# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in: STK3405/4405 — Introduction to risk and reliability analysis
Day of examination: Friday December 8, 2017
Examination hours: 09.00 – 13.00
This problem set consists of 4 pages.
Appendices: None
Permitted aids: Calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

## Problem 1

In this problem we consider measures of reliability importance. Let $(C, \phi)$ be a binary monotone system with component set $C = \{1, \dots, n\}$ and structure function $\phi$. The Birnbaum measure of the reliability importance of component $i \in C$ at time $t \geq 0$ is defined as:

$$\begin{aligned} I_B^{(i)}(t) &= \mathrm{P}(\text{Component } i \text{ is critical for the system at time } t) \\ &= \mathrm{P}(\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1) \\ &= \mathrm{E}[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))], \end{aligned}$$

where $\boldsymbol{X}(t) = (X_1(t), \dots, X_n(t))$ denotes the vector of component state variables at time $t \geq 0$. We assume that $\mathrm{P}(X_i(t) = 1) = p_i(t)$, $i = 1, \dots, n$, and let $\boldsymbol{p}(t) = (p_1(t), \dots, p_n(t))$ denote the vector of component reliabilities at time $t \geq 0$. We let $h(t) = \mathrm{P}(\phi(\boldsymbol{X}(t)) = 1) = \mathrm{E}[\phi(\boldsymbol{X}(t))]$ denote the reliability of the system at time $t \geq 0$. If $X_1(t), \dots, X_n(t)$ are stochastically independent, we may write $h(t) = h(\boldsymbol{p}(t))$.

(a) Let $T_S$ denote the lifetime of the system, and let $T_i$ denote the lifetime of component $i$, $i = 1, \dots n$. Explain briefly that for $t \geq 0$ we have $T_S > t$ if and only if $\phi(\boldsymbol{X}(t)) = 1$, and use this to show that:

MUST EXPLAIL both WAYS

$$I_B^{(i)}(t) = \mathrm{P}(T_S > t | T_i > t) - \mathrm{P}(T_S > t | T_i \leq t), \quad t \geq 0, \quad i = 1, \dots, n.$$

EXPLAIN THE TRANSITION

(b) Assume that $X_1(t), \dots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i)}(t) = \frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)}, \quad t \geq 0, \quad i = 1, \dots, n.$$

*(Continued on page 2.)*

## Problem 1a:

$\rightarrow T_S :=$ lifetime of $\phi$
$\quad T_i :=$ '' '' $i$ $\rightarrow$ for $t \geq 0$, $T_S > t \longleftrightarrow \phi(\vec{X}(t)) = 1$

This means that the lifetime of the system (TS) has not reached its limit at time (t), when t>0, because the system is still functioning at time (t). Conversely, if the system is functioning at time (t), then the lifetime of the system (TS) has not reached its limit, for t>0.

$\rightarrow I_B^{(i)} = P(T_S > t \mid T_i > t) - P(T_S > t \mid T_i \leq t)$, $t > 0$, $i = 1, \dots, n$

$$I_B^{(i)} = \frac{d[h(\vec{p}(t))]}{dp_i} = h_+(1_i, \vec{p}(t)) - h_-(0_i, \vec{p}(t))$$
$$= E[\phi(1_i, \vec{X}(t))] - E[\phi(0_i, \vec{X}(t))]$$
$$= P(\phi(1_i, \vec{X}(t)) = 1) - P(\phi(0_i, \vec{X}(t)) = 0)$$
$$= P(\phi(\vec{X}(t)) = 1 \mid X_i = 1) = 1) - P(\phi(\vec{X} = 1 \mid X_i = 0) = 0)$$
$$= P(T_S > t \mid T_i > t) - P(T_S > t \mid T_i \leq t) \quad \blacksquare$$

## Problem 1b:

$$h(\vec{p}(t)) = p_i \cdot h(1_i, \vec{p}(t)) + (1 - p_i) \cdot h(0_i, \vec{p}(t))$$

$$\therefore \frac{d[h(\vec{p}(t))]}{dp_i} = h(1_i, \vec{p}(t)) - h(0_i, \vec{p}(t))$$
$$= E[\phi(1_i, \vec{X}(t)) - \phi(0_i, \vec{X}(t))] = I_B^{(i)}(t) \quad \blacksquare$$

