

# CHAPTER 13 VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

## 13.1 CURVES IN SPACE AND THEIR TANGENTS

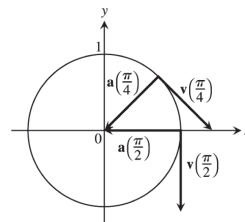
1.  $x = t + 1$  and  $y = t^2 - 1 \Rightarrow y = (x - 1)^2 - 1 = x^2 - 2x$ ;  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$  and  $\mathbf{a} = 2\mathbf{j}$  at  $t = 1$

2.  $x = \frac{t}{t+1}$  and  $y = \frac{1}{t} \Rightarrow x = \frac{\frac{1}{y}}{\frac{1}{y} + 1} = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$ ;  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{1}{(t+1)^2}\mathbf{i} - \frac{1}{t^2}\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{2}{(t+1)^3}\mathbf{i} + \frac{2}{t^3}\mathbf{j}$   
 $\Rightarrow \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$  and  $\mathbf{a} = -16\mathbf{i} - 16\mathbf{j}$  at  $t = -\frac{1}{2}$

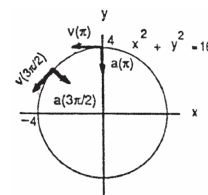
3.  $x = e^t$  and  $y = \frac{2}{9}e^{2t} \Rightarrow y = \frac{2}{9}x^2$ ;  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^t\mathbf{i} + \frac{4}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{a} = e^t\mathbf{i} + \frac{8}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$  and  $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$  at  $t = \ln 3$

4.  $x = \cos 2t$  and  $y = 3 \sin 2t \Rightarrow x^2 + \frac{1}{9}y^2 = 1$ ;  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j}$   
 $\Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-4 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j} \Rightarrow \mathbf{v} = 6\mathbf{j}$  and  $\mathbf{a} = -4\mathbf{i}$  at  $t = 0$

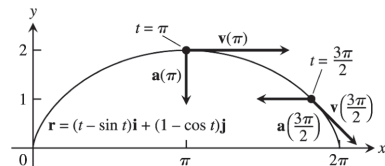
5.  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$  and  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j}$   
 $\Rightarrow$  for  $t = \frac{\pi}{4}$ ,  $\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$  and  $\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ ;  
 for  $t = \frac{\pi}{2}$ ,  $\mathbf{v}\left(\frac{\pi}{2}\right) = -\mathbf{j}$  and  $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{i}$



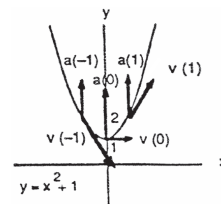
6.  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin \frac{t}{2})\mathbf{i} + (2 \cos \frac{t}{2})\mathbf{j}$  and  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-\cos \frac{t}{2})\mathbf{i} + (-\sin \frac{t}{2})\mathbf{j}$   
 $\Rightarrow$  for  $t = \pi$ ,  $\mathbf{v}(\pi) = -2\mathbf{i}$  and  $\mathbf{a}(\pi) = -\mathbf{j}$ ;  
 for  $t = \frac{3\pi}{2}$ ,  $\mathbf{v}\left(\frac{3\pi}{2}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$  and  $\mathbf{a}\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



7.  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$  and  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$   
 $\Rightarrow$  for  $t = \pi$ ,  $\mathbf{v}(\pi) = 2\mathbf{i}$  and  $\mathbf{a}(\pi) = -\mathbf{j}$ ;  
 for  $t = \frac{3\pi}{2}$ ,  $\mathbf{v}\left(\frac{3\pi}{2}\right) = \mathbf{i} - \mathbf{j}$  and  $\mathbf{a}\left(\frac{3\pi}{2}\right) = -\mathbf{i}$



8.  $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$  and  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow$  for  $t = -1$ ,  $\mathbf{v}(-1) = \mathbf{i} - 2\mathbf{j}$  and  $\mathbf{a}(-1) = 2\mathbf{j}$ ; for  $t = 0$ ,  $\mathbf{v}(0) = \mathbf{i}$  and  $\mathbf{a}(0) = 2\mathbf{j}$ ;  
 for  $t = 1$ ,  $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$  and  $\mathbf{a}(1) = 2\mathbf{j}$



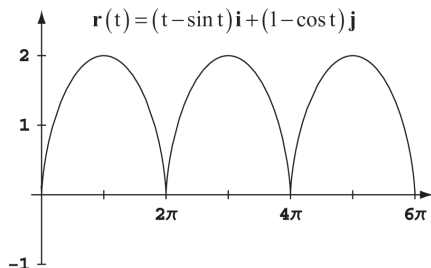
9.  $\mathbf{r} = (t+1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$ ; Speed:  $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$ ;  
 Direction:  $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + 2(1)\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$
10.  $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$ ;  
 Speed:  $|\mathbf{v}(1)| = \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$ ;  
 Direction:  $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1) = 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$
11.  $\mathbf{r} = (2 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j}$ ;  
 Speed:  $|\mathbf{v}\left(\frac{\pi}{2}\right)| = \sqrt{(-2 \sin \frac{\pi}{2})^2 + (3 \cos \frac{\pi}{2})^2 + 4^2} = 2\sqrt{5}$ ;  
 Direction:  $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{|\mathbf{v}\left(\frac{\pi}{2}\right)|} = \left(-\frac{2}{2\sqrt{5}} \sin \frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}} \cos \frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}\right)$
12.  $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j} + \frac{4}{3}\mathbf{k}$   
 $\Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2 \sec^2 t \tan t)\mathbf{j}$ ;  
 Speed:  $|\mathbf{v}\left(\frac{\pi}{6}\right)| = \sqrt{\left(\sec \frac{\pi}{6} \tan \frac{\pi}{6}\right)^2 + \left(\sec^2 \frac{\pi}{6}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$ ;  
 Direction:  $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{|\mathbf{v}\left(\frac{\pi}{6}\right)|} = \frac{(\sec \frac{\pi}{6} \tan \frac{\pi}{6})\mathbf{i} + (\sec^2 \frac{\pi}{6})\mathbf{j} + \frac{4}{3}\mathbf{k}}{2} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$
13.  $\mathbf{r} = (2 \ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ;  
 Speed:  $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$ ;  
 Direction:  $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$
14.  $\mathbf{r} = (e^{-t})\mathbf{i} + (2 \cos 3t)\mathbf{j} + (2 \sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\mathbf{i} - (6 \sin 3t)\mathbf{j} + (6 \cos 3t)\mathbf{k}$   
 $\Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (e^{-t})\mathbf{i} - (18 \cos 3t)\mathbf{j} - (18 \sin 3t)\mathbf{k}$ ;  
 Speed:  $|\mathbf{v}(0)| = \sqrt{(-e^0)^2 + (-6 \sin 3(0))^2 + (6 \cos 3(0))^2} = \sqrt{37}$ ;  
 Direction:  $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\mathbf{i} - 6 \sin 3(0)\mathbf{j} + 6 \cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$
15.  $\mathbf{v} = 3\mathbf{i} + \sqrt{3}\mathbf{j} + 2t\mathbf{k}$  and  $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\mathbf{j}$  and  $\mathbf{a}(0) = 2\mathbf{k} \Rightarrow |\mathbf{v}(0)| = \sqrt{3^2 + (\sqrt{3})^2 + 0^2} = \sqrt{12}$  and  
 $|\mathbf{a}(0)| = \sqrt{2^2} = 2$ ;  $\mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

16.  $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\mathbf{j}$  and  $\mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$  and  $\mathbf{a}(0) = -32\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$  and  $|\mathbf{a}(0)| = \sqrt{(-32)^2} = 32; \mathbf{v}(0) \cdot \mathbf{a}(0) = \left(\frac{\sqrt{2}}{2}\right)(-32) = -16\sqrt{2} \Rightarrow \cos \theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}$
17.  $\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-1/2}\mathbf{k}$  and  $\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j}$  and  $\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \Rightarrow |\mathbf{v}(0)| = 1$  and  $|\mathbf{a}(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}; \mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
18.  $\mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$  and  $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$  and  $\mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$  and  $|\mathbf{a}(0)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}; \mathbf{v}(0) \cdot \mathbf{a}(0) = \frac{2}{9} - \frac{2}{9} = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
19.  $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}; t_0 = 0 \Rightarrow \mathbf{v}(t_0) = \mathbf{i} + \mathbf{k}$  and  $\mathbf{r}(t_0) = P_0 = (0, -1, 1) \Rightarrow x = 0 + t = t, y = -1, \text{ and } z = 1 + t$  are parametric equations of the tangent line
20.  $\mathbf{r}(t) = t^2\mathbf{i} + (2t - 1)\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j} + 3t^2\mathbf{k}; t_0 = 2 \Rightarrow \mathbf{v}(2) = 4\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}$  and  $\mathbf{r}(t_0) = P_0 = (4, 3, 8) \Rightarrow x = 4 + 4t, y = 3 + 2t, \text{ and } z = 8 + 12t$  are parametric equations of the tangent line
21.  $\mathbf{r}(t) = (\ln t)\mathbf{i} + \frac{t-1}{t+2}\mathbf{j} + (t \ln t)\mathbf{k} \Rightarrow \mathbf{v}(t) = \frac{1}{t}\mathbf{i} + \frac{3}{(t+2)^2}\mathbf{j} + (\ln t + 1)\mathbf{k}; t_0 = 1 \Rightarrow \mathbf{v}(1) = \mathbf{i} + \frac{1}{3}\mathbf{j} + \mathbf{k}$  and  $\mathbf{r}(t_0) = P_0 = (0, 0, 0) \Rightarrow x = 0 + t = t, y = 0 + \frac{1}{3}t = \frac{1}{3}t, \text{ and } z = 0 + t = t$  are parametric equations of the tangent line
22.  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2 \cos 2t)\mathbf{k}; t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(t_0) = -\mathbf{i} - 2\mathbf{k}$  and  $\mathbf{r}(t_0) = P_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t, y = 1, \text{ and } z = 0 - 2t = -2t$  are parametric equations of the tangent line
23. (a)  $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j};$   
 (i)  $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$  constant speed;  
 (ii)  $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$  yes, orthogonal;  
 (iii) counterclockwise movement;  
 (iv) yes,  $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$
- (b)  $\mathbf{v}(t) = -(2 \sin 2t)\mathbf{i} + (2 \cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4 \cos 2t)\mathbf{i} - (4 \sin 2t)\mathbf{j};$   
 (i)  $|\mathbf{v}(t)| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2 \Rightarrow$  constant speed;  
 (ii)  $\mathbf{v} \cdot \mathbf{a} = 8 \sin 2t \cos 2t - 8 \cos 2t \sin 2t = 0 \Rightarrow$  yes, orthogonal;  
 (iii) counterclockwise movement;  
 (iv) yes,  $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$
- (c)  $\mathbf{v}(t) = -\sin\left(t - \frac{\pi}{2}\right)\mathbf{i} + \cos\left(t - \frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos\left(t - \frac{\pi}{2}\right)\mathbf{i} - \sin\left(t - \frac{\pi}{2}\right)\mathbf{j};$   
 (i)  $|\mathbf{v}(t)| = \sqrt{\sin^2\left(t - \frac{\pi}{2}\right) + \cos^2\left(t - \frac{\pi}{2}\right)} = 1 \Rightarrow$  constant speed;

- (ii)  $\mathbf{v} \cdot \mathbf{a} = \sin\left(t - \frac{\pi}{2}\right) \cos\left(t - \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right) \sin\left(t - \frac{\pi}{2}\right) = 0 \Rightarrow$  yes, orthogonal;  
 (iii) counterclockwise movement;  
 (iv) no,  $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$  instead of  $\mathbf{i} + 0\mathbf{j}$
- (d)  $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ;  
 (i)  $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$  constant speed;  
 (ii)  $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$  yes, orthogonal;  
 (iii) clockwise movement;  
 (iv) yes,  $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$
- (e)  $\mathbf{v}(t) = -(2t \sin t)\mathbf{i} + (2t \cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2 \sin t + 2t \cos t)\mathbf{i} + (2 \cos t - 2t \sin t)\mathbf{j}$ ;  
 (i)  $|\mathbf{v}(t)| = \sqrt{(-2t \sin t)^2 + (2t \cos t)^2} = \sqrt{4t^2(\sin^2 t + \cos^2 t)} = 2|t| = 2t, t \geq 0 \Rightarrow$  variable speed;  
 (ii)  $\mathbf{v} \cdot \mathbf{a} = 4(t \sin^2 t + t^2 \sin t \cos t) + 4(t \cos^2 t - t^2 \cos t \sin t) = 4t \neq 0$  in general  $\Rightarrow$  not orthogonal in general;  
 (iii) counterclockwise movement;  
 (iv) yes,  $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

24. Let  $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  denote the position vector of the point  $(2, 2, 1)$  and let,  $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$  and  $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ . Then  $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ . Note that  $(2, 2, 1)$  is a point on the plane and  $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$  is normal to the plane. Moreover,  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal unit vectors with  $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are parallel to the plane. Therefore,  $\mathbf{r}(t)$  identifies a point that lies in the plane for each  $t$ . Also, for each  $t$ ,  $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$  is a unit vector. Starting at the point  $\left(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1\right)$  the vector  $\mathbf{r}(t)$  traces out a circle of radius 1 and center  $(2, 2, 1)$  in the plane  $x + y - 2z = 2$ .
25. The velocity vector is tangent to the graph of  $y^2 = 2x$  at the point  $(2, 2)$ , has length 5, and a positive  $\mathbf{i}$  component. Now,  $y^2 = 2x \Rightarrow 2y \frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx}\bigg|_{(2,2)} = \frac{2}{2 \cdot 2} = \frac{1}{2} \Rightarrow$  the tangent vector lies in the direction of the vector  $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$  the velocity vector is  $\mathbf{v} = \frac{5}{\sqrt{1 + \frac{1}{4}}} \left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)} \left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

26. (a)



- (b)  $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$  and  $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ ;  $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2 \cos t \Rightarrow |\mathbf{v}|^2$  is at a max when  $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$ , etc., and at these values of  $t$ ,  $|\mathbf{v}|^2 = 4 \Rightarrow \max |\mathbf{v}| = \sqrt{4} = 2$ ;  $|\mathbf{v}|^2$  is at a min when  $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$ , etc., and at these values of  $t$ ,  $|\mathbf{v}|^2 = 0 \Rightarrow \min |\mathbf{v}| = 0$ ;  
 $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$  for every  $t \Rightarrow \max |\mathbf{a}| = \min |\mathbf{a}| = \sqrt{1} = 1$

27.  $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2 \cdot 0 = 0 \Rightarrow \mathbf{r} \cdot \mathbf{r}$  is a constant  $\Rightarrow |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$  is constant
28. (a)  $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \left( \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{v} \times \frac{d\mathbf{w}}{dt} \right)$   
 $= \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}$
- (b)  $\frac{d}{dt} \left[ \mathbf{r} \cdot \left( \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) \right] = \frac{d\mathbf{r}}{dt} \cdot \left( \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left( \frac{d^2\mathbf{r}}{dt^2} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left( \frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right) = \mathbf{r} \cdot \left( \frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right)$ , since  $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$   
 and  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{B}) = 0$  for any vectors  $\mathbf{A}$  and  $\mathbf{B}$
29. (a)  $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c \frac{df}{dt}\mathbf{i} + c \frac{dg}{dt}\mathbf{j} + c \frac{dh}{dt}\mathbf{k}$   
 $= c \left( \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right) = c \frac{d\mathbf{u}}{dt}$
- (b)  $\mathbf{u} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \Rightarrow f(t)\mathbf{u} = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$   
 $\Rightarrow \frac{d}{dt}(f(t)\mathbf{u}) = \left[ \frac{df}{dt}x(t) + f(t)\frac{dx}{dt} \right]\mathbf{i} + \left[ \frac{df}{dt}y(t) + f(t)\frac{dy}{dt} \right]\mathbf{j} + \left[ \frac{df}{dt}z(t) + f(t)\frac{dz}{dt} \right]\mathbf{k}$   
 $= \frac{df}{dt}[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] + f(t)\left[ \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right] = \frac{df}{dt}\mathbf{u} + f(t)\frac{d\mathbf{u}}{dt}$
30. Let  $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$  and  $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$ .  
 Then  $\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k}$   
 $\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f_1'(t) + g_1'(t)]\mathbf{i} + [f_2'(t) + g_2'(t)]\mathbf{j} + [f_3'(t) + g_3'(t)]\mathbf{k}$   
 $= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] + [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt};$   
 $\mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k}$   
 $\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f_1'(t) - g_1'(t)]\mathbf{i} + [f_2'(t) - g_2'(t)]\mathbf{j} + [f_3'(t) - g_3'(t)]\mathbf{k}$   
 $= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] - [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$
31. Suppose  $\mathbf{r}$  is continuous at  $t = t_0$ . Then  $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t \rightarrow t_0} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$   
 $= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t \rightarrow t_0} f(t) = f(t_0), \lim_{t \rightarrow t_0} g(t) = g(t_0), \text{ and } \lim_{t \rightarrow t_0} h(t) = h(t_0) \Leftrightarrow f, g, \text{ and } h \text{ are}$   
 continuous at  $t = t_0$ .
32.  $\lim_{t \rightarrow t_0} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \lim_{t \rightarrow t_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lim_{t \rightarrow t_0} f_1(t) & \lim_{t \rightarrow t_0} f_2(t) & \lim_{t \rightarrow t_0} f_3(t) \\ \lim_{t \rightarrow t_0} g_1(t) & \lim_{t \rightarrow t_0} g_2(t) & \lim_{t \rightarrow t_0} g_3(t) \end{vmatrix} = \lim_{t \rightarrow t_0} \mathbf{r}_1(t) \times \lim_{t \rightarrow t_0} \mathbf{r}_2(t)$   
 $= \mathbf{A} \times \mathbf{B}$
33.  $\mathbf{r}'(t_0)$  exists  $\Rightarrow f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$  exists  $\Rightarrow f'(t_0), g'(t_0), h'(t_0)$  all exist  $\Rightarrow f, g, \text{ and } h$  are  
 continuous at  $t = t_0 \Rightarrow \mathbf{r}(t)$  is continuous at  $t = t_0$

34.  $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  with  $a, b, c$  real constants  $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

35-38. Example CAS commands:

Maple:

```
> with( plots );
r := t -> [sin(t)-t*cos(t),cos(t)+t*sin(t),t^2];
t0 := 3*Pi/2;
l0 := 0;
hi := 6*Pi;
P1 := spacecurve( r(t), t=l0..hi, axes=boxed, thickness=3 );
display( P1, title="#35(a) (Section 13.1)" );
Dr := unapply( diff(r(t),t), t ); # (b)
Dr(t0); # (c)
q1 := expand( r(t0) + Dr(t0)*(t-t0) );
T := unapply( q1, t );
P2 := spacecurve( T(t), t=l0..hi, axes=boxed, thickness=3, color=black );
display( [P1,P2], title="#35(d) (Section 13.1)" );
```

39-40. Example CAS commands:

Maple:

```
a := 'a'; b := 'b';
r := (a,b,t) -> [cos(a*t),sin(a*t),b*t];
Dr := unapply( diff(r(a,b,t),t), (a,b,t) );
t0 := 3*Pi/2;
q1 := expand( r(a,b,t0) + Dr(a,b,t0)*(t-t0) );
T := unapply( q1, (a,b,t) );
l0 := 0;
hi := 4*Pi;
P := NULL;
for a in [ 1, 2, 4, 6 ] do
    P1 := spacecurve( r(a,1,t), t=l0..hi, thickness=3 );
    P2 := spacecurve( T(a,1,t), t=l0..hi, thickness=3, color=black );
    P := P, display( [P1,P2], axes=boxed, title=sprintf("#39 (Section 13.1)\n a=%a",a) );
end do;
display( [P], insequence=true );
```

35-40. Example CAS commands:

Mathematica: (assigned functions, parameters, and intervals will vary)

The x-y-z components for the curve are entered as a list of functions of t. The unit vectors **i**, **j**, **k** are not inserted.

If a graph is too small, highlight it and drag out a corner or side to make it larger.

