STK3405 – Week 48

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Exam STK2400 - 2006

STK2400 - 2006, problem 1

A binary monotone system, (C, ϕ), with component set $C = \{1, \ldots, n\}$ and structure function ϕ, is said to be a *threshold system* if the structure function can be written as:

$$\phi(\boldsymbol{X}) = I(\sum_{i=1}^{n} a_i X_i \geq b),$$

where $a_1, \ldots, a_n$ and b are non-negative constants, and $\boldsymbol{X} = (X_1, \ldots, X_n)$ is the vecor of component state variables.

The constants $a_1, \ldots, a_n$ are called the *component weights* of the system, while the constant b is called the *threshold value* of the system.

STK2400 - 2006, problem 1 (a)

Let (C, ϕ) be a threshold system where all component weights are 1 and where the threshold value is k, where k is an interger such that $1 \leq k \leq n$. What kind of system is this?

SOLUTION: In this case the structure function ϕ is:

$$\phi(\boldsymbol{X}) = I(\sum_{i=1}^{n} X_i \geq k).$$

Thus, the system is functioning if and only if at least k of the n components are functioning. Such a system is called a k-out-of-n system.

STK2400 - 2006, problem 1 (b)

Let (C, ϕ_1) and (C, ϕ_2) be two threshold systems with common component positive weights $a_1, \ldots, a_n$, and with threshold values b_1 og b_2 respectively given by:

$$
\begin{aligned}
b_1 &= \sum_{i=1}^{n} a_i \\
b_2 &= \min_{1 \leq i \leq n} a_i.
\end{aligned}
$$

What kind of systems are (C, ϕ_1) and (C, ϕ_2)?

STK2400 - 2006, problem 1 (b)

SOLUTION: We observe that:

$$\phi_1(\boldsymbol{X}) = I(\sum_{i=1}^{n} a_i X_i \geq \sum_{i=1}^{n} a_i)$$

Hence, (C, ϕ_1) is functioning if and only if *all* components are functioning. Thus, (C, ϕ_1) is a *series system*.

We observe that:

$$\phi_2(\boldsymbol{X}) = I(\sum_{i=1}^{n} a_i X_i \geq \min_{1 \leq i \leq n} a_i)$$

Hence, (C, ϕ_2) is functioning if and only if *at least one* component is functioning. Thus, (C, ϕ_2) is a *parallel system*.

STK2400 - 2006, problem 1 (c)

Let (C, ϕ) be a threshold system and let (C, ϕ^D) denote the dual system. Show that (C, ϕ^D) is a threshold system as well.

SOLUTION: We consider the structure function ϕ^D:

$$\begin{aligned}
\phi^D(\boldsymbol{X}^D) = 1 - \phi(\boldsymbol{1} - \boldsymbol{X}^D) &= 1 - I(\sum_{i=1}^n a_i(1 - X_i^D) \geq b) \\
&= 1 - I(\sum_{i=1}^n a_i - \sum_{i=1}^n a_i X_i^D \geq b) \\
&= 1 - I(\sum_{i=1}^n a_i X_i^D \leq \sum_{i=1}^n a_i - b) \\
&= I(\sum_{i=1}^n a_i X_i^D > \sum_{i=1}^n a_i - b) \\
&= I(\sum_{i=1}^n a_i X_i^D \geq b^D)
\end{aligned}$$

STK2400 - 2006, problem 1 (c)

Summarizing this we have that:

$$\phi^D(\boldsymbol{X}^D) = I(\sum_{i=1}^{n} a_i X_i^D \geq b^D)$$

where $b^D > \sum_{i=1}^{n} a_i - b$ is a suitably number. More specifically, b^D can be found as follows. Let the set $\mathcal{X}$ be defined as:

$$\mathcal{X} = \{\boldsymbol{x} \in \{0,1\}^n : \sum_{i=1}^{n} a_i x_i > \sum_{i=1}^{n} a_i - b\}$$

Since $\mathcal{X}$ only contains a *finite* number of elements, we can define:

$$b^D = \min_{\boldsymbol{x} \in \mathcal{X}} \sum_{i=1}^{n} a_i x_i.$$

Hence, the dual system is a threshold system with the same component weights as the original system, and with threshold b^D.

