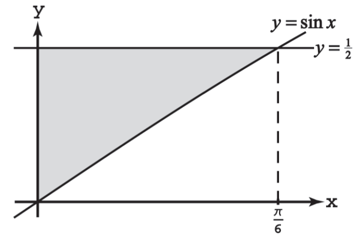

42. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} 6x\, dx\, dy$



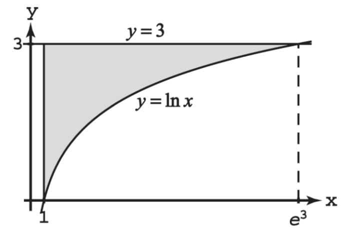

43. $\int_0^1 \int_{e^y}^{e} xy\, dx\, dy$



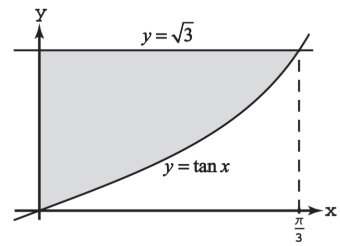
44. $\int_0^{1/2}\int_0^{\sin^{-1} y} xy^2\, dx\, dy$



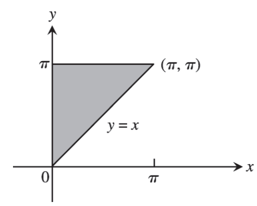

45. $\int_1^{e^3}\int_{\ln x}^{3} (x+y)\, dy\, dx$




46. $\int_0^{\pi/3}\int_{\tan x}^{\sqrt{3}} \sqrt{x\,y}\, dy\, dx$



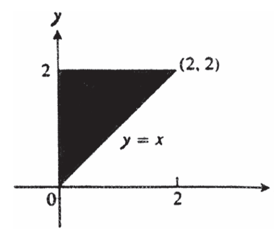

47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = 2$




48. $\int_0^{2}\int_x^{2} 2y^2 \sin xy\, dy\, dx = \int_0^{2}\int_0^{y} 2y^2 \sin xy\, dx\, dy$

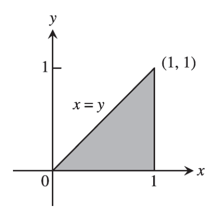
$= \int_0^{2} \left[-2y\cos xy\right]_0^{y} dy = \int_0^{2} \left(-2y\cos y^2 + 2y\right) dy$

$= \left[-\sin y^2 + y^2\right]_0^{2} = 4 - \sin 4$



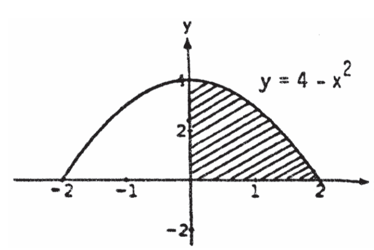

49. $\int_0^{1}\int_y^{1} x^2 e^{xy}\, dx\, dy = \int_0^{1}\int_0^{x} x^2 e^{xy}\, dy\, dx = \int_0^{1} \left[xe^{xy}\right]_0^{x} dx$

$= \int_0^{1} \left(xe^{x^2} - x\right) dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^{1} = \frac{e-2}{2}$

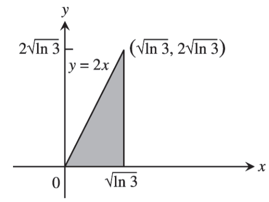



50. $\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^{4}\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$

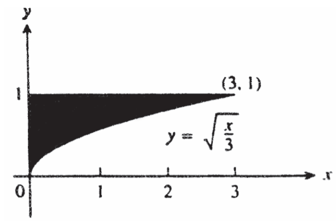
$= \int_0^{4} \left[\frac{x^2 e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}} dy = \int_0^{4} \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{4}\right]_0^{4} = \frac{e^8-1}{4}$



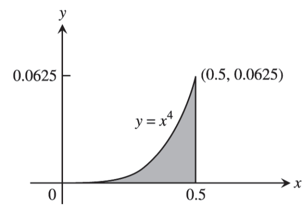
51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy = \int_0^{\sqrt{\ln 3}}\int_0^{2x} e^{x^2}\,dy\,dx$
$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\,dx = \left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$



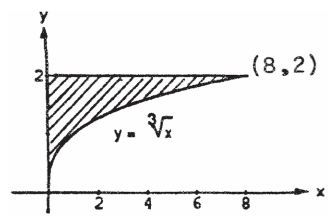

52. $\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx = \int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy$
$= \int_0^1 3y^2e^{y^3}\,dy = \left[e^{y^3}\right]_0^1 = e - 1$



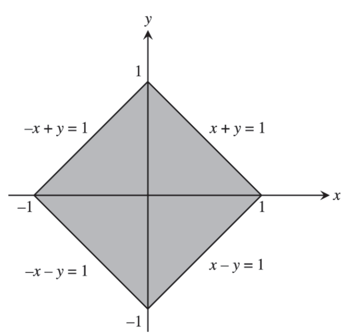

53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dx\,dy$
$= \int_0^{1/2}\int_0^{x^4} \cos\left(16\pi x^5\right)dy\,dx = \int_0^{1/2} x^4\cos\left(16\pi x^5\right)dx$
$= \left[\frac{\sin\left(16\pi x^5\right)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$



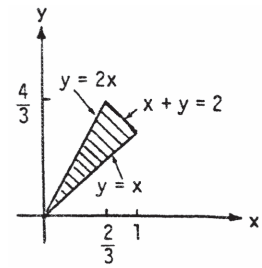

54. $\int_0^8\int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2\int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$
$= \int_0^2 \frac{y^3}{y^4+1}\,dy = \frac{1}{4}\left[\ln\left(y^4+1\right)\right]_0^2 = \frac{\ln 17}{4}$




55. $\iint\limits_R \left(y - 2x^2\right)dA$
$= \int_{-1}^0\int_{-x-1}^{x+1}\left(y-2x^2\right)dy\,dx + \int_0^1\int_{x-1}^{1-x}\left(y-2x^2\right)dy\,dx$
$= \int_{-1}^0\left[\frac{1}{2}y^2 - 2x^2y\right]_{-x-1}^{x+1}dx + \int_0^1\left[\frac{1}{2}y^2 - 2x^2y\right]_{x-1}^{1-x}dx$
$= \int_{-1}^0\left(\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right)dx$
$+ \int_0^1\left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right]dx$
$= -4\int_{-1}^0\left(x^3 + x^2\right)dx + 4\int_0^1\left(x^3 + x^2\right)dx$
$= -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$
$= 4\left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4} - \frac{1}{3}\right) = 8\left(\frac{3}{12} - \frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$



56. $\iint_R xy\, dA = \int_0^{2/3}\int_x^{2x} xy\, dy\, dx + \int_{2/3}^1\int_x^{2-x} xy\, dy\, dx$




$= \int_0^{2/3}\left[\frac{1}{2}xy^2\right]_x^{2x} dx + \int_{2/3}^1\left[\frac{1}{2}xy^2\right]_x^{2-x} dx$

$= \int_0^{2/3}\left(2x^3 - \frac{1}{2}x^3\right) dx + \int_{2/3}^1\left[\frac{1}{2}x(2-x)^2 - \frac{1}{2}x^3\right] dx$

$= \int_0^{2/3}\frac{3}{2}x^3\, dx + \int_{2/3}^1\left(2x - x^2\right) dx$

$= \left[\frac{3}{8}x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3}x^3\right]_{2/3}^1 = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right) + \left(1 - \frac{2}{3}\right) - \left[\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}$

57. $V = \int_0^1\int_x^{2-x}\left(x^2 + y^2\right) dy\, dx = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1\left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$

$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$

58. $V = \int_{-2}^1\int_x^{2-x^2} x^2\, dy\, dx = \int_{-2}^1\left[x^2y\right]_x^{2-x^2} dx = \int_{-2}^1\left(2x^2 - x^4 - x^3\right) dx = \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^1$

$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60}\right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60}\right) = \frac{189}{60} = \frac{63}{20}$

59. $V = \int_{-4}^1\int_{3x}^{4-x^2}(x+4)\, dy\, dx = \int_{-4}^1\left[xy + 4y\right]_{3x}^{4-x^2} dx = \int_{-4}^1\left[x\left(4 - x^2\right) + 4\left(4 - x^2\right) - 3x^2 - 12x\right] dx$

$= \int_{-4}^1\left(-x^3 - 7x^2 - 8x + 16\right) dx = \left[-\frac{1}{4}x^4 - \frac{7}{3}x^3 - 4x^2 + 16x\right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12\right) - \left(\frac{64}{3} - 64\right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}$

60. $V = \int_0^2\int_0^{\sqrt{4-x^2}}(3 - y)\, dy\, dx = \int_0^2\left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2\left[3\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$

$= \left[\frac{3}{2}x\sqrt{4-x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2 = 6\left(\frac{\pi}{2}\right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$

61. $V = \int_0^2\int_0^3\left(4 - y^2\right) dx\, dy = \int_0^2\left[4x - y^2x\right]_0^3 dy = \int_0^2\left(12 - 3y^2\right) dy = \left[12y - y^3\right]_0^2 = 24 - 8 = 16$

62. $V = \int_0^2\int_0^{4-x^2}\left(4 - x^2 - y\right) dy\, dx = \int_0^2\left[\left(4 - x^2\right)y - \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_0^2\frac{1}{2}\left(4 - x^2\right)^2 dx = \int_0^2\left(8 - 4x^2 + \frac{x^4}{2}\right) dx$

$= \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480 - 320 + 96}{30} = \frac{128}{15}$

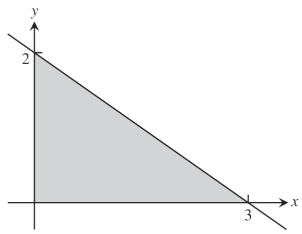
63. $V = \int_0^2\int_0^{2-x}\left(12 - 3y^2\right) dy\, dx = \int_0^2\left[12y - y^3\right]_0^{2-x} dx = \int_0^2\left[24 - 12x - (2-x)^3\right] dx = \left[24x - 6x^2 + \frac{(2-x)^4}{4}\right]_0^2 = 20$

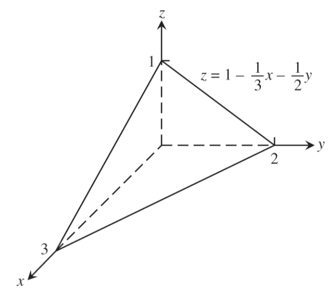
64. $V = \int_{-1}^0\int_{-x-1}^{x+1}(3 - 3x)\, dy\, dx + \int_0^1\int_{x-1}^{1-x}(3 - 3x)\, dy\, dx = 6\int_{-1}^0\left(1 - x^2\right) dx + 6\int_0^1\left(1 - x^2\right) dx = 4 + 2 = 6$

65. $V=\int_1^2\int_{-1/x}^{1/x}(x+1)\,dy\,dx=\int_1^2\left[xy+y\right]_{-1/x}^{1/x}\,dx=\int_1^2\left[1+\frac{1}{x}-\left(-1-\frac{1}{x}\right)\right]dx=2\int_1^2\left(1+\frac{1}{x}\right)dx=2\left[x+\ln x\right]_1^2$
$=2(1+\ln 2)$

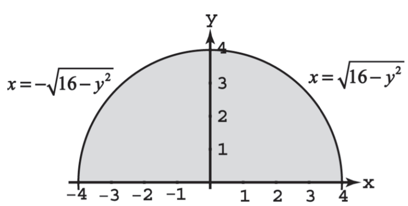
66. $V=4\int_0^{\pi/3}\int_0^{\sec x}\left(1+y^2\right)dy\,dx=4\int_0^{\pi/3}\left[y+\frac{y^3}{3}\right]_0^{\sec x}\,dx=4\int_0^{\pi/3}\left(\sec x+\frac{\sec^3 x}{3}\right)dx$
$=\frac{2}{3}\left[7\ln\left|\sec x+\tan x\right|+\sec x\tan x\right]_0^{\pi/3}=\frac{2}{3}\left[7\ln\left(2+\sqrt{3}\right)+2\sqrt{3}\right]$

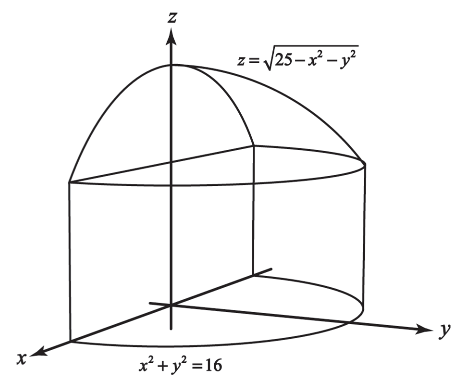
67.




68.




69. $\int_1^\infty\int_{e^{-x}}^1\frac{1}{x^3y}\,dy\,dx=\int_1^\infty\left[\frac{\ln y}{x^3}\right]_{e^{-x}}^1\,dx=\int_1^\infty-\left(\frac{-x}{x^3}\right)dx=-\lim_{b\to\infty}\left[\frac{1}{x}\right]_1^b=-\lim_{b\to\infty}\left(\frac{1}{b}-1\right)=1$

70. $\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx=\int_{-1}^1\left[y^2+y\right]_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}\,dx=\int_{-1}^1\frac{2}{\sqrt{1-x^2}}\,dx=4\lim_{b\to1^-}\left[\sin^{-1}x\right]_0^b=4\lim_{b\to1^-}[\sin^{-1}b-0]$
$=2\pi$

71. $\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{\left(x^2+1\right)\left(y^2+1\right)}\,dx\,dy=2\int_0^\infty\left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)dy=2\pi\lim_{b\to\infty}\int_0^b\frac{1}{y^2+1}\,dy$
$=2\pi\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)=(2\pi)\left(\frac{\pi}{2}\right)=\pi^2$

72. $\int_0^\infty\int_0^\infty xe^{-(x+2y)}\,dx\,dy=\int_0^\infty e^{-2y}\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b\,dy=\int_0^\infty e^{-2y}\lim_{b\to\infty}\left(-be^{-b}-e^{-b}+1\right)dy$
$=\int_0^\infty e^{-2y}\,dy=\frac{1}{2}\lim_{b\to\infty}\left(-e^{-2b}+1\right)=\frac{1}{2}$

73. $\iint\limits_R f(x, y)\, dA \approx \frac{1}{4} f\left(-\frac{1}{2}, 0\right) + \frac{1}{8} f(0, 0) + \frac{1}{8} f\left(\frac{1}{4}, 0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0 + \frac{1}{4}\right) = -\frac{3}{32}$

74. $\iint\limits_R f(x, y)\, dA \approx \frac{1}{4}\left[f\left(\frac{7}{4}, \frac{11}{4}\right) + f\left(\frac{9}{4}, \frac{11}{4}\right) + f\left(\frac{7}{4}, \frac{13}{4}\right) + f\left(\frac{9}{4}, \frac{13}{4}\right)\right] = \frac{1}{16}(29 + 31 + 33 + 35) = \frac{128}{16} = 8$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$.

Thus, $\iint\limits_R f(x, y)\, dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\, dy\, dx = \int_0^{\sqrt{3}} \left[\left(4 - x^2\right) - \frac{x}{\sqrt{3}} \sqrt{4-x^2}\right] dx = \left[4x - \frac{x^3}{3} + \frac{\left(4-x^2\right)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}}$

$= \frac{20\sqrt{3}}{9}$

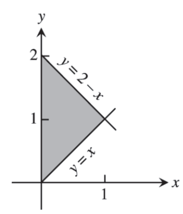
76. $\int_2^\infty \int_0^2 \frac{1}{\left(x^2-x\right)(y-1)^{2/3}}\, dy\, dx = \int_2^\infty \left[\frac{3(y-1)^{1/3}}{x^2-x}\right]_0^2 dx = \int_2^\infty \left(\frac{3}{x^2-x} + \frac{3}{x^2-x}\right) dx = 6 \int_2^\infty \frac{dx}{x(x-1)} = 6 \lim\limits_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$

$= 6 \lim\limits_{b\to\infty} \left[\ln(x-1) - \ln x\right]_2^b = 6 \lim\limits_{b\to\infty} \left[\ln(b-1) - \ln b - \ln 1 + \ln 2\right] = 6\left[\lim\limits_{b\to\infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2\right] = 6 \ln 2$

77. $V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$

$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$

$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$




78. $\int_0^2 \left(\tan^{-1} \pi x - \tan^{-1} x\right) dx = \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2}\, dy\, dx = \int_0^2 \int_{y/\pi}^{y} \frac{1}{1+y^2}\, dx\, dy + \int_2^{2\pi} \int_{y/\pi}^{2} \frac{1}{1+y^2}\, dx\, dy$

$= \int_0^2 \frac{\left(1-\frac{1}{\pi}\right) y}{1+y^2}\, dy + \int_2^{2\pi} \frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\, dy = \left(\frac{\pi-1}{2\pi}\right)\left[\ln\left(1+y^2\right)\right]_0^2 + \left[2 \tan^{-1} y + \frac{1}{2\pi} \ln\left(1+y^2\right)\right]_2^{2\pi}$

$= \left(\frac{\pi-1}{2\pi}\right) \ln 5 + 2 \tan^{-1} 2\pi - \frac{1}{2\pi} \ln\left(1 + 4\pi^2\right) - 2 \tan^{-1} 2 + \frac{1}{2\pi} \ln 5$

$= 2 \tan^{-1} 2\pi - 2 \tan^{-1} 2 - \frac{1}{2\pi} \ln\left(1 + 4\pi^2\right) + \frac{\ln 5}{2}$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points $(x, y)$ such that $4 - x^2 - 2y^2 \geq 0$ or $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

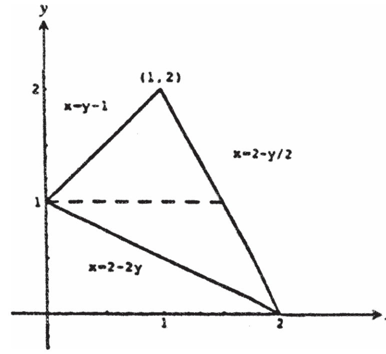
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points $(x, y)$ such that $x^2 + y^2 - 9 \leq 0$ or $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition $R$ into two triangles with the line $y=1$. The integral of $f$ over $R$ could then be written as a sum of integrals that could be evaluated by integrating first with respect to $x$ and then with respect to $y$:

$$\iint_R f(x, y)\, dA$$
$$=\int_0^1\int_{2-2y}^{2-(y/2)} f(x, y)\, dx\, dy+\int_1^2\int_{y-1}^{2-(y/2)} f(x, y)\, dx\, dy.$$

Partitioning $R$ with the line $x=1$ would let us write the integral of $f$ over $R$ as a sum of iterated integrals with order $dy\, dx$.




83. $$\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy=\int_{-b}^{b}\int_{-b}^{b} e^{-y^2}e^{-x^2}\, dx\, dy=\int_{-b}^{b} e^{-y^2}\left(\int_{-b}^{b} e^{-x^2}\, dx\right) dy=\left(\int_{-b}^{b} e^{-x^2}\, dx\right)\left(\int_{-b}^{b} e^{-y^2}\, dy\right)$$
$$=\left(\int_{-b}^{b} e^{-x^2}\, dx\right)^2=\left(2\int_0^b e^{-x^2}\, dx\right)^2=4\left(\int_0^b e^{-x^2}\, dx\right)^2;$$ taking limits as $b\to\infty$ gives the stated result.

84. $$\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx=\int_0^3\int_0^1 \frac{x^2}{(y-1)^{2/3}}\, dx\, dy=\int_0^3 \frac{1}{(y-1)^{2/3}}\left[\frac{x^3}{3}\right]_0^1 dy=\frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}}$$
$$=\frac{1}{3}\lim_{b\to 1^-}\int_0^b \frac{dy}{(y-1)^{2/3}}+\frac{1}{3}\lim_{b\to 1^+}\int_b^3 \frac{dy}{(y-1)^{2/3}}=\lim_{b\to 1^-}\left[(y-1)^{1/3}\right]_0^b+\lim_{b\to 1^+}\left[(y-1)^{1/3}\right]_b^3$$
$$=\left[\lim_{b\to 1^-}(b-1)^{1/3}-(-1)^{1/3}\right]-\left[\lim_{b\to 1^+}(b-1)^{1/3}-(2)^{1/3}\right]=(0+1)-\left(0-\sqrt[3]{2}\right)=1+\sqrt[3]{2}$$

85-88. Example CAS commands:

Maple:

```
f:= (x,y) -> 1/x/y;
q1:= Int( Int (f(x,y) y=1..x), x=1..3 );
evalf( q1);
value( q1);
evalf( value(q1) );
```

89-94. Example CAS commands:

<u>Maple</u>:

```
f:= (x,y) -> exp(x^2);
c,d:= 0,1;
g1:= y -> 2*y;
g2:= y -> 4;
q5: = Int( Int( f(x,y), x=g1(y)..g2(y) ),y=c..d );
value( q5 );
plot3d( 0, (x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0]
    scaling=constrained, title="#89(Section 15.2)" );
r5 := Int( Int( f(x,y), y=0..x / 2), x =0..2 ) + Int( Int( f(x, y=0..1), x = 2..4 );
```

```
value( r5 );
value( q5-t5 );
```

85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_]:= 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with ImplicitPlot and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x == 2y,x == 4, y == 0, y == 1},{x,0,4.1}, {y,0,1.1}];
f[x_, y_]:= Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1}, {x, 2y, 4}]
```

Another way to show a region is with the FilledPlot command. This assumes that functions are given as y=f(x).

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2,9},{x,0,3}, AxesLabels → {x,y}];
f[x_, y_]:= x Cos[y^2]
Integrate[f[x, y], [y, 0, 9], {x, 0, Sqrt[y]}]
```

85. $\int_1^3 \int_1^x \frac{1}{xy}\, dy\, dx \approx 0.603$

86. $\int_0^1 \int_0^1 e^{-\left(x^2+y^2\right)}\, dy\, dx \approx 0.558$

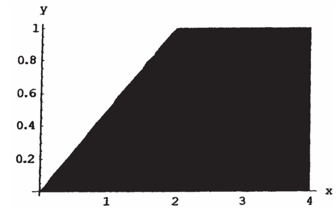
87. $\int_0^1 \int_0^1 \tan^{-1} xy\, dy\, dx \approx 0.233$

88. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx \approx 3.142$

89. Evaluate the integrals:

$$\int_0^1 \int_{2y}^4 e^{x^2} \, dx \, dy$$
$$= \int_0^2 \int_0^{x/2} e^{x^2} \, dy \, dx + \int_2^4 \int_0^1 e^{x^2} \, dy \, dx$$
$$= -\tfrac{1}{4} + \tfrac{1}{4}\left(e^4 - 2\sqrt{\pi}\ \mathrm{erfi}(2) + 2\sqrt{\pi}\ \mathrm{erfi}(4)\right)$$
$$\approx 1.1494 \times 10^6$$

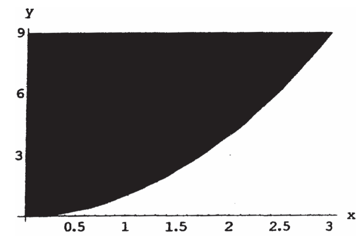
The following graph was generated using Mathematica.




90. Evaluate the integrals:

$$\int_0^3 \int_{x^2}^9 x \cos\left(y^2\right) dy \, dx = \int_0^9 \int_0^{\sqrt{y}} x \cos\left(y^2\right) dx \, dy$$
$$= \frac{\sin(81)}{4} \approx -0.157472$$

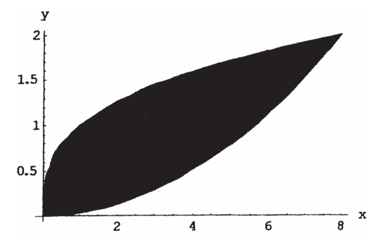
The following graph was generated using Mathematica.




91. Evaluate the integrals:

$$\int_0^2 \int_{y^3}^{4\sqrt{2y}} \left(x^2 y - xy^2\right) dx \, dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} \left(x^2 y - xy^2\right) dy \, dx$$
$$= \frac{67{,}520}{693} \approx 97.4315$$

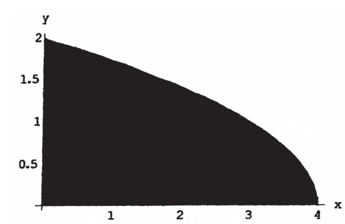
The following graph was generated using Mathematica.




92. Evaluate the integrals:

$$\int_0^2 \int_0^{4-y^2} e^{xy} \, dx \, dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy} \, dy \, dx$$
$$\approx 20.5648$$

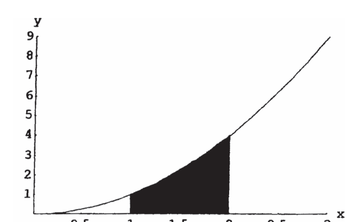
The following graph was generated using Mathematica.




93. Evaluate the integrals:

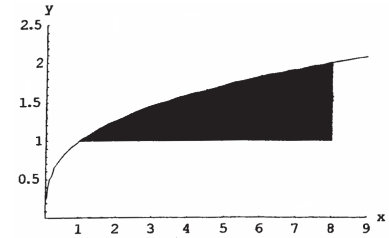
$$\int_1^2 \int_0^{x^2} \frac{1}{x+y} \, dy \, dx$$
$$= \int_0^1 \int_1^2 \frac{1}{x+y} \, dx \, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y} \, dx \, dy$$
$$= -1 + \ln\left(\frac{27}{4}\right) \approx 0.909543$$

The following graph was generated using Mathematica.



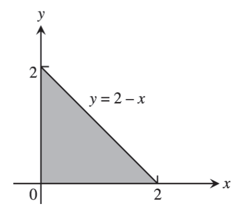
94. Evaluate the integrals:
$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$$
$\approx 0.866649$

The following graph was generated using Mathematica.

