MATH112 FORMULAS

Notation

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$ is the factorial.

$\mathbb{N}$ is the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

$\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Q}$ is the set of rational numbers, $\mathbb{Q} = \{\frac{a}{b},\ \ a, b \in \mathbb{Z},\ b \neq 0\}$.

$\mathbb{I}$ is the set of irrational numbers, which cannot be written as a fraction of two integers, for example $\sqrt{2}$, $\pi = 3,14...$, $e = 2,71...$

$\mathbb{R}$ is the set of real numbers, $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I}$. The real numbers may be thought of as all points on an infinitely long number line.

$A =$ area, $l =$arc length, $V =$volume

1 ELEMENTARY ALGEBRA

1.1 Absolute value

Definition:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

Properties

$$\begin{aligned}
&1)\ \ |-a| = |a| \\
&2)\ \ \sqrt{a^2} = |a| \\
&3)\ \ |a \cdot b| = |a| \cdot |b| \\
&4)\ \ ||a| - |b|| \leq |a+b| \leq |a| + |b| \\
&5)\ \ ||a| - |b|| \leq |a-b| \leq |a| + |b|
\end{aligned}$$

1.2 Degree property

For any $m, n \in \mathbb{Z}$ or for any $a, b > 0$ and $m, n \in \mathbb{Q}$:

$$\begin{aligned}
&1)\ \ a^0 = 1 \\
&2)\ \ a^{-m} = \frac{1}{a^m} \quad (a \neq 0) \\
&3)\ \ a^m a^n = a^{m+n} \\
&4)\ \ \frac{a^m}{a^n} = a^{m-n}
\end{aligned}$$

$$
\begin{array}{ll}
5) & (a^m)^n = a^{mn} \\
6) & (ab)^m = a^m b^m \\
7) & \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)
\end{array}
$$

For $a > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$:

$$
\begin{array}{ll}
1) & a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \\
2) & a^{\frac{1}{n}} = \sqrt[n]{a}
\end{array}
$$

1.3 Quadratic propositions

$$
\begin{array}{ll}
1) & (a+b)^2 = a^2 + 2\,ab + b^2 \\
2) & (a-b)^2 = a^2 - 2\,ab + b^2 \\
3) & (a+b)\,(a-b) = a^2 - b^2
\end{array}
$$

1.4 Quadratic equation

The quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, has:

- two different roots

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{if the } \textit{discriminant } D = b^2 - 4ac > 0,$$

- one root (with multiplicity 2)

$$x_1 = x_2 = -\frac{b}{2a} \quad \text{if } D = 0,$$

- no real roots if $D < 0$.

The roots x_1, x_2 of the quadratic equation $ax^2 + bx + c = 0$ satisfy

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1\, x_2 = \frac{c}{a}.$$

Quadratic factorization
If the equation $ax^2 + bx + c = 0$ has two roots x_1, x_2 then

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Perfect square

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}.$$

1.5 Polynomial division

Polynomial division is an algorithm for dividing a polynomial by another polynomial of the same or lower degree.

For any polynomials $f(x)$ and $g(x)$, with $g(x)$ not identical to zero, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

with $r(x)$ having lower degree than $g(x)$.
Example

$$f(x) = x^4 + 5x^3 + 6x^2 - 4x + 11, \quad g(x) = x^2 - 3x + 1.$$

Applying the algorithm

$$\begin{array}{ccc|ccccc}
 & & & x^2 & +8x & +29 & & \\
\hline
x^2 & -3x & +1 & x^4 & +5x^3 & +6x^2 & -4x & +11 \\
 & & & -x^4 & +3x^3 & -x^2 & & \\
\cline{4-7}
 & & & & 8x^3 & +5x^2 & -4x & \\
 & & & & -8x^3 & +24x^2 & -8x & \\
\cline{5-8}
 & & & & & 29x^2 & -12x & +11 \\
 & & & & & -29x^2 & +87x & -29 \\
\cline{6-8}
 & & & & & & 75x & -18
\end{array}$$

we get $q(x) = x^2 + 8x + 29, \quad r(x) = 75x - 18.$

$$\frac{x^4 + 5x^3 + 6x^2 - 4x + 11}{x^2 - 3x + 1} = x^2 + 8x + 29 + \frac{75x - 18}{x^2 - 3x + 1}$$

1.6 Simple fractions

The rational expression $\dfrac{P_n(x)}{Q_m(x)}$, $n < m$, where $P_n(x)$, $Q_m(x)$ are polynomials of degrees n and m, respectively, can be represented as the finite sum of the simple fractions of following types:

$$\frac{A}{(x-a)^s}, \qquad s \in \mathbb{N}, \quad A, a \in \mathbb{R}$$

$$\frac{Bx + C}{(x^2 + px + q)^r}, \quad r \in \mathbb{N}, \quad B, C, p, q \in \mathbb{R},$$

where $x^2 + px + q$ is an irreducible quadratic polynomial, i.e. its discriminant $D < 0$.

Examples

$$1)\quad \frac{P_k(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, \quad k \in \{0,1\}$$

$$2)\quad \frac{P_l}{(x-a)(x^2+px+q)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+px+q}, \quad l \in \{0,1,2\}$$

$$3)\quad \frac{P_m}{(x-a)^2(x^2+px+q)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{Bx+C}{x^2+px+q}, \quad m \in \{0,1,2,3\}$$

$$4)\quad \frac{P_n}{(x-a)(x^2+px+q)^2} = \frac{A}{x-a} + \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2}, \quad n \in \{0,1,2,3,4\}$$

1.7 The logarithm

Definition: The *logarithm* of x to the *base* a is the power to which one needs to raise a in order to get x. Symbolically, if $x = a^b$ then $\log_a(x) = b, \quad x > 0,\ a > 0,\ a \neq 1$.

The logarithm to the base e is called the *natural logarithm* and denoted by $\ln(x)$. The logarithm to the base 10 is called the *common logarithm* and denoted by $\log(x)$.

