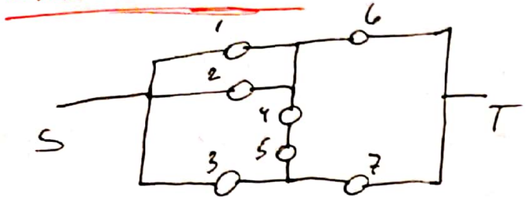
Exam 2015:



(1a) The minimal path sets: $\coprod_{j=1}^{p} \prod_{i \in P_j} x_i \xrightarrow{} \min_{1 \le i \le m} \rightarrow \rho_j(\vec{x}^{P_j})$

$\hookrightarrow \{1,6\}; \{3,7\}; \{2,6\}; \{1,4,5,7\}; \{2,4,5,7\};$
$\{3,4,5,6\}$ //

The minimal cut sets: $\prod_{j=1}^{k} \coprod_{i \in K_j} x_i \rightarrow \max \rightarrow k_j(\vec{x}^{K_j})$

$\hookrightarrow \{1,2,3\}; \{6,7\}; \{1,2,4,7\}; \{1,2,5,7\};$
$\{3,4,6\}; \{3,5,6\}$ // ■

(1b) multiplication $(2^p - 1)$ $\quad 2^6 - 1 \rightarrow$ fastest

enumeration $(2^m - 1)$ $\quad 2^7 - 1 \rightarrow$ slowest ■

(1c) $h(\vec{p})$? . after parallel reduction on 1 and 2, and series reduction on 4, 5:

$\hookrightarrow h_{1'} = p_1 \amalg p_2 = p_1 + p_2 - p_1 p_2 = p_{1'}$

$\hookrightarrow h_{4'} = p_4 \cdot p_5 = p_{4'}$

• Choose component 4' for pivoting method as following:

$$h(\vec{p}) = p_{4'} \cdot h_+(1_4, \vec{p}) + (1 - p_{4'}) \cdot h_-(0_4, \vec{p})$$

where:

$$h_+(1_4, \vec{p}) = \left((p_1 + p_2 - p_1 p_2) \amalg p_3\right) \cdot (p_6 + p_7 - p_6 p_7)$$
$$= \left(p_1 + p_2 - p_1 p_2 + p_3 - (p_1 + p_2 - p_1 p_2) \cdot p_3\right) \cdot (p_6 + p_7 - p_6 p_7)$$

$$h_-(0_4, \vec{p}) = (p_1 + p_2 - p_1 p_2) \cdot p_6 \amalg p_3 \cdot p_7$$
$$= \left((p_1 + p_2 - p_1 p_2) \cdot p_6 + p_3 p_7\right) - \left((p_1 + p_2 - p_1 p_2) \cdot p_6 \cdot p_3 p_7\right)$$

$$\therefore h(\vec{p}) = p_4 p_5 \cdot \left[\left(p_1 + p_2 - p_1 p_2 + p_3 - (p_1 + p_2 - p_1 p_2) \cdot p_3\right)(p_6 + p_7 - p_6 p_7)\right]$$
$$+ (1 - p_4 p_5) \cdot \left[\left((p_1 + p_2 - p_1 p_2) \cdot p_6 + p_3 p_7\right) - (p_1 + p_2 - p_1 p_2) \cdot p_6 p_3 p_7\right] ■$$

(7d) $I_B^{(4)} = ? = \dfrac{d\left[p_4 p_5 \cdot h_{+4} + (1-p_4 p_5)\cdot h_{-4}\right]}{d p_4} = p_5 \cdot h_{+4} - p_5 \cdot h_{-4} =$

$$= p_5\Big(\big(p_1+p_2-p_1p_2+p_3-(p_1+p_2-p_1p_2)p_3\big)\cdot\big(p_6+p_7-p_6p_7\big)\Big)$$

$$-p_5\Big(\big(p_1+p_2-p_1p_2\big)p_6+p_3p_7-\big(p_1+p_2-p_1p_2\big)\cdot p_3p_6p_7\Big)$$

