# A Computational Methodology for Modelling the Dynamics of Statistical Arbitrage

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To my parents, Arnold and Carol.

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# Abstract

Recent years have seen the emergence of a multi-disciplinary research area known as “Computational Finance”. In many cases the data generating processes of financial and other economic time-series are at best imperfectly understood. By allowing restrictive assumptions about price dynamics to be relaxed, recent advances in computational modelling techniques offer the possibility to discover new “patterns” in market activity.

This thesis describes an integrated “statistical arbitrage” framework for identifying, modelling and exploiting small but consistent regularities in asset price dynamics. The methodology developed in the thesis combines the flexibility of emerging techniques such as neural networks and genetic algorithms with the rigour and diagnostic techniques which are provided by established modelling tools from the fields of statistics, econometrics and time-series forecasting.

The modelling methodology which is described in the thesis consists of three main parts. The first part is concerned with constructing combinations of time-series which contain a significant predictable component, and is a generalisation of the econometric concept of cointegration. The second part of the methodology is concerned with building predictive models of the mispricing dynamics and consists of low-bias estimation procedures which combine elements of neural and statistical modelling. The third part of the methodology controls the risks posed by model selection and performance instability through actively encouraging diversification across a "portfolio of models". A novel population-based algorithm for joint optimisation of a set of trading strategies is presented, which is inspired both by genetic and evolutionary algorithms and by modern portfolio theory.

Throughout the thesis the performance and properties of the algorithms are validated by means of experimental evaluation on synthetic data sets with known characteristics. The effectiveness of the methodology is demonstrated by extensive empirical analysis of real data sets, in particular daily closing prices of FTSE 100 stocks and international equity indices.

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# Introduction: A Methodology for Statistical Arbitrage

This part of the thesis introduces the concept of statistical arbitrage and examines the issues which will be developed in the rest of the thesis. Chapter 1 consists of a brief introduction which outlines the scope, motivation and organisation of the thesis as well as summarising the major contributions which it makes to the current state of the art. Chapter 2 describes the recent advances in computational modelling and computational finance which provide the background to the thesis. Chapter 3 describes the opportunities for statistical arbitrage which are presented by the advances in modelling methodology, assessing the strengths and weaknesses of existing modelling methods and highlighting the outstanding issues which the methodology presented in the thesis is designed to address. Finally Chapter 4 presents an overview of the methodology and a “route map” to the rest of the thesis.

# *1. Introduction*

## 1.1 Scope

Recent years have seen the emergence of a multi-disciplinary research area known as “Computational Finance”, driven on the one hand by advances in computing power, data availability and computational modelling methods, and on the other by the competitive and continually evolving nature of the financial markets. Whilst the competitive drive to achieve high profitability at low levels of risk creates a demand for improved models of market dynamics, the improvements in hardware, software, data and methodology create a fertile platform for developing the supply of such “market beating” models.

In this thesis we investigate the opportunities that are presented by specific modelling tools from the field of machine learning, namely neural networks and genetic algorithms, and exploit these in the context of a specific area of investment finance, namely statistical arbitrage. The thesis develops a methodological framework for exploiting these recent advances in computational modelling as a means of identifying and forecasting predictable components in the dynamics of combinations of asset prices. The resulting models present opportunities for identifying statistically under- and over-priced assets, extending the traditional “riskless” arbitrage framework to encompass empirical as well as theoretical relationships between assets.

The thesis contributes to the state of the art by enhancing specific aspects of modelling methodology and developing tools, such as variable and model selection algorithms, which support the modelling and forecasting of noisy and nonstationary time series. It applies these tools within the context of a modelling framework inspired by techniques from statistics, econometrics, and time-series modelling. The starting point of the modelling framework is the econometric concept of “cointegration”. The machine learning techniques are employed to relax the requirements and assumptions of cointegration modelling and hence open up the possibility of improving the extent to which future innovations in the time-series can be both forecasted and exploited.

The thesis is cross-disciplinary in nature, taking ideas from statistics and machine learning and presenting a synthesis of these ideas in the form of novel methodologies for performing modelling and forecasting in noisy and nonstationary environments. As such it is likely to be of interest to statisticians, econometricians, operational researchers, computer scientists and members of the machine learning community. The methodologies developed will be of particular interest to researchers and practitioners of time-series forecasting, computational finance and quantitative marketing.  The methodologies derived provide insights into both the opportunities and obstacles which are presented by the use of advanced modelling techniques in investment finance and other application domains with similar properties. Throughout the thesis the performance and properties of the algorithms are validated by means of experimental evaluation on synthetic data sets with known characteristics. Further support for the effectiveness of the algorithms is provided by extensive empirical analysis of real data sets, in particular daily closing prices for equity of the companies which constituted the FTSE 100 index during the period 1997-1999.

## 1.2 Motivation

The philosophy espoused in this thesis is that recent improvements in computational modelling techniques open up new opportunities to identify and model regularities in asset price dynamics. The exploitation of the predictive information provided by such models is referred to as “statistical arbitrage”, reflecting the fact that it can be considered as an example of a broader class of “arbitrage” strategies in which systematic components in asset price dynamics are exploited by market participants known as “arbitrageurs”.

There is a competitive “arms race” between arbitrageurs, which causes the opportunities provided by theoretically-motivated “riskless” strategies to be both self-limiting and restricted to relatively privileged market players who are geared to trade quickly, at low cost, and with sufficient financial leverage to make the exercise worthwhile. In contrast, “statistical arbitrage” opportunities, which are based on empirical regularities with no direct theoretical underpinning, are likely to be both more persistent and more prevalent in financial markets. More persistent because risk-free arbitrage opportunities are rapidly eliminated by market activity. More prevalent because in principle they may occur between any set of assets rather than solely in cases where a suitable risk-free hedging strategy can be implemented.

Due to the highly competitive nature of financial markets, there is an ongoing debate about whether markets are so “efficient” that no predictable components can possibly exist. The position taken in this thesis is that whilst there is no *a priori* reason why regularities in asset price dynamics should not exist it is likely that any easily identifiable effects will soon be eliminated or “arbitraged away” in the very process of being exploited. However, by looking at the markets from *new* angles and using *new* tools it may be possible to identify new opportunities in the form of previously undiscovered “patterns” in market activity.

It is this perspective that motivates this thesis, in aiming to harness recent developments in computational modelling to the task of financial forecasting. In particular the thesis explores the exciting opportunities which are offered by so-called “machine learning” methods such as Neural Networks and Genetic Algorithms.

**Neural Networks** (NNs) are computational models inspired by the learning mechanisms of the human brain; the fundamental concept of neural networks is that they “learn by example”. Typically a neural network is presented a set of “training examples” each of which matches a stimulus, or input pattern, to a desired response, or output pattern. The original, untrained, neural network, when presented with an example stimulus, will generate a random response. A so-called “learning algorithm” is then applied, in order to calculate the error, or difference between desired and actual output, and then to adjust the internal network parameters in such a way that this error is reduced. Eventually the network will converge to a stable configuration which represents the learned solution to the problem represented by the training examples. From a modelling perspective, the novelty of NNs lies in their ability to model arbitrary input-output relationships with few (if any) *a priori* restrictive assumptions about the specific nature of the relationship.

**Genetic Algorithms** (GAs) are computational models inspired by the mechanisms of evolution and natural selection. A genetic algorithm is a means by which a population of, initially random, candidate solutions to a problem is evolved towards better and better solutions. Each new generation of solutions is derived from the previous “parent” generation by mechanisms such as cross-breeding and mutation. In each generation the “parents” are selected according to their “fitness”; the fittest solutions will thus have an increased representation in the new generation and advantageous traits will be propagated through the population. From a modelling perspective, the novelty of GAs lies in the fact that they impose few restrictions on the nature of the “fitness function” which determines the optimisation criteria.

With any new modelling methodology, a clear understanding of the strengths and weaknesses of the methodology is critical in deciding how it should be applied to a particular problem domain, and indeed whether or not it is suitable for application in that domain. Whilst the strengths of the new techniques have been extensively demonstrated in other problem domains, attempts to apply them in the financial domain have been largely unsuccessful.

A major motivation for applying the new techniques in financial forecasting is that the data generating processes of financial and other economic time-series are at best imperfectly understood. By allowing the restrictive assumptions of more established time-series modelling techniques to be relaxed, the flexibility of the emerging techniques offers the possibility of improving forecasting ability and identifying opportunities which would otherwise be overlooked. However, in order to facilitate the successful uptake of the new techniques in this important application area it is first necessary to provide modelling tools which overcome the obstacles posed by the inherent properties of the financial domain - such as a typically large stochastic component ("noise") in observed data and a tendency for the underlying relationships to be time-varying (nonstationary) in nature. These difficulties are exacerbated by weaknesses in the methodology which supports the emerging techniques, which to a large extent lacks the diagnostic tools and modelling frameworks which are an essential component of more traditional time-series modelling approaches.

The objective of this thesis is therefore to identify and exploit the opportunities which are offered by the *strengths* of the emerging modelling techniques, whilst at the same time minimising the effect of their *weaknesses*, through the development of an integrated modelling framework which places the new methods in partnership with, and in the context of, established statistical techniques. The methodology should include tools for the identification of statistical mispricings in sets of asset prices, modelling the predictable component in the mispricing dynamics, and exploiting the predictive forecasts through a suitable asset allocation strategy.

The task of identifying and exploiting statistical arbitrage opportunities in the financial markets can be considered one of the purest and most challenging tests of a predictive modelling methodology, whilst at the same time being of huge practical significance. The highly competitive nature of financial markets means that any predictable component in asset price dynamics is likely to be very small, accounting for no more than a few percentage points of the total variability in the asset prices. Furthermore, the performance of a statistical arbitrage strategy is related solely to the predictive ability of the underlying model, as the taking of offsetting long and short positions in the market eliminates the (favourable) performance contamination which is caused by any tendency for prices to drift upwards over time. The potential rewards of a successful statistical arbitrage methodology are nevertheless enormous: as Figure 1.1 illustrates, even an ability to predict only 1% of daily asset price variability can provide a source of consistent profits at acceptably low levels of risk.