Properties of logarithm

$$1)\quad a^{\log_a(x)} = x$$

$$2)\quad \log_a(a) = 1$$

$$3)\quad \log_a(1) = 0$$

$$4)\quad \log_a(x\,y) = \log_a(x) + \log_a(y)$$

$$5)\quad \log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$$

$$6)\quad \log_a(x^k) = k\,\log_a(x)$$

$$7)\quad \log_a(x) < \log_a(y) \Leftrightarrow \begin{cases} x < y & \text{if} \quad a > 1, \\ x > y & \text{if} \quad 0 < a < 1 \end{cases}$$

1.8 Arithmetical and Geometrical series

Arithmetical series

$$1)\quad a_n = a_{n-1} + d = a_1 + (n-1)\,d, \quad d = \text{difference}$$

$$2)\quad \sum_{k=1}^{n} a_k = \frac{a_1 + a_n}{2}\,n = \frac{2a_1 + d(n-1)}{2}\,n$$

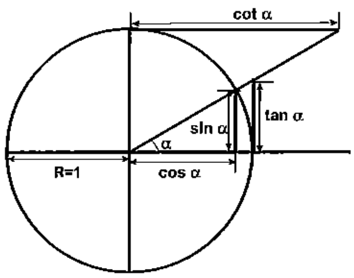
Geometrical series

$$1)\quad b_n = b_{n-1}\cdot q = b_0\, q^n, \quad q = \text{quotient}$$

$$2)\quad \sum_{k=0}^{n-1} b_k = \frac{b_n\, q - b_0}{q-1} = b_0\,\frac{1-q^n}{1-q}, \quad q \neq 1$$

$$3)\quad \sum_{k=0}^{\infty} b_k = \frac{b_0}{1-q}, \quad -1 < q < 1$$

1.9 Trigonometry



Sign of trigonometric functions in each quadrant

	sin	cos	tan	cot
I	+	+	+	+
II	+	-	-	-
III	-	-	+	+
IV	-	+	-	-

1) $y = \cos(x)$ is the even 2π-periodic function, i.e.

$$\cos(-x) = \cos(x)$$
$$\cos(x + 2\pi n) = \cos(x),\ n \in Z$$

2) $y = \sin(x)$ is the odd 2π-periodic function, i.e.

$$\begin{aligned} &\sin(-x) = -\sin(x) \\ &\sin(x + 2\pi n) = \sin(x),\ n \in Z \end{aligned}$$

3) $y = \tan(x)$ is the odd π-periodic function, i.e.

$$\begin{aligned} &\tan(-x) = -\tan(x) \\ &\tan(x + \pi n) = \tan(x),\ x \neq \pi k,\ n, k \in Z \end{aligned}$$

4) $y = \cot(x)$ is the odd π-periodic function, i.e.

$$\begin{aligned} &\cot(-x) = -\cot(x) \\ &\cot(x + \pi n) = \cot(x),\ x \neq \frac{(2k+1)\pi}{2},\ n, k \in Z \end{aligned}$$

Trigonometric identities

$$\begin{aligned} &1) \quad \cos^2\alpha + \sin^2\alpha = 1 \\ &2) \quad \tan\alpha\, \cot\alpha = 1 \quad (\alpha \neq \frac{\pi n}{2},\ n \in Z) \\ &3) \quad \tan^2\alpha = \frac{1}{\cos^2\alpha} - 1 \quad (\alpha \neq \frac{\pi(2n+1)}{2},\ n \in Z) \\ &4) \quad \cot^2\alpha = \frac{1}{\sin^2\alpha} - 1 \quad (\alpha \neq \frac{\pi n}{2},\ n \in Z) \end{aligned}$$

Angle-addition and substraction formulas

$$\begin{aligned} &1) \quad \sin(\alpha \pm \beta) = \sin\alpha\, \cos\beta \pm \cos\alpha\, \sin\beta \\ &2) \quad \cos(\alpha \pm \beta) = \cos\alpha\, \cos\beta \mp \sin\alpha\, \sin\beta \\ &3) \quad \tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\ \tan\beta} \\ &4) \quad \cot(\alpha \pm \beta) = \frac{\cot\alpha\ \cot\beta \mp 1}{\cot\beta \pm \cot\alpha} \end{aligned}$$

Some values of trigonometric functions

α°	$arc(\alpha)$	sin	cos	tan	cot
0°	0	0	1	0	$\times$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	$\times$	0
180°	π	0	-1	0	$\times$
270°	$\frac{3\pi}{2}$	-1	0	$\times$	0
360°	2π	0	1	0	$\times$

Shift and periodicity formulas

	sin	cos	tan	cot
$-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
$\frac{\pi}{2}\pm\alpha$	$\cos\alpha$	$\mp\sin\alpha$	$\mp\tan\alpha$	$\mp\cot\alpha$
$\pi\pm\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$
$\frac{3\pi}{2}\pm\alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$\mp\cot\alpha$	$\mp\tan\alpha$
$2\pi\pm\alpha$	$\pm\sin\alpha$	$\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$

The trigonometric functions in terms of the other functions

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$		$\pm\sqrt{1-\cos^2\alpha}$	$\frac{\tan\alpha}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\pm\sqrt{1+\cot^2\alpha}}$
$\cos\alpha$	$\pm\sqrt{1-\sin^2\alpha}$		$\frac{1}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}$
$\tan\alpha$	$\frac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}}$	$\frac{\pm\sqrt{1-\cos^2\alpha}}{\cos\alpha}$		$\frac{1}{\cot\alpha}$
$\cot\alpha$	$\frac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha}$	$\frac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}}$	$\frac{1}{\tan\alpha}$	

Sum-to-product identities

$$1)\quad \sin\alpha + \sin\beta = 2\ \sin\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$$
$$2)\quad \sin\alpha - \sin\beta = 2\ \cos\frac{\alpha+\beta}{2}\ \sin\frac{\alpha-\beta}{2}$$
$$3)\quad \cos\alpha + \cos\beta = 2\ \cos\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$$
$$4)\quad \cos\alpha - \cos\beta = -2\ \sin\frac{\alpha+\beta}{2}\ \sin\frac{\alpha-\beta}{2}$$
$$5)\quad \cos\alpha \pm \sin\beta = 2\ \sin(\frac{\pi}{4} - \frac{\alpha\mp\beta}{2})\ \sin(\frac{\pi}{4} + \frac{\alpha\pm\beta}{2})$$
$$6)\quad \cot\alpha \pm \tan\beta = \frac{\cos(\alpha\mp\beta)}{\sin\alpha\ \cos\beta}$$

Linear combination

1) $a\ \cos\alpha + b\sin\alpha = \sqrt{a^2+b^2}\ \cos(\alpha-\varphi)$, $\quad a^2+b^2 \neq 0$, where $\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$

2) $a\ \cos\alpha + b\sin\alpha = \sqrt{a^2+b^2}\ \sin(\alpha+\psi)$, $\quad a^2+b^2 \neq 0$, where $\cos\psi = \frac{b}{\sqrt{a^2+b^2}}$, $\sin\psi = \frac{a}{\sqrt{a^2+b^2}}$

Double and half-angle formulas

$$1)\quad \sin(2\alpha) = 2\ \sin\alpha\ \cos\alpha = \frac{2\ \tan\alpha}{1+\tan^2\alpha}$$
$$2)\quad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$
$$3)\quad \tan(2\alpha) = \frac{2\ \tan\alpha}{1-\tan^2\alpha}$$
$$4)\quad \cot(2\alpha) = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$
$$5)\quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$
$$6)\quad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$
$$7)\quad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$
$$8)\quad \cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$$

Product-to-sum identities

$$1)\quad \sin\alpha\ \sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$2)\quad \cos\alpha\ \cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$

$$3)\quad \sin\alpha\ \cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))$$

1.10 Inverse trigonometric functions

The inverse trigonometric functions $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arccot} x$ are defined in the following way:

$\arcsin x = y$ such that $\sin y = x$ for $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,
$\arccos x = y$ such that $\cos y = x$ for $-1 \le x \le 1$, $0 \le y \le \pi$,
$\arctan x = y$ such that $\tan y = x$ for $-\infty < x < +\infty$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$,
$\operatorname{arccot} x = y$ such that $\cot y = x$ for $-\infty \le x \le \infty$, $0 \le y \le \pi$.