Only the components of  $\mathbf{r}[t]$  and values for  $t_0$ ,  $t_{\min}$ , and  $t_{\max}$  require alteration for each problem.

```
Clear[r, v, t, x, y, z]
```

```
r[t_]:= { Sin[t]-t Cos[t], Cos[t]+t Sin[t], t^2 }
```

```
t0= 3π/2; tmin= 0; tmax= 6π;
```

```
ParametricPlot3D[Evaluate[r[t]], {t, tmin, tmax}, AxesLabel -> {x, y, z}];
```

```
v[t_]= r'[t]
```

```
tanline[t_]= v[t0]t + r[t0]
```

```
ParametricPlot3D[Evaluate[ {r[t], tanline[t]} ], {t, tmin, tmax}, AxesLabel -> {x, y, z}];
```

For 39 and 40, the curve can be defined as a function of t, a, and b. Leave a space between a and t and b and t.

```
Clear[r, v, t, x, y, z, a, b]
```

```
r[t_,a_,b_]:= { Cos[a t], Sin[a t], b t }
```

```
t0= 3π/2; tmin= 0; tmax= 4π;
```

```
v[t_,a_,b_]= D[r[t, a, b], t]
```

```
tanline[t_,a_,b_]= v[t0, a, b]t + r[t0, a, b]
```

```
pa1=ParametricPlot3D[Evaluate[ {r[t, 1, 1], tanline[t, 1, 1]} ], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
```

```
pa2=ParametricPlot3D[Evaluate[ {r[t, 2, 1], tanline[t, 2, 1]} ], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
```

```
pa4=ParametricPlot3D[Evaluate[ {r[t, 4, 1], tanline[t, 4, 1]} ], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
```

```
pa6=ParametricPlot3D[Evaluate[ {r[t, 6, 1], tanline[t, 6, 1]} ], {t,tmin, tmax}, AxesLabel -> {x, y, z}];
```

```
Show[GraphicsRow[ {pa1, pa2, pa4, pa6} ]]
```

## 13.2 INTEGRALS OF VECTOR FUNCTIONS; PROJECTILE MOTION

- $$\int_0^1 [t^3 \mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}] dt = \left[ \frac{t^4}{4} \right]_0^1 \mathbf{i} + \left[ 7t \right]_0^1 \mathbf{j} + \left[ \frac{t^2}{2} + t \right]_0^1 \mathbf{k} = \frac{1}{4} \mathbf{i} + 7\mathbf{j} + \frac{3}{2} \mathbf{k}$$
- $$\int_1^2 \left[ (6-6t)\mathbf{i} + 3\sqrt{t}\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k} \right] dt = \left[ 6t - 3t^2 \right]_1^2 \mathbf{i} + \left[ 2t^{3/2} \right]_1^2 \mathbf{j} + \left[ -4t^{-1} \right]_1^2 \mathbf{k} = -3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j} + 2\mathbf{k}$$
- $$\int_{-\pi/4}^{\pi/4} \left[ (\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\mathbf{k} \right] dt = \left[ -\cos t \right]_{-\pi/4}^{\pi/4} \mathbf{i} + \left[ t + \sin t \right]_{-\pi/4}^{\pi/4} \mathbf{j} + \left[ \tan t \right]_{-\pi/4}^{\pi/4} \mathbf{k} = \left( \frac{\pi + 2\sqrt{2}}{2} \right) \mathbf{j} + 2\mathbf{k}$$
- $$\begin{aligned} \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt &= \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}] dt \\ &= \left[ \sec t \right]_0^{\pi/3} \mathbf{i} + \left[ -\ln(\cos t) \right]_0^{\pi/3} \mathbf{j} + \left[ -\frac{1}{2} \cos 2t \right]_0^{\pi/3} \mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4} \mathbf{k} \end{aligned}$$
- $$\int_1^4 \left( \frac{1}{t} \mathbf{i} + \frac{1}{5-t} \mathbf{j} + \frac{1}{2t} \mathbf{k} \right) dt = \left[ \ln t \right]_1^4 \mathbf{i} + \left[ -\ln(5-t) \right]_1^4 \mathbf{j} + \left[ \frac{1}{2} \ln t \right]_1^4 \mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$$

6.  $\int_0^1 \left( \frac{2}{\sqrt{1-t^2}} \mathbf{i} + \frac{\sqrt{3}}{1+t^2} \mathbf{k} \right) dt = \left[ 2 \sin^{-1} t \right]_0^1 \mathbf{i} + \left[ \sqrt{3} \tan^{-1} t \right]_0^1 \mathbf{k} = \pi \mathbf{i} + \frac{\pi\sqrt{3}}{4} \mathbf{k}$
7.  $\int_0^1 \left( te^{t^2} \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k} \right) dt = \left[ \frac{1}{2} e^{t^2} \right]_0^1 \mathbf{i} - \left[ e^{-t} \right]_0^1 \mathbf{j} + \left[ t \right]_0^1 \mathbf{k} = \frac{e-1}{2} \mathbf{i} + \frac{e-1}{e} \mathbf{j} + \mathbf{k}$
8.  $\int_1^{\ln 3} \left( te^t \mathbf{i} + e^t \mathbf{j} + \ln t \mathbf{k} \right) dt = \left[ te^t - e^t \right]_1^{\ln 3} \mathbf{i} - \left[ e^t \right]_1^{\ln 3} \mathbf{j} + \left[ t \ln t - t \right]_1^{\ln 3} \mathbf{k}$   
 $= 3(\ln 3 - 1) \mathbf{i} + (3 - e) \mathbf{j} + (\ln 3(\ln(\ln 3) - 1) + 1) \mathbf{k}$
9.  $\int_0^{\pi/2} \left[ (\cos t) \mathbf{i} - (\sin 2t) \mathbf{j} + (\sin^2 t) \mathbf{k} \right] dt = \int_0^{\pi/2} \left[ (\cos t) \mathbf{i} - (\sin 2t) \mathbf{j} + \left( \frac{1}{2} - \frac{1}{2} \cos 2t \right) \mathbf{k} \right] dt$   
 $= \left[ \sin t \right]_0^{\pi/2} \mathbf{i} + \left[ \frac{1}{2} \cos t \right]_0^{\pi/2} \mathbf{j} + \left[ \frac{1}{2} t - \frac{1}{4} \sin 2t \right]_0^{\pi/2} \mathbf{k} = \mathbf{i} - \mathbf{j} + \frac{\pi}{4} \mathbf{k}$
10.  $\int_0^{\pi/4} \left[ (\sec t) \mathbf{i} + (\tan^2 t) \mathbf{j} - (t \sin t) \mathbf{k} \right] dt = \int_0^{\pi/4} \left[ (\sec t) \mathbf{i} + (\sec^2 t - 1) \mathbf{j} - (t \sin t) \mathbf{k} \right] dt$   
 $= \left[ \ln(\sec t + \tan t) \right]_0^{\pi/4} \mathbf{i} + \left[ \tan t - t \right]_0^{\pi/4} \mathbf{j} + \left[ t \cos t - \sin t \right]_0^{\pi/4} \mathbf{k} = \ln(1 + \sqrt{2}) \mathbf{i} + \left( 1 - \frac{\pi}{4} \right) \mathbf{j} + \left( \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \mathbf{k}$
11.  $\mathbf{r} = \int (-t \mathbf{i} - t \mathbf{j} - t \mathbf{k}) dt = -\frac{t^2}{2} \mathbf{i} - \frac{t^2}{2} \mathbf{j} - \frac{t^2}{2} \mathbf{k} + \mathbf{C}; \quad \mathbf{r}(0) = 0 \mathbf{i} - 0 \mathbf{j} - 0 \mathbf{k} + \mathbf{C} = \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k}$   
 $\Rightarrow \mathbf{r} = \left( -\frac{t^2}{2} + 1 \right) \mathbf{i} + \left( -\frac{t^2}{2} + 2 \right) \mathbf{j} + \left( -\frac{t^2}{2} + 3 \right) \mathbf{k}$
12.  $\mathbf{r} = \int \left[ (180t) \mathbf{i} + (180t - 16t^2) \mathbf{j} \right] dt = 90t^2 \mathbf{i} + \left( 90t^2 - \frac{16}{3} t^3 \right) \mathbf{j} + \mathbf{C}; \quad \mathbf{r}(0) = 90(0)^2 \mathbf{i} + \left[ 90(0)^2 - \frac{16}{3} (0)^3 \right] \mathbf{j} + \mathbf{C} = 100 \mathbf{j}$   
 $\Rightarrow \mathbf{C} = 100 \mathbf{j} \Rightarrow \mathbf{r} = 90t^2 \mathbf{i} + \left( 90t^2 - \frac{16}{3} t^3 + 100 \right) \mathbf{j}$
13.  $\mathbf{r} = \int \left[ \left( \frac{3}{2} (t+1)^{1/2} \right) \mathbf{i} + e^{-t} \mathbf{j} + \left( \frac{1}{t+1} \right) \mathbf{k} \right] dt = (t+1)^{3/2} \mathbf{i} - e^{-t} \mathbf{j} + \ln(t+1) \mathbf{k} + \mathbf{C};$   
 $\mathbf{r}(0) = (0+1)^{3/2} \mathbf{i} - e^{-0} \mathbf{j} + \ln(0+1) \mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r} = \left[ (t+1)^{3/2} - 1 \right] \mathbf{i} + \left( 1 - e^{-t} \right) \mathbf{j} + [1 + \ln(t+1)] \mathbf{k}$
14.  $\mathbf{r} = \int \left[ \left( t^3 + 4t \right) \mathbf{i} + t \mathbf{j} + 2t^2 \mathbf{k} \right] dt = \left( \frac{t^4}{4} + 2t^2 \right) \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{2t^3}{3} \mathbf{k} + \mathbf{C}; \quad \mathbf{r}(0) = \left( \frac{0^4}{4} + 2(0)^2 \right) \mathbf{i} + \frac{0^2}{2} \mathbf{j} + \frac{2(0)^3}{3} \mathbf{k} + \mathbf{C} = \mathbf{i} + \mathbf{j}$   
 $\Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left( \frac{t^4}{4} + 2t^2 + 1 \right) \mathbf{i} + \left( \frac{t^2}{2} + 1 \right) \mathbf{j} + \frac{2t^3}{3} \mathbf{k}$
15.  $\frac{d\mathbf{r}}{dt} = \int (-32t \mathbf{k}) dt = -32t \mathbf{k} + \mathbf{C}_1; \quad \frac{d\mathbf{r}}{dt}(0) = 8 \mathbf{i} + 8 \mathbf{j} \Rightarrow -32(0) \mathbf{k} + \mathbf{C}_1 = 8 \mathbf{i} + 8 \mathbf{j} \Rightarrow \mathbf{C}_1 = 8 \mathbf{i} + 8 \mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = 8 \mathbf{i} + 8 \mathbf{j} - 32t \mathbf{k};$   
 $\mathbf{r} = \int (8 \mathbf{i} + 8 \mathbf{j} - 32t \mathbf{k}) dt = 8t \mathbf{i} + 8t \mathbf{j} - 16t^2 \mathbf{k} + \mathbf{C}_2; \quad \mathbf{r}(0) = 100 \mathbf{k} \Rightarrow 8(0) \mathbf{i} + 8(0) \mathbf{j} - 16(0)^2 \mathbf{k} + \mathbf{C}_2 = 100 \mathbf{k}$   
 $\Rightarrow \mathbf{C}_2 = 100 \mathbf{k} \Rightarrow \mathbf{r} = 8t \mathbf{i} + 8t \mathbf{j} + (100 - 16t^2) \mathbf{k}$



16.  $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = -(\mathbf{i}t + \mathbf{j}t + \mathbf{k}t) + \mathbf{C}_1$ ;  $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$   
 $\Rightarrow \frac{d\mathbf{r}}{dt} = -(\mathbf{i}t + \mathbf{j}t + \mathbf{k}t)$ ;  $\mathbf{r} = \int -(\mathbf{i}t + \mathbf{j}t + \mathbf{k}t) dt = -\left(\frac{t^2}{2}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}\right) + \mathbf{C}_2$ ;  $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$   
 $\Rightarrow -\left(\frac{0^2}{2}\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{0^2}{2}\mathbf{k}\right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$   
 $\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$
17.  $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3\mathbf{i}t - \mathbf{j}t + \mathbf{k}t + \mathbf{C}_1$ ; the particle travels in the direction of the vector  $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$  (since it travels in a straight line), and at time  $t = 0$  it has speed 2  
 $\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}$   
 $\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2$ ;  $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$   
 $\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$
18.  $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2\mathbf{i}t + \mathbf{j}t + \mathbf{k}t + \mathbf{C}_1$ ; the particle travels in the direction of the vector  $(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  (since it travels in a straight line), and at time  $t = 0$  it has speed 2  
 $\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$   
 $\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{6}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2$ ;  $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$   
 $\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
19.  $x = (v_0 \cos \alpha)t \Rightarrow (21\text{ km})\left(\frac{1000\text{ m}}{1\text{ km}}\right) = (840\text{ m/s})(\cos 60^\circ)t \Rightarrow t = \frac{21,000\text{ m}}{(840\text{ m/s})(\cos 60^\circ)} = 50\text{ seconds}$
20.  $R = \frac{v_0^2}{g} \sin 2\alpha$  and maximum  $R$  occurs when  $\alpha = 45^\circ \Rightarrow 24.5\text{ km} = \left(\frac{v_0^2}{9.8\text{ m/s}^2}\right)(\sin 90^\circ)$   
 $\Rightarrow v_0 = \sqrt{(9.8)(24,500)\text{ m}^2/\text{s}^2} = 490\text{ m/s}$
21. (a)  $t = \frac{2v_0 \sin \alpha}{g} = \frac{2(500\text{ m/s})(\sin 45^\circ)}{9.8\text{ m/s}^2} \approx 72.2\text{ seconds}$ ;  $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{(500\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 90^\circ) \approx 25,510.2\text{ m}$   
 (b)  $x = (v_0 \cos \alpha)t \Rightarrow 5000\text{ m} = (500\text{ m/s})(\cos 45^\circ)t \Rightarrow t = \frac{5000\text{ m}}{(500\text{ m/s})(\cos 45^\circ)} \approx 14.14\text{ s}$ ; thus,  
 $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500\text{ m/s})(\sin 45^\circ)(14.14\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(14.14\text{ s})^2 \approx 4020\text{ m}$   
 (c)  $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{((500\text{ m/s})(\sin 45^\circ))^2}{2(9.8\text{ m/s}^2)} \approx 6378\text{ m}$
22.  $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32\text{ ft} + (32\text{ ft/sec})(\sin 30^\circ)t - \frac{1}{2}(32\text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$ ;  
 the ball hits the ground when  $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$  or  $t = 2 \Rightarrow t = 2\text{ sec}$  since  $t > 0$ ; thus,  
 $x = (v_0 \cos \alpha)t \Rightarrow x = (32\text{ ft/sec})(\cos 30^\circ)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4\text{ ft}$

23. (a)  $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 10 \text{ m} = \left( \frac{v_0^2}{9.8 \text{ m/s}^2} \right) (\sin 90^\circ) \Rightarrow v_0^2 = 98 \text{ m}^2/\text{s}^2 \Rightarrow v_0 \approx 9.9 \text{ m/s};$   
 (b)  $6 \text{ m} \approx \frac{(9.9 \text{ m/s}^2)}{9.8 \text{ m/s}^2} (\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87^\circ \text{ or } 143.12^\circ \Rightarrow \alpha \approx 18.4^\circ \text{ or } 71.6^\circ$
24.  $v_0 = 5 \times 10^6 \text{ m/s}$  and  $x = 40 \text{ cm} = 0.4 \text{ m}$ ; thus  $x = (v_0 \cos \alpha)t \Rightarrow 0.4 \text{ m} = (5 \times 10^6 \text{ m/s})(\cos 0^\circ)t$   
 $\Rightarrow t = 0.08 \times 10^{-6} \text{ s} = 8 \times 10^{-8} \text{ s}$ ; also,  $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$   
 $\Rightarrow y = (5 \times 10^6 \text{ m/s})(\sin 0^\circ)(8 \times 10^{-8} \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(8 \times 10^{-8} \text{ s})^2 = -3.136 \times 10^{-14} \text{ m}$  or  $-3.136 \times 10^{-12} \text{ cm}$ .  
 Therefore, it drops  $3.136 \times 10^{-12} \text{ cm}$ .
25.  $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 16,000 \text{ m} = \frac{(400 \text{ m/s})^2}{9.8 \text{ m/s}^2} \sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5^\circ \text{ or } 2\alpha \approx 101.5^\circ \Rightarrow \alpha \approx 39.3^\circ$   
 or  $50.7^\circ$
26. (a)  $R = \frac{(2v_0)^2}{g} \sin 2\alpha = \frac{4v_0^2}{g} \sin 2\alpha = 4 \left( \frac{v_0^2}{g} \sin 2\alpha \right)$  or 4 times the original range.  
 (b) Now, let the initial range be  $R = \frac{v_0^2}{g} \sin 2\alpha$ . Then we want the factor  $p$  so that  $pv_0$  will double the range  
 $\Rightarrow \frac{(pv_0)^2}{g} \sin 2\alpha = 2 \left( \frac{v_0^2}{g} \sin 2\alpha \right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$  or about 141%. The same percentage will  
 approximately double the height:  $\frac{(pv_0 \sin \alpha)^2}{2g} = \frac{2(v_0 \sin \alpha)^2}{2g} \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ .
27. The projectile reaches its maximum height when its vertical component of velocity is zero  
 $\Rightarrow \frac{dy}{dt} = v_0 \sin \alpha - gt = 0 \Rightarrow t = \frac{v_0 \sin \alpha}{g} \Rightarrow y_{\max} = (v_0 \sin \alpha) \left( \frac{v_0 \sin \alpha}{g} \right) - \frac{1}{2}g \left( \frac{v_0 \sin \alpha}{g} \right)^2 = \frac{(v_0 \sin \alpha)^2}{g} - \frac{(v_0 \sin \alpha)^2}{2g}$   
 $= \frac{(v_0 \sin \alpha)^2}{2g}$ . To find the flight time we find the time when the projectile lands:  $(v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 0$   
 $\Rightarrow t(v_0 \sin \alpha - \frac{1}{2}gt) = 0 \Rightarrow t = 0$  or  $t = \frac{2v_0 \sin \alpha}{g}$ . Since  $t = 0$  is the time when the projectile is fired, then  
 $t = \frac{2v_0 \sin \alpha}{g}$  is the time when the projectile strikes the ground. The range is the value of the horizontal  
 component when  $t = \frac{2v_0 \sin \alpha}{g} \Rightarrow R = x = (v_0 \cos \alpha) \left( \frac{2v_0 \sin \alpha}{g} \right) = \frac{v_0^2}{g} (2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g} \sin 2\alpha$ . The range is  
 largest when  $2\alpha = 1 \Rightarrow \alpha = 45^\circ$ .
28. When marble  $A$  is located  $R$  units downrange, we have  $x = (v_0 \cos \alpha)t \Rightarrow R = (v_0 \cos \alpha)t \Rightarrow t = \frac{R}{v_0 \cos \alpha}$ . At that  
 time the height of marble  $A$  is  $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = (v_0 \sin \alpha) \left( \frac{R}{v_0 \cos \alpha} \right) - \frac{1}{2}g \left( \frac{R}{v_0 \cos \alpha} \right)^2$   
 $\Rightarrow y = R \tan \alpha - \frac{1}{2}g \left( \frac{R^2}{v_0^2 \cos^2 \alpha} \right)$ . The height of marble  $B$  at the same time  $t = \frac{R}{v_0 \cos \alpha}$  seconds is  
 $h = R \tan \alpha - \frac{1}{2}gt^2 = R \tan \alpha - \frac{1}{2}g \left( \frac{R^2}{v_0^2 \cos^2 \alpha} \right)$ . Since the heights are the same, the marbles collide regardless of  
 the initial velocity  $v_0$ .

$$\begin{aligned}
 29. \quad \frac{d\mathbf{r}}{dt} &= \int (-g\mathbf{j})dt = -gt\mathbf{j} + \mathbf{C}_1 \quad \text{and} \quad \frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \\
 &\Rightarrow \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}; \quad \mathbf{r} = \int [(v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}]dt \\
 &= (v_0 t \cos \alpha)\mathbf{i} + \left(v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2 \quad \text{and} \quad \mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0)\cos \alpha]\mathbf{i} + \left[v_0(0)\sin \alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2 \\
 &= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t \cos \alpha)\mathbf{i} + \left(y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t \cos \alpha \quad \text{and} \\
 &y = y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \text{The maximum height is } y &= \frac{(v_0 \sin \alpha)^2}{2g} \quad \text{and this occurs for } x = \frac{v_0^2}{2g} \sin 2\alpha = \frac{v_0^2 \sin \alpha \cos \alpha}{g}. \quad \text{These equations} \\
 &\text{describe parametrically the points on a curve in the } xy\text{-plane associated with the maximum heights on the} \\
 &\text{parabolic trajectories in terms of the parameter (launch angle) } \alpha. \quad \text{Eliminating the parameter } \alpha, \text{ we have} \\
 x^2 &= \frac{v_0^4 \sin^2 \alpha \cos^2 \alpha}{g^2} = \frac{(v_0^4 \sin^2 \alpha)(1 - \sin^2 \alpha)}{g^2} = \frac{v_0^4 \sin^2 \alpha}{g^2} - \frac{v_0^4 \sin^4 \alpha}{g^2} = \frac{v_0^2}{g} (2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0 \\
 &\Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{4g^2} \Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2}, \quad \text{where } x \geq 0.
 \end{aligned}$$

31. (a) At the time  $t$  when the projectile hits the line  $OR$

$$\text{we have } \tan \beta = \frac{y}{x}; \quad x = [v_0 \cos(\alpha - \beta)]t \quad \text{and}$$

$$y = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2}gt^2 < 0 \quad \text{since } R \text{ is}$$

below level ground. Therefore let

$$|y| = \frac{1}{2}gt^2 - [v_0 \sin(\alpha - \beta)]t > 0 \quad \text{so that}$$

$$\tan \beta = \frac{\left[\frac{1}{2}gt^2 - v_0 \sin(\alpha - \beta)t\right]}{[v_0 \cos(\alpha - \beta)]t} = \frac{\left[\frac{1}{2}gt - v_0 \sin(\alpha - \beta)\right]}{v_0 \cos(\alpha - \beta)}$$

$$\Rightarrow v_0 \cos(\alpha - \beta) \tan \beta = \frac{1}{2}gt - v_0 \sin(\alpha - \beta)$$

$$\Rightarrow t = \frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g},$$

which is the time when the projectile hits the downhill slope. Therefore,

$$x = [v_0 \cos(\alpha - \beta)] \left[ \frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g} \right] = \frac{2v_0^2}{g} [\cos^2(\alpha - \beta) \tan \beta + \sin(\alpha - \beta) \cos(\alpha - \beta)].$$

$$\text{If } x \text{ is maximized, then } OR \text{ is maximized: } \frac{dx}{d\alpha} = \frac{2v_0^2}{g} [-\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta)] = 0$$

$$\Rightarrow -\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta) = 0 \Rightarrow \tan \beta = \cot 2(\alpha - \beta) \Rightarrow 2(\alpha - \beta) = 90^\circ - \beta$$

$$\Rightarrow \alpha - \beta = \frac{1}{2}(90^\circ - \beta) \Rightarrow \alpha = \frac{1}{2}(90^\circ + \beta) = \frac{1}{2} \text{ of } \angle AOR.$$

- (b) At the time  $t$  when the projectile hits  $OR$  we have

$$\tan \beta = \frac{y}{x}; \quad x = [v_0 \cos(\alpha + \beta)]t \quad \text{and}$$

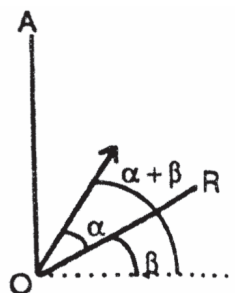
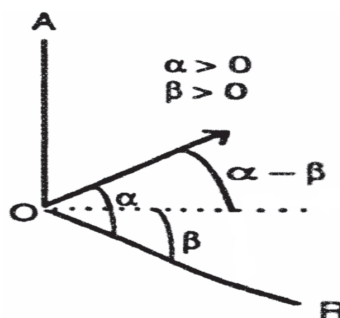
$$y = [v_0 \sin(\alpha + \beta)]t - \frac{1}{2}gt^2$$

$$\Rightarrow \tan \beta = \frac{[v_0 \sin(\alpha + \beta)]t - \frac{1}{2}gt^2}{[v_0 \cos(\alpha + \beta)]t} = \frac{[v_0 \sin(\alpha + \beta) - \frac{1}{2}gt]}{v_0 \cos(\alpha + \beta)}$$

$$\Rightarrow v_0 \cos(\alpha + \beta) \tan \beta = v_0 \sin(\alpha + \beta) - \frac{1}{2}gt$$

$$\Rightarrow t = \frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g}, \quad \text{which is the}$$

time when the projectile hits the uphill slope.