STK2400 - 2006, problem 1 (d)

In the remaining part of this problem we let (C, ϕ_b) denote a threshold system where $C = \{1, \ldots, 5\}$, $b \in \{1, \ldots, 8\}$, and where:

$$\phi_b(\boldsymbol{X}) = I(3X_1 + 2X_2 + X_3 + X_4 + X_5 \geq b).$$

Moreover, we denote the reliability of (C, ϕ_b) by h_b, $b = 1, \ldots, 8$.

STK2400 - 2006, problem 1 (d)

Find the minimal path and cut sets of (C, ϕ_3).

$$\phi_3(\boldsymbol{X}) = I(3X_1 + 2X_2 + X_3 + X_4 + X_5 \geq 3).$$

Minimal path sets:

$$\{1\}, \quad \{2,3\}, \quad \{2,4\}, \quad \{2,5\}, \quad \{3,4,5\}$$

Minimal cut sets:

$$\{1,2,3\}, \quad \{1,2,4\}, \quad \{1,2,5\}, \quad \{1,3,4,5\}$$

STK2400 - 2006, problem 1 (e)

Show that:

$$\phi_3(1_1, \boldsymbol{X}) = 1,$$
$$\phi_3(0_1, \boldsymbol{X}) = X_2 \cdot (X_2 \amalg X_4 \amalg X_5) + (1 - X_2) \cdot (X_1 X_4 X_5).$$

STK2400 - 2006, problem 1 (e)

SOLUTION: By letting $X_1 = 1$ we get that:

$$\begin{aligned}\phi_3(1_1, \boldsymbol{X}) &= I(3 \cdot 1 + 2X_2 + X_3 + X_4 + X_5 \geq 3) \\ &= I(2X_2 + X_3 + X_4 + X_5 \geq 0) = 1.\end{aligned}$$

Moreover, by letting $X_1 = 0$ and by pivoting with respect to component 2 we get that:

$$\begin{aligned}\phi_3(0_1, \boldsymbol{X}) &= I(3 \cdot 0 + 2X_2 + X_3 + X_4 + X_5 \geq 3) \\ &= X_2 \cdot I(2 \cdot 1 + X_3 + X_4 + X_5 \geq 3) \\ &+ (1 - X_2) \cdot I(2 \cdot 0 + X_3 + X_4 + X_5 \geq 3) \\ &= X_2 \cdot I(X_3 + X_4 + X_5 \geq 1) + (1 - X_2) \cdot I(X_3 + X_4 + X_5 \geq 3) \\ &= X_2 \cdot (X_3 \amalg X_4 \amalg X_5) + (1 - X_2) \cdot (X_3 X_4 X_5),\end{aligned}$$

where the last equality follows from the result in (b) (or alternatively by (a))

STK2400 - 2006, problem 1 (f)

Assume that $X_1, \ldots X_5$ are stochastically independent and that $P(X_i = 1) = p_i$, $i = 1, \ldots, 5$. Moreover, let $\boldsymbol{p} = (p_1, \ldots, p_n)$. Find the reliability $h_3 = h_3(\boldsymbol{p})$ of the system (C, ϕ_3).