Alternative notation

$\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$,
$\arctan x = \tan^{-1} x$, $\operatorname{arccot} x = \cot^{-1} x$.

Correspondences between the inverse trigonometric functions

	$\arcsin x$	$\arccos x$	$\arctan x$	$\operatorname{arccot} x$
$\arcsin x$	$-\arcsin(-x)$	$\frac{\pi}{2}-\arccos x$	$\arctan\frac{x}{\sqrt{1-x^2}}$	$\frac{\pi}{2}-\operatorname{arccot}\frac{x}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{\pi}{2}-\arcsin x$	$\pi-\arccos(-x)$	$\frac{\pi}{2}-\arctan\frac{x}{\sqrt{1-x^2}}$	$\operatorname{arccot}\frac{x}{\sqrt{1-x^2}}$
$\arctan x$	$\arcsin(\frac{x}{\sqrt{1+x^2}})$	$\frac{\pi}{2}-\arccos(\frac{x}{\sqrt{1+x^2}})$	$-\arctan(-x)$	$\frac{\pi}{2}-\operatorname{arccot} x$
$\operatorname{arccot} x$	$\frac{\pi}{2}-\arcsin(\frac{x}{\sqrt{1+x^2}})$	$\arccos(\frac{x}{\sqrt{1+x^2}})$	$\frac{\pi}{2}-\arctan x$	$\pi-\operatorname{arccot}(-x)$

Trigonometric equations

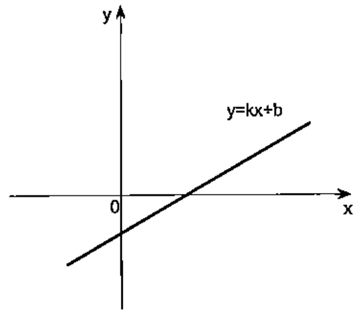
$$1)\ \sin x = c,\ c \in [-1,1] \quad \Leftrightarrow \quad x = \begin{cases} \arcsin c + 2\pi k, \\ (\pi - \arcsin c) + 2\pi k, \end{cases} \quad k \in \mathbb{Z}$$

$$2)\ \cos x = c,\ c \in [-1,1] \quad \Leftrightarrow \quad x = \begin{cases} \arccos c + 2\pi k, \\ -\arccos c + 2\pi k, \end{cases} \quad k \in \mathbb{Z}$$

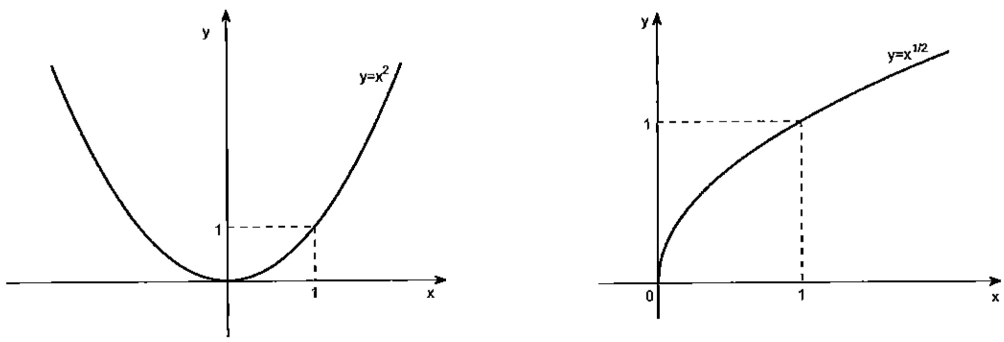
$$3)\ \tan x = c \quad \Leftrightarrow \quad x = \arctan c + \pi k, \quad k \in \mathbb{Z}$$

1.11 Graphs of some elementary functions

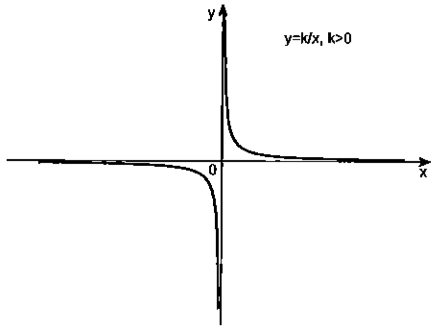
Linear function $y = kx + b$

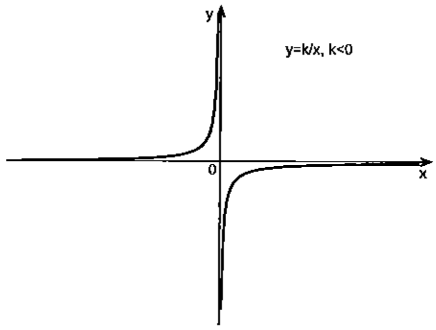


Power function $y = x^n$

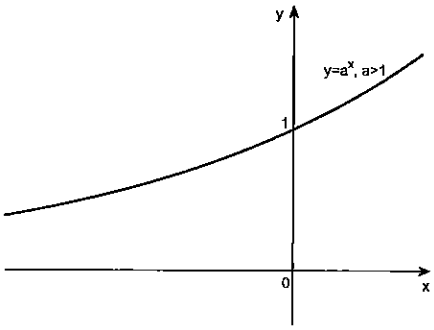


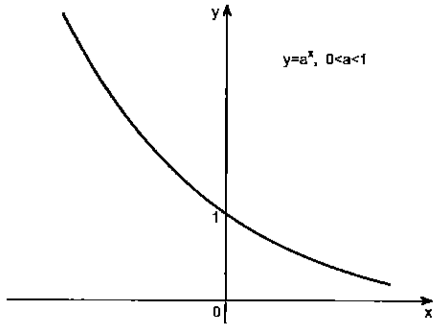
Linear-fractional function $y = \dfrac{k}{x}$ $(k \neq 0,\ x \neq 0$



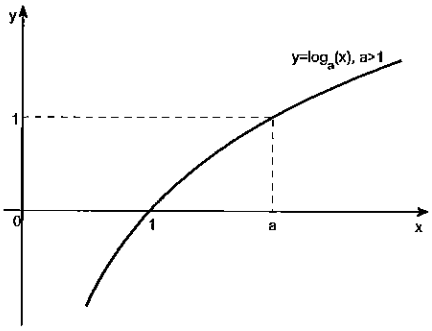


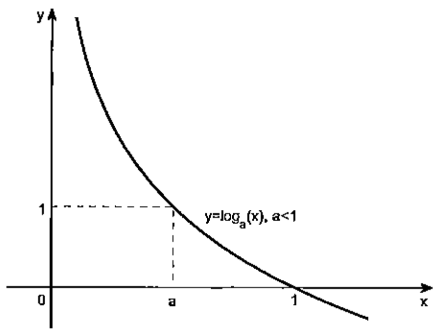
Exponential function $y = a^x$ $(a > 0,\ a \neq 1)$