Therefore,  $x = [v_0 \cos(\alpha + \beta)] \left[ \frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g} \right]$   
 $= \frac{2v_0^2}{g} [\sin(\alpha + \beta) \cos(\alpha + \beta) - \cos^2(\alpha + \beta) \tan \beta]$ . If  $x$  is maximized, then  $OR$  is maximized:  
 $\frac{dx}{d\alpha} = \frac{2v_0^2}{g} [\cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta] = 0 \Rightarrow \cos 2(\alpha + \beta) + \sin 2(\alpha + \beta) \tan \beta = 0$   
 $\Rightarrow \cot 2(\alpha + \beta) + \tan \beta = 0 \Rightarrow \cot 2(\alpha + \beta) = -\tan \beta = \tan(-\beta) \Rightarrow 2(\alpha + \beta) = 90^\circ - (-\beta) = 90^\circ + \beta$   
 $\Rightarrow \alpha = \frac{1}{2}(90^\circ - \beta) = \frac{1}{2}$  of  $\angle AOR$ . Therefore  $v_0$  would bisect  $\angle AOR$  for maximum range uphill.

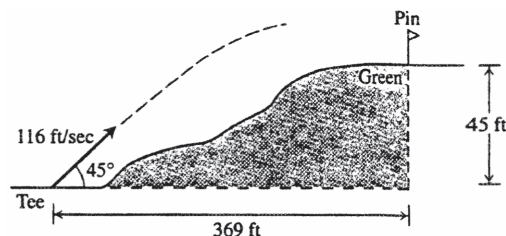
32.  $v_0 = 116$  ft/sec,  $\alpha = 45^\circ$ , and  $x = (v_0 \cos \alpha)t$

$$\Rightarrow 369 = (116 \cos 45^\circ)t \Rightarrow t \approx 4.50 \text{ sec};$$

$$\text{also } y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

$$\Rightarrow y = (116 \sin 45^\circ)(4.50) - \frac{1}{2}(32)(4.50)^2 \approx 45.11 \text{ ft.}$$

It will take the ball 4.50 sec to travel 369 ft. At that time the ball will be 45.11 ft in the air and will hit the green past the pin.



33. (a) (Assuming that "x" is zero at the point of impact:)

$$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}; \text{ where } x(t) = (35 \cos 27^\circ)t \text{ and } y(t) = 4 + (35 \sin 27^\circ)t - 16t^2.$$

- (b)  $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} + 4 = \frac{(35 \sin 27^\circ)^2}{64} + 4 \approx 7.945$  feet, which is reached at  $t = \frac{v_0 \sin \alpha}{g} = \frac{35 \sin 27^\circ}{32} \approx 0.497$  seconds.

- (c) For the time, solve  $y = 4 + (35 \sin 27^\circ)t - 16t^2 = 0$  for  $t$ , using the quadratic formula

$$t = \frac{35 \sin 27^\circ + \sqrt{(-35 \sin 27^\circ)^2 + 256}}{32} \approx 1.201 \text{ sec. Then the range is about } x(1.201) = (35 \cos 27^\circ)(1.201) \approx 37.453 \text{ feet.}$$

- (d) For the time, solve  $y = 4 + (35 \sin 27^\circ)t - 16t^2 = 7$  for  $t$ , using the quadratic formula

$$t = \frac{35 \sin 27^\circ + \sqrt{(-35 \sin 27^\circ)^2 - 192}}{32} \approx 0.254 \text{ and } 0.740 \text{ seconds. At those times the ball is about } x(0.254) = (35 \cos 27^\circ)(0.254) \approx 7.921 \text{ feet and } x(0.740) = (35 \cos 27^\circ)(0.740) \approx 23.077 \text{ feet from the impact point, or about } 37.453 - 7.921 \approx 29.532 \text{ feet and } 37.453 - 23.077 \approx 14.376 \text{ feet from the landing spot.}$$

- (e) Yes. It changes things because the ball won't clear the net ( $y_{\max} \approx 7.945$ ).

34.  $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40^\circ)t \approx 0.766 v_0 t$  and  $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40^\circ)t - 16t^2 \approx 6.5 + 0.643 v_0 t - 16t^2$ ; now the shot went 73.833 ft  $\Rightarrow 73.833 = 0.766 v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$  sec; the shot lands when  $y = 0 \Rightarrow 0 = 6.5 + (0.643)(96.383) - 16\left(\frac{96.383}{v_0}\right)^2 \Rightarrow 0 \approx 68.474 - \frac{148,635}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148,635}{68.474}} \approx 46.6$  ft/sec, the shot's initial speed

35. Flight time = 1 sec and the measure of the angle of elevation is about  $64^\circ$  (using a protractor) so that

$$t = \frac{2v_0 \sin \alpha}{g} \Rightarrow 1 = \frac{2v_0 \sin 64^\circ}{32} \Rightarrow v_0 \approx 17.80 \text{ ft/sec. Then } y_{\max} = \frac{(17.80 \sin 64^\circ)^2}{2(32)} \approx 4.00 \text{ ft and } R = \frac{v_0^2}{g} \sin 2\alpha$$

$\Rightarrow R = \frac{(17.80)^2}{32} \sin 128^\circ \approx 7.80 \text{ ft} \Rightarrow$  the engine traveled about 7.80 ft in 1 sec  $\Rightarrow$  the engine velocity was about 7.80 ft/sec

36. (a)  $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$ ; where  $x(t) = (145 \cos 23^\circ - 14)t$  and  $y(t) = 2.5 + (145 \sin 23^\circ)t - 16t^2$ .
- (b)  $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} + 2.5 = \frac{(145 \sin 23^\circ)^2}{64} + 2.5 \approx 52.655$  feet, which is reached at  $t = \frac{v_0 \sin \alpha}{g} = \frac{145 \sin 23^\circ}{32} \approx 1.771$  seconds.
- (c) For the time, solve  $y = 2.5 + (145 \sin 23^\circ)t - 16t^2 = 0$  for  $t$ , using the quadratic formula
- $$t = \frac{145 \sin 23^\circ + \sqrt{(145 \sin 23^\circ)^2 + 160}}{32} \approx 3.585 \text{ sec.}$$
- Then the range at  $t \approx 3.585$  is about
- $$x = (145 \cos 23^\circ - 14)(3.585) \approx 428.311 \text{ feet.}$$
- (d) For the time, solve  $y = 2.5 + (145 \sin 23^\circ)t - 16t^2 = 20$  for  $t$ , using the quadratic formula
- $$t = \frac{145 \sin 23^\circ + \sqrt{(145 \sin 23^\circ)^2 - 1120}}{32} \approx 0.342 \text{ and } 3.199 \text{ seconds.}$$
- At those times the ball is about
- $$x(0.342) = (145 \cos 23^\circ - 14)(0.342) \approx 40.860 \text{ feet from home plate and}$$
- $$x(3.199) = (145 \cos 23^\circ - 14)(3.199) \approx 382.195 \text{ feet from home plate.}$$
- (e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

37.  $\frac{d^2 \mathbf{r}}{dt^2} + k \frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$  and  $\mathbf{Q}(t) = -g\mathbf{j} \Rightarrow \int P(t) dt = kt \Rightarrow v(t) = e^{\int P(t) dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{1}{v(t)} \int v(t) \mathbf{Q}(t) dt$

$$= -ge^{-kt} \int e^{kt} \mathbf{j} dt = -ge^{-kt} \left[ \frac{e^{kt}}{k} \mathbf{j} + \mathbf{C}_1 \right] = -\frac{g}{k} \mathbf{j} + \mathbf{C} e^{-kt}, \text{ where } \mathbf{C} = -g\mathbf{C}_1; \text{ apply the initial condition:}$$

$$\frac{d\mathbf{r}}{dt} \Big|_{t=0} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} = -\frac{g}{k} \mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0 \cos \alpha)\mathbf{i} + \left( \frac{g}{k} + v_0 \sin \alpha \right) \mathbf{j}$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = \left( v_0 e^{-kt} \cos \alpha \right) \mathbf{i} + \left( -\frac{g}{k} + e^{-kt} \left( \frac{g}{k} + v_0 \sin \alpha \right) \right) \mathbf{j}, \quad \mathbf{r} = \int \left[ \left( v_0 e^{-kt} \cos \alpha \right) \mathbf{i} + \left( -\frac{g}{k} + e^{-kt} \left( \frac{g}{k} + v_0 \sin \alpha \right) \right) \mathbf{j} \right] dt$$

$$= \left( -\frac{v_0}{k} e^{-kt} \cos \alpha \right) \mathbf{i} + \left( -\frac{gt}{k} - \frac{e^{-kt}}{k} \left( \frac{g}{k} + v_0 \sin \alpha \right) \right) \mathbf{j} + \mathbf{C}_2; \text{ apply the initial condition:}$$

$$\mathbf{r}(0) = \mathbf{0} = \left( -\frac{v_0}{k} \cos \alpha \right) \mathbf{i} + \left( -\frac{g}{k^2} - \frac{v_0 \sin \alpha}{k} \right) \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left( \frac{v_0}{k} \cos \alpha \right) \mathbf{i} + \left( \frac{g}{k^2} + \frac{v_0 \sin \alpha}{k} \right) \mathbf{j}$$

$$\Rightarrow \mathbf{r}(t) = \left( \frac{v_0}{k} (1 - e^{-kt}) \cos \alpha \right) \mathbf{i} + \left( \frac{v_0}{k} (1 - e^{-kt}) \sin \alpha + \frac{g}{k^2} (1 - kt - e^{-kt}) \right) \mathbf{j}$$

38. (a)  $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$ ; where  $x(t) = \left( \frac{152}{0.12} \right) (1 - e^{-0.12t}) (\cos 20^\circ)$  and
- $$y(t) = 3 + \left( \frac{152}{0.12} \right) (1 - e^{-0.12t}) (\sin 20^\circ) + \left( \frac{32}{0.12^2} \right) (1 - 0.12t - e^{-0.12t})$$
- (b) Solve graphically using a calculator or CAS: At  $t \approx 1.484$  seconds the ball reaches a maximum height of about 40.435 feet.
- (c) Use a graphing calculator or CAS to find that  $y = 0$  when the ball has traveled for  $\approx 3.126$  seconds. The range is about  $x(3.126) = \left( \frac{152}{0.12} \right) (1 - e^{-0.12(3.126)}) (\cos 20^\circ) \approx 372.311$  feet.
- (d) Use a graphing calculator or CAS to find that  $y = 30$  for  $t \approx 0.689$  and 2.305 seconds, at which times the ball is about  $x(0.689) \approx 94.454$  feet and  $x(2.305) \approx 287.621$  feet from home plate.
- (e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

39. (a)  $\int_a^b k\mathbf{r}(t) dt = \int_a^b [kf(t)\mathbf{i} + kg(t)\mathbf{j} + kh(t)\mathbf{k}] dt = \int_a^b [kf(t)] dt \mathbf{i} + \int_a^b [kg(t)] dt \mathbf{j} + \int_a^b [kh(t)] dt \mathbf{k}$   
 $= k \left( \int_a^b f(t) dt \mathbf{i} + \int_a^b g(t) dt \mathbf{j} + \int_a^b h(t) dt \mathbf{k} \right) = k \int_a^b \mathbf{r}(t) dt$
- (b)  $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)] dt = \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]) dt$   
 $= \int_a^b ([f_1(t) \pm f_2(t)]\mathbf{i} + [g_1(t) \pm g_2(t)]\mathbf{j} + [h_1(t) \pm h_2(t)]\mathbf{k}) dt$   
 $= \int_a^b [f_1(t) \pm f_2(t)] dt \mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)] dt \mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)] dt \mathbf{k}$   
 $= \left[ \int_a^b f_1(t) dt \mathbf{i} \pm \int_a^b f_2(t) dt \mathbf{i} \right] + \left[ \int_a^b g_1(t) dt \mathbf{j} \pm \int_a^b g_2(t) dt \mathbf{j} \right] + \left[ \int_a^b h_1(t) dt \mathbf{k} \pm \int_a^b h_2(t) dt \mathbf{k} \right]$   
 $= \int_a^b \mathbf{r}_1(t) dt \pm \int_a^b \mathbf{r}_2(t) dt$
- (c) Let  $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ . Then  $\int_a^b \mathbf{C} \cdot \mathbf{r}(t) dt = \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)] dt$   
 $= c_1 \int_a^b f(t) dt + c_2 \int_a^b g(t) dt + c_3 \int_a^b h(t) dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t) dt;$   
 $\int_a^b \mathbf{C} \times \mathbf{r}(t) dt = \int_a^b [c_2h(t) - c_3g(t)]\mathbf{i} + [c_3f(t) - c_1h(t)]\mathbf{j} + [c_1g(t) - c_2f(t)]\mathbf{k} dt$   
 $= \left[ c_2 \int_a^b h(t) dt - c_3 \int_a^b g(t) dt \right] \mathbf{i} + \left[ c_3 \int_a^b f(t) dt - c_1 \int_a^b h(t) dt \right] \mathbf{j} + \left[ c_1 \int_a^b g(t) dt - c_2 \int_a^b f(t) dt \right] \mathbf{k}$   
 $= \mathbf{C} \times \int_a^b \mathbf{r}(t) dt$
40. (a) Let  $u$  and  $\mathbf{r}$  be continuous on  $[a, b]$ . Then  $\lim_{t \rightarrow t_0} u(t)\mathbf{r}(t) = \lim_{t \rightarrow t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$   
 $= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r}$  is continuous for every  $t_0$  in  $[a, b]$ .
- (b) Let  $u$  and  $\mathbf{r}$  be differentiable. Then  $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$   
 $= \left( \frac{du}{dt}f(t) + u(t)\frac{df}{dt} \right) \mathbf{i} + \left( \frac{du}{dt}g(t) + u(t)\frac{dg}{dt} \right) \mathbf{j} + \left( \frac{du}{dt}h(t) + u(t)\frac{dh}{dt} \right) \mathbf{k}$   
 $= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \frac{du}{dt} + u(t) \left( \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right) = \mathbf{r} \frac{du}{dt} + u \frac{d\mathbf{r}}{dt}$
41. (a) If  $\mathbf{R}_1(t)$  and  $\mathbf{R}_2(t)$  have identical derivatives on  $I$ , then  $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$   
 $= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt}, \frac{dg_1}{dt} = \frac{dg_2}{dt}, \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1, g_1(t) = g_2(t) + c_2, h_1(t) = h_2(t) + c_3$   
 $\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$ , where  
 $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ .
- (b) Let  $\mathbf{R}(t)$  be an antiderivative of  $\mathbf{r}(t)$  on  $I$ . Then  $\mathbf{R}'(t) = \mathbf{r}(t)$ . If  $\mathbf{U}(t)$  is an antiderivative of  $\mathbf{r}(t)$  on  $I$ , then  $\mathbf{U}'(t) = \mathbf{r}(t)$ . Thus  $\mathbf{U}'(t) = \mathbf{R}'(t)$  on  $I \Rightarrow \mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$ .
42.  $\frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \frac{d}{dt} \int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}] d\tau = \frac{d}{dt} \int_a^t f(\tau) d\tau \mathbf{i} + \frac{d}{dt} \int_a^t g(\tau) d\tau \mathbf{j} + \frac{d}{dt} \int_a^t h(\tau) d\tau \mathbf{k}$   
 $= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t)$ . Since  $\frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{r}(t)$ , we have that  $\int_a^t \mathbf{r}(\tau) d\tau$  is an antiderivative of  $\mathbf{r}$ .

If  $\mathbf{R}$  is any antiderivative of  $\mathbf{r}$ , then  $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau) d\tau + \mathbf{C}$  by Exercise 41(b). Then  $\mathbf{R}(a) = \int_a^a \mathbf{r}(\tau) d\tau + \mathbf{C} = \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau) d\tau = \mathbf{R}(b) - \mathbf{R}(a)$ .

43. (a)  $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$ ; where  $x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152 \cos 20^\circ - 17.6)$  and
- $$y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20^\circ) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$$
- (b) Solve graphically using a calculator or CAS: At  $t \approx 1.527$  seconds the ball reaches a maximum height of about 41.893 feet.
- (c) Use a graphing calculator or CAS to find that  $y = 0$  when the ball has traveled for  $\approx 3.181$  seconds. The range is about  $x(3.181) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(3.181)})(152 \cos 20^\circ - 17.6) \approx 351.734$  feet.
- (d) Use a graphing calculator or CAS to find that  $y = 35$  for  $t \approx 0.877$  and 2.190 seconds, at which times the ball is about  $x(0.877) \approx 106.028$  feet and  $x(2.190) \approx 251.530$  feet from home plate.
- (e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that  $y = 20$  at  $t \approx 0.376$  and 2.716 seconds. Then define
- $$x(w) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(2.716)})(152 \cos 20^\circ + w), \text{ and solve } x(w) = 380 \text{ to find } w \approx 12.846 \text{ ft/sec.}$$

44.  $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g} \Rightarrow \frac{3}{4} y_{\max} = \frac{3(v_0 \sin \alpha)^2}{8g}$  and  $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin \alpha)^2}{8g} = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$   
 $\Rightarrow 3(v_0 \sin \alpha)^2 = (8gv_0 \sin \alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin \alpha)t + 3(v_0 \sin \alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin \alpha = 0$  or  
 $2gt - v_0 \sin \alpha = 0 \Rightarrow t = \frac{3v_0 \sin \alpha}{2g}$  or  $t = \frac{v_0 \sin \alpha}{2g}$ . Since the time it takes to reach  $y_{\max}$  is  $t_{\max} = \frac{v_0 \sin \alpha}{g}$ , then  
the time it takes the projectile to reach  $\frac{3}{4}$  of  $y_{\max}$  is the shorter time  $t = \frac{v_0 \sin \alpha}{2g}$  or half the time it takes to reach the maximum height.