## Problem 1c:

**Joint reliability:**

$$I_B^{(i)}(t) = E[\phi(1_i, 1_j, \vec{X}(t))] - E[\phi(1_i, 0_j, \vec{X}(t))] - E[\phi(0_i, 1_j, \vec{X}(t))] + E[\phi(0_i, 0_j, \vec{X}(t))]$$

**For $I_B^{(i)}(t) > 0$:**

$$E[\phi(1_i, 1_j, \vec{X}(t))] - E[\phi(0_i, 1_j, \vec{X}(t))] > E[\phi(1_i, 0_j, \vec{X}(t))] - E[\phi(0_i, 0_j, \vec{X}(t))]$$

This means that the component i is more important when component j is functioning together than when only i is alone.

$$E[\phi(1_i, 1_j, \vec{X}(t))] - E[\phi(1_i, 0_j, \vec{X}(t))] > E[\phi(0_i, 1_j, \vec{X}(t))] - E[\phi(0_i, 0_j, \vec{X}(t))]$$

This means that the component (j) is more important if the component (i) is functioning than if (i) is failed.

**For $I_B^{(i)} < 0$:** This change the inequality above from > to < and the interpretation is the oposite about the importance of one component with the other.

The conclusion is that when the Birnbaum reliability is greater than 0, the components increase the importance of each other when functioning together, and when the Birnbaum reliability is lower than 0, this reduces the importance of the components when functioning with each other.

We now introduce the Birnbaum measure of the *joint* reliability importance of the components $i, j \in C$ at time $t \geq 0$ defined by:

$$I_B^{(i,j)}(t) = \mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t)) \\ - \phi(0_i, 1_j, \boldsymbol{X}(t)) + \phi(0_i, 0_j, \boldsymbol{X}(t))].$$

(c) Explain briefly if $I_B^{(i,j)}(t) > 0$, this implies that: JUST ADD WHAT WE WANT OUT OF THE EQUATION

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))],$$
$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))],$$

while the opposite inequalities hold if $I_B^{(i,j)}(t) < 0$. Use this to give a practical interpretation of the sign of $I_B^{(i,j)}(t)$.

(d) Show that for $i, j \in C$ and $t \geq 0$ we have:

$$I_B^{(i,j)}(t) = \mathrm{P}(T_S > t | T_i > t, T_j > t) - \mathrm{P}(T_S > t | T_i > t, T_j \leq t) \\ - \mathrm{P}(T_S > t | T_i \leq t, T_j > t) + \mathrm{P}(T_S > t | T_i \leq t, T_j \leq t).$$

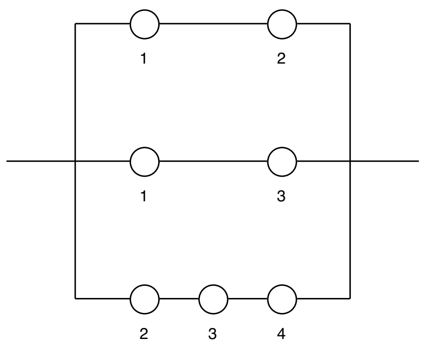
(e) Assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i,j)}(t) = \frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t) \partial p_j(t)}, \quad t \geq 0, \quad i, j = 1, \ldots, n.$$

FACTORIZE OUT AND DIFFERENTIATE

(f) We still assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent, and let $i, j \in C$ and $t \geq 0$. Moreover, assume that $0 < p_i(t) < 1$ for all $i \in C$ and that $n \geq 3$. Show that $I_B^{(i,j)}(t) > 0$ if $(C, \phi)$ is a series system, while $I_B^{(i,j)}(t) < 0$ if $(C, \phi)$ is a parallel system. Give a brief comment to this result.

## Problem 2




In this problem we consider a binary monotone system $(C, \phi)$. The system is shown in the block diagram in the figure above. The component set of the system is $C = \{1, 2, 3, 4\}$.

*(Continued on page 3.)*

We let $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ denote the vector of component state variables and assume throughout this problem that $X_1, X_2, X_3, X_4$ are stochastically independent. Moreover, we let $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ denote the vector of component reliabilities where $p_i = P(X_i = 1)$, $i = 1, 2, 3, 4$. We assume that $0 < p_i < 1$, $i = 1, 2, 3, 4$.