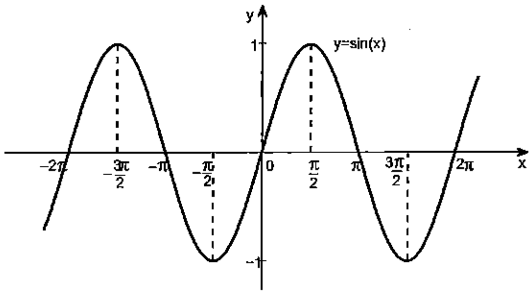


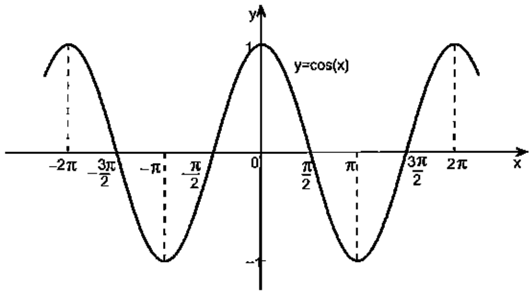
Logarithmic function $y = \log_a(x)$ $(a > 0,\ a \neq 1,\ x > 0)$

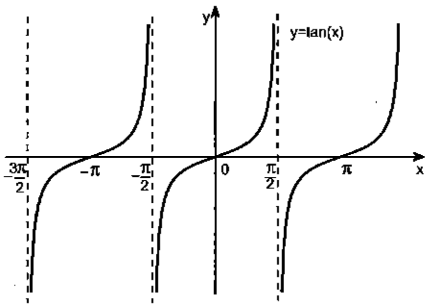


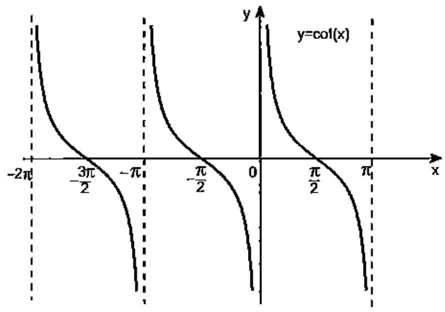


Trigonometric functions

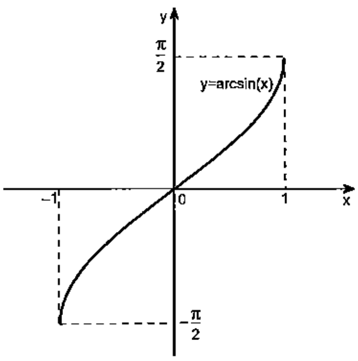


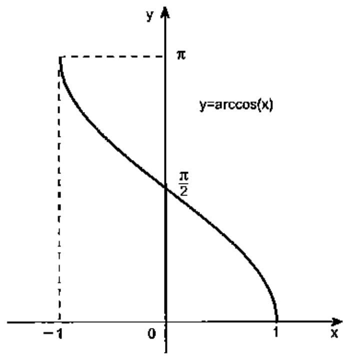


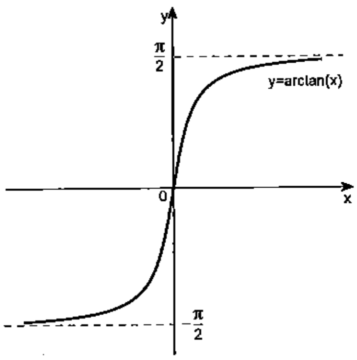


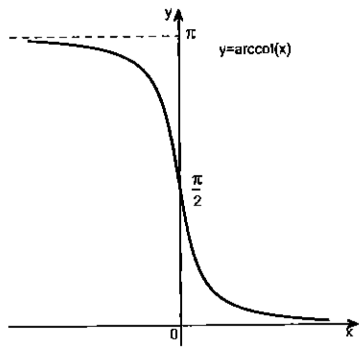


Arc trigonometric functions






y
$\frac{\pi}{2}$
y=arctan(x)
0
x
$\frac{\pi}{2}$


y
π
y=arccot(x)
$\frac{\pi}{2}$
0
x

2 CALCULUS I

2.1 Limits

Limit laws

Let L, M, c and k be real numbers and $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$.

Then

1) $\lim\limits_{x\to c}(f(x) \pm g(x)) = L \pm M$

2) $\lim\limits_{x\to c} kf(x) = kL$

3) $\lim\limits_{x\to c} f(x)g(x) = LM$

4) $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$

5) $\lim\limits_{x\to c} (f(x))^q = L^q$, $q \in \mathbb{Q}$

6) *L'Hospital's rule.* Assume that

   (i) $g(x) \neq 0$ and $f(x)$, $g(x)$ have continuous derivatives in a deleted neighborhood about c (or ∞)

   (ii) $f(x), g(x) \to 0$ (or ∞) as $x \to c$ (or $x \to \infty$)

   then

$$\lim_{\substack{x \to c \\ (x\to\infty)}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to c \\ (x\to\infty)}} \frac{f'(x)}{g'(x)}$$

   if the limit on the right hand side exists.

Useful limits

1) $\lim\limits_{\substack{x \to c \\ (x\to\infty)}} k = k$

2) $\lim\limits_{x\to\infty} x^n = \infty$

3) $\lim_{x \to -\infty} x^n = \begin{cases} \infty \text{ if } n \text{ is even,} \\ -\infty \text{ if } n \text{ is odd} \end{cases}$

4) $\lim_{x \to \infty} \frac{1}{x^n} = 0$

5) $\lim_{x \to \infty} \frac{a_m x^m + ... a_0}{b_n x^n + ... + b_0} = \begin{cases} 0 \text{ if } m < n, \\ \frac{a_m}{b_n} \text{ if } m = n \\ \infty \text{ if } m > n \end{cases}$

6) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

7) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

8) $\lim_{x \to \infty} x^{1/x} = 1,$

where c, k, n, m are constants, $n, m > 0$ and $a_m, b_n \neq 0$.

2.2 Continuity

Definition: A function $y = f(x)$ is *continuous at* x_0 if x_0 belongs to the domain of $f(x)$ and $\lim_{x \to x_0} f(x) = f(x_0)$.

A function f is continuous in an interval I if f is continuous at every point of I. We call f continuous if f is continuous in its domain.

