UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

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This problem set consists of 9 pages.

Appendices: None

Permitted aids: Accepted calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Let $B(t,T)$ denote the price at time t of a a zero coupon bond with maturity time T. The return of the bond over a period $[s,t]$ with $0 \leq s < t \leq T$ is given by the formula

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)}.$$

Moreover the formula for the price of the bond is given by $B(t,T) = e^{-r(T-t)}$. Hence, we get

$$R(s,t) = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}} = e^{r(t-s)} - 1.$$

and

$$r = \frac{\log(1 + R(s,t))}{t-s}.$$

In this problem we have $t - s = 1/4$ and $R(s,t) = 0.04$, which yields

$$r = \frac{\log(1+0.04)}{1/4} = 4\log(1.04) \approx 0.1569 = 15.69\%.$$

b (weight 10p)

Suppose that

$$C^E(0) - P^E(0) > S(0) - Ke^{-rT}.$$

At time 0

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- Buy one share for $S(0)$.
- Buy one put option for $P^E(0)$.
- Write and sell one call option for $C^E(0)$.
- Invest/borrow the amount $C^E(0) - P^E(0) - S(0)$, depending on the sign, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting (or paying) the amount
$$\left(C^E(0) - P^E(0) - S(0)\right) e^{rT}.$$
- Sell the share for K, either by:
  - exercising the put option if $S(T) \le K$
  - settling the short position in the call option if $S(T) > K$.

This will give a total profit of

$$\left(C^E(0) - P^E(0) - S(0)\right) e^{rT} + K > 0,$$

which is positive by assumption. Hence, we have a sure risk-less profit.

c (weight 10p)

In this strategy you buy a call option and a put option with the same strike K and the same expiry time T. The profit of the straddle as a function of the final price of the stock S_T is given by

$$P(S_T) = (S_T - K)^+ + (K - S_T)^+ - C^E(0) - P^E(0).$$

In this case, the table of profits is given by

S_T	Profit
$S_T < K$	$K - S_T - C^E(0) - P^E(0)$
$S_T > K$	$S_T - K - C^E(0) - P^E(0)$

Problem 2

Let B denote the price process for the bank account. We have that $B(0) = 1$ and B(1) $= 1 + r = \frac{11}{10}$. The discounted price process for the risky asset is given by

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$S_1^*(0) = S_1(0)/B(0) = 5$ and $S_1^*(1) = S_1(1)/B(1) = (6,4,3)^T$. A risk neutral probability measure $Q{=}(Q_1,Q_2,Q_3)^T$ must satisfy the following system of equations

$$\begin{aligned} 5 &= S_1^*(0) = \mathbb{E}_Q\left[S_1^*(1)\right] = 6Q_1 + 4Q_2 + 3Q_3, \qquad (1) \\ 1 &= Q_1 + Q_2 + Q_3. \end{aligned}$$

We have that $Q_1 = 1 - Q_2 - Q_3$ and substituting this value in equation (1) we obtain

$$1 = 2Q_2 + 3Q_3 \iff Q_2 = \frac{1-3Q_3}{2}.$$

Moreover,

$$Q_1 = 1 - \frac{1-3Q_3}{2} - Q_3 = \frac{2-1+3Q_3-2Q_3}{2} = \frac{1+Q_3}{2}.$$

Hence, setting $Q_3 = \lambda$, we get $Q_\lambda = \left(\frac{1+\lambda}{2}, \frac{1-3\lambda}{2}, \lambda\right)^T$. Finally, as $Q_1 > 0, Q_2 > 0$, and $Q_3 > 0$ we have the following conditions on the parameter λ

$$\begin{aligned} Q_1 &= \frac{1+\lambda}{2} > 0 \iff \lambda > -1, \\ Q_2 &= \frac{1-3\lambda}{2} > 0 \iff \lambda < \frac{1}{3}, \\ Q_3 &= \lambda > 0, \end{aligned}$$

which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. In addition, by the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market.

a (weight 10p)

A contingent claim $X = (X_1, X_2, X_3)^T$ is attainable if there exists a portfolio $H = (H_0, H_1)^T$such that $X = H_0 B(1) + H_1 S_1(1)$. This translates to the following system of equations

$$\begin{aligned} X_1 &= \frac{11}{10}H_0 + \frac{33}{5}H_1, \\ X_2 &= \frac{11}{10}H_0 + \frac{22}{5}H_1, \\ X_3 &= \frac{11}{10}H_0 + \frac{33}{10}H_1. \end{aligned}$$

From the first equation we get that $\frac{11}{10}H_0 = X_1 - \frac{33}{5}H_1$. Substituting in the second and third equations we obtain

$$\begin{aligned} X_2 &= X_1 - \frac{11}{5}H_1, \\ X_3 &= X_1 - \frac{33}{10}H_1. \end{aligned}$$

(Continued on page 4.)