(a) Find the minimal path and cut sets of $(C, \phi)$.

(b) Show that the structure function of the system can be expressed as:

$$\begin{aligned}\phi(\boldsymbol{X}) = X_4[X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3] \\ + (1 - X_4)[X_1X_2 + X_1X_3 - X_1X_2X_3],\end{aligned}$$

FROM HERE TO $h(\vec{p})$ if and only if $\{X_i\}$ are independent.

and use this to find the reliability of the system, $h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})]$.

You may use that the Birnbaum measure of the reliability importance of component $i \in C$ is given by:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, 2, 3, 4,$$

and that the Birnbaum measure of the joint reliability importance of the components $i, j \in C$ is given by:

$$I_B^{(i,j)} = \frac{\partial^2 h(\boldsymbol{p})}{\partial p_i \partial p_j}, \quad i, j = 1, 2, 3, 4.$$

(c) Show that:

$$I_B^{(4)} = p_2p_3 - p_1p_2p_3.$$

(d) Show that $I_B^{(1,4)} < 0$ and that $I_B^{(i,4)} > 0$, $i = 2, 3$. Give a brief comment to these results.

## Problem 3

If $X_1, X_2, \ldots$ is an infinite sequence of independent identically distributed stochastic variables where $\mathrm{E}[X_i] = \mu < \infty$, it can be shown that:

$$P(\bar{X}_n \to \mu) = 1,$$

der $\bar{X}_n = (X_1 + \cdots + X_n)/n$, $n = 1, 2, \ldots$.

Let $\{S(t)\}$ be a stochastic process where $S(t)$ denotes the state of the process at time $t \geq 0$. We say that $\{S(t)\}$ is *a pure jump process* if $S(t)$ can be expressed as:

$$S(t) = S(0) + \sum_{j=1}^{\infty} \mathrm{I}(T_j \leq t) J_j, \qquad t \geq 0,$$

where $0 = T_0 < T_1 < T_2 < \cdots$ is a sequence of stochastic points of time, and $J_1, J_2, \ldots$ is a sequence of stochastic *jumps*.

*(Continued on page 4.)*

# Problem 1d and e:

$$h(\vec{p}(\tau)) = p_i \cdot p_j [h(1_i, 1_j, \vec{p}(\tau))] + p_i \cdot (1-p_j) \cdot [h(1_i, 0_j, \vec{p}(\tau))] + (1-p_i) \cdot p_j \cdot [h(0_i, 1_j, \vec{p}(\tau))] + (1-p_i) \cdot (1-p_j) [h(0_i, 0_j, \vec{p}(\tau))]$$

$$\frac{d^2 h[\vec{p}(\tau)]}{d p_i p_j} = h(1_i, 1_j, \vec{p}(\tau)) - h(1_i, 0_j, \vec{p}(\tau)) - h(0_i, 1_j, \vec{p}(\tau)) + h(0_i, 0_j, \vec{p}(\tau))$$

$$= E[\phi(1_i, 1_j, \vec{X}(\tau))] - E[\phi(1_i, 0_j, \vec{X}(\tau))] - E[\phi(0_i, 1_j, \vec{X}(\tau))] + E[\phi(0_i, 0_j, \vec{X}(\tau))]$$

$$= P(\phi(\vec{X}(\tau)) = 1 \mid X_i = 1, X_j = 1) = 1) - P(\phi(\vec{X}(\tau)) = 1 \mid X_i = 1, X_j = 0) = 1) - P(\phi(\vec{X}(\tau)) = 1 \mid X_i = 0, X_j = 1) = 1) + P(\phi(\vec{X}(\tau)) = 1 \mid X_i = 0, X_j = 0) = 1)$$

$$\therefore P(T_S > \tau \mid T_i > \tau, T_j > \tau) - P(T_S > \tau \mid T_i > \tau, T_j < \tau) - P(T_S > \tau \mid T_i < \tau, T_j > \tau) + P(T_S > \tau \mid T_i < \tau, T_j < \tau)$$

$$= I_B^{(i)(j)}(\tau)$$

$\blacksquare$

# Problem 1f:

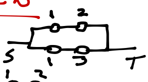
From exercise 1c, we know that when the Birnbaum reliability is greater than 0, the importance of the components are greater when they are working together, this means that the system is in series. The oposite, when the Birnbaum reliability is less than 0, the importance of the components are weaker when working together, then they are in parallel.