### 13.3 ARC LENGTH IN SPACE

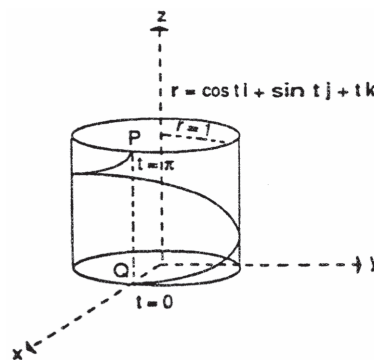
- $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \sqrt{5}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (\sqrt{5})^2}$   
 $= \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} = 3$ ;  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(-\frac{2}{3} \sin t\right)\mathbf{i} + \left(\frac{2}{3} \cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$  and  $\text{Length} = \int_0^\pi |\mathbf{v}| dt = \int_0^\pi 3 dt$   
 $= [3t]_0^\pi = 3\pi$
- $\mathbf{r} = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j} + 5\mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{144 \cos^2 2t + 144 \sin^2 2t + 25} = 13$ ;  
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$  and  $\text{Length} = \int_0^\pi |\mathbf{v}| dt = \int_0^\pi 13 dt = [13t]_0^\pi = 13\pi$
- $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}$ ;  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$  and  
 $\text{Length} = \int_0^8 \sqrt{1+t} dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$

4.  $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$ ;  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$  and  
 Length  $= \int_0^3 \sqrt{3} \, dt = \left[ \sqrt{3}t \right]_0^3 = 3\sqrt{3}$
5.  $\mathbf{r} = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\mathbf{j} + (3\sin^2 t \cos t)\mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{(9\cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} = 3|\cos t \sin t|$ ;  
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3|\cos t \sin t|}\mathbf{j} + \frac{3\sin^2 t \cos t}{3|\cos t \sin t|}\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}$ , if  $0 \leq t \leq \frac{\pi}{2}$ , and  
 Length  $= \int_0^{\pi/2} 3|\cos t \sin t| \, dt = \int_0^{\pi/2} 3 \cos t \sin t \, dt = \int_0^{\pi/2} \frac{3}{2} \sin 2t \, dt = \left[ -\frac{3}{4} \cos 2t \right]_0^{\pi/2} = \frac{3}{2}$
6.  $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2$ ;  
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\mathbf{i} - \frac{6t^2}{21t^2}\mathbf{j} - \frac{9t^2}{21t^2}\mathbf{k} = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$  and Length  $= \int_1^2 21t^2 \, dt = \left[ 7t^3 \right]_1^2 = 49$
7.  $\mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + (\sqrt{2}t^{1/2})\mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (\sqrt{2}t)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1$ , if  $t \geq 0$ ;  
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \frac{\cos t - t \sin t}{t+1} \right)\mathbf{i} + \left( \frac{\sin t + t \cos t}{t+1} \right)\mathbf{j} + \left( \frac{\sqrt{2}t^{1/2}}{t+1} \right)\mathbf{k}$  and Length  $= \int_0^\pi (t+1) \, dt = \left[ \frac{t^2}{2} + t \right]_0^\pi = \frac{\pi^2}{2} + \pi$
8.  $\mathbf{r} = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t \cos t - \sin t)\mathbf{i} + (\cos t - t \sin t - \cos t)\mathbf{j}$   
 $= (t \cos t)\mathbf{i} - (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (-t \sin t)^2} = \sqrt{t^2} = |t| = t$  if  $\sqrt{2} \leq t \leq 2$ ;  
 $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \frac{t \cos t}{t} \right)\mathbf{i} - \left( \frac{t \sin t}{t} \right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$  and Length  $= \int_{\sqrt{2}}^2 t \, dt = \left[ \frac{t^2}{2} \right]_{\sqrt{2}}^2 = 1$
9. Let  $P(t_0)$  denote the point. Then  $\mathbf{v} = (5 \cos t)\mathbf{i} - (5 \sin t)\mathbf{j} + 12\mathbf{k}$  and  $26\pi = \int_0^{t_0} \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} \, dt$   
 $= \int_0^{t_0} 13 \, dt = 13t_0 \Rightarrow t_0 = 2\pi$ , and the point is  $P(2\pi) = (5 \sin 2\pi, 5 \cos 2\pi, 24\pi) = (0, 5, 24\pi)$
10. Let  $P(t_0)$  denote the point. Then  $\mathbf{v} = (12 \cos t)\mathbf{i} + (12 \sin t)\mathbf{j} + 5\mathbf{k}$  and  
 $-13\pi = \int_0^{t_0} \sqrt{144 \cos^2 t + 144 \sin^2 t + 25} \, dt = \int_0^{t_0} 13 \, dt = 13t_0 \Rightarrow t_0 = -\pi$ , and the point is  
 $P(-\pi) = (12 \sin(-\pi), -12 \cos(-\pi), -5\pi) = (0, 12, -5\pi)$
11.  $\mathbf{r} = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2 + 3^2}$   
 $= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5 \, d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$



12.  $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t \cos t)\mathbf{i} + (\cos t - \cos t + t \sin t)\mathbf{j}$   
 $= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = t$ , since  $\frac{\pi}{2} \leq t \leq \pi \Rightarrow s(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2}$   
 $\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$
13.  $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t \mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} = \sqrt{3e^{2t}} = \sqrt{3}e^t \Rightarrow s(t) = \int_0^t \sqrt{3}e^\tau \, d\tau = \sqrt{3}e^t - \sqrt{3}$   
 $\Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - (\sqrt{3}e^{-\ln 4} - \sqrt{3}) = \frac{3\sqrt{3}}{4}$
14.  $\mathbf{r} = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7 \, d\tau = 7t$   
 $\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$
15.  $\mathbf{r} = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2} = \sqrt{4+4t^2}$   
 $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1+t^2} \, dt = \left[ 2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right) \right]_0^1 = \sqrt{2} + \ln(1+\sqrt{2})$

16. Let the helix make one complete turn from  $t = 0$  to  $t = 2\pi$ . Note that the radius of the cylinder is 1  
 $\Rightarrow$  the circumference of the base is  $2\pi$ . When  $t = 2\pi$ , the point  $P$  is  
 $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$  the cylinder is  $2\pi$  units high. Cut the cylinder along  $PQ$  and flatten. The resulting rectangle has a width equal to the circumference of the cylinder  $= 2\pi$  and a height equal to  $2\pi$ , the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from  $t = 0$  to  $t = 2\pi$  is its diagonal.



17. (a)  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$ ,  $0 \leq t \leq 2\pi \Rightarrow x = \cos t$ ,  $y = \sin t$ ,  $z = 1 - \cos t$   
 $\Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ , a right circular cylinder with the  $z$ -axis as the axis and radius  $= 1$ .  
 Therefore  $P(\cos t, \sin t, 1 - \cos t)$  lies on the cylinder  $x^2 + y^2 = 1$ ;  $t = 0 \Rightarrow P(1, 0, 0)$  is on the curve;  
 $t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$  is on the curve;  $t = \pi \Rightarrow R(-1, 0, 2)$  is on the curve. Then  $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  
 $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{k}$  is a vector normal to the plane of  $P$ ,  $Q$ , and  $R$ . Then  
 the plane containing  $P$ ,  $Q$ , and  $R$  has an equation  $2x + 2z = 2(1) + 2(0)$  or  $x + z = 1$ . Any point on the curve will satisfy this equation since  $x + z = \cos t + (1 - \cos t) = 1$ . Therefore, any point on the curve lies on the intersection of the cylinder  $x^2 + y^2 = 1$  and the plane  $x + z = 1 \Rightarrow$  the curve is an ellipse.

$$(b) \quad \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t}$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1 + \sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j}, \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{T}(\pi) = -\mathbf{j}, \mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$$

$$(c) \quad \mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}; \quad \mathbf{n} = \mathbf{i} + \mathbf{k} \text{ is}$$

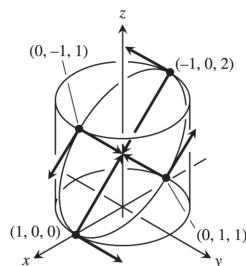
normal to the plane  $x + z = 1$

$$\Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t = 0$$

$$\Rightarrow \mathbf{a} \text{ is orthogonal to } \mathbf{n}$$

$$\Rightarrow \mathbf{a} \text{ is parallel to the plane;}$$

$$\mathbf{a}(0) = -\mathbf{i} + \mathbf{k}, \mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j}, \mathbf{a}(\pi) = \mathbf{i} - \mathbf{k}, \mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$$



$$(d) \quad |\mathbf{v}| = \sqrt{1 + \sin^2 t} \quad (\text{See part (b)}) \Rightarrow L = \int_0^{2\pi} \sqrt{1 + \sin^2 t} \, dt$$

$$(e) \quad L \approx 7.64 \quad (\text{by Mathematica})$$

$$18. (a) \quad \mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + 4\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2} \, dt = \left[ 4\sqrt{2}t \right]_0^{\pi/2} = 2\pi\sqrt{2}$$

$$(b) \quad \mathbf{r} = \left(\cos \frac{t}{2}\right)\mathbf{i} + \left(\sin \frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2}\sin \frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2}\cos \frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2}\sin \frac{t}{2}\right)^2 + \left(\frac{1}{2}\cos \frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2} \, dt = \left[ \frac{\sqrt{2}}{2}t \right]_0^{4\pi} = 2\pi\sqrt{2}$$

$$(c) \quad \mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1 + 1}$$

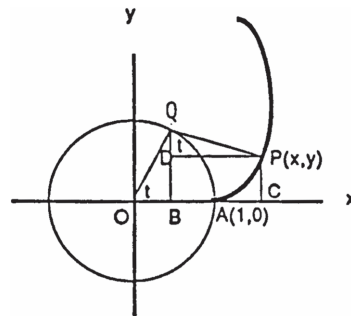
$$= \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^0 \sqrt{2} \, dt = \left[ \sqrt{2}t \right]_{-2\pi}^0 = 2\pi\sqrt{2}$$

$$19. \quad \angle PQB = \angle QOB = t \text{ and } PQ = \text{arc } (AQ) = t \text{ since}$$

$PQ = \text{length of the unwound string} = \text{length of arc } (AQ)$ ;

thus  $x = OB + BC = OB + DP = \cos t + t \sin t$ , and

$y = PC = QB - QD = \sin t - t \cos t$



$$20. \quad \mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + t \cos t + \sin t)\mathbf{i} + (\cos t - (t(-\sin t) + \cos t))\mathbf{j}$$

$$= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = |t| = t, t \geq 0$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t \cos t}{t}\mathbf{i} + \frac{t \sin t}{t}\mathbf{j} = \cos t \mathbf{i} + \sin t \mathbf{j}$$

$$21. \quad \mathbf{v} = \frac{d}{dt}(x_0 + t u_1)\mathbf{i} + \frac{d}{dt}(y_0 + t u_2)\mathbf{j} + \frac{d}{dt}(z_0 + t u_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u}, \text{ so}$$

$$s(t) = \int_0^t |\mathbf{v}| \, dt = \int_0^t |\mathbf{u}| \, d\tau = \int_0^t 1 \, d\tau = t$$

$$\begin{aligned}
 22. \quad \mathbf{r}(t) &= t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}(t)| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}. \quad (0, 0, 0) \Rightarrow t = 0 \\
 &\text{and } (2, 4, 8) \Rightarrow t = 2. \text{ Thus } L = \int_0^2 |\mathbf{v}(t)| \, dt = \int_0^2 \sqrt{1 + 4t^2 + 9t^4} \, dt. \text{ Using Simpson's rule with } n = 10 \text{ and} \\
 \Delta x &= \frac{2-0}{10} = 0.2 \\
 \Rightarrow L &\approx \frac{0.2}{3} (|\mathbf{v}(0)| + 4|\mathbf{v}(0.2)| + 2|\mathbf{v}(0.4)| + 4|\mathbf{v}(0.6)| + 2|\mathbf{v}(0.8)| + 4|\mathbf{v}(1)| + 2|\mathbf{v}(1.2)| + 4|\mathbf{v}(1.4)| + 2|\mathbf{v}(1.6)| \\
 &\quad + 4|\mathbf{v}(1.8)| + |\mathbf{v}(2)|) \\
 &\approx \frac{0.2}{3} (1 + 4(1.0837) + 2(1.3676) + 4(1.8991) + 2(2.6919) + 4(3.7417) + 2(5.0421) + 4(6.5890) + 2(8.3800) \\
 &\quad + 4(10.4134) + 12.6886) = \frac{0.2}{3} (143.5594) \approx 9.5706
 \end{aligned}$$

### 13.4 CURVATURE AND NORMAL VECTORS OF A CURVE

- $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t,$   
 since  $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \quad \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$   
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t.$
- $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t,$   
 since  $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} + \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad \frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$   
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t.$
- $\mathbf{r} = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1+t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$   
 $= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}; \quad \frac{d\mathbf{T}}{dt} = \frac{-t}{(\sqrt{1+t^2})^3}\mathbf{i} - \frac{1}{(\sqrt{1+t^2})^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{(\sqrt{1+t^2})^3}\right)^2 + \left(\frac{-1}{(\sqrt{1+t^2})^3}\right)^2} = \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2}$   
 $\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{3/2}}$
- $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = |t| = t,$   
 since  $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}; \quad \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$   
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t} \cdot 1 = \frac{1}{t}$

5. (a)  $\kappa(x) = \frac{1}{|\mathbf{v}(x)|} \cdot \left| \frac{d\mathbf{T}(x)}{dt} \right|$ . Now,  $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow |\mathbf{v}(x)| = \sqrt{1 + [f'(x)]^2}$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(1 + [f'(x)]^2\right)^{-1/2} \mathbf{i} + f'(x) \left(1 + [f'(x)]^2\right)^{-1/2} \mathbf{j}. \text{ Thus } \frac{d\mathbf{T}}{dt}(x) = \frac{-f'(x)f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}} \mathbf{i} + \frac{f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}} \mathbf{j}$$

$$\Rightarrow \left| \frac{d\mathbf{T}(x)}{dt} \right| = \sqrt{\left[ \frac{-f'(x)f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}} \right]^2 + \left[ \frac{f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}} \right]^2} = \sqrt{\frac{[f''(x)]^2(1 + [f'(x)]^2)}{\left(1 + [f'(x)]^2\right)^3}} = \frac{|f''(x)|}{1 + [f'(x)]^2}$$

Thus  $\kappa(x) = \frac{1}{\left(1 + [f'(x)]^2\right)^{1/2}} \cdot \frac{|f''(x)|}{1 + [f'(x)]^2} = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}}$

(b)  $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|\sec^2 x|}{\left[1 + (-\tan x)^2\right]^{3/2}} = \frac{\sec^2 x}{\sec^3 x}$

$= \frac{1}{\sec x} = \cos x$ , since  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) Note that  $f''(x) = 0$  at an inflection point.

6. (a)  $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \mathbf{i} + \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \mathbf{j}$

$$\frac{d\mathbf{T}}{dt} = \frac{\dot{y}(\ddot{x} - \dot{x}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \mathbf{i} + \frac{\dot{x}(\ddot{y} - \dot{y}\ddot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left[ \frac{\dot{y}(\ddot{x} - \dot{x}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right]^2 + \left[ \frac{\dot{x}(\ddot{y} - \dot{y}\ddot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right]^2} = \sqrt{\frac{(\dot{y}^2 + \dot{x}^2)(\ddot{y} - \dot{y}\ddot{x})^2}{(\dot{x}^2 + \dot{y}^2)^3}} = \frac{|\ddot{y} - \dot{y}\ddot{x}|}{\dot{x}^2 + \dot{y}^2}$$

$$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \cdot \frac{|\ddot{y} - \dot{y}\ddot{x}|}{\dot{x}^2 + \dot{y}^2} = \frac{|\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(b)  $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}$ ,  $0 < t < \pi \Rightarrow x = t$  and  $y = \ln(\sin t) \Rightarrow \dot{x} = 1, \ddot{x} = 0; \dot{y} = \frac{\cos t}{\sin t} = \cot t, \ddot{y} = -\csc^2 t$

$$\Rightarrow \kappa = \frac{|\csc^2 t - 0|}{(1 + \cot^2 t)^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$$

(c)  $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$  and  $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1 + \sinh^2 t} = \frac{1}{\cosh t}$

$$= \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t, \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t|}{(\operatorname{sech}^2 t + \tanh^2 t)} = |\operatorname{sech} t| = \operatorname{sech} t$$

7. (a)  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$  is tangent to the curve at the point  $(f(t), g(t))$ ;

$\mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0$ ;  $-\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0$ ; thus,  $\mathbf{n}$  and  $-\mathbf{n}$  are both normal to the curve at the point

(b)  $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$  points toward the concave side of the curve;  $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$  and

$$|\mathbf{n}| = \sqrt{4e^{4t} + 1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1 + 4e^{4t}}} \mathbf{i} + \frac{1}{\sqrt{1 + 4e^{4t}}} \mathbf{j}$$

(c)  $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4 - t^2}} \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4 - t^2}} \mathbf{j}$  points toward the concave side of the curve;

$$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|} \text{ and } |\mathbf{n}| = \sqrt{1 + \frac{t^2}{4 - t^2}} = \frac{2}{\sqrt{4 - t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2} \left( \sqrt{4 - t^2} \mathbf{i} + t \mathbf{j} \right)$$

8. (a)  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$  points toward the concave side of the curve when  $t < 0$  and  $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$  points toward the concave side when  $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}(t^2\mathbf{i} - \mathbf{j})$  for  $t < 0$  and  $\mathbf{N} = \frac{1}{\sqrt{1+t^4}}(-t^2\mathbf{i} + \mathbf{j})$  for  $t > 0$
- (b) From part (a),  $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}\mathbf{i} + \frac{t^2}{\sqrt{1+t^4}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4t^6+4t^2}{(1+t^4)^3}}$
- $= \frac{2|t|}{1+t^4}$ ;  $\mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1+t^4}{2|t|} \left( \frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j} \right) = \frac{-t^3}{|t|\sqrt{1+t^4}}\mathbf{i} + \frac{t}{|t|\sqrt{1+t^4}}\mathbf{j}$ ;  $t \neq 0$ .  $\mathbf{N}$  does not exist at  $t = 0$ , where the curve has a point of inflection;  $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$  so the curvature  $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{ds} \cdot \frac{dt}{ds}\right| = 0$  at  $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$  is undefined. Since  $x = t$  and  $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$ , the curve is the cubic power curve which is concave down for  $x = t < 0$  and concave up for  $x = t > 0$ .
9.  $\mathbf{r} = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2} = \sqrt{25} = 5$
- $\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5}\cos t\right)\mathbf{i} - \left(\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5}\sin t\right)\mathbf{i} - \left(\frac{3}{5}\cos t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(-\frac{3}{5}\cos t\right)^2} = \frac{3}{5}$
- $\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ ;  $\kappa = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$
10.  $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t$ , if  $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ ,  $t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
- $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ ;  $\kappa = \frac{1}{t} \cdot 1 = \frac{1}{t}$
11.  $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$
- $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2e^{2t}} = e^t \sqrt{2}$ ;  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$
- $\Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1$
- $\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$ ;  $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{e^t \sqrt{2}} \cdot 1 = \frac{1}{e^t \sqrt{2}}$
12.  $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} - (12\sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\mathbf{i} - \left(\frac{24}{13}\cos 2t\right)\mathbf{j}$
- $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13}\sin 2t\right)^2 + \left(-\frac{24}{13}\cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}$ ;  $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{13} \cdot \frac{24}{13} = \frac{24}{169}$ .

$$13. \quad \mathbf{r} = \left(\frac{t^3}{3}\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j}, t > 0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2 + 1}, \text{ since } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t}{\sqrt{t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{t^2 + 1}}\mathbf{j}$$

$$\Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2 + 1)^{3/2}}\mathbf{i} - \frac{t}{(t^2 + 1)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{(t^2 + 1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2 + 1)^{3/2}}\right)^2} = \sqrt{\frac{1 + t^2}{(t^2 + 1)^3}} = \frac{1}{t^2 + 1}$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1}{\sqrt{t^2 + 1}}\mathbf{i} - \frac{t}{\sqrt{t^2 + 1}}\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t\sqrt{t^2 + 1}} \cdot \frac{1}{t^2 + 1} = \frac{1}{t(t^2 + 1)^{3/2}}.$$

$$14. \quad \mathbf{r} = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, 0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\mathbf{i} + (3\sin^2 t \cos t)\mathbf{j}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\cos t \sin t, \text{ since } 0 < t < \frac{\pi}{2}$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{3\cos t \sin t} \cdot 1 = \frac{1}{3\cos t \sin t}.$$

$$15. \quad \mathbf{r} = t\mathbf{i} + \left(a \cosh \frac{t}{a}\right)\mathbf{j}, a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh \frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2 \left(\frac{t}{a}\right)} = \sqrt{\cosh^2 \left(\frac{t}{a}\right)} = \cosh \frac{t}{a}$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a} \operatorname{sech} \frac{t}{a} \tanh \frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a} \operatorname{sech}^2 \frac{t}{a}\right)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{a^2} \operatorname{sech}^2 \left(\frac{t}{a}\right) \tanh^2 \left(\frac{t}{a}\right) + \frac{1}{a^2} \operatorname{sech}^4 \left(\frac{t}{a}\right)} = \frac{1}{a} \operatorname{sech} \left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{j};$$

$$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\cosh \frac{t}{a}} \cdot \frac{1}{a} \operatorname{sech} \left(\frac{t}{a}\right) = \frac{1}{a} \operatorname{sech}^2 \left(\frac{t}{a}\right).$$

$$16. \quad \mathbf{r} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2} \cosh t$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t\right)\mathbf{k}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2} \operatorname{sech}^4 t + \frac{1}{2} \operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}} \operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k};$$

$$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2} \cosh t} \cdot \frac{1}{\sqrt{2}} \operatorname{sech} t = \frac{1}{2} \operatorname{sech}^2 t.$$

$$17. \quad y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a; \text{ from Exercise 5(a), } \kappa(x) = \frac{|2a|}{(1 + 4a^2x^2)^{3/2}} = |2a|(1 + 4a^2x^2)^{-3/2}$$

$\Rightarrow \kappa'(x) = -\frac{3}{2}|2a|(1 + 4a^2x^2)^{-5/2}(8a^2x)$ ; thus,  $\kappa'(x) = 0 \Rightarrow x = 0$ . Now,  $\kappa'(x) > 0$  for  $x < 0$  and  $\kappa'(x) < 0$  for  $x > 0$  so that  $\kappa(x)$  has an absolute maximum at  $x = 0$  which is the vertex of the parabola. Since  $x = 0$  is the only critical point for  $\kappa(x)$ , the curvature has no minimum value.

