# STK3405 – Week 38

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# Section 4.2.

**The multiplication method**

# The multiplication method

By expanding either the formula based on the minimal path sets, or the formula based on the minimal cut sets, and using that $X_i^r = X_i$, $i = 1, \ldots, n$, $r = 1, 2, \ldots$, we eventually get an expression of the form:

$$\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$$

where for all $A \subseteq C$, $\delta(A)$ denotes the coefficient of the term associated with $\prod_{i \in A} X_i$. The $\delta$-function is called the *signed domination function* of the structure.

By taking the expectation on both sides, and assuming that the component state variables are independent, we obtain:

$$h(\boldsymbol{p}) = E[\phi(\boldsymbol{X})] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} E[X_i] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i$$

# The multiplication method (cont.)

Introduce the following version of the structure function of a binary monotone system $(C, \phi)$:

$$\phi(B) = \phi(\mathbf{1}^B, \mathbf{0}^{\bar{B}}), \text{ for all } B \subseteq C.$$

It can then be shown that:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \text{ for all } A \subseteq C.$$

# Properties of the signed domination

**Theorem**

*Let $(C, \phi)$ be a binary monotone system, and let $A \subseteq C$.*

*If $\phi(A) = 0$, then $\delta(A) = 0$ as well.*

**Theorem**

*Let $(C, \phi)$ be a binary monotone system, let $A \subseteq C$, and let $i \in A$.*

*If $\phi(B \cup i) = \phi(B)$ for all $B \subseteq A \setminus i$, then $\delta(A) = 0$.*

**Theorem**

*Let $(C, \phi)$ be a binary monotone system, let $P \subseteq C$ be a minimal path set.*

*Then $\delta(P) = 1$.*

# The multiplication method (cont.)

Consider a binary monotone system $(C, \phi)$ where $C = \{1, 2, 3\}$, with minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. Using the multiplication method we obtain:

$$\begin{aligned}
\phi(\boldsymbol{X}) &= (X_1 X_2) \amalg (X_1 X_3) = 1 - (1 - X_1 X_2)(1 - X_1 X_3) \\
&= 1 - (1 - X_1 X_2 - X_1 X_3 + X_1^2 X_2 X_3) \\
&= X_1 X_2 + X_1 X_3 - X_1 X_2 X_3,
\end{aligned}$$

where we have used that $X_1^2 = X_1$.

Thus, $\delta(\{1,2\}) = \delta(\{1,3\}) = 1$, $\delta(\{1,2,3\}) = -1$, while $\delta(A) = 0$ for all other subsets of $C$.

# The multiplication method (cont.)

Note that these coefficients can be obtained using the signed domination formula and the theorems as well since:

$$\delta(\{1,2\}) = 1, \quad \text{since } \{1,2\} \text{ is a minimal path set}$$

$$\delta(\{1,3\}) = 1, \quad \text{since } \{1,3\} \text{ is a minimal path set}$$

$$\begin{aligned}\delta(\{1,2,3\}) &= (-1)^{|\{1,2,3\}|-|\{1,2\}|}\phi(\{1,2\}) + (-1)^{|\{1,2,3\}|-|\{1,3\}|}\phi(\{1,2\}) \\ &+ (-1)^{|\{1,2,3\}|-|\{1,2,3\}|}\phi(\{1,2,3\}) = -1 - 1 + 1 = -1.\end{aligned}$$

Having derived the formula for the structure function $\phi$, we immediately obtain the reliability function:

$$h(\boldsymbol{p}) = p_1p_2 + p_1p_3 - p_1p_2p_3.$$

# Section 4.3.

**The inclusion-exclusion method**

# The inclusion-exclusion method

Consider a binary monotone system $(C, \phi)$ with minimal path sets $P_1, \ldots, P_p$. We then introduce the events

$$E_j = \{\text{All of the components in } P_j \text{ are functioning}\}, \quad j = 1, \ldots, p.$$

Since the system is functioning if and only if at least one of the minimal path sets is functioning, we have:

$$\phi = 1 \quad \text{if and only if} \quad \bigcup_{j=1}^{p} E_j \text{ holds true.}$$