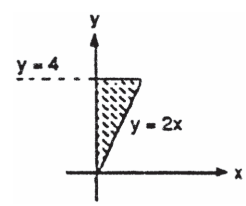


## 15.3 AREA BY DOUBLE INTEGRATION

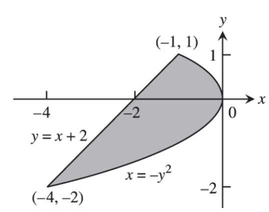
1. $\int_0^2 \int_0^{2-x} dy\,dx = \int_0^2 (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$
or $\int_0^2 \int_0^{2-y} dx\,dy = \int_0^2 (2-y)\,dy = 2$



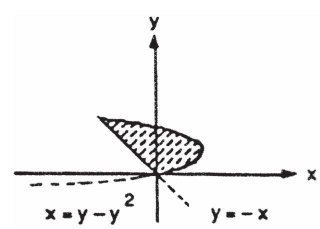

2. $\int_0^2 \int_{2x}^4 dy\,dx = \int_0^2 (4-2x)\,dx = \left[4x - x^2\right]_0^2 = 4,$
or $\int_0^4 \int_0^{y/2} dx\,dy = \int_0^4 \frac{y}{2}\,dy = 4$



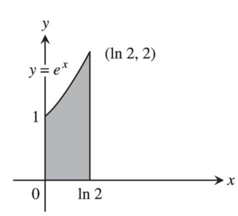

3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx\,dy = \int_{-2}^1 \left(-y^2 - y + 2\right) dy$
$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



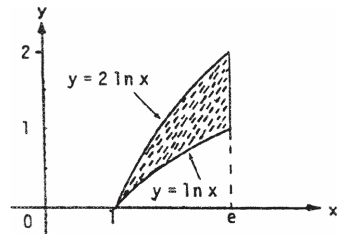

4. $\int_0^2 \int_{-y}^{y-y^2} dx\,dy = \int_0^2 \left(2y - y^2\right) dy = \left[y^2 - \frac{y^3}{3}\right]_0^2$
$= 4 - \frac{8}{3} = \frac{4}{3}$



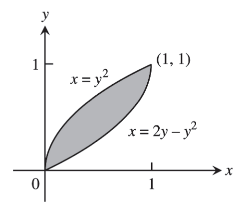

5. $\int_0^{\ln 2} \int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = \left[e^x\right]_0^{\ln 2} = 2 - 1 = 1$



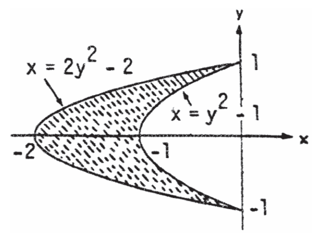
6. $\int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx = \left[x\ln x - x\right]_1^e$
$= (e-e)-(0-1) = 1$



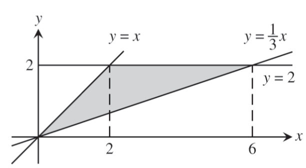

7. $\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 \left(2y-2y^2\right) dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1$
$= \frac{1}{3}$



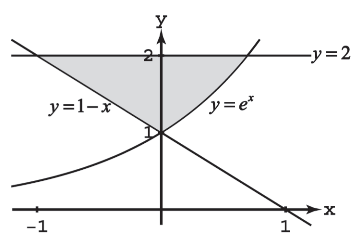

8. $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^1 \left(y^2 - 1 - 2y^2 + 2\right) dy$
$= \int_{-1}^1 \left(1-y^2\right) dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



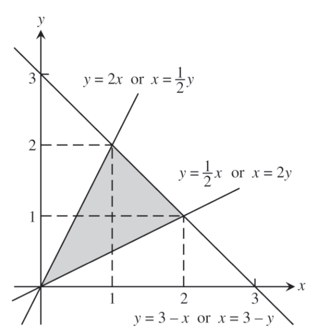

9. $\int_0^2 \int_y^{3y} 1\,dx\,dy = \int_0^2 [x]_y^{3y}\,dy$
$= \int_0^2 (2y)\,dy = \left[y^2\right]_0^2 = 4$



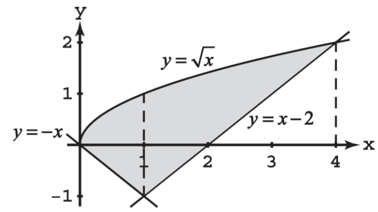

10. $\int_1^2 \int_{1-y}^{\ln y} 1\,dx\,dy = \int_0^2 [x]_{1-y}^{\ln y}\,dy$
$= \int_1^2 (\ln y - 1 + y)\,dy = \left[y\ln y - 2y + \frac{y^2}{2}\right]_1^2$
$= 2\ln 2 - \frac{1}{2}$



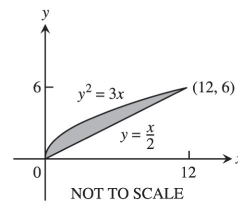

11. $\int_0^1 \int_{x/2}^{2x} 1\,dy\,dx + \int_1^2 \int_{x/2}^{3-x} 1\,dy\,dx$
$= \int_0^1 [y]_{x/2}^{2x}\,dx + \int_1^2 [y]_{x/2}^{3-x}\,dx$
$= \int_0^1 \left(\frac{3}{2}x\right) dx + \int_1^2 \left(3 - \frac{3}{2}x\right) dx$
$= \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}$



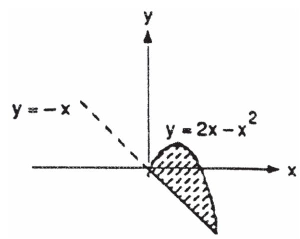
12. $\int_0^1\int_{-x}^{\sqrt{x}} 1\,dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} 1\,dy\,dx$
$= \int_0^1 [y]_{-x}^{\sqrt{x}}\,dx + \int_1^4 [y]_{x-2}^{\sqrt{x}}\,dx$
$= \int_0^1 \left(\sqrt{x}+x\right)dx + \int_1^4 \left(\sqrt{x}-x+2\right)dx$
$= \left[\frac{2}{3}x^{3/2}+\frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_1^4 = \frac{13}{3}$



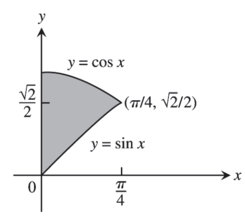

13. $\int_0^6\int_{y^2/3}^{2y} dx\,dy = \int_0^6\left(2y-\frac{y^2}{3}\right)dy = \left[y^2-\frac{y^3}{9}\right]_0^6$
$= 36 - \frac{216}{9} = 12$



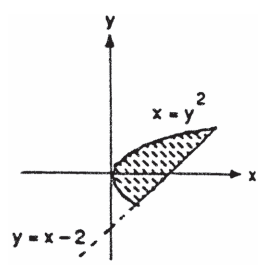

14. $\int_0^3\int_{-x}^{2x-x^2} dy\,dx = \int_0^3\left(3x-x^2\right)dx = \left[\frac{3}{2}x^2-\frac{1}{3}x^3\right]_0^3$
$= \frac{27}{2} - 9 = \frac{9}{2}$



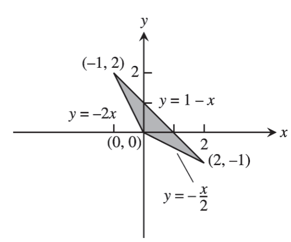

15. $\int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx$
$= \int_0^{\pi/4}(\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2}-1$




16. $\int_{-1}^2\int_{y^2}^{y+2} dx\,dy = \int_{-1}^2\left(y+2-y^2\right)dy$
$= \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^2 = \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$
$= 5 - \frac{1}{2} = \frac{9}{2}$



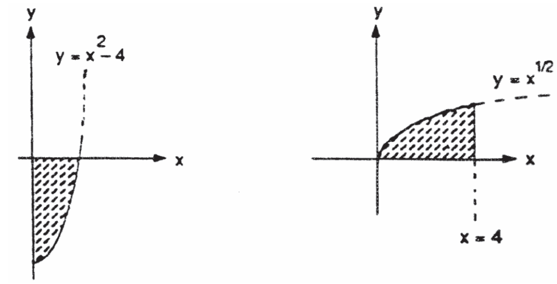

17. $\int_{-1}^0\int_{-2x}^{1-x} dy\,dx + \int_0^2\int_{-x/2}^{1-x} dy\,dx$
$= \int_{-1}^0 (1+x)\,dx + \int_0^2\left(1-\frac{x}{2}\right)dx$
$= \left[x+\frac{x^2}{2}\right]_{-1}^0 + \left[x-\frac{x^2}{4}\right]_0^2 = -\left(-1+\frac{1}{2}\right)+(2-1) = \frac{3}{2}$



18. $\int_0^2 \int_{x^2-4}^0 dy\,dx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx$

$= \int_0^2 \left(4-x^2\right) dx + \int_0^4 x^{1/2}\,dx$

$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{2}{3}x^{3/2}\right]_0^4 = \left(8 - \frac{8}{3}\right) + \frac{16}{3} = \frac{32}{3}$




19. (a) average $= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx = \frac{1}{\pi^2} \int_0^\pi \left[-\cos(x+y)\right]_0^\pi\,dx = \frac{1}{\pi^2} \int_0^\pi \left[-\cos(x+\pi) + \cos x\right] dx$

$= \frac{1}{\pi^2}\left[-\sin(x+\pi) + \sin x\right]_0^\pi = \frac{1}{\pi^2}\left[(-\sin 2\pi + \sin \pi) - (-\sin \pi + \sin 0)\right] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)} \int_0^\pi \int_0^{\pi/2} \sin(x+y)\,dy\,dx = \frac{2}{\pi^2} \int_0^\pi \left[-\cos(x+y)\right]_0^{\pi/2} dx = \frac{2}{\pi^2} \int_0^\pi \left[-\cos\left(x+\frac{\pi}{2}\right) + \cos x\right] dx$

$= \frac{2}{\pi^2}\left[-\sin\left(x+\frac{\pi}{2}\right) + \sin x\right]_0^\pi = \frac{2}{\pi^2}\left[\left(-\sin\frac{3\pi}{2} + \sin \pi\right) - \left(-\sin\frac{\pi}{2} + \sin 0\right)\right] = \frac{4}{\pi^2}$

20. average value over the square $= \int_0^1 \int_0^1 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4} = 0.25;$

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 \left(x - x^3\right) dx$

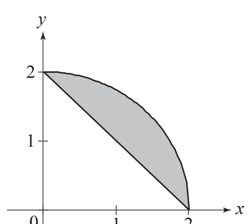
$= \frac{2}{\pi}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2\pi} \approx 0.159.$ The average value over the square is larger.

21. average height $= \frac{1}{4} \int_0^2 \int_0^2 \left(x^2 + y^2\right) dy\,dx = \frac{1}{4} \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{4x}{3}\right]_0^2 = \frac{8}{3}$

22. average $= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy}\,dy\,dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2} dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(\ln 2 + \ln\ln 2 - \ln\ln 2\right) dx$

$= \left(\frac{1}{\ln 2}\right) \int_{\ln 2}^{2\ln 2} \frac{dx}{x} = \left(\frac{1}{\ln 2}\right)\left[\ln x\right]_{\ln 2}^{2\ln 2} = \left(\frac{1}{\ln 2}\right)(\ln 2 + \ln\ln 2 - \ln\ln 2) = 1$

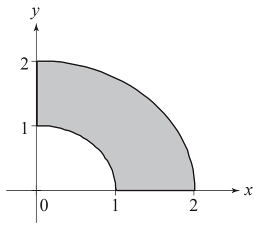
23. The region $R$ is shaded in the following figure.




$\iint_R dA = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} 1\,dy\,dx = \int_0^2 \left(\sqrt{4-x^2} - (2-x)\right) dx = \left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + \frac{x^2}{2} - 2x\right)\Bigg]_0^2 = \pi - 2,$ where

we use integration by parts with $u = \sqrt{4-x^2}$ and $dv = 1/2$ to find $\int \sqrt{4-x^2}\,dx$. Geometrically, the region $R$ is a quarter of a circle of radius 2 with a triangle of area 2 removed, giving area $\pi - 2$.

24. The area of the region $R$ is 4 times the shaded in the following figure.




The area integral will be easy to compute in polar coordinates, but in rectangular coordinates the calculation is awkward.

$$\iint_R dA = 4\left[\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} 1\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} 1\,dy\,dx\right] = 4\left[\left(\frac{\sqrt{3}}{2}+\frac{\pi}{12}\right)+\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\right] = 4\frac{3\pi}{4} = 3\pi$$

(As in Exercise 23, use integration by parts to evaluate the integrals $\int \sqrt{4-x^2}\,dx$ and $\int \sqrt{1-x^2}\,dx$.)

Geometrically the area is the difference between the area of a circle of radius 2 and the area of a circle of radius 1, or $4\pi - \pi = 3\pi$.

25. $$\int_{-5}^{5}\int_{-2}^{0} \frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000\left(1-e^{-2}\right)\int_{-5}^{5}\frac{dx}{1+\frac{|x|}{2}} = 10{,}000\left(1-e^{-2}\right)\left[\int_{-5}^{0}\frac{dx}{1-\frac{x}{2}}+\int_0^5\frac{dx}{1+\frac{x}{2}}\right]$$

$$= 10{,}000\left(1-e^{-2}\right)\left[-2\ln\left(1-\tfrac{x}{2}\right)\right]_{-5}^{0} + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\tfrac{x}{2}\right)\right]_0^5$$

$$= 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\tfrac{5}{2}\right)\right] + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\tfrac{5}{2}\right)\right] = 40{,}000\left(1-e^{-2}\right)\ln\left(\tfrac{7}{2}\right) \approx 43{,}329$$

26. $$\int_0^1\int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_0^1 \left[100(y+1)x\right]_{y^2}^{2y-y^2} dy = \int_0^1 100(y+1)\left(2y-2y^2\right)dy = 200\int_0^1\left(y-y^3\right)dy$$

$$= 200\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 = (200)\left(\tfrac{1}{4}\right) = 50$$

27. Let $(x_i, y_i)$ be the location of the weather station in county $i$ for $i = 1, \ldots, 254$. The average temperature in Texas at time $t_0$ is approximately $\dfrac{\sum_{i=1}^{254} T(x_i, y_i)\Delta A_i}{A}$, where $T(x_i, y_i)$ is the temperature at time $t_0$ at the weather station in county $i$, $\Delta A_i$ is the area of country $i$, and $A$ is the area of Texas.

28. Let $y = f(x)$ be a nonnegative, continuous function on $[a, b]$, then $A = \iint_R dA = \int_a^b \int_0^{f(x)} dy\,dx = \int_a^b [y]_0^{f(x)}\,dx$

$$= \int_a^b f(x)\,dx$$

29. Since $f$ is continuous on $R$, if $m \le f(x, y) \le M$, property 3(b) of double integrals gives us $\iint_R m\,dA \le \iint_R f(x, y)\,dA \le \iint_R M\,dA$ and hence $mA(R) \le \iint_R f(x, y)\,dA \le MA(R)$.

30. If $f(x, y)$ is positive at some point $P$ in $R$ or on the boundary of $R$ then by the continuity of $f$ there is a disk of positive radius around $P$ (or if $P$ is on the boundary, the intersection of such a disk with $R$) on which $f(x, y)$ is positive. This sub-region will make a positive contribution to the area $\iint_R f(x, y)\,dA$, and since $f(x, y)$ is never negative, $\iint_R f(x, y)\,dA$ will be greater than 0. This contradicts our assumption that $\iint_R f(x, y)\,dA = 0$, so $f(x, y)$ is positive nowhere on $R$ and is thus equal to 0 at every point of $R$.

## 15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le r \le 9$

2. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 4$

3. $y = x \Rightarrow \theta = \frac{\pi}{4},\ y = -x \Rightarrow \theta = \frac{3\pi}{4},\ y = 1 \Rightarrow r = \csc\theta \Rightarrow \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le r \le \csc\theta$

4. $x = 1 \Rightarrow r = \sec\theta,\ y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le \sec\theta$

5. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3}\sec\theta,\ y = 2 \Rightarrow r = 2\csc\theta;$
$2\sqrt{3}\sec\theta = 2\csc\theta \Rightarrow \theta = \frac{\pi}{6} \Rightarrow 0 \le \theta \le \frac{\pi}{6},\ 1 \le r \le 2\sqrt{3}\sec\theta;\ \ \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sqrt{3}\csc\theta$

6. $x^2 + y^2 = 2^2 \Rightarrow r = 2,\ x = 1 \Rightarrow r = \sec\theta;\ \ 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ \sec\theta \le r \le 2$

7. $x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$

8. $x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta \Rightarrow 0 \le \theta \le \pi,\ 0 \le r \le 2\sin\theta$

9. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx = \int_0^{\pi}\int_0^1 r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$

10. $\int_0^1\int_0^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dx\,dy = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}$

11. $\int_0^2\int_0^{\sqrt{4-y^2}} \left(x^2 + y^2\right) dx\,dy = \int_0^{\pi/2}\int_0^2 r^3\,dr\,d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_0^{2\pi}\int_0^a r\,dr\,d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2$

13. $\int_0^6\int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\,dr\,d\theta = 72\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

14. $\int_0^2 \int_0^x y\, dy\, dx = \int_0^{\pi/4} \int_0^{2\sec\theta} r^2 \sin\theta\, dr\, d\theta = \frac{8}{3}\int_0^{\pi/4} \tan\theta \sec^2\theta\, d\theta = \frac{4}{3}$

15. $\int_1^{\sqrt{3}} \int_1^x dy\, dx = \int_{\pi/6}^{\pi/4} \int_{\csc\theta}^{\sqrt{3}\sec\theta} r\, dr\, d\theta = \int_{\pi/6}^{\pi/4} \left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right) d\theta = \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$

16. $\int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dy\, dx = \int_{\pi/4}^{\pi/2} \int_2^{2\csc\theta} r\, dr\, d\theta = \int_{\pi/6}^{\pi/4} \left(2\csc^2\theta - 2\right) d\theta = \left[-2\cot\theta - \frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$

17. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx = \int_\pi^{3\pi/2} \int_0^1 \frac{2r}{1+r}\, dr\, d\theta = 2\int_\pi^{3\pi/2} \int_0^1 \left(1 - \frac{1}{1+r}\right) dr\, d\theta = 2\int_\pi^{3\pi/2} (1 - \ln 2)\, d\theta = (1 - \ln 2)\pi$

18. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\, dy\, dx = 4\int_0^{\pi/2} \int_0^1 \frac{2r}{\left(1+r^2\right)^2}\, dr\, d\theta = 4\int_0^{\pi/2} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_0^{\pi/2} d\theta = \pi$

19. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^{\pi/2} \int_0^{\ln 2} re^r\, dr\, d\theta = \int_0^{\pi/2} (2\ln 2 - 1)\, d\theta = \frac{\pi}{2}(2\ln 2 - 1)$

20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2 + y^2 + 1\right) dx\, dy = 4\int_0^{\pi/2} \int_0^1 \ln\left(r^2 + 1\right) r\, dr\, d\theta = 2\int_0^{\pi/2} (\ln 4 - 1)\, d\theta = \pi(\ln 4 - 1)$

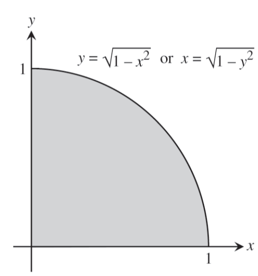
21. $\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\, dy\, dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_0^{\sqrt{2}} d\theta$

$= \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3}\cos\theta + \frac{4\sqrt{2}}{3}\sin\theta\right) d\theta = \left[\frac{2\sqrt{2}}{3}\sin\theta - \frac{4\sqrt{2}}{3}\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2\left(1+\sqrt{2}\right)}{3}$

22. $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}\, dy\, dx = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\, r\, dr\, d\theta = \int_0^{\pi/4} \left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta = \int_0^{\pi/4} \left(\frac{1}{2}\cos^2\theta - \frac{1}{8}\sec^2\theta\right) d\theta$

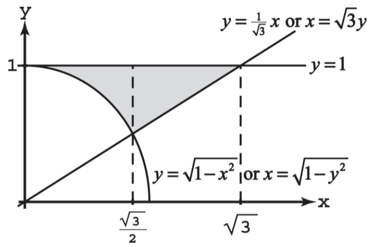
$= \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

23. $\int_0^1 \int_0^{\sqrt{1-x^2}} x\, y\, dy\, dx$ or $\int_0^1 \int_0^{\sqrt{1-y^2}} x\, y\, dy\, dx$



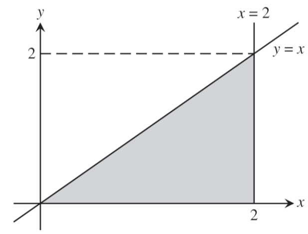
24. $\int_{1/2}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{3}y} x\,dx\,dy$ or

$\int_{0}^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^{1} x\,dy\,dx+\int_{\sqrt{3}/2}^{\sqrt{3}}\int_{x/\sqrt{3}}^{1} x\,dy\,dx$

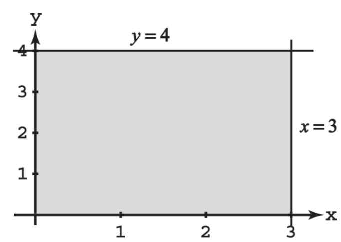



25. $\int_0^2\int_0^x y^2\left(x^2+y^2\right)dy\,dx$ or

$\int_0^2\int_y^2 y^2\left(x^2+y^2\right)dx\,dy$




26. $\int_0^3\int_0^4\left(x^2+y^2\right)^3 dy\,dx$ or $\int_0^4\int_0^3\left(x^2+y^2\right)^3 dx\,dy$




27. $\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta=2\int_0^{\pi/2}(2-\sin 2\theta)\,d\theta=2(\pi-1)$

28. $A=2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta=\int_0^{\pi/2}\left(2\cos\theta+\cos^2\theta\right)d\theta=\frac{8+\pi}{4}$

29. $A=2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta=144\int_0^{\pi/6}\cos^2 3\theta\,d\theta=12\pi$

30. $A=\int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta=\frac{8}{9}\int_0^{2\pi}\theta^2\,d\theta=\frac{64\pi^3}{27}$

31. $A=\int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta=\frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\right)d\theta=\frac{3\pi}{8}+1$

32. $A=4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta=2\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta=\frac{3\pi}{2}-4$

33. average $=\frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta=\frac{4}{3\pi a^2}\int_0^{\pi/2}a^3 d\theta=\frac{2a}{3}$

34. average $=\frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^a r^2\,dr\,d\theta=\frac{4}{3\pi a^2}\int_0^{\pi/2}a^3 d\theta=\frac{2a}{3}$

35. average $=\frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\,dx=\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta=\frac{a}{3\pi}\int_0^{2\pi}d\theta=\frac{2a}{3}$

36. average $=\frac{1}{\pi}\iint\limits_R\left[(1-x)^2+y^2\right]dy\,dx=\frac{1}{\pi}\int_0^{2\pi}\int_0^1\left[(1-r\cos\theta)^2+r^2\sin^2\theta\right]r\,dr\,d\theta$

$=\frac{1}{\pi}\int_0^{2\pi}\int_0^1\left(r^3-2r^2\cos\theta+r\right)dr\,d\theta=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{3}{4}-\frac{2\cos\theta}{3}\right)d\theta=\frac{1}{\pi}\left[\frac{3}{4}\theta-\frac{2\sin\theta}{3}\right]_0^{2\pi}=\frac{3}{2}$

37. $\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right)r\,dr\,d\theta=\int_0^{2\pi}\int_1^{\sqrt{e}}2\ln r\,dr\,d\theta=2\int_0^{2\pi}\left[r\ln r-r\right]_1^{e^{1/2}}d\theta=2\int_0^{2\pi}\sqrt{e}\left[\left(\frac{1}{2}-1\right)+1\right]d\theta=2\pi\left(2-\sqrt{e}\right)$

38. $\int_0^{2\pi}\int_1^{e}\left(\frac{\ln r^2}{r}\right)dr\,d\theta=\int_0^{2\pi}\int_1^{e}\left(\frac{2\ln r}{r}\right)dr\,d\theta=\int_0^{2\pi}\left[(\ln r)^2\right]_1^e d\theta=\int_0^{2\pi}d\theta=2\pi$

39. $V=2\int_0^{\pi/2}\int_1^{1+\cos\theta}r^2\cos\theta\,dr\,d\theta=\frac{2}{3}\int_0^{\pi/2}\left(3\cos^2\theta+3\cos^3\theta+\cos^4\theta\right)d\theta$

$=\frac{2}{3}\left[\frac{15\theta}{8}+\sin 2\theta+3\sin\theta-\sin^3\theta+\frac{\sin 4\theta}{32}\right]_0^{\pi/2}=\frac{4}{3}+\frac{5\pi}{8}$

40. $V=4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}}r\sqrt{2-r^2}\,dr\,d\theta=-\frac{4}{3}\int_0^{\pi/4}\left[(2-2\cos 2\theta)^{3/2}-2^{3/2}\right]d\theta$

$=\frac{2\pi\sqrt{2}}{3}-\frac{32}{3}\int_0^{\pi/4}\left(1-\cos^2\theta\right)\sin\theta\,d\theta=\frac{2\pi\sqrt{2}}{3}-\frac{32}{3}\left[\frac{\cos^3\theta}{3}-\cos\theta\right]_0^{\pi/4}=\frac{6\pi\sqrt{2}+40\sqrt{2}-64}{9}$

41. (a) $I^2=\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy=\int_0^{\pi/2}\int_0^\infty\left(e^{-r^2}\right)r\,dr\,d\theta=\int_0^{\pi/2}\left[\lim\limits_{b\to\infty}\int_0^b re^{-r^2}dr\right]d\theta$

$=-\frac{1}{2}\int_0^{\pi/2}\lim\limits_{b\to\infty}\left(e^{-b^2}-1\right)d\theta=\frac{1}{2}\int_0^{\pi/2}d\theta=\frac{\pi}{4}\Rightarrow I=\frac{\sqrt{\pi}}{2}$

(b) $\lim\limits_{x\to\infty}\int_0^x\frac{2e^{-t^2}}{\sqrt{\pi}}dt=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt=\left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right)=1,$ from part (a)

42. $\int_0^\infty\int_0^\infty\frac{1}{\left(1+x^2+y^2\right)^2}dx\,dy=\int_0^{\pi/2}\int_0^\infty\frac{r}{\left(1+r^2\right)^2}dr\,d\theta=\frac{\pi}{2}\lim\limits_{b\to\infty}\int_0^b\frac{r}{\left(1+r^2\right)^2}dr=\frac{\pi}{4}\lim\limits_{b\to\infty}\left[-\frac{1}{1+r^2}\right]_0^b=\frac{\pi}{4}\lim\limits_{b\to\infty}\left(1-\frac{1}{1+b^2}\right)$

$=\frac{\pi}{4}$

43. Over the disk $x^2+y^2\le\frac{3}{4}$: $\iint\limits_R\frac{1}{1-x^2-y^2}dA=\int_0^{2\pi}\int_0^{\sqrt{3}/2}\frac{r}{1-r^2}dr\,d\theta=\int_0^{2\pi}\left[-\frac{1}{2}\ln\left(1-r^2\right)\right]_0^{\sqrt{3}/2}d\theta$

$=\int_0^{2\pi}\left(-\frac{1}{2}\ln\frac{1}{4}\right)d\theta=(\ln 2)\int_0^{2\pi}d\theta=\pi\ln 4$

Over the disk $x^2+y^2\le 1$: $\iint\limits_R\frac{1}{1-x^2-y^2}dA=\int_0^{2\pi}\int_0^1\frac{r}{1-r^2}dr\,d\theta=\int_0^{2\pi}\left[\lim\limits_{a\to 1^-}\int_0^a\frac{r}{1-r^2}dr\right]d\theta$

$=\int_0^{2\pi}\lim\limits_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right]d\theta=2\pi\cdot\lim\limits_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right]=2\pi\cdot\infty,$ so the integral does not exist over $x^2+y^2\le 1$

44. The area in polar coordinates is given by $A=\int_\alpha^\beta\int_0^{f(\theta)} r\,dr\,d\theta=\int_\alpha^\beta\left[\frac{r^2}{2}\right]_0^{f(\theta)}d\theta=\frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta=\int_\alpha^\beta\frac{1}{2}r^2 d\theta,$ where $r=f(\theta)$