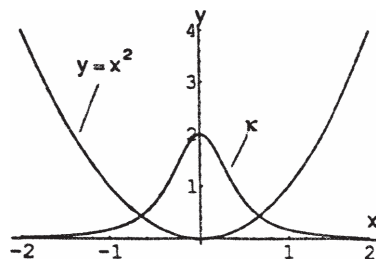
18.  $\mathbf{r} = (a \cos t)\mathbf{i} + (b \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (b \cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a \cos t)\mathbf{i} - (b \sin t)\mathbf{j}$
- $$\Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & b \cos t & 0 \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = |ab| = ab, \text{ since } a > b > 0; \kappa(t) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$
- $$= ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-3/2}; \quad \kappa'(t) = -\frac{3}{2}(ab)(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}(2a^2 \sin t \cos t - 2b^2 \sin t \cos t)$$
- $$= -\frac{3}{2}(ab)(a^2 - b^2)(\sin t)(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}; \text{ thus, } \kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi \text{ identifying points}$$
- on the major axis, or  $t = \frac{\pi}{2}, \frac{3\pi}{2}$  identifying points on the minor axis. Furthermore,  $\kappa'(t) < 0$  for  $0 < t < \frac{\pi}{2}$  and for  $\pi < t < \frac{3\pi}{2}$ ;  $\kappa'(t) > 0$  for  $\frac{\pi}{2} < t < \pi$  and  $\frac{3\pi}{2} < t < 2\pi$ . Therefore, the points associated with  $t = 0$  and  $t = \pi$  on the major axis give absolute maximum curvature and the points associated with  $t = \frac{\pi}{2}$  and  $t = \frac{3\pi}{2}$  on the minor axis give absolute minimum curvature.
19.  $\kappa = \frac{a}{a^2 + b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2 + b^2}{(a^2 + b^2)^2}; \frac{d\kappa}{da} = 0 \Rightarrow -a^2 + b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b$  since  $a, b \geq 0$ . Now,  $\frac{d\kappa}{da} > 0$  if  $a < b$  and  $\frac{d\kappa}{da} < 0$  if  $a > b \Rightarrow \kappa$  is at a maximum for  $a = b$  and  $\kappa(b) = \frac{b}{b^2 + b^2} = \frac{1}{2b}$  is the maximum value of  $\kappa$ .
20. (a) From Example 5, the curvature of the helix  $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}$ ,  $a, b \geq 0$  is  $\kappa = \frac{a}{a^2 + b^2}$ ; also  $|\mathbf{v}| = \sqrt{a^2 + b^2}$ . For the helix  $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 4\pi$ ,  $a = 3$  and  $b = 1 \Rightarrow \kappa = \frac{3}{3^2 + 1^2} = \frac{3}{10}$  and  $|\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10} \sqrt{10} dt = \left[ \frac{3}{\sqrt{10}} t \right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$
- (b)  $y = x^2 \Rightarrow x = t$  and  $y = t^2$ ,  $-\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2}$ ;
- $$\mathbf{T} = \frac{1}{\sqrt{1 + 4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1 + 4t^2}}\mathbf{j}; \frac{d\mathbf{T}}{dt} = \frac{-4t}{(1 + 4t^2)^{3/2}}\mathbf{i} + \frac{2}{(1 + 4t^2)^{3/2}}\mathbf{j}; \left| \frac{d\mathbf{T}}{dt} \right| = \frac{\sqrt{16t^2 + 4}}{(1 + 4t^2)^3} = \frac{2}{1 + 4t^2}. \text{ Thus } \kappa = \frac{1}{\sqrt{1 + 4t^2}} \cdot \frac{2t}{1 + 4t^2}$$
- $$= \frac{2}{(\sqrt{1 + 4t^2})^3}. \text{ Then } K = \int_{-\infty}^{\infty} \frac{2}{(\sqrt{1 + 4t^2})^3} \left( \sqrt{1 + 4t^2} \right) dt = \int_{-\infty}^{\infty} \frac{2}{1 + 4t^2} dt = \lim_{a \rightarrow -\infty} \int_a^0 \frac{2}{1 + 4t^2} dt + \lim_{b \rightarrow \infty} \int_0^b \frac{2}{1 + 4t^2} dt$$
- $$= \lim_{a \rightarrow -\infty} \left[ \tan^{-1} 2t \right]_a^0 + \lim_{b \rightarrow \infty} \left[ \tan^{-1} 2t \right]_0^b = \lim_{a \rightarrow -\infty} \left( -\tan^{-1} 2a \right) + \lim_{b \rightarrow \infty} \left( \tan^{-1} 2b \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$
21.  $\mathbf{r} = t\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (\cos t)^2} = \sqrt{1 + \cos^2 t} \Rightarrow \left| \mathbf{v} \left( \frac{\pi}{2} \right) \right| = \sqrt{1 + \cos^2 \left( \frac{\pi}{2} \right)} = 1$ ;
- $$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + (\cos t)\mathbf{j}}{\sqrt{1 + \cos^2 t}} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{\sin t \cos t}{(1 + \cos^2 t)^{3/2}}\mathbf{i} + \frac{-\sin t}{(1 + \cos^2 t)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\sin t|}{1 + \cos^2 t}; \left| \frac{d\mathbf{T}}{dt} \right|_{t=\frac{\pi}{2}} = \frac{\left| \sin \frac{\pi}{2} \right|}{1 + \cos^2 \left( \frac{\pi}{2} \right)} = \frac{1}{1} = 1. \text{ Thus}$$
- $$\kappa \left( \frac{\pi}{2} \right) = \frac{1}{1} \cdot 1 = 1 \Rightarrow \rho = \frac{1}{1} = 1 \text{ and the center is } \left( \frac{\pi}{2}, 0 \right) \Rightarrow \left( x - \frac{\pi}{2} \right)^2 + y^2 = 1$$
22.  $\mathbf{r} = (2 \ln t)\mathbf{i} - \left( t + \frac{1}{t} \right)\mathbf{j} \Rightarrow \mathbf{v} = \left( \frac{2}{t} \right)\mathbf{i} - \left( 1 - \frac{1}{t^2} \right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\frac{4}{t^2} + \left( 1 - \frac{1}{t^2} \right)^2} = \frac{t^2 + 1}{t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{t^2 + 1}\mathbf{i} - \frac{t^2 - 1}{t^2 + 1}\mathbf{j}$ 

$$\frac{d\mathbf{T}}{dt} = \frac{-2(t^2 - 1)}{(t^2 + 1)^2}\mathbf{i} - \frac{4t}{(t^2 + 1)^2}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\frac{4(t^2 - 1)^2 + 16t^2}{(t^2 + 1)^4}} = \frac{2}{t^2 + 1}. \text{ Thus } \kappa = \frac{1}{|\mathbf{v}|} \cdot \left| \frac{d\mathbf{T}}{dt} \right| = \frac{t^2}{t^2 + 1} \cdot \frac{2}{t^2 + 1} = \frac{2t^2}{(t^2 + 1)^2}$$

$\Rightarrow \kappa(1) = \frac{2}{2^2} = \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$ . The circle of curvature is tangent to the curve at  $P(0, -2) \Rightarrow$  circle has same tangent as the curve  $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$  is tangent to the circle  $\Rightarrow$  the center lies on the  $y$ -axis. If  $t \neq 1 (t > 0)$ , then  $(t-1)^2 > 0 \Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$  since  $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -(t + \frac{1}{t}) < -2 \Rightarrow y < -2$  on both sides of  $(0, -2) \Rightarrow$  the curve is concave down  $\Rightarrow$  center of circle of curvature is  $(0, -4)$   
 $\Rightarrow x^2 + (y+4)^2 = 4$  is an equation of the circle of curvature

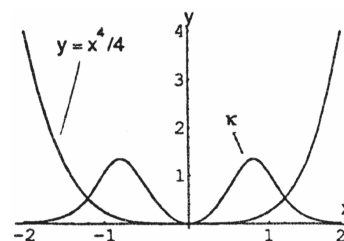
23.  $y = x^2 \Rightarrow f'(x) = 2x$  and  $f''(x) = 2$

$$\Rightarrow \kappa = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$



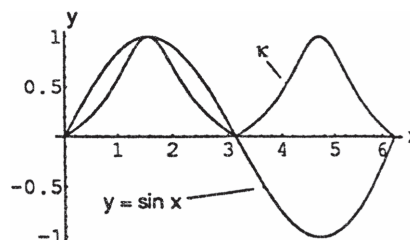
24.  $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$  and  $f''(x) = 3x^2$

$$\Rightarrow \kappa = \frac{|3x^2|}{(1+(x^3)^2)^{3/2}} = \frac{3x^2}{(1+x^6)^{3/2}}$$



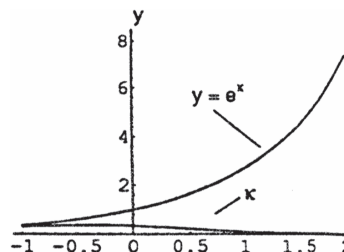
25.  $y = \sin x \Rightarrow f'(x) = \cos x$  and  $f''(x) = -\sin x$

$$\Rightarrow \kappa = \frac{|-\sin x|}{(1+\cos^2 x)^{3/2}} = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$$



26.  $y = e^x \Rightarrow f'(x) = e^x$  and  $f''(x) = e^x$

$$\Rightarrow \kappa = \frac{|e^x|}{(1+(e^x)^2)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$$



27. We will use the formula  $\kappa = \frac{1}{|\mathbf{v}(a)|} \left| \frac{d\mathbf{T}}{dt}(a) \right|$  to find the curvature at the point  $(a, a^2)$ .

By Example 4 in Section 13.4,

$$\mathbf{v}(t) = \sqrt{1+4t^2} \quad \text{and} \quad \frac{d\mathbf{T}}{dt} = 2(1+4t^2)^{-3/2}(-2t\mathbf{i} + \mathbf{j}).$$

At  $t = a$  this gives  $\kappa = \frac{1}{|\mathbf{v}(a)|} \left| \frac{d\mathbf{T}}{dt}(a) \right| = \frac{2}{\sqrt{1+4a^2}} (1+4a^2)^{-3/2} \sqrt{1+4a^2} = \frac{2}{(1+4a^2)^{3/2}}$ . Thus the radius of



the osculating circle is  $r = \frac{1}{2}(1 + 4a^2)^{3/2}$ . To show that the given formulas for center and radius are correct we

must first show that the distance between  $(a, a^2)$  and  $(-4a^3, 3a^2 + \frac{1}{2})$  is  $r$ . This distance is

$$\sqrt{(-4a^3 - a)^2 + (3a^2 + \frac{1}{2} - a^2)^2} = \sqrt{16a^6 + 12a^4 + 3a^2 + \frac{1}{4}} = \frac{1}{2}\sqrt{(1 + 4a^2)^3} \text{ as required.}$$

Finally we must show that the line containing the points  $(-4a^3, 3a^2 + \frac{1}{2})$  and  $(a, a^2)$  is perpendicular to the tangent line at  $(a, a^2)$ ,

which has slope  $2a$ . This requires that  $\frac{3a^2 + \frac{1}{2} - a^2}{-4a^3 - a} = -\frac{1}{2a}$ , which is correct.

28. By Exercise 27, for  $a = 1$ , the center of the osculating circle is at  $(-4, \frac{7}{2})$  and its radius is  $\frac{5\sqrt{5}}{2}$ . A

parametrization of this circle is  $x(\theta) = -4 + \frac{5\sqrt{5}}{2} \cos \theta$ ,  $y(\theta) = \frac{7}{2} + \frac{5\sqrt{5}}{2} \sin \theta$ .

29-36. Example CAS commands:

Maple:

```
with( plots );
r := t -> [3*cos(t), 5*sin(t)];
lo := 0;
hi := 2*Pi;
t0 := Pi/4;
P1 := plot( [r(t)[], t=lo..hi] );
display( P1, scaling=constrained, title="#29(a) (Section 13.4)" );
CURVATURE := (x,y,t) -> simplify(abs(diff(x,t)*diff(y,t,t)-diff(y,t)*diff(x,t,t))/
(diff(x,t)^2+diff(y,t)^2)^(3/2));
kappa := eval(CURVATURE(r(t)[],t),t=t0);
UnitNormal := (x,y,t) -> expand( [-diff(y,t), diff(x,t)]/sqrt(diff(x,t)^2+diff(y,t)^2) );
N := eval( UnitNormal(r(t)[],t), t=t0 );
C := expand( r(t0) + N/kappa );
OscCircle := (x-C[1])^2+(y-C[2])^2 = 1/kappa^2;
evalf( OscCircle );
P2 := implicitplot( (x-C[1])^2+(y-C[2])^2 = 1/kappa^2, x=-7..4, y=-4..6, color=blue );
display( [P1,P2], scaling=constrained, title="#27(e) (Section 13.4)" );
```

Mathematica: (assigned functions and parameters may vary)

In Mathematica, the dot product can be applied either with a period "." or with the word, "Dot".

Similarly, the cross product can be applied either with a very small "x" (in the palette next to the arrow) or with the word, "Cross". However, the Cross command assumes the vectors are in three dimensions.

For the purposes of applying the cross product command, we will define the position vector  $\mathbf{r}$  as a three dimensional vector with zero for its z-component. For graphing, we will use only the first two components.

```

Clear[r, t, x, y]
r[t_]= {3 Cos[t], 5 Sin[t]}
t0= π/4; tmin= 0; tmax= 2π;
r2[t_]= {r[t][[1]], r[t][[2]]}
pp=ParametricPlot[r2[t], {t, tmin, tmax}];
mag[v_]=Sqrt[v.v]
vel[t_]= r'[t]
speed[t_]=mag[vel[t]]
acc[t_]= vel'[t]

curv[t_]= mag[Cross[vel[t],acc[t]]]/speed[t]^3//Simplify
unittan[t_]= vel[t]/speed[t]//Simplify
unitnorm[t_]= unittan'[t]/ mag[unittan'[t]]
ctr= r[t0]+ (1 / curv[t0]) unitnorm[t0]//Simplify
{a,b}= {ctr[[1]], ctr[[2]]}

```

To plot the osculating circle, load a graphics package and then plot it, and show it together with the original curve.

```

<< Graphics`ImplicitPlot`
pc=ImplicitPlot[(x - a)^2 + (y - b)^2 == 1/curv[t0]^2, {x, -8, 8}, {y, -8, 8}]
radius=Graphics[Line[{ {a, b}, r2[t0]}]]
Show[pp, pc, radius, AspectRatio -> 1]

```

### 13.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

- $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2}$   
 $= \sqrt{a^2 + b^2} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0; \quad \mathbf{a} = (-a \cos t)\mathbf{i} + (-a \sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a \cos t)^2 + (-a \sin t)^2} = \sqrt{a^2} = |a|$   
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\mathbf{N} = |a|\mathbf{N}$
- $\mathbf{r} = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0;$   
 $\mathbf{a} = \mathbf{0} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$
- $\mathbf{r} = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \Rightarrow a_T = \frac{1}{2}(5 + 4t^2)^{-1/2}(8t)$   
 $= 4t(5 + 4t^2)^{-1/2} \Rightarrow a_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}; \quad \mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2$   
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2} = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$

4.  $\mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + 2t\mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (2t)^2} = \sqrt{5t^2 + 1} \Rightarrow a_T = \frac{1}{2}(5t^2 + 1)^{-1/2} (10t) = \frac{5t}{\sqrt{5t^2 + 1}}$   
 $\Rightarrow a_T(0) = 0; \mathbf{a} = (-2 \sin t - t \cos t)\mathbf{i} + (2 \cos t - t \sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$   
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(2\sqrt{2})^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$
5.  $\mathbf{r} = t^2\mathbf{i} + (t + \frac{1}{3}t^3)\mathbf{j} + (t - \frac{1}{3}t^3)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + (1 + t^2)\mathbf{j} + (1 - t^2)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + (1 + t^2)^2 + (1 - t^2)^2}$   
 $= \sqrt{2(t^4 + 2t^2 + 1)} = \sqrt{2}(1 + t^2) \Rightarrow a_T = 2t\sqrt{2} \Rightarrow a_T(0) = 0; \mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2$   
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$
6.  $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$   
 $\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (\sqrt{2}e^t)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2;$   
 $\mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$   
 $= (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6}$   
 $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$
7.  $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$   
 $\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$   
 $\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j};$   
 $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k} \Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}, \text{ the normal to the osculating plane; } \mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k}$   
 $\Rightarrow P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right) \text{ lies on the osculating plane} \Rightarrow 0\left(x - \frac{\sqrt{2}}{2}\right) + 0\left(y - \frac{\sqrt{2}}{2}\right) + (z - (-1)) = 0 \Rightarrow z = -1 \text{ is the}$   
 $\text{osculating plane; } \mathbf{T} \text{ is normal to the normal plane} \Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0$   
 $\Rightarrow -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0 \Rightarrow -x + y = 0 \text{ is the normal plane; } \mathbf{N} \text{ is normal to the rectifying plane}$   
 $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = -1 \Rightarrow x + y = \sqrt{2} \text{ is the rectifying}$   
 $\text{plane.}$

$$\begin{aligned}
 8. \quad \mathbf{r} &= (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \\
 &\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{j} \\
 &\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \text{ thus } \mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \text{ and } \mathbf{N}(0) = -\mathbf{i}
 \end{aligned}$$

$$\Rightarrow \mathbf{B}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}, \text{ the normal to the osculating plane; } \mathbf{r}(0) = \mathbf{i} \Rightarrow P(1, 0, 0) \text{ lies on the}$$

osculating plane  $\Rightarrow 0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y-z=0$  is the osculating plane;  $\mathbf{T}$  is normal to the normal plane  $\Rightarrow 0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y+z=0$  is the normal plane;  $\mathbf{N}$  is normal to the rectifying plane  $\Rightarrow -1(x-1) + 0(y-0) + 0(z-0) = 0 \Rightarrow x=1$  is the rectifying plane.

$$9. \text{ By Exercise 9 in Section 13.4, } \mathbf{T} = \left(\frac{3}{5}\cos t\right)\mathbf{i} + \left(-\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \text{ and } \mathbf{N} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} \text{ so that}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}. \text{ Also } \mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k}$$

$$\begin{aligned}
 \Rightarrow \mathbf{a} &= (-3\sin t)\mathbf{i} + (-3\cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} \text{ and } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} \\
 &= (12\cos t)\mathbf{i} - (12\sin t)\mathbf{j} - 9\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (12\cos t)^2 + (-12\sin t)^2 + (-9)^2 = 225. \text{ Thus}
 \end{aligned}$$

$$\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225} = \frac{4(-9\sin^2 t - 9\cos^2 t)}{225} = \frac{-36}{225} = -\frac{4}{25}$$

$$10. \text{ By Exercise 10 in Section 13.4, } \mathbf{T} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \text{ and } \mathbf{N} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \text{ thus}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t)\mathbf{k} = \mathbf{k}. \text{ Also } \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$$

$$\Rightarrow \mathbf{a} = (t(-\sin t) + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-t\cos t - \sin t - \sin t)\mathbf{i} + (-t\sin t + \cos t + \cos t)\mathbf{j}$$

$$= (-t\cos t - 2\sin t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j}. \text{ Thus } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ -t\sin t + \cos t & t\cos t + \sin t & 0 \end{vmatrix}$$

$$= [(t\cos t)(t\cos t + \sin t) - (t\sin t)(-t\sin t + \cos t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (t^2)^2 = t^4.$$

$$\text{Thus } \tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + \cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - \sin t & 0 \end{vmatrix}}{t^4} = \frac{0}{t^4} = 0$$

11. By Exercise 11 in Section 13.4,  $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$  and  $\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$ ; Thus

$$\begin{aligned}\mathbf{B} = \mathbf{T} \times \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix} = \left[ \left( \frac{\cos^2 t - 2\cos t \sin t + \sin^2 t}{2} \right) + \left( \frac{\sin^2 t + 2\sin t \cos t + \cos^2 t}{2} \right) \right] \mathbf{k} \\ &= \left[ \left( \frac{1 - \sin(2t)}{2} \right) + \left( \frac{1 + \sin(2t)}{2} \right) \right] \mathbf{k} = \mathbf{k}. \text{ Also, } \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \\ \Rightarrow \mathbf{a} &= [e^t(-\sin t - \cos t) + e^t(\cos t - \sin t)]\mathbf{i} + [e^t(\cos t - \sin t) + e^t(\sin t + \cos t)]\mathbf{j} \\ &= (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = -2e^t(\cos t + \sin t)\mathbf{i} + 2e^t(-\sin t + \cos t)\mathbf{j}.\end{aligned}$$

$$\text{Thus } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix} = 2e^{2t}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (2e^{2t})^2 = 4e^{4t}.$$

$$\text{Thus } \tau = \frac{\begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \\ -2e^t(\cos t + \sin t) & 2e^t(-\sin t + \cos t) & 0 \end{vmatrix}}{4e^{4t}} = 0$$

12. By Exercise 12 in Section 13.4,  $\mathbf{T} = \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$  and  $\mathbf{N} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}$  Thus

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{12}{13} \cos 2t & -\frac{12}{13} \sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix} = \left(\frac{5}{13} \cos 2t\right)\mathbf{i} - \left(\frac{5}{13} \sin 2t\right)\mathbf{j} - \frac{12}{13}\mathbf{k}.$$

$$\text{Also, } \mathbf{v} = (12 \cos 2t)\mathbf{i} - (12 \sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{a} = (-24 \sin 2t)\mathbf{i} - (24 \cos 2t)\mathbf{j} \text{ and } \frac{d\mathbf{a}}{dt} = (-48 \cos 2t)\mathbf{i} + (48 \sin 2t)\mathbf{j}$$

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \end{vmatrix} = (120 \cos 2t)\mathbf{i} - (120 \sin 2t)\mathbf{j} - 288\mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (120 \cos 2t)^2 + (-120 \sin 2t)^2 + (-288)^2 = 120^2 (\cos^2 2t + \sin^2 2t) + 288^2 = 97344. \text{ Thus}$$

$$\tau = \frac{\begin{vmatrix} 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \\ -48 \cos 2t & 48 \sin 2t & 0 \end{vmatrix}}{97344} = \frac{5(-24 \cdot 48)}{97344} = -\frac{10}{169}$$

13. By Exercise 13 in Section 13.4,  $\mathbf{T} = \frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j}$  and  $\mathbf{N} = \frac{1}{\sqrt{t^2+1}}\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\mathbf{j}$  so that

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = -\mathbf{k}. \text{ Also, } \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{a} = 2t\mathbf{i} + \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = 2\mathbf{i} \text{ so that } \begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$$

14. By Exercise 14 in Section 13.4,  $\mathbf{T} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j}$  and  $\mathbf{N} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$  so that

$$\begin{aligned}\mathbf{B} = \mathbf{T} \times \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k}. \text{ Also, } \mathbf{v} = (-3\cos^2 t \sin t)\mathbf{i} + (3\sin^2 t \cos t)\mathbf{j} \\ \Rightarrow \mathbf{a} &= \frac{d}{dt}(-3\cos^2 t \sin t)\mathbf{i} + \frac{d}{dt}(3\sin^2 t \cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{d}{dt}\left[\frac{d}{dt}(-3\cos^2 t \sin t)\right]\mathbf{i} + \frac{d}{dt}\left[\frac{d}{dt}(3\sin^2 t \cos t)\right]\mathbf{j} \\ \Rightarrow &\begin{vmatrix} -3\cos^2 t \sin t & 3\sin^2 t \cos t & 0 \\ \frac{d}{dt}(-3\cos^2 t \sin t) & \frac{d}{dt}(3\sin^2 t \cos t) & 0 \\ \frac{d}{dt}\left[\frac{d}{dt}(-3\cos^2 t \sin t)\right] & \frac{d}{dt}\left[\frac{d}{dt}(3\sin^2 t \cos t)\right] & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0\end{aligned}$$

15. By Exercise 15 in Section 13.4,  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{j}$  and  $\mathbf{N} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{j}$  so that

$$\begin{aligned}\mathbf{B} = \mathbf{T} \times \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0 \\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \mathbf{k}. \text{ Also, } \mathbf{v} = \mathbf{i} + \left(\sinh \frac{t}{a}\right)\mathbf{j} \Rightarrow \mathbf{a} = \left(\frac{1}{a} \cosh \frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{1}{a^2} \sinh\left(\frac{t}{a}\right)\mathbf{j} \text{ so} \\ \text{that } &\begin{vmatrix} 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a} \cosh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a^2} \sinh\left(\frac{t}{a}\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0\end{aligned}$$

16. By Exercise 16 in Section 13.4,  $\mathbf{T} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k}$  and  $\mathbf{N} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}$  so that

$$\begin{aligned}\mathbf{B} = \mathbf{T} \times \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \tanh t & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k}. \text{ Also, } \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \\ \mathbf{a} &= (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} \text{ and } \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} \\ &= (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + (\cosh^2 t - \sinh^2 t)\mathbf{k} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = \sinh^2 t + \cosh^2 t + 1. \text{ Thus} \\ \tau &= \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{2\cosh^2 t}.\end{aligned}$$

17. Yes. If the car is moving along a curved path, then  $\kappa \neq 0$  and  $a_N = \kappa |\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \neq \mathbf{0}$ .