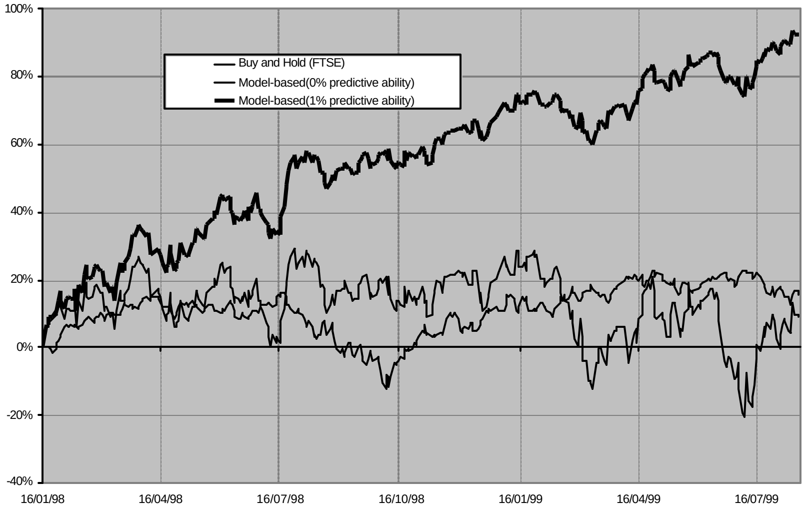


Figure 1.1: Illustration of the potential of statistical arbitrage strategies. The chart shows the accumulated return of holding the FTSE 100 index over the period 16th January 1998 – 17th August 1999, together with the returns of two strategies based on (idealised) predictive models constructed in such a way as to explain 0% and 1% of the variability of the FTSE (i.e. with $R^2$ statistics of 0% and 1%). Transaction costs are not included.

## 1.3 Thesis Overview

The prime contribution of this thesis is the development of a computational methodology for modelling and exploiting the dynamics of statistical arbitrage. A preview of the methodology developed in this thesis is presented in schematic form in Figure 1.2. The methodology consists of three main stages which are (i) constructing time-series of statistical mispricings, (ii) forecasting the mispricing dynamics, and (iii) exploiting the forecasts through a statistical arbitrage asset allocation strategy. These three strands correspond to the modelling phases of intelligent pre-processing, predictive modelling, and risk-averse decision-making.

The three components are intimately linked to each other, in each case the methodology has been arrived at following a process of reviewing the research literature in appropriate fields, assessing the strengths and weaknesses of existing methods, and developing tools and techniques for addressing the methodological gaps thus identified. In each phase, certain assumptions of more traditional modelling approaches are relaxed in a manner aimed at maximising predictive ability and reducing particular sources of forecasting error. The focus of the methodological developments is a consequence of the relaxation of these assumptions, the objective being to develop tools which support model development under the new, relaxed, assumptions.

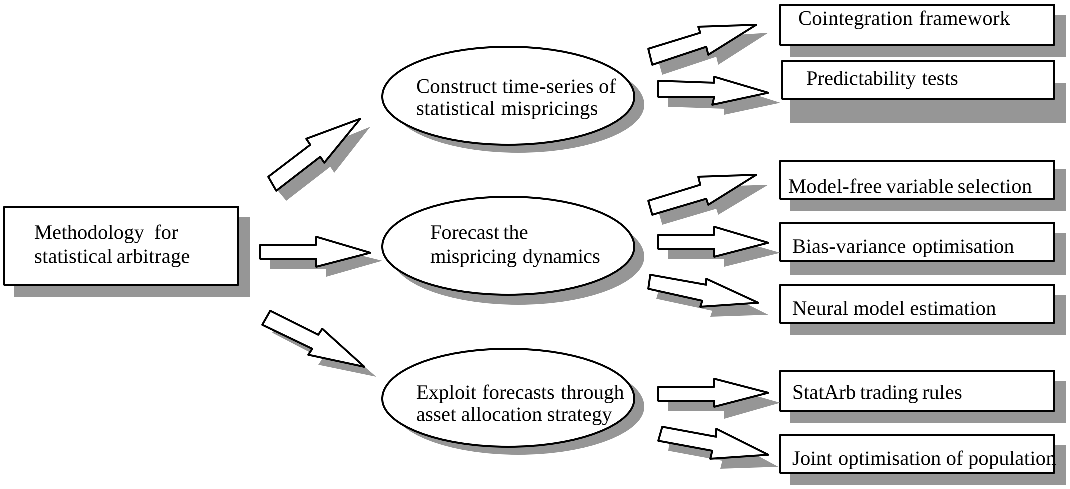

Methodology for statistical arbitrage
Construct time-series of statistical mispricings
Forecast the mispricing dynamics
Exploit forecasts through asset allocation strategy
Cointegration framework
Predictability tests
Model-free variable selection
Bias-variance optimisation
Neural model estimation
StatArb trading rules
Joint optimisation of population

Figure 1.2: Schematic breakdown of the methodology presented in this thesis, representing a computational methodology for forecasting the dynamics of statistical arbitrage. Each of the three phases aims to reduce forecasting errors by relaxing certain underlying assumptions of more traditional forecasting techniques.

The first component of the methodology uses the econometric concept of *cointegration* as a means of creating time-series of statistical "mispricings" which can form the basis of statistical arbitrage models. The objective of this phase is to identify particular combinations of assets such that the combined time-series exhibits a significant predictable component, and is furthermore uncorrelated from underlying movements in the market as a whole.

The second component of the methodology is concerned with modelling the dynamics of the mispricing process whereby statistical mispricings tend to be reduced through an "error correction" effect. As in the first phase, the approach aims to improve forecasting ability by relaxing certain underlying assumptions, in this case the usual assumptions of linearity and independence are relaxed to allow for both direct nonlinear dependencies and indirect nonlinear interaction effects. In order to support this approach, methodology is developed to support the model-free identification of exogenous influences as a means of variable selection. Further methodology is developed which aims to optimise the bias-variance tradeoff by means of neural network model estimation algorithms which combine elements of neural and statistical modelling.

The third and final component of the methodology has the objective of controlling the effects of both model and asset risk through the use of model combination techniques. The fact that the *future* performance of models cannot be known for certain is used to motivate firstly a "portfolio of models" approach, and secondly a population-based algorithm which encourages diversification between the information sets upon which individual models are based.

Earlier versions of the methodology described in this thesis formed the basis of the modelling work in equity and fixed-income markets which was conducted in the ESPRIT project "High-frequency Arbitrage Trading", (HAT) and have been described in Burgess (1995c, 1997, 1998, 1998b) and Burgess and Refenes (1995, 1996).

## 1.4 Contributions

In addition to the development of the modelling framework as a whole, the thesis also contributes to the state of the art in a number of different ways.

Literature review and evaluation: the thesis presents a detailed review of techniques for modelling and forecasting noisy and nonstationary time-series, and evaluates their strengths and weaknesses in the context of financial forecasting.

Development of novel modelling algorithms: the thesis presents novel developments and algorithms which address the methodological gaps identified in the development of the statistical arbitrage framework. Subdivided by the stage of the overall framework to which they refer, the major developments are:

Stage 1

- generalisation of the concept of cointegration to include statistical mispricing between sets of asset prices; use of cointegration framework for constructing statistical mispricings; extensions to deal with time-varying relationships and high-dimensional datasets;

- development of novel variance ratio tests for potential predictability in cointegration residuals; evaluation of empirical distribution of these tests using monte-carlo simulations; corrections for bias induced by sample-size, dimensionality, and stepwise selection;

- development of novel “implicit statistical arbitrage” rules for trading mean-reversion in combinations of asset prices.

Stage 2

- development of statistical tests for selecting the subset of variables which should be used as the input variables to a neural network model. The methodology is based upon non-parametric statistical tests and is related to neural networks by means of an “equivalent kernels” perspective;

- novel statistical methodology for neural model estimation, encompassing both variable and architecture selection, based upon degrees of freedom calculation for neural networks and analysis of variance testing; monte-carlo comparative analysis of the effect of noise and spurious variables on constructive, deconstructive and regularisation-based neural estimation algorithms;

- formulation of “conditional statistical arbitrage models”, based on mispricing-correction models (MCMs) which are a generalisation of traditional error-correction models (ECMs), and which have the ability to capture both nonlinear dependencies and interaction effects in the mispricing dynamics.

Stage 3

- novel analysis of the risks involved in the model selection process, in particular selection risk, criterion risk, and the inefficiency of multi-stage optimisation procedures;

- development of the “portfolio of models” approach, based upon a synthesis of evolutionary and genetic optimisation and portfolio theory, which is a novel model combination procedure that explicitly incorporates the trade-off between the two objectives of profit maximisation and risk minimisation;

- development of a population-based algorithm for joint optimisation both of models within a portfolio, and of model components within a compound model; novel approach in which fitness of models is conditioned upon remainder of population in order to actively encourage diversification within the portfolio of models.

Simulations and real-world applications*:* The accuracy and value of the methodology presented in the thesis are rigorously validated in artificial simulations using data with known properties. The methodologies are also applied to real-world data sets, in particular the daily closing prices of FTSE 100 stocks over the period 1997-1999, and are shown to result in significant opportunities for statistical arbitrage. Specific real-world experiments include:

- evaluation of implicit statistical arbitrage models for FTSE 100 constituents;
- evaluation of adaptive statistical arbitrage models for German Dax and French Cac stock market indices;
- evaluation of conditional statistical arbitrage models for FTSE 100 constituents;
- evaluation of "portfolio of models" approach to trading statistical arbitrage models of FTSE 100 constituents;
- evaluation of population-based algorithm for joint optimisation of a portfolio of equity index models.

In general, the methodology and ideas presented in the thesis serve to place machine learning techniques in the context of established techniques in statistics, econometrics and engineering control, and to demonstrate the synergies which can be achieved by combining elements from these traditionally disparate areas. The methodologies which are described serve to broaden the range of problems which can successfully be addressed, by harnessing together the "best" ideas from a number of fields.

## 1.5 Organisation

The thesis is divided into five parts.

The first part of the thesis consists of a brief introduction, followed by an extensive review and analysis of computational modelling techniques from a statistical arbitrage perspective and an outline of the methodological developments described in the remainder of the thesis. The subsequent three parts of the thesis describe the three components which together comprise our methodology for statistical arbitrage: Part I describes the cointegration framework which is used to identify and construct statistical mispricings, Part II describes the neural network methodology which is used to forecast the mispricing dynamics, and Part III describes the use of model combination approaches as a means of diversifying both model and asset risk. The final part consists of our conclusions and bibliography.

The organisation of the chapters within the thesis is given below.

**Introduction:**

Chapter 1 consists of a brief introduction which outlines the scope, motivation and organisation of the thesis as well as summarising the main contributions which it makes to the state of the art.

Chapter 2 presents a review of recent developments in computational modelling which are relevant to statistical arbitrage modelling, particularly concentrating on particular aspects of time-series modelling, econometrics, machine learning and computational finance.

Chapter 3 assesses the opportunities for statistical arbitrage which are presented by the advances in computational modelling, assessing the strengths and weaknesses of existing modelling techniques and highlighting the outstanding issues which the methodology in the thesis is designed to address.

Chapter 4 presents an overview of our methodology, and a "route map" to the rest of the thesis.

**Part I: A Cointegration Framework for Statistical Arbitrage**

Chapter 5 introduces the cointegration framework which is used to generate potential statistical mispricings, and describes extensions for dealing with time-varying and high-dimensional cases.

