STK3405 – Week 34b

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Section 2.3

Dual systems

Dual systems

Definition

Let ϕ be a structure function of a binary monotone system of order n. We then define the *dual structure function*, ϕ^D for all $\boldsymbol{y} \in \{0, 1\}^n$ as:

$$\phi^D(\boldsymbol{y}) = 1 - \phi(\mathbf{1} - \boldsymbol{y}).$$

Furthermore, if $\boldsymbol{X}$ is the component state vector of a binary monotone system, we define the dual component state vector $\boldsymbol{X}^D$ as:

$$\boldsymbol{X}^D = (X_1^D, \ldots, X_n^D) = (1 - X_1, \ldots, 1 - X_n) = \mathbf{1} - \boldsymbol{X}$$

Dual systems (cont.)

Note:

- The relation between ϕ and ϕ^D is a relation between two *functions*
- The relation between $\boldsymbol{X}$ and $\boldsymbol{X}^D$ is a relation between two *stochastic vectors*

We also introduce the dual component set $C^D = \{1^D, \ldots, n^D\}$, where the dual component i^D is functioning if the component i is failed, while i^D is failed if the component i is functioning.

We have the following relation between the two stochastic variables $\phi(\boldsymbol{X})$ and $\phi^D(\boldsymbol{X}^D)$:

$$\phi^D(\boldsymbol{X}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{X}^D) = 1 - \phi(\boldsymbol{X}).$$

Hence, the dual system is functioning if and only if the original system is failed and vice versa.

Examples of dual systems

Let ϕ be the structure function of a system of order 3 such that:

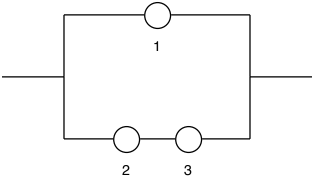
$$\phi(\boldsymbol{y}) = y_1 \amalg (y_2 \cdot y_3),$$

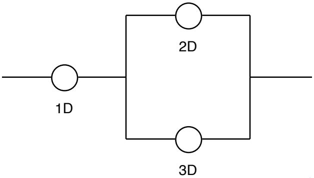
The dual structure function is then given by:

$$\begin{aligned}
\phi^D(\boldsymbol{y}) &= 1 - \phi(\mathbf{1} - \boldsymbol{y}) \\
&= 1 - (1 - y_1) \amalg ((1 - y_2) \cdot (1 - y_3)) \\
&= 1 - [1 - (1 - (1 - y_1))(1 - (1 - y_2) \cdot (1 - y_3))] \\
&= 1 - [1 - y_1 \cdot (1 - (1 - y_2) \cdot (1 - y_3))] \\
&= y_1 \cdot (y_2 \amalg y_3)
\end{aligned}$$

Examples of dual systems (cont.)

$$\phi(\boldsymbol{y}) = y_1 \amalg (y_2 \cdot y_3), \qquad \phi^D(\boldsymbol{y}) = y_1 \cdot (y_2 \amalg y_3)$$





Examples of dual systems (cont.)

Let (C, ϕ) be a series system of order n:

$$\phi(\boldsymbol{y}) = \prod_{i=1}^{n} y_i.$$

The dual structure function is then given by:

$$\begin{aligned}\phi^D(\boldsymbol{y}) &= 1 - \phi(\boldsymbol{1} - \boldsymbol{y}) \\ &= 1 - \prod_{i=1}^{n}(1 - y_i) = \coprod_{i=1}^{n} y_i.\end{aligned}$$

Thus, (C^D, ϕ^D) is a parallel system of order n.

Examples of dual systems (cont.)

Let (C, ϕ) be a parallel system of order n:

$$\phi(\boldsymbol{y}) = \coprod_{i=1}^{n} y_i.$$

The dual structure function is then given by:

$$\begin{aligned}
\phi^D(\boldsymbol{y}) &= 1 - \phi(\boldsymbol{1} - \boldsymbol{y}) = 1 - \coprod_{i=1}^{n} (1 - y_i) \\
&= 1 - (1 - \prod_{i=1}^{n} (1 - (1 - y_i)) = \prod_{i=1}^{n} y_i.
\end{aligned}$$

Thus, (C^D, ϕ^D) is a series system of order n.

Dual systems (cont.)

Theorem

Let ϕ be the structure function of a binary monotone system, and let ϕ^D be the corresponding dual structure function. Then we have:

$$(\phi^D)^D = \phi.$$

That is, the dual of the dual system is equal to the original system.