SOLUTION: From (e) we get that:

$$\begin{aligned}\phi_3(\boldsymbol{X}) &= X_1\phi_3(1_1, \boldsymbol{X}) + (1 - X_1)\phi_3(0_1, \boldsymbol{X}) \\ &= X_1 + (1 - X_1)[X_2(X_3 \amalg X_4 \amalg X_5) + (1 - X_2)(X_3X_4X_5)].\end{aligned}$$

The reliability $h_3 = E[\phi_3(\boldsymbol{X})]$ is then given by:

$$h_3(\boldsymbol{p}) = p_1 + (1 - p_1)[p_2(p_3 \amalg p_4 \amalg p_5) + (1 - p_2)(p_3p_4p_5)].$$

STK2400 - 2006, problem 1 (g)

The Birnbaum measure for the reliability importance of the ith component in the system (C, ϕ_b) is defined as:

$$I_B^{(i)}(\phi_b) = \frac{\partial h_b(\boldsymbol{p})}{\partial p_i}, \; i = 1, \ldots, 5.$$

Find $I_B^1(\phi_3)$.

SOLUTION: By pivoting with respect to component 1 we get:

$$\begin{aligned} I_B^{(i)}(\phi_3) &= \frac{\partial}{\partial p_1}[p_1 h_3(1_1, \boldsymbol{p}) + (1 - p_1) h_3(0_1, \boldsymbol{p})] \\ &= h_3(1_1, \boldsymbol{p}) - h_3(0_1, \boldsymbol{p}) \\ &= 1 - [p_2(p_3 \amalg p_4 \amalg p_5) + (1 - p_2)(p_3 p_4 p_5)]. \end{aligned}$$

STK2400 - 2006, problem 1 (h)

Assume that $p_1 < \cdots < p_5$. Which component has the highest reliability importance in (C, ϕ_1)? What would the conclusion be if we consider (C, ϕ_8) instead? Give a comment to this.

SOLUTION: From (b) we know that (C, ϕ_1) is a *parallel system*. In a parallel system the component with the highest reliability is the most important one. Thus, component 5 is the most important component in this system.

From (b) we also know that (C, ϕ_8) is a *series system*. In a series system the component with the lowest reliability is the most important one. Thus, component 1 is the most important component in this system.

Comment: We observe that the ranking of the components with respect to their reliability importance is very sensitive to the threshold value of the system.

STK2400 - 2006, problem 1 (i)

In the remaining part of this problem we assume that $p_1 = \cdots = p_5 = p$, and introduce:

$$S = 3X_1 + 2X_2 + X_3 + X_4 + X_5.$$

Explain briefly why $S \in \{0, 1, \ldots, 8\}$. Find the probability distribution of S expressed in terms of p.

SOLUTION: Since $X_1, \ldots, X_5$ are binary and since all component weights are integers, we know that S must be an integer as well. The smallest value S can have is 0, (when $X_1 = \cdots = X_5 = 0$). The largest value S can have is 8, (when $X_1 = \cdots = X_5 = 1$).

By varying $\boldsymbol{X}$ over all 5-dimensional binary vectors, we find that the set of possible values for S is $\{0, 1, \ldots, 8\}$.

E.g., $S = 1$ if exactly one of the variables X_3, X_4, X_5 equals 1, while the other 4 variables are 0.

STK2400 - 2006, problem 1 (i)

The probability distribution of S can be found by e.g., using:

$$P(S = s)$$

$$= \sum_{\boldsymbol{x}} I(3x_1 + 2x_2 + x_3 + x_4 + x_5 = s) p^{\sum_{i=1}^5 x_i} (1 - p)^{5 - \sum_{i=1}^5 x_i},$$

where the sum is taken over all 5-dimensional binary vectors.

STK2400 - 2006, problem 1 (i)

After simplification of these expressions we get:

$$
\begin{aligned}
\Pr(S=0) &= (1-p)^5 \\
\Pr(S=1) &= 3p(1-p)^4 \\
\Pr(S=2) &= p(1-p)^4 + 3p^2(1-p)^3 \\
\Pr(S=3) &= p(1-p)^4 + 3p^2(1-p)^3 + p^3(1-p)^2 \\
\Pr(S=4) &= 3p^2(1-p)^3 + 3p^3(1-p)^2 \\
\Pr(S=5) &= p^2(1-p)^3 + 3p^3(1-p)^2 + p^4(1-p) \\
\Pr(S=6) &= 3p^3(1-p)^2 + p^4(1-p) \\
\Pr(S=7) &= 3p^4(1-p) \\
\Pr(S=8) &= p^5.
\end{aligned}
$$

STK2400 - 2006, problem 1 (j)

Explain why:

$$h_b = h_b(p) = P(S \geq b),\ b = 1, \ldots, 8,$$

and use this in order to find h_5.