18.  $|\mathbf{v}| \text{ constant} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow \mathbf{a} = a_N \mathbf{N}$  is orthogonal to  $\mathbf{T} \Rightarrow$  the acceleration is normal to the path

19.  $\mathbf{a} \perp \mathbf{v} \Rightarrow \mathbf{a} \perp \mathbf{T} \Rightarrow a_T = 0 \Rightarrow \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow |\mathbf{v}|$  is constant

20.  $\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N}$ , where  $a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(10) = 0$  and  $a_N = \kappa|\mathbf{v}|^2 = 100\kappa \Rightarrow \mathbf{a} = 0\mathbf{T} + 100\kappa \mathbf{N}$ . Now, from Exercise 5(a) Section 13.4, we find for  $y = f(x) = x^2$  that  $\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}} = \frac{2}{[1+(2x)^2]^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$ ; also,
- $$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \text{ is the position vector of the moving mass } \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j}).$$

At  $(0, 0)$ :  $\mathbf{T}(0) = \mathbf{i}$ ,  $\mathbf{N}(0) = \mathbf{j}$  and  $\kappa(0) = 2 \Rightarrow \mathbf{F} = m\mathbf{a} = m(100\kappa)\mathbf{N} = 200m \mathbf{j}$ ;

At  $(\sqrt{2}, 2)$ :  $\mathbf{T}(\sqrt{2}) = \frac{1}{3}(\mathbf{i} + 2\sqrt{2}\mathbf{j}) = \frac{1}{3}\mathbf{i} + \frac{2\sqrt{2}}{3}\mathbf{j}$ ,  $\mathbf{N}(\sqrt{2}) = -\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$ , and  $\kappa(\sqrt{2}) = \frac{2}{27}$   
 $\Rightarrow \mathbf{F} = m\mathbf{a} = m(100\kappa)\mathbf{N} = \left(\frac{200}{27}m\right)\left(-\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}\right) = -\frac{400\sqrt{2}}{81}m\mathbf{i} + \frac{200}{81}m\mathbf{j}$

21.  $\mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v} \times \mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0$ . Since the curve is a plane curve,  $\tau = 0$ .

22.  $a_N = 0 \Rightarrow \kappa|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$  (since the particle is moving, we cannot have zero speed)  $\Rightarrow$  the curvature is zero so the particle is moving along a straight line

23. From Example 1,  $|\mathbf{v}| = t$  and  $a_N = t$  so that  $a_N = \kappa|\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}$ ,  $t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$

24. If a plane curve is sufficiently differentiable the torsion is zero as the following argument shows:

$$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \Rightarrow \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = f'''(t)\mathbf{i} + g'''(t)\mathbf{j} \Rightarrow \tau = \frac{\begin{vmatrix} f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \\ f'''(t) & g'''(t) & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

25.  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$ ;  $\mathbf{v} \cdot \mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$   
 $\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k}$  and  $\mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0, g(a) = 0$  and  $C = 0 \Rightarrow h(t) = 0$ .

26. From Example 2,  $\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2}$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}}[-(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}]; \quad \frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2 + b^2}}[-(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}]$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-a \sin t}{\sqrt{a^2 + b^2}} & \frac{a \cos t}{\sqrt{a^2 + b^2}} & \frac{b}{\sqrt{a^2 + b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{b \sin t}{\sqrt{a^2 + b^2}}\mathbf{i} - \frac{b \cos t}{\sqrt{a^2 + b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2 + b^2}}\mathbf{k}$$

$$\Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2 + b^2}}[(b \cos t)\mathbf{i} + (b \sin t)\mathbf{j}] \Rightarrow \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = -\frac{b}{\sqrt{a^2 + b^2}} \Rightarrow \tau = -\frac{1}{|\mathbf{v}|} \left( \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right) = \left( -\frac{1}{\sqrt{a^2 + b^2}} \right) \left( -\frac{b}{\sqrt{a^2 + b^2}} \right) = \frac{b}{a^2 + b^2},$$

which is consistent with the result in Example 2.

27-30. Example CAS commands:

Maple:

```

with( LinearAlgebra );
r := <t*cos(t) | t*sin(t) | t >;
t0 := sqrt(3);
rr := eval( r, t=t0 );
v := map( diff, r, t );
vv := eval( v, t=t0 );
a := map( diff, v, t );
aa := eval( a, t=t0 );
s := simplify(Norm( v, 2 )) assuming t::real;
ss := eval( s, t=t0 );
T := v/s;
TT := vv/ss ;
q1 := map( diff, simplify(T), t );
NN := simplify(eval( q1/Norm(q1,2), t=t0 ));
BB := CrossProduct( TT, NN );
kappa := Norm(CrossProduct(vv,aa),2)/ss^3;
tau := simplify( Determinant(< vv, aa, eval(map(diff,a,t),t=t0) >)/Norm(CrossProduct(vv,aa),2)^3 );
a_t := eval( diff( s, t ), t=t0 );
a_n := evalf[4]( kappa*ss^2 );

```

Mathematica: (assigned functions and value for t0 will vary)

```

Clear[t, v, a, t]
mag[vector_]:=Sqrt[vector.vector]
Print["The position vector is ",r[t_]={t Cos[t], t Sin[t], t}]
Print["The velocity vector is ",v[t_]=r'[t]]
Print["The acceleration vector is ",a[t_]=v'[t]]
Print["The speed is ",speed[t_]= mag[v[t]]//Simplify]
Print["The unit tangent vector is ",utan[t_]= v[t]/speed[t]//Simplify]
Print["The curvature is ",curv[t_]= mag[Cross[v[t],a[t]]] / speed[t]^3//Simplify]
Print["The torsion is ",torsion[t_]= Det[{v[t], a[t], a'[t]}] / mag[Cross[v[t],a[t]]]^2//Simplify]
Print["The unit normal vector is ",unorm[t_]= utan'[t] / mag[utan'[t]]//Simplify]
Print["The unit binormal vector is ",ubinorm[t_]= Cross[utan[t],unorm[t]]//Simplify]
Print["The tangential component of the acceleration is ",at[t_]=a[t].utan[t]//Simplify]
Print["The normal component of the acceleration is ",an[t_]=a[t].unorm[t]//Simplify]

```

You can evaluate any of these functions at a specified value of t.

```

t0= Sqrt[3]
{ utan[t0], unorm[t0], ubinorm[t0] }
N[ { utan[t0], unorm[t0], ubinorm[t0] } ]
{ curv[t0], torsion[t0] }

```



$$N[\{\text{curv}[t0], \text{torsion}[t0]\}]$$

$$\{\text{at}[t0], \text{an}[t0]\}$$

$$N[\{\text{at}[t0], \text{an}[t0]\}]$$

To verify that the tangential and normal components of the acceleration agree with the formulas in the book:

$$\text{at}[t] == \text{speed}[t] // \text{Simplify}$$

$$\text{an}[t] == \text{curv}[t] \text{ speed}[t]^2 // \text{Simplify}$$

### 13.6 VELOCITY AND ACCELERATION IN POLAR COORDINATES

1.  $\frac{d\theta}{dt} = 3 = \dot{\theta} \Rightarrow \ddot{\theta} = 0$ ,  $r = a(1 - \cos \theta) \Rightarrow \dot{r} = a \sin \theta \frac{d\theta}{dt} = 3a \sin \theta \Rightarrow \ddot{r} = 3a \cos \theta \frac{d\theta}{dt} = 9a \cos \theta$   
 $\mathbf{v} = (3a \sin \theta) \mathbf{u}_r + (a(1 - \cos \theta))(3) \mathbf{u}_\theta = (3a \sin \theta) \mathbf{u}_r + 3a(1 - \cos \theta) \mathbf{u}_\theta$   
 $\mathbf{a} = (9a \cos \theta - a(1 - \cos \theta)(3)^2) \mathbf{u}_r + (a(1 - \cos \theta) \cdot 0 + 2(3a \sin \theta)(3)) \mathbf{u}_\theta$   
 $= (9a \cos \theta - 9a + 9a \cos \theta) \mathbf{u}_r + (18a \sin \theta) \mathbf{u}_\theta = 9a(2 \cos \theta - 1) \mathbf{u}_r + (18a \sin \theta) \mathbf{u}_\theta$
2.  $\frac{d\theta}{dt} 2t = \dot{\theta} \Rightarrow \ddot{\theta} = 2$ ,  $r = a \sin 2\theta \Rightarrow \dot{r} = a \cos 2\theta \cdot 2 \frac{d\theta}{dt} = 4ta \cos 2\theta \Rightarrow \ddot{r} = 4ta(-\sin 2\theta \cdot 2 \frac{d\theta}{dt}) + 4a \cos 2\theta$   
 $= -16t^2 a \sin 2\theta + 4a \cos 2\theta$ ;  $\mathbf{v} = (4ta \cos 2\theta) \mathbf{u}_r + (a \sin 2\theta)(2t) \mathbf{u}_\theta = (4ta \cos 2\theta) \mathbf{u}_r + (2ta \sin 2\theta) \mathbf{u}_\theta$   
 $\mathbf{a} = \left[ (-16t^2 a \sin 2\theta + 4a \cos 2\theta) - (a \sin 2\theta)(2t)^2 \right] \mathbf{u}_r + \left[ (a \sin 2\theta)(2) + 2(4ta \cos 2\theta)(2t) \right] \mathbf{u}_\theta$   
 $= \left[ -16t^2 a \sin 2\theta + 4a \cos 2\theta - 4t^2 a \sin 2\theta \right] \mathbf{u}_r + \left[ 2a \sin 2\theta + 16t^2 a \cos 2\theta \right] \mathbf{u}_\theta$   
 $= \left[ -20t^2 a \sin 2\theta + 4a \cos 2\theta \right] \mathbf{u}_r + \left[ 2a \sin 2\theta + 16t^2 a \cos 2\theta \right] \mathbf{u}_\theta$   
 $= 4a(\cos 2\theta - 5t^2 \sin 2\theta) \mathbf{u}_r + 2a(\sin 2\theta + 8t^2 \cos 2\theta) \mathbf{u}_\theta$
3.  $\frac{d\theta}{dt} = 2 = \dot{\theta} \Rightarrow \ddot{\theta} = 0$ ,  $r = e^{a\theta} \Rightarrow \dot{r} = e^{a\theta} \cdot a \frac{d\theta}{dt} = 2a e^{a\theta} \Rightarrow \ddot{r} = 2a e^{a\theta} \cdot a \frac{d\theta}{dt} = 4a^2 e^{a\theta}$   
 $\mathbf{v} = (2a e^{a\theta}) \mathbf{u}_r + (e^{a\theta})(2) \mathbf{u}_\theta = (2a e^{a\theta}) \mathbf{u}_r + (2e^{a\theta}) \mathbf{u}_\theta$   
 $\mathbf{a} = \left[ (4a^2 e^{a\theta}) - (e^{a\theta})(2)^2 \right] \mathbf{u}_r + \left[ (e^{a\theta})(0) + 2(2a e^{a\theta})(2) \right] \mathbf{u}_\theta = [4a^2 e^{a\theta} - 4e^{a\theta}] \mathbf{u}_r + [0 + 8a e^{a\theta}] \mathbf{u}_\theta$   
 $= 4e^{a\theta}(a^2 - 1) \mathbf{u}_r + (8a e^{a\theta}) \mathbf{u}_\theta$
4.  $\theta = 1 - e^{-t} \Rightarrow \dot{\theta} = e^{-t} \Rightarrow \ddot{\theta} = -e^{-t}$ ,  $r = a(1 + \sin t) \Rightarrow \dot{r} = a \cos t \Rightarrow \ddot{r} = -a \sin t$   
 $\mathbf{v} = (a \cos t) \mathbf{u}_r + (a(1 + \sin t))(e^{-t}) \mathbf{u}_\theta = (a \cos t) \mathbf{u}_r + a e^{-t}(1 + \sin t) \mathbf{u}_\theta$   
 $\mathbf{a} = \left[ (-a \sin t) - (a(1 + \sin t))(e^{-t})^2 \right] \mathbf{u}_r + \left[ (a(1 + \sin t))(-e^{-t}) + 2(a \cos t)(e^{-t}) \right] \mathbf{u}_\theta$   
 $= \left[ -a \sin t - a e^{-2t}(1 + \sin t) \right] \mathbf{u}_r + \left[ -a e^{-t}(1 + \sin t) + 2a e^{-t} \cos t \right] \mathbf{u}_\theta$   
 $= -a(\sin t + e^{-2t}(1 + \sin t)) \mathbf{u}_r + a e^{-t}(-(1 + \sin t) + 2 \cos t) \mathbf{u}_\theta$   
 $= -a(\sin t + e^{-2t}(1 + \sin t)) \mathbf{u}_r + a e^{-t}(2 \cos t - 1 - \sin t) \mathbf{u}_\theta$

$$\begin{aligned}
 5. \quad \theta = 2t &\Rightarrow \dot{\theta} = 2 \Rightarrow \ddot{\theta} = 0, \quad r = 2 \cos 4t \Rightarrow \dot{r} = -8 \sin 4t \Rightarrow \ddot{r} = -32 \cos 4t \\
 \mathbf{v} &= (-8 \sin 4t) \mathbf{u}_r + (2 \cos 4t)(2) \mathbf{u}_\theta = -8(\sin 4t) \mathbf{u}_r + 4(\cos 4t) \mathbf{u}_\theta \\
 \mathbf{a} &= \left( (-32 \cos 4t) - (2 \cos 4t)(2)^2 \right) \mathbf{u}_r + \left( (2 \cos 4t) \cdot 0 + 2(-8 \sin 4t)(2) \right) \mathbf{u}_\theta \\
 &= (-32 \cos 4t - 8 \cos 4t) \mathbf{u}_r + (0 - 32 \sin 4t) \mathbf{u}_\theta = -40(\cos 4t) \mathbf{u}_r - 32(\sin 4t) \mathbf{u}_\theta
 \end{aligned}$$

$$6. \quad e = \frac{r_0 v_0^2}{GM} - 1 \Rightarrow v_0^2 = \frac{GM(e+1)}{r_0} \Rightarrow v_0 = \sqrt{\frac{GM(e+1)}{r_0}};$$

$$\text{Circle: } e = 0 \Rightarrow v_0 = \sqrt{\frac{GM}{r_0}}$$

$$\text{Ellipse: } 0 < e < 1 \Rightarrow \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$$

$$\text{Parabola: } e = 1 \Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$$

$$\text{Hyperbola: } e > 1 \Rightarrow v_0 > \sqrt{\frac{2GM}{r_0}}$$

$$7. \quad r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}} \text{ which is constant since } G, M, \text{ and } r \text{ (the radius of orbit) are constant}$$

$$\begin{aligned}
 8. \quad \Delta A &= \frac{1}{2} |\mathbf{r}(t + \Delta t) \times \mathbf{r}(t)| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) + \frac{1}{\Delta t} \mathbf{r}(t) \times \mathbf{r}(t) \right| \\
 &= \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t \rightarrow 0} \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{d\mathbf{r}}{dt} \times \mathbf{r}(t) \right| = \frac{1}{2} |\mathbf{r}(t) \times \frac{d\mathbf{r}}{dt}| = \frac{1}{2} |\mathbf{r} \times \dot{\mathbf{r}}|
 \end{aligned}$$

$$\begin{aligned}
 9. \quad T &= \left( \frac{2\pi a^2}{r_0^2 v_0^2} \right) \sqrt{1 - e^2} \Rightarrow T^2 = \left( \frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) (1 - e^2) = \left( \frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[ 1 - \left( \frac{r_0 v_0^2}{GM} - 1 \right)^2 \right] \text{ (from Equation 5)} \\
 &= \left( \frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[ -\frac{r_0^2 v_0^4}{G^2 M^2} + 2 \left( \frac{r_0 v_0^2}{GM} \right) \right] = \left( \frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[ \frac{2GM r_0 v_0^2 - r_0^2 v_0^4}{G^2 M^2} \right] = \frac{(4\pi^2 a^4)(2GM - r_0 v_0^2)}{r_0 G^2 M^2} = \left( 4\pi^2 a^4 \right) \left( \frac{2GM - r_0 v_0^2}{2r_0 GM} \right) \left( \frac{2}{GM} \right) \\
 &= \left( 4\pi^2 a^4 \right) \left( \frac{1}{2a} \right) \left( \frac{2}{GM} \right) \text{ (from Equation 10)} \Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}
 \end{aligned}$$

$$10. \quad \text{For each of the planets listed we form the ratio } \frac{T^2 / a^3}{(4\pi^2) / (GM)}. \text{ The values we obtain are}$$

Mercury	1.00225
Venus	1.00288
Mars	1.00252
Saturn	1.00019

These values are all close to 1, so they support Kepler's third law.

$$11. \quad \text{Solve Kepler's third law for } a \text{ and double this result: } 2 \cdot \left( \frac{(365.256 \text{ days})^2}{(4\pi^2) / (GM)} \right)^{1/3} \approx 29.925 \times 10^{10} \text{ m}$$

$$12. \quad \text{Solve Kepler's third law for } a \text{ and double this result: } 2 \cdot \left( \frac{(84 \text{ years})^2}{(4\pi^2) / (GM)} \right)^{1/3} \approx 573.95 \times 10^{10} \text{ m}$$

13. Assuming Earth has a circular orbit with radius  $150 \times 10^6$  km, the rate of change of area is

$$\frac{\pi(150 \times 10^6 \text{ km})^2}{365.256 \text{ days}} \approx 2.24 \times 10^9 \text{ km}^2/\text{sec}.$$

14. Solving Kepler's third law for  $T$  we find  $T = \sqrt{\frac{4\pi^2}{GM}}(77.8 \times 10^{10} \text{ m})^3 \approx 11.857$  years.

15. Solving Kepler's third law for the mass  $M$  of the body around which Io is orbiting we find

$$M = 4\pi^2 \frac{a^3}{T^2 G} = 4\pi^2 \frac{(0.042 \times 10^{10} \text{ m})^3}{(1.769 \text{ days})^2 G} \approx 1.876 \times 10^{27} \text{ kg}.$$

16. To solve this we need a value for the mass of Earth, which is approximately  $M = 5.972 \times 10^{24}$  kg. Solving Kepler's third law for the orbital radius we get

$$a = (4\pi^2)^{-1/3} (T^2 GM)^{1/3} = (4\pi^2)^{-1/3} \left( (2.36055 \times 10^6 \text{ sec}) G (5.972 \times 10^{24} \text{ kg}) \right)^{1/3} \approx 3.831 \times 10^5 \text{ km}.$$

Since Earth's radius is about 6371, the orbit of the moon is about  $383,143 - 6371 = 376.772$  km from the surface, assuming a circular orbit for the moon.