45. average $=\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a\left[(r\cos\theta-h)^2+r^2\sin^2\theta\right]r\;dr\;d\theta=\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a\left(r^3-2r^2h\cos\theta+rh^2\right)dr\;d\theta$

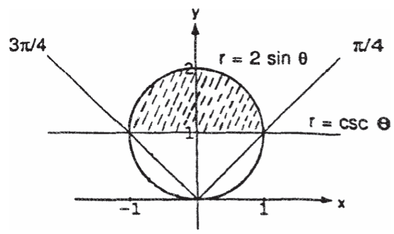
$=\frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{a^4}{4}-\frac{2a^3h\cos\theta}{3}+\frac{a^2h^2}{2}\right)d\theta=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{a^2}{4}-\frac{2ah\cos\theta}{3}+\frac{h^2}{2}\right)d\theta=\frac{1}{\pi}\left[\frac{a^2\theta}{4}-\frac{2ah\sin\theta}{3}+\frac{h^2\theta}{2}\right]_0^{2\pi}$

$=\frac{1}{2}\left(a^2+2h^2\right)$

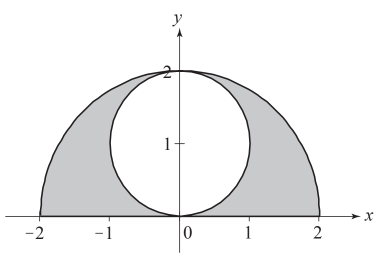
46. $A=\int_{\pi/4}^{3\pi/4}\int_{\csc\theta}^{2\sin\theta} r\;dr\;d\theta$

$=\frac{1}{2}\int_{\pi/4}^{3\pi/4}\left(4\sin^2\theta-\csc^2\theta\right)d\theta$

$=\frac{1}{2}\left[2\theta-\sin 2\theta+\cot\theta\right]_{\pi/4}^{3\pi/4}=\frac{\pi}{2}$




47. The region $R$ is shaded in the graph below.



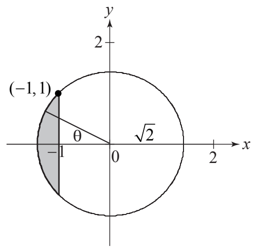

The polar equation of the outer circle is just $r=2$. The inner circle is $x^2+(y-1)^2=1$ or $x^2+y^2=2y$. This is equivalent to $r^2=2r\sin\theta$ or $r=2\sin\theta$. The integrand is $r$ in polar coordinates, so

$$\iint_R\sqrt{x^2+y^2}\,dA=\int_0^\pi\int_{2\sin\theta}^2 r\cdot r\;dr\;d\theta$$

$$=\int_0^\pi\left.\frac{r^3}{3}\right]_{2\sin\theta}^2 d\theta=\int_0^\pi\frac{8}{3}\left(1-\sin^3\theta\right)d\theta$$

Write the integrand as $\frac{8}{3}\left(1-\sin\theta\left(1-\cos^2\theta\right)\right)$. The indefinite integral is then $\frac{8}{3}\left(\theta+\cos\theta-\frac{1}{3}\cos^3\theta\right)$ and the definite integral is $\frac{8}{3}\left(\theta+\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big]_0^\pi=\frac{8}{9}(3\pi-4)$

48. The region $R$ is shaded in the graph below.




As $\theta$ ranges from $3\pi/4$ to $5\pi/4$ the ray at angle $\theta$ enters $R$ at $r = \sec\theta$ and leaves $R$ at $r = \sqrt{2}$. Thus

$$\iint_R \left(x^2+y^2\right)^{-2} dA = \int_{3\pi/4}^{5\pi/4}\int_{\sec\theta}^{\sqrt{2}} r^{-4}\cdot r\,dr\,d\theta$$

$$= \int_{3\pi/4}^{5\pi/4} -\frac{1}{2}r^{-2}\Big]_{\sec\theta}^{\sqrt{2}} d\theta = \int_{3\pi/4}^{5\pi/4}\frac{1}{4}\left(2\cos^2\theta-1\right)d\theta = \frac{1}{4}\int_{3\pi/4}^{5\pi/4}\cos 2\theta\,d\theta = \frac{1}{8}\sin 2\theta\Big]_{3\pi/4}^{5\pi/4} = \frac{1}{4}$$

49-52. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
    a,b := 0,1;
    f1 := x -> x;
    f2 := x -> 1;
    plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180],
          title="#49(a) (Section 15.4" );                                  # (a)
    q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );                    # (b)
    q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
    q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
    q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
    theta1 := solve( q3, theta );
    theta2 := solve( q1, theta );
    r1 := 0;
    r2 := solve( q4, r );


    plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
      title="#49(c) (Section 15.4)" );

    fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] ));       # (d)
    q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
    value( q5 );
```

Mathematica: (functions and bounds will vary)

For 49 and 50, begin by drawing the region of integration with the **FilledPlot** command.

```
Clear[x, y, r, t]
<<Graphics`FilledPlot`
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio → 1, AxesLabel → {x,y}];
```

The picture demonstrates that r goes from 0 to the line y=1 or r = 1/Sin[t], while t goes from π/4 to π/2.

```
f:= y / (x^2 + y^2)
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, π/4, π/2}, {r, 0, 1/Sin[t]}]
```

For 51 and 52, drawing the region of integration with the ImplicitPlot command.

```
Clear[x, y]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==y, x==2 − y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];
```

The picture shows that as t goes from 0 to π/4, r goes from 0 to the line $x=2-y$. **Solve** will find the bound for r.

```
bdr=Solve[r Cos[t]==2 − r Sin[t], r]//Simplify
f:=Sqrt[x + y]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, π/4}, {r, 0, bdr[[1, 1, 2]]}]
```

## 15.5 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

1. $\int_0^1\int_0^{1-x}\int_{x+z}^1 F(x, y, z)\,dy\,dz\,dx = \int_0^1\int_0^{1-x}\int_{x+z}^1 dy\,dz\,dx = \int_0^1\int_0^{1-x}(1-x-z)\,dz\,dx$

$= \int_0^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx = \int_0^1\frac{(1-x)^2}{2}\,dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$

2. $\int_0^1\int_0^2\int_0^3 dz\,dy\,dx = \int_0^1\int_0^2 3\,dy\,dx = \int_0^1 6\,dx = 6,\ \int_0^2\int_0^1\int_0^3 dz\,dx\,dy,\ \int_0^3\int_0^2\int_0^1 dx\,dy\,dz,\ \int_0^2\int_0^3\int_0^1 dx\,dz\,dy,$

$\int_0^3\int_0^1\int_0^2 dy\,dx\,dz,\ \int_0^1\int_0^3\int_0^2 dy\,dz\,dx$

3. $\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx = \int_0^1\int_0^{2-2x}\left(3-3x-\frac{3}{2}y\right)dy\,dx$

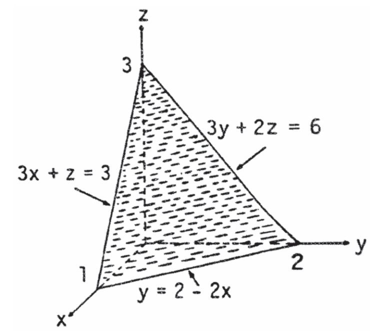
$= \int_0^1\left[3(1-x)\cdot 2(1-x)-\frac{3}{4}\cdot 4(1-x)^2\right]dx$

$= 3\int_0^1(1-x)^2\,dx = \left[-(1-x)^3\right]_0^1 = 1,$

$\int_0^2\int_0^{1-y/2}\int_0^{3-3x-3y/2} dz\,dx\,dy,\ \int_0^1\int_0^{3-3x}\int_0^{2-2x-2z/3} dy\,dz\,dx,$

$\int_0^3\int_0^{1-z/3}\int_0^{2-2x-2z/3} dy\,dx\,dz,\ \int_0^2\int_0^{3-3y/2}\int_0^{1-y/2-z/3} dx\,dz\,dy,$

$\int_0^3\int_0^{2-2z/3}\int_0^{1-y/2-z/3} dx\,dy\,dz$




4. $\int_0^2\int_0^3\int_0^{\sqrt{4-x^2}} dz\,dy\,dx = \int_0^2\int_0^3\sqrt{4-x^2}\,dy\,dx = \int_0^2 3\sqrt{4-x^2}\,dx = \frac{3}{2}\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_0^2 = 6\sin^{-1}1 = 3\pi,$

$\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy,\ \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 dy\,dz\,dx,\ \int_0^2\int_0^{\sqrt{4-z^2}}\int_0^3 dy\,dx\,dz,\ \int_0^2\int_0^3\int_0^{\sqrt{4-z^2}} dx\,dy\,dz,$

$\int_0^3\int_0^2\int_0^{\sqrt{4-z^2}} dx\,dz\,dy$

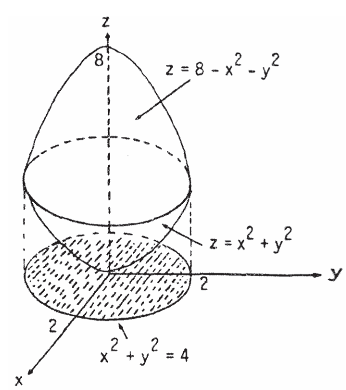
5. $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$

$= 4\int_0^2\int_0^{\sqrt{4-x^2}}\left[8-2\left(x^2+y^2\right)\right]dy\,dx$

$= 8\int_0^2\int_0^{\sqrt{4-x^2}}\left(4-x^2-y^2\right)dy\,dx = 8\int_0^{\pi/2}\int_0^2\left(4-r^2\right)r\,dr\,d\theta$

$= 8\int_0^{\pi/2}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta = 32\int_0^{\pi/2}d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi,$

$\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy,$

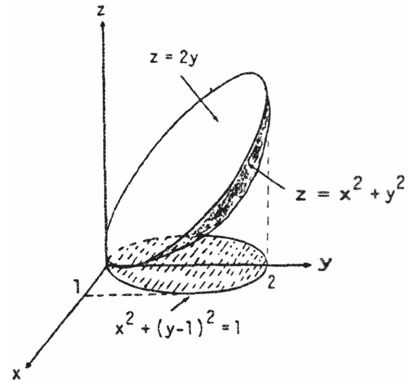



$\int_{-2}^2\int_{y^2}^4\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^2\int_4^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy,\ \int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz,$

$\int_{-2}^2\int_{x^2}^4\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^2\int_4^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx,\ \int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$

6. The projection of $D$ onto the $xy$-plane has the boundary $x^2+y^2=2y \Rightarrow x^2+(y-1)^2=1$, which is a circle. Therefore the two integrals are:

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy \text{ and } \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz\,dy\,dx$$




7. $\int_0^1\int_0^1\int_0^1\left(x^2+y^2+z^2\right) dz\ dy\ dx = \int_0^1\int_0^1\left(x^2+y^2+\frac{1}{3}\right) dy\ dx = \int_0^1\left(x^2+\frac{2}{3}\right) dx = 1$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\ dx\ dy = \int_0^{\sqrt{2}}\int_0^{3y}\left(8-2x^2-4y^2\right) dx\,dy = \int_0^{\sqrt{2}}\left[8x-\frac{2}{3}x^3-4xy^2\right]_0^{3y} dy$
$= \int_0^{\sqrt{2}}\left(24y-18y^3-12y^3\right) dy = \left[12y^2-\frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24-30=-6$

9. $\int_1^e\int_1^{e^2}\int_1^{e^3}\frac{1}{xyz}\,dx\,dy\,dz = \int_1^e\int_1^{e^2}\left[\frac{\ln x}{yz}\right]_1^{e^3} dy\,dz = \int_1^e\int_1^{e^2}\frac{3}{yz}\,dy\,dz = 3\int_1^e\left[\frac{\ln y}{z}\right]_1^{e^2} dz = \int_1^e\frac{6}{z}\,dz = 6$

10. $\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx = \int_0^1\int_0^{3-3x}(3-3x-y)\,dy\,dx = \int_0^1\left[(3-3x)^2-\frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1(1-x)^2\ dx$
$= -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$

11. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dx\,dy\,dz = \int_0^{\pi/6}\int_0^1 5y\sin z\ dy\,dz = \frac{5}{2}\int_0^{\pi/6}\sin z\ dz = \frac{5\left(2-\sqrt{3}\right)}{4}$

12. $\int_{-1}^1\int_0^1\int_0^2(x+y+z)\ dy\ dx\ dz = \int_{-1}^1\int_0^1=\left[xy+\frac{1}{2}y^2+zy\right]_0^2 dx\ dz = \int_{-1}^1\int_0^1(2x+2+2z)\ dx\ dz$
$= \int_{-1}^1\left[x^2+2x+2zx\right]_0^1\ dz = \int_{-1}^1(3+2z)\ dz = \left[3z+z^2\right]_{-1}^1 = 6$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\ dy\ dx = \int_0^3\int_0^{\sqrt{9-x^2}}\sqrt{9-x^2}\ dy\ dx = \int_0^3\left(9-x^2\right) dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 18$

14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2x+y)\,dx\,dy = \int_0^2\left[x^2+xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2\left(4-y^2\right)^{1/2}(2y)\ dy$
$= \left[-\frac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$

15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx = \int_0^1 \int_0^{2-x} (2-x-y)\, dy\, dx = \int_0^1 \left[ (2-x)^2 - \frac{1}{2}(2-x)^2 \right] dx = \frac{1}{2} \int_0^1 (2-x)^2\, dx$
$= \left[ -\frac{1}{6}(2-x)^3 \right]_0^1 = -\frac{1}{6} + \frac{8}{6} = \frac{7}{6}$

16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x^2} x\left(1-x^2-y\right) dy\, dx = \int_0^1 x\left[ \left(1-x^2\right)^2 - \frac{1}{2}\left(1-x^2\right) \right] dx = \int_0^1 \frac{1}{2} x\left(1-x^2\right)^2 dx$
$= \left[ -\frac{1}{12}(1-x^2)^3 \right]_0^1 = \frac{1}{12}$

17. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)\, du\, dv\, dw = \int_0^\pi \int_0^\pi [\sin(w+v+\pi) - \sin(w+v)]\, dv\, dw$
$= \int_0^\pi [(-\cos(w+2\pi) + \cos(w+\pi)) + (\cos(w+\pi) - \cos w)]\, dw$
$= [-\sin(w+2\pi) + \sin(w+\pi) - \sin w + \sin(w+\pi)]_0^\pi = 0$

18. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e s\, e^s \ln r \frac{(\ln t)^2}{t} dt\, dr\, ds = \int_0^1 \int_1^{\sqrt{e}} \left(s\, e^s \ln r\right) \left[ \frac{1}{3}(\ln t)^3 \right]_1^e dr\, ds = \int_0^1 \int_1^{\sqrt{e}} \frac{s\, e^s}{3} \ln r\, dr\, ds$
$= \int_0^1 \frac{s\, e^s}{3} [r \ln r - r]_1^{\sqrt{e}} ds = \frac{2-\sqrt{e}}{6} \int_0^1 s\, e^s\, ds = \frac{2-\sqrt{e}}{6} \left[ s\, e^s - e^s \right]_0^1 = \frac{2-\sqrt{e}}{6}$

19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x\, dx\, dt\, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b \to -\infty} \left(e^{2t} - e^b\right) dt\, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t}\, dt\, dv$
$= \int_0^{\pi/4} \left( \frac{1}{2} e^{2 \ln \sec v} - \frac{1}{2} \right) dv = \int_0^{\pi/4} \left( \frac{\sec^2 v}{2} - \frac{1}{2} \right) dv = \left[ \frac{\tan v}{2} - \frac{v}{2} \right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$

20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\, dp\, dq\, dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\, dq\, dr = \int_0^7 \frac{1}{3(r+1)} \left[ -\left(4-q^2\right)^{3/2} \right]_0^2 dr = \frac{8}{3} \int_0^7 \frac{1}{r+1}\, dr = \frac{8 \ln 8}{3} = 8 \ln 2$

21. (a) $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy\, dz\, dx$ (b) $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy\, dx\, dz$ (c) $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$
(d) $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$ (e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz\, dx\, dy$

22. (a) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$ (b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$ (c) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\, dy\, dz$
(d) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\, dz\, dy$ (e) $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\, dx\, dy$

23. $V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\, dy\, dx = \int_0^1 \int_{-1}^1 y^2\, dy\, dx = \frac{2}{3} \int_0^1 dx = \frac{2}{3}$

24. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx = \int_0^1 \int_0^{1-x} (2-2z)\, dz\, dx = \int_0^1 \left[ 2z - z^2 \right]_0^{1-x} dx = \int_0^1 \left(1-x^2\right) dx = \left[ x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$

25. $V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\, dy\, dx = \int_0^4 \left[ 2\sqrt{4-x} - \left(\frac{4-x}{2}\right) \right] dx = \left[ -\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2 \right]_0^4$
$= \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) = \frac{32}{3} - 4 = \frac{20}{3}$

26. $V = 2\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\, dy\, dx = -2\int_0^1 \int_{-\sqrt{1-x^2}}^0 y\, dy\, dx = \int_0^1 \left(1 - x^2\right) dx = \frac{2}{3}$

27. $V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\, dy\, dx = \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\, dx = \int_0^1 \left[ 6(1-x)^2 - \frac{3}{4}\cdot 4(1-x)^2 \right] dx$
$= \int_0^1 3(1-x)^2\, dx = \left[ -(1-x)^3 \right]_0^1 = 1$

28. $V = \int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dz\, dy\, dx = \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) dy\, dx = \int_0^1 \left(\cos \frac{\pi x}{2}\right)(1-x)\, dx$
$= \int_0^1 \cos\left(\frac{\pi x}{2}\right) dx - \int_0^1 x \cos\left(\frac{\pi x}{2}\right) dx = \left[ \frac{2}{\pi} \sin \frac{\pi x}{2} \right]_0^1 - \frac{4}{\pi^2} \int_0^{\pi/2} u \cos u\, du = \frac{2}{\pi} - \frac{4}{\pi^2} \left[ \cos u + u \sin u \right]_0^{\pi/2}$
$= \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2} - 1\right) = \frac{4}{\pi^2}$

29. $V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\, dy\, dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\, dx = 8\int_0^1 \left(1 - x^2\right) dx = \frac{16}{3}$

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dy\, dx = \int_0^2 \int_0^{4-x^2} \left(4 - x^2 - y\right) dy\, dx = \int_0^2 \left[ \left(4-x^2\right)^2 - \frac{1}{2}\left(4-x^2\right)^2 \right] dx$
$= \frac{1}{2}\int_0^2 \left(4-x^2\right)^2 dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \frac{128}{15}$

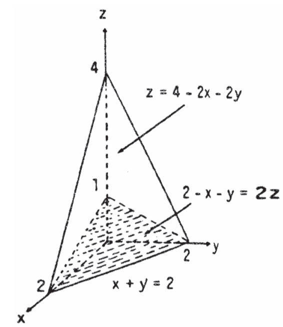
31. $V = \int_0^4 \int_0^{\sqrt{16-y^2}/2} \int_0^{4-y} dx\, dz\, dy = \int_0^4 \int_0^{\sqrt{16-y^2}/2} (4-y)\, dz\, dy = \int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\, dy$
$= \int_0^4 2\sqrt{16-y^2}\, dy - \frac{1}{2}\int_0^4 y\sqrt{16-y^2}\, dy = \left[ y\sqrt{16-y^2} + 16 \sin^{-1} \frac{y}{4} \right]_0^4 + \left[ \frac{1}{6}\left(16-y^2\right)^{3/2} \right]_0^4$
$= 16\left(\frac{\pi}{2}\right) - \frac{1}{6}(16)^{3/2} = 8\pi - \frac{32}{3}$

32. $V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz\, dy\, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x)\, dy\, dx = 2\int_{-2}^2 (3-x)\sqrt{4-x^2}\, dx$
$= 3\int_{-2}^2 2\sqrt{4-x^2}\, dx - 2\int_{-2}^2 x\sqrt{4-x^2}\, dx = 3\left[ x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \left[ \frac{2}{3}\left(4-x^2\right)^{3/2} \right]_{-2}^2$
$= 12 \sin^{-1} 1 - 12 \sin^{-1}(-1) = 12\left(\frac{\pi}{2}\right) - 12\left(-\frac{\pi}{2}\right) = 12\pi$

33. $\int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx = \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2}\right) dy\,dx$

$= \int_0^2 \left[3\left(1-\frac{x}{2}\right)(2-x) - \frac{3}{4}(2-x)^2\right] dx = \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4}\right] dx$

$= \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4}\right]_0^2 = (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2$




34. $V = \int_0^4 \int_z^8 \int_z^{8-z} dx\,dy\,dz = \int_0^4 \int_z^8 (8-2z)\,dy\,dz = \int_0^4 (8-2z)(8-z)\,dz = \int_0^4 \left(64 - 24z + 2z^2\right) dz$

$= \left[64z - 12z^2 + \frac{2}{3}z^3\right]_0^4 = \frac{320}{3}$

35. $V = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz\,dy\,dx = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} (x+2)\,dy\,dx = \int_{-2}^2 (x+2)\sqrt{4-x^2}\,dx$

$= \int_{-2}^2 2\sqrt{4-x^2}\,dx + \int_{-2}^2 x\sqrt{4-x^2}\,dx = \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[-\frac{1}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^2 = 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi$

36. $V = 2\int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz\,dx\,dy = 2\int_0^2 \int_0^{1-y^2} \left(x^2+y^2\right) dx\,dy = 2\int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y^2} dy$

$= 2\int_0^1 \left(1-y^2\right)\left[\frac{1}{3}\left(1-y^2\right)^2 + y^2\right] dy = 2\int_0^1 \left(1-y^2\right)\left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4\right) dy = \frac{2}{3}\int_0^1 \left(1-y^6\right) dy$

$= \frac{2}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = \frac{4}{7}$

37. average $= \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 \left(x^2+9\right) dz\,dy\,dx = \frac{1}{8}\int_0^2 \int_0^2 \left(2x^2+18\right) dy\,dx = \frac{1}{8}\int_0^2 \left(4x^2+36\right) dx = \frac{31}{3}$

38. average $= \frac{1}{2}\int_0^1 \int_0^1 \int_0^2 (x+y-z)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2x-1)\,dx = 0$

39. average $= \int_0^1 \int_0^1 \int_0^1 \left(x^2+y^2+z^2\right) dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

40. average $= \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 xyz\,dz\,dy\,dx = \frac{1}{4}\int_0^2 \int_0^2 xy\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = 1$

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dy\,dx\,dz = \int_0^4 \int_0^2 \frac{x\cos\left(x^2\right)}{\sqrt{z}}\,dx\,dz = \int_0^4 \left(\frac{\sin 4}{2}\right) z^{-1/2}\,dz$

$= \left[(\sin 4)z^{1/2}\right]_0^4 = 2\sin 4$

42. $\int_0^1\int_0^1\int_{x^2}^1 12xz\, e^{zy^2}\, dy\, dx\, dz = \int_0^1\int_0^1\int_0^{\sqrt{y}} 12xz\, e^{zy^2} dx\, dy\, dz = \int_0^1\int_0^1 6yz\, e^{zy^2} dy\, dz = \int_0^1\left[3e^{zy^2}\right]_0^1 dz$

$= 3\int_0^1\left(e^z - 1\right) dz = 3\left[e^z - z\right]_0^1 = 3e - 6$

43. $\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3}\frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\, dx\, dy\, dz = \int_0^1\int_{\sqrt[3]{z}}^1\frac{4\pi\sin\left(\pi y^2\right)}{y^2}\, dy\, dz = \int_0^1\int_0^{y^3}\frac{4\pi\sin\left(\pi y^2\right)}{y^2}\, dz\, dy$

$= \int_0^1 4\pi y\sin\left(\pi y^2\right) dy = \left[-2\cos\left(\pi y^2\right)\right]_0^1 = -2(-1) + 2(1) = 4$

44. $\int_0^2\int_0^{4-x^2}\int_0^x\frac{\sin 2z}{4-z}\, dy\, dz\, dx = \int_0^2\int_0^{4-x^2}\frac{x\sin 2z}{4-z}\, dz\, dx = \int_0^4\int_0^{\sqrt{4-z}}\left(\frac{\sin 2z}{4-z}\right)x\, dx\, dz = \int_0^4\left(\frac{\sin 2z}{4-z}\right)\frac{1}{2}(4-z)\, dz$

$= \left[-\frac{1}{4}\cos 2z\right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$

45. $\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\, dy\, dx = \frac{4}{15} \Rightarrow \int_0^1\int_0^{4-a-x^2}\left(4 - x^2 - y - a\right) dy\, dx = \frac{4}{15}$

$\Rightarrow \int_0^1\left[\left(4-a-x^2\right)^2 - \frac{1}{2}\left(4-a-x^2\right)^2\right] dx = \frac{4}{15} \Rightarrow \frac{1}{2}\int_0^1\left(4-a-x^2\right)^2 dx = \frac{4}{15} \Rightarrow \int_0^1\left[(4-a)^2 - 2x^2(4-a) + x^4\right] dx$

$= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3}x^3(4-a) + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$

$\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow \left[3(4-a)+1\right]\left[(4-a)-1\right] = 0 \Rightarrow 4 - a = -\frac{1}{3}$ or $4 - a = 1 \Rightarrow a = \frac{13}{3}$ or $a = 3$

46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3.$

47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points $(x, y, z)$ such that $4x^2 + 4y^2 + z^2 - 4 \le 0$ or $4x^2 + 4y^2 + z^2 \le 4,$ which is a solid ellipsoid centered at the origin.

48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points $(x, y, z)$ such that $1 - x^2 - y^2 - z^2 \ge 0$ or $x^2 + y^2 + z^2 \le 1,$ which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1 );
value( q1 );
```

Mathematica: (functions and bounds will vary)

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,–1,1}, {y,–Sqrt[1 – x^2], Sqrt[1 – x^2]}, {z, 0, 1}]
```

