STK3100 Exercises, Week 13

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Exercise 9.1

Let $\text{cor}\left(Y_{i1}^{A},Y_{i2}^{A}\right)=\text{cor}\left(Y_{i1}^{B},Y_{i2}^{B}\right)=\rho$, all other correlations equal to zero, and $\text{Var}Y_{i}^{A}=\text{Var}Y_{i}^{B}=\sigma^{2}$ for all i. Define

$$\begin{aligned}
B &= \frac{1}{2}\left[\left(\overline{Y}_{1}^{A}+\overline{Y}_{2}^{A}\right)-\left(\overline{Y}_{1}^{B}+\overline{Y}_{2}^{B}\right)\right]\\
W &= \frac{1}{2}\left[\left(\overline{Y}_{1}^{A}-\overline{Y}_{2}^{A}\right)+\left(\overline{Y}_{1}^{B}-\overline{Y}_{2}^{B}\right)\right]
\end{aligned}$$

Then

$$\begin{aligned}
\text{Var}B &= \frac{1}{4}\text{Var}\left(\overline{Y}_{1}^{A}+\overline{Y}_{2}^{A}\right)+\frac{1}{4}\text{Var}\left(\overline{Y}_{1}^{B}+\overline{Y}_{2}^{B}\right)\\
\text{Var}W &= \frac{1}{4}\text{Var}\left(\overline{Y}_{1}^{A}-\overline{Y}_{2}^{A}\right)+\frac{1}{4}\text{Var}\left(\overline{Y}_{1}^{B}-\overline{Y}_{2}^{B}\right)
\end{aligned}$$

Since $\text{Var}\left(\overline{Y}_{1}^{A}\pm\overline{Y}_{2}^{A}\right)=\text{Var}\left(\overline{Y}_{1}^{B}\pm\overline{Y}_{2}^{B}\right)=2n^{-1}\sigma^{2}\left(1\pm\rho\right)$, $\text{Var}B=n^{-1}\sigma^{2}\left(1+\rho\right)$ and $\text{Var}W=n^{-1}\sigma^{2}\left(1-\rho\right)$ as claimed.

Exercise 9.22

The mixed effect probit model can be written using two normal latent variables u and w:

$$\begin{aligned}
p\left(u\right) &= \phi\left(u;0,\Sigma\right)\\
p\left(w\mid u\right) &= \phi\left(w;\beta^{T}x+z^{T}u,1\right)\\
p\left(y\mid w\right) &= 1_{[w\geq0]}
\end{aligned}$$

The marginal likelihood $p\left(y\right)$ is found by integrating out u and w. First we find $p\left(w\right)=\int p\left(w\mid u\right)p\left(u\right)du$. To do this, we will only require $p\left(z^{T}u\right)=\phi\left(z^{T}u,0,z^{T}\Sigma z\right)$. Then $p\left(w\right)=\int\phi\left(u;\beta^{T}x+v,1\right)\phi\left(v,0,z^{T}\Sigma z\right)dv$. Now we show that this equals $\phi\left(w;\beta^{T}x,1+z^{T}\Sigma z\right)$. To this end, let $\mu=\beta^{T}x$ and $\sigma^{2}=z^{T}\Sigma z$. Then

$$
\begin{aligned}
\phi(w;\mu+v,1)\,\phi\left(v,0,\sigma^2\right) &\overset{v}{\propto} \\
\exp\left[\frac{-\left(\sigma^2+1\right)v^2+2\sigma^2(u-\mu)v+\sigma^2(\mu-w)^2}{2\sigma^2}\right] \\
\exp\left[\frac{-v^2+2\sigma^2(w-\mu)/\left(\sigma^2+1\right)v-\sigma^2(\mu-w)^2/\left(\sigma^2+1\right)}{2\sigma^2/\left(\sigma^2+1\right)}\right]
\end{aligned}
$$

Notice that

$$
A=\exp\left[\frac{-v^2+2\sigma^2(w-\mu)/\left(\sigma^2+1\right)v-\sigma^4(\mu-w)^2/\left(\sigma^2+1\right)^2}{2\sigma^2/\left(\sigma^2+1\right)}\right]
$$

is proportional to a normal in v, so its integral is 1. (Notice the difference between A and the last line of the equation stack!)

We will take out the term $-\frac{(\mu-w)^2}{2}$ and add the term $-\frac{\sigma^2(\mu-w)^2}{2\left(\sigma^2+1\right)}$. The residual becomes

$$
\begin{aligned}
\frac{-(\mu-w)^2\left(\sigma^2+1\right)+\sigma^2(\mu-w)^2}{2\left(\sigma^2+1\right)} &= \\
\frac{-(\mu-w)^2}{2\left(\sigma^2+1\right)} &=
\end{aligned}
$$

Taking the exponential of this we recover the wished for normal in u.