# The inclusion-exclusion method (cont.)

We then use the *inclusion-exclusion* formula and get:

$$
\begin{aligned}
h = P(\bigcup_{j=1}^{p} E_j) \\
&= P(E_1) + P(E_2) + \cdots + P(E_p) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots - P(E_{p-1} \cap E_p) \\
&+ P(E_1 \cap E_2 \cap E_3) + \cdots + P(E_{p-2} \cap E_{p-1} \cap E_p) \\
&\cdots \\
&+ (-1)^{p-1} P(E_1 \cap \cdots \cap E_p).
\end{aligned}
$$

NOTE: The number of terms is $2^p - 1$. However, typically many of the terms can be merged as they correspond to the same component set.

# The inclusion-exclusion method (cont.)

NOTE: An event of form $E_{i_1} \cap \cdots \cap E_{i_r}$ occurs if and only if all the components in the set $P_{i_1} \cup \cdots \cup P_{i_r}$ are functioning. When the component state variables are independent, we get:

$$P(E_{i_1} \cap \cdots \cap E_{i_r}) = \prod_{i \in P_{i_1} \cup \cdots \cup P_{i_r}} p_i.$$

# The inclusion-exclusion method (cont.)

Consider a binary monotone system $(C, \phi)$ where $C = \{1, 2, 3, 4\}$ with minimal path sets $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3, 4\}$.

Assuming that the component state variables are independent, we then get:

$$
\begin{aligned}
P(E_1) = p_1 p_2, \quad P(E_2) = p_1 p_3, \quad P(E_3) = p_2 p_3 p_4 \\
P(E_1 \cap E_2) = p_1 p_2 p_3, \quad P(E_1 \cap E_3) = P(E_2 \cap E_3) = p_1 p_2 p_3 p_4 \\
P(E_1 \cap E_2 \cap E_3) = p_1 p_2 p_3 p_4.
\end{aligned}
$$

Hence, the reliability of the system is:

$$
\begin{aligned}
h(\boldsymbol{p}) &= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - 2 p_1 p_2 p_3 p_4 + p_1 p_2 p_3 p_4 \\
&= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - p_1 p_2 p_3 p_4.
\end{aligned}
$$

# The inclusion-exclusion method (cont.)

NOTE: The final expression for the reliability function is exactly the same as we get using the multiplication methods:

$$h(\boldsymbol{p}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i,$$

where $\delta$ denotes the signed domination function. However, the steps we take in order to get this expression is different.

By using the inclusion-exclusion formula we see that all terms in the expansion correspond to sets $A \subseteq C$ which are unions of minimal path sets.

If a set $A \subseteq C$ is *not* a union of minimal path sets, it follows that $\delta(A) = 0$.

# The inclusion-exclusion method (cont.)

If $P_{i_1}, \ldots, P_{i_r}$ is a collection of minimal path sets such that:

$$\bigcup_{j=1}^{r} P_{i_j} = A,$$

the collection is said to be a *formation* of the set $A$.

The formation is *odd* if $r$ is odd, and *even* if $r$ is even.

An odd formation contributes with a coefficient $+1$, while an even formation contributes with a coefficient $-1$.

# The inclusion-exclusion method (cont.)

Simplifying the expansion, all terms corresponding to formations of the same set are merged. Hence, it follows that we have:

**Theorem**

*Let $(C, \phi)$ be a binary monotone system with minimal path sets $P_1, \ldots, P_p$, and let $\delta$ denote the signed domination function of the system. Then for all $A \subseteq C$ we have:*

$$\begin{aligned}\delta(A) = &\ \textit{The number of odd formations of } A \\ &- \textit{The number of even formations of } A.\end{aligned}$$

*In particular $\delta(A) = 0$ if $A$ is not a union of minimal path sets.*

As a corollary we obtain the following result:

**Corollary**

*If $(C, \phi)$ is a binary monotone system which is not coherent, then $\delta(C) = 0$.*

# An upper bound on the system reliability

Skipping all higher order terms and keeping the probabilities of the individual events $E_1, \ldots, E_p$ only, we get an *upper bound* on the system reliability:

$$h \leq \sum_{j=1}^{p} P(E_j).$$

Given that $p$ is moderate, and that we are given the minimal path sets, this upper bound is easy to calculate.