(7e) $J_B^{(4)} = ? = \dfrac{1}{2^{m-1}} \cdot \sum_{(\cdot_i,\vec{x})} \left[\phi[1_i,\vec{x}] - \phi[0_i,\vec{x}]\right]$

method 1: By symetri, the component reliabilities for $p_1 = p_2 = p_3 = p_6 = p_7 = \frac{1}{2}$ and use it on $I_B^{(4)}$:

$$J_B^{(4)} = \frac{1}{2}\left(\left(\overbrace{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}}^{3/4}+\frac{1}{2}-\left(\overbrace{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}}^{3/4}\right)\frac{1}{2}\right)\cdot\left(\overbrace{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}}^{3/4}\right)\right)$$

$$-\frac{1}{2}\left(\left(\overbrace{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}}^{3/4}\right)\frac{1}{2}+\frac{1}{4}-\left(\overbrace{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}}^{3/4}\right)\cdot\frac{1}{8}\right)$$

$$= \frac{1}{2}\left(\left(\frac{3}{4}+\frac{1}{2}-\overbrace{\left(\frac{3}{4}\right)\cdot\frac{1}{2}}^{3/8}\right)\cdot\left(\frac{3}{4}\right)\right)-\frac{1}{2}\left(\left(\frac{3}{4}\right)\cdot\frac{1}{2}+\frac{1}{4}\right.$$

$$\left.-\overbrace{\left(\frac{3}{4}\right)\cdot\frac{1}{8}}^{3/32}\right) = \frac{1}{2}\left(\left(\overbrace{\frac{3}{4}+\frac{1}{2}-\frac{3}{8}}^{7/8}\right)\cdot\frac{3}{4}\right)-\frac{1}{2}\cdot\left(\overbrace{\frac{3}{8}+\frac{1}{4}-\frac{3}{32}}^{17/64}\right) =$$

$$= \frac{1}{2}\cdot\left(\frac{6+4-3}{8}\right)\cdot\frac{3}{4}-\frac{1}{2}\cdot\frac{12+5-3}{32} = \frac{21}{64}-\frac{17}{64} =$$

$$= \frac{4}{64} = \frac{1}{16}$$

method 2: find all critical vectors for component 4:

$\{1,4,5,7\}$; $\{2,4,5,7\}$; $\{3,4,5,6\}$; $\{1,2,4,5,7\}$

$\therefore$ 4 critical vectors

From $J_B^{(4)} = \dfrac{4}{2^{7-1}} = \dfrac{4}{64} = \dfrac{1}{16}$

(7f) $J_B^{(1)} = ?$. assume $p_2=p_3=p_4=p_5=p_6=p_7=1/2$ and use it on $I_B^{(1)} = \dfrac{d[h\vec{p}]}{d p_1}$

$$\Leftrightarrow p_4p_5\cdot\left[\left(1-p_2-(1-p_2)\cdot p_3\right)\cdot\left(p_6+p_7-p_6p_7\right)\right]$$

$$+(1-p_4p_5)\cdot\left[\left((1-p_2)\cdot p_6\right)-(1-p_2)\cdot p_6p_3p_7\right] =$$

$$= \frac{1}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right)\cdot\frac{3}{4}\right]+\frac{3}{4}\cdot\left[\frac{1}{4}-\frac{1}{16}\right] =$$

$$= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{3}{16} =$$

$$= \frac{3}{64}+\frac{9}{64} = \frac{12}{64} = \frac{3}{16}$$

Exam 2015

2a we have $\{X_1, X_2, \dots, X_n\}$ independent random variables and want to prove that they are associated. we prove that by induction.

First: for $n=1$, the proposition is true by the theorem 6.1.4 (ii) which says that a random variable is also associated;

Second: for $n=m-1$, we have a set $\{X_1, \dots, X_{m-1}\}$ of associated random variables which is the union of the set $\{X_1\}$ and $\{X_{m-1}\}$, then by Theorem 6.1.4 (iv) it is associated, because the union of 2 independent sets of associated random variables is a set of associated random variables. ■

2b associated component states:

$p_i = E[X_i]$

$$\prod_{i=1}^{m} p_i = \prod_{i=1}^{m} E[X_i] \leq E\left[\prod_{i=1}^{m} X_i\right] \leq h$$

$$\leq E\left[\coprod_{i=1}^{m} X_i\right] \leq \coprod_{i=1}^{m} E[X_i] = \coprod_{i=}^{m} p_i$$