Properties

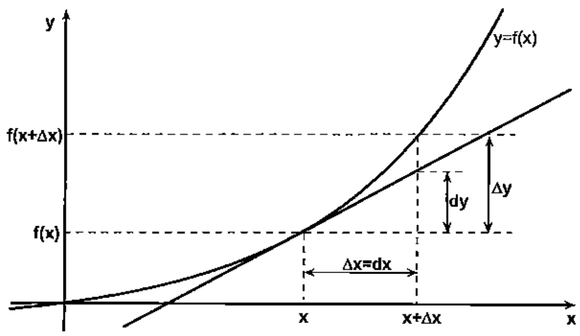
Let f and g be continuous then the following functions are continues (where they are defined)

$$f(x) \pm g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)} \ (g(x) \neq 0), \quad f(g(x)).$$

2.3 Differentiation

Definition: The derivative $f'(x)$ of a function $y = f(x)$ is defined by

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}.$$



Alternative notation: $y' = f'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}f(x)$
Second derivative: $f''(x) = [f'(x)]'$

Differentiation rules

Assume u and v are differentiable functions of x.

1) $\dfrac{d}{dx}(c) = 0$, $c = const$

2) $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

3) $\dfrac{d}{dx}(cu) = c\,\dfrac{du}{dx}$, $c = const$

4) $\dfrac{d}{dx}(uv) = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$

5) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$

6) Chain rule: $\dfrac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Useful derivatives

1) $\frac{d}{dx}(c) = 0$, c =const

2) $\frac{d}{dx}(x^r) = rx^{r-1}$

3) $\frac{d}{dx}(e^x) = e^x$

4) $\frac{d}{dx}(a^x) = a^x \ln a$

5) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

6) $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

7) $\frac{d}{dx}(\sin x) = \cos x$

8) $\frac{d}{dx}(\cos x) = -\sin x$

9) $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$

10) $\frac{d}{dx}(\cot x) = \frac{-1}{\sin^2 x}$

11) $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

12) $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$

13) $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

14) $\frac{d}{dx}(\text{arccot}x) = \frac{-1}{1+x^2}$

2.4 Indefinite integral

Definition: $F(x) = \int f(x)dx$ means that $F'(x) = f(x)$.
Any *primitive function* of $f(x)$ can be written $F(x) + C$, where C is an arbitrary constant.

General properties

1) $\int k \cdot u(x)dx = k \int u(x)dx$, k=const

2) $\int (u(x) \pm v(x))dx = \int u(x)dx \pm \int v(x)dx$

3) Integration by parts: $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

4) Substitution: $\int f(g(x)) \cdot g'(x)dx = \int f(t)dt$, where $t = g(x)$, $dt = g'(x)dx$

Useful integration formulas

1) $\int kdu = ku + C$, $k = \text{const}$

2) $\int u^n du = \dfrac{u^{n+1}}{n+1} + C$, $n \neq -1$

3) $\int \dfrac{du}{u} = \ln|u| + C$

4) $\int \sin u du = -\cos u + C$

5) $\int \cos u du = \sin u + C$

6) $\int \dfrac{du}{\cos^2 u} = \tan u + C$

7) $\int \dfrac{du}{\sin^2 u} = -\cot u + C$

8) $\int e^u du = e^u + C$

9) $\int a^u du = \dfrac{a^u}{\ln a} + C$ $(a \neq 1,\ a > 0)$

10) $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

11) $\int \dfrac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

12) $\int \dfrac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + C$

13) $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \ln|u + \sqrt{a^2 + u^2}| + C$

2.5 Definite integral

The fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a), \text{ where } F'(x) = f(x).$$

Some properties

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a \leq c \leq b$

4) $f(x) \leq g(x) \quad \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

5) Integration by parts: $\int_a^b u'(x)v(x)dx = u(x)v(x) \mid_a^b - \int_a^b u(x)v'(x)dx$

6) Substitution: $\int_a^b f(g(x)) \cdot g'(x)dx = \int_\alpha^\beta f(t)dt$, where $t = g(x)$, $dt = g'(x)dx$, $\alpha = g(a)$, $\beta = g(b)$

2.6 Applications of integrals

Curves in a function form $\mathbf{y = f(x)}$

$$A = \int_a^b f(x)dx \ \text{ if } \ f(x) \geq 0 \ \text{ on } \ [a,b] \ \text{(Fig.1)}$$

$$A = \int_a^b (f(x) - g(x))dx \ \text{ if } \ f(x) \geq g(x) \ \text{ on } \ [a,b] \ \text{(Fig.2)}$$

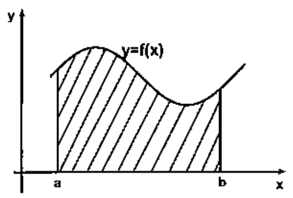


Fig. 1

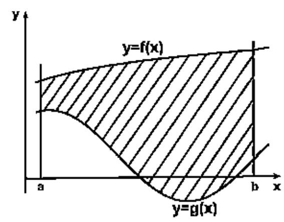


Fig. 2

$$l = \int_a^b \sqrt{1 + (f'(x))^2}dx$$
$$V_x = \pi \int_a^b (f(x))^2 dx,$$

when one rotates the area bounded by the curve $y = f(x)$, x-axis and the lines $x = a$, $x = b$ about x-axis.

Curves in a parametric form $\left\{ \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right.$ $\mathbf{a \leq t \leq b}$

$$A = \int_a^b y(t)\dot{x}(t)dt \ y \geq 0$$
$$l = \int_a^b \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}dt$$

Curves in polar coordinates $\mathbf{x = r\cos\theta, \ y = r\sin\theta}$

Curve $C: \ r = r(\theta), \ \alpha \leq \theta \leq \beta$

$$A = \frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta \quad \text{(see Fig.3)}$$
$$l = \int_\alpha^\beta \sqrt{(r(\theta))^2 + (r'(\theta))^2}d\theta$$

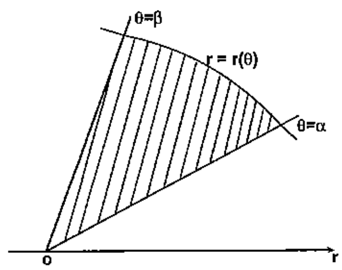


Fig. 3

2.7 Improper integrals

Integrals with infinite limits of integration

1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,$$

where c is any real number.

Integrals of functions that become infinite at a point within the interval of integration

1) If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$$

2) If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int\limits_a^b f(x)dx = \lim_{c \to b^-} \int\limits_a^c f(x)dx.$$

3) If $f(x)$ is discontinuous at c, where $a < c < b$ and continuous on $[a, c) \bigcup (c, b]$, then

$$\int\limits_b^a f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx.$$

2.8 Infinite sequences of real numbers

Sequence convergence laws

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that

$$\lim_{n\to\infty} a_n = A,\ \lim_{n\to\infty} b_n = B, \quad A, B \in \mathbb{R}.$$

1) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$

2) $\lim\limits_{n\to\infty} ka_n = kA$, $k = \text{const}$

3) $\lim\limits_{n\to\infty} a_n b_n = AB$

4) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ $(B \neq 0)$

5) *The sandwich theorem.* Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$, for all $n \geq N$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ also.