# An lower bound on the system reliability

We then consider the minimal cut sets of the system $K_1, \ldots, K_k$, and introduce:

$$F_j = \{\text{All the components in } K_j \text{ are failed}\}, \quad j = 1, \ldots, k.$$

Since the system is failed if and only if all the components in at least one cut set are failed, we have:

$$1 - h = P(\bigcup_{j=1}^{k} F_j).$$

An upper bound on $1 - h$ is then given by:

$$1 - h \leq \sum_{j=1}^{k} P(F_j).$$

Combining the upper and lower bounds, we get:

$$1 - \sum_{j=1}^{k} P(F_j) \leq h \leq \sum_{j=1}^{p} P(E_j).$$

# Bounds on the system reliability (cont.)

If the components are independent, we get:

$$1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i) \leq h(\boldsymbol{p}) \leq \sum_{j=1}^{p} \prod_{i \in P_j} p_i.$$

If the component reliabilites are close to 1, the lower bound turns out to be very good, while the upper bound will be crude and possibly greater than 1.

If the component reliabilites are close to 0, the lower bound will be crude and possibly less than 0, while the upper bound turns out to be very good.

# Bounds on the system reliability (cont.)

In order to avoid bounds outside of the interval $[0, 1]$, one would typically replace the bounds by:

$$\max(1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i),\ 0) \leq h(\boldsymbol{p}) \leq \min(\sum_{j=1}^{p} \prod_{i \in P_j} p_i,\ 1).$$

# Section 4.4.

**Computing the reliability of directed network systems**

# Directed network systems

### Definition

A Source-to-$K$-terminal-system (SKT-system) is a system defined relative to a directed network where the system functions if and only if a node $S$ (called the source) can send information to a given set of $K$ nodes $T_1, \ldots, T_K$ (called the terminals).

The components of the system are the *directed edges* of the network, while the *nodes* are assumed to be *functioning perfectly* with probability one.

# Directed network systems (cont.)

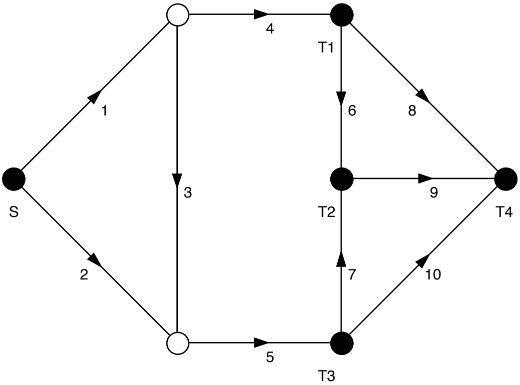


Figure: An S4T system with components $1, 2, \ldots, 10$. The node $S$ is the source, while the nodes $T_1$, $T_2$, $T_3$, $T_4$ are the terminals.

# Directed network systems (cont.)

### Theorem

*Let $\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$ be the structure function of an SKT system.*

- *If $A$ can be expressed as a union of minimal path sets, and the subgraph spanned by $A$ does not contain any directed cycle, we have:*

$$\delta(A) = (-1)^{|A| - v(A) + 1}$$

  *where $v(A)$ denotes the number of nodes in the subgraph spanned by $A$.*

- *In the opposite case we have:*

$$\delta(A) = 0$$

# Example of a directed network system

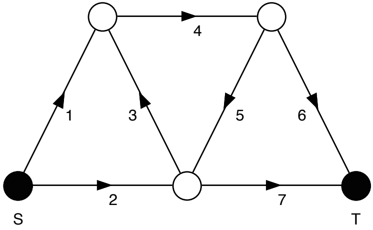


Figure: An S1T system $(C, \phi)$ where $C = \{1, \ldots, 7\}$.

The system is functioning if the source $S$ can send signals to the terminal $T$ through the network.

The component state variables are independent and $P(X_i = 1) = p_i$ for $i \in C$.