```
N[%]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, 2π}, {r, 0, 1},{z, 0, 1}]
N[%]
```

## 15.6 MOMENTS AND CENTERS OF MASS

1. $M = \int_0^1 \int_x^{2-x^2} 3\,dy\,dx = 3\int_0^1 \left(2 - x^2 - x\right) dx = \frac{7}{2};\ M_y = \int_0^1 \int_x^{2-x^2} 3x\,dy\,dx = 3\int_0^1 \left[xy\right]_x^{2-x^2} dx$
$= 3\int_0^1 \left(2x - x^3 - x^2\right) dx = \frac{5}{4};\ M_x = \int_0^1 \int_x^{2-x^2} 3y\,dy\,dx = \frac{3}{2}\int_0^1 \left[y^2\right]_x^{2-x^2} dx = \frac{3}{2}\int_0^1 \left(4 - 5x^2 + x^4\right) dx = \frac{19}{5}$
$\Rightarrow \bar{x} = \frac{5}{14}$ and $\bar{y} = \frac{38}{35}$

2. $M = \delta \int_0^3 \int_0^3 dy\,dx = \delta \int_0^3 3\,dx = 9\delta;\ I_x = \delta \int_0^3 \int_0^3 y^2\,dy\,dx = \delta \int_0^3 \left[\frac{y^3}{3}\right]_0^3 dx = 27\delta;$
$I_y = \delta \int_0^3 \int_0^3 x^2\,dy\,dx = \delta \int_0^3 \left[x^2 y\right]_0^3 dx = \delta \int_0^3 3x^2\,dx = 27\delta$

3. $M = \int_0^2 \int_{y^2/2}^{4-y} dx\,dy = \int_0^2 \left(4 - y - \frac{y^2}{2}\right) dy = \frac{14}{3};\ M_y = \int_0^2 \int_{y^2/2}^{4-y} x\,dx\,dy = \frac{1}{2}\int_0^2 \left[x^2\right]_{y^2/2}^{4-y} dy$
$= \frac{1}{2}\int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4}\right) dy = \frac{128}{3};\ M_x = \int_0^2 \int_{y^2/2}^{4-y} y\,dx\,dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2}\right) dy = \frac{10}{3} \Rightarrow \bar{x} = \frac{64}{35}$ and $\bar{y} = \frac{5}{7}$

4. $M = \int_0^3 \int_0^{3-x} dy\,dx = \int_0^3 (3 - x)\,dx = \frac{9}{2};\ M_y = \int_0^3 \int_0^{3-x} x\,dy\,dx = \int_0^3 \left[xy\right]_0^{3-x} dx = \int_0^3 \left(3x - x^2\right) dx = \frac{9}{2}$
$\Rightarrow \bar{x} = 1$ and $\bar{y} = 1$, by symmetry

5. $M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \frac{\pi a^2}{4};\ M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^a \left[xy\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a x\sqrt{a^2 - x^2}\,dx = \frac{a^3}{3}$
$\Rightarrow \bar{x} = \bar{y} = \frac{4a}{3\pi}$, by symmetry

6. $M = \int_0^\pi \int_0^{\sin x} dy\,dx = \int_0^\pi \sin x\,dx = 2;\ M_x = \int_0^\pi \int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^\pi \left[y^2\right]_0^{\sin x} dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx$
$= \frac{1}{4}\int_0^\pi (1 - \cos 2x)\,dx = \frac{\pi}{4} \Rightarrow \bar{x} = \frac{\pi}{2}$ and $\bar{y} = \frac{\pi}{8}$

7. $I_x = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx = \int_{-2}^2 \left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{2}{3}\int_{-2}^2 \left(4 - x^2\right)^{3/2} dx = 4\pi;\ I_y = 4\pi$, by symmetry;
$I_o = I_x + I_y = 8\pi$

8. $I_y = \int_\pi^{2\pi} \int_0^{\left(\sin^2 x\right)/x^2} x^2\,dy\,dx = \int_\pi^{2\pi} \left(\sin^2 x - 0\right) dx = \frac{1}{2}\int_\pi^{2\pi} (1 - \cos 2x)\,dx = \frac{\pi}{2}$

9. $M=\int_{-\infty}^{0}\int_{0}^{e^x} dy\,dx=\int_{-\infty}^{0} e^x\,dx=\lim\limits_{b\to-\infty}\int_{b}^{0} e^x\,dx=1-\lim\limits_{b\to-\infty} e^b=1;\ M_y=\int_{-\infty}^{0}\int_{0}^{e^x} x\,dy\,dx=\int_{-\infty}^{0} xe^x dx$
$=\lim\limits_{b\to-\infty}\int_{b}^{0} xe^x dx=\lim\limits_{b\to-\infty}\left[xe^x-e^x\right]_b^0=-1-\lim\limits_{b\to-\infty}\left(be^b-e^b\right)=-1;\ M_x=\int_{-\infty}^{0}\int_{0}^{e^x} y\,dy\,dx=\frac{1}{2}\int_{-\infty}^{0} e^{2x}\,dx$
$=\frac{1}{2}\lim\limits_{b\to-\infty}\int_{b}^{0} e^{2x} dx=\frac{1}{4}\Rightarrow \overline{x}=-1$ and $\overline{y}=\frac{1}{4}$

10. $M_y=\int_{0}^{\infty}\int_{0}^{e-x^2/2} x\,dy\,dx=\lim\limits_{b\to\infty}\int_{0}^{b} xe^{-x^2/2}\,dx=-\lim\limits_{b\to\infty}\left[\frac{1}{e^{x^2/2}}-1\right]_0^b=1$

11. $M=\int_{0}^{2}\int_{-y}^{y-y^2}(x+y)\,dx\,dy=\int_{0}^{2}\left[\frac{x^2}{2}+xy\right]_{-y}^{y-y^2} dy=\int_{0}^{2}\left(\frac{y^4}{2}-2y^3+2y^2\right)dy=\left[\frac{y^5}{10}-\frac{y^4}{2}+\frac{2y^3}{3}\right]_0^2=\frac{8}{15};$
$I_x=\int_{0}^{2}\int_{-y}^{y-y^2} y^2(x+y)\,dx\,dy=\int_{0}^{2}\left[\frac{x^2y^2}{2}+xy^3\right]_{-y}^{y-y^2} dy=\int_{0}^{2}\left(\frac{y^6}{2}-2y^5+2y^4\right)dy=\frac{64}{105};$

12. $M=\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy=5\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dy=\frac{5}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(12-4y^2-16y^4\right)dy=23\sqrt{3}$

13. $M=\int_{0}^{1}\int_{x}^{2-x}(6x+3y+3)\,dy\,dx=\int_{0}^{1}\left[6xy+\frac{3}{2}y^2+3y\right]_x^{2-x} dx=\int_{0}^{1}\left(12-12x^2\right)dx=8;$
$M_y=\int_{0}^{1}\int_{x}^{2-x} x(6x+3y+3)\,dy\,dx=\int_{0}^{1}\left(12x-12x^3\right)dx=3;\ M_x=\int_{0}^{1}\int_{x}^{2-x} y(6x+3y+3)\,dy\,dx$
$=\int_{0}^{1}\left(14-6x-6x^2-2x^3\right)dx=\frac{17}{2}\Rightarrow \overline{x}=\frac{3}{8}$ and $\overline{y}=\frac{17}{16}$

14. $M=\int_{0}^{1}\int_{y^2}^{2y-y^2}(y+1)\,dx\,dy=\int_{0}^{1}\left(2y-2y^3\right)dy=\frac{1}{2};\ M_x=\int_{0}^{1}\int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy=\int_{0}^{1}\left(2y^2-2y^4\right)dy=\frac{4}{15};$
$M_y=\int_{0}^{1}\int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy=\int_{0}^{1}\left(2y^2-2y^4\right)dy=\frac{4}{15}\Rightarrow \overline{x}=\frac{8}{15}$ and $\overline{y}=\frac{8}{15}; I_x=\int_{0}^{1}\int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy$
$=2\int_{0}^{1}\left(y^3-y^5\right)dy=\frac{1}{6}$

15. $M=\int_{0}^{1}\int_{0}^{6}(x+y+1)\,dx\,dy=\int_{0}^{1}(6y+24)\,dy=27;\ M_x=\int_{0}^{1}\int_{0}^{6} y(x+y+1)\,dx\,dy=\int_{0}^{1} y(6y+24)\,dy=14;$
$M_y=\int_{0}^{1}\int_{0}^{6} x(x+y+1)\,dx\,dy=\int_{0}^{1}(18y+90)\,dy=99\Rightarrow \overline{x}=\frac{11}{3}$ and $\overline{y}=\frac{14}{27};$
$I_y=\int_{0}^{1}\int_{0}^{6} x^2(x+y+1)\,dx\,dy=216\int_{0}^{1}\left(\frac{y}{3}+\frac{11}{6}\right)dy=432$

16. $M=\int_{-1}^{1}\int_{x^2}^{1}(y+1)\,dy\,dx=-\int_{-1}^{1}\left(\frac{x^4}{2}+x^2-\frac{3}{2}\right)dx=\frac{32}{15};\ M_x=\int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{5}{6}-\frac{x^6}{3}-\frac{x^4}{2}\right)dx=\frac{48}{35};$
$M_y=\int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{3x}{2}-\frac{x^5}{2}-x^3\right)dx=0\Rightarrow \overline{x}=0$ and $\overline{y}=\frac{9}{14};$
$I_y=\int_{-1}^{1}\int_{x^2}^{1} x^2(y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{3x^2}{2}-\frac{x^6}{2}-x^4\right)dx=\frac{16}{35}$

17. $M=\int_{-1}^{1}\int_{0}^{x^2}(7y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{7x^4}{2}+x^2\right)dx=\frac{31}{15};\ M_x=\int_{-1}^{1}\int_{0}^{x^2}y(7y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{7x^6}{3}+\frac{x^4}{2}\right)dx=\frac{13}{15};$

$M_y=\int_{-1}^{1}\int_{0}^{x^2}x(7y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{7x^5}{2}+x^3\right)dx=0\Rightarrow\bar{x}=0$ and $\bar{y}=\frac{13}{31};$

$I_y=\int_{-1}^{1}\int_{0}^{x^2}x^2(7y+1)\,dy\,dx=\int_{-1}^{1}\left(\frac{7x^6}{2}+x^4\right)dx=\frac{7}{5}$

18. $M=\int_{0}^{20}\int_{-1}^{1}\left(1+\frac{x}{20}\right)dy\,dx=\int_{0}^{20}\left(2+\frac{x}{10}\right)dx=60;\ M_x=\int_{0}^{20}\int_{-1}^{1}y\left(1+\frac{x}{20}\right)dy\,dx=\int_{0}^{20}\left[\left(1+\frac{x}{20}\right)\left(\frac{y^2}{2}\right)\right]_{-1}^{1}dx=0;$

$M_y=\int_{0}^{20}\int_{-1}^{1}x\left(1+\frac{x}{20}\right)dy\,dx=\int_{0}^{20}\left(2x+\frac{x^2}{10}\right)dx=\frac{2000}{3}\Rightarrow\bar{x}=\frac{100}{9}$ and $\bar{y}=0;$

$I_x=\int_{0}^{20}\int_{-1}^{1}y^2\left(1+\frac{x}{20}\right)dy\,dx=\frac{2}{3}\int_{0}^{20}\left(1+\frac{x}{20}\right)dx=20$

19. $M=\int_{0}^{1}\int_{-y}^{y}(y+1)\,dx\,dy=\int_{0}^{1}\left(2y^2+2y\right)dy=\frac{5}{3};\ M_x=\int_{0}^{1}\int_{-y}^{y}y(y+1)\,dx\,dy=2\int_{0}^{1}\left(y^3+y^2\right)dy=\frac{7}{6};$

$M_y=\int_{0}^{1}\int_{-y}^{y}x(y+1)\,dx\,dy=\int_{0}^{1}0\,dy=0\Rightarrow\bar{x}=0$ and $\bar{y}=\frac{7}{10};\ I_x=\int_{0}^{1}\int_{-y}^{y}y^2(y+1)\,dx\,dy$

$=\left(\int_{0}^{1}2y^4+2y^3\,dy\right)=\frac{9}{10};\ I_y=\int_{0}^{1}\int_{-y}^{y}x^2(y+1)\,dx\,dy=\frac{1}{3}\int_{0}^{1}\left(2y^4+2y^3\right)dy=\frac{3}{10}\Rightarrow I_o=I_x+I_y=\frac{6}{5}$

20. $M=\int_{0}^{1}\int_{-y}^{y}\left(3x^2+1\right)dx\,dy=\int_{0}^{1}\left(2y^3+2y\right)dy=\frac{3}{2};\ M_x=\int_{0}^{1}\int_{-y}^{y}y\left(3x^2+1\right)dx\,dy=\int_{0}^{1}\left(2y^4+2y^2\right)dy=\frac{16}{15};$

$M_y=\int_{0}^{1}\int_{-y}^{y}x\left(3x^2+1\right)dx\,dy=0\Rightarrow\bar{x}=0$ and $\bar{y}=\frac{32}{45};\ I_x=\int_{0}^{1}\int_{-y}^{y}y^2\left(3x^2+1\right)dx\,dy=\int_{0}^{1}\left(2y^5+2y^3\right)dy=\frac{5}{6};$

$I_y=\int_{0}^{1}\int_{-y}^{y}x^2\left(3x^2+1\right)dx\,dy=2\int_{0}^{1}\left(\frac{3}{5}y^5+\frac{1}{3}y^3\right)dy=\frac{11}{30}\Rightarrow I_o=I_x+I_y=\frac{6}{5}$

21. $I_x=\int_{0}^{a}\int_{0}^{b}\int_{0}^{c}\left(y^2+z^2\right)dz\,dy\,dx=\int_{0}^{a}\int_{0}^{b}\left(cy^2+\frac{c^3}{3}\right)dy\,dx=\int_{0}^{a}\left(\frac{cb^3}{3}+\frac{c^3b}{3}\right)dx=\frac{abc\left(b^2+c^2\right)}{3}=\frac{M}{3}\left(b^2+c^2\right)$ where $M=abc;\ I_y=\frac{M}{3}\left(a^2+c^2\right)$ and $I_z=\frac{M}{3}\left(a^2+b^2\right),$ by symmetry

22. The plane $z=\frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x=\int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}\left(y^2+z^2\right)dz\,dy\,dx$

$=\int_{-3}^{3}\int_{-2}^{4}\left[\frac{8y^2}{3}-\frac{2y^3}{3}+\frac{8(2-y)^3}{81}+\frac{64}{81}\right]dy\,dx=\int_{-3}^{3}\frac{104}{3}\,dx=208;\ I_y=\int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}\left(x^2+z^2\right)dz\,dy\,dx$

$=\int_{-3}^{3}\int_{-2}^{4}\left[\frac{(4-2y)^3}{81}+\frac{x^2(4-2y)}{3}+\frac{4x^2}{3}+\frac{64}{81}\right]dy\,dx=\int_{-3}^{3}\left(12x^2+\frac{32}{3}\right)dx=280;$

$I_z=\int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}\left(x^2+y^2\right)dz\,dy\,dx=\int_{-3}^{3}\int_{-2}^{4}\left(x^2+y^2\right)\left(\frac{8}{3}-\frac{2y}{3}\right)dy\,dx=12\int_{-3}^{3}\left(x^2+2\right)dx=360$

23. $M=4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}dz\,dy\,dx=4\int_{0}^{1}\int_{0}^{1}\left(4-4y^2\right)dy\,dx=16\int_{0}^{1}\frac{2}{3}\,dx=\frac{32}{3};\ M_{xy}=4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}z\,dz\,dy\,dx$

$=2\int_{0}^{1}\int_{0}^{1}\left(16-16y^4\right)dy\,dx=\frac{128}{5}\int_{0}^{1}dx=\frac{128}{5}\Rightarrow\bar{z}=\frac{12}{5},$ and $\bar{x}=\bar{y}=0,$ by symmetry;

$I_x = 4\int_0^1\int_0^1\int_{4y^2}^4 \left(y^2+z^2\right) dz\,dy\,dx = 4\int_0^1\int_0^1\left[\left(4y^2+\frac{64}{3}\right)-\left(4y^4+\frac{64y^6}{3}\right)\right] dy\,dx = 4\int_0^1 \frac{1976}{105}\,dx = \frac{7904}{105}$;

$I_y = 4\int_0^1\int_0^1\int_{4y^2}^4 \left(x^2+z^2\right) dz\,dy\,dx = 4\int_0^1\int_0^1\left[\left(4x^2+\frac{64}{3}\right)-\left(4x^2y^4+\frac{64y^6}{3}\right)\right] dy\,dx = 4\int_0^1\left(\frac{8}{3}x^2+\frac{128}{7}\right) dx = \frac{4832}{63}$;

$I_z = 4\int_0^1\int_0^1\int_{4y^2}^4 \left(x^2+y^2\right) dz\,dy\,dx = 16\int_0^1\int_0^1\left(x^2-x^2y^2+y^2-y^4\right) dy\,dx = 16\int_0^1\left(\frac{2x^2}{3}+\frac{2}{15}\right) dx = \frac{256}{45}$

24. (a) $M = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{2-x} dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}(2-x)\,dy\,dx = \int_{-2}^2(2-x)\sqrt{4-x^2}\,dx = 4\pi$;

$M_{yz} = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{2-x} x\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}x(2-x)\,dy\,dx = \int_{-2}^2 x(2-x)\sqrt{4-x^2}\,dx = -2\pi$;

$M_{xz} = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{2-x} y\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}y(2-x)\,dy\,dx = \frac{1}{2}\int_{-2}^2(2-x)\left[\frac{4-x^2}{4}-\frac{4-x^2}{4}\right] dx = 0$

$\Rightarrow \overline{x} = -\frac{1}{2}$ and $\overline{y} = 0$

(b) $M_{xy} = \int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{2-x} z\,dz\,dy\,dx = \frac{1}{2}\int_{-2}^2\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}(2-x)^2\,dy\,dx = \frac{1}{2}\int_{-2}^2(2-x)^2\sqrt{4-x^2}\,dx$

$= 5\pi \Rightarrow \overline{z} = \frac{5}{4}$

25. (a) $M = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^4 dz\,dy\,dx = 4\int_0^{\pi/2}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^2\left(4r-r^3\right) dr\,d\theta = 4\int_0^{\pi/2} 4\,d\theta = 8\pi$;

$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r}{2}\left(16-r^4\right) dr\,d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{8}{3}$, and $\overline{x} = \overline{y} = 0$,
by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_0^{2\pi}\int_0^{\sqrt{c}}\int_{r^2}^c r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{c}}\left(cr-r^3\right) dr\,d\theta = \int_0^{2\pi}\frac{c^2}{4}\,d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2}$,
since $c > 0$

26. $M = 8$; $M_{xy} = \int_{-1}^1\int_3^5\int_{-1}^1 z\,dz\,dy\,dx = \int_{-1}^1\int_3^5\left[\frac{z^2}{2}\right]_{-1}^1 dy\,dx = 0$; $M_{yz} = \int_{-1}^1\int_3^5\int_{-1}^1 x\,dz\,dy\,dx = 2\int_{-1}^1\int_3^5 x\,dy\,dx$

$= 4\int_{-1}^1 x\,dx = 0$; $M_{xz} = \int_{-1}^1\int_3^5\int_{-1}^1 y\,dz\,dy\,dx = 2\int_{-1}^1\int_3^5 y\,dy\,dx = 16\int_{-1}^1 dx = 32 \Rightarrow \overline{x} = 0, \overline{y} = 4, \overline{z} = 0$;

$I_x = \int_{-1}^1\int_3^5\int_{-1}^1\left(y^2+z^2\right) dz\,dy\,dx = \int_{-1}^1\int_3^5\left(2y^2+\frac{2}{3}\right) dy\,dx = \frac{2}{3}\int_{-1}^1 100\,dx = \frac{400}{3}$;

$I_y = \int_{-1}^1\int_3^5\int_{-1}^1\left(x^2+z^2\right) dz\,dy\,dx = \int_{-1}^1\int_3^5\left(2x^2+\frac{2}{3}\right) dy\,dx = \frac{4}{3}\int_{-1}^1\left(3x^2+1\right) dx = \frac{16}{3}$;

$I_z = \int_{-1}^1\int_3^5\int_{-1}^1\left(x^2+y^2\right) dz\,dy\,dx = 2\int_{-1}^1\int_3^5\left(x^2+y^2\right) dy\,dx = 2\int_{-1}^1\left(2x^2+\frac{98}{3}\right) dx = \frac{400}{3}$

27. The plane $y+2z=2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2\int_{-2}^4\int_{-1}^{(2-y)/2}\left[(y-6)^2+z^2\right] dz\,dy\,dx$

$= \int_{-2}^2\int_{-2}^4\left[\frac{(y-6)^2(4-y)}{2}+\frac{(2-y)^3}{24}+\frac{1}{3}\right] dy\,dx$; let $t = 2-y \Rightarrow I_L = 4\int_{-2}^4\left(\frac{13t^3}{24}+5t^2+16t+\frac{49}{3}\right) dt = 1386$;

$M = \frac{1}{2}(3)(6)(4) = 36$

28. The plane $y+2z=2$ is the top of the wedge $\Rightarrow I_L=\int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2}\left[(x-4)^2+y^2\right]dz\,dy\,dx$
$=\frac{1}{2}\int_{-2}^{2}\int_{-2}^{4}\left(x^2-8x+16+y^2\right)(4-y)\,dy\,dx=\int_{-2}^{2}\left(9x^2-72x+162\right)dx=696;\ M=\frac{1}{2}(3)(6)(4)=36$

29. (a) $M=\int_0^2\int_0^{2-x}\int_0^{2-x-y}2x\,dz\,dy\,dx=\int_0^2\int_0^{2-x}\left(4x-2x^2-2xy\right)dy\,dx=\int_0^2\left(x^3-4x^2+4x\right)dx=\frac{4}{3}$

(b) $M_{xy}=\int_0^2\int_0^{2-x}\int_0^{2-x-y}2xz\,dz\,dy\,dx=\int_0^2\int_0^{2-x}x(2-x-y)^2\,dy\,dx=\int_0^2\frac{x(2-x)^3}{3}\,dx=\frac{8}{15};\ M_{xz}=\frac{8}{15}$ by symmetry; $M_{yz}=\int_0^2\int_0^{2-x}\int_0^{2-x-y}2x^2\,dz\,dy\,dx=\int_0^2\int_0^{2-x}2x^2(2-x-y)\,dy\,dx=\int_0^2\left(2x-x^2\right)^2dx=\frac{16}{15}$
$\Rightarrow \overline{x}=\frac{4}{5}$, and $\overline{y}=\overline{z}=\frac{2}{5}$

30. (a) $M=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}kxy\,dz\,dy\,dx=k\int_0^2\int_0^{\sqrt{x}}xy\left(4-x^2\right)dy\,dx=\frac{k}{2}\int_0^2\left(4x^2-x^4\right)dx=\frac{32k}{15}$

(b) $M_{yz}=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}kx^2y\,dz\,dy\,dx=k\int_0^2\int_0^{\sqrt{x}}x^2y\left(4-x^2\right)dy\,dx=\frac{k}{2}\int_0^2\left(4x^3-x^5\right)dx=\frac{8k}{3}\Rightarrow\overline{x}=\frac{5}{4};$
$M_{xz}=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}kxy^2\,dz\,dy\,dx=k\int_0^2\int_0^{\sqrt{x}}xy^2\left(4-x^2\right)dy\,dx=\frac{k}{3}\int_0^2\left(4x^{5/2}-x^{9/2}\right)dx=\frac{256\sqrt{2}k}{231}$
$\Rightarrow\overline{y}=\frac{40\sqrt{2}}{77};\ M_{xy}=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}kxyz\,dz\,dy\,dx=\int_0^2\int_0^{\sqrt{x}}xy\left(4-x^2\right)^2dy\,dx=\frac{k}{4}\int_0^2\left(16x^2-8x^4+x^6\right)dx$
$=\frac{256k}{105}\Rightarrow\overline{z}=\frac{8}{7}$

31. (a) $M=\int_0^1\int_0^1\int_0^1(x+y+z+1)\,dz\,dy\,dx=\int_0^1\int_0^1\left(x+y+\frac{3}{2}\right)dy\,dx=\int_0^1(x+2)\,dx=\frac{5}{2}$

(b) $M_{xy}=\int_0^1\int_0^1\int_0^1 z(x+y+z+1)\,dz\,dy\,dx=\frac{1}{2}\int_0^1\int_0^1\left(x+y+\frac{5}{3}\right)dy\,dx=\frac{1}{2}\int_0^1\left(x+\frac{13}{6}\right)dx=\frac{4}{3}$
$\Rightarrow M_{xy}=M_{yz}=M_{xz}=\frac{4}{3}$, by symmetry $\Rightarrow\overline{x}=\overline{y}=\overline{z}=\frac{8}{15}$

(c) $I_z=\int_0^1\int_0^1\int_0^1\left(x^2+y^2\right)(x+y+z+1)\,dz\,dy\,dx=\int_0^1\int_0^1\left(x^2+y^2\right)\left(x+y+\frac{3}{2}\right)dy\,dx$
$=\int_0^1\left(x^3+2x^2+\frac{1}{3}x+\frac{3}{4}\right)dx=\frac{11}{6}\Rightarrow I_x=I_y=I_z=\frac{11}{6}$, by symmetry

32. The plane $y+2z=2$ is the top of the wedge.

(a) $M=\int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}(x+1)\,dz\,dy\,dx=\int_{-1}^{1}\int_{-2}^{4}(x+1)\left(2-\frac{y}{2}\right)dy\,dx=18$

(b) $M_{yz}=\int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}x(x+1)\,dz\,dy\,dx=\int_{-1}^{1}\int_{-2}^{4}x(x+1)\left(2-\frac{y}{2}\right)dy\,dx=6;$
$M_{xz}=\int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}y(x+1)\,dz\,dy\,dx=\int_{-1}^{1}\int_{-2}^{4}y(x+1)\left(2-\frac{y}{2}\right)dy\,dx=0;$
$M_{xy}=\int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}z(x+1)\,dz\,dy\,dx=\frac{1}{2}\int_{-1}^{1}\int_{-2}^{4}(x+1)\left(\frac{y^2}{2}-y\right)dy\,dx=0\Rightarrow\overline{x}=\frac{1}{3}$, and $\overline{y}=\overline{z}=0$

(c) $I_x = \int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}(x+1)\left(y^2+z^2\right) dz\, dy\, dx = \int_{-1}^{1}\int_{-2}^{4}(x+1)\left[2y^2+\frac{1}{3}-\frac{y^3}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\, dx = 45;$

$I_y = \int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}(x+1)\left(x^2+z^2\right) dz\, dy\, dx = \int_{-1}^{1}\int_{-2}^{4}(x+1)\left[2x^2+\frac{1}{3}-\frac{x^2 y}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\, dx = 15;$

$I_z = \int_{-1}^{1}\int_{-2}^{4}\int_{-1}^{(2-y)/2}(x+1)\left(x^2+y^2\right) dz\, dy\, dx = \int_{-1}^{1}\int_{-2}^{4}(x+1)\left(2-\frac{y}{2}\right)\left(x^2+y^2\right) dy\, dx = 42$

33. $M = \int_{-1}^{1}\int_{z-1}^{1-z}\int_{0}^{\sqrt{z}}(2y+5)\, dy\, dx\, dz = \int_{0}^{1}\int_{z-1}^{1-z}\left(z+5\sqrt{z}\right) dx\, dz = \int_{0}^{1}2\left(z+5\sqrt{z}\right)(1-z)\, dz$

$= 2\int_{0}^{1}\left(5z^{1/2}+z-5z^{3/2}-z^2\right) dz = 2\left[\frac{10}{3}z^{3/2}+\frac{1}{2}z^2-2z^{5/2}-\frac{1}{3}z^3\right]_0^1 = 2\left(\frac{9}{3}-\frac{3}{2}\right) = 3$

34. $M = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2\left(x^2+y^2\right)}^{16-2\left(x^2+y^2\right)}\sqrt{x^2+y^2}\; dz\, dy\, dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\sqrt{x^2+y^2}\left[16-4\left(x^2+y^2\right)\right] dy\, dx$

$= 4\int_{0}^{2\pi}\int_{0}^{2} r\left(4-r^2\right) r\, dr\, d\theta = 4\int_{0}^{2\pi}\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 d\theta = 4\int_{0}^{2\pi}\frac{64}{15}\, d\theta = \frac{512\pi}{15}$

35. (a) $\overline{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint\limits_R x\delta(x, y, z)\, dx\, dy\, dz = 0 \Rightarrow M_{yz} = 0$

(b) $I_L = \iiint\limits_D |\mathbf{v}-h\mathbf{i}|^2\, dm = \iiint\limits_D |(x-h)\mathbf{i}+y\mathbf{j}|^2\, dm = \iiint\limits_D \left(x^2-2xh+h^2+y^2\right) dm$

$= \iiint\limits_D \left(x^2+y^2\right) dm - 2h\iiint\limits_D x\, dm + h^2\iiint\limits_D dm = I_x - 0 + h^2 m = I_{\text{c.m.}} + h^2 m$

36. $I_L = I_{\text{c.m.}} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

37. (a) $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \Rightarrow I_z = I_{\text{c.m.}} + abc\left(\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}\right)^2 \Rightarrow I_{\text{c.m.}} = I_z - \frac{abc\left(a^2+b^2\right)}{4}$

$= \frac{abc\left(a^2+b^2\right)}{3} - \frac{abc\left(a^2+b^2\right)}{4} = \frac{abc\left(a^2+b^2\right)}{12};\quad R_{\text{c.m.}} = \sqrt{\frac{I_{\text{c.m.}}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$

(b) $I_L = I_{\text{c.m.}} + abc\left(\sqrt{\frac{a^2}{4}+\left(\frac{b}{2}-2b\right)^2}\right)^2 = \frac{abc\left(a^2+b^2\right)}{12} + \frac{abc\left(a^2+9b^2\right)}{4} = \frac{abc\left(4a^2+28b^2\right)}{12} = \frac{abc\left(a^2+7b^2\right)}{3};$

$R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$

38. $M = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} dz\, dy\, dx = \int_{-3}^{3}\int_{-2}^{4}\frac{2}{3}(4-y)\, dy\, dx = \int_{-3}^{3}\frac{2}{3}\left[4y-\frac{y^2}{2}\right]_{-2}^{4} dx = 12\int_{-3}^{2} dx = 72;\quad \overline{x} = \overline{y} = \overline{z} = 0$

from Exercise $22 \Rightarrow I_x = I_{\text{c.m.}} + 72\left(\sqrt{0^2+0^2}\right)^2 = I_{\text{c.m.}} \Rightarrow I_L = I_{\text{c.m.}} + 72\left(\sqrt{16+\frac{16}{9}}\right)^2 = 208 + 72\left(\frac{160}{9}\right) = 1488$

## 15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left[r\left(2-r^2\right)^{1/2}-r^2\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{r^3}{3}\right]_0^1 d\theta$
$= \int_0^{2\pi}\left(\frac{2^{3/2}}{3}-\frac{2}{3}\right) d\theta = \frac{4\pi\left(\sqrt{2}-1\right)}{3}$