Proof: For all $\boldsymbol{y} \in \{0,1\}^n$ we have:

$$\begin{aligned}(\phi^D)^D(\boldsymbol{y}) &= 1 - \phi^D(\mathbf{1} - \boldsymbol{y}) \\ &= 1 - [1 - \phi(\mathbf{1} - (\mathbf{1} - \boldsymbol{y}))] \\ &= \phi(\boldsymbol{y}).\end{aligned}$$

Reliability of binary monotone systems

Reliability of binary monotone systems

Let (C, ϕ) be a binary monotone system, and let $i \in C$.

$$p_i = P(X_i = 1) = \text{The } \textit{reliability} \text{ of a component } i$$

Since the state variable X_i is binary, we have for all $i \in C$:

$$\mathrm{E}[X_i] = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1) = p_i$$

Thus, the reliability of component i is equal to the expected value of its component state variable, X_i.

Reliability of binary monotone systems (cont.)

$$h = P(\phi(\boldsymbol{X}) = 1) = \text{The } \textit{reliability} \text{ of the system}$$

Since ϕ is binary, we have:

$$\mathrm{E}[\phi(\boldsymbol{X})] = 0 \cdot P(\phi(\boldsymbol{X}) = 0) + 1 \cdot P(\phi(\boldsymbol{X}) = 1) = P(\phi(\boldsymbol{X}) = 1) = h.$$

Thus, the reliability of the system is equal to the expected value of the structure function, $\phi(\boldsymbol{X})$.

From this it immediately follows that the reliability of a system, at least in principle, can be calculated as:

$$h = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$

Independent components

We now focus on the case where the component state variables can be assumed to be *independent* and introduce $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$. We note that:

$$P(X_i = x_i) = \begin{cases} p_i & \text{if } x_i = 1, \\ 1 - p_i & \text{if } x_i = 0. \end{cases}$$

Since x_i is either 0 or 1, $P(X_i = x_i)$ can be written in the following more compact form:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1 - x_i}.$$

The reliability function

Thus, when the component state variables are independent, their joint distribution can be written as:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$

Hence, we get the following expression for the system reliability:

$$h = h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

The function $h(\boldsymbol{p})$ is called *the reliability function* of the system.

Reliability of a series system

Consider a series system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} \mathrm{E}[X_i] = \prod_{i=1}^{n} p_i,$$

where the third equality follows since $X_1, X_2, \ldots, X_n$ are independent.

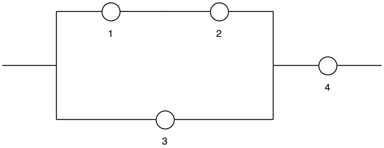
Reliability of a parallel system

Consider a parallel system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\coprod_{i=1}^{n} X_i] = \mathrm{E}[1 - \prod_{i=1}^{n}(1 - X_i)]$$
$$= 1 - \prod_{i=1}^{n}(1 - E[X_i]) = \coprod_{i=1}^{n} \mathrm{E}[X_i] = \coprod_{i=1}^{n} p_i,$$

where the fourth equality follows since $X_1, X_2, \ldots, X_n$ are independent.

Reliability of a mixed system



Assuming independent component states the system reliability becomes:

$$
\begin{aligned}
h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] &= \mathrm{E}[[(X_1 \cdot X_2) \amalg X_3] \cdot X_4] \\
&= \mathrm{E}[(X_1 \cdot X_2) \amalg X_3] \cdot \mathrm{E}[X_4] \\
&= [\mathrm{E}[X_1 \cdot X_2] \amalg \mathrm{E}[X_3]] \cdot \mathrm{E}[X_4] \\
&= [(\mathrm{E}[X_1] \cdot \mathrm{E}[X_2]) \amalg \mathrm{E}[X_3]] \cdot \mathrm{E}[X_4] \\
&= [(p_1 \cdot p_2) \amalg p_3] \cdot p_4.
\end{aligned}
$$

Component level changes vs. system level changes

Theorem

Let $h(\boldsymbol{p})$ be the reliability function of a monotone system (C, ϕ) of order n. Then for all $\boldsymbol{p}, \boldsymbol{p}' \in [0, 1]^n$ we have:

(i) $h(\boldsymbol{p} \amalg \boldsymbol{p}') \geq h(\boldsymbol{p}) \amalg h(\boldsymbol{p}')$,

(ii) $h(\boldsymbol{p} \cdot \boldsymbol{p}') \leq h(\boldsymbol{p}) \cdot h(\boldsymbol{p}')$

If (C, ϕ) is coherent, equality holds in (i) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0, 1]^n$ if and only if (C, ϕ) is a parallel system.

If (C, ϕ) is coherent, equality holds in (ii) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0, 1]^n$ if and only if (C, ϕ) is a series system.

Component level vs. system level (cont.)