# Problem 2a:

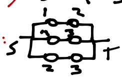
The minimal path sets: $\{1,2\}; \{1,3\}; \{2,3,4\}$

The minimal cut sets: $\{1,2\}; \{1,3\}; \{2,3\}; \{1,4\}$

# Problem 2b:

$\phi(0_4, \vec{x})$: 


$$:= x_1 \cdot x_2 \amalg x_1 x_3 = x_1 x_2 + x_1 x_3 - x_1^2 x_2 x_3 = x_1 x_2 + x_1 x_3 - x_1 x_2 x_3$$

$\phi(1_4, \vec{x})$: 


$$:= x_1 x_2 \amalg x_1 x_3 \amalg x_2 x_3 = x_1 x_2 + x_1 x_3 - \cancel{x_1 x_2 x_3} + x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 + \cancel{x_1 x_2 x_3}$$

$$= x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 x_1 x_2 x_3$$

$$\phi(\vec{x}) = x_4 \cdot \phi(1_4, \vec{x}) + (1 - x_4) \cdot \phi(0_4, \vec{x})$$

$$E[\phi(\vec{X})] = p_4 \cdot h(1_4, \vec{p}) + (1 - p_4) \cdot h(0_4, \vec{p}) = h(\vec{p})$$

$\blacksquare$

# Problem 2c:

$$I_B^{(4)} = \frac{d[h(\vec{p})]}{d p_4} = h_+(1_4, \vec{p}) - h_-(0_4, \vec{p}) = \cancel{p_1 p_2} + \cancel{p_1 p_3} + p_2 p_3 - \cancel{2} p_1 p_2 p_3 - \cancel{p_1 p_2} - \cancel{p_1 p_3} + \cancel{p_1 p_2 p_3}$$

$$= p_2 p_3 - p_1 p_2 p_3$$

$$I_B^{(j,4)} \Rightarrow I_B^{(1,4)} = \frac{d[p_2 p_3 - p_1 p_2 p_3]}{d p_1} = -p_2 p_3 < 0 \,//$$

$$I_B^{(2,4)} = \frac{d[p_2 p_3 - p_1 p_2 p_3]}{d p_2} = p_3 - p_1 p_3 > 0 \,//$$

$$I_B^{(3,4)} = \frac{d[p_2 p_3 - p_1 p_2 p_3]}{p_3} = p_2 - p_1 p_2 > 0 \,//$$

$\blacksquare$

Since the Birnbaum reliability of 1, 4 are less thab 0, this means that each component weaken each other when functioning together. The oposite makes the component strengthen each other when functioning together.

We introduce:

$$N(t) = \sum_{j=1}^{\infty} I(T_j \leq t) = \text{The number of jumps in [0, t]}.$$

The process $\{S(t)\}$ is said to be *regular* if $P(N(t) < \infty) = 1$ for all $t > 0$.

We then let $\Delta_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$.

(a) Show that if the sequence $\{\Delta_j\}$ contains an infinite subsequence, $\{\Delta_{k_j}\}$, of independent, identically distributed stochastic variables such that $E[\Delta_{k_j}] = d > 0$, then $\{S(t)\}$ us regular.

(b) Explain why regularity is important for simulations of pure jump processes.

**Problem 3a:**

from def.: $\{S_T\}$ is regular $\longleftrightarrow$ $T_\infty = \infty$ almost surely

and $\longleftrightarrow$ $\sum_{j=1}^{\infty} \Delta T_j$ is divergent with probability $= 1$

Then $P\left(\lim_{m\to\infty} \frac{\sum_{j=1}^{m} \Delta T_j}{m} = d\right) = 1$

Hence $\sum_{j=1}^{\infty} \Delta T_{k_j} < \sum_{j=1}^{\infty} \Delta T_j = T_\infty$ ■

**Problem 3b:**

Regularity is importante for simulations of pure jump processes because we need to ensure that the number of events in the process is finite and it is satisfied with the regular pure process which has finite events with probability 1.