2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3\left[r\left(18-r^2\right)^{1/2}-\frac{r^3}{3}\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(18-r^2\right)^{3/2}-\frac{r^4}{12}\right]_0^3 d\theta = \frac{9\pi\left(8\sqrt{2}-7\right)}{2}$

3. $\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^3} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3r+24r^3\right) dr\, d\theta = \int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/(2\pi)} d\theta$
$= \frac{3}{2}\int_0^{2\pi}\left(\frac{\theta^2}{4\pi^2}+\frac{4\theta^4}{16\pi^4}\right) d\theta = \frac{3}{2}\left[\frac{\theta^3}{12\pi^2}+\frac{\theta^5}{20\pi^4}\right]_0^{2\pi} = \frac{17\pi}{5}$

4. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\frac{1}{2}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right] r\, dr\, d\theta = 4\int_0^{\pi}\int_0^{\theta/\pi}\left(4r-r^3\right) dr\, d\theta$
$= 4\int_0^{\pi}\left[2r^2-\frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi}\left(\frac{2\theta^2}{\pi^2}-\frac{\theta^4}{4\pi^4}\right) d\theta = \frac{37\pi}{15}$

5. $\int_0^{2\pi}\int_0^1\int_r^{\left(2-r^2\right)^{-1/2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^1\left[r\left(2-r^2\right)^{-1/2}-r^2\right] dr\, d\theta = 3\int_0^{2\pi}\left[-\left(2-r^2\right)^{1/2}-\frac{r^3}{3}\right]_0^1 d\theta$
$= 3\int_0^{2\pi}\left(\sqrt{2}-\frac{4}{3}\right) d\theta = \pi\left(6\sqrt{2}-8\right)$

6. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta+\frac{r}{12}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4}+\frac{1}{24}\right) d\theta = \frac{\pi}{3}$

7. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\theta = \int_0^{2\pi}\int_0^3\frac{z^4}{324} dz\, d\theta = \int_0^{2\pi}\frac{3}{20} d\theta = \frac{3\pi}{10}$

8. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\, dr\, d\theta\, dz = \int_{-1}^1\int_0^{2\pi} 2(1+\cos\theta)^2\, d\theta\, dz = \int_{-1}^1 6\pi\, d\theta = 12\pi$

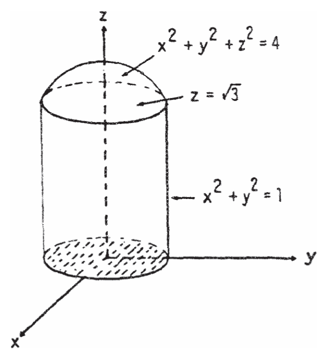
9. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right) r\, d\theta\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left[\frac{r^2\theta}{2}+\frac{r^2\sin 2\theta}{4}+z^2\theta\right]_0^{2\pi} r\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left(\pi r^3+2\pi r z^2\right) dr\, dz$
$= \int_0^1\left[\frac{\pi r^4}{4}+\pi r^2 z^2\right]_0^{\sqrt{z}} dz = \int_0^1\left(\frac{\pi z^2}{4}+\pi z^3\right) dz = \left[\frac{\pi z^3}{12}+\frac{\pi z^4}{4}\right]_0^1 = \frac{\pi}{3}$

10. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)r\, d\theta\, dz\, dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\, dz\, dr = 2\pi\int_0^2\left[r\left(4-r^2\right)^{1/2}-r^2+2r\right] dr$
$= 2\pi\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}-\frac{r^3}{3}+r^2\right]_0^2 = 2\pi\left[-\frac{8}{3}+4+\frac{1}{3}(4)^{3/2}\right] = 8\pi$

11. (a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$

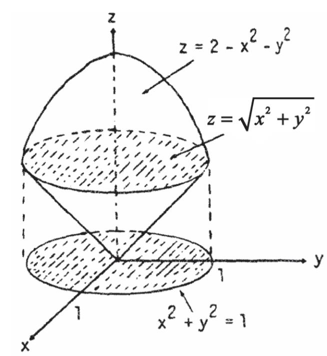
(c) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$




12. (a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^z r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\, dr\, dz\, d\theta$

(c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\, d\theta\, dz\, dr$




13. $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

14. $\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^3\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^4\cos\theta\, dr\, d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{2}{5}$

15. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

16. $\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

17. $\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r,\theta,z)\, dz\, r\, dr\, d\theta$

18. $\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

19. $\int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{2-r\sin\theta} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

20. $\int_{\pi/4}^{\pi/2}\int_0^{\cos\theta}\int_0^{2-r\sin\theta} f(r,\theta,z)\, dz\, r\, dr\, d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right) d\theta$

$= 2\int_0^{\pi}\int_0^{\pi}\sin^2\phi\, d\phi\, d\theta = \int_0^{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} d\theta = \int_0^{\pi}\pi\, d\theta = \pi^2$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 4\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left[2\sin^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta=\frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi}\,d\theta$
$=\frac{1}{96}\int_0^{2\pi}\left(2^4-0\right)d\theta=\frac{16}{96}\int_0^{2\pi}d\theta=\frac{1}{6}(2\pi)=\frac{\pi}{3}$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta=\frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta=\frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi}+\frac{2}{3}\int_0^{\pi}\sin\phi\,d\phi\right)d\theta$
$=\frac{5}{6}\int_0^{3\pi/2}\left[-\cos\phi\right]_0^{\pi}\,d\theta=\frac{5}{3}\int_0^{3\pi/2}d\theta=\frac{5\pi}{2}$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/3}\left(8-\sec^3\phi\right)\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\sec^2\phi\right]_0^{\pi/3}d\theta$
$=\int_0^{2\pi}\left[(-4-2)-\left(-8-\frac{1}{2}\right)\right]d\theta=\frac{5}{2}\int_0^{2\pi}d\theta=5\pi$

26. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta=\frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4}\tan\phi\sec^2\phi\,d\phi\,d\theta=\frac{1}{4}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4}d\theta$
$=\frac{1}{8}\int_0^{2\pi}d\theta=\frac{\pi}{4}$

27. $\int_0^2\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\,d\phi\,d\theta\,d\rho=\int_0^2\int_{-\pi}^{0}\rho^3\left[-\frac{\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2}d\theta\,d\rho=\int_0^2\int_{-\pi}^{0}\frac{\rho^3}{2}d\theta\,d\rho=\int_0^2\frac{\rho^3\pi}{2}d\rho$
$=\left[\frac{\pi\rho^4}{8}\right]_0^2=2\pi$

28. $\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\rho\,d\phi=2\pi\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi=\frac{2\pi}{3}\int_{\pi/6}^{\pi/3}\left[\rho^3\sin\phi\right]_{\csc\phi}^{2\csc\phi}d\phi$
$=\frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\csc^2\phi\,d\phi=\frac{28\pi}{3\sqrt{3}}$

29. $\int_0^1\int_0^{\pi}\int_0^{\pi/4}12\rho\sin^3\phi\,d\phi\,d\theta\,d\rho=\int_0^1\int_0^{\pi}\left(12\rho\left[\frac{-\sin^2\phi\cos\phi}{3}\right]_0^{\pi/4}+8\rho\int_0^{\pi/4}\sin\phi\,d\phi\right)d\theta\,d\rho$
$=\int_0^1\int_0^{\pi}\left(-\frac{2\rho}{\sqrt{2}}-8\rho\left[\cos\phi\right]_0^{\pi/4}\right)d\theta\,d\rho=\int_0^1\int_0^{\pi}\left(8\rho-\frac{10\rho}{\sqrt{2}}\right)d\theta\,d\rho=\pi\int_0^1\left(8\rho-\frac{10\rho}{\sqrt{2}}\right)d\rho=\pi\left[4\rho^2-\frac{5\rho^2}{\sqrt{2}}\right]_0^1$
$=\frac{\left(4\sqrt{2}-5\right)\pi}{\sqrt{2}}$

30. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2}5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi=\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\left(32-\csc^5\phi\right)\sin^3\phi\,d\theta\,d\phi$
$=\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\left(32\sin^3\phi-\csc^2\phi\right)d\theta\,d\phi=\pi\int_{\pi/6}^{\pi/2}\left(32\sin^3\phi-\csc^2\phi\right)d\phi$
$=\pi\left[-\frac{32\sin^2\phi\cos\phi}{3}\right]_{\pi/6}^{\pi/2}+\frac{64\pi}{3}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi+\pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}=\pi\left(\frac{32\sqrt{3}}{24}\right)-\frac{64\pi}{3}\left[\cos\phi\right]_{\pi/6}^{\pi/2}-\pi\left(\sqrt{3}\right)$
$=\frac{\sqrt{3}}{3}\pi+\left(\frac{64\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{33\pi\sqrt{3}}{3}=11\pi\sqrt{3}$

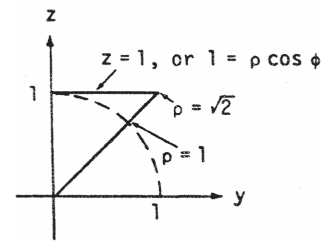
31. (a) $x^2+y^2=1 \Rightarrow \rho^2\sin^2\phi=1$, and $\rho\sin\phi=1 \Rightarrow \rho=\csc\phi$;

thus $\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta+\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(b) $\int_0^{2\pi}\int_1^{2}\int_0^{\sin^{-1}(1/\rho)}\rho^2\sin\phi\, d\phi\, d\rho\, d\theta+\int_0^{2\pi}\int_0^{2}\int_0^{\pi/6}\rho^2\sin\phi\, d\phi\, d\rho\, d\theta$

32. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(b) $\int_0^{2\pi}\int_0^{1}\int_0^{\pi/4}\rho^2\sin\phi\, d\phi\, d\rho\, d\theta+\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\cos^{-1}(1/\rho)}^{\pi/4}\rho^2\sin\phi\, d\phi\, d\rho\, d\theta$




33. $V=\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left(8-\cos^3\phi\right)\sin\phi\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi+\frac{\cos^4\phi}{4}\right]_0^{\pi/2}d\theta$

$=\frac{1}{3}\int_0^{2\pi}\left(8-\frac{1}{4}\right)d\theta=\left(\frac{31}{12}\right)(2\pi)=\frac{31\pi}{6}$

34. $V=\int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left(3\cos\phi+3\cos^2\phi+\cos^3\phi\right)\sin\phi\, d\phi\, d\theta$

$=\frac{1}{3}\int_0^{2\pi}\left[-\frac{3}{2}\cos^2\phi-\cos^3\phi-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}d\theta=\frac{1}{3}\int_0^{2\pi}\left(\frac{3}{2}+1+\frac{1}{4}\right)d\theta=\frac{11}{12}\int_0^{2\pi}d\theta=\left(\frac{11}{12}\right)(2\pi)=\frac{11\pi}{6}$

35. $V=\int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi}d\theta$

$=\frac{1}{12}(2)^4\int_0^{2\pi}d\theta=\frac{4}{3}(2\pi)=\frac{8\pi}{3}$

36. $V=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(1-\cos\phi)^3\sin\phi\, d\phi\, d\theta=\frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi/2}d\theta$

$=\frac{1}{12}\int_0^{2\pi}d\theta=\frac{1}{12}(2\pi)=\frac{\pi}{6}$

37. $V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\cos^3\phi\sin\phi\, d\phi\, d\theta=\frac{8}{3}\int_0^{2\pi}\left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2}d\theta$

$=\left(\frac{8}{3}\right)\left(\frac{1}{16}\right)\int_0^{2\pi}d\theta=\frac{1}{6}(2\pi)=\frac{\pi}{3}$

38. $V=\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\sin\phi\, d\phi\, d\theta=\frac{8}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/3}^{\pi/2}d\theta=\frac{4}{3}\int_0^{2\pi}d\theta=\frac{8\pi}{3}$

39. (a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(b) $8\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{4-r^2}}dz\, r\, dr\, d\theta$

(c) $8\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}dz\, dy\, dx$

40. (a) $\int_0^{\pi/2}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}} dz\ r\ dr\ d\theta$ (b) $\int_0^{\pi/2}\int_0^{\pi/4}\int_r^3 \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$

(c) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = 9\int_0^{\pi/2}\int_0^{\pi/4} \sin\phi\ d\phi\ d\theta = -9\int_0^{\pi/2}\left(\frac{1}{\sqrt{2}}-1\right) d\theta = \frac{9\pi\left(2-\sqrt{2}\right)}{4}$

41. (a) $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^2 \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$ (b) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} dz\ r\ dr\ d\theta$

(c) $V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz\ dy\ dx$

(d) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\left[r\left(4-r^2\right)^{1/2} - r\right] dr\ d\theta = \int_0^{2\pi}\left[-\frac{\left(4-r^2\right)^{3/2}}{3} - \frac{r^2}{2}\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(-\frac{1}{3}-\frac{3}{2}+\frac{4^{3/2}}{3}\right) d\theta$

$= \frac{5}{6}\int_0^{2\pi} d\theta = \frac{5\pi}{3}$

42. (a) $I_z = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r^2\ dz\ r\ dr\ d\theta$

(b) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1\left(\rho^2 \sin^2\phi\right)\left(\rho^2 \sin\phi\right) d\rho\ d\phi\ d\theta,$

since $r^2 = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin\phi$

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin^3\phi\ d\phi\ d\theta = \frac{1}{5}\int_0^{2\pi}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\sin\phi\ d\phi\right) d\theta = \frac{2}{15}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2} d\theta$

$= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$

43. $V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\ r\ dr\ d\theta = 4\int_0^{\pi/2}\int_0^1\left(5r - 4r^3 - r^5\right) dr\ d\theta = 4\int_0^{\pi/2}\left(\frac{5}{2}-1-\frac{1}{6}\right) d\theta = 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}$

44. $V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\ r\ dr\ d\theta = 4\int_0^{\pi/2}\int_0^1\left(r - r^2 + r\sqrt{1-r^2}\right) dr\ d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{3}-\frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$

$= 4\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$

45. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz\ r\ dr\ d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\left(-r^2 \sin\theta\right) dr\ d\theta = \int_{3\pi/2}^{2\pi}\left(-9\cos^3\theta\right)(\sin\theta)\ d\theta$

$= \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$

46. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^r dz\ r\ dr\ d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2\ dr\ d\theta = \frac{2}{3}\int_{\pi/2}^{\pi}\left(-27\cos^3\theta\right) d\theta$

$= -18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi}\cos\theta\ d\theta\right) = -12\left[\sin\theta\right]_{\pi/2}^{\pi} = 12$

47. $V = \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\sin\theta} r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$

$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2}-1\right]d\theta = -\frac{1}{3}\int_0^{\pi/2}\left(\cos^3\theta-1\right)d\theta = -\frac{1}{3}\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/2}+\frac{2}{3}\int_0^{\pi/2}\cos\theta\, d\theta\right)+\left[\frac{\theta}{3}\right]_0^{\pi/2}$

$= -\frac{2}{9}\left[\sin\theta\right]_0^{\pi/2}+\frac{\pi}{6} = \frac{-4+3\pi}{18}$

48. $V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2}\left[-\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$

$= \int_0^{\pi/2}\left[-\left(1-\cos^2\theta\right)^{3/2}+1\right]d\theta = \int_0^{\pi/2}\left(1-\sin^3\theta\right)d\theta = \left[\theta+\frac{\sin^2\theta\cos\theta}{3}\right]_0^{\pi/2} - \frac{2}{3}\int_0^{\pi/2}\sin\theta\, d\theta$

$= \frac{\pi}{2}+\frac{2}{3}\left[\cos\theta\right]_0^{\pi/2} = \frac{\pi}{2}-\frac{2}{3} = \frac{3\pi-4}{6}$

49. $V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\frac{a^3}{3}\sin\phi\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\right)d\theta$

$= \frac{2\pi a^3}{3}$

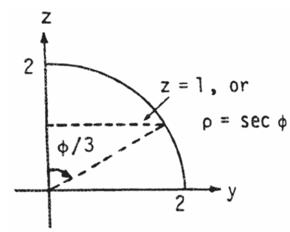
50. $V = \int_0^{\pi/6}\int_0^{\pi/2}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6}\int_0^{\pi/2}\sin\phi\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6} d\theta = \frac{a^3\pi}{18}$

51. $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\left(8\sin\phi-\tan\phi\sec^2\phi\right)d\phi\, d\theta$

$= \frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$

$= \frac{1}{3}\int_0^{2\pi}\left[-4-\frac{1}{2}(3)+8\right]d\theta = \frac{1}{3}\int_0^{2\pi}\frac{5}{2}\, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}$




52. $V = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_{\sec\phi}^{2\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4}\left(8\sec^3\phi-\sec^3\phi\right)\sin\phi\, d\phi\, d\theta$

$= \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sec^3\phi\ \sin\phi\, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\tan\phi\sec^2\phi\, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{14}{3}\int_0^{\pi/2} d\theta = \frac{7\pi}{3}$

53. $V = 4\int_0^{\pi/2}\int_0^{1}\int_0^{r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r^3\, dr\, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$

54. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r\, dr\, d\theta = 2\int_0^{\pi/2} d\theta = \pi$

55. $V = 8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{r} dz\, r\, dr\, d\theta = 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\, dr\, d\theta = 8\left(\frac{2\sqrt{2}-1}{3}\right)\int_0^{\pi/2} d\theta = \frac{4\pi\left(2\sqrt{2}-1\right)}{3}$

56. $V=8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta=8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\sqrt{2-r^2}\ dr\ d\theta=8\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}} d\theta=8\int_0^{\pi/2} d\theta=\frac{4\pi}{3}$

57. $V=\int_0^{2\pi}\int_0^2\int_0^{4-r\sin\theta} dz\ r\ dr\ d\theta=\int_0^{2\pi}\int_0^2\left(4r-r^2\sin\theta\right) dr\ d\theta=8\int_0^{2\pi}\left(1-\frac{\sin\theta}{3}\right) d\theta=16\pi$

58. $V=\int_0^{2\pi}\int_0^2\int_0^{4-r\cos\theta-r\sin\theta} dz\ r\ dr\ d\theta=\int_0^{2\pi}\int_0^2\left[4r-r^2(\cos\theta+\sin\theta)\right] dr\ d\theta=\frac{8}{3}\int_0^{2\pi}(3-\cos\theta-\sin\theta)\ d\theta$ $=16\pi$

59. The paraboloids intersect when $4x^2+4y^2=5-x^2-y^2\Rightarrow x^2+y^2=1$ and $z=4$
$\Rightarrow V=4\int_0^{\pi/2}\int_0^1\int_{4r^2}^{5-r^2} dz\ r\ dr\ d\theta=4\int_0^{\pi/2}\int_0^1\left(5r-5r^3\right) dr\ d\theta=20\int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta=5\int_0^{\pi/2} d\theta=\frac{5\pi}{2}$

60. The paraboloid intersects the $xy$-plane when $9-x^2-y^2=0\Rightarrow x^2+y^2=9\Rightarrow V=4\int_0^{\pi/2}\int_1^3\int_0^{9-r^2} dz\ r\ dr\ d\theta$
$=4\int_0^{\pi/2}\int_1^3\left(9r-r^3\right) dr\ d\theta=4\int_0^{\pi/2}\left[\frac{9r^2}{2}-\frac{r^4}{4}\right]_1^3 d\theta=4\int_0^{\pi/2}\left(\frac{81}{4}-\frac{17}{4}\right) d\theta=64\int_0^{\pi/2} d\theta=32\pi$

61. $V=8\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\ r\ dr\ d\theta=8\int_0^{2\pi}\int_0^1 r\left(4-r^2\right)^{1/2} dr\ d\theta=8\int_0^{2\pi}\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}\right]_0^1 d\theta=-\frac{8}{3}\int_0^{2\pi}\left(3^{3/2}-8\right) d\theta$
$=\frac{4\pi\left(8-3\sqrt{3}\right)}{3}$

62. The sphere and paraboloid intersect when $x^2+y^2+z^2=2$ and $z=x^2+y^2\Rightarrow z^2+z-2=0$
$\Rightarrow(z+2)(z-1)=0\Rightarrow z=1$ or $z=-2\Rightarrow z=1$ since $z\ge 0$. Thus, $x^2+y^2=1$ and the volume is given by the triple integral $V=4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta=4\int_0^{\pi/2}\int_0^1\left[r\left(2-r^2\right)^{1/2}-r^3\right] dr\ d\theta$
$=4\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{r^4}{4}\right]_0^1 d\theta=4\int_0^{\pi/2}\left(\frac{2\sqrt{2}}{3}-\frac{7}{12}\right)d\theta=\frac{\pi\left(8\sqrt{2}-7\right)}{6}$

63. average $=\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\int_{-1}^1 r^2 dz\ dr\ d\theta=\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 2r^2 dr\ d\theta=\frac{1}{3\pi}\int_0^{2\pi} d\theta=\frac{2}{3}$

64. average $=\frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\ dz\ dr\ d\theta=\frac{3}{4\pi}\int_0^{2\pi}\int_0^1 2r^2\sqrt{1-r^2} dr\ d\theta$
$=\frac{3}{2\pi}\int_0^{2\pi}\left[\frac{1}{8}\sin^{-1} r-\frac{1}{8}r\sqrt{1-r^2}\left(1-2r^2\right)\right]_0^1 d\theta=\frac{3}{16\pi}\int_0^{2\pi}\left(\frac{\pi}{2}+0\right) d\theta=\frac{3}{32}\int_0^{2\pi} d\theta=\left(\frac{3}{32}\right)(2\pi)=\frac{3\pi}{16}$

65. average $=\frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin\phi\ d\rho\ d\phi\ d\theta=\frac{3}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\ d\phi\ d\theta=\frac{3}{8\pi}\int_0^{2\pi} d\theta=\frac{3}{4}$

66. average $= \frac{1}{\left(\frac{2\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[ \frac{\sin^2\phi}{2} \right]_0^{\pi/2} d\theta$

$= \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}$

67. $M = 4\int_0^{\pi/2} \int_0^1 \int_0^r dz \, r \, dr \, d\theta = 4\int_0^{\pi/2} \int_0^1 r^2 \, dr \, d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z \, dz \, r \, dr \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3 dr \, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

68. $M = \int_0^{\pi/2} \int_0^2 \int_0^r dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^2 dr \, d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x \, dz \, r \, dr \, d\theta$

$= \int_0^{\pi/2} \int_0^2 r^3 \cos\theta \, dr \, d\theta = 4\int_0^{\pi/2} \cos\theta \, d\theta = 4$; $M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin\theta \, dr \, d\theta$

$= 4\int_0^{\pi/2} \sin\theta \, d\theta = 4$; $M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3 \, dr \, d\theta = 2\int_0^{\pi/2} d\theta = \pi \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,

$\overline{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

69. $M = \frac{8\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

$= 4\int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta = 4\int_0^{2\pi} \left[ \frac{\sin^2\phi}{2} \right]_{\pi/3}^{\pi/2} d\theta = 4\int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8}\right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$

$\Rightarrow \overline{z} = \frac{M_{xy}}{M} = (\pi)\left(\frac{3}{8\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

70. $M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2-\sqrt{2}}{2} d\theta = \frac{\pi a^3 \left(2-\sqrt{2}\right)}{3}$;

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$