↳ theorem 2.2.4: $\prod_{i=1}^{m} x_i \leq \phi(\vec{x}) \leq \coprod_{i=1}^{m} x_i$

associated component states. ■

2c The minimal path or cut sets are non-decreasing, then by theorem 6.1.4 (iii) they are associated.

$$\prod_{j=1}^{k} P\left(k_j\left(\vec{X}^{K_j}\right)=1\right) = \prod_{j=1}^{k} E\left[k_j\left(\vec{X}^{K_j}\right)\right] \leq$$

$$E\left[\prod_{j=1}^{k} k_j\left(\vec{X}^{K_j}\right)\right] = P\left(\phi(\vec{X})=1\right) = E\left[\coprod_{j=1}^{p} p_j\left(\vec{X}^{P_j}\right)\right]$$

$$\leq \coprod_{j=1}^{p} E\left[p_j\left(\vec{X}^{P_j}\right)\right] = \coprod_{j=1}^{p} P\left(p_j\left(\vec{X}^{P_j}\right)=1\right)$$ ■

↳ They are not explicit. ■

2d

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \overset{p_i = E[X_i]}{=} \prod_{j=1}^{k} \coprod_{i \in K_j} E[X_i]$$

$$\overset{\{X_i\} \text{ independent}}{=} \prod_{j=1}^{k} E\left[\coprod_{i \in K_j} X_i\right] = \prod_{j=1}^{k} P\left(k_j(\vec{X}^{K_j}) = 1\right)$$

$E\left[\coprod_{i \in K_j} X_i\right] = P\left(k_j(\vec{X}^{K_j}) = 1\right)$

$$\leq E\left[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i\right] = P\left(\phi(\vec{X}) = 1\right)$$

Theorem 6.1.4(iii)

$$= E\left[\coprod_{j=1}^{p} \prod_{i \in P_j} X_i\right] \leq \coprod_{j=1}^{p} P\left(p_j(\vec{X}^{P_j}) = 1\right)$$

$$= \coprod_{j=1}^{p} E\left[\prod_{i \in P_j} X_i\right] = \coprod_{j=1}^{p} \prod_{i \in P_j} E[X_i]$$

$\{X_i\}$ independent

$$= \coprod_{j=1}^{p} \prod_{i \in P_j} p_i \quad \blacksquare$$

$p_i = E[X_i]$

$E\left[\prod_{i \in P_j} X_i\right] = P\left(p_j(\vec{X}^{P_j}) = 1\right)$

2e

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \overset{p_i = E[X_i]}{=} \prod_{j=1}^{k} \coprod_{i \in K_j} E[X_i] \overset{\{X_i\} \text{ independent}}{=} \prod_{j=1}^{k} E\left[\coprod_{i \in K_j} X_i\right] =$$

$$\overset{\text{non-overlapping}}{=} E\left[\coprod_{i \in K_1} X_i\right] \cdot \prod_{j=2}^{k} E\left[\coprod_{i \in K_j} X_i\right] \overset{\text{T.6.1.4 iii}}{\leq} E\left[\coprod_{i \in K_1} X_i\right] \cdot E\left[\prod_{j=2}^{k} \coprod_{i \in K_j} X_i\right]$$

$$< E\left[\coprod_{i \in K_1} X_i \cdot \prod_{j=2}^{k} \coprod_{i \in K_j} X_i\right] \overset{K_1, K_j \text{ overlapped}}{=} E\left[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i\right]$$

$$\boxed{Cov\left(\coprod_{i \in K_1} X_i, \prod_{j=2}^{k} \coprod_{i \in K_j} X_i\right) > 0}$$

$$= P\left(\phi(\vec{X}) = 1\right) \quad \blacksquare$$

Exam 2015

2f Consider a 3 out of 4 system with $p_i = p$ for $i = 1, \dots, 4$. The minimal path sets are: $\{1,2,3\}$; $\{1,2,4\}$; $\{1,3,4\}$; $\{2,3,4\}$ and the minimal cut sets are: $\{1,2\}$; $\{1,3\}$; $\{1,4\}$; $\{2,3\}$; $\{2,4\}$; $\{3,4\}$. Then the lower bound (d) is:

$$\prod_{j=1}^{6} \underbrace{\coprod_{i \in K_j} p}_{2p-p^2} = \boxed{(2p-p^2)^6 = l_d}$$

and the lower bound (b):

$$\prod_{i=1}^{4} p = \boxed{p^4 = l_b}$$

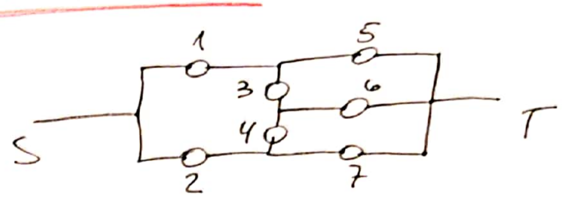
the reliability of the system:

$$h = \sum_{i=3}^{4} \binom{4}{i} p^i (1-p)^{n-i} = \boxed{4p^3 \cdot (1-p)^1 + p^4 = h(p)}$$

$\therefore$ we choose $\frac{1}{10} = p$, then

$$l_d = \left(\frac{2}{10} - \frac{1}{100}\right)^6 = \left(\frac{19}{100}\right)^6 < l_b = \left(\frac{1}{10}\right)^4 < h(p) = \frac{4}{1000} \cdot \frac{9}{10} + \frac{1}{10000} = \frac{37}{10000}$$

Exam 2016:



1a The minimal path sets:

$\{1,5\}$; $\{1,3,6\}$; $\{1,3,4,7\}$; $\{2,7\}$; $\{2,4,6\}$; $\{2,3,4,5\}$

The minimal cut sets:

$\{1,2\}$;

Exam 2018:

2a (C, ϕ) with $C = \{1, 2, 3\}$

$$\sum_{i=2}^{n=3} \binom{3}{i} p^i (1-p)^{n-i}$$

$h(\vec{p})$ $P(S \geq k)$

$$\phi(\vec{X}) = I\left(\sum_{i=1}^{3} X_i \geq 2\right)$$

because out of 3 components, at least 2 must be functioning.

. we see that the structure function represents a system $\boxed{k \text{ out of } n \text{ system}}$, where $k=2$ and $n=3$:

Pivotal decomposition:

$\phi(\vec{X}) = X_1 \cdot \phi(1_1, \vec{x}) + (1-X_1)\phi(0_1, \vec{x})$

$= X_1 \cdot \left(\sum_{i=2}^{3} \geq 1\right) + (1-X_1)\cdot\left(\sum_{i=2}^{3} x_i \geq 2\right)$

$= X_1 (X_2 \amalg X_3) + (1-X_1)\cdot(X_2 X_3)$

$= X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3$

or k-out-of n system:

$\phi(\vec{X}) = x_1x_2 \amalg x_1x_3 \amalg x_2x_3$

$= (x_1x_2 + x_1x_3 - x_1x_2x_3) \amalg x_2x_3$

$= x_1x_2 + x_1x_3 - x_1x_2x_3 + x_2x_3 - x_1x_2x_3$

$\to = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$ ■

2b $X_i = Y_0 \cdot Y_i, \quad i = 1, 2, 3$

we see that:

$$\boxed{\begin{aligned} X_1 &= Y_0 Y_1 = \theta \cdot q \\ X_2 &= Y_0 \cdot Y_2 = \theta \cdot q \\ X_3 &= Y_0 \cdot Y_3 = \theta \cdot q \\ X_4 &= Y_0 \cdot Y_4 = \theta \cdot q \end{aligned}}$$

6.1.4: i => subset of a set of associated rv's

ii => a independent rv. is associated

iii => non-decreasing functions of associated rv's

iv => union of 2 independent sets of associated rv's

where the following component, for example X_2, depends of Y_0 which also depend the previous component (X_1), then they are associated.

and

by Theorem 6.1.4 (iii), non-decreasing functions of independent rv's $\to Y_0, \dots, Y_4$