Chapter 6 describes a range of predictability tests which are designed to identify potential predictability in the mispricing time-series. Novel tests based upon the variance ratio approach are described and evaluated against standard tests by means of Monte-Carlo simulation.

Chapter 7 describes an empirical evaluation of the first part of the framework, in the form of a test of a set of “implicit statistical arbitrage” models - so-called because there is no explicit forecasting model at this stage. The high-dimensional version of the methodology is evaluated with respect to models of the daily closing prices of FTSE 100 stocks and the adaptive version of the methodology is evaluated on the French CAC 40 and German DAX 30 stock market indices.

**Part II: Forecasting the Mispricing Dynamics using Neural Networks.**

Chapter 8 examines the factors which contribute to forecasting error and characterises Neural Networks as a “low bias” modelling methodology. An “equivalent kernels” perspective is used to highlight the similarities between neural modelling and recent developments in non-parametric statistics.

Chapter 9 describes a methodology in which non-parametric statistical tests are used to pre-select the variables which should be included in the neural modelling procedure. The tests are capable of identifying both nonlinear dependencies and interaction effects and thus avoid discarding variables which would be rejected by standard linear tests.

Chapter 10 describes a statistical methodology for automatically optimising the specification of neural network forecasting models. The basic approach combines aspects of both variable selection and architecture selection, is based on the significance tests of the previous chapter and is used as the basis of both constructive and deconstructive algorithms.

Chapter 11 describes an empirical evaluation of the first two stages of the methodology used in combination. The neural model estimation algorithms are used to generate conditional statistical arbitrage models in which both exogenous and time-series effects can be captured without being explicitly prespecified by the modeller. The resulting models are evaluated on daily closing price data for the constituents of the FTSE 100 index.

**Part III: Diversifying Risk by Combining a Portfolio of Statistical Arbitrage Models**

Chapter 12 describes a “portfolio of models” approach which avoids the need for model selection and thus reduces the risk of relying on a single model. The methodology is evaluated by applying it to the set of conditional statistical arbitrage models from Chapter 11.

Chapter 13 describes a population-based algorithm which explicitly encourages decorrelation within the portfolio of models, thus generating a set of complementary models and enhancing the opportunities for risk-diversification.

**Conclusions and Bibliography**

Chapter 14 describes the conclusions of the thesis and discusses avenues for future developments.

A list of references is included following the conclusions in Chapter 14.

## 1.6 Summary

In this chapter we have presented a brief introduction which outlines the scope, motivation and organisation of the thesis as well as summarising the major contributions which it makes to the current state of the art. In the following chapter we present the background to the methodological developments which are described in the thesis, introducing the concept of statistical arbitrage and describing the recent advances in computational modelling and computational finance which provide the background to the thesis.

# *2. Background*

In this chapter we present the background to our methodology. The first section develops the concept of "statistical arbitrage" as a means of motivating the use of predictive modelling in investment finance and outlines the modelling challenges which are involved. The second section reviews the recent developments in computational modelling which form the basis of our methodology for statistical arbitrage. The third section reviews other applications of these techniques in the area of computational finance.

## 2.1 Statistical Arbitrage

In this section we develop the concept of "statistical arbitrage" as a generalisation of more traditional "riskless" arbitrage which motivates the use of predictive modelling within investment finance. We first review the basic concepts of traditional "riskless" arbitrage, in which relationships between financial asset prices are theoretically motivated, noting that practical implementation of arbitrage strategies will necessarily lead to at least a small amount of risk-taking. We then extend the arbitrage concept to cover the more general situation of "statistical arbitrage", which attempts to exploit small but consistent regularities in asset price dynamics through the use of a suitable framework for predictive modelling, and outline the requirements that such a framework should fulfil.

### 2.1.1 The Arbitrage Perspective

As mentioned in the motivation section, the philosophy espoused in this thesis is that recent improvements in computational modelling techniques open up new opportunities in financial forecasting, and that this is particularly the case in the area of "arbitrage" – namely identifying and exploiting regularities or "patterns" in asset price dynamics[1].

The view that market prices do not *automatically* reflect all currently available information, but do so only to the extent to which this information is firstly recognised, and secondly acted upon by participants in the markets might be termed the "relative efficiency" hypothesis. In

[1] see Weisweiller (1986) and Wong (1993) for overviews of arbitrage strategies

cases where regularities in market prices can be identified, they will attract the attention of speculators, in this case more properly termed "arbitrageurs". Arbitrage acts as an error-correction, or negative feedback, effect in that the buying and selling activity of arbitrageurs will tend to eliminate (or "arbitrage away") the very regularities (or "arbitrage opportunities") which the arbitrageur is attempting to exploit.

Consider the hypothetical case where the price of a given asset is known to rise on a particular day of the week, say Friday. Arbitrageurs would then tend to buy the asset on Thursdays (thus raising the price) and "lock in" a profit by selling at the close of business on Fridays (thus lowering the price), with the net effect being to move some of Friday's price rise to Thursday. However, the cleverest amongst the arbitrageurs would spot this new opportunity and tend to buy earlier, on Wednesdays, and the process would continue until no one single day would exhibit a significant tendency for price rises.

Thus the competitive essence of financial markets entails that predictable regularities in asset price dynamics are difficult to identify, and in fact it is largely the action of arbitrageurs (in the broadest sense) that ensures that market prices are as "efficient" as they are at reflecting all available information. In a general sense, market efficiency can be seen to equate to market (almost-)unpredictability. The fact that arbitrageurs can be considered as living "at the margins" of the markets explains the motivation behind the continual search for new arbitrage opportunities.

The ideal arbitrage strategy is one which generates positive profits, at zero risk, and requires zero financing. Perhaps surprisingly, such opportunities do exist in financial markets, even if never quite in the ideal form. The following section describes the general structure of strategies for "riskless arbitrage" and provides a particular example of such a strategy in the case of the UK equity market.

### 2.1.2 Riskless Arbitrage

The basic concept of riskless arbitrage is very simple: if the future cash-flows of an asset can be replicated by a combination of other assets then the price of forming the replicating portfolio should be approximately the same as the price of the original asset. More specifically, in an efficient market there will exist no riskless arbitrage opportunities which allow traders to obtain profits by buying and selling equivalent assets at prices that differ by more than the "transaction costs" involved in making the trades. Thus the no-arbitrage condition can be represented in a general form as:

$$\left|\, \text{payoff}\left(X_t - SA\left(X_t\right)\right) \right| < TransactionCost \tag{2.1}$$

where $X_t$ is an arbitrary asset (or combination of assets), $SA\left(X_t\right)$ is a "synthetic asset" which is constructed to replicate the payoff of $X_t$ and $TransactionCost$ represents the net costs involved in constructing (buying) the synthetic asset and selling the "underlying" $X_t$ (or vice versa). This general relationship forms the basis of the "no-arbitrage" pricing approach used in the pricing of financial "derivatives" such as options, forwards and futures[2]; the key idea being that the price of the derivative can be obtained by calculating the cost of the appropriate replicating portfolio (or "synthetic asset" in our terminology). We refer to the deviation $X_t - SA\left(X_t\right)$ as the *mispricing* between the two (sets of) assets.

Although differing significantly in detail, there is a common structure to all riskless arbitrage strategies. Riskless arbitrage strategies can be broken down into the following three components:

- construction of fair-price relationships between assets (through theoretical derivation of riskless hedging (replication) portfolio);
- identification of specific arbitrage opportunities (when prices deviate from fair price relationship);
- implementation of appropriate "lock in" transactions (buy the underpriced asset(s), sell the overpriced asset(s)).

A specific example of riskless arbitrage is index arbitrage in the UK equities market. Index arbitrage (see for example Hull (1993)) occurs between the equities constituting a particular market index, and the associated futures contract on the index itself. Typically the futures contract $F_t$ will be defined so as to pay a value equal to the level of the index at some future "expiration date" $T$. Denoting the current (spot) stock prices as $S_t^i$, the no-arbitrage relationship, specialising the general case in Eqn. (2.1), is given by:

$$\left| F_t - \sum_i w_i S_t^i e^{(r-q_i)(T-t)} \right| < Cost \tag{2.2}$$

where $w_i$ is the weight of stock $i$ in determining the market index, $r$ is the riskfree interest rate, and $q_i$ is the dividend rate for stock $i$. In the context of Eqn. (2.1) the weighted combination of constituent equities can be considered as the synthetic asset which replicates the index futures contract.

The strategy based on Eqn. (2.2.) involves monitoring the so-called "basis" $F_t - \sum_i w_i S_t^i e^{(r-q_i)(T-t)}$, which represents the deviation from the fair-price relationship. When the basis exceeds the transaction costs of a particular trader, the arbitrageur can "lock in" a riskless profit by selling the (overpriced) futures contract $F_t$ and buying the (underpriced) combination of constituent equities. In the converse case, where the negative value of the basis exceeds the transaction costs, the arbitrageur would *buy* the futures contract and *sell* the combination of equities.

The introduction of the SETS order-driven system for the London market has made index arbitrage activity particularly easy to identify because it results in significant discontinuities in the FTSE 100 index when all constituents are traded simultaneously, as can be seen in Figure 2.1. When the magnitude of the mispricing between the spot and future grows, there are frequently large corrections in the basis which are caused by index arbitrage activity.

[2] see Hull (1993) for a good introduction to derivative securities and standard no-arbitrage relationships

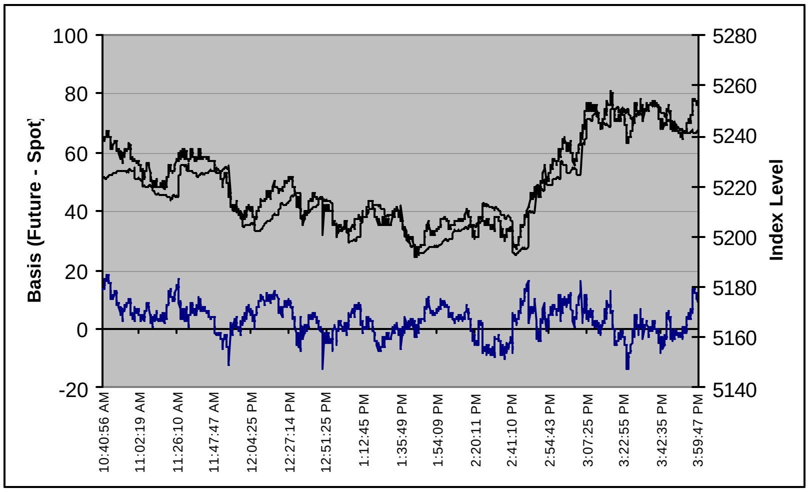


Figure 2.1: Illustration of index arbitrage opportunities in the UK equity market; the data consists of 3200 prices for the FTSE 100 Index (in bold) and the derivative futures contract expiring Sept 98; the lower curve shows the so-called "basis", the deviation from the theoretical fair price relationship between the two series; the data sample covers the period from 10.40am to 4pm on September 15th 1998.