$\Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right) \left[ \frac{3}{\pi a^3 \left(2-\sqrt{2}\right)} \right] = \left(\frac{3a}{8}\right)\left(\frac{2+\sqrt{2}}{2}\right) = \frac{3\left(2+\sqrt{2}\right)a}{16}$, and $\overline{x} = \overline{y} = 0$, by symmetry

71. $M = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r^{3/2} dr \, d\theta = \frac{64}{5} \int_0^{2\pi} d\theta = \frac{128\pi}{5}$; $M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} z \, dz \, r \, dr \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \int_0^4 r^2 \, dr \, d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{5}{6}$, and $\overline{x} = \overline{y} = 0$, by symmetry

72. $M = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \int_0^1 2r\sqrt{1-r^2} dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left[ -\frac{2}{3}\left(1-r^2\right)^{3/2} \right]_0^1 d\theta$

$= \frac{2}{3} \int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}$; $M_{yz} = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \cos\theta \, dz \, dr \, d\theta = 2\int_{-\pi/3}^{\pi/3} \int_0^1 r^2 \sqrt{1-r^2} \cos\theta \, dr \, d\theta$

$= 2\int_{-\pi/3}^{\pi/3} \left[ \frac{1}{8} \sin^{-1} r - \frac{1}{8} r\sqrt{1-r^2} \left(1-2r^2\right) \right]_0^1 \cos\theta \, d\theta = \frac{\pi}{8} \int_{-\pi/3}^{\pi/3} \cos\theta \, d\theta = \frac{\pi}{8} \left[\sin\theta\right]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right)\left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8}$

$\Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}$, and $\overline{y} = \overline{z} = 0$, by symmetry

73. We orient the cone with its vertex at the origin and axis along the $z$-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the $x$-axis which is through the vertex and parallel to the base of the cone $\Rightarrow I_x = \int_0^{2\pi}\int_0^1\int_r^1 \left(r^2 \sin^2\theta + z^2\right) dz\, r\, dr\, d\theta$
$= \int_0^{2\pi}\int_0^1 \left(r^3\sin^2\theta - r^4\sin^2\theta + \frac{r}{3} - \frac{r^4}{3}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{20} + \frac{1}{10}\right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$

74. $I_z = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a 2r^3\sqrt{a^2-r^2}\, dr\, d\theta = 2\int_0^{2\pi}\left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta$
$= 2\int_0^{2\pi} \frac{2}{15}a^5\, d\theta = \frac{8\pi a^5}{15}$

75. $I_z = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^{h}\left(x^2+y^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^a\int_{\frac{hr}{a}}^{h} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a \left(hr^3 - \frac{hr^4}{a}\right) dr\, d\theta$
$= \int_0^{2\pi} h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta = \int_0^{2\pi} h\left(\frac{a^4}{4} - \frac{a^5}{5a}\right) d\theta = \frac{ha^4}{20}\int_0^{2\pi} d\theta = \frac{\pi h a^4}{10}$

76. (a) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^5\, dr\, d\theta = \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}$;
$M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} z^2\, dz\, r\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^3\, dz\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{16}\int_0^{2\pi} d\theta = \frac{\pi}{8}$

(b) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^4\, dr\, d\theta = \frac{1}{5}\int_0^{2\pi} d\theta = \frac{2\pi}{5}$;
$M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2\, dz\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{14}\int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^4\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{7}\int_0^{2\pi} d\theta = \frac{2\pi}{7}$

77. (a) $M = \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$;
$M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r - r^4\right) dr\, d\theta = \frac{1}{10}\int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 zr^3\, dz\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r^3 - r^5\right) dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12}$

(b) $M = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta = \frac{\pi}{5}$ from part (a); $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^3\, dz\, r\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1 \left(r - r^5\right) dr\, d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 z^2 r^3\, dz\, dr\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r^3 - r^6\right) dr\, d\theta = \frac{1}{28}\int_0^{2\pi} d\theta = \frac{\pi}{14}$

78. (a) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$;
$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^6 \sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{a^7}{7}\int_0^{2\pi}\int_0^{\pi}\left(1 - \cos^2\phi\right)\sin\phi\, d\phi\, d\theta = \frac{a^7}{7}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$
$= \frac{4a^7}{21}\int_0^{2\pi} d\theta = \frac{8a^7\pi}{21}$

(b) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi} \frac{(1-\cos 2\phi)}{2} d\phi \, d\theta = \frac{\pi a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}$;

$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5 \sin^4\phi \, d\rho \, d\phi \, d\theta = \frac{a^6}{6}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi \, d\phi \, d\theta$

$= \frac{a^6}{6}\int_0^{2\pi}\left(\left[\frac{-\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi \, d\phi\right) d\theta = \frac{a^6}{8}\int_0^{2\pi}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi} d\theta = \frac{\pi a^6}{16}\int_0^{2\pi} d\theta = \frac{a^6\pi^2}{8}$

79. $M = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^{a} \frac{h}{a} r\sqrt{a^2-r^2} \, dr \, d\theta = \frac{h}{a}\int_0^{2\pi}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^{a} d\theta = \frac{h}{a}\int_0^{2\pi}\frac{a^3}{3} d\theta$

$= \frac{2ha^2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} z \, dz \, r \, dr \, d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\int_0^{a}\left(a^2 r - r^3\right) dr \, d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) d\theta = \frac{a^2h^2\pi}{4}$

$\Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4}\right)\left(\frac{3}{2ha^2\pi}\right) = \frac{3}{8}h$, and $\bar{x} = \bar{y} = 0$, by symmetry

80. Let the base radius of the cone be $a$ and the height $h$, and place the cone's axis of symmetry along the $z$-axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_{\left(\frac{h}{a}\right)r}^{h} z \, dz \, r \, dr \, d\theta$

$= \frac{1}{2}\int_0^{2\pi}\int_0^{a}\left(h^2 r - \frac{h^2}{a^2}r^3\right) dr \, d\theta = \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^{a} d\theta = \frac{h^2}{2}\int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^2}{4}\right) d\theta = \frac{h^2a^2}{8}\int_0^{2\pi} d\theta = \frac{h^2a^2\pi}{4}$

$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2a^2\pi}{4}\right)\left(\frac{3}{\pi a^2 h}\right) = \frac{3}{4}h$, and $\bar{x} = \bar{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

81. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where $\rho$ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant $k$: $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(k\rho - kR)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[k\frac{\rho^4}{4} - kR\frac{\rho^3}{3}\right]_0^{R} \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} k\left(\frac{R^4}{4} - \frac{R^4}{3}\right)\sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\left(-\frac{k}{12}R^4\left[-\cos\phi\right]_0^{\pi}\right) d\theta = \int_0^{2\pi}\left(-\frac{k}{6}R^4\right) d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$

$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4}\rho + \frac{3M}{\pi R^4}R$. At the center of the planet $\rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4}\right)R = \frac{3M}{\pi R^3}$.

82. The mass of the planet's atmosphere to an altitude $h$ above the surface of the planet is the triple integral

$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^{h} \mu_0 e^{-c(\rho - R)}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_R^{h}\int_0^{2\pi}\int_0^{\pi} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$

$= \int_R^{h}\int_0^{2\pi}\left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta \, d\rho = 2\int_R^{h}\int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho}\rho^2 d\theta \, d\rho = 4\pi\mu_0 e^{cR}\int_R^{h} e^{-c\rho}\rho^2 d\rho$

$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^{h}$ (by parts)

$= 4\pi\mu_0 e^{cR}\left(-\frac{h^2e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right)$.

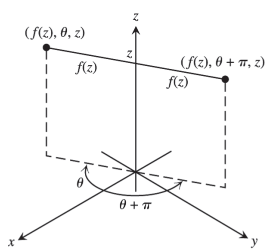
The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

83. (a) A plane perpendicular to the $x$-axis has the form $x=a$ in rectangular coordinates $\Rightarrow r\cos\theta=a$ $\Rightarrow r=\frac{a}{\cos\theta}\Rightarrow r=a\sec\theta$, in cylindrical coordinates.

(b) A plane perpendicular to the $y$-axis has the form $y=b$ in rectangular coordinates $\Rightarrow r\sin\theta=b$ $\Rightarrow r=\frac{b}{\sin\theta}\Rightarrow r=b\csc\theta$, in cylindrical coordinates.

84. $ax+by=c\Rightarrow a(r\cos\theta)+b(r\sin\theta)=c\Rightarrow r(a\cos\theta+b\sin\theta)=c\Rightarrow r=\frac{c}{a\cos\theta+b\sin\theta}$.

85. The equation $r=f(z)$ implies that the point $(r,\theta,z)=(f(z),\theta,z)$ will lie on the surface for all $\theta$. In particular $(f(z),\theta+\pi,z)$ lies on the surface whenever $(f(z),\theta,z)$ does $\Rightarrow$ the surface is symmetric with respect to the $z$-axis.



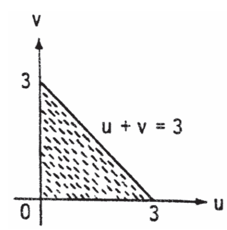

86. The equation $\rho=f(\phi)$ implies that the point $(\rho,\phi,\theta)=(f(\phi),\phi,\theta)$ lies on the surface for all $\theta$. In particular, if $(f(\phi),\phi,\theta)$ lies on the surface, then $(f(\phi),\phi,\theta+\pi)$ lies on the surface, so the surface is symmetric with respect to the $z$-axis.

## 15.8 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x-y=u$ and $2x+y=v\Rightarrow 3x=u+v$ and $y=x-u\Rightarrow x=\frac{1}{3}(u+v)$ and $y=\frac{1}{3}(-2u+v)$;

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3}\end{vmatrix}=\frac{1}{9}+\frac{2}{9}=\frac{1}{3}$$

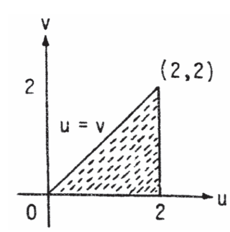
(b) The line segment $y=x$ from $(0,0)$ to $(1,1)$ is $x-y=0\Rightarrow u=0$; the line segment $y=-2x$ from $(0,0)$ to $(1,-2)$ is $2x+y=0\Rightarrow v=0$; the line segment $x=1$ from $(1,1)$ to $(1,-2)$ is $(x-y)+(2x+y)=3\Rightarrow u+v=3$. The transformed region is sketched at the right.




2. (a) $x+2y=u$ and $x-y=v\Rightarrow 3y=u-v$ and $x=v+y\Rightarrow y=\frac{1}{3}(u-v)$ and $x=\frac{1}{3}(u+2v)$;

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{vmatrix}=-\frac{1}{9}-\frac{2}{9}=-\frac{1}{3}$$

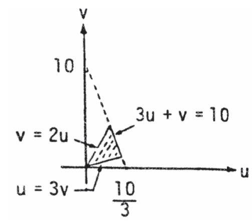
(b) The triangular region in the $xy$-plane has vertices $(0,0),(2,0)$, and $\left(\frac{2}{3},\frac{2}{3}\right)$. The line segment $y=x$ from $(0,0)$ to $\left(\frac{2}{3},\frac{2}{3}\right)$ is $x-y=0\Rightarrow v=0$; the line segment $y=0$ from $(0,0)$ to $(2,0)$ $\Rightarrow u=v$; the line segment $x+2y=2$ from $\left(\frac{2}{3},\frac{2}{3}\right)$ to $(2,0)\Rightarrow u=2$. The transformed region is sketched at the right.



3. (a) $3x+2y=u$ and $x+4y=v \Rightarrow -5x=-2u+v$ and $y=\frac{1}{2}(u-3x) \Rightarrow x=\frac{1}{5}(2u-v)$ and $y=\frac{1}{10}(3v-u)$;

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{2}{5} & -\frac{1}{5}\\ -\frac{1}{10} & \frac{3}{10}\end{vmatrix}=\frac{6}{50}-\frac{1}{50}=\frac{1}{10}$$

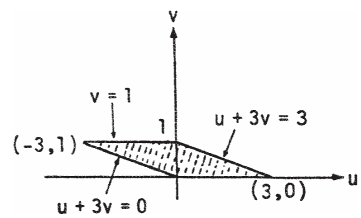
(b) The $x$-axis $y=0 \Rightarrow u=3v$;
the $y$-axis $x=0 \Rightarrow v=2u$;
the line $x+y=1 \Rightarrow \frac{1}{5}(2u-v)+\frac{1}{10}(3v-u)=1$
$\Rightarrow 2(2u-v)+(3v-u)=10 \Rightarrow 3u+v=10$. The transformed region is sketched at the right.




4. (a) $2x-3y=u$ and $-x+y=v \Rightarrow -x=u+3v$ and $y=v+x \Rightarrow x=-u-3v$ and $y=-u-2v$;

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1 & -3\\ -1 & -2\end{vmatrix}=2-3=-1$$

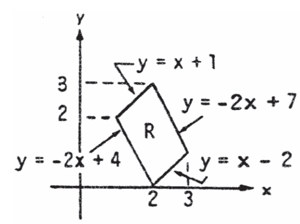
(b) The line $x=-3 \Rightarrow -u-3v=-3$ or $u+3v=3$;
$x=0 \Rightarrow u+3v=0$; $y=x \Rightarrow v=0$; $y=x+1$
$\Rightarrow v=1$. The transformed region is the parallelogram sketched at the right.




5. $\int_0^4 \int_{y/2}^{(y/2)+1}\left(x-\frac{y}{2}\right) dx\, dy=\int_0^4\left[\frac{x^2}{2}-\frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy=\frac{1}{2}\int_0^4\left[\left(\frac{y}{2}+1\right)^2-\left(\frac{y}{2}\right)^2-\left(\frac{y}{2}+1\right)y+\left(\frac{y}{2}\right)y\right] dy$

$=\frac{1}{2}\int_0^4 (y+1-y)\, dy=\frac{1}{2}\int_0^4 dy=\frac{1}{2}(4)=2$

6. $\iint_R \left(2x^2-xy-y^2\right) dx\, dy=\iint_R (x-y)(2x+y)\, dx\, dy$
$=\iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv=\frac{1}{3}\iint_G uv\, du\, dv$; We find the boundaries of $G$ from the boundaries of $R$, shown in the accompanying figure:




| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $y=-2x+4$ | $\frac{1}{3}(-2u+v)=-\frac{2}{3}(u+v)+4$ | $v=4$ |
| $y=-2x+7$ | $\frac{1}{3}(-2u+v)=-\frac{2}{3}(u+v)+7$ | $v=7$ |
| $y=x-2$ | $\frac{1}{3}(-2u+v)=\frac{1}{3}(u+v)-2$ | $u=2$ |
| $y=x+1$ | $\frac{1}{3}(-2u+v)=\frac{1}{3}(u+v)+1$ | $u=-1$ |

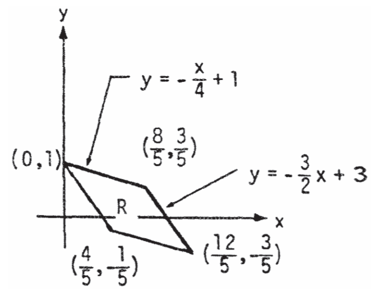
$\Rightarrow \frac{1}{3}\iint_G uv\, du\, dv=\frac{1}{3}\int_{-1}^2\int_4^7 uv\, dv\, du=\frac{1}{3}\int_{-1}^2 u\left[\frac{v^2}{2}\right]_4^7 du=\frac{11}{2}\int_{-1}^2 u\, du=\left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^2=\left(\frac{11}{4}\right)(4-1)=\frac{33}{4}$

7. $\iint\limits_R \left(3x^2+14xy+8y^2\right)\,dx\,dy$

$=\iint\limits_R (3x+2y)(x+4y)\,dx\,dy$

$=\iint\limits_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\frac{1}{10}\iint\limits_G uv\,du\,dv;$

We find the boundaries of $G$ from the boundaries of $R$, shown in the accompanying figure:




| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $y=-\frac{3}{2}x+1$ | $\frac{1}{10}(3v-u)=-\frac{3}{10}(2u-v)+1$ | $u=2$ |
| $y=-\frac{3}{2}x+3$ | $\frac{1}{10}(3v-u)=-\frac{3}{10}(2u-v)+3$ | $u=6$ |
| $y=-\frac{1}{4}x$ | $\frac{1}{10}(3v-u)=-\frac{1}{20}(2u-v)$ | $v=0$ |
| $y=-\frac{1}{4}x+1$ | $\frac{1}{10}(3v-u)=-\frac{1}{20}(2u-v)+1$ | $v=4$ |

$\Rightarrow \frac{1}{10}\iint\limits_G uv\,du\,dv=\frac{1}{10}\int_2^6\int_0^4 uv\,dv\,du=\frac{1}{10}\int_2^6 u\left[\frac{v^2}{2}\right]_0^4 du=\frac{4}{5}\int_2^6 u\,du=\left(\frac{4}{5}\right)\left[\frac{u^2}{2}\right]_2^6=\left(\frac{4}{5}\right)(18-2)=\frac{64}{5}$

8. $\iint\limits_R 2(x-y)\,dx\,dy=\iint\limits_G (-2v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\iint\limits_G (-2v)\,du\,dv;$ the region $G$ is sketched in Exercise 4

$\Rightarrow \iint\limits_G (-2v)\,du\,dv=\int_0^1\int_{-3v}^{3-3v}(-2v)\,du\,dv=\int_0^1(-2v)(3-3v+3v)\,dv=\int_0^1(-6v)\,dv=\left[-3v^2\right]_0^1=-3$

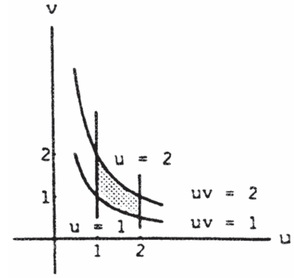
9. $x=\frac{u}{v}$ and $y=uv\Rightarrow \frac{y}{x}=v^2$ and $xy=u^2$; $\frac{\partial(x,y)}{\partial(u,v)}=J(u,v)=\begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix}=v^{-1}u+v^{-1}u=\frac{2u}{v};$

$y=x\Rightarrow uv=\frac{u}{v}\Rightarrow v=1$, and $y=4x\Rightarrow v=2$; $xy=1\Rightarrow u=1$, and $xy=9\Rightarrow u=3$; thus

$\iint\limits_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dx\,dy=\int_1^3\int_1^2(v+u)\left(\frac{2u}{v}\right)dv\,du=\int_1^3\int_1^2\left(2u+\frac{2u^2}{v}\right)dv\,du=\int_1^3\left[2uv+2u^2\ln v\right]_1^2 du$

$=\int_1^3\left(2u+2u^2\ln 2\right)du=\left[u^2+\frac{2}{3}u^2\ln 2\right]_1^3=8+\frac{2}{3}(26)(\ln 2)=8+\frac{52}{3}(\ln 2)$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)}=J(u,v)=\begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix}=u$, and the region $G$ is sketched at the right



(b) $x=1 \Rightarrow u=1$, and $x=2 \Rightarrow u=2$; $y=1 \Rightarrow uv=1 \Rightarrow v=\frac{1}{u}$, and $y=2 \Rightarrow uv=2 \Rightarrow v=\frac{2}{u}$; thus,

$$\int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u\,dv\,du = \int_1^2 \int_{1/u}^{2/u} uv\,dv\,du = \int_1^2 u\left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u\left(\frac{2}{u^2} - \frac{1}{2u^2}\right) du$$

$$= \frac{3}{2}\int_1^2 u\left(\frac{1}{u^2}\right) du = \frac{3}{2}\left[\ln u\right]_1^2 = \frac{3}{2}\ln 2;\quad \int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \left[\frac{1}{x}\cdot\frac{y^2}{2}\right]_1^2 dx = \frac{3}{2}\int_1^2 \frac{dx}{x} = \frac{3}{2}\left[\ln x\right]_1^2 = \frac{3}{2}\ln 2$$

11. $x = ar\cos\theta$ and $y = ar\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr;$

$$I_0 = \iint_R \left(x^2+y^2\right) dA = \int_0^{2\pi}\int_0^1 r^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\left|J(r,\theta)\right| dr\,d\theta$$

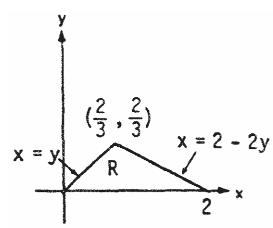
$$= \int_0^{2\pi}\int_0^1 abr^3\left(a^2\cos^2\theta + b^2\sin^2\theta\right) dr\,d\theta = \frac{ab}{4}\left(\int_0^{2\pi} a^2\cos^2\theta + b^2\sin^2\theta\right) d\theta$$

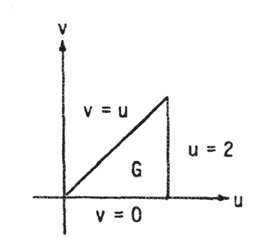
$$= \frac{ab}{4}\left[\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4}\right]_0^{2\pi} = \frac{ab\pi\left(a^2+b^2\right)}{4}$$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab;\quad A = \iint_R dy\,dx = \iint_G ab\,du\,dv = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du = 2ab\int_{-1}^{1}\sqrt{1-u^2}\,du$

$$= 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1}u\right]_{-1}^{1} = ab\left[\sin^{-1}1 - \sin^{-1}(-1)\right] = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = ab\pi$$

13. The region of integration $R$ in the $xy$-plane is sketched in the figure at the right. The boundaries of the image $G$ are obtained as follows, with $G$ sketched at the right:




| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x=y$ | $\frac{1}{3}(u+2v) = \frac{1}{3}(u-v)$ | $v=0$ |
| $x=2-2y$ | $\frac{1}{3}(u+2v) = 2-\frac{2}{3}(u-v)$ | $u=2$ |
| $y=0$ | $0=\frac{1}{3}(u-v)$ | $v=u$ |

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3}\int_y^{2-2y}(x+2y)e^{(y-x)}\,dx\,dy = \int_0^2\int_0^u ue^{-v}\left|-\frac{1}{3}\right|dv\,du$

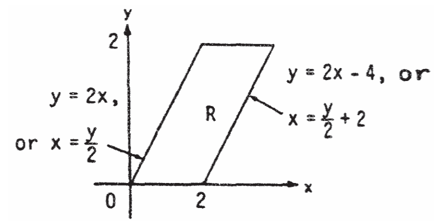
$= \frac{1}{3}\int_0^2 u\left[-e^{-v}\right]_0^u du = \frac{1}{3}\int_0^2 u\left(1-e^{-u}\right)du = \frac{1}{3}\left[u\left(u+e^{-u}\right)-\frac{u^2}{2}+e^{-u}\right]_0^2 = \frac{1}{3}\left[2\left(2+e^{-2}\right)-2+e^{-2}-1\right]$

$= \frac{1}{3}\left(3e^{-2}+1\right) \approx 0.4687$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u+v) - v = 2u$

and $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next,

$u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration $R$ in the $xy$-plane are transformed to the boundaries of $G$:




| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x = \frac{y}{2}$ | $u + \frac{v}{2} = \frac{v}{2}$ | $u = 0$ |
| $x = \frac{y}{2} + 2$ | $u + \frac{v}{2} = \frac{v}{2} + 2$ | $u = 2$ |
| $y = 0$ | $v = 0$ | $v = 0$ |
| $y = 2$ | $v = 2$ | $v = 2$ |

$\Rightarrow \int_0^2\int_{y/2}^{(y/2)+2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy = \int_0^2\int_0^2 v^3(2u)e^{4u^2}\,du\,dv = \int_0^2 v^3\left[\frac{1}{4}e^{4u^2}\right]_0^2 dv = \frac{1}{4}\int_0^2 v^3\left(e^{16}-1\right)dv$

$= \frac{1}{4}\left(e^{16}-1\right)\left[\frac{v^4}{4}\right]_0^2 = e^{16}-1$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus

$\int_1^2\int_{1/y}^{y}\left(x^2+y^2\right)dx\,dy + \int_2^4\int_{y/4}^{4/y}\left(x^2+y^2\right)dx\,dy = \int_1^2\int_1^2\left(\frac{u^2}{v^2}+u^2v^2\right)\left(\frac{2u}{v}\right)du\,dv = \int_1^2\int_1^2\left(\frac{2u^3}{v^3}+2u^3v\right)du\,dv$

$= \int_1^2\left[\frac{u^4}{2v^3}+\frac{1}{2}u^4v\right]_1^2 dv = \int_1^2\left(\frac{15}{2v^3}+\frac{15v}{2}\right)dv = \left[-\frac{15}{4v^2}+\frac{15v^2}{4}\right]_1^2 = \frac{225}{16}$

16. $x = u^2 - v^2$ and $y = 2uv$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4\left(u^2+v^2\right)$;

$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4\left(1-\left(u^2-v^2\right)\right) \Rightarrow u = \pm 1$; $y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0$ or $v = 0$;

$x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v$ or $u = -v$; This gives us four triangular regions, but only the one in the quadrant where both $u$, $v$ are positive maps into the region $R$ in the $xy$-plane.

$$\int_0^1\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}\,dx\,dy=\int_0^1\int_0^u\sqrt{\left(u^2-v^2\right)^2+(2uv)^2}\cdot 4\left(u^2+v^2\right)dv\,du=4\int_0^1\int_0^u\left(u^2+v^2\right)^2dv\,du$$
$$=4\int_1^2\left[u^4v+\tfrac{2}{3}u^2v^3+\tfrac{1}{5}v^5\right]_0^u du=\tfrac{112}{15}\int_1^2 u^5du=\tfrac{112}{15}\left[\tfrac{1}{6}u^6\right]_1^2=\tfrac{56}{45}$$

17. (a) $x=u\cos v$ and $y=u\sin v\Rightarrow\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\end{vmatrix}=u\cos^2 v+u\sin^2 v=u$

(b) $x=u\sin v$ and $y=u\cos v\Rightarrow\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\sin v & u\cos v\\ \cos v & -u\sin v\end{vmatrix}=-u\sin^2 v-u\cos^2 v=-u$