Useful convergent sequences

1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3) $\lim_{n\to\infty} a^{1/n} = 1 \ (a > 0)$

4) $\lim_{n\to\infty} a^n = 0 \ (|a| < 1)$

5) $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$

6) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

2.9 Infinite series

Series convergence laws

Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then

1) $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$

2) $\sum_{n=1}^{\infty} k \cdot a_n = k \cdot A$, $k = \text{const.}$

Useful convergent series

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if} \ \ p > 1, \\ \text{diverges} & \text{if} \ \ p \le 1 \end{cases}$

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if} \ \ |r| < 1, \\ \text{diverges} & \text{if} \ \ |r| \ge 1 \end{cases}$

Divergence test

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \ne 0$ or fails to exist.

Convergence tests

Integral test. Let $\{a_n\}$ be a nonnegative sequence. Suppose that $a_n = f(n)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq N$, $N \in \mathbb{N}$. Then $\sum\limits_{n=N}^{\infty} a_n$ and $\int\limits_{N}^{\infty} f(n)dx$ either both converge or both diverge.

Comparison test. Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} c_n$ and $\sum\limits_{n=1}^{\infty} d_n$ be series with nonnegative terms.

a) If $a_n \leq c_n$ for all $x \geq N$, $N \in \mathbb{N}$ and $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

b) If $a_n \geq d_n$ for all $x \geq N$, $N \in \mathbb{N}$ and $\sum\limits_{n=1}^{\infty} d_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Limit comparison. Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$, $N \in \mathbb{N}$.

a) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ either both converge or both diverge.

b) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

c) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Ratio test. Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p.$$

a) If $p < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

b) If $p > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

c) If $p = 1$, then the test is inconclusive.

Root test. Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = p.$$

a) If $p < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

b) If $p > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

c) If $p = 1$, then the test is inconclusive.

Alternating series test (Leibniz's theorem). The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} u_n$ converges if the following three conditions are satisfied:

i) $u_n \geq 0$, for all $n \in \mathbb{N}$,

ii) $u_n \geq u_{n+1}$ for all $n \geq N$, $N \in \mathbb{N}$,

iii) $\lim\limits_{n \to \infty} u_n = 0$.

Absolute convergence test. If $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Taylor series

Let $y = f(x)$ be an infinitely differentiable function on some interval containing a as an interior point. Then the Taylor series of $f(x)$ at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

If $a = 0$, then the series is called a Maclaurin series for $f(x)$.

Useful series

1) $\dfrac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k, \quad |x| < 1$

2) $\dfrac{1}{1+x} = \sum\limits_{k=0}^{\infty} (-1)^k x^k, \quad |x| < 1$

3) $e^x = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!}, \quad x \in R$

4) $\sin(x) = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad x \in R$

5) $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad x \in R$

6) $\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}, \quad -1 < x \leq 1$

2.10 Vectors

The vector $\mathbf{u}$ from $A(a_1, a_2, a_3)$ to $B(b_1, b_2, b_3)$ is

$$\mathbf{u} = \overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Magnitude:	$\lvert\mathbf{u}\rvert = \sqrt{u_1^2 + u_2^2 + u_3^2}$
Addition:	$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
Scalar multiplication:	$k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

The dot product

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

or

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The cross product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n},$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{n}$ is a unit vector perpendicular to the plane of $\mathbf{uv}$ vectors by the right-hand rule.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

2.11 Lines and planes in space

Parametric equations for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$L : \begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \,, \\ z = z_0 + tv_3 \end{cases} \quad -\infty < t < \infty.$$

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

3 CALCULUS II

3.1 Partial Derivation

Chain Rule
Let $w = f(x, y, z)$ then

1) $x = x(t),\ y = y(t),\ z = z(t):\quad \dfrac{dw}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}.$

2) $x = x(u, v),\ y = y(u, v),\ z = z(u, v):$

$$\frac{dw}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} + \frac{\partial f}{\partial z}\frac{dz}{du}, \quad \frac{dw}{dv} = \frac{\partial f}{\partial x}\frac{dx}{dv} + \frac{\partial f}{\partial y}\frac{dy}{dv} + \frac{\partial f}{\partial z}\frac{dz}{dv}.$$

The *total differential* of f is $df = f_x dx + f_y dy + f_z dz$.

The *directional derivative* of f at $P = (x, y, z)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$, is the number

$$(D_{\mathbf{u}} f)|_P = \lim_{s \to 0} \frac{f(x + su_1, y + su_2, z + su_3) - f(x, y, z)}{s}$$

The *gradient* of f is $grad(f) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$. The alternative notation $grad(f) = \nabla f$.

Properties
1) ∇f is normal to the level curve $f(x, y, z) = C$.

2) $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

3) $D_{\mathbf{u}} f$ has its maximum value in the direction $\mathbf{u} = \dfrac{\nabla f}{|\nabla f|}$, and it is equal $|\nabla f|$.

4) Algebraic properties:

$$\nabla(kf) = k\nabla f \text{ for any } k = const, \quad \nabla(f \pm g) = \nabla f \pm \nabla g,$$

$$\nabla(fg) = f\nabla g + g\nabla f, \qquad \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}.$$

3.2 Tangent plane and normal line

The *tangent plane* to $f(x, y, z) = c$ at $P = (x_0, y_0, z_0)$ is

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

If the surface is given by $z = f(x, y)$, the tangent plane at $P = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

The *normal line* to $f(x, y, z) = c$ at $P = (x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P)t, \ y = y_0 + f_y(P)t, \ z = z_0 + f_z(P)t.$$

The tangent plain to a parameterized surface $S : r(u, v) = x(u, v)\mathrm{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ at $P = (x_0, y_0, z_0)$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where the normal vector $\mathbf{n} = (A, B, C) = \mathbf{r}_u \times \mathbf{r}_v$,

$$\mathbf{r}_u = \frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}, \quad \mathbf{r}_v = \frac{d\mathbf{r}}{dv} = \frac{dx}{dv}\mathbf{i} + \frac{dy}{dv}\mathbf{j} + \frac{dz}{dv}\mathbf{k}.$$

3.3 Extreme Values and Saddle Points

Derivative test for local extreme values for functions of two variables
Let (a, b) be a critical point of the function $f(x, y)$ and $D(a, b)$ is given as

$$D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}.$$

1) f has a local maximum at (a, b) if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$,

2) f has a local minimum at (a, b) if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$,

3) f has a saddle point at (a, b) if $D(a, b) < 0$,

4) The test is inconclusive if $D(a, b) = 0$.

Global extreme points
Let $f(x, y)$ is defined on a closed domain R. The global extremums of $f(x, y)$ can occur only at boundary points of R or at critical points of the function in R.
Method of Lagrange Multipliers
The necessary condition for maximum or minimum of $f(x, y, z)$ with the constrain $g(x, y, z) = 0$ is

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0.$$

If there are two constrains $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, then the condition is

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0, \ g_2(x, y, z) = 0.$$

Global extreme points of $f(x, y, z)$ with constrains
Global maximum(minimum) value of $f(x, y)$ with constrains can be found as the largest(smallest) value that f takes in critical points defined by Lagrange method.