Riskless (or near-riskless) arbitrage is clearly an important subject in its own right, and many complex arbitrage relationships exist, particularly with the recent growth of financial derivatives such as options, futures, swaps and exotic options. However such strategies are inherently self-limiting - as competition amongst arbitrageurs grows, the magnitude and duration of mispricings decreases. As the profits available through arbitrage decrease, the amount of capital employed in order to achieve significant profits must increase. Furthermore only the arbitrageurs who are geared to trade very rapidly and at low transaction costs will be in a position to achieve arbitrage profits. For both of these reasons large banks and other privileged financial institutions tend to dominate arbitrage activity.

In practice, even arbitrage which is technically "riskless" will still involve a certain level of risk. This risk is introduced by numerous factors: uncertain future dividend rates $q_i$; market volatility during the short time required to carry out the lock-in trades (slippage); failure to "fill" all legs of the trade, thus leaving a residual "unhedged" risk. A very important source of risk in arbitrage activity is "basis risk" caused by fluctuations in the difference between spot and futures prices prior to the expiration date.

"Basis risk" is indeed a source of risk, particularly when Exchange regulations or internal accounting practices require that positions be regularly "marked to market" at current prices. However, it is also a source of opportunity in that favourable fluctuations may move the basis *beyond* the fair price value (i.e. from positive to negative, or vice versa) and thus allow an arbitrageur to realise higher profits by reversing the trade at some point prior to expiration.

Thus practical arbitrage strategies are much more complex than may at first appear. In fact, the majority of arbitrage strategies are at least implicitly reliant on the statistical properties of the “mispricing” or deviation from the fair price relationship. From this perspective the attraction of index arbitrage strategies lies in a favourable property of the mispricing dynamics – namely a tendency for the basis risk to “mean revert” or fluctuate around a stable level. In the following section we describe how this recognition, that practical arbitrage strategies both involve risk and rely upon favourable statistical properties of the mispricing dynamics, leads to a more general class of arbitrage strategies, namely “statistical arbitrage”.

### 2.1.3 Elements of Statistical Arbitrage

The discussion above shows that from a statistical perspective, the mispricing time-series can be considered as a “synthetic asset” which exhibits strong mean-reversion, and hence a certain degree of potentially predictable behaviour. The premise of so-called “statistical arbitrage” (see for example Wong (1993)) is that, in many cases, statistical regularities in combinations of asset prices can be exploited as the basis of profitable trading strategies, irrespective of the presence or absence of a theoretical fair-price relationship between the set of assets involved.

Whilst clearly subject to a higher degree of risk than “true” arbitrage strategies, such statistical arbitrage opportunities are likely to be both more persistent and more prevalent in financial markets. More persistent because risk-free arbitrage opportunities are rapidly eliminated by market activity. More prevalent because in principle they may occur between any set of assets rather than solely in cases where a suitable risk-free hedging strategy can be implemented. An example of a set of assets which present possible opportunities for statistical arbitrage are shown in Figure 2.2 below.

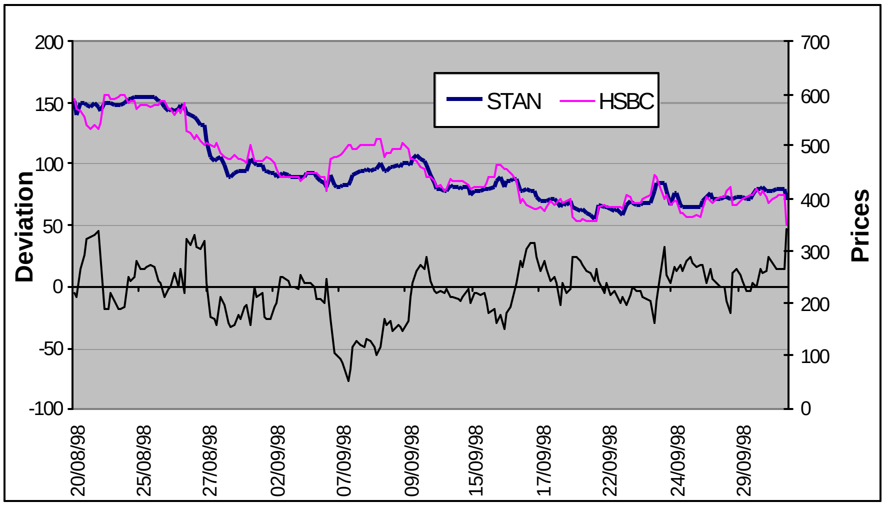


Figure 2.2: Illustration of potential statistical arbitrage opportunities in the UK equity market; the chart shows equity prices for Standard Chartered and HSBC, sampled on an hourly basis from August 20th to September 30th, 1998.

Comparing the illustration of potential statistical arbitrage opportunities in Figure 2.2, to the riskless arbitrage opportunities shown in Figure 2.1, there is clearly a marked similarity. In both cases the two prices "move together" in the long term, with temporary deviations from the long term correlation which exhibit a strong mean-reversion pattern. Note however that in the "statistical arbitrage" case the magnitude of the deviations is greater (around +/- 10% as opposed to <0.5%) and so is the time-period over which the price corrections occur (days or weeks as opposed to seconds or minutes).

From this perspective, we consider statistical arbitrage as a generalised case of riskless arbitrage in which the relative returns, $payoff\left(X_t - SA(X_t)\right)$, are no longer completely hedged (i.e. immunised) against the underlying sources of economic uncertainty (or "risk factors") which drive asset price movements in general. Given a set of economic variables, assets or portfolios of assets which correspond to risk factors $F = \left\{F_1, \ldots, F_{n_F}\right\}$ we can quantify the extent to which risk is introduced to the "statistical mispricing" portfolio $X_t - SA(X_t, w)$ in terms of the set of sensitivities $s = \left\{s_1, \ldots, s_{n_F}\right\}$ where $s_i = \dfrac{d\,\mathrm{E}\left[payoff\left(X_t - SA_t(X, w)\right)\right]}{dF_i}$ and $w$ are the parameters which define the synthetic asset. Thus in the case of riskless arbitrage, the future relative return,

$payoff\left(X_t - SA(X_t, w)\right)$, is dependent purely on the magnitude of the mispricing $X_t - SA(X_t, w)$ and the sensitivity to all risk factors is zero, i.e. $\forall_i : s_i = 0$.

In a case where an appropriate set of risk factors can be pre-defined, perhaps the most natural approach to developing statistical arbitrage strategies is to introduce risk in a controlled manner through specifying a desired set of factor sensitivities *s*. This is the approach taken in Burgess (1996), in which the "risk factors" common to a set of money-market futures contracts are statistically determined through the use of Principal Component Analysis (PCA) [Jolliffe, 1986] of the returns (price changes) in the different contracts. The synthetic asset $SA(X_t, w, F, s)$ is then defined as the combination of assets which, together with a particular asset (or portfolio) $X_t$ exhibits a specified risk profile *s* with respect to the set of risk factors $F$. In the case of Burgess (1996) the risk factors are defined by the principal components of the set of asset returns and the sensitivity profiles are of the form $s_j = 1, \forall_{i \neq j} : s_i = 0$, i.e. portfolios defined to have unit exposure to a selected factor *j* whilst immunised to all other factors (a "factor bet" on factor *j*).

However, whilst this approach is sufficient to control the degree of risk exposure which is introduced into the deviation time-series, it does not directly support the identification of mispricing time-series which contain mean-reverting or other (potentially nonlinear) deterministic components in their dynamics. This is because a low (or otherwise controlled) exposure to risk factors does not in itself imply the presence of a non-random component in the dynamics. For instance, Burgess (1996) reports a situation where the third principle component portfolio (*j*=3) shows evidence of being predictable with respect to nonlinear but not linear techniques and the first two principle component portfolios (*j*=1,2) show no evidence of either mean-reversion or other non-random behaviour.

In comparison to the "explicit factor modelling" approach described above, the methodology described in this thesis is based upon an alternative approach to the construction of synthetic assets. In this approach, the introduction of risk is treated not from a risk factor perspective but instead in an aggregate manner in which it is the "tracking error" between the asset or portfolio $X_t$ and the synthetic asset $SA(X_t, w)$ which is of concern. The parameters of the synthetic asset are taken to be those, $w^*$, which minimise the variance of the price deviation series, i.e. $w^* = \underset{w}{\arg\min}\ \mathrm{var}\left(X_t - SA_t(X, w)\right)$.

In the limiting case of perfect tracking, where $\mathrm{var}\left(X_t - SA_t(X, w^*)\right) = 0$, then the factor exposures (to *any* set of factors) are also known for certain in that they must all be zero, i.e. $\forall_i : s_i = 0$. In the more general case, where $\mathrm{var}\left(X_t - SA_t(X, w^*)\right) > 0$, the factor sensitivities $s_i$ are only *implicitly* defined in that the minimisation of residual price variance (with respect to the synthetic asset parameters $w$) will tend to minimise the *aggregate* exposure to the risk factors, without imposing any specific constraints upon permissible values of the *individual* sensitivities. As discussed in Chapter 5, this additional flexibility has the advantage of encouraging exposure to those risk factors (be they market wide or specific to an individual asset) which contain a non-random (or more specifically, mean-reverting) component. Furthermore, as we discuss in Chapter 3, a suitable set of econometric tools, upon which to base a methodology for identifying models of this type, already exist in the form of tools for modelling *cointegration* relationships within sets of time-series.

Thus in an analogous manner to that in which riskless arbitrage strategies were earlier decomposed into three components, the three components required for statistical arbitrage models can be considered as:

- construction of statistical fair-price relationships between assets such that the deviations or "statistical mispricings" have a potentially predictable component (through time-series analysis of historical asset price movements);

- identification of statistical arbitrage opportunities (forecasting of changes in <u>combinations</u> of asset prices);

- implementation of appropriate trading strategy (buy assets which are <u>forecasted</u> to outperform, sell assets which are <u>forecasted</u> to underperform).

These three components correspond to the three stages of our statistical arbitrage methodology, which will be described in detail in the later parts of the thesis. The statistical arbitrage equivalent of the no-arbitrage condition in Eqn. (2.1) can be expressed as:

$$\left| \mathrm{E}\left[ payoff\left(X_t - SA\left(X_t\right)\right)\right] \right| < TransactionCost \tag{2.3}$$

where $\mathrm{E}[\ ]$ is the expectation operator. In the context of Eqn (2.3), the challenge for computational modelling becomes clear: firstly, given an asset (or portfolio) $X_t$ to identify an appropriate combination of assets to form the corresponding statistical hedge, or "synthetic asset", $SA(X_t)$; secondly to create models capable of predicting the changes in the "statistical mispricing" between the two portfolios, $\mathrm{E}\left[ payoff\left( X_t - SA(X_t)\right)\right]$; thirdly to construct a trading strategy which is capable of exploiting the predictive information in a manner which overcomes the transaction costs. These three tasks correspond both to the general requirements listed above and also to the three stages of our methodology for statistical arbitrage. Before moving on in the rest of the thesis to describe the details of the methodology, the remainder of this chapter presents a review of recent advances in computational modelling which form the basis upon which our methodology has been developed.