From the first equation we get that $H_1 = \frac{5}{11}(X_1 - X_2)$ and substituting in the third equation we finally get

$$X_3 = X_1 - \frac{33}{10}\frac{5}{11}(X_1 - X_2) \Longleftrightarrow X_1 - 3X_2 + 2X_3 = 0.$$

b (weight 10p)

The contingent claim $X = (1,5,2)^T$ is not attainable because

$$X_1 - 3X_2 + 2X_3 = 1 - 15 + 4 \neq 0.$$

Hence, there is an interval of arbitrage free prices $[V_-(X), V_+(X)]$, where $V_-(X)$ is the lower hedging price of X and $V_+(X)$ is the upper hedging price of X. Moreover, we know that

$$V_-(X) = \inf_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right]\right\},$$

and

$$V_+(X) = \sup_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right]\right\}.$$

We have that

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \frac{10}{11}\mathbb{E}_{Q_\lambda}[X] = \frac{10}{11}\left\{1\frac{1+\lambda}{2} + 5\frac{1-3\lambda}{2} + 2\lambda\right\} = \frac{5}{11}(6-10\lambda).$$

The previous computation yields

$$\begin{aligned}V_-(X) &= \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\frac{5}{11}(6-10\lambda)\right\} = \frac{5}{11}\left(6-10\frac{1}{3}\right) = \frac{5}{11}\frac{8}{3} = \frac{40}{33},\\ V_+(X) &= \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\frac{5}{11}(6-10\lambda)\right\} = \frac{5}{11}6 = \frac{30}{11}.\end{aligned}$$

Problem 3

a (weight 10p)

We first compute the partitions associated to $S_1(0)$, $S_1(1)$ and $S_1(2)$. We have

$$\begin{aligned}\pi_{S_1(0)} &= \{S_1(0) = 3\} = \{\Omega\},\\ \pi_{S_1(1)} &= \{\{S_1(1) = 2\}, \{S_1(1) = 4\}\} = \{\{\omega_2,\omega_4\}, \{\omega_1,\omega_3\}\} =: \{A_{1,1}, A_{1,2}\},\\ \pi_{S_1(2)} &= \{\{S_1(2) = 1\}, \{S_1(2) = 3\}, \{S_1(2) = 7\}\} = \{\{\omega_4\}, \{\omega_2,\omega_3\}, \{\omega_1\}\} =: \{A_{2,1}, A_{2,2}, A_{2,3}\}.\end{aligned}$$

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The partitions associated to $(S(0), S(1))$ and to $(S(0), S(1), S(2))$ are given by

$$\begin{aligned}
\pi_{(S_1(0),S_1(1))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} = \{\Omega \cap A_{1,1}, \Omega \cap A_{1,1}\} = \{A_{1,1}, A_{1,2}\}, \\
\pi_{(S_1(0),S_1(1),S_1(2))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} \cap \pi_{S_1(2)} = \pi_{S_1(0),S_1(1)} \cap \pi_{S_1(2)} \\
&= \{A_{1,1} \cap A_{2,1}, A_{1,1} \cap A_{2,2}, A_{1,1} \cap A_{2,3}, A_{1,2} \cap A_{2,1}, A_{1,2} \cap A_{2,2}, A_{1,2} \cap A_{2,3}\} \\
&= \{\{\omega_4\}, \{\omega_2\}, \emptyset, \emptyset, \{\omega_3\}, \{\omega_1\}\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.
\end{aligned}$$

The filtrations are given by

$$\begin{aligned}
\mathcal{F}_0 &= \sigma(S_1(0)) = \sigma(\{\Omega\}) = \{\emptyset, \Omega\}, \\
\mathcal{F}_1 &= \sigma(S_1(0), S_1(1)) = \sigma(\{A_{1,1}, A_{1,2}\}) = \{\emptyset, \Omega, A_{1,1}, A_{1,2}\}, \\
\mathcal{F}_2 &= \sigma(S_1(0), S_1(1), S_1(2)) = \sigma(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),
\end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

b (weight 20p)

Since $M = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup,$M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_W \mathbb{E}[U(W)]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$,$U(u) = 2u^{1/2}$,