3.4 Coordinate conversion formulas

Cylindrical to rectangular:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Spherical to rectangular:

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

Spherical to cylindrical:

$$\begin{aligned} r &= \rho\sin\phi \\ z &= \rho\cos\phi \\ \theta &= \theta \end{aligned}$$

Corresponding formulas for dV in triple integrals:

$$dV = dxdydz = dzrdrd\theta = \rho^2 \sin\phi d\rho d\phi d\theta$$

3.5 Multiple integrals

Double integral $\iint\limits_R f(x,y)dA$ or $\iint\limits_R f(x,y)dxdy$

Fubini's formula Let $f(x,y)$ be continuous on a region R

1) If $R:\ a \leq x \leq b,\ \alpha(x) \leq y \leq \beta(x)$, then

$$\iint\limits_R f(x,y)dxdy = \int\limits_a^b \int\limits_{\alpha(x)}^{\beta(x)} f(x,y)dydx$$

2) If $R:\ \gamma(y) \leq x \leq \delta(y),\ c \leq y \leq d$, then

$$\iint\limits_R f(x,y)dxdy = \int\limits_c^d \int\limits_{\gamma(y)}^{\delta(y)} f(x,y)dxdy$$

Application of double integral

$A(R) = \iint\limits_R dxdy$

$V(D) = \iint\limits_R (f(x,y) - g(x,y))dxdy$, where $g(x,y) \leq z \leq f(x,y)$ (see Fig.4)

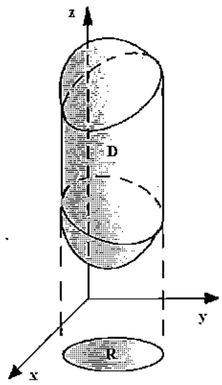


Fig. 4

Integration by substitution

$$\iint_{R_{xy}} f(x,y)dxdy = \iint_{R_{uv}} f(x(u,v),y(u,v))|J(u,v)|dudv$$

where

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

For example, $J = r$ for polar coordinates.

Triple integral $\iiint_D f(x,y,z)dV$ or $\iiint_D f(x,y,z)dxdydz$

Fubini's formula Let $f(x,y,z)$ be continuous on a region D given as

$$\gamma(y,z) \le x \le \delta(y,z),\ \alpha(z) \le y \le \beta(z),\ a \le z \le b.$$

$$\iiint_D f(x,y,z)dxdydz = \int_a^b \int_{\alpha(z)}^{\beta(z)} \int_{\gamma(y,z)}^{\delta(y,z)} f(x,y,z)dx\ dy\ dz.$$

Application of triple integral

$V(D) = \iiint_D dxdydz$ (see Fig. 4)

Integration by substitution

$$\iiint_{D_{xyz}} f(x,y,z)dxdydz = \iiint_{D_{uvw}} f(x(u,v,w),y(u,v,w),z(u,v,w))|J(u,v,w)|dudvdw$$

where

$$J(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

For example, $J = r$ for cylindrical coordinates and $J = \rho^2 \sin \Phi$ for spherical coordinates.

3.6 Integration in Vector Fields

Line integrals $\int\limits_C f(x,y,z)ds$

Curve $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \le t \le b$. Then

$$d\mathbf{r} = (dx, dy, dz), \quad ds = |d\mathbf{r}| = \sqrt{(x')^2 + (y')^2 + (z')^2}dt.$$

Line integral is $\int\limits_C f(x,y,z)ds = \int\limits_a^b f(x(t), y(t), z(t))\sqrt{(x')^2 + (y')^2 + (z')^2}dt.$

The length of the curve C is $l = \int\limits_C ds = \int\limits_a^b \sqrt{(x')^2 + (y')^2 + (z')^2}dt.$

The mass of the curve C with density $\sigma = \sigma(x,y,z)$ is $M = \int\limits_C \sigma(x,y,z)ds.$

Let $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be a closed curve in the plane, $\mathbf{n}$ be the outward-pointing unit normal vector to the curve C, and $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a vector field. Then the *flux across the curve* C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{n} ds = \int\limits_C (M\frac{dy}{ds} - N\frac{dx}{ds})ds = \oint\limits_C Mdy - Ndx.$$

Let $\mathbf{T}$ be a unit tangent vector, $\mathbf{n}$ be a outward-pointing unit normal vector to the curve C, $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ be a vector field. Then the *flow along* C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{T} ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \int\limits_a^b Mdx + Ndy + Pdz = \int\limits_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt.$$

The *circulation* of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around a closed positive oriented curve C is $\oint\limits_C \mathbf{F} \cdot \mathbf{T} ds.$

The *work* done by a force $\mathbf{F}$ over C is $W = \int\limits_C \mathbf{F} \cdot \mathbf{T} ds$.

Fundamental theorem of line integrals

If $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla f$, then $\int\limits_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A and B and

$$\int\limits_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Component test for conservative fields

1) $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

2) $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Green's theorem in the plane

1) The circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a closed positive oriented curve C is given as

$$\oint\limits_C (\mathbf{F} \cdot \mathbf{T}) ds = \oint\limits_C M dx + N dy = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

where the region R is enclosed by C (see Fig. 5).

2) Flux of the field $F = M\vec{i} + N\vec{j}$ across closed positive oriented curve C is given as

$$\oint\limits_C (\mathbf{F} \cdot \mathbf{n}) ds = \oint\limits_C M dy - N dx = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

where the region R is enclosed by C (see Fig. 5).

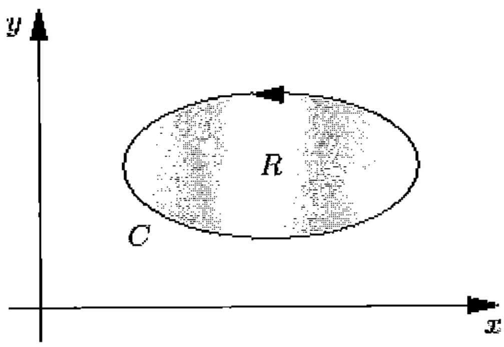


Fig. 5

Surface integrals $\iint\limits_S g(x,y,z)d\sigma$

1)**Let surface S be given by** $f(x,y,z)=c$, R be the "shadow" of S on a coordinate plane, $\mathbf{p}$ be a unit vector normal to R such that $\nabla f\cdot\mathbf{p}\neq 0$.
Then the surface area differential $d\sigma$ is

$$d\sigma=\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}dA$$

and the surface integral is

$$\iint\limits_S g(x,y,z)d\sigma=\iint\limits_R g(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}dA.$$

The flux of a three-dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is

$$Flux=\iint\limits_S \mathbf{F}\cdot\mathbf{n}d\sigma,$$

where $\mathbf{n}$ is one of the two fields $\mathbf{n}=\pm\frac{\nabla f}{|\nabla f|}$, depending on which one gives the preferred direction.
2) **Let S be a smooth surface given by**

$$r(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k},\quad a\le u\le b,\quad c\le v\le d.$$