## 2.2 Recent Advances in Computational Modelling

In this section we review the recent advances in computational modelling which comprise the platform upon which our statistical arbitrage methodology is based. The review brings together various strands of work from a number of academic disciplines, including time series forecasting, statistical modelling, econometrics, machine learning and computational finance.

### 2.2.1 Time Series Representation

In this section we review alternative methods for representing and modelling time-series. Perhaps the two key complications in modelling financial time-series are the inherent properties of "**noise**" and "**nonstationarity**" and to a large extent these properties are determined by the choice of time-series representation which is adopted. In recognition of the importance which the choice of problem representation can have in determining the likely success of any subsequent modelling or forecasting procedure, researchers in forecasting and econometrics have developed a number of modelling frameworks which are designed to exploit time-series with different types of characteristics.

The choice of target or "dependent" variable is perhaps the single most important decision in the entire modelling process. We can consider any random variable $y_t$ as being the result of a data-generating process which is partially deterministic and partially stochastic:

$$y_t = g(\mathbf{z}_t) + e(\mathbf{v}_t)_t \tag{2.4}$$

where the deterministic component is a function ‘g’ of the vector of influential variables $\mathbf{z}_t = \left\{ z_{1,t}, z_{2,t}, \ldots, z_{n_Z,t} \right\}$ and the stochastic component is drawn from a distribution ‘ε’ which may vary as a function of variables $\mathbf{v}_t = \left\{ v_{1,t}, v_{2,t}, \ldots, v_{n_V,t} \right\}$. Under the assumptions that the stochastic term is homoskedastic and normally distributed with zero mean and variance $s_y^2$, the data-generating process reduces to:

$$y_t = g(\mathbf{z}_t) + \mathrm{N}(0, s_y^2) \qquad (2.5)$$

The choice as the target or “dependent” variable implicitly limits the maximum possible performance of a forecasting procedure to the proportion of the variance of $y_t$ which is due to the deterministic and hence potentially predictable component:

$$R^2(y_t, f(\mathbf{x_t})) =_{def} 1 - \frac{\mathrm{E}\left[\left(f(\mathbf{x}_t) - y_t\right)^2\right]}{\mathrm{E}\left[\left(g(\mathbf{z}_t) - m_y\right)^2\right] + s_y^2} \leq 1 - \frac{s_y^2}{\mathrm{E}\left[\left(g(\mathbf{z}_t) - m_y\right)^2\right] + s_y^2} = 1 - n \qquad (2.6)$$

where the performance measure $R^2\left(y_t, f(\mathbf{x_t})\right)$ is the proportion of variance (correctly) absorbed by the model and $n$ is the “noise content” of the time-series. Note that the degree to which the inequality can approach the equality is dependent on how closely the forecasting model $f(\mathbf{x_t})$ approximates the true deterministic component $g(\mathbf{z}_t)$ - an issue which is returned to later in this review.

In many forecasting applications, the issue of representation is overlooked, largely because the target series is often taken for granted. However in an application such as statistical arbitrage there are many possible combinations of assets which could be used as target series, and identifying combinations of time-series with the greatest possible potentially predictable component is a key objective of the first stage of our statistical arbitrage methodology.

The second key property which is determined by the adopted representation is that of “nonstationarity”, or time-variation in the statistical properties. At some level, all forecasting techniques are utterly reliant on the assumption that the future will be related to the past. This assumption is typically phrased as one of **stationarity**, or stability of the statistical properties of the system over time. Some common technical conditions for stationarity of a single time-series are presented below:

$$E[\ y_t\ ] = \mu \qquad (C.1)$$

$$E[\ (y_t - \mu)^2\ ] = \sigma_y^2 = \gamma(0) \qquad (C.2)$$

$$E[\ (y_t - \mu)(y_{t-\tau} - \mu)\ ] = \gamma(\tau) \qquad \tau = 1, 2, .... \qquad (C.3)$$

$$pdf(\ y_t\ , y_{t+1}\ , y_{t+2}\ , y_{t+3}\ , \ldots\ ) = pdf(\ y_{t+\tau}\ , y_{t+\tau+1}\ , y_{t+\tau+2}\ , y_{t+\tau+3}\ , \ldots\ ) \qquad (C.4)$$

Conditions (C.1-C.3) relate in turn to stationarity of the mean, variance and auto-covariances of the series. The fourth condition for "strict stationarity" requires that the entire joint distribution of a sequence of observations be independent of the reference time from which they are taken. A fuller treatment of these issues can be found in any time-series forecasting textbook, for instance (Harvey, 1993).

The observation that successful application of statistical modelling tools is often dependent upon the assumption of stationarity is easily illustrated; consider a time-series which exhibits non-stationarity in mean, i.e. violates condition (C.1), as shown in Figure 2.3. Within an autoregressive modelling framework, where lagged values of the series are used as the basis for future forecasts, not only would forecasting involve the model being queried in a region where it had seen no previous examples, but it would also be expected to give a previously unseen response.

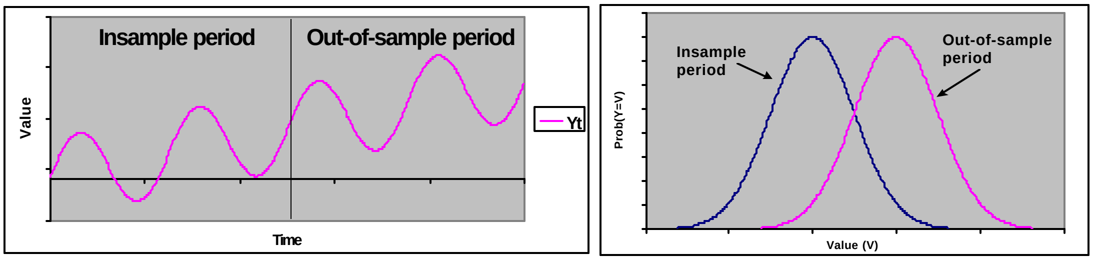


Figure 2.3: An example time-series which shows non-stationarity in mean; the left hand chart shows the time series $y_t$; the right hand chart illustrates the fact that the idealised probability distributions differ during the in-sample and out-of-sample periods.

In the more general case, most types of model are more suited to tasks of *interpolation* (queries within the range of past data) rather than *extrapolation* (queries outside the range of known data) and thus the effect of nonstationarity is to cause a degradation in the performance of the associated model. The most common solution to the problems posed by nonstationarity is to attempt to transform the representation of the data to achieve weak stationarity (Spanos, 1986), i.e. satisfaction of (C.1) and (C.2).

Figure 2.4 illustrates different classes of time-series from the viewpoint of the transformations that are required to achieve stationarity.

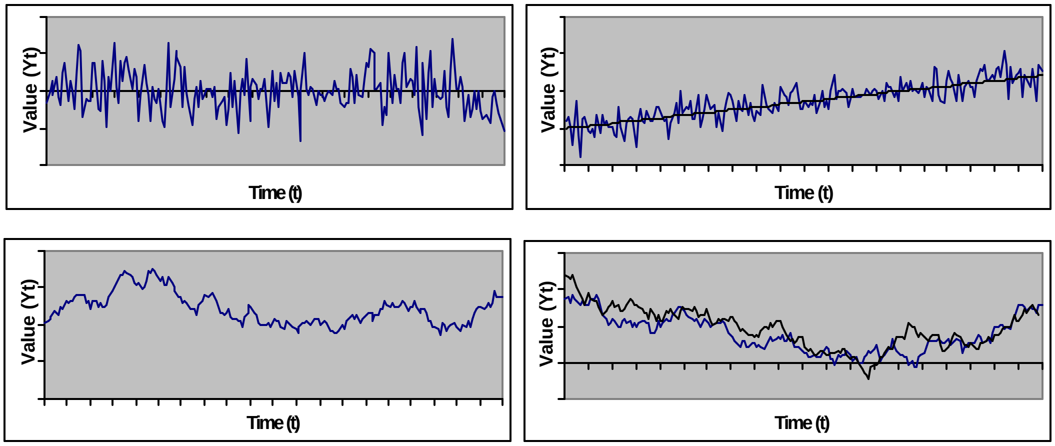


Figure 2.4: Time-series with different characteristics, particularly with regard to stationarity: (top left) stationary time-series; (top right) trend-stationary time-series; (bottom left) integrated time-series; (bottom left) cointegrated time-series

A stationary series, such as that shown in the top-left chart of Figure 2.4, is a suitable candidate for direct inclusion as either a dependent or independent variable in a forecasting model without creating a risk of extrapolation.

The chart in the top-right of Figure 2.4 is an example of a "trend stationary" variable, that is it is stationary around a known trend which is a deterministic function of time. A stationary representation of such a variable can be obtained by "de-trending" the variable relative to the known deterministic trend

Series such as that in the bottom left chart of Figure 2.4 are known as "integrated series" because they can be viewed as the integration (sum) of a stationary time-series. If the first-differences $\left(\Delta y_t = y_{t+1} - y_t\right)$ of the series are stationary then the series is integrated of order 1, i.e. y ~ I(1) and may also be referred to as being "difference stationary". From this perspective, a stationary time-series is referred to as being integrated of order zero, or I(0). In recent years there has been a growing recognition that time-series may be neither completely stationary, i.e. I(0), nor completely nonstationary, i.e. I(1), but instead may be *fractionally integrated*, i.e. most typically I(d) where 0<d<1 (Granger and Joyeux, 1980).

The two series in the bottom-right chart of Figure 2.4 represent a so-called "cointegrated set" of variables. The concept of cointegration, initially proposed over 15 years ago by Granger

(1983), has attracted much interest in recent years. As the name suggests, it is closely related to the concept of *integration* but refers to a relationship between a number of time-series rather than a property of an individual time-series. If two or more time-series are integrated of order $d$ and yet a linear combination of the time-series exists which is integrated of order $b < d$, then the time-series are said to be cointegrated of order $d, b;$ following the terminology of Harvey (1993), the vector $\mathbf{y}_t \sim \mathrm{CI}(d,b)$ iff

all components of $\mathbf{y}_t$ are I(d); and

there exists a non-null vector, a **,** such that $a^{\mathbf{T}}\mathbf{y}$ is I($d$-$b$) with $b > 0$

A wide range of methods have been proposed both for estimating the "cointegrating vector" a including Ordinary Least Squares (OLS) regression [Engle (1993), (Engle and Granger, 1987)], Augmented Least Squares (with leads and lagged terms, possibly nonlinear), (Stock, 1987), Canonical Cointegration (Park, 1989), Principal Components [Stock and Watson (1989); Phillips and Ouliaris (1988)], Three-Step estimation (Engle and Yoo, 1991), and Differenced Vector Auto-Regression (Johansen, 1988, 1991). The most popular method of testing for cointegration (see Section 2.2.2) is that introduced by Granger (1983) and is based upon the concept of a "cointegrating regression". In this approach a particular time-series (the "target series") $y_{0,t}$ is regressed upon the remainder of the set of time-series (the "cointegrating series") $y_{1,t},\ldots y_{n,t}$:

$$y_{0,t} = a + b_1 y_{1,t} + b_2 y_{2,t} \ldots\ldots + b_n y_{n,t} + d_t \tag{2.7}$$

if the series are cointegrated then the "deviation" term $d_t$ will be a stationary variable. The usual motivation for this approach is that stationary time-series tend to have smaller variance than nonstationary time-series; thus the OLS regression procedure of minimising the variance of the residuals will tend to maximise the probability of identifying any cointegration between the set of time-series. As a means of creating combinations of time-series which are statistically "well behaved", the concept of cointegration will serve as an important motivation for the first part of our statistical arbitrage methodology.