18. (a) $x=u\cos v,\ y=u\sin v,\ z=w\Rightarrow\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\cos v & -u\sin v & 0\\ \sin v & u\cos v & 0\\ 0 & 0 & 1\end{vmatrix}=u\cos^2 v+u\sin^2 v=u$

(b) $x=2u-1,\ y=3v-4,\ z=\frac{1}{2}(w-4)\Rightarrow\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & \frac{1}{2}\end{vmatrix}=(2)(3)\left(\frac{1}{2}\right)=3$

19. $\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}$

$$=(\cos\phi)\begin{vmatrix}\rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}+(\rho\sin\phi)\begin{vmatrix}\sin\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}$$
$$=\left(\rho^2\cos\phi\right)\left(\sin\phi\cos\phi\cos^2\theta+\sin\phi\cos\phi\sin^2\theta\right)+\left(\rho^2\sin\phi\right)\left(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta\right)$$
$$=\rho^2\sin\phi\cos^2\phi+\rho^2\sin^3\phi=\left(\rho^2\sin\phi\right)\left(\cos^2\phi+\sin^2\phi\right)=\rho^2\sin\phi$$

20. Let $u=g(x)\Rightarrow J(x)=\frac{du}{dx}=g'(x)\Rightarrow\int_a^b f(u)\,du=\int_{g(a)}^{g(b)} f\left(g(x)\right)g'(x)\,dx$ in accordance with Theorem 7 in Section 5.6. Note that $g'(x)$ represents the Jacobian of the transformation $u=g(x)$ or $x=g^{-1}(u)$.

21. $\int_0^3\int_0^4\int_{y/2}^{1+(y/2)}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz=\int_0^3\int_0^4\left[\frac{x^2}{2}-\frac{xy}{2}+\frac{xz}{3}\right]_{y/2}^{1+(y/2)}dy\,dz=\int_0^3\int_0^4\left[\frac{1}{2}(y+1)-\frac{y}{2}+\frac{z}{3}\right]dy\,dz$

$$=\int_0^3\left[\frac{(y+1)^2}{4}-\frac{y^2}{4}+\frac{yz}{3}\right]_0^4 dz=\int_0^3\left(\frac{9}{4}+\frac{4z}{3}-\frac{1}{4}\right)dz=\int_0^3\left(2+\frac{4z}{3}\right)dz=\left[2z+\frac{2z^2}{3}\right]_0^3=12$$

22. $J(u,v,w)=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$; the transformation takes the ellipsoid region $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ in $xyz$-space into the spherical region $u^2+v^2+w^2\le 1$ in $uvw$-space $\left(\text{which has volume } V=\frac{4}{3}\pi\right)\Rightarrow V=\iiint\limits_R dx\,dy\,dz$

$$=\iiint\limits_G abc\,du\,dv\,dw=\frac{4\pi abc}{3}$$

23. $J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; for $R$ and $G$ as in Exercise 22, $\iiint\limits_R |xyz|\, dx\, dy\, dz \iiint\limits_G a^2b^2c^2 uvw\, dw\, dv\, du$

$= 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta$

$= \frac{4a^2b^2c^2}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^3\phi\cos\phi\, d\phi\, d\theta = \frac{a^2b^2c^2}{3}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{a^2b^2c^2}{6}$

24. $u = x,\ v = xy,$ and $w = 3z \Rightarrow x = u,\ y = \frac{v}{u},$ and $z = \frac{1}{3}w \Rightarrow J(u, v, w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$;

$\iiint\limits_D \left(x^2y + 3xyz\right) dx\, dy\, dz = \iiint\limits_G \left[u^2\left(\frac{v}{u}\right) + 3u\left(\frac{v}{u}\right)\left(\frac{w}{3}\right)\right] \left|J(u, v, w)\right| du\, dv\, dw = \frac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \frac{vw}{u}\right) du\, dv\, dw$

$= \frac{1}{3}\int_0^3\int_0^2 (v + vw\ln 2)\, dv\, dw = \frac{1}{3}\int_0^3 (1 + w\ln 2)\left[\frac{v^2}{2}\right]_0^2 dw = \frac{2}{3}\int_0^3 (1 + w\ln 2)\, dw = \frac{2}{3}\left[w + \frac{w^2}{2}\ln 2\right]_0^3$

$= \frac{2}{3}\left(3 + \frac{9}{2}\ln 2\right) = 2 + 3\ln 2 = 2 + \ln 8$

25. The first moment about the $xy$-coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation in Exercise 23 is, $M_{xy} = \iiint\limits_D z\, dz\, dy\, dx = \iiint\limits_G cw\, |J(u, v, w)|\, du\, dv\, dw$

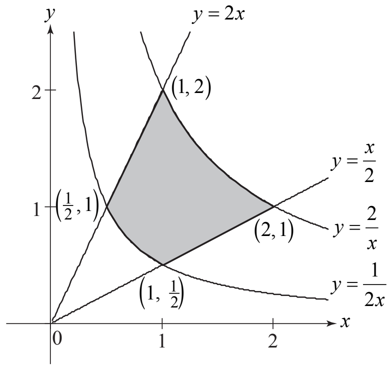
$= abc^2 \iiint\limits_G w\, du\, dv\, dw = \left(abc^2\right)\cdot\left(M_{xy} \text{ of the hemisphere } x^2 + y^2 + z^2 = 1,\ z \ge 0\right) = \frac{abc^2\pi}{4}$;

the mass of the semi-ellipsoid is $\frac{2abc\pi}{3} \Rightarrow \overline{z} = \left(\frac{abc^2\pi}{4}\right)\left(\frac{3}{2abc\pi}\right) = \frac{3}{8}c$

26. A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of $r$. That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r\cos\theta,\ y = y$ and $z = r\sin\theta$, we have $V = \iiint\limits_G r\, dy\, d\theta\, dr = \int_a^b\int_0^{2x}\int_0^{f(r)} r\, dy\, d\theta\, dr = \int_a^b\int_0^{2\pi}\left[r\ y\right]_0^{f(r)} d\theta\, dr = \int_a^b\int_0^{2\pi} r\ f(r)\, d\theta\, dr$

$= \int_a^b \left[r\theta f(r)\right]_0^{2\pi} dr = \int_a^b 2\pi r f(r) dr$. In the last integral, $r$ is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing $x$ instead of $r$ we have $V = \int_a^b 2\pi x f(x)\, dx$, which is the same result obtained using the shell method.

27. The region $R$ is shaded in the graph below.




Solving explicitly for the transformation that gives $x$ and $y$ in terms of $u$ and $v$ yields a complicated expression for $\frac{\partial(x, y)}{\partial(u,v)}$. However, its reciprocal, $\frac{\partial(u, v)}{\partial(x, y)}$ is relatively easy to compute.

Since $u(x, y) = xy$ and $v(x, y) = y/x$, $J(x, y) = \left|\begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}\right| = 2\frac{y}{x} = 2v$. Thus $J(u,v) = 1/2v$. In the $uv$-plane the region corresponding to $R$ is $G: \frac{1}{2} \le u \le 2,\ \frac{1}{2} \le v \le 2$. Thus $v$ is positive and $|J(u, v)| = 1/2v$.

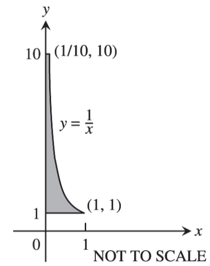
$$\iint_R dA = \int_{1/2}^{2}\int_{1/2}^{2} \frac{1}{2v}\, du\, dv = \int_{1/2}^{2}\left(\left[\frac{\ln u}{2}\right]_{1/2}^{2}\right) dv = \int_{1/2}^{2} \ln 2\, dv = \frac{3}{2}\ln 2$$

28. Under the given transformation, $y^2 = uv$, so

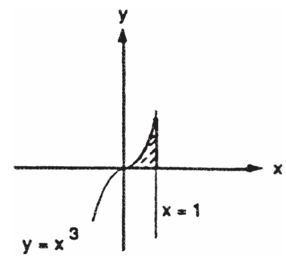
$$\iint_R y^2 dA = \int_{1/2}^{2}\int_{1/2}^{2} \frac{uv}{2v}\, du\, dv = \int_{1/2}^{2}\left(\left[\frac{u^2}{4}\right]_{1/2}^{2}\right) dv = \int_{1/2}^{2} \frac{15}{16}\, dv = \frac{45}{32}$$

## CHAPTER 15 PRACTICE EXERCISES

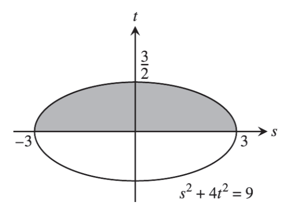
1. $\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy = \int_1^{10}\left[e^{xy}\right]_0^{1/y} dy$
$= \int_1^{10}(e-1)\,dy = 9e-9$



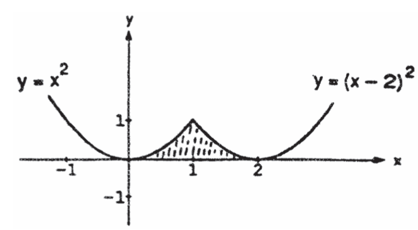

2. $\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx = \int_0^1 x\left[e^{y/x}\right]_0^{x^3} dx$
$= \int_0^1\left(xe^{x^2} - x\right)dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



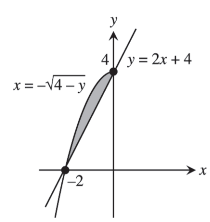

3. $\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt = \int_0^{3/2}\left[ts\right]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} dt$
$= \int_0^{3/2} 2t\sqrt{9-4t^2}\,dt = \left[-\frac{1}{6}\left(9-4t^2\right)^{3/2}\right]_0^{3/2}$
$= -\frac{1}{6}\left(0^{3/2} - 9^{3/2}\right) = \frac{27}{6} = \frac{9}{2}$




4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_{\sqrt{y}}^{2-\sqrt{y}} dy$
$= \frac{1}{2}\int_0^1 y\left(4-4\sqrt{y}+y-y\right)dy$
$= \int_0^1\left(2y-2y^{3/2}\right)dy = \left[y^2 - \frac{4y^{5/2}}{5}\right]_0^1 = \frac{1}{5}$




5. $\int_{-2}^0\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^0\left(-x^2-2x\right)dx$
$= \left[-\frac{x^3}{3} - x^2\right]_{-2}^0 = -\left(\frac{8}{3}-4\right) = \frac{4}{3}$
$\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy = \int_0^4\left(\frac{y-4}{2} + \sqrt{4-y}\right)dy$
$= \left[\frac{y^2}{2} - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^4$
$= 4-8+\frac{2}{3}\cdot 4^{3/2} = -4+\frac{16}{3} = \frac{4}{3}$



6. $\int_0^1 \int_y^{\sqrt{y}} \sqrt{x}\, dx\, dy = \int_0^1 \left[\frac{2}{3}x^{3/2}\right]_y^{\sqrt{y}} dy$

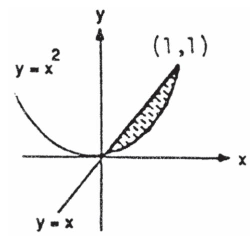
$= \frac{2}{3}\int_0^1 \left(y^{3/4} - y^{3/2}\right) dy = \frac{2}{3}\left[\frac{4}{7}y^{7/4} - \frac{2}{5}y^{5/2}\right]_0^1$

$= \frac{2}{3}\left(\frac{4}{7} - \frac{2}{5}\right) = \frac{4}{35}$

$\int_0^1 \int_{x^2}^{x} \sqrt{x}\, dy\, dx = \int_0^1 x^{1/2}\left(x - x^2\right) dx$

$= \int_0^1 \left(x^{3/2} - x^{5/2}\right) dx$

$= \left[\frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^1 = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}$



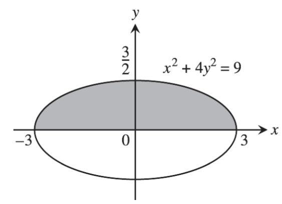

7. $\int_{-3}^{3} \int_0^{(1/2)\sqrt{9-x^2}} y\, dy\, dx = \int_{-3}^{3} \left[\frac{y^2}{2}\right]_0^{(1/2)\sqrt{9-x^2}} dx$

$= \int_{-3}^{3} \frac{1}{8}\left(9 - x^2\right) dx = \left[\frac{9x}{8} - \frac{x^3}{24}\right]_{-3}^{3}$

$= \left(\frac{27}{8} - \frac{27}{24}\right) - \left(-\frac{27}{8} + \frac{27}{24}\right) = \frac{27}{6} = \frac{9}{2}$

$\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\, dx\, dy = \int_0^{3/2} 2y\sqrt{9-4y^2}\, dy$

$= \left[-\frac{1}{4}\cdot\frac{2}{3}\left(9-4y^2\right)^{3/2}\right]_0^{3/2} = \frac{1}{6}\cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}$



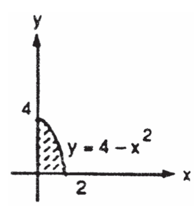

8. $\int_0^2 \int_0^{4-x^2} 2x\, dy\, dx = \int_0^2 [2xy]_0^{4-x^2} dx$

$= \int_0^2 \left(2x\left(4-x^2\right)\right) dx = \int_0^2 \left(8x - 2x^3\right) dx$

$= \left[4x^2 - \frac{x^4}{2}\right]_0^2 = 16 - \frac{16}{2} = 8$

$\int_0^4 \int_0^{\sqrt{4-y}} 2x\, dx\, dy = \int_0^4 \left[x^2\right]_0^{\sqrt{4-y}} dy$

$= \int_0^4 (4-y)\, dy = \left[4y - \frac{y^2}{2}\right]_0^4 = 16 - \frac{16}{2} = 8$




9. $\int_0^1 \int_{2y}^{2} 4\cos\left(x^2\right) dx\, dy = \int_0^2 \int_0^{x/2} 4\cos\left(x^2\right) dy\, dx = \int_0^2 2x\cos\left(x^2\right) dx = \left[\sin\left(x^2\right)\right]_0^2 = \sin 4$

10. $\int_0^2 \int_{y/2}^{1} e^{x^2} dx\, dy = \int_0^1 \int_0^{2x} e^{x^2} dy\, dx = \int_0^1 2xe^{x^2} dx = \left[e^{x^2}\right]_0^1 = e - 1$

11. $\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy\, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\, dy = \frac{\ln 17}{4}$

12. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin(\pi x^2)}{x^2}\, dx\, dy = \int_0^1 \int_0^{x^3} \frac{2\pi \sin(\pi x^2)}{x^2}\, dy\, dx = \int_0^1 2\pi x \sin(\pi x^2)\, dx = \left[-\cos(\pi x^2)\right]_0^1 = -(-1)-(-1) = 2$

13. $A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy\, dx = \int_{-2}^0 \left(-x^2 - 2x\right) dx = \frac{4}{3}$

14. $A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx\, dy = \int_1^4 \left(\sqrt{y} - 2 + y\right) dy = \frac{37}{6}$

15. $V = \int_0^1 \int_x^{2-x} \left(x^2+y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12}\right) + \frac{2^4}{12} = \frac{4}{3}$

16. $V = \int_{-3}^2 \int_x^{6-x^2} x^2\, dy\, dx = \int_{-3}^2 \left[x^2 y\right]_x^{6-x^2} dx = \int_{-3}^2 \left(6x^2 - x^4 - x^3\right) dx = \frac{125}{4}$

17. average value $= \int_0^1 \int_0^1 xy\, dy\, dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\, dx = \frac{1}{4}$

18. average value $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 \left(x - x^3\right) dx = \frac{1}{2\pi}$

19. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\left(1+x^2+y^2\right)^2}\, dy\, dx = \int_0^{2\pi} \int_0^1 \frac{2r}{\left(1+r^2\right)^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$

20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right) dx\, dy = \int_0^{2\pi} \int_0^1 r \ln\left(r^2+1\right) dr\, d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u\, du\, d\theta = \frac{1}{2} \int_0^{2\pi} \left[u \ln u - u\right]_1^2 d\theta$
$= \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1)\, d\theta = [\ln(4) - 1]\pi$

21. $\left(x^2+y^2\right)^2 - \left(x^2-y^2\right) = 0 \Rightarrow r^4 - r^2 \cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{\left(1+r^2\right)^2}\, dr\, d\theta$
$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2\left(1+r^2\right)}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta$
$= \frac{1}{2}\left[\theta - \frac{\tan\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$

22. (a) $\iint_R \frac{1}{\left(1+x^2+y^2\right)^2}\, dx\, dy = \int_0^{\pi/3} \int_0^{\sec\theta} \frac{r}{\left(1+r^2\right)^2}\, dr\, d\theta = \int_0^{\pi/3} \left[-\frac{1}{2\left(1+r^2\right)}\right]_0^{\sec\theta} d\theta = \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2\left(1+\sec^2\theta\right)}\right] d\theta$
$= \frac{1}{2} \int_0^{\pi/3} \frac{\sec^2\theta}{1+\sec^2\theta}\, d\theta;\ \left[\begin{array}{c} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{array}\right] \rightarrow \frac{1}{2} \int_0^{\sqrt{3}} \frac{du}{2+u^2} = \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}}\right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4} \tan^{-1} \sqrt{\frac{3}{2}}$

(b) $\iint\limits_R \frac{1}{\left(1+x^2+y^2\right)^2}\, dx\, dy = \int_0^{\pi/2}\int_0^{\infty} \frac{r}{\left(1+r^2\right)^2}\, dr\, d\theta = \int_0^{\pi/2} \lim\limits_{b\to\infty}\left[-\frac{1}{2\left(1+r^2\right)}\right]_0^b d\theta = \int_0^{\pi/2} \lim\limits_{b\to\infty}\left[\frac{1}{2}-\frac{1}{2\left(1+b^2\right)}\right] d\theta$

$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

23. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x+y+z)\, dx\, dy\, dz = \int_0^{\pi}\int_0^{\pi}\left[\sin(z+y+\pi)-\sin(z+y)\right] dy\, dz$

$= \int_0^{\pi}\left[-\cos(z+2\pi)+\cos(z+\pi)-\cos z+\cos(z+\pi)\right] dz = 0$

24. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dz\, dy\, dx = \int_{\ln 6}^{\ln 7}\int_0^{\ln 2} e^{(x+y)}\, dy\, dx = \int_{\ln 6}^{\ln 7} e^x\, dx = 1$

25. $\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)\, dz\, dy\, dx = \int_0^1\int_0^{x^2}\left(\frac{3x^2}{2}-\frac{3y^2}{2}\right) dy\, dx = \int_0^1\left(\frac{3x^4}{2}-\frac{x^6}{2}\right) dx = \frac{8}{35}$

26. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\, dy\, dz\, dx = \int_1^e\int_1^x \frac{1}{z}\, dz\, dx = \int_1^e \ln x\, dx = \left[x\ln x - x\right]_1^e = 1$

27. $V = 2\int_0^{\pi/2}\int_{-\cos y}^{0}\int_0^{-2x} dz\, dx\, dy = 2\int_0^{\pi/2}\int_{-\cos y}^{0} (-2x)\, dx\, dy = 2\int_0^{\pi/2}\cos^2 y\, dy = 2\left[\frac{y}{2}+\frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi}{2}$

28. $V = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2} dz\, dy\, dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}\left(4-x^2\right) dy\, dx = 4\int_0^2\left(4-x^2\right)^{3/2} dx$

$= \left[x\left(4-x^2\right)^{3/2} + 6x\sqrt{4-x^2} + 24\sin^{-1}\frac{x}{2}\right]_0^2 = 24\sin^{-1}1 = 12\pi$

29. average $= \frac{1}{3}\int_0^1\int_0^3\int_0^1 30xz\sqrt{x^2+y}\, dz\, dy\, dx = \frac{1}{3}\int_0^1\int_0^3 15x\sqrt{x^2+y}\, dy\, dx = \frac{1}{3}\int_0^3\int_0^1 15x\sqrt{x^2+y}\, dx\, dy$

$= \frac{1}{3}\int_0^3\left[5\left(x^2+y\right)^{3/2}\right]_0^1 dy = \frac{1}{3}\int_0^3\left[5(1+y)^{3/2}-5y^{3/2}\right] dy = \frac{1}{3}\left[2(1+y)^{5/2}-2y^{5/2}\right]_0^3 = \frac{1}{3}\left[2(4)^{5/2}-2(3)^{5/2}-2\right]$

$= \frac{1}{3}\left[2\left(31-3^{5/2}\right)\right]$

30. average $= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3\sin\phi\, d\rho\, d\phi\, d\theta = \frac{3a}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = \frac{3a}{8\pi}\int_0^{2\pi} d\theta = \frac{3a}{4}$

31. (a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\, dz\, dx\, dy$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 3\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(c) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{2}}\left[r\left(4-r^2\right)^{1/2}-r^2\right] dr\, d\theta = 3\int_0^{2\pi}\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}-\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$

$= \int_0^{2\pi}\left(-2^{3/2}-2^{3/2}+4^{3/2}\right) d\theta = \left(8-4\sqrt{2}\right)\int_0^{2\pi} d\theta = 2\pi\left(8-4\sqrt{2}\right)$

32. (a) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2 dz\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\, dz\, r\, dr\, d\theta$

(b) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\, dz\, r\, dr\, d\theta = 84\int_0^{\pi/2}\int_0^1 r^6\sin^2\theta\cos\theta\, dr\, d\theta = 12\int_0^{\pi/2}\sin^2\theta\cos\theta\, d\theta = 4$

33. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}(\sec\phi)(\sec\phi\tan\phi)\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{1}{6}\int_0^{2\pi} d\theta = \frac{\pi}{3}$

34. (a) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}}(6+4y)\, dz\, dy\, dx$ (b) $\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\, dz\, r\, dr\, d\theta$

(c) $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi}(6+4\rho\sin\phi\sin\theta)\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta$

(d) $\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^1\left(6r^2+4r^3\sin\theta\right) dr\, d\theta = \int_0^{\pi/2}\left[2r^3+r^4\sin\theta\right]_0^1 d\theta$

$= \int_0^{\pi/2}(2+\sin\theta)\, d\theta = \left[2\theta-\cos\theta\right]_0^{\pi/2} = \pi+1$

35. $\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx$

36. (a) Bounded on the top and bottom by the sphere $x^2+y^2+z^2=4$, on the right by the right circular cylinder $(x-1)^2+y^2=1$, on the left by the plane $y=0$

(b) $\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

37. (a) $V = \int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2\left(r\sqrt{8-r^2}-2r\right) dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(8-r^2\right)^{3/2}-r^2\right]_0^2 d\theta$

$= \int_0^{2\pi}\left[-\frac{1}{3}(4)^{3/2}-4+\frac{1}{3}(8)^{3/2}\right] d\theta = \int_0^{2\pi}\frac{4}{3}\left(-2-3+2\sqrt{8}\right) d\theta = \frac{4}{3}\left(4\sqrt{2}-5\right)\int_0^{2\pi} d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

(b) $V = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{\sqrt{8}}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\left(2\sqrt{2}\sin\phi-\sec^3\phi\sin\phi\right) d\phi\, d\theta$

$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\left(2\sqrt{2}\sin\phi-\tan\phi\sec^2\phi\right) d\phi\, d\theta = \frac{8}{3}\int_0^{2\pi}\left[-2\sqrt{2}\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{8}{3}\int_0^{2\pi}\left(-2-\frac{1}{2}+2\sqrt{2}\right) d\theta = \frac{8}{3}\int_0^{2\pi}\left(\frac{-5+4\sqrt{2}}{2}\right) d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

38. $I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^2(\rho\sin\phi)^2\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^2\rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{32}{5}\int_0^{2\pi}\int_0^{\pi/3}\left(\sin\phi-\cos^2\phi\sin\phi\right) d\phi\, d\theta = \frac{32}{5}\int_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

39. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi}\int_0^{\pi}\int_a^b \delta(\rho\sin\phi)^2\left(\rho^2\sin\phi\right)d\rho\, d\phi\, d\theta$

$= \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta = \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\int_0^{\pi}\left(\sin\phi-\cos^2\phi\sin\phi\right)d\phi\, d\theta$

$= \frac{\delta\left(b^5-a^5\right)}{5}\int_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi}d\theta = \frac{4\delta\left(b^5-a^5\right)}{15}\int_0^{2\pi}d\theta = \frac{8\pi\delta\left(b^5-a^5\right)}{15}$

40. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}(\rho\sin\phi)^2\left(\rho^2\sin\phi\right)d\rho d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}\rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^5\sin^3\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^6(1+\cos\phi)\sin\phi\, d\phi\, d\theta;\ \begin{bmatrix} u = 1-\cos\phi \\ du = \sin\phi\, d\phi \end{bmatrix}$

$\to \frac{1}{5}\int_0^{2\pi}\int_0^{2}u^6(2-u)\,du\, d\theta = \frac{1}{5}\int_0^{2\pi}\left[\frac{2u^7}{7}-\frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5}\int_0^{2\pi}\left(\frac{1}{7}-\frac{1}{8}\right)2^8\, d\theta = \frac{1}{5}\int_0^{2\pi}\frac{2^3\cdot 2^5}{56}\, d\theta = \frac{32}{35}\int_0^{2\pi}d\theta = \frac{64\pi}{35}$

41. $M = \int_1^2\int_{2/x}^2 dy\, dx = \int_1^2\left(2-\frac{2}{x}\right)dx = 2-\ln 4;\ M_y = \int_1^2\int_{2/x}^2 x\, dy\, dx = \int_1^2 x\left(2-\frac{2}{x}\right)dx = 1;$

$M_x = \int_1^2\int_{2/x}^2 y\, dy\, dx = \int_1^2\left(2-\frac{2}{x^2}\right)dx = 1 \Rightarrow \overline{x} = \overline{y} = \frac{1}{2-\ln 4}$

42. $M = \int_0^4\int_{-2y}^{2y-y^2}dx\, dy = \int_0^4\left(4y-y^2\right)dy = \frac{32}{3};\ M_x = \int_0^4\int_{-2y}^{2y-y^2}y\, dx\, dy = \int_0^4\left(4y^2-y^3\right)dy = \left[\frac{4y^3}{3}-\frac{y^4}{4}\right]_0^4 = \frac{64}{3};$

$M_y = \int_0^4\int_{-2y}^{2y-y^2}x\, dx\, dy = \int_0^4\left[\frac{\left(2y-y^2\right)^2}{2}-2y^2\right]dy = \left[\frac{y^5}{10}-\frac{y^4}{2}\right]_0^4 = -\frac{128}{5} \Rightarrow \overline{x} = \frac{M_y}{M} = -\frac{12}{5}$ and $\overline{y} = \frac{M_x}{M} = 2$

43. $I_o = \int_0^2\int_{2x}^4\left(x^2+y^2\right)(3)\, dy\, dx = 3\int_0^2\left(4x^2+\frac{64}{3}-\frac{14x^3}{3}\right)dx = 104$

44. (a) $I_o = \int_{-2}^2\int_{-1}^1\left(x^2+y^2\right)dy\, dx = \int_{-2}^2\left(2x^2+\frac{2}{3}\right)dx = \frac{40}{3}$

(b) $I_x = \int_{-a}^a\int_{-b}^b y^2 dy\, dx = \int_{-a}^a\frac{2b^3}{3}\, dx = \frac{4ab^3}{3};$

$I_y = \int_{-b}^b\int_{-a}^a x^2 dx\, dy = \int_{-b}^b\frac{2a^3}{3}\, dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y = \frac{4ab^3}{3}+\frac{4a^3b}{3} = \frac{4ab\left(b^2+a^2\right)}{3}$

45. $M = \delta\int_0^3\int_0^{2x/3}dy\, dx = \delta\int_0^3\frac{2x}{3}\, dx = 3\delta;\ I_x = \delta\int_0^3\int_0^{2x/3}y^2 dy\, dx = \frac{8\delta}{81}\int_0^3 x^3 dx = \left(\frac{8\delta}{81}\right)\left(\frac{3^4}{4}\right) = 2\delta$