Then the surface area differential $d\sigma$ is

$$d\sigma=|r_u\times r_v|dudv,$$

where

$$\mathbf{r}_u=\frac{d\mathbf{r}}{du}=\frac{dx}{du}\mathbf{i}+\frac{dy}{du}\mathbf{j}+\frac{dz}{du}\mathbf{k},\quad \mathbf{r}_v=\frac{d\mathbf{r}}{dv}=\frac{dx}{dv}\mathbf{i}+\frac{dy}{dv}\mathbf{j}+\frac{dz}{dv}\mathbf{k},$$

and the area of S is

$$A=\int\limits_c^d\int\limits_a^b|r_u\times r_v|dudv.$$

The parametric surface integral is given by

$$\iint\limits_S g(x,y,z)d\sigma=\int\limits_c^d\int\limits_a^b g(x(u,v),y(u,v),z(u,v))|r_u\times r_v|dudv.$$

The flux of a vector field $\mathbf{F}$ across an oriented smooth surface S in the direction of $\mathbf{n}$ is

$$Flux = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma,$$

where $\mathbf{n}$ is one of the two fields $\mathbf{n} = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$, depending on which one gives the preferred direction.

Denote by ∇ the operator given by $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$. For instance, when ∇ applied to a scalar function $f = f(x, y, z)$, it gives the gradient of f

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The divergence (flux density) of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The *curl vector* (circulation density) of a vector field $\mathbf{F}$ is

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

Stoke's theorem

The circulation of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma \quad \text{(see Fig. 6)}.$$

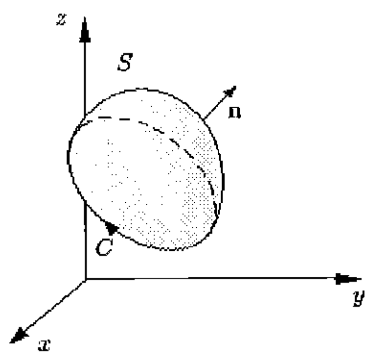


Fig. 6

Gauss' theorem (Divergence theorem)
The flux of a vector field $\mathbf{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal vector $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F} dV \quad \text{(see Fig. 7)}.$$

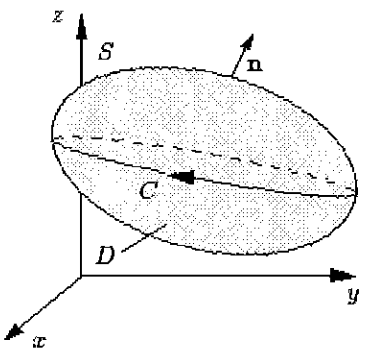
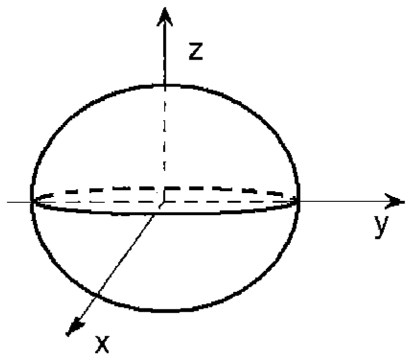


Fig. 7

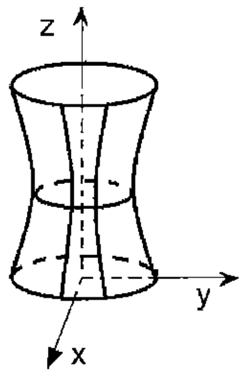
3.7 Second degree surfaces in standard form

Ellipsoid



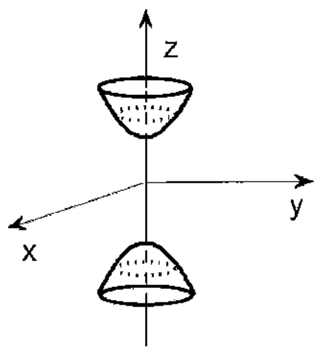
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Elliptic Hyperboloid of one sheet



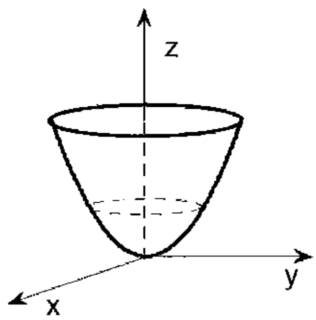
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Elliptic Hyperboloid of two sheets



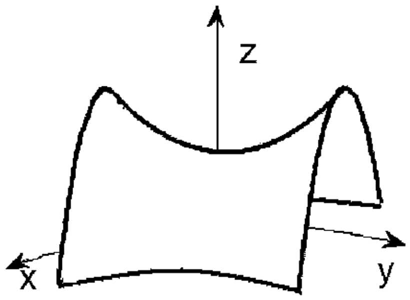
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

Elliptic Paraboloid



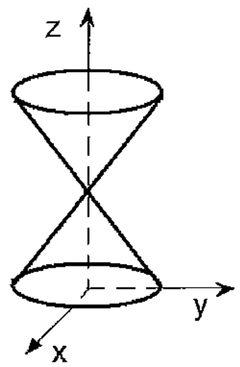
$$z=\frac{x^2}{2p}+\frac{y^2}{2q}$$

Hyperbolic Paraboloid



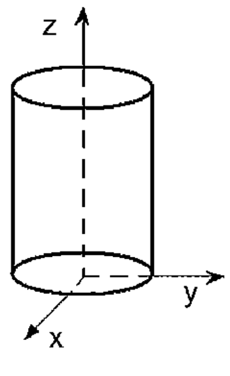
$$z = \frac{x^2}{2p} - \frac{y^2}{2q}$$

Elliptic Cone



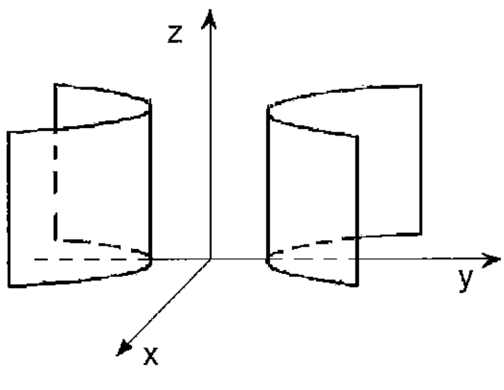
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Elliptic Cylinder



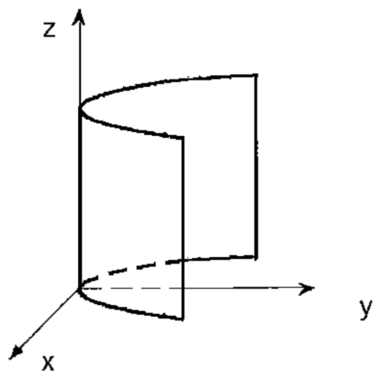
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hyperbolic Cylinder



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parabolic Cylinder



$$x^2 = 2py$$