The class into which a time-series or set of time-series fall, whether stationary, mean-nonstationary (integrated), or cointegrated, has important implications both for the modelling approach which should be adopted and the nature of any potentially predictable components that the time-series may contain. Having characterised different types of time-series by the transformations which must be performed upon them to attain stationarity, we discuss in the next section a range of statistical tests which serve one of two purposes: firstly to identify the type of the time-series (stationary, nonstationary, cointegrated) and secondly, having established the type of the time-series, to identify the presence of any potentially predictable component in the time-series dynamics.

## 2.2.2 Time-series Identification

In this section we consider four main categories of time-series identification tests, based on the type of behaviour which they are intended to identify, namely autoregressive tests, unit root tests, variance ratio tests and cointegration tests. Autoregressive tests are designed to be applied to stationary time-series, and are designed to distinguish time-series which contain a predictable component from time-series which are pure noise. Unit root tests are designed to differentiate between stationary and non-stationary (integrated) time-series. Variance ratio tests are designed to distinguish random walk time-series from non-random walks. Cointegration tests are designed to identify whether a set of mean-nonstationary time-series contain one or more linear combinations which are themselves stationary (i.e. cointegration of order 1,1.)

### Identification of Autoregressive Dynamics

The first set of tests are based on the short-term dynamics of the time-series and are derived from the Auto-Correlation Function (ACF), otherwise known as the *correlogram*:

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y}) \Big/ [T - (k+1)]}{\sum_{t=1}^{T} (y_t - \bar{y})^2 \Big/ [T-1]} \tag{2.8}$$

Autocorrelations <u>significantly</u> different to zero indicate that future values of the time-series are <u>significantly</u> related to past values, and hence the presence of a predictable component in the time-series dynamics. The auto-correlation function and the related Partial Auto-Correlation Function (PACF) form the basis of standard techniques such as ARMA and ARIMA modelling. So-called ‘portmanteau’ tests such as those of Box and Pierce (1970) and Lyung and Box (1978) are designed to test the joint null hypothesis: $H_0 : r_1 = r_2 = \ldots = r_k = 0$, against the general alternative $H_1$: not all $r_j = 0$. Box-Pierce-Lyung portmanteau tests are often used as tests for non-random components, particularly as diagnostic tests applied to the residual errors of a regression or time-series model. In the context of statistical arbitrage modelling, such diagnostic tests for autoregressive behaviour in a time-series may play a useful role in identifying a potentially predictable component in the dynamics of the statistical mispricing.

### Identification of Deterministic Trends

In the case of time-series with a known deterministic trend, the standard procedure is to "de-trend" the variable by subtracting the deterministic trend and build forecasting models based upon the stationary variable which is thus created. Detrending of variables can also be achieved by including a suitable time-trend within a regression (Frisch and Waugh, 1933). However, if the time-trend is not deterministic but is actually stochastic (such as a random walk) it is likely to appear to be (spuriously) significant [Nelson and Kang (1984), Phillips (1986)] and furthermore will induce apparent periodicity in the de-trended series (Nelson and Kang, 1981). In practice, even financial time-series which appear to contain a trend are generally treated as stochastically trending (i.e. integrated) variables in order to avoid the risk of inducing spurious periodicity. For similar reasons, the deterministic detrending approach is not considered further in this thesis.

### Tests for Stationarity

Stationarity tests are used to differentiate between integrated, (mean-)nonstationary, time-series and stationary time-series. The most common such test is the Dickey-Fuller test which is performed by regressing the one-step-ahead-values of a time-series on the current value:

$$y_{t+1} = a + by_t + e_t \tag{2.9}$$

If the estimated coefficient $b$ is not significantly less than one, then the null hypothesis of nonstationarity is not rejected and the time-series should be transformed by differencing before being modelled. The fact that the null hypothesis of nonstationarity implies a $b$ of one is the reason why this test is often referred to as a "unit root" test. The regression may be extended by including autoregressive terms which absorb any short-term dynamics in the differenced series, the differenced form of this Augmented-Dickey-Fuller or ADF test thus involves the following regression:

$$y_{t+1} - y_t = \Delta y_t = a + b y_t + \sum_{j=1}^{p-1} b_j \Delta y_{t-j} + e_t \qquad (2.10)$$

and nonstationarity is rejected only if the $b$ coefficient is significantly less than zero. The test-statistic which is used for the DF or ADF test in this form is the t-statistic associated with the $b$ coefficient. Under the null hypothesis of nonstationarity the t-statistic follows a non-standard distribution which was first studied by Dickey (1976) and Fuller (1976), and is tabulated in Fuller (1976). Alternative stationarity tests have been proposed, for example by Philips (1987), Phillips and Perron (1988) and Hall (1989), but the available evidence suggests that these do not improve upon the original tests [Schwert (1989), Pantula (1991)].

Even in cases where nonstationarity can be rejected, as is sometimes the case in some financial time-series, it is still common to difference the data before modelling. This serves two purposes, firstly the quantity of interest is more often *returns* (price changes) of financial assets as opposed to the *price levels*, secondly it serves to reduce the level of autocorrelation in the resulting time-series, thus more closely approximating the assumption of independent identically distributed (i.i.d.) errors upon which test diagnostics are predicated, thereby reducing the danger of achieving spurious results.

Additionally, the DF and ADF tests can be considered as tests for a particular type of predictable component in the dynamics of a time-series. This is made clear from the form of Eqn (2.10): if the $b$ coefficient is significantly negative then this implies a significantly negative relationship between *future* changes and *current* levels, indicative of a significant mean-reversion component in the time-series.

### Tests for Deviation from Random Walk Behaviour

Whilst the "unit root" tests are intended to distinguish between stationary and nonstationary time-series, another family of tests are designed to identify the situation where a time-series deviates from a particular type of nonstationary dynamics in which increments in the time-series are completely uncorrelated:

$$y_{t+1} = y_t + e_t \tag{2.11}$$

So-called *variance ratio* tests follow from the fact that the variance of the innovations in such a "random walk" series is a linear function of the period over which the increments are measured. A simple intuition for this property is presented in Figure 2.5 below.

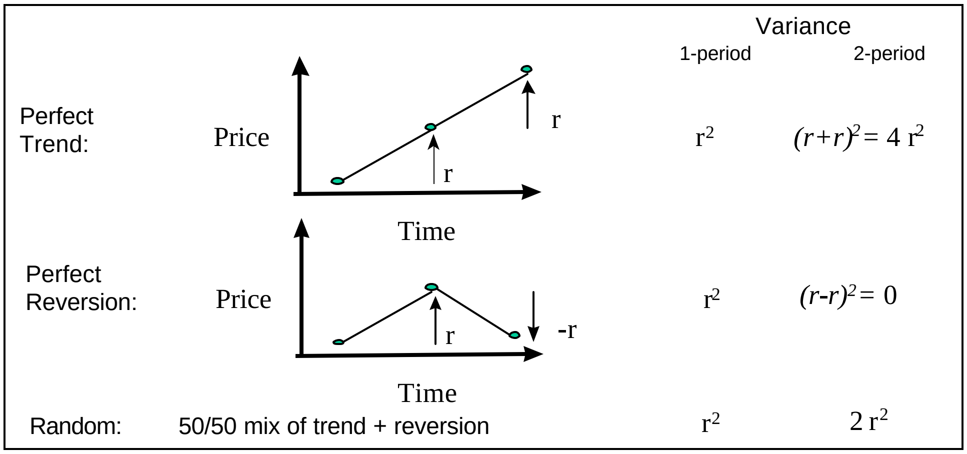


Figure 2.5: The relationship between variance and time for a simple diffusion process: in the limiting case where all steps are in the same direction the variance of the series will grow as a function of time-squared, at the other extreme of pure-reversion the variance of the series will be independent of time (and close to zero). A random diffusion will be a weighted combination of both behaviours and will exhibit variance which grows linearly with time.

The linear growth of variance as a function of the increment period has been used as the basis of statistical tests for deviations from random walk behaviour by a number of authors since Lo and McKinley (1988) and Cochrane (1988).

The variance ratio statistic can be defined as the normalised ratio of long term variance (calculated over period $\tau$) to single-period variance and is thus:

$$\mathrm{VR}(t) = \frac{\sum_t \left(\Delta^t y_t - \overline{\Delta^t y}\right)^2}{t \sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2} \tag{2.12}$$

As with autocorrelation statistics, the variance ratio statistics can be viewed collectively, thus forming the "variance ratio function" or VRF of the time-series. A positive gradient to the variance ratio function (VRF) indicates positive autocorrelation and hence trending behaviour; conversely a negative gradient to the VRF indicates negative autocorrelations and mean-reverting or cyclical behaviour. Figure 2.6 shows example time-series with different characteristics, together with their associated VRFs.

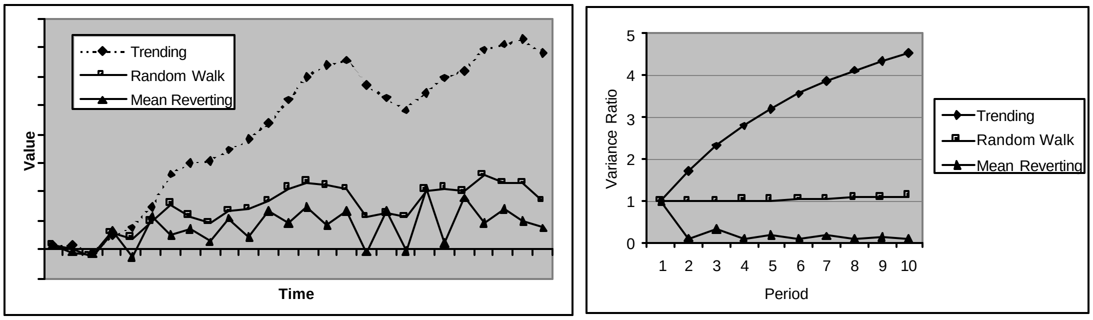


Figure 2.6: Example time-series with different characteristics (left), and their VRFs or variance ratio functions (right). For the random walk, variance grows linearly with the period $\tau$ and hence the VRF remains close to one; for a trending series the variance grows at a greater than linear rate and so the VRF rises above one as the period over which the differences are calculated increases; for the mean-reverting series the converse is true: the variance grows sublinearly and hence the VRF falls below one.

