UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK3405/STK4405 — Elementary introduction to risk and reliability analysis.
Day of examination:	Wednesday 19. December 2018.
Examination hours:	14.30 – 18.30.
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subpoints will be equally weighted in the marking.

Problem 1

Consider the binary monotone system (C, ϕ) shown in Figure 1. The component set of the system is $C = \{1, 2, \ldots, 6\}$. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_6)$ denote the vector of component state variables, and assume throughout this problem that $X_1, X_2, \ldots, X_6$ are stochastically independent. Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 6$. We assume that $0 < p_i < 1$ for $i = 1, 2, \ldots, 6$.

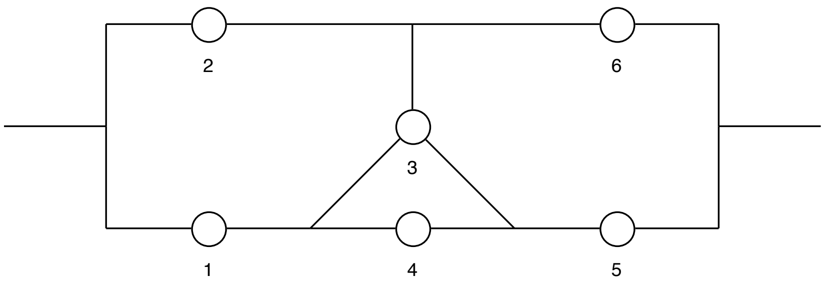


Figure 1: A binary monotone system of 6 components.

a) Find the minimal path and cut sets of the system.

b) Use the result in a) to find an expression for the structure function of

(Continued on page 2.)

Problem 1a:

The minimal path sets: {1,4,5}; {1,3,6}; {1,3,5}; {2,6}; {2,3,5}

The minimal cut sets: {1,2}; {1,3,6}; {2,3,5}; {6,5}; {3,4,6}; {2,3,5}

Problem 1b:

Using the minimal path sets: {1,4,5}; {1,3,6}; {1,3,5}; {2,6}; {2,3,5}

$$\phi(\vec{x}) = \prod_{j=1}^{p} \coprod_{i\in P_j} x_i = (x_1 \amalg x_4 \amalg x_5)\cdot(x_1 \amalg x_3 \amalg x_6)\cdot(x_1 \amalg x_3 x_5)\cdot(x_2 \amalg x_6)\cdot(x_2 \amalg x_3 \amalg x_5)$$

Remember: $\coprod x_i = 1 - \prod(1-x_i)$

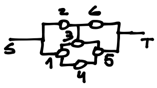
or using the minimal cut sets: {1,2}; {1,3,6}; {2,3,5}; {6,5}; {3,4,6}; {2,3,5}

$$\phi(\vec{x}) = \coprod_{j=1}^{k} \prod_{i\in K_j} x_i = (x_1x_2)\amalg(x_1x_3x_6)\amalg(x_2x_3x_5)\amalg(x_6x_5)\amalg(x_3x_4x_6)\amalg(x_2x_3x_5)$$

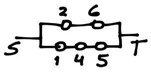
$$\therefore\ P(\phi(\vec{x})=1) = E[\phi(\vec{x})] = h(\vec{p})$$

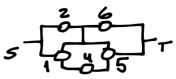
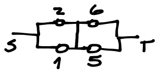
→ Possible to use any of the 2 options above

Problem 1c: — Use factor algorithm



→ pivoting in respect of component 3

$\phi(0_3,\vec{x}) :=$  $:= (x_1x_4x_5)\amalg(x_2x_6) = x_1x_4x_5 + x_2x_6 - x_1x_2x_4x_5x_6$

$\phi(1_3,\vec{x}) =$  := component 4 is irrelevant :=  $:= (x_1\amalg x_2)\cdot(x_5\amalg x_6) = (x_1+x_2-x_1x_2)\cdot(x_5+x_6-x_5x_6)$

$$\therefore\ \phi(\vec{x}) = x_3\cdot\phi(1_3,\vec{x}) + (1-x_3)\,\phi(0_3,\vec{x})$$

$E[\phi(\vec{x})] = h(\vec{p}) = p_3\,\underbrace{h(1_3,\vec{p})}_{E[\phi(1_3,\vec{x})]} + (1-p_3)\,\underbrace{h(0_3,\vec{p})}_{E[\phi(0_3,\vec{x})]}$ where $P(\vec{X}=\vec{x}) = p_i$ for $\forall i \in C$

Problem 1d:

The definition of the Birnbaum measure for the reliability importance of a component is the probability of the system fails when the component ith fails, in other words, the probability that the component ith is critical.

$$I_B^{(i)} := P(\phi(1_i,\vec{x}) - \phi(0_i,\vec{x}) = 1) := P(\text{component } i^{th} \text{ is critical})$$

Problem 1e:

$$I_B^{(3)} = \frac{d[h(\vec{p})]}{dp_3} = (p_1+p_2-p_1p_2)(p_5+p_6-p_5p_6) - p_1p_4p_5 - p_2p_6 + p_1p_2p_4p_5p_6$$