$\to \boxed{X_1, X_2, X_3, X_4}$

are associated. ■

2c $h(\theta, q) = \theta q^2 (3 - 2q) = 3\theta q^2 - 2\theta q^3$

$h(\theta, q) = E[(\phi \vec{x})] =$

$= P(Y_0 = 1) \cdot E[\phi(\vec{x}) \mid Y_0 = 1] + P(Y_0 = 0) \cdot E[\phi(\vec{x}) \mid Y_0 = 0]$

$= Y_0 \cdot (Y_1Y_2 + Y_1Y_3 + Y_2Y_3 - 2Y_1Y_2Y_3) + (1 - Y_0) \cdot 0$

$= 3\theta q^2 - 2\theta q^3$

$= \theta q^2 (3 - 2q)$ ■

2d $\tilde{h}(\theta, q) = E[\phi(\vec{x})] = E[x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3] = (\theta q)^2 + (\theta q)^2 + (\theta q)^2 - 2(\theta q)^3$

$= \theta^2 q^2 (3 - 2\theta q)$ ■

2e $\theta = \frac{1}{2}$

$h = \frac{3}{2} \cdot q^2 - q^3$

$\tilde{h} = \frac{1}{4} q^2 \cdot (3 - q) = \frac{3}{4} q^2 - \frac{1}{4} q^3$

$d = h - \tilde{h}$

$d = \frac{3}{2} \cdot q^2 - q^3 - \frac{3}{4} q^2 + \frac{1}{4} q^3$

$d = \frac{3}{4} q^2 - \frac{3}{4} q^3 = \left[\frac{3}{4} q^2\right] \cdot (1 - q)$

$d > 0 \therefore h > \tilde{h}$ ■

2f $\Theta = \frac{3}{4}$

$$h = \frac{3}{4}q^2(3-2q) = \frac{9}{4}q^2 - \frac{6}{4}q^3 = \frac{9}{4}q^2 - \frac{3}{2}q^3$$

$$\tilde{h} = \frac{9}{16}q^2\left(3 - 2\cdot\frac{3}{4}q\right) = \frac{27}{16}q^2 - \frac{27}{32}q^3$$

$$d = h - \tilde{h} = \frac{9}{4}q^2 - \frac{27}{16}q^2 - \frac{3}{2}q^3 + \frac{27}{32}q^3$$

$$= \frac{3}{32}q^2\cdot[6-7q] \quad \therefore d>0 \leftrightarrow 7q<6 \leftrightarrow q<\frac{6}{7}$$

↳ an incorrect assumption of the dependency of q can lead us to underestimate the reliability of the system. If $q<\frac{6}{7}$ is ignored, can lead us to overestimate the reliability of the system. ■

3a $$\mathrm{Var}(\hat{\Theta}_j) = \frac{1}{N_j^2}\sum_{r=1}^{N_j}\mathrm{Var}(\phi|S=s_j) = \frac{1}{N_j}\cdot\mathrm{Var}(\phi|S=s_j)$$

$$\therefore \mathrm{Var}(\hat{h}_{CMC}) = \sum_{r=1}^{k}\mathrm{Var}(\hat{\Theta}_j)\cdot\left[P(S=s_j)\right]^2$$

3b Insert $N_j \approx N\cdot P(S=s_j)$ in 3a:

$$\mathrm{Var}(\hat{\Theta}_j) = \sum_{r=1}^{k}\frac{1}{N\cdot \cancel{P(S=s_j)}}\cdot\mathrm{Var}(\phi|S=s_j)\cdot\left[P(S=s_j)\right]^{\cancel{2}}$$

$$= \frac{1}{N}\sum_{r=1}^{k}\mathrm{Var}(\phi|S=s_j)\cdot\left[P(S=s_j)\right]$$

$$= \frac{1}{N}E\left[\mathrm{Var}(\phi|S)\right]$$

$$\mathrm{Var}(\phi) = \mathrm{Var}(E[\phi|S]) + E[\mathrm{Var}(\phi|S)]$$

$$= \frac{1}{N}\Big[\mathrm{Var}(\phi) - \underbrace{\mathrm{Var}(E[\phi|S])}_{>0}\Big]$$

$\boxed{\text{as much info } S \text{ has from } \phi}$

$$\therefore \underbrace{V(\hat{h}_{CMC})}_{\text{conditional variance}} \le \mathrm{Var}(\hat{h}_{MC})$$

3d ① S must have as much information of ϕ as possible. ② S must have a distribution derived analytically in polynomial time. ③ S variables limited by n. ④ possible to sample from distribution of X given S.