## CHAPTER 13 PRACTICE EXERCISES

1.  $\mathbf{r}(t) = (4 \cos t) \mathbf{i} + (\sqrt{2} \sin t) \mathbf{j}$

$$\Rightarrow x = 4 \cos t \text{ and } y = \sqrt{2} \sin t \Rightarrow \frac{x^2}{16} + \frac{y^2}{2} = 1;$$

$$\mathbf{v} = (-4 \sin t) \mathbf{i} + (\sqrt{2} \cos t) \mathbf{j} \text{ and}$$

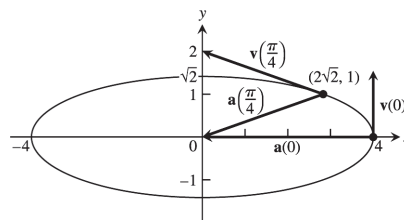
$$\mathbf{a} = (-4 \cos t) \mathbf{i} - (\sqrt{2} \sin t) \mathbf{j};$$

$$\mathbf{r}(0) = 4 \mathbf{i}, \mathbf{v}(0) = \sqrt{2} \mathbf{j}, \mathbf{a}(0) = -4 \mathbf{i};$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2} \mathbf{i} + \mathbf{j}, \mathbf{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2} \mathbf{i} + \mathbf{j}, \mathbf{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2} \mathbf{i} - \mathbf{j}; |\mathbf{v}| = \sqrt{16 \sin^2 t + 2 \cos^2 t}$$

$$\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = \frac{14 \sin t \cos t}{\sqrt{16 \sin^2 t + 2 \cos^2 t}}; \text{ at } t = 0: a_T = 0, a_N = \sqrt{|\mathbf{a}|^2 - 0} = 4, \mathbf{a} = 0\mathbf{T} + 4\mathbf{N} = 4\mathbf{N},$$

$$\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4}{2} = 2; \text{ at } t = \frac{\pi}{4}: a_T = \frac{7}{\sqrt{8+1}} = \frac{7}{3}, a_N = \sqrt{9 - \frac{49}{9}} = \frac{4\sqrt{2}}{3}, \mathbf{a} = \frac{7}{3}\mathbf{T} + \frac{4\sqrt{2}}{3}\mathbf{N}, \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4\sqrt{2}}{27}$$



2.  $\mathbf{r}(t) = (\sqrt{3} \sec t) \mathbf{i} + (\sqrt{3} \tan t) \mathbf{j} \Rightarrow x = \sqrt{3} \sec t \text{ and } y = \sqrt{3} \tan t \Rightarrow \frac{x^2}{3} - \frac{y^2}{3} = \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 3;$

$$\mathbf{v} = (\sqrt{3} \sec t \tan t) \mathbf{i} + (\sqrt{3} \sec^2 t) \mathbf{j} \text{ and}$$

$$\mathbf{a} = (\sqrt{3} \sec t \tan^2 t + \sqrt{3} \sec^3 t) \mathbf{i} - (2\sqrt{3} \sec^2 t \tan t) \mathbf{j};$$

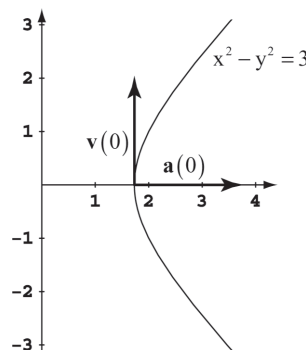
$$\mathbf{r}(0) = \sqrt{3} \mathbf{i}, \mathbf{v}(0) = \sqrt{3} \mathbf{j}, \mathbf{a}(0) = \sqrt{3} \mathbf{j};$$

$$|\mathbf{v}| = \sqrt{3 \sec^2 t \tan^2 t + 3 \sec^4 t}$$

$$\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = \frac{6 \sec^2 t \tan^3 t + 18 \sec^4 t \tan t}{2\sqrt{3 \sec^2 t \tan^2 t + 3 \sec^4 t}};$$

$$\text{at } t = 0: a_T = 0, a_N = \sqrt{|\mathbf{a}|^2 - 0} = \sqrt{3},$$

$$\mathbf{a} = 0\mathbf{T} + \sqrt{3}\mathbf{N} = \sqrt{3}\mathbf{N}, \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$



3.  $\mathbf{r} = \frac{1}{\sqrt{1+t^2}} \mathbf{i} + \frac{t}{\sqrt{1+t^2}} \mathbf{j} \Rightarrow \mathbf{v} = -t(1+t^2)^{-3/2} \mathbf{i} + (1+t^2)^{-3/2} \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left[-t(1+t^2)^{-3/2}\right]^2 + \left[(1+t^2)^{-3/2}\right]^2} = \frac{1}{1+t^2}.$

We want to maximize  $|\mathbf{v}|: \frac{d|\mathbf{v}|}{dt} = \frac{-2t}{(1+t^2)^2}$  and  $\frac{d|\mathbf{v}|}{dt} = 0 \Rightarrow \frac{-2t}{(1+t^2)^2} = 0 \Rightarrow t = 0$ . For  $t < 0$ ,  $\frac{-2t}{(1+t^2)^2} > 0$ ; for  $t > 0$ ,

$$\frac{-2t}{(1+t^2)^2} < 0 \Rightarrow |\mathbf{v}|_{\max} \text{ occurs when } t = 0 \Rightarrow |\mathbf{v}|_{\max} = 1$$

4.  $\mathbf{r} = (e^t \cos t) \mathbf{i} + (e^t \sin t) \mathbf{j} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j}$

$$\Rightarrow \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \mathbf{j} = (-2e^t \sin t) \mathbf{i} + (2e^t \cos t) \mathbf{j}.$$

Let  $\theta$  be the angle between  $\mathbf{r}$  and  $\mathbf{a}$ . Then  $\theta = \cos^{-1} \left( \frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{r}| |\mathbf{a}|} \right) = \cos^{-1} \left( \frac{-2e^{2t} \sin t \cos t + 2e^{2t} \sin t \cos t}{\sqrt{(e^t \cos t)^2 + (e^t \sin t)^2} \sqrt{(-2e^t \sin t)^2 + (2e^t \cos t)^2}} \right)$

$$= \cos^{-1} \left( \frac{0}{2e^{2t}} \right) = \cos^{-1} 0 = \frac{\pi}{2} \text{ for all } t$$

$$5. \quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j} \text{ and } \mathbf{a} = 5\mathbf{i} + 15\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 25; \quad |\mathbf{v}| = \sqrt{3^2 + 4^2} = 5 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{25}{5^3} = \frac{1}{5}$$

$$6. \quad \kappa = \frac{|y''|}{[1+(y')^2]^{3/2}} = e^x (1+e^{2x})^{-3/2} \Rightarrow \frac{d\kappa}{dx} = e^x (1+e^{2x})^{-3/2} + e^x \left[ -\frac{3}{2}(1+e^{2x})^{-5/2} (2e^{2x}) \right]$$

$$= e^x (1+e^{2x})^{-3/2} - 3e^{3x} (1+e^{2x})^{-5/2} = e^x (1+e^{2x})^{-5/2} [(1+e^{2x}) - 3e^{2x}] = e^x (1+e^{2x})^{-5/2} (1-2e^{2x});$$

$$\frac{d\kappa}{dx} = 0 \Rightarrow (1-2e^{2x}) = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = -\ln 2 \Rightarrow x = -\frac{1}{2} \ln 2 = -\ln \sqrt{2} \Rightarrow y = \frac{1}{\sqrt{2}}; \text{ therefore } \kappa \text{ is at a}$$

$$\text{maximum at the point } \left( -\ln \sqrt{2}, \frac{1}{\sqrt{2}} \right)$$

$$7. \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \text{ and } \mathbf{v} \cdot \mathbf{i} = y \Rightarrow \frac{dx}{dt} = y. \text{ Since the particle moves around the unit circle}$$

$$x^2 + y^2 = 1, \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x. \text{ Since } \frac{dx}{dt} = y \text{ and } \frac{dy}{dt} = -x,$$

$$\text{we have } \mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow \text{at } (1, 0), \mathbf{v} = -\mathbf{j} \text{ and the motion is clockwise.}$$

$$8. \quad 9y = x^3 \Rightarrow 9 \frac{dy}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3} x^2 \frac{dx}{dt}. \text{ If } \mathbf{r} = x\mathbf{i} + y\mathbf{j}, \text{ where } x \text{ and } y \text{ are differentiable functions of } t,$$

$$\text{then } \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}. \text{ Hence } \mathbf{v} \cdot \mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4 \text{ and } \mathbf{v} \cdot \mathbf{j} = \frac{dy}{dt} = \frac{1}{3} x^2 \frac{dx}{dt} = \frac{1}{3} (3)^2 (4) = 12 \text{ at } (3, 3).$$

$$\text{Also, } \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} \text{ and } \frac{d^2y}{dt^2} = \left( \frac{2}{3}x \right) \left( \frac{dx}{dt} \right) + \left( \frac{1}{3}x^2 \right) \frac{d^2x}{dt^2}. \text{ Hence } \mathbf{a} \cdot \mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2 \text{ and}$$

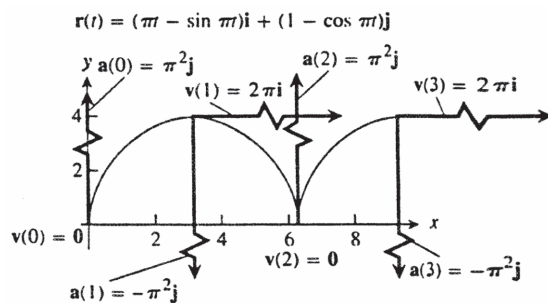
$$\mathbf{a} \cdot \mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26 \text{ at the point } (x, y) = (3, 3).$$

$$9. \quad \frac{d\mathbf{r}}{dt} \text{ orthogonal to } \mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = \frac{1}{2} \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \frac{1}{2} \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2} \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) \Rightarrow \mathbf{r} \cdot \mathbf{r} = K, \text{ a constant. If } \mathbf{r} = x\mathbf{i} + y\mathbf{j}, \text{ where } x$$

$$\text{and } y \text{ are differentiable functions of } t, \text{ then } \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K, \text{ which is the equation of a circle}$$

$$\text{centered at the origin.}$$

10. (a)



$$(b) \quad \mathbf{v} = (\pi - \pi \cos \pi t)\mathbf{i} + (\pi \sin \pi t)\mathbf{j}$$

$$\Rightarrow \mathbf{a} = (\pi^2 \sin \pi t)\mathbf{i} + (\pi^2 \cos \pi t)\mathbf{j};$$

$$\mathbf{v}(0) = \mathbf{0} \text{ and } \mathbf{a}(0) = \pi^2 \mathbf{j};$$

$$\mathbf{v}(1) = 2\pi \mathbf{i} \text{ and } \mathbf{a}(1) = -\pi^2 \mathbf{j};$$

$$\mathbf{v}(2) = \mathbf{0} \text{ and } \mathbf{a}(2) = \pi^2 \mathbf{j};$$

$$\mathbf{v}(3) = 2\pi \mathbf{i}; \text{ and } \mathbf{a}(3) = -\pi^2 \mathbf{j}$$

(c) Forward speed at the topmost point is  $|\mathbf{v}(1)| = |\mathbf{v}(3)| = 2\pi$  ft/sec; since the circle makes  $\frac{1}{2}$  revolution per second, the center moves  $\pi$  ft parallel to the  $x$ -axis each second  $\Rightarrow$  the forward speed of  $C$  is  $\pi$  ft/sec.

11.  $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 6.5 + (44 \text{ ft/sec})(\sin 45^\circ)(3 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(3 \text{ sec})^2 = 6.5 + 66\sqrt{2} - 144$   
 $\approx -44.16 \text{ ft} \Rightarrow$  the shot put is on the ground. Now,  $y = 0 \Rightarrow 6.5 + 22\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 2.13 \text{ sec}$  (the positive root)  $\Rightarrow x \approx (44 \text{ ft/sec})(\cos 45^\circ)(2.13 \text{ sec}) \approx 66.27 \text{ ft}$  or about 66 ft, 3 in. from the stopboard

12.  $y_{\max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} = 7 \text{ ft} + \frac{[(80 \text{ ft/sec})(\sin 45^\circ)]^2}{(2)(32 \text{ ft/sec}^2)} \approx 57 \text{ ft}$

13.  $x = (v_0 \cos \alpha)t$  and  $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \tan \phi = \frac{y}{x} = \frac{(v_0 \sin \alpha)t - \frac{1}{2}gt^2}{(v_0 \cos \alpha)t} = \frac{(v_0 \sin \alpha) - \frac{1}{2}gt}{v_0 \cos \alpha}$   
 $\Rightarrow v_0 \cos \alpha \tan \phi = v_0 \sin \alpha - \frac{1}{2}gt \Rightarrow t = \frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g}$ , which is the time when the golf ball hits the upward slope. At this time  $x = (v_0 \cos \alpha) \left( \frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g} \right) = \left( \frac{2}{g} \right) (v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi)$ . Now

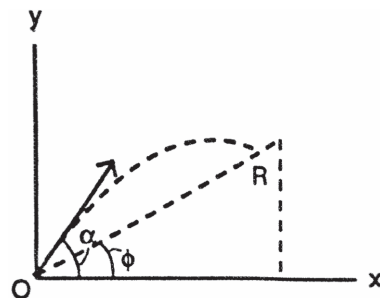
$$OR = \frac{x}{\cos \phi} \Rightarrow OR = \left( \frac{2}{g} \right) \left( \frac{v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi}{\cos \phi} \right)$$

$$= \left( \frac{2v_0^2 \cos \alpha}{g} \right) \left( \frac{\sin \alpha}{\cos \phi} - \frac{\cos \alpha \tan \phi}{\cos \phi} \right) = \left( \frac{2v_0^2 \cos \alpha}{g} \right) \left( \frac{\sin \alpha \cos \phi - \cos \alpha \sin \phi}{\cos^2 \phi} \right)$$

$$= \left( \frac{2v_0^2 \cos \alpha}{g \cos^2 \phi} \right) [\sin(\alpha - \phi)].$$
 The distance  $OR$  is maximized when  $x$

is maximized:  $\frac{dx}{d\alpha} = \left( \frac{2v_0^2}{g} \right) (\cos 2\alpha + \sin 2\alpha \tan \phi) = 0$

$$\Rightarrow (\cos 2\alpha + \sin 2\alpha \tan \phi) = 0 \Rightarrow \cot 2\alpha + \tan \phi = 0 \Rightarrow \cot 2\alpha = \tan(-\phi) \Rightarrow 2\alpha = \frac{\pi}{2} + \phi \Rightarrow \alpha = \frac{\phi}{2} + \frac{\pi}{4}$$



14. (a)  $x = v_0(\cos 40^\circ)t$  and  $y = 6.5 + v_0(\sin 40^\circ)t - \frac{1}{2}gt^2 = 6.5 + v_0(\sin 40^\circ)t - 16t^2$ ;  $x = 262 \frac{5}{12} \text{ ft}$  and

$$y = 0 \text{ ft} \Rightarrow 262 \frac{5}{12} = v_0(\cos 40^\circ)t \text{ or } v_0 = \frac{262.4167}{(\cos 40^\circ)t} \text{ and } 0 = 6.5 + \left[ \frac{262.4167}{(\cos 40^\circ)t} \right] (\sin 40^\circ)t - 16t^2$$

$$\Rightarrow t^2 = 14.1684 \Rightarrow t \approx 3.764 \text{ sec. Therefore, } 262.4167 \approx v_0(\cos 40^\circ)(3.764 \text{ sec})$$

$$\Rightarrow v_0 \approx \frac{262.4167}{(\cos 40^\circ)(3.764 \text{ sec})} \Rightarrow v_0 \approx 91 \text{ ft/sec}$$

(b)  $y_{\max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} \approx 6.5 + \frac{(91)(\sin 40^\circ)^2}{(2)(32)} \approx 60 \text{ ft}$

15.  $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (2t)^2}$   
 $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1+t^2} dt = \left[ t\sqrt{1+t^2} + \ln \left| t + \sqrt{1+t^2} \right| \right]_0^{\pi/4} = \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln \left( \frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}} \right)$

16.  $\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 3t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (3t^{1/2})^2}$   
 $= \sqrt{9+9t} = 3\sqrt{1+t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1+t} dt = \left[ 2(1+t)^{3/2} \right]_0^3 = 14$

$$\begin{aligned}
17. \quad \mathbf{r} &= \frac{4}{9}(1+t)^{3/2} \mathbf{i} + \frac{4}{9}(1-t)^{3/2} \mathbf{j} + \frac{1}{3}t \mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1+t)^{1/2} \mathbf{i} - \frac{2}{3}(1-t)^{1/2} \mathbf{j} + \frac{1}{3} \mathbf{k} \\
\Rightarrow |\mathbf{v}| &= \sqrt{\left[\frac{2}{3}(1+t)^{1/2}\right]^2 + \left[-\frac{2}{3}(1-t)^{1/2}\right]^2 + \left(\frac{1}{3}\right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1+t)^{1/2} \mathbf{i} - \frac{2}{3}(1-t)^{1/2} \mathbf{j} + \frac{1}{3} \mathbf{k} \\
\Rightarrow \mathbf{T}(0) &= \frac{2}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} + \frac{1}{3} \mathbf{k}; \quad \frac{d\mathbf{T}}{dt} = \frac{1}{3}(1+t)^{-1/2} \mathbf{i} + \frac{1}{3}(1-t)^{-1/2} \mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt}(0) \right| = \frac{\sqrt{2}}{3} \\
\Rightarrow \mathbf{N}(0) &= \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j}; \quad \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}} \mathbf{i} + \frac{1}{3\sqrt{2}} \mathbf{j} + \frac{4}{3\sqrt{2}} \mathbf{k}; \\
\mathbf{a} &= \frac{1}{3}(1+t)^{-1/2} \mathbf{i} + \frac{1}{3}(1-t)^{-1/2} \mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} \text{ and } \mathbf{v}(0) = \frac{2}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} + \frac{1}{3} \mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} \\
&= -\frac{1}{9} \mathbf{i} + \frac{1}{9} \mathbf{j} + \frac{4}{9} \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3}; \\
\dot{\mathbf{a}} &= -\frac{1}{6}(1+t)^{-3/2} \mathbf{i} + \frac{1}{6}(1-t)^{-3/2} \mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6} \mathbf{i} + \frac{1}{6} \mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6} \\
18. \quad \mathbf{r} &= (e^t \sin 2t) \mathbf{i} + (e^t \cos 2t) \mathbf{j} + 2e^t \mathbf{k} \Rightarrow \mathbf{v} = (e^t \sin 2t + 2e^t \cos 2t) \mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t) \mathbf{j} + 2e^t \mathbf{k} \\
\Rightarrow |\mathbf{v}| &= \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t \\
\Rightarrow \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{3} \sin 2t + \frac{2}{3} \cos 2t\right) \mathbf{i} + \left(\frac{1}{3} \cos 2t - \frac{2}{3} \sin 2t\right) \mathbf{j} + \frac{2}{3} \mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} + \frac{2}{3} \mathbf{k}; \\
\frac{d\mathbf{T}}{dt} &= \left(\frac{2}{3} \cos 2t - \frac{4}{3} \sin 2t\right) \mathbf{i} + \left(-\frac{2}{3} \sin 2t - \frac{4}{3} \cos 2t\right) \mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3} \mathbf{i} - \frac{4}{3} \mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt}(0) \right| = \frac{2}{3} \sqrt{5} \\
\Rightarrow \mathbf{N}(0) &= \frac{\left(\frac{2}{3} \mathbf{i} - \frac{4}{3} \mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}} \mathbf{i} - \frac{2}{\sqrt{5}} \mathbf{j}; \quad \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}} \mathbf{i} + \frac{2}{3\sqrt{5}} \mathbf{j} - \frac{5}{3\sqrt{5}} \mathbf{k}; \\
\mathbf{a} &= (4e^t \cos 2t - 3e^t \sin 2t) \mathbf{i} + (-3e^t \cos 2t - 4e^t \sin 2t) \mathbf{j} + 2e^t \mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \text{ and } \\
\mathbf{v}(0) &= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{64 + 16 + 100} = 6\sqrt{5} \text{ and } \\
|\mathbf{v}(0)| &= 3 \Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9}; \\
\dot{\mathbf{a}} &= (4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t) \mathbf{i} + (-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t) \mathbf{j} + 2e^t \mathbf{k} \\
&= (-2e^t \cos 2t - 11e^t \sin 2t) \mathbf{i} + (-11e^t \cos 2t + 2e^t \sin 2t) \mathbf{j} + 2e^t \mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i} - 11\mathbf{j} + 2\mathbf{k} \\
\Rightarrow \tau(0) &= \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9}
\end{aligned}$$

$$19. \quad \mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j};$$

$$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{(1+e^{4t})^{3/2}}\mathbf{i} + \frac{2e^{2t}}{(1+e^{4t})^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{17}\mathbf{j};$$

$$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k}; \quad \mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 8 \text{ and } |\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}};$$

$$\dot{\mathbf{a}} = 4e^{2t}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

$$20. \quad \mathbf{r} = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6 \sinh 2t)\mathbf{i} + (6 \cosh 2t)\mathbf{j} + 6\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{36 \sinh^2 2t + 36 \cosh^2 2t + 36} = 6\sqrt{2} \cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} 2t\right)\mathbf{k}$$

$$\Rightarrow \mathbf{T}(\ln 2) = \frac{5}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k}; \quad \frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}} \operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}} \operatorname{sech} 2t \tanh 2t\right)\mathbf{k}$$