The rationale for testing for deviations from random walk behaviour is that such deviations are indications of a potentially predictable component in the dynamics of a time-series. Lo and MacKinley (1988) showed that a particular form of the variance ratio statistic is essentially equivalent to a linear combination of autocorrelation coefficients:

$$\mathrm{VR}(t) \equiv \sum_{j=1}^{q-1} \frac{2(q-j)}{q} r_j \tag{2.13}$$

where $r_j$ are the autocorrelation coefficients calculated according to Eqn (2.8). Lo and MacKinley (1989) used Monte Carlo simulation to investigate the finite sample performance of the variance ratio against different alternative hypotheses and concluded that it is more powerful than either the Dickey-Fuller or Box-Pierce tests in two of the three cases considered and equally powerful in the third case. Faust (1992) noted that the VR statistic can be considered as a specific example of a class of “filter variance ratio” statistics which consist of linear combinations of autocorrelation coefficients and that such statistics can be shown to be optimal tests for a wide class of time-series dynamics. Chow and Denning (1993) examined the performance of a multiple variance ratio test using Monte-Carlo simulations and found that the test is as reliable as the Dickey-Fuller and Phillips-Perron unit root tests when testing a random walk against an autoregressive AR(1) alternative, whilst being more powerful when testing against two nonstationary alternatives which contain a predictable component.

As a powerful means of identifying deviations from random behaviour, variance ratio tests are an important inspiration behind our methodology for testing for potentially predictable behaviour in the dynamics of statistical mispricing time-series.

### Tests for Cointegration

As noted in the previous section, cointegration is a property of a set of time-series which are individually nonstationary (integrated) but which contain one or more stationary components. The implications of cointegration for statistical arbitrage modelling are twofold. The first of these is that the presence of cointegration in a set of asset prices implies the existence of a long run relationship between the assets which could be considered as the statistical equivalent of a "fair price" relationship; deviations from this relationship could then be considered as "statistical mispricings". The second major implication of cointegration is the presence of a significant predictable component in the time-series dynamics – it already having been noted above that the property of stationarity implies a mean-reverting component in the dynamics of a time-series.

As noted in Section 2.2.1, the most popular form of cointegration testing is that introduced by Granger (1983) and is based upon the concept of a "cointegrating regression". In this approach a "target" time-series $y_{0,t}$ is regressed upon the "cointegrating series" $y_{1,t}, \dots y_{n,t}$. The presence (or absence) of cointegration is detected by testing the deviation term $d_t$ which is represented by the residual errors of the cointegrating regression:

$$d_t = y_{0,t} - \left[a + b_1 y_{1,t} + b_2 y_{2,t} \ldots\ldots + b_n y_{n,t}\right] = \mathrm{a}^{\mathrm{T}} \mathbf{y}_{\mathbf{t}} \tag{2.14}$$

If the set of variables $\mathbf{y_t} = \left\{ y_{0,t}, 1, y_{1,t}, \ldots, y_{n,t} \right\}$ are cointegrated then statistical tests will indicate that $d_t$ is stationary and the parameter vector $\mathrm{a} = \left(1, -a, -b_1, -b_2, \ldots, -b_n\right)$ is referred to as the *cointegrating vector*.

In order to correct for spurious mean-reversion which is induced by the cointegrating regression itself, the stationarity tests used in the identification of cointegrating relationships are <u>modified</u> versions of standard tests for stationarity. Many different cointegration tests have been proposed in the econometric literature, for example Dickey-Fuller and Augmented Dickey-Fuller (Said and Dickey, 1984), Cointegrating Regression Durbin Watson (Engle and

Granger, 1987), adjusted ADF (Phillips, 1987), spurious regressors (Park, Ouliaris and Choi, 1988), maximum likelihood (Johansen and Juselius, 1990). These tests reflect both the variety of estimation methods which have been proposed, and also minor variations in the null and alternative hypotheses between which the tests are designed to distinguish.

An early comparison of a range of cointegration tests was conducted by Engle and Granger (1987) who recommend two tests in particular the Dickey-Fuller (DF) and the Cointegrating Regression Durbin Watson (CRDW). Further comparisons of cointegration tests are contained in Gregory (1991) and Hargreaves (1994b), the latter conducting a Monte-Carlo comparison of different estimation methods/cointegration tests and concluding that no one technique can be said to dominate the others.

Standard tests for cointegration make a "hard" distinction between stationary and nonstationary processes, with a null hypothesis of no cointegration (nonstationarity) and an alternative hypothesis of full cointegration (stationarity)[3]. As noted in the previous section, there is now a growing recognition that time-series may be neither completely stationary nor completely nonstationary but instead may be fractionally integrated (Granger and Joyeux, 1980); in the multi-variate case this perspective has led to the development of so-called *fractional cointegration models* [Baillie and Bollerslev (1994); Dueker and Startz (1995)] with the objective of identifying linear combinations which have a lower (and potentially fractional) degree of integration than any of the (also potentially fractionally integrated) original series. Whilst the hard distinction between stationary and nonstationary series may be useful when the primary objective is **descriptive**, it is less appropriate from our predictive modelling perspective, when the objective is to **forecast** the time-series. From our perspective of aiming to exploit general deterministic components in asset price dynamics, an important motivation for a multi-variate approach is that whilst a set of time-series might appear to be true random walks and hence unpredictable on an individual basis, a suitable combination of the time-series may contain a stationary (or other non-random) component - even if in combination with a nonstationary (random) component. This issue is returned to in Chapter 3 as it constitutes an important aspect of our methodology for statistical arbitrage.

[3] Harris and Inder (1994) reverse the standard test procedure by using a null hypothesis of cointegration and an alternative hypothesis of no cointegration. However they still maintain the hard distinction between cointegrated and non-cointegrated, stationary and nonstationary, series.

Having reviewed in this section a number of families of tests which are designed to distinguish between different types of dynamics and to identify the presence of potentially predictable components, we move on, in the following section, to consider techniques for capturing the predictability in appropriate time-series models and hence making forecasts of future dynamics.

## 2.2.3 Statistical Forecasting and Model Selection

In the previous sections we have reviewed the division of time-series into different classes according to their stationarity characteristics (stationary, nonstationary/integrated, cointegrated), tests to distinguish between these different classes, and tests to identify the presence of potentially predictable components in the time-series dynamics (autoregressive, mean reversion, deviations from random walk). In this section we move on to consider both the modelling methods which aim to capture the predictable component of the dynamics, and the model selection techniques which are used to identify the "best" model from amongst various alternative model specifications.

**Smoothing Methods**

The most basic family of forecasting procedures are the so-called smoothing methods such as the Exponentially Weighted Moving Average (EWMA), a simple ad-hoc technique which forecasts future values through a recurrence relation:

$$\hat{y}_{t+t|t} = f_t = ay_t + (1-a)f_{t-1} \tag{2.15}$$

i.e. at time 't' the forecasted future values $\hat{y}_{t+t|t}$ (over all forecast horizons $t$) are simply equal to a weighted sum $f_t$ of the most recent observation $y_t$ and the previous forecast $f_{t-1}$. The name arises because it is easy to show that the recurrence relation in Eqn (2.15) is equivalent to constructing an exponentially weighted average of past observations. Originally an ad-hoc procedure, the EWMA was shown by Muth (1960) to provide optimal forecasts (in the minimum mean-squared-error sense) when the underlying time-series is a random walk contaminated with stationary noise. The model deals with time-variation in the level of the series by constructing an adaptive estimate of the "local mean" which is updated with each new observation. Holt (1957) and Winters (1960) generalised this exponential smoothing approach to more complex models containing components relating to trend and seasonality. Discussions of the optimality of these approaches, together with further generalisations, are provided by Theil and Wage (1964) and Nerlove and Wage (1964).

A different approach, but which leads to almost identical models, is due to Brown (1963), who suggested the use of *discounted least squares*, in which the increased relevance of more-recent observations is explicitly taken into account through the use of an exponentially-weighted decay function. This approach leads to a recurrence relation for updating estimates of both intercept and slope which correspond to a “local trend”. Further generalisations are discussed in Brown (1963) and more recent papers, of which a review is provided by Gardner (1985).

In the case of these smoothing techniques, it is common for parameters such as the smoothing parameter $a$ in Eqn. (2.15) to be pre-specified according to an *a priori* belief regarding the underlying properties of the time-series. Alternatively a small number of plausible alternatives may be compared, and the “best” model selected on the basis of one or more metrics of forecasting accuracy such as (one-step ahead) mean-squared-error (MSE), mean-squared percentage error (MSPE), mean-absolute error (MAE) or mean-absolute-percentage error (MAPE). Definitions and discussions of such metrics can be found in any forecasting text, for example in Chapter 12 of Diebold (1998).

### Modelling stationary time-series

The classic approach to modelling stationary time-series is based on the recognition that any stationary stochastic time series can be approximated by a combination of so-called *auto-regressive* (AR) and *moving-average* (MA) processes, i.e. so-called ARMA processes of the form:

$$y_t = m + \sum_{i=1..p} f_i y_{t-i} + \sum_{j=1..q} q_j e_{t-j} + e_t \qquad (2.16)$$

A model with *p>0, q=0* includes only lagged values of the dependent variable as explanatory variables and is known as a pure auto-regressive model, denoted AR(p). A model with *p=0, q>0*, includes only past innovation terms as explanatory variables and is referred to as a pure moving-average model, MA(q). The general case is known as an auto-regressive moving-average model, and denoted ARMA(p,q).

This approach developed initially from the work of E. Slutsky (1927) and G.U. Yule (1921). It was shown by Mann and Wald (1943) that pure AR models can be correctly estimated using standard OLS regression. For pure MA or mixed (ARMA) models, more sophisticated methods are required which are based upon nonlinear least squares estimation (Box and Jenkins, 1970) or maximum likelihood methods (Newbold, 1974). A detailed discussion and improved procedures can be found in any standard time-series or econometric text, for instance Harvey (1993) or Greene (1993). ARMA modelling can be extended to allow for exogenous variables, when it becomes known as ARMAX [Hatanaka (1975), Wallis (1977)].

Model selection for ARMA models is sometimes based upon diagnostic tests for (residual) structure such as the Box-Pierce and Box-Ljung statistics, and statistical tests of variable significance such as t-statistics, model restriction F-statistics, and stepwise regression procedures. More common however is the use of statistical model selection criteria within a modelling framework such as the Box-Jenkins framework discussed below.