# Example of a directed network system (cont.)

The minimal path sets of this system are $P_1 = \{1, 4, 6\}$, $P_2 = \{1, 4, 5, 7\}$, $P_3 = \{2, 3, 4, 6\}$ and $P_4 = \{2, 7\}$. We calculate the reliability of this system by using the inclusion-exclusion formula. Since there are 4 minimal path sets, this formula will consist of $2^4 - 1 = 15$ terms before we simplify:

$$
\begin{aligned}
h(\boldsymbol{p}) = {} & p_1p_4p_6 + p_1p_4p_5p_7 + p_2p_3p_4p_6 + p_2p_7 \\
& - p_1p_4p_5p_6p_7 - p_1p_2p_3p_4p_6 - p_1p_2p_4p_6p_7 \\
& - p_1p_2p_3p_4p_5p_6p_7 - p_1p_2p_4p_5p_7 - p_2p_3p_4p_6p_7 \\
& + p_1p_2p_3p_4p_5p_6p_7 + p_1p_2p_4p_5p_6p_7 \\
& + p_1p_2p_3p_4p_6p_7 + p_1p_2p_3p_4p_5p_6p_7 \\
& - p_1p_2p_3p_4p_5p_6p_7.
\end{aligned}
$$

# Example of a directed network system (cont.)

By merging similar terms we obtain:

$$
\begin{aligned}
h(\boldsymbol{p}) = {} & p_1 p_4 p_6 + p_1 p_4 p_5 p_7 + p_2 p_3 p_4 p_6 + p_2 p_7 \\
& - p_1 p_4 p_5 p_6 p_7 - p_1 p_2 p_3 p_4 p_6 - p_1 p_2 p_4 p_6 p_7 \\
& - p_1 p_2 p_4 p_5 p_7 - p_1 p_2 p_4 p_5 p_7 \\
& + p_1 p_2 p_4 p_5 p_6 p_7 + p_1 p_2 p_3 p_4 p_6 p_7 .
\end{aligned}
$$

Here $\delta(A)$ is either $+1$, $-1$ or zero for all $A \subseteq C$.

Since the network contains a directed cycle $\{3, 4, 5\}$, $\delta(C) = 0$.

# Example of a directed network system (cont.)

$$
\begin{aligned}
\delta(\{1,4,6\}) &= (-1)^{3-4+1} = +1, \\
\delta(\{1,4,5,7\}) &= (-1)^{4-5+1} = +1, \\
&\cdots\cdots \\
\delta(\{1,4,5,6,7\}) &= (-1)^{5-5+1} = -1, \\
\delta(\{1,2,3,4,6\}) &= (-1)^{5-5+1} = -1, \\
\delta(\{1,2,4,6,7\}) &= (-1)^{5-5+1} = -1, \\
&\cdots\cdots \\
\delta(\{1,2,4,5,6,7\}) &= (-1)^{6-5+1} = +1, \\
\delta(\{1,2,3,4,6,7\}) &= (-1)^{6-5+1} = +1.
\end{aligned}
$$

# Linear consecutive $k$-out-of-$n$ systems

Let $(C, \phi)$ be a linear consecutive 2-out-of-5 system where $C = \{1, \ldots, 5\}$.

**Minimal path sets:** $P_1 = \{1, 2\}$, $P_2 = \{2, 3\}$, $P_3 = \{3, 4\}$, $P_4 = \{4, 5\}$.

By using e.g., the inclusion-exclusion formula it is easy to see that the reliability of this system, assuming independent component state variables, is:

$$\begin{aligned} h(\boldsymbol{p}) &= p_1 p_2 + p_2 p_3 + p_3 p_4 + p_4 p_5 \\ &- p_1 p_2 p_3 - p_2 p_3 p_4 - p_3 p_4 p_5 - p_1 p_2 p_4 p_5 \\ &+ p_1 p_2 p_3 p_4 p_5. \end{aligned}$$

It can be shown that this system is *not* an SKT-system.

Linear consecutive $k$-out-of-$n$ systems share the property with SKT-systems that the signed domination function is either $+1$, $-1$ or zero for all subsets of the component set.