$$\Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$$

$$\Rightarrow \mathbf{N}(\ln 2) = \frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{k}; \quad \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{5}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{8}{17\sqrt{2}}\mathbf{k};$$

$$\mathbf{a} = (12 \cosh 2t)\mathbf{i} + (12 \sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\mathbf{i} + \frac{45}{2}\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k}$$

$$= \frac{45}{4}\mathbf{i} + \frac{51}{4}\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix} = -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2} \text{ and } |\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2}$$

$$\Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867}; \quad \dot{\mathbf{a}} = (24 \sinh 2t)\mathbf{i} + (24 \cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{32}{867}$$

$$21. \quad \mathbf{r} = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6 \cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6 \sin t)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6 \sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36 \sin^2 t}$$

$$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(25 + 100t + 100t^2 + 36 \sin^2 t)^{-1/2}(100 + 200t + 72 \sin t \cos t) \Rightarrow a_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10;$$



$$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (6\cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6\cos t)^2} = \sqrt{100 + 36\cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$$

$$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$$

$$22. \quad \mathbf{r} = (2+t)\mathbf{i} + (t+2t^2)\mathbf{j} + (1+t^2)\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1+4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1+4t)^2 + (2t)^2} = \sqrt{2+8t+20t^2}$$

$$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(2+8t+20t^2)^{-1/2}(8+40t) \Rightarrow a_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2}; \quad \mathbf{a} = 4\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20}$$

$$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - (2\sqrt{2})^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$$

$$23. \quad \mathbf{r} = (\sin t)\mathbf{i} + (\sqrt{2}\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - (\sqrt{2}\sin t)\mathbf{j} + (\cos t)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + (-\sqrt{2}\sin t)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{k};$$

$$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}}\sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}}\sin t\right)^2} = 1$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}}\sin t\right)\mathbf{k}; \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\cos t & -\sin t & \frac{1}{\sqrt{2}}\cos t \\ -\frac{1}{\sqrt{2}}\sin t & -\cos t & -\frac{1}{\sqrt{2}}\sin t \end{vmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k};$$

$$\mathbf{a} = (-\sin t)\mathbf{i} - (\sqrt{2}\cos t)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix} = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}; \quad \dot{\mathbf{a}} = (-\cos t)\mathbf{i} + (\sqrt{2}\sin t)\mathbf{j} - (\cos t)\mathbf{k}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\cos t)(\sqrt{2}) - (\sqrt{2}\sin t)(0) + (\cos t)(-\sqrt{2})}{4} = 0$$

$$24. \quad \mathbf{r} = \mathbf{i} + (5\cos t)\mathbf{j} + (3\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5\sin t)\mathbf{j} + (3\cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5\cos t)\mathbf{j} - (3\sin t)\mathbf{k}$$

$$\Rightarrow \mathbf{v} \cdot \mathbf{a} = 25\sin t \cos t - 9\sin t \cos t = 16\sin t \cos t; \quad \mathbf{v} \cdot \mathbf{a} = 0 \Rightarrow 16\sin t \cos t = 0 \Rightarrow \sin t = 0 \text{ or } \cos t = 0$$

$$\Rightarrow t = 0, \frac{\pi}{2} \text{ or } \pi$$

$$25. \quad \mathbf{r} = 2\mathbf{i} + \left(4\sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 2(1) + \left(4\sin \frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4\sin \frac{t}{2} \Rightarrow \sin \frac{t}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{t}{2} = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{3} \text{ (for the first time)}$$

$$26. \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2+9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14} \Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k},$$

which is normal to the normal plane  $\Rightarrow \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$  or  $x+2y+3z=6$  is an equation of the normal plane. Next we calculate  $\mathbf{N}(1)$  which is normal to the rectifying plane. Now,  $\mathbf{a} = 2\mathbf{j} + 6t\mathbf{k}$

$$\Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}};$$

$$\frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left. \frac{d^2s}{dt^2} \right|_{t=1} = \frac{1}{2} (1 + 4t^2 + 9t^4)^{-1/2} (8t + 36t^3) \Big|_{t=1} = \frac{22}{\sqrt{14}}, \text{ so } \mathbf{a} = \frac{d^2s}{dt^2} \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}$$

$$\Rightarrow 2\mathbf{j} + 6\mathbf{k} = \frac{22}{\sqrt{14}} \left( \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}} \right) + \frac{\sqrt{19}}{7\sqrt{14}} (\sqrt{14})^2 \mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}} \left( -\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k} \right)$$

$$\Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) = 0 \text{ or } 11x + 8y - 9z = 10 \text{ is an equation of the rectifying plane. Finally,}$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left( \frac{\sqrt{14}}{2\sqrt{19}} \right) \left( \frac{1}{\sqrt{14}} \right) \left( \frac{1}{7} \right) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}} (3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0 \text{ or}$$

$$3x - 3y + z = 1 \text{ is an equation of the osculating plane.}$$

27.  $\mathbf{r} = e^t \mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t \mathbf{i} + (\cos t)\mathbf{j} - \left( \frac{1}{1-t} \right) \mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}; \mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0)$  is on the line  
 $\Rightarrow x = 1+t, y = t, \text{ and } z = -t$  are parametric equations of the line

28.  $\mathbf{r} = (\sqrt{2} \cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sqrt{2} \sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + \mathbf{k}$   
 $\Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2} \sin \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \cos \frac{\pi}{4})\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$  is a vector tangent to the helix when  $t = \frac{\pi}{4} \Rightarrow$  the  
 tangent line is parallel to  $\mathbf{v}\left(\frac{\pi}{4}\right)$ ; also  $\mathbf{r}\left(\frac{\pi}{4}\right) = (\sqrt{2} \cos \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \sin \frac{\pi}{4})\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow$  the point  $(1, 1, \frac{\pi}{4})$  is on the  
 line  $\Rightarrow x = 1-t, y = 1+t, \text{ and } z = \frac{\pi}{4} + t$  are parametric equations of the line

29.  $x^2 = (v_0^2 \cos^2 \alpha) t^2$  and  $(y + \frac{1}{2} g t^2)^2 = (v_0^2 \sin^2 \alpha) t^2 \Rightarrow x^2 + (y + \frac{1}{2} g t^2)^2 = v_0^2 t^2$

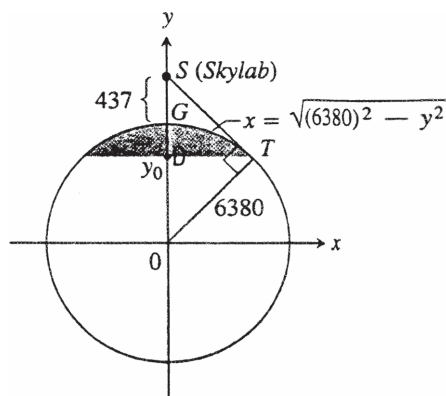
30.  $\ddot{s} = \frac{d}{dt} \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x} \ddot{x} + \dot{y} \ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2 = \ddot{x}^2 + \ddot{y}^2 - \frac{(\dot{x} \ddot{x} + \dot{y} \ddot{y})^2}{\dot{x}^2 + \dot{y}^2} = \frac{(\dot{x}^2 + \dot{y}^2)(\ddot{x}^2 + \ddot{y}^2) - (\dot{x} \ddot{x} + \dot{y} \ddot{y})^2}{\dot{x}^2 + \dot{y}^2}$   
 $= \frac{\dot{x}^2 \ddot{y}^2 + \dot{y}^2 \ddot{x}^2 - 2\dot{x} \ddot{x} \dot{y} \ddot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{(\dot{x} \ddot{y} - \dot{y} \ddot{x})^2}{\dot{x}^2 + \dot{y}^2} \Rightarrow \sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2} = \frac{|\dot{x} \ddot{y} - \dot{y} \ddot{x}|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x} \ddot{y} - \dot{y} \ddot{x}|} = \frac{1}{\kappa} = \rho$

31.  $s = a\theta \Rightarrow \theta = \frac{s}{a} \Rightarrow \phi = \frac{s}{a} + \frac{\pi}{2} \Rightarrow \frac{d\phi}{ds} = \frac{1}{a} \Rightarrow \kappa = \left| \frac{1}{a} \right| = \frac{1}{a}$  since  $a > 0$

32. (1)  $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO}$   
 $\Rightarrow \frac{y_0}{6380} = \frac{6380}{6380+437} \Rightarrow y_0 = \frac{6380^2}{6817}$   
 $\Rightarrow y_0 \approx 5971 \text{ km};$

(2)  $VA = \int_{5971}^{6380} 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy$   
 $= 2\pi \int_{5971}^{6380} \sqrt{6380^2 - y^2} \left( \frac{6380}{\sqrt{6380^2 - y^2}} \right) dy$   
 $= 2\pi \int_{5971}^{6380} 6380 dy = 2\pi [6380y]_{5971}^{6380}$   
 $= 16,395,469 \text{ km}^2 \approx 1.639 \times 10^7 \text{ km}^2;$

(3) percentage visible  $\approx \frac{16,395,469 \text{ km}^2}{4\pi(6380 \text{ km})^2} \approx 3.21\%$



# CHAPTER 13 ADDITIONAL AND ADVANCED EXERCISES

1. (a)  $\mathbf{r}(\theta) = (a \cos \theta)\mathbf{i} + (a \sin \theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}] \frac{d\theta}{dt};$   
 $|\mathbf{v}| = \sqrt{2gz} = \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{a^2 + b^2} \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gz}}{\sqrt{a^2 + b^2}} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + b^2}} \Rightarrow \frac{d\theta}{dt} \Big|_{\theta=2\pi} = \frac{\sqrt{4\pi gb}}{\sqrt{a^2 + b^2}} = 2\sqrt{\frac{\pi gb}{a^2 + b^2}}$
- (b)  $\frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + b^2}} \Rightarrow \frac{d\theta}{\sqrt{\theta}} = \frac{\sqrt{2gb}}{\sqrt{a^2 + b^2}} dt \Rightarrow 2\theta^{1/2} = \frac{\sqrt{2gb}}{\sqrt{a^2 + b^2}} t + C; \quad t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0 \Rightarrow 2\theta^{1/2} = \frac{\sqrt{2gb}}{\sqrt{a^2 + b^2}} t$   
 $\Rightarrow \theta = \frac{gbt^2}{2(a^2 + b^2)}; \quad z = b\theta \Rightarrow z = \frac{gb^2 t^2}{2(a^2 + b^2)}$
- (c)  $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}] \frac{d\theta}{dt} = [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}] \left( \frac{gbt}{a^2 + b^2} \right),$  from part (b)  
 $\Rightarrow \mathbf{v}(t) = \left[ \frac{(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2 + b^2}} \right] \left( \frac{gbt}{\sqrt{a^2 + b^2}} \right) = \frac{gbt}{\sqrt{a^2 + b^2}} \mathbf{T};$   
 $\frac{d^2\mathbf{r}}{dt^2} = [(-a \cos \theta)\mathbf{i} - (a \sin \theta)\mathbf{j}] \left( \frac{d\theta}{dt} \right)^2 + [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}] \frac{d^2\theta}{dt^2}$   
 $= \left( \frac{gbt}{a^2 + b^2} \right)^2 [(-a \cos \theta)\mathbf{i} - (a \sin \theta)\mathbf{j}] + [(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}] \left( \frac{gb}{a^2 + b^2} \right)$   
 $= \left[ \frac{(-a \sin \theta)\mathbf{i} + (a \cos \theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2 + b^2}} \right] \left( \frac{gb}{\sqrt{a^2 + b^2}} \right) + a \left( \frac{gbt}{a^2 + b^2} \right) [(-\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j}] = \frac{gb}{\sqrt{a^2 + b^2}} \mathbf{T} + a \left( \frac{gbt}{a^2 + b^2} \right)^2 \mathbf{N}$  (there is no component in the direction of  $\mathbf{B}$ ).
2. (a)  $\mathbf{r}(\theta) = (a\theta \cos \theta)\mathbf{i} + (a\theta \sin \theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(a \cos \theta - a\theta \sin \theta)\mathbf{i} + (a \sin \theta + a\theta \cos \theta)\mathbf{j} + b\mathbf{k}] \frac{d\theta}{dt};$   
 $|\mathbf{v}| = \sqrt{2gz} = \left| \frac{d\mathbf{r}}{dt} \right| = (a^2 + a^2\theta^2 + b^2)^{1/2} \left( \frac{d\theta}{dt} \right) \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + a^2\theta^2 + b^2}}$
- (b)  $s = \int_0^t |\mathbf{v}| dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2} \frac{d\theta}{dt} dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2} d\theta = \int_0^\theta (a^2 + a^2u^2 + b^2)^{1/2} du$   
 $= \int_0^\theta a \sqrt{\frac{a^2 + b^2}{a^2} + u^2} du = a \int_0^\theta \sqrt{c^2 + u^2} du, \text{ where } c = \frac{\sqrt{a^2 + b^2}}{|a|} \Rightarrow s = a \left[ \frac{u}{2} \sqrt{c^2 + u^2} + \frac{c^2}{2} \ln \left| u + \sqrt{c^2 + u^2} \right| \right]_0^\theta$   
 $= \frac{a}{2} \left( \theta \sqrt{c^2 + \theta^2} + c^2 \ln \left| \theta + \sqrt{c^2 + \theta^2} \right| - c^2 \ln c \right)$
3.  $r = \frac{(1+e)r_0}{1+e \cos \theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2}; \quad \frac{dr}{d\theta} = 0 \Rightarrow \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2} = 0 \Rightarrow (1+e)r_0(e \sin \theta) = 0 \Rightarrow \sin \theta = 0$   
 $\Rightarrow \theta = 0 \text{ or } \pi.$  Note that  $\frac{dr}{d\theta} > 0$  when  $\sin \theta > 0$  and  $\frac{dr}{d\theta} < 0$  when  $\sin \theta < 0$ . Since  $\sin \theta < 0$  on  $-\pi < \theta < 0$  and  $\sin \theta > 0$  on  $0 < \theta < \pi$ ,  $r$  is a minimum when  $\theta = 0$  and  $r(0) = \frac{(1+e)r_0}{1+e \cos 0} = r_0$
4. (a)  $f(x) = x - 1 - \frac{1}{2} \sin x = 0 \Rightarrow f(0) = -1$  and  $f(2) = 2 - 1 - \frac{1}{2} \sin 2 \geq \frac{1}{2}$  since  $|\sin 2| \leq 1$ ; since  $f$  is continuous on  $[0, 2]$ , the Intermediate Value Theorem implies there is a root between 0 and 2  
 (b) Root  $\approx 1.4987011335179$
5. (a)  $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$  and  $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta = (\dot{r})[(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}] + (r\dot{\theta})[(-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}] \Rightarrow \mathbf{v} \cdot \mathbf{i} = \dot{x}$  and  
 $\mathbf{v} \cdot \mathbf{i} = \dot{r} \cos \theta - r\dot{\theta} \sin \theta \Rightarrow \dot{x} = \dot{r} \cos \theta - r\dot{\theta} \sin \theta; \quad \mathbf{v} \cdot \mathbf{j} = \dot{y}$  and  
 $\mathbf{v} \cdot \mathbf{j} = \dot{r} \sin \theta + r\dot{\theta} \cos \theta \Rightarrow \dot{y} = \dot{r} \sin \theta + r\dot{\theta} \cos \theta$

(b)  $\mathbf{u}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \dot{x} \cos \theta + \dot{y} \sin \theta = (\dot{r} \cos \theta - r\dot{\theta} \sin \theta)(\cos \theta) + (\dot{r} \sin \theta + r\dot{\theta} \cos \theta)(\sin \theta)$   
 by part (a),  $\Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \dot{r}$ ; therefore,  $\dot{r} = \dot{x} \cos \theta + \dot{y} \sin \theta$ ;  $\mathbf{u}_\theta = -(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}$   
 $\Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = -\dot{x} \sin \theta + \dot{y} \cos \theta = (\dot{r} \cos \theta - r\dot{\theta} \sin \theta)(-\sin \theta) + (\dot{r} \sin \theta + r\dot{\theta} \cos \theta)(\cos \theta)$  by part (a)  
 $\Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = r\dot{\theta}$ ; therefore,  $r\dot{\theta} = -\dot{x} \sin \theta + \dot{y} \cos \theta$

6.  $r = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta) \frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta) \left( \frac{d\theta}{dt} \right)^2 + f'(\theta) \frac{d^2\theta}{dt^2}$ ;

$$\mathbf{v} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\theta}{dt} \mathbf{u}_\theta = \left( \cos \theta \frac{dr}{dt} - r \sin \theta \frac{d\theta}{dt} \right) \mathbf{i} + \left( \sin \theta \frac{dr}{dt} + r \cos \theta \frac{d\theta}{dt} \right) \mathbf{j}$$

$$\Rightarrow |\mathbf{v}| = \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right]^{1/2} = \left[ (f')^2 + f^2 \right]^{1/2} \left( \frac{d\theta}{dt} \right); \quad |\mathbf{v} \times \mathbf{a}| = |\dot{x} \ddot{y} - \dot{y} \ddot{x}|,$$

where  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then  $\frac{dx}{dt} = (-r \sin \theta) \frac{d\theta}{dt} + (\cos \theta) \frac{dr}{dt}$

$$\Rightarrow \frac{d^2x}{dt^2} = (-2 \sin \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \cos \theta) \left( \frac{d\theta}{dt} \right)^2 - (r \sin \theta) \frac{d^2\theta}{dt^2} + (\cos \theta) \frac{d^2r}{dt^2}; \quad \frac{dy}{dt} = (r \cos \theta) \frac{d\theta}{dt} + (\sin \theta) \frac{dr}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = (2 \cos \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \sin \theta) \left( \frac{d\theta}{dt} \right)^2 + (r \cos \theta) \frac{d^2\theta}{dt^2} + (\sin \theta) \frac{d^2r}{dt^2}. \text{ Then, after much algebra } |\mathbf{v} \times \mathbf{a}|$$

$$= r^2 \left( \frac{d\theta}{dt} \right)^3 + r \frac{d^2\theta}{dt^2} \frac{dr}{dt} - r \frac{d\theta}{dt} \frac{d^2r}{dt^2} + 2 \frac{d\theta}{dt} \left( \frac{dr}{dt} \right)^2 = \left( \frac{d\theta}{dt} \right)^3 \left( f^2 - f \cdot f'' + 2(f')^2 \right) \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{f^2 - f \cdot f'' + 2(f')^2}{[(f')^2 + f^2]^{3/2}}$$

7. (a) Let  $r = 2 - t$  and  $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$  and  $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$ . The halfway point is  $(1, 3) \Rightarrow t = 1$ ;

$$\mathbf{v} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\theta}{dt} \mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta; \quad \mathbf{a} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[ r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] \mathbf{u}_\theta$$

$$\Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$$

(b) It takes the beetle 2 min to crawl to the origin  $\Rightarrow$  the rod has revolved 6 radians

$$\Rightarrow L = \int_0^6 \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_0^6 \sqrt{\left(2 - \frac{\theta}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}} d\theta$$

$$= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}} d\theta = \frac{1}{3} \int_0^6 \sqrt{(\theta - 6)^2 + 1} d\theta = \frac{1}{3} \left[ \frac{(\theta - 6)}{2} \sqrt{(\theta - 6)^2 + 1} + \frac{1}{2} \ln \left| \theta - 6 + \sqrt{(\theta - 6)^2 + 1} \right| \right]_0^6$$

$$= \sqrt{37} - \frac{1}{6} \ln(\sqrt{37} - 6) \approx 6.5 \text{ in.}$$

8. (a)  $x = r \cos \theta \Rightarrow dx = \cos \theta dr - r \sin \theta d\theta$ ;  $y = r \sin \theta \Rightarrow dy = \sin \theta dr + r \cos \theta d\theta$ ; thus

$$dx^2 = \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2 \text{ and}$$

$$dy^2 = \sin^2 \theta dr^2 + 2r \sin \theta \cos \theta dr d\theta + r^2 \cos^2 \theta d\theta^2 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

(c)  $r = e^\theta \Rightarrow dr = e^\theta d\theta$

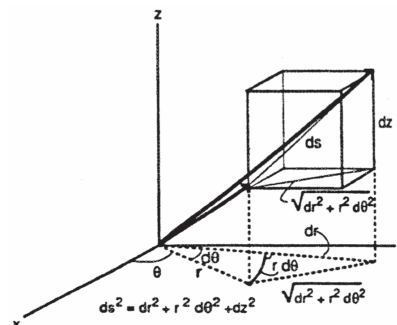
$$\Rightarrow L = \int_0^{\ln 8} \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$$

$$= \int_0^{\ln 8} \sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}} d\theta$$

$$= \int_0^{\ln 8} \sqrt{3} e^\theta d\theta = \left[ \sqrt{3} e^\theta \right]_0^{\ln 8}$$

$$= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$$

(b)



9. (a)  $\mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{vmatrix} = \mathbf{k} \Rightarrow a \text{ right-handed frame of unit vectors}$

(b)  $\frac{d\mathbf{u}_r}{d\theta} = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} = \mathbf{u}_\theta$  and  $\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j} = -\mathbf{u}_r$

(c) From Eq. (7),  $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \Rightarrow \mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta) + \ddot{z}\mathbf{k}$   
 $= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$

10.  $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times m\mathbf{a};$

$\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3}\mathbf{r} = m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \left(-\frac{c}{|\mathbf{r}|^3}\mathbf{r}\right) = -\frac{c}{|\mathbf{r}|^3}(\mathbf{r} \times \mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} = \text{constant vector}$