Assume $p_1, p_2, p_4, p_5, p_6 = 1/2$, then use it on $I_B^{(3)}$:

$$\left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right) - \frac{1}{8} - \frac{1}{4} + \frac{1}{32} = \frac{18-4-8+1}{32} = \frac{7}{32} = I_B^{(3)}$$

Problem 1f:

by symmetri: $I_B^{(1)} = I_B^{(5)}$, $I_B^{(2)} = I_B^{(6)}$ and $I_B^{(3)}, I_B^{(4)}$ less important the system, and explain briefly how this can be used to find the system reliability. A detailed calculation is not required.

c) Use the factoring algorithm to derive the reliability of the system in a different way from the one in b).

d) What is the definition of the Birnbaum measure for the reliability importance of a component?

e) What is the reliability importance of component 3 according to the Birnbaum measure? How can you use this result to find the structural importance of component 3?

f) Assume that $p_i = p$ for $i = 1, 2, \ldots, 6$, i.e., that all the components have the same component reliability. What can you say about the reliability importance of the other 5 components?

Problem 2

Consider a binary monotone system (C, ϕ), where $C = \{1, 2, 3\}$ and where the structure function ϕ is given by:

$$\phi(\boldsymbol{X}) = \mathrm{I}(\sum_{i=1}^{3} X_i \geq 2).$$

Here $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the vector of component state variables and $\mathrm{I}(\cdot)$ denotes the indicator function.

a) Show that the structure function ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3.$$

In the following we assume that:

$$X_i = Y_0 \cdot Y_i, \quad i = 1, 2, 3,$$

where Y_0, Y_1, Y_2, Y_3 are independent binary stochastic variables and:

$$P(Y_0 = 1) = \theta,\ P(Y_1 = 1) = P(Y_2 = 1) = P(Y_3 = 1) = q,$$

where $0 < \theta < 1$ and $0 < q < 1$.

b) Explain why this implies that X_1, X_2, X_3 are associated stochastic variables.

We then introduce $h = \mathrm{E}[\phi(\boldsymbol{X})] = P(\phi(\boldsymbol{X}) = 1)$.

c) Show that:

$$h = h(\theta, q) = \theta q^2(3 - 2q).$$

(Continued on page 3.)

Problem 2a:

Assume that we ignore the dependence between the X_is, and instead computes the system reliability as if X_1, X_2, X_3 are independent and:

$$P(X_i = 1) = \theta q, \quad i = 1, 2, 3.$$

Let $\tilde{h}$ denote the system reliability we then get.

d) Show that:

$$\tilde{h} = \tilde{h}(\theta, q) = \theta^2 q^2 (3 - 2\theta q).$$

e) Assume that $\theta = \frac{1}{2}$. Show that we then have $\tilde{h} < h$ for all $0 < q < 1$.

f) Assume instead that $\theta = \frac{3}{4}$. What can you say about the relationship between $\tilde{h}$ and h in this case?

Problem 3

Let (C, ϕ) be a binary monotone system, and let $\boldsymbol{X}$ denote the vector of component state variables. In this problem we consider how the system reliability $h = P(\phi(\boldsymbol{X}) = 1)$ can be estimated using Monte Carlo simulation. The simplest Monte Carlo estimate is:

$$\hat{h}_{MC} = \frac{1}{N} \sum_{r=1}^{N} \phi(\boldsymbol{X}_r),$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are data generated from the distribution of $\boldsymbol{X}$.

In order to improve this estimate we let $S = S(\boldsymbol{X})$ be a stochastic variable with values in the set $\{s_1, \ldots, s_k\}$. We assume that the distribution of S is known, and introduce:

$$\theta_j = E[\phi | S = s_j], \quad j = 1, \ldots, k.$$

We then use Monte Carlo simulation in order to estimate $\theta_1, \ldots, \theta_k$, and generate data from the conditional distribution of $\boldsymbol{X}$ given S. We let $\{\boldsymbol{X}_{r,j} : r = 1, \ldots, N_j\}$ denote the vectors generated from the distribution of $\boldsymbol{X}$ given that $S = s_j$, $j = 1, \ldots, k$, and get the following estimates:

$$\hat{\theta}_j = \frac{1}{N_j} \sum_{r=1}^{N_j} \phi(\boldsymbol{X}_{r,j}), \quad j = 1, \ldots, k.$$

These estimates are then combined into the following estimate of the system reliability:

$$\hat{h}_{CMC} = \sum_{j=1}^{k} \hat{\theta}_j P(S = s_j).$$

(Continued on page 4.)

a) Show that $E[\hat{h}_{CMC}] = h$ and that the variance of the estimate is given by:

$$\operatorname{Var}(\hat{h}_{CMC}) = \sum_{j=1}^{k} \frac{1}{N_j} \operatorname{Var}(\phi | S = s_j)[P(S = s_j)]^2$$

b) Assume that $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$. Show that we then have:

$$\operatorname{Var}(\hat{h}_{CMC}) \approx \frac{1}{N}(\operatorname{Var}(\phi) - \operatorname{Var}[E(\phi | S)]),$$

and explain briefly why this implies that $\operatorname{Var}(\hat{h}_{CMC}) \leq \operatorname{Var}(\hat{h}_{MC})$.

c) What should one take into account when choosing S?

END