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$P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)^T$,we have that

$$\begin{aligned} i &= U'(u) = u^{-1/2} \Longleftrightarrow I(i) = u = i^{-2}, \\ L &= \left(\frac{\frac{1}{8}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}}, \frac{\frac{3}{8}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}}\right)^T = \left(\frac{1}{2}, 1, \frac{3}{2}, 1\right)^T, \\ B(2) &= 1, \end{aligned}$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-2}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-2}\right] = \left(\widehat{\lambda}\right)^{-2} \mathbb{E}_Q\left[L^{-2}\right].$$

Therefore, we get

$$\widehat{\lambda} = \left(\frac{\mathbb{E}_Q\left[L^{-2}\right]}{v}\right)^{1/2}, \qquad \widehat{W} = v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]},$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[2\left(v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]}\right)^{1/2}\right] = 2v^{1/2}\frac{\mathbb{E}\left[L^{-1}\right]}{\mathbb{E}_Q\left[L^{-2}\right]^{1/2}} = 2v^{1/2}\mathbb{E}_Q\left[L^{-2}\right]^{1/2},$$

where we have used that $\mathbb{E}\left[L^{-1}\right] = \mathbb{E}\left[LL^{-2}\right] = \mathbb{E}_Q\left[L^{-2}\right]$. Moreover, computing $L^{-2} = \left(4, 1, \frac{4}{9}, 1\right)^T$ and

$$\mathbb{E}_Q\left[L^{-2}\right] = 4\frac{1}{8} + 1\frac{1}{4} + \frac{4}{9}\frac{3}{8} + 1\frac{1}{4} = \frac{7}{6},$$

we obtain $\mathbb{E}\left[U\left(\widehat{W}\right)\right] = 2\left(\frac{7}{6}v\right)^{1/2}$ and $\widehat{W} = \left(\frac{24}{7}v, \frac{6}{7}v, \frac{8}{21}v, \frac{6}{7}v\right)^T$.

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process V satisfies V(2) $= \widehat{W}$. We first compute the discounted increments of the risky asset

$$\begin{aligned} \Delta S_1^*(2) &= \Delta S_1(2) = (3, 1, -1, -1)^T, \\ \Delta S_1^*(1) &= \Delta S_1(1) = (1, -1, 1, -1)^T. \end{aligned}$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that $\frac{\widehat{W}}{B(2)} = \widehat{W} = \widehat{V}^*(1) + \widehat{H}_1(2)\Delta S_1^*(2)$.

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- Assuming that $\omega \in A_{1,1} = \{\omega_2, \omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}
\frac{6}{7}v = \widehat{W_2} &= \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2) \times 1, \\
\frac{6}{7}v = \widehat{W_4} &= \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4) \times (-1), \\
\widehat{V}^*(1,\omega_2) &= \widehat{V}^*(1,\omega_4), \\
\widehat{H}_1(2,\omega_2) &= \widehat{H}_1(2,\omega_4),
\end{aligned}$$

which, using that $r = 0$, yield

$$\begin{aligned}
\widehat{V}^*(1,\omega_2) &= \widehat{V}^*(1,\omega_4) = V(1,\omega_2) = V(1,\omega_4) = \frac{6}{7}v, \\
\widehat{H}_1(2,\omega_2) &= \widehat{H}_1(2,\omega_4) = 0.
\end{aligned}$$

- Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}
\frac{24}{7}v = \widehat{W_1} &= \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1) \times 3, \\
\frac{24}{63}v = \widehat{W_3} &= \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3) \times (-1), \\
\widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_3), \\
\widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_3),
\end{aligned}$$

which, using that $r = 0$, yield

$$\begin{aligned}
\widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_3) = V(1,\omega_1) = V(1,\omega_3) = \frac{8}{7}v, \\
\widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_3) = \frac{16}{21}v.
\end{aligned}$$

- For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\,\Delta S_1^*(1)$ and we get the following two equations

$$\begin{aligned}
\frac{6}{7}v = \widehat{V}^*(1,\omega) &= \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1}) \\
\frac{8}{7}v = \widehat{V}^*(1,\omega) &= \widehat{V}^*(0) + \widehat{H}_1(1) \times (1), \qquad (\text{for } \omega \in A_{1,2})
\end{aligned}$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = V(0) = v, \qquad \widehat{H}_1(1) = \frac{1}{7}v.$$

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- Finally we compute $\widehat{H}_0(1)$ and $\widehat{H}_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)\, S_1^*(0) = \frac{4}{7}v,$$

and

$$\begin{aligned}\widehat{H}_0(2,\omega) &= \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)\, S_1^*(1,\omega) \\ &= \begin{cases} \frac{6}{7}v - 0 \times 2 = \frac{6}{7}v & \text{if} \quad \omega \in A_{1,1} \\ \frac{8}{7}v - \frac{16}{21}v \times 4 = -\frac{40}{21}v & \text{if} \quad \omega \in A_{1,2} \end{cases}\end{aligned}$$