SOLUTION: We have that

$$\begin{aligned}
h_b(p) &= \mathrm{E}[\phi_b(\boldsymbol{X})] \\
&= E[I(3X_1 + 2X_2 + X_3 + X_4 + X_5 \geq b)] \\
&= P(3X_1 + 2X_2 + X_3 + X_4 + X_5 \geq b) \\
&= P(S \geq b).
\end{aligned}$$

STK2400 - 2006, problem 1 (j)

Hence, we get:

$$
\begin{aligned}
h_5(p) &= P(S \geq 5) \\
&= P(S = 5) + P(S = 6) + P(S = 7) + P(S = 8) \\
&= p^2(1-p)^3 + 3p^3(1-p)^2 + p^4(1-p) \\
&+ 3p^3(1-p)^2 + p^4(1-p) \\
&+ 3p^4(1-p) \\
&+ p^5 \\
&= p^2(1-p)^3 + 6p^3(1-p)^2 + 5p^4(1-p) + p^5
\end{aligned}
$$

STK2400 - 2006, problem 1 (k)

For which values of b do we have $h_b(p) \geq p$ for all $p \in [0, 1]$?
For which values of b do we have $h_b(p) \leq p$ for all $p \in [0, 1]$?

SOLUTION: Let $h = h(p)$ be the reliability polynomial for a system where all components have reliability p. The *S-form* theorem says that if the system does not have any cut sets of size 1 or any path set of size 1, there exists a $p_0 \in (0, 1)$ such that:

$$\begin{aligned} h(p) &= p, \text{ for } p = p_0, \\ h(p) &< p, \text{ for } 0 < p < p_0, \\ h(p) &> p, \text{ for } p_0 < p < 1. \end{aligned}$$

Furthermore, if the system has at least one path set of size 1, we have:

$$h(p) \geq p, \text{ for } 0 \leq p \leq 1.$$

Moreover, if the system has at least one cut set of size 1, we have:

$$h(p) \leq p, \text{ for } 0 \leq p \leq 1.$$

STK2400 - 2006, problem 1 (k)

In our case component 1 is *in parallel* with the rest of the system if $a_1 \geq b$, i.e., if $b \leq 3$.

Moreover, component 1 is *in series* with the rest of the system if $\sum_{i=2}^{5} a_i < b$, i.e., if $b > a_2 + a_3 + a_4 + a_5 = 2 + 1 + 1 + 1 = 5$.

If $3 < b \leq 5$, no components will be in series or parallel with the rest of the system.

Hence, it follows that:

$$
\begin{aligned}
h_b(p) &\geq p, \text{ for } 0 \leq p \leq 1 \text{ if } b = 1, 2, 3, \\
h_b(p) &\leq p, \text{ for } 0 \leq p \leq 1 \text{ if } b = 6, 7, 8.
\end{aligned}
$$

STK2400 - 2006, problem 2

Define what it means that stochastic variable $T_1, \ldots T_n$ are *associated*. Explain how this can be used in reliability analysis.

SOLUTION: The stochastic variables $T_1, \ldots T_n$ are said to be *associated* if:

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all pairs of binary non-decreasing functions Γ and Δ, where $\boldsymbol{T} = (T_1, \ldots, T_n)$.

The concept of associated stochastic variables is a form of *positive* dependence. In cases where we cannot assume that the component state variables are independent, we may argue that these variables instead are associated. Using the theory of associated random variables we can derive upper and lower bounds on the system reliability.