Finally,

$$
\begin{aligned}
p(y\mid x) &= \int p(y\mid w)\,p(w)\,du \\
&= \int_0^\infty \phi\left(w;\beta^T x,1+z^T\Sigma z\right)du \\
&= 1-\Phi\left(0;\beta^T x,1+z^T\Sigma z\right) \\
&= 1-\Phi\left(-\frac{\beta^T x}{\sqrt{1+z^T\Sigma z}}\right) \\
&= \Phi\left(\frac{\beta^T x}{\sqrt{1+z^T\Sigma z}}\right)
\end{aligned}
$$

Exercise 9.24

We want to show that

$$
\operatorname{Cov}\left(Y_{ij},Y_{ik}\right)=\exp\left[\left(x_{ij}+x_{ik}\right)\beta\right]\left[\exp\left(\sigma_u^2\right)\left(\exp\left(\sigma_u^2\right)-1\right)\right]
$$

for the Poisson GLM with a random intercept.

Recall the definition of the Poisson GLM with random intercept.

$$
E\left(Y_{ij}\mid u_i\right)=\exp\left(x_{ij}\beta+u_i\right).
$$

From this it follows that

$$
\begin{aligned}
E\left(Y_{ij}Y_{ik}\mid u_i\right) &= E\left[\exp\left(x_{ij}\beta+u_i\right)\exp\left(x_{ik}\beta+u_i\right)\right] \\
&= E\left\{\exp\left[\left(x_{ij}+x_{ik}\right)\beta+4u_i\right]\right\} \\
&= \exp\left(\left(x_{ij}+x_{ik}\right)\beta+2\sigma_u^2\right)
\end{aligned}
$$

On the other hand

$$\begin{aligned} E\left(Y_{ij} \mid u_i\right) &= E\left[\exp\left(x_{ij}\beta + u_i\right)\right] \\ &= \exp\left(x_{ij}\beta + \frac{1}{2}\sigma_u^2\right) \end{aligned}$$

Hence

$$\begin{aligned} E\left(Y_{ij}Y_{ik} \mid u_i\right) - E\left(Y_{ij} \mid u_i\right) E\left(Y_{ik} \mid u_i\right) &= \\ \exp\left(\left(x_{ij} + x_{ik}\right)\beta + 2\sigma_u^2\right) - \exp\left(\left(x_{ij} + x_{ik}\right)\beta + \sigma_u^2\right) &= \\ \exp\left[\left(x_{ij} + x_{ik}\right)\beta\right]\left\{\exp\left(\sigma_u^2\right)\left[\exp\left(\sigma_u^2\right) - 1\right]\right\} \end{aligned}$$

Since none of the factors are ever zero, the correlation is positive.

Exercise 9.29

The book did not define V_i, at least not very clearly. This forces us to guess what it is. I guess it is $\frac{1}{\psi}R(\alpha)$ for the relevant section of $R(\alpha)$, which is I nonetheless.

Recall that $\frac{\partial \mu_{ij}}{\partial \beta_k} = \frac{\partial \mu_{ij}}{\partial \eta_{ij}}\frac{\partial \eta_{ij}}{\partial \beta_k} = \frac{\partial \mu_{ij}}{\partial \eta_{ij}}x_{ijk}$. Hence

$$\begin{aligned} D_{ijk} &= \partial \mu_{ij}/\partial \beta_k \\ &= x_{ijk}\frac{\partial \mu_{ij}}{\partial \eta_{ij}} \\ &= \left[\Delta_i X_i\right]_{jk} \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n} \boldsymbol{D_i^T V_i^{-1}}\left(\boldsymbol{y_i} - \boldsymbol{\mu_i}\right) &= 0 \\ &\Updownarrow \\ \frac{1}{\psi}\sum_{i=1}^{n} \boldsymbol{X_i^T \Delta_i}\left(\boldsymbol{y_i} - \boldsymbol{\mu_i}\right) &= 0 \end{aligned}$$

This is the same as equation (4.11) in chapter 4, with $V^{-1} = \frac{1}{\psi}I$.

Exam 2016, problem 3

a)

The ith observation vector is

$$Y_i = X_i\beta + Z_i b + \epsilon,$$

where $X_i = \begin{bmatrix} X_{i1} & X_{i2} & \cdots & X_{ik} \end{bmatrix}$ are the fixed effects covariate vectors belonging to the ith group and $Z_i = \begin{bmatrix} Z_{i1} & Z_{i2} & \cdots & Z_{ik} \end{bmatrix}$ the random effects covariate vector. The vector β are the fixed effects, while $b \sim N(0, \Sigma)$ are the random effects. The residual $\epsilon_i \sim N\left(0, \sigma^2 I\right)$ are uncorrelated residual variances that cannot be explained either by the fixed or random effects.

b)

The distribution of Y_i is normal with mean $\beta_0 + X_i\beta_1$ and variance $\psi X_i^2 + \sigma^2$. It's not possible to calculate these from the output, as we do not know X_i.

c)

Since $Y_{ij} - X_{ij}\beta_1 \mid X_{ij}b_i \sim N\left(X_{ij}b_i, \sigma^2 I\right)$ and $b \sim N\left(0, \psi^2\right)$, we can find the distribution of b by Bayes' rule. The posterior of $b_i \mid Y_i$ is normal with mean $\overline{Y_{ij} - X_{ij}\beta}$ and variance ψ^2/n. To estimate them we can use the signed residuals.