46. $M = \int_0^1\int_{x^2}^x(x+1)\, dy\, dx = \int_0^1\left(x-x^3\right)dx = \frac{1}{4};\ M_x = \int_0^1\int_{x^2}^x y(x+1)\, dy\, dx = \frac{1}{2}\int_0^1\left(x^3-x^5+x^2-x^4\right)dx = \frac{13}{120};$

$M_y = \int_0^1\int_{x^2}^x x(x+1)\, dy\, dx = \int_0^1\left(x^2-x^4\right)dx = \frac{2}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{13}{30}; I_x = \int_0^1\int_{x^2}^x y^2(x+1)\, dy\, dx$

$= \frac{1}{3}\int_0^1\left(x^4-x^7+x^3-x^6\right)dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}};\ I_y = \int_0^1\int_{x^2}^x x^2(x+1)\, dy\, dx = \int_0^1\left(x^3-x^5\right)dx = \frac{1}{12}$

47. $M=\int_{-1}^{1}\int_{-1}^{1}\left(x^2+y^2+\frac{1}{3}\right)dy\,dx=\int_{-1}^{1}\left(2x^2+\frac{4}{3}\right)dx=4$; $M_x=\int_{-1}^{1}\int_{-1}^{1}y\left(x^2+y^2+\frac{1}{3}\right)dy\,dx=\int_{-1}^{1}0\,dx=0$;
$M_y=\int_{-1}^{1}\int_{-1}^{1}x\left(x^2+y^2+\frac{1}{3}\right)dy\,dx=\int_{-1}^{1}\left(2x^3+\frac{4}{3}x\right)dx=0$

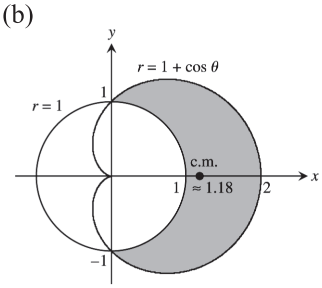
48. Place the $\Delta ABC$ with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points $A$ and $C$ is $y=\frac{h}{a}x$; the line through the points $C$ and $B$ is $y=\frac{h}{a-b}(x-b)$. Thus, $M=\int_0^h\int_{ay/h}^{(a-b)y/h+b}\delta\,dx\,dy$
$=b\delta\int_0^h\left(1-\frac{y}{h}\right)dy=\frac{\delta bh}{2}$; $I_x=\int_0^h\int_{ay/h}^{(a-b)y/h+b}y^2\delta\,dx\,dy=b\delta\int_0^h\left(y^2-\frac{y^3}{h}\right)dy=\frac{\delta bh^3}{12}$

49. $M=\int_{-\pi/3}^{\pi/3}\int_0^3 r\,dr\,d\theta=\frac{9}{2}\int_{-\pi/3}^{\pi/3}d\theta=3\pi$; $M_y=\int_{-\pi/3}^{\pi/3}\int_0^3 r^2\cos\theta\,dr\,d\theta=9\int_{-\pi/3}^{\pi/3}\cos\theta\,d\theta=9\sqrt{3}\Rightarrow\overline{x}=\frac{3\sqrt{3}}{\pi}$, and $\overline{y}=0$ by symmetry

50. $M=\int_0^{\pi/2}\int_1^3 r\,dr\,d\theta=4\int_0^{\pi/2}d\theta=2\pi$; $M_y=\int_0^{\pi/2}\int_1^3 r^2\cos\theta\,dr\,d\theta=\frac{26}{3}\int_0^{\pi/2}\cos\theta\,d\theta=\frac{26}{3}\Rightarrow\overline{x}=\frac{13}{3\pi}$, and $\overline{y}=\frac{13}{3\pi}$ by symmetry

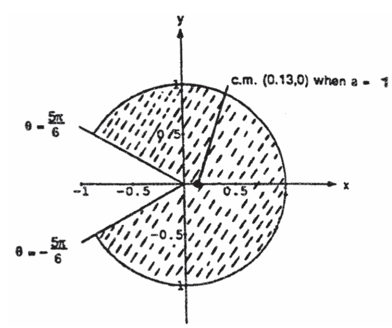
51. (a) $M=2\int_0^{\pi/2}\int_1^{1+\cos\theta}r\,dr\,d\theta$
$=\int_0^{\pi/2}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta=\frac{8+\pi}{4}$;
$M_y=\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}(r\cos\theta)\,r\,dr\,d\theta$
$=\int_{-\pi/2}^{\pi/2}\left(\cos^2\theta+\cos^3\theta+\frac{\cos^4\theta}{3}\right)d\theta$
$=\frac{32+15\pi}{24}\Rightarrow\overline{x}=\frac{15\pi+32}{6\pi+48}$, and $\overline{y}=0$ by symmetry

(b)



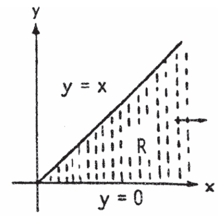

52. (a) $M=\int_{-\alpha}^{\alpha}\int_0^a r\,dr\,d\theta=\int_{-\alpha}^{\alpha}\frac{a^2}{2}\,d\theta=a^2\alpha$;
$M_y=\int_{-\alpha}^{\alpha}\int_0^a(r\cos\theta)\,r\,dr\,d\theta$
$=\int_{-\alpha}^{\alpha}\frac{a^3\cos\theta}{3}\,d\theta=\frac{2a^3\sin\alpha}{3}$
$\Rightarrow\overline{x}=\frac{2a\sin\alpha}{3\alpha}$, and $\overline{y}=0$ by symmetry;
$\lim_{\alpha\to\pi^-}\overline{x}=\lim_{\alpha\to\pi^-}\frac{2a\sin\alpha}{3\alpha}=0$

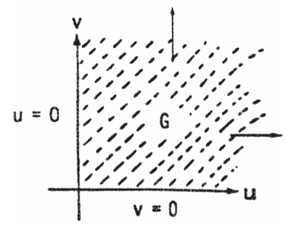
(b) $\overline{x}=\frac{2a}{5\pi}$ and $\overline{y}=0$



53. $x = u + y$ and $y = v \Rightarrow x = u + v$ and $y = v$

$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$; the boundary of the image $G$ is obtained from the boundary of $R$ as follows:




| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $y = x$ | $v = u + v$ | $u = 0$ |
| $y = 0$ | $v = 0$ | $v = 0$ |

$\Rightarrow \int_0^\infty \int_0^x e^{-sx} f(x-y, y)\, dy\, dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v)\, du\, dv$

54. If $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$ where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$, then $x = \frac{\delta s - \beta t}{\alpha\delta - \beta\gamma}$, $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$, and

$J(s,t) = \frac{1}{(\alpha\delta-\beta\gamma)^2} \begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(s^2+t^2\right)} \frac{1}{\sqrt{ac-b^2}}\, ds\, dt = \frac{1}{\sqrt{ac-b^2}} \int_0^{2\pi} \int_0^\infty re^{-r^2}\, dr\, d\theta$

$= \frac{1}{2\sqrt{ac-b^2}} \int_0^{2\pi} d\theta = \frac{\pi}{\sqrt{ac-b^2}}$. Therefore, $\frac{\pi}{\sqrt{ac-b^2}} = 1 \Rightarrow ac - b^2 = \pi^2$.

## CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $V = \int_{-3}^{2} \int_{x}^{6-x^2} x^2 dy\, dx$ (b) $V = \int_{-3}^{2} \int_{x}^{6-x^2} \int_0^{x^2} dz\, dy\, dx$

(c) $V = \int_{-3}^{2} \int_{x}^{6-x^2} x^2 dy\, dx = \int_{-3}^{2} \int_{x}^{6-x^2} \left(6x^2 - x^4 - x^3\right) dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4}\right]_{-3}^{2} = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$.
Then $9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the $xy$-plane

$\Rightarrow V = \int_0^{2\pi} \int_0^4 \int_{-\sqrt{25-r^2}}^{-3} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^4 \left(r\sqrt{25-r^2} - 3r\right) dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(25-r^2\right)^{3/2} - \frac{3}{2}r^2\right]_0^4 d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{26}{3}\, d\theta = \frac{52\pi}{3}$

3. Using cylindrical coordinates, $V = \int_0^{2\pi} \int_0^1 \int_0^{2-r(\cos\theta+\sin\theta)} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \left(2r - r^2\cos\theta - r^2\sin\theta\right) dr\, d\theta$

$= \int_0^{2\pi} \left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta = \left[\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 2\pi$

4. $V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta = 4\int_0^{\pi/2}\int_0^1\left(r\sqrt{2-r^2}-r^3\right)dr\ d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{r^4}{4}\right]_0^1 d\theta$

$= 4\int_0^{\pi/2}\left(-\frac{1}{3}-\frac{1}{4}+\frac{2\sqrt{2}}{3}\right)d\theta = \left(\frac{8\sqrt{2}-7}{3}\right)\int_0^{\pi/2} d\theta = \frac{\pi\left(8\sqrt{2}-7\right)}{6}$

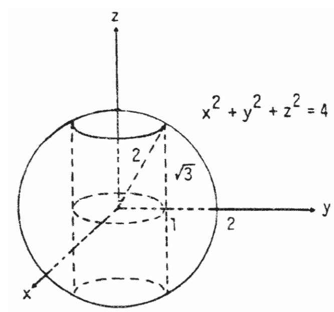
5. The surfaces intersect when $3-x^2-y^2 = 2x^2+2y^2 \Rightarrow x^2+y^2=1$. Thus the volume is

$V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{2x^2+2y^2}^{3-x^2-y^2} dz\ dy\ dx = 4\int_0^{\pi/2}\int_0^1\int_{2r^2}^{3-r^2} dz\ r\ dr\ d\theta = 4\int_0^{\pi/2}\int_0^1\left(3r-3r^3\right)dr\ d\theta = 3\int_0^{\pi/2} d\theta = \frac{3\pi}{2}$

6. $V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi}\rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \frac{64}{3}\int_0^{\pi/2}\int_0^{\pi/2}\sin^4\phi\ d\phi\ d\theta$

$= \frac{64}{3}\int_0^{\pi/2}\left[-\frac{\sin^3\phi\cos\phi}{4}\Big|_0^{\pi/2}+\frac{3}{4}\int_0^{\pi/2}\sin^2\phi\ d\phi\right]d\theta = 16\int_0^{\pi/2}\left[\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right]_0^{\pi/2} d\theta = 4\pi\int_0^{\pi/2} d\theta = 2\pi^2$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.




(b) $V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\ dr\ dz\ d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}\left(3-z^2\right)dz\ d\theta = 2\sqrt{3}\int_0^{2\pi} d\theta = 4\sqrt{3}\pi$

8. $V = \int_0^{\pi}\int_0^{3\sin\theta}\int_0^{\sqrt{9-r^2}} dz\ r\ dr\ d\theta = \int_0^{\pi}\int_0^{3\sin\theta} r\sqrt{9-r^2}\,dr\ d\theta = \int_0^{\pi}\left[-\frac{1}{3}\left(9-r^2\right)^{3/2}\right]_0^{3\sin\theta} d\theta$

$= \int_0^{\pi}\left[-\frac{1}{3}\left(9-9\sin^2\theta\right)^{3/2}+\frac{1}{3}(9)^{3/2}\right]d\theta = 9\int_0^{\pi}\left[1-\left(1-\sin^2\theta\right)^{3/2}\right]d\theta = 9\int_0^{\pi}\left(1-\cos^3\theta\right)d\theta$

$= \int_0^{\pi}\left(1-\cos\theta+\sin^2\theta\cos\theta\right)d\theta = 9\left[\theta-\sin\theta+\frac{\sin^3\theta}{3}\right]_0^{\pi} = 9\pi$

9. The surfaces intersect when $x^2+y^2 = \frac{x^2+y^2+1}{2} \Rightarrow x^2+y^2=1$. Thus the volume in cylindrical coordinates is

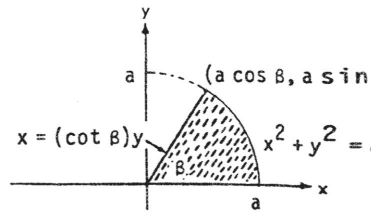
$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\left(r^2+1\right)/2} dz\ r\ dr\ d\theta = 4\int_0^{\pi/2}\int_0^1\left(\frac{r}{2}-\frac{r^3}{2}\right)dr\ d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{4}-\frac{r^4}{8}\right]_0^1 d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

10. $V = \int_0^{\pi/2}\int_1^2\int_0^{r^2\sin\theta\cos\theta} dz\ r\ dr\ d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin\theta\cos\theta\ dr\ d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_1^2\sin\theta\cos\theta\ d\theta$

$= \frac{15}{4}\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{15}{8}$

11. $\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\,dx = \int_0^{\infty}\int_a^b e^{-xy}\,dy\,dx = \int_a^b\int_0^{\infty} e^{-xy}\,dx\,dy = \int_a^b\left(\lim_{t\to\infty}\int_0^t e^{-xy}\,dx\right)dy = \int_a^b \lim_{t\to\infty}\left[-\frac{e^{-xy}}{y}\right]_0^t dy$
$= \int_a^b \lim_{t\to\infty}\left(\frac{1}{y}-\frac{e^{-yt}}{y}\right)dy = \int_a^b \frac{1}{y}\,dy = \left[\ln y\right]_a^b = \ln\left(\frac{b}{a}\right)$

12. (a) The region of integration is sketched at the right




$\Rightarrow \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln\left(x^2+y^2\right)dx\,dy$
$= \int_0^{\beta}\int_0^{a} r\ln\left(r^2\right)dr\,d\theta;$
$\begin{bmatrix} u = r^2 \\ du = 2r\,dr \end{bmatrix} \to \frac{1}{2}\int_0^{\beta}\int_0^{a^2} \ln u\,du\,d\theta$
$= \frac{1}{2}\int_0^{\beta}\left[u\ln u - u\right]_0^{a^2} d\theta = \frac{1}{2}\int_0^{\beta}\left[2a^2\ln a - a^2 - \lim_{t\to 0} t\ln t\right]d\theta = \frac{a^2}{2}\int_0^{\beta}(2\ln a - 1)\,d\theta = a^2\beta\left(\ln a - \frac{1}{2}\right)$

(b) $\int_0^{a\cos\beta}\int_0^{(\tan\beta)x} \ln\left(x^2+y^2\right)dy\,dx + \int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}} \ln\left(x^2+y^2\right)dy\,dx$

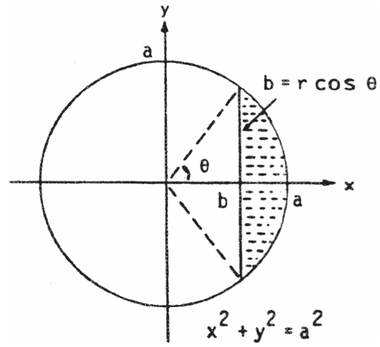
13. $\int_0^x\int_0^u e^{m(x-t)}f(t)\,dt\,du = \int_0^x\int_t^x e^{m(x-t)}f(t)\,du\,dt = \int_0^x (x-t)e^{m(x-t)}f(t)\,dt;$ also
$\int_0^x\int_0^v\int_0^u e^{m(x-t)}f(t)\,dt\,du\,dv = \int_0^x\int_t^x\int_t^v e^{m(x-t)}f(t)\,du\,dv\,dt = \int_0^x\int_t^x (v-t)e^{m(x-t)}f(t)\,dv\,dt$
$= \int_0^x\left[\frac{1}{2}(v-t)^2 e^{m(x-t)}f(t)\right]_t^x dt = \int_0^x \frac{(x-t)^2}{2}e^{m(x-t)}f(t)\,dt$

14. $\int_0^1 f(x)\left(\int_0^x g(x-y)f(y)\,dy\right)dx = \int_0^1\int_0^x g(x-y)f(x)f(y)\,dy\,dx = \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy$
$= \int_0^1 f(y)\left(\int_y^1 g(x-y)f(x)\,dx\right)dy;$
$\int_0^1\int_0^1 g(|x-y|)\,f(x)f(y)\,dx\,dy = \int_0^1\int_0^x g(x-y)f(x)f(y)\,dy\,dx + \int_0^1\int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \int_0^1\int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \underbrace{\int_0^1\int_y^1 g(x-y)f(y)f(x)\,dx\,dy}_{\text{simply interchange } x \text{ and } y \text{ variable names}} = 2\int_0^1\int_y^1 g(x-y)f(x)f(y)\,dx\,dy,$
and the statement now follows.

15. $I_o(a) = \int_0^a\int_0^{x/a^2}\left(x^2+y^2\right)dy\,dx = \int_0^a\left[x^2y+\frac{y^3}{3}\right]_0^{x/a^2}dx = \int_0^a\left(\frac{x^3}{a^2}+\frac{x^3}{3a^6}\right)dx = \left[\frac{x^4}{4a^2}+\frac{x^4}{12a^6}\right]_0^a = \frac{a^2}{4}+\frac{1}{12}a^{-2};$
$I_o'(a) = \frac{1}{2}a - \frac{1}{6}a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}.$ Since $I_o''(a) = \frac{1}{2}+\frac{1}{2}a^{-4} > 0,$ the value of $a$ does provide a <u>minimum</u> for the polar moment of inertia $I_o(a)$.

16. $I_o = \int_0^2\int_{2x}^4\left(x^2+y^2\right)(3)\,dy\,dx = 3\int_0^2\left(4x^2-\frac{14x^3}{3}+\frac{64}{3}\right)dx = 104$

17. $M = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta}\left(\frac{a^2}{2}-\frac{b^2}{2}\sec^2\theta\right)d\theta$
$= a^2\theta - b^2\tan\theta = a^2\cos^{-1}\left(\frac{b}{a}\right) - b^2\left(\frac{\sqrt{a^2-b^2}}{b}\right)$
$= a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2};$
$I_o = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r^3\,dr\,d\theta = \frac{1}{4}\int_{-\theta}^{\theta}\left(a^4+b^4\sec^4\theta\right)d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta}\left[a^4+b^4\left(1+\tan^2\theta\right)\left(\sec^2\theta\right)\right]d\theta$
$= \frac{1}{4}\left[a^4\theta - b^4\tan\theta - \frac{b^4\tan^3\theta}{3}\right]_{-\theta}^{\theta} = \frac{a^4\theta}{2} - \frac{b^4\tan\theta}{2} - \frac{b^4\tan^3\theta}{6} = \frac{1}{2}a^4\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}b^3\sqrt{a^2-b^2} - \frac{1}{6}b^3\left(a^2-b^2\right)^{3/2}$




18. $M = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} dx\,dy = \int_{-2}^{2}\left(1-\frac{y^2}{4}\right)dy = \left[y - \frac{y^3}{12}\right]_{-2}^{2} = \frac{8}{3};\; M_y = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} x\,dx\,dy$
$= \int_{-2}^{2}\left[\frac{x^2}{2}\right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^{2}\frac{3}{32}\left(4-y^2\right)dy = \frac{3}{32}\int_{-2}^{2}\left(16-8y^2+y^4\right)dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$
$= \frac{3}{16}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \left(\frac{3}{16}\right)\left(\frac{32\cdot 8}{15}\right) = \frac{48}{15} = \frac{3}{32}\int_{-2}^{2}\left(16-8y^2+y^4\right)dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$ and $\overline{y} = 0$ by symmetry

19. $= \left[\frac{1}{2ab}e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba}e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{b^2a^2}-1\right) + \frac{1}{2ab}\left(e^{a^2b^2}-1\right) = \frac{1}{ab}\left(e^{a^2b^2}-1\right)$

20. $\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{\partial^2 F(x,y)}{\partial x\,\partial y}\,dx\,dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right]dx = \left[F(x_1,y) - F(x_0,y)\right]_{y_0}^{y_1}$
$= F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$

21. (a) (i) Fubini's Theorem
(ii) Treating $G(y)$ as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

(b) $\int_0^{\ln 2}\int_0^{\pi/2} e^x\cos y\,dy\,dx = \left(\int_0^{\ln 2} e^x dx\right)\left(\int_0^{\pi/2}\cos y\,dy\right) = \left(e^{\ln 2} - e^0\right)\left(\sin\frac{\pi}{2} - \sin 0\right) = (1)(1) = 1$

(c) $\int_1^2\int_{-1}^1 \frac{x}{y^2}\,dx\,dy = \left(\int_1^2 \frac{1}{y^2}\,dy\right)\left(\int_{-1}^1 x\,dx\right) = \left[-\frac{1}{y}\right]_{-1}^{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = \left(-\frac{1}{2}+1\right)\left(\frac{1}{2}-\frac{1}{2}\right) = 0$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1x + u_2y;$ the area of the region of integration is $\frac{1}{2}$
$\Rightarrow$ average $= 2\int_0^1\int_0^{1-x}\left(u_1x+u_2y\right)dy\,dx = 2\int_0^1\left[u_1x(1-x) + \frac{1}{2}u_2(1-x)^2\right]dx$
$= 2\left[u_1\left(\frac{x^2}{2}-\frac{x^3}{3}\right) - \left(\frac{1}{2}u_2\right)\frac{(1-x)^3}{3}\right]_0^1 = 2\left(\frac{1}{6}u_1 + \frac{1}{6}u_2\right) = \frac{1}{3}\left(u_1+u_2\right)$

(b) average $= \frac{1}{\text{area}}\iint_R \left(u_1x+u_2y\right)dA = \frac{u_1}{\text{area}}\iint_R x\,dA + \frac{u_2}{\text{area}}\iint_R y\,dA = u_1\left(\frac{M_y}{M}\right) + u_2\left(\frac{M_x}{M}\right) = u_1\overline{x} + u_2\overline{y}$

23. (a) $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_0^{\pi/2} \int_0^\infty \left(e^{-r^2}\right) r\, dr\, d\theta = \int_0^{\pi/2} \left[\lim_{b\to\infty} \int_0^b re^{-r^2} dr\right] d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \lim_{b\to\infty} \left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty \left(y^2\right)^{-1/2} e^{-y^2} (2y)\, dy = 2\int_0^\infty e^{-y^2} dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$, where $y = \sqrt{t}$

24. $Q = \int_0^{2\pi} \int_0^R kr^2(1-\sin\theta)\, dr\, d\theta = \frac{kR^3}{3}\int_0^{2\pi} (1-\sin\theta)\, d\theta = \frac{kR^3}{3}\left[\theta + \cos\theta\right]_0^{2\pi} = \frac{2\pi kR^3}{3}$

25. For a height $h$ in the bowl the volume of water is $V = \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{x^2+y^2}^{h} dz\, dy\, dx$

$= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \left(h - x^2 - y^2\right) dy\, dx = \int_0^{2\pi} \int_0^{\sqrt{h}} \left(h - r^2\right) r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4}\, d\theta = \frac{h^2\pi}{2}.$

Since the top of the bowl has area $10\pi$, then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is $10\pi$ from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth $w$ then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.

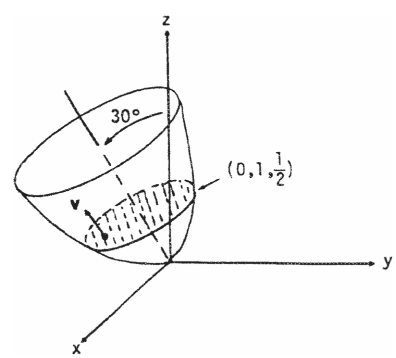
26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted $30°$, a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies

$b = \mathbf{v}\cdot\mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow a = -\sqrt{1-b^2} = -\frac{1}{2}$

$\Rightarrow \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$

$\Rightarrow -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) = 0$




$\Rightarrow z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$ is an equation of the plane of the water level. Therefore the volume of water is

$V = \iint_R \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} dz\, dy\, dx$, where $R$ is the interior of the ellipse $x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 + \frac{2}{\sqrt{3}} = 0$. When $x = 0$,

then $y = \alpha$ or $y = \beta$, $\alpha = \frac{\frac{2}{\sqrt{3}} + \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$ and $\beta = \frac{\frac{2}{\sqrt{3}} - \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$

$\Rightarrow V = \int_\alpha^\beta \int_{-\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}}^{\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}} \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} 1\, dz\, dx\, dy$

(b) $x = 0 \Rightarrow z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \Rightarrow \frac{dz}{dy} = 1 \Rightarrow$ the tangent line has slope 1 or a $45°$ slant $\Rightarrow$ at $45°$ and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2+y^2=1$ from $z=1$ to $\infty \Rightarrow \iiint\limits_D z\left(r^2+z^2\right)^{-5/2}\,dV$

$$=\int_0^{2x}\int_0^1\int_1^\infty \frac{z}{\left(r^2+z^2\right)^{5/2}}\,dz\,r\,dr\,d\theta=\lim_{a\to\infty}\int_0^{2\pi}\int_0^1\int_1^a \frac{r\,z}{\left(r^2+z^2\right)^{5/2}}\,dz\,dr\,d\theta=\lim_{a\to\infty}\int_0^{2\pi}\int_0^1\left[\left(-\tfrac{1}{3}\right)\frac{r}{\left(r^2+z^2\right)^{3/2}}\right]_1^a dr\,d\theta$$

$$=\lim_{a\to\infty}\int_0^{2\pi}\int_0^1\left[\left(-\tfrac{1}{3}\right)\frac{r}{\left(r^2+a^2\right)^{3/2}}+\left(\tfrac{1}{3}\right)\frac{r}{\left(r^2+1\right)^{3/2}}\right]dr\,d\theta=\lim_{a\to\infty}\int_0^{2\pi}\left[\tfrac{1}{3}\left(r^2+a^2\right)^{-1/2}-\tfrac{1}{3}\left(r^2+1\right)^{-1/2}\right]_0^1 d\theta$$

$$=\lim_{a\to\infty}\int_0^{2\pi}\left[\tfrac{1}{3}\left(1+a^2\right)^{-1/2}-\tfrac{1}{3}\left(2^{-1/2}\right)-\tfrac{1}{3}\left(a^2\right)^{-1/2}+\tfrac{1}{3}\right]d\theta=\lim_{a\to\infty}2\pi\left[\tfrac{1}{3}\left(1+a^2\right)^{-1/2}-\tfrac{1}{3}\left(\tfrac{\sqrt{2}}{2}\right)-\tfrac{1}{3}\left(\tfrac{1}{a}\right)+\tfrac{1}{3}\right]$$

$$=2\pi\left[\tfrac{1}{3}-\left(\tfrac{1}{3}\right)\tfrac{\sqrt{2}}{2}\right].$$

28. Let's see?

The length of the "unit" line segment is: $L=2\int_0^1 dx=2.$

The area of the unit circle is : $A=4\int_0^1\int_0^{\sqrt{1-x^2}}dy\,dx=\pi.$

The volume of the unit sphere is : $V=8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}dz\,dy\,dx=\tfrac{4}{3}\pi.$

Therefore, the hypervolume of the unit 4-sphere should be:

$V_{\text{hyper}}=16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}}dw\,dz\,dy\,dx.$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$V_{\text{hyper}}=16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_a^{\sqrt{1-x^2-y^2-z^2}}dw\,dz\,dy\,dx=16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{1-x^2-y^2-z^2}\,dz\,dy\,dx$$

$$=16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{1-x^2-y^2}\sqrt{1-\frac{z^2}{1-x^2-y^2}}\,dz\,dy\,dx;\quad\left[\begin{array}{l}\frac{z}{\sqrt{1-x^2-y^2}}=\cos\theta\\ dz=-\sqrt{1-x^2-y^2}\sin\theta\,d\theta\end{array}\right]$$

$$=16\int_0^1\int_0^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)\int_{\pi/2}^0\left(-\sqrt{1-\cos^2\theta}\sin\theta\right)d\theta\,dy\,dx=16\int_0^1\int_0^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)\int_{\pi/2}^0\left(-\sin^2\theta\right)d\theta\,dy\,dx$$

$$=16\int_0^1\int_0^{\sqrt{1-x^2}}\tfrac{\pi}{4}\left(1-x^2-y^2\right)dy\,dx=4\pi\int_0^1\left(\sqrt{1-x^2}-x^2\sqrt{1-x^2}-\tfrac{1}{3}\left(1-x^2\right)^{3/2}\right)dx$$

$$=4\pi\int_0^1\sqrt{1-x^2}\left[\left(1-x^2\right)-\tfrac{1-x^3}{3}\right]dx=\tfrac{8}{3}\pi\int_0^1\left(1-x^2\right)^{3/2}dx;\quad\left[\begin{array}{c}x=\cos\theta\\ dx=-\sin\theta\,d\theta\end{array}\right]$$

$$=-\tfrac{8}{3}\pi\int_{\pi/2}^0\sin^4\theta\,d\theta=-\tfrac{8}{3}\pi\int_{\pi/2}^0\left(\tfrac{1-\cos 2\theta}{2}\right)^2d\theta=-\tfrac{2}{3}\pi\int_{\pi/2}^0\left(1-2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$=-\tfrac{2}{3}\pi\int_{\pi/2}^0\left(\tfrac{3}{2}-2\cos 2\theta+\tfrac{\cos 4\theta}{2}\right)d\theta=\tfrac{\pi^2}{2}$$