Fábio Rodrigues Pereira

fabior @ math. uio. no

Problem 1a: why $Y = e^{\sigma U + \mu}$ implies $Y \sim lognorm(\mu, \sigma)$

- Suppose that X is a normal stochastic variable with μ_x and σ_x. Then, it can standardized by:

$\hookrightarrow U \sim N(0,1)$

$$U = \frac{X - \mu_x}{\sigma_x} \longrightarrow \boxed{X = U \cdot \sigma_x + \mu_x}$$

- we know that $Y = e^{\sigma U + \mu}$, Then:

$$\ln(Y) = \sigma_x U + \mu_x = X$$

- where :

  $\mu_x :=$ The mean of X ;

  $\sigma_x :=$ The standard deviation of X ;

  $U :=$ The standardized form of X.

$\hookrightarrow$ Then $\ln(Y) = X$ which implies by lognormal definition that a stochastic variable Y is lognormal distributed with parameters μ_x and σ_x if $X = \ln(Y)$ is normally distributed with μ_x and σ_x.

Problem 1b:

$\left\{\begin{array}{l}\text{assume } Y \sim \text{lognormal}(\mu, \sigma)\\[4pt]\text{let } y_\alpha : \quad P(Y \leq y_\alpha) = \alpha \% \text{ where } 0 \leq \alpha \leq 100\\[4pt]\text{show } y_\alpha = e^{\sigma \mu_\frac{\alpha}{2} + \mu} \text{ where } \mu_\alpha \text{ is the } \alpha \% \text{ in the}\\[4pt]\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{standard normal distribution}\end{array}\right.$

- assume $Y \sim \text{lognormal}(\mu, \sigma)$, $Y = e^{\sigma u + \mu}$ or $\ln(Y) = \sigma u + \mu$,

  hence:
  $$\alpha \% = P(Y \leq y_\alpha)$$

  $$\alpha \% = P\left(\ln(Y) \leq \boxed{\ln(y_\alpha)}\right)$$

  $$\alpha \% = P\left(\sigma u + \mu \leq \boxed{\sigma \mu_\alpha + \mu}\right)$$

then: $\ln(y_\alpha) = \sigma \mu_\alpha + \mu$

$$y_\alpha = e^{\sigma \mu_\alpha + \mu}$$

Problem 1c :

Show $\dfrac{y_{90}}{y_{50}} = \dfrac{y_{50}}{y_{10}}$

From 1b, we know that $y_\alpha = e^{\sigma \mu_\alpha + \mu}$, then :

$$\dfrac{e^{\sigma \cdot \mu_{90} + \mu}}{e^{\sigma \cdot \mu_{50} + \mu}} = \dfrac{e^{\sigma \cdot \mu_{50} + \mu}}{e^{\sigma \cdot \mu_{10} + \mu}}$$

$$\dfrac{e^{\sigma \cdot \mu_{90}} \cdot \cancel{e^{\mu}}}{e^{\sigma \cdot \mu_{50}} \cdot \cancel{e^{\mu}}} = \dfrac{e^{\sigma \cdot \mu_{50}} \cdot \cancel{e^{\mu}}}{e^{\sigma \cdot \mu_{10}} \cdot \cancel{e^{\mu}}}$$

$$e^{\sigma \cdot \cancel{\mu_{90}}^{\nearrow -\mu_{10}} + \sigma \mu_{10}} = e^{2\sigma \cdot \cancel{\mu_{50}}^{\nearrow 0}}$$

$$e^0 = e^0 = 1 \qquad \blacksquare$$

By symmetry, $\mu_{90} = -\mu_{10}$

Problem 1d : assume $y_{10} = 5$ and $y_{90} = 20$.

Find $y_{50} = ?$

From 1c, we have $\dfrac{y_{90}}{y_{50}} = \dfrac{y_{50}}{y_{10}}$, then $\dfrac{20}{y_{50}} = \dfrac{y_{50}}{5}$

$$\therefore \; y_{50} = 10$$

Determine μ and σ for this distribution. Assume $\mu_{90} = 1,28155$

$$y_{90} = e^{\sigma \cdot \mu_{90} + \mu} \longrightarrow \boxed{20 = e^{\sigma \cdot 1,28155 + \mu}}$$

$$y_{50} = e^{\sigma \cdot \mu_{50} + \mu} \longrightarrow 10 = e^{\sigma \cdot 0 + \mu} \longrightarrow 10 = e^{\mu} \longrightarrow \boxed{\mu = \ln(10)}$$

$$\therefore \; \mu = 2.30259$$

Then $20 = e^{\sigma \cdot 1,28155} \cdot e^{\ln(10)}$

$$20 = e^{\sigma \cdot 1,28155} \cdot 10$$

$$e^{\sigma \cdot 1,28155} = 2$$

$$\sigma = \frac{\ln(2)}{1,28155} = 0,54087$$