Problem 4

a (weight 10p)

The conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$- measurable random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B], \qquad B \in \mathcal{G}.$$

In order to prove that, if Y is $\mathcal{G}$-measurable then

$$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}],$$

we have to prove first that $Y\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable and secondly that

$$\mathbb{E}[XY\mathbf{1}_B] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B], \qquad B \in \mathcal{G}. \tag{2}$$

Let $\{A_1, ..., A_m\}$ be the partition that generates $\mathcal{G}$. That $Y\mathbb{E}[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable r.v. follows from the fact that the product of $\mathcal{G}$-measurable r.v is a $\mathcal{G}$-measurable r.v, because it is constant over the subsets of the partition generating $\mathcal{G}$.

To prove (2), first note that by the linearity of the conditional expectation we can assume that $Y = \mathbf{1}_{A_i}$ for some $i \in \{1, ..., m\}$ (Recall that an arbitrary $\mathcal{G}$-measurable r.v. is of the form $\sum_{i=1}^m a_i \mathbf{1}_{A_i}$ with $a_i \in \mathbb{R}$). For all $B \in \mathcal{G}$, we have

$$\begin{aligned}\mathbb{E}[XY\mathbf{1}_B] &= \mathbb{E}[X\mathbf{1}_{A_i}\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_{A_i\cap B}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_{A_i\cap B}] \\ &= \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_{A_i}\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]Y\mathbf{1}_B],\end{aligned}$$

which proves (2). In the third equality we have used the definition of conditional expectation and the fact that $A_i \cap B \in \mathcal{G}$.

b (weight 10p)

We say that a process $X = \{X(t)\}_{t=0,...,T}$ is a martingale with respect to the filtration $\mathbb{F}$ under the probability measure P if X is $\mathbb{F}$-adapted, that is, X(t) is $\mathcal{F}_t$-measurable for all $t = 0, ..., T$, and

$$\mathbb{E}[X(t+s)|\mathcal{F}_t] = X(t), \qquad t, s \geq 0.$$

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Or, equivalently,

$$\mathbb{E}\left[X(t+1)|\mathcal{F}_t\right] = X(t), \qquad t \geq 0.$$

To prove that the process $G = \{G(t)\}_{t=0,\ldots,T}$ defined by

$$\begin{aligned} G(0) &= 0, \\ G(t) &= \sum_{u=1}^{t} H(u)\left(Z(u) - Z(u-1)\right), \end{aligned}$$

is a martingale first we have to prove that $G(t)$ is $\mathbb{F}$-adapted. First note that if X and Y are $\mathcal{G}$-measurable with respect to an algebra $\mathcal{G}$ on Ω, then XY and $X+Y$ are $\mathcal{G}$-measurable. The process H is predictable and, in particular, adapted to $\mathbb{F}$. The process Z is adapted to $\mathbb{F}$ because it is an $\mathbb{F}$-martingale, moreover $Z(u-1)$ is also $\mathcal{F}_u$-measurable because $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$. Therefore, $H(u)(Z(u) - Z(u-1))$ is $\mathcal{F}_u$-measurable for $u \leq t$. As $\mathbb{F}$ is a filtration, $\mathcal{F}_u \subseteq \mathcal{F}_t$, and we can conclude that $G(t)$ is $\mathcal{F}_t$-measurable and, hence, G is $\mathbb{F}$-adapted. The result

To prove the martingale property, first note that

$$G(t+1) = G(t) + H(t+1)\left(Z(t+1) - Z(t)\right).$$

Then,

$$\begin{aligned} \mathbb{E}\left[G(t+1)|\mathcal{F}_t\right] &= \mathbb{E}\left[G(t) + H(t+1)\left(Z(t+1) - Z(t)\right)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[G(t)|\mathcal{F}_t\right] + \mathbb{E}\left[H(t+1)\left(Z(t+1) - Z(t)\right)|\mathcal{F}_t\right] \\ &= G(t) + H(t+1)\,\mathbb{E}\left[\left(Z(t+1) - Z(t)\right)|\mathcal{F}_t\right] \\ &= G(t), \end{aligned}$$

where in the second equality we have used the linearity of conditional expectation, in the third equality we have used that $H(t+1)$ is $\mathcal{F}_t$-measurable and the property proved in the previous section, and in the fourth equality we have used that Z is a martingale and, therefore,

$$\mathbb{E}\left[Z(t+1)|\mathcal{F}_t\right] = Z(t) \Longleftrightarrow \mathbb{E}\left[\left(Z(t+1) - Z(t)\right)|\mathcal{F}_t\right] = 0.$$