Basic notations in Statistics:

Population set	sample set	
$X, Y \ldots$	$x, y, \ldots$	stochastic variables
$\mu_x, \mu_y, \ldots$	$\bar{x}, \bar{y}, \ldots$	mean
$\sigma_x, \sigma_y, \ldots$	$s_x, s_y, \ldots$	the standard deviation
$\rho_x, \rho_y, \ldots$	$r_x, r_y, \ldots$	correlation coefficient
N	n	# of elements
$Z = (X - \mu_x)/\sigma_x$ where $X :=$ value of an element	$z = (x - \bar{x})/s_x$ where $x :=$ value of an element	the normal random variable of a standard normal distribution

Standard Normal Distribution

- Every normal random variable X can be transformed into a normal random variable of a standard normal distribution:

$$Z = (X - \mu_x)/\sigma_x$$

The normal Distribution

- Z, z: standardized score;
  - ↳ indicates how many σ an element X is from μ;
  - ↳ the result, i.e. $z = 1$, represents an element X is 1 σ greater than μ.

- Element X: normal random variable in the normal equation:

$$Y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{(x-\mu_x)^2/2\sigma_x^2} \rightarrow X \sim N(\mu_x, \sigma_x^2)$$

↳ normal random variable $\rightarrow Y \sim N(\mu_Y, \sigma_Y^2)$

↳ Probability density function of the normal distribution

↳ The probability, that a random variable assumes a value between a and b, is equal to the area under the density function bounded by a and b.

Example: 

$\therefore P(x \leq x_i) =$ area in $(-\infty, x_i]$

→ cumulative probability:

↳ The probability that the value of a random variable falls within a specified range.

$$\therefore P(x \leq x_i) = \sum_{j=1}^{i} f(x_j) \cdot \Delta x_j = \int_{-\infty}^{i} f(x)\,dx$$

(i.e.): $P(X \leq 1) = P(X = 0) + P(X = 1) = 0{,}25 + 0{,}50 = 0{,}75$

Problem 1/: time-series model $\forall Y \sim \text{lognormal}(\mu, \sigma)$

$\{W_n\}$ independent standard normally distributed variables

New

$$\{U_n\}: \begin{cases} U_1 = W_1 \\ U_n = a \cdot U_{n-1} + b \cdot W_n \\ n = 2, 3, \dots \end{cases}$$

stochastic dependent

$\Rightarrow$ determine a, b : $U_n \sim N(0,1)$ and $\operatorname{cov}(U_n, U_{n-1}) = c$, $n = 2, 3, \dots$,
where c is a suitable constant : $|c| \leq 1$

adequada

correlation: measures the degree to which two variables move in relation to each other.

$$\rho_{X,Y} = \frac{E[XY] - E[X] \cdot E[Y]}{\sqrt{E[X^2] - E[X]^2} \cdot \sqrt{E[Y^2] - E[Y]^2}}$$

correlation coefficient: calculates the strength of the relationship between the relative movements of two variables.

↳ between $-1, 1$ where 0 means no relationship, -1 means perfect negative correlation and 1 perfecte positive correlation.

$$\rho_{X,Y} = \operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

covariance: measures the directional relationship between two variables.

↳ Positive: means they move together;

↳ Negative: means they move inversely.

$$\operatorname{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X] \cdot E[Y]$$

Discrete random variables: $(X,Y) \to (x_i, y_i)$ for $i = 1, \dots, n$ with equal $p_i = \frac{1}{n}$:

$$\operatorname{cov}(X,Y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_X)(y_i - \mu_Y)$$

with equal p_i :

$$\operatorname{cov}(X,Y) = \sum_{i=1}^{n} p_i (x_i - \mu_X)(y_i - \mu_Y)$$

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