Proof: We have:

$$\begin{aligned} h(\boldsymbol{p} \amalg \boldsymbol{p}') &- h(\boldsymbol{p}) \amalg h(\boldsymbol{p}') \\ &= E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y})] - E[\phi(\boldsymbol{X})] \amalg E[\phi(\boldsymbol{Y})] \\ &= E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y}) - \phi(\boldsymbol{X}) \amalg \phi(\boldsymbol{Y})], \end{aligned}$$

where the last expectation must be non-negative since by the corresponding result for structure functions we know that:

$$\phi(\boldsymbol{x} \amalg \boldsymbol{y}) - \phi(\boldsymbol{x}) \amalg \phi(\boldsymbol{y}) \geq 0, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n.$$

This completes the proof of (i). The proof of (ii) is similar.

Component level vs. system level (cont.)

We now consider the case where (C, ϕ) is coherent and show that equality in (i) holds for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a parallel system.

Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be chosen so that $0 < p_i < 1, 0 < p_i' < 1$ for $i = 1, \ldots, n$:

$$P(\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{Y} = \boldsymbol{y}) > 0, \text{ for all } \boldsymbol{x} \in \{0,1\}^n \text{ and } \boldsymbol{y} \in \{0,1\}^n.$$

From this it follows that:

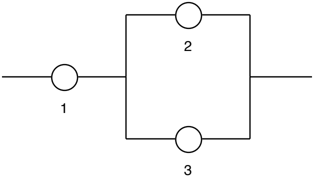
$$E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y}) - \phi(\boldsymbol{X}) \amalg \phi(\boldsymbol{Y})] = 0$$

if and only if

$$\phi(\boldsymbol{x} \amalg \boldsymbol{y}) - \phi(\boldsymbol{x}) \amalg \phi(\boldsymbol{y}) = 0 \text{ for all } \boldsymbol{x} \in \{0,1\}^n \text{ and } \boldsymbol{y} \in \{0,1\}^n.$$

By the corresponding result for structure functions this holds if and only if (C, ϕ) is a parallel system. The other equivalence is proved similarly.

Component level vs. system level (cont.)



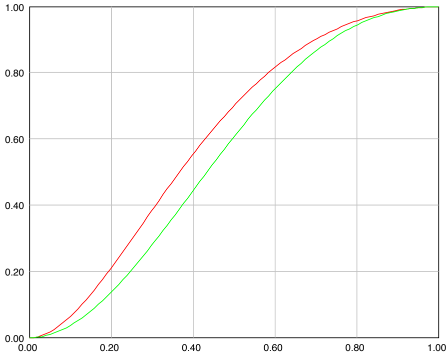
Let (C, ϕ) be a system with independent component state variables with $P(X_i = 1) = p$ for all $i \in C$, and where $\phi(\boldsymbol{x}) = x_1 \cdot (x_2 \amalg x_3)$.

We then get that $h(p) = p \cdot (p \amalg p) = p \cdot (p + p - p^2) = 2p^2 - p^3$.

Hence, for all $0 \leq p \leq 1$, we have:

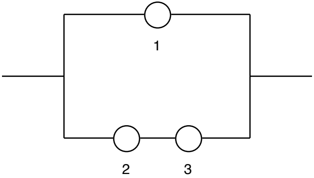
$$\begin{aligned} h(p \amalg p) &= 2(p \amalg p)^2 - (p \amalg p)^3 \\ &\geq h(p) \amalg h(p) = (2p^2 - p^3) \amalg (2p^2 - p^3) \end{aligned}$$

Component level vs. system level (cont.)



- Red curve: $h(p \amalg p) = 2(p \amalg p)^2 - (p \amalg p)^3$
- Green curve: $h(p) \amalg h(p) = (2p^2 - p^3) \amalg (2p^2 - p^3)$

Component level vs. system level (cont.)



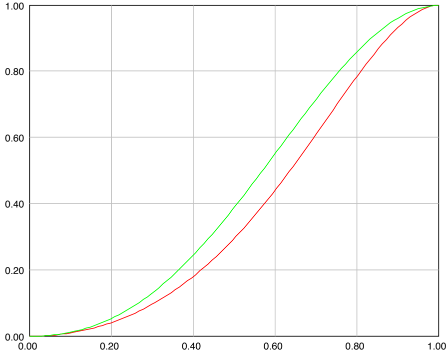
Let (C, ϕ) be a system with independent component state variables with $P(X_i = 1) = p$ for all $i \in C$, and where $\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3)$.

We then get that $h(p) = p \amalg (p \cdot p) = p \amalg p^2 = p + p^2 - p^3$.

Hence, for all $0 \leq p \leq 1$, we have:

$$\begin{aligned} h(p \cdot p) &= p^2 + p^4 - p^6 \\ &\leq h(p) \cdot h(p) = (p + p^2 - p^3)^2 \end{aligned}$$

Component level vs. system level (cont.)



- Red curve: $h(p \cdot p) = p^2 + p^4 - p^6$
- Green curve: $h(p) \cdot h(p) = (p + p^2 - p^3)^2$