Multivariate forms of ARMA models also exist, with the vector autoregression or VAR model being particularly popular due to the difficulties in estimating multivariate models with moving-average terms. VAR models have been studied extensively since being first popularised by Sims (1980).

### Modelling nonstationary time-series

The most common approach to modelling difference-stationary time-series is the ARIMA modelling framework of Box and Jenkins (1970). This framework deals with nonstationary time-series by repeatedly *differencing* the data until the resulting series is stationary, thus identifying the “order of integration” within the data. ARMA modelling is then applied to the stationary variable thus obtained. Data with degree of integration I(1) need only be differenced a single time and the resulting ARIMA(p,1,q) model has the form:

$$\Delta y_t = m + \sum_{i=1..p} f_i \Delta y_{t-i} + \sum_{j=1..q} q_j e_{t-j} + e_t \qquad (2.17)$$

Box and Jenkins included the ARIMA modelling process within an integrated framework for *model identification*. This framework uses model selection criteria to choose between alternative specifications of ARIMA models with varying numbers of AR and MA terms $p$ and $q$ respectively.

Statistical model selection criteria are based on a recognition of the fact that the insample fit of a model will *automatically* improve when additional terms are added to a model, even in the case where the additional terms have no real forecasting ability. This effect is a purely mechanical consequence of adding free parameters into a model and is sometimes referred to as "overfitting" or "data mining"[4]. The objective of the model selection criteria is to adjust for these spurious effects and thus allow a fair comparison between models of different complexity. The criteria work by adjusting the insample goodness of fit by a penalty term based upon the number of "degrees of freedom" (or free parameters) in the model. The basic "adjusted $R^2$ " measure is a standard tool of regression modelling which is designed to allow comparison between models with different numbers of explanatory variables. More sophisticated criteria such as the AIC introduced by Akaike (1974b) and the BIC of Schwartz (1978) apply increasingly severe penalties to model complexity.

Discussions of the relative pros and cons of different selection criteria can be found in many places, including pages 85-91 of (Diebold, 1998), and chapter 17 of (Efron and Tibshirani, 1993). In particular the more heavily penalising BIC has the property of **consistency** (i.e. can be shown to select the correct model in the asymptotic limit, as the sample size $n \to \infty$) and the AIC has the property of **asymptotic efficiency** (will select models with performance which converges towards the optimal performance at least as rapidly as any other criterion, as sample size increases). Diebold (1998) notes that when the selection criteria differ as to the optimal model, many authors recommend use of the more parsimonious model selected by BIC, and that this approach is supported by empirical comparisons such as that conducted by Engle and Brown (1986).

Model selection issues play an important role in both the second and third parts of our methodology, which are concerned with the estimation of forecasting models and the diversification of model risk respectively, and hence are discussed more extensively later in the thesis.

[4] Overfitting effects become increasingly important as model complexity grows, as is discussed in the following section on "low bias" modelling approaches. Note that "data mining" in its more usual sense refers to the identification of meaningful, rather than spurious, relationships between a set of variables.

### Modelling cointegrated time-series

A useful corollary of the existence of cointegration in a given set of variables $\mathbf{y}$ is provided by the "Granger Representation Theorem" [Granger, (1983), Engle and Granger (1987)]. The important implication of this theorem is that the presence of cointegration between a set of variables motivates the use of Error-Correction Models (ECMs) [Sargan, 1964] of the form:

$$\Delta y_{0,t} = m + \sum_{i=1..p_0} f_{0,i} \Delta y_{0,t-i} + \sum_{k=1..n} \sum_{j=1..p_k} f_{k,j} \Delta y_{k,t-j} - g d_t + e_t \qquad (2.18)$$

where $d_t$ is the (stationary) cointegration residual calculated according to Eqn. (2.14).

The motivation behind the ECM in Eqn (2.18) is to capture the autoregressive dynamics of $\Delta y_{0,t}$, the lagged cross-correlations between $\Delta y_{0,t}$ and changes in the cointegrating variables $\Delta y_{1,t}, \ldots, \Delta y_{n,t}$ and also the mean-reverting or "error-correcting" effect induced by the cointegration residual $d_t$.

The key idea of the ECM is that the cointegration residual represents a misalignment within the set of time-series and that the cointegrating effect will act so as to reduce this misalignment. Thus the current sign and magnitude of the cointegration residual contain predictive information about changes in the individual time-series. This two-step procedure, of estimating the long-run or cointegration model, and then the ECM which governs the dynamics by which the long-run relationship is restored, is due to Engle and Granger (1987). Multivariate forms of the ECM are known as Vector Error-Correction Models (VECMs).

### State Space Form and Adaptive Models

State space representations of ARMA and ARIMA models were first developed by Harrison and Stevens (1971, 1976) and were based upon the workings of the Kalman Filter [Kalman (1960), Kalman and Bucy (1961)] developed by control engineers interested in recursive methods for multi-sensor systems such as tracking systems or servo-controls. In the engineering context, the number of quantities to be estimated was generally small and the quantity of data generally large. Various developments in maximum likelihood estimation and model selection methodology were required in order to adapt the techniques to statistical and econometric modelling. A review of these developments is contained in section 2.1.3 of Harvey (1989).

The recursive nature of state space models was used by Brown *et al* (1975) in their development of “cusum” test statistics to check for nonstationarity in the form of unstable regression parameters over time. A link between ARIMA and state-space modelling was also provided by Akaike (1974) who showed that any ARIMA process could be expressed in state-space form. A development of this approach was used by Harvey and Philips (1979) as the basis of a methodology for estimating ARIMA models using exact maximum likelihood methods, as opposed to the approximate methods which had previously been used. Standard OLS regression turns out to be a special case of a more general adaptive model which allows for different types of time-varying regression parameters, see for example (Duncan and Horn, 1972) or section 4.5 of (Harvey, 1993).

### Volatility Modelling

Whilst the majority of the models discussed in this section have been concerned with estimating the expected value, or conditional first moment, of a time-series, much recent effort has been devoted to modelling the conditional  variance, or second moment, of time-series. There are now a whole family of models based on the Auto-Regressive Conditional Heteroskedasticity (ARCH) model of Engle (1982) which is an ARIMA-like methodology for modelling the **squared** innovations in a time-series. Generalisations of this model include the Generalised Auto-Regressive Conditional Heteroskedasticity model (GARCH), of Bollerslev (1986), which allows moving-average terms in the squared innovation regression, the (G)ARCH-in-Mean models of Engle, Lilien and Robins (1987), and the EGARCH framework of Nelson (1991).

In the previous sections we have reviewed methods from time-series modelling, statistics and econometrics regarding the representation, identification and modelling of stochastic time-series. The modelling methods in these fields typically impose restrictive assumptions or “biases”, such as linearity and the lack of interaction effects, concerning the functional form of the underlying data generating process. In the following section we review recent developments in the field of machine learning, and neural networks in particular, from the perspective of the potential which they offer for “low bias” modelling of financial time-series dynamics.

### 2.2.4 Low-Bias Modelling

In cases where the restrictive assumptions of traditional techniques are not completely correct, the use of flexible modelling techniques may allow more accurate forecasting models to be built by relaxing these underlying assumptions in a controlled manner. In this section we review the recent emergence of computationally intensive and data intensive modelling techniques which dispense with the traditional assumptions of linearity and independence and can be seen as flexible developments of regression modelling.

Researchers in the fields of nonparametric statistics and machine learning have recently proposed a number of generalisations of traditional statistical methods for classification and regression analysis. Such techniques include the generalised additive models of Hastie and Tibshirani (1990), projection pursuit regression (Friedman and Stuetzle, 1981), classification and regression trees (Breiman et al., 1984), multivariate adaptive regression spline models (Friedman, 1991), interaction splines (Wahba, 1990) and non-parametric regression techniques (Haerdle, 1990). In practice, the most widely studied and applied class of flexible computational models is that referred to as "neural networks", a review of which is presented below.

#### Neural networks

The idea of computational modelling using "neural networks" has existed since the earliest days of computing, with pioneering research on computing using neural-like networks carried out by McCulloch and Pitts (1943) and on learning through synaptic modification by Hebb (1949). The area attracted much interest throughout the 1950's and 1960's, with important early work by Minsky (1954, 1959) and the invention of the *perceptron*, a simplified forerunner of today's neural networks, by Rosenblatt (1959, 1962). This interest declined, however, following a mathematical demonstration of inherent limitations in perceptron-like models (Minsky and Papert, 1969); for instance, the self-learning systems of the time were shown to be incapable of learning a simple logical "exclusive or" (XOR) relationship. This identification of seemingly insuperable technical obstacles had a significant dampening effect on the whole field.

There was a resurgence of interest in the field in the late 1980's following the discovery (Rumelhart et al, 1986) of a more powerful learning rule which overcame the obstacles posed by Minsky and Papert. This "generalised delta rule", was essentially a gradient descent algorithm which, through the use of the chain rule of derivatives, allowed the use of neural networks with more than one layer of adjustable parameters. In the wake of this development it arose that similar learning rules had been developed independently by a number of earlier researchers including Werbos (1974) and Parker (1985).

Figure 2.7 illustrates the method by which the response of a neural network is generated by performing a so-called "forward pass":

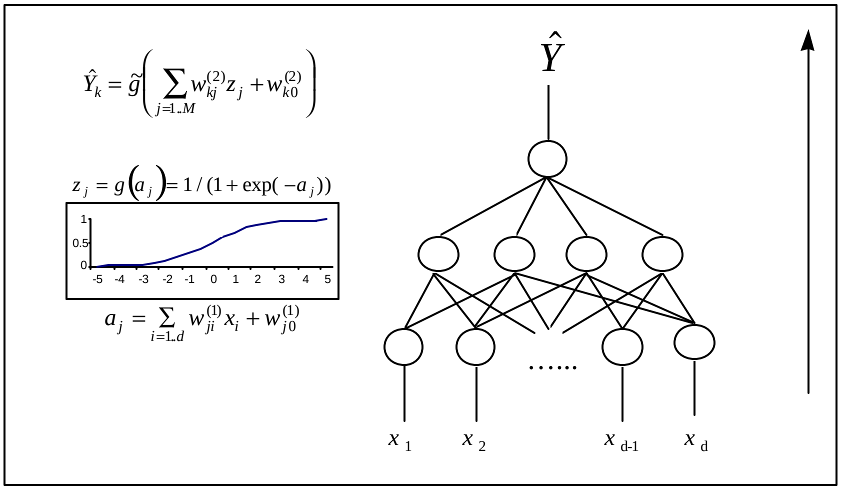

$\hat{Y}_k = \tilde{g}\left( \sum_{j=1..M} w_{kj}^{(2)} z_j + w_{k0}^{(2)} \right)$
$z_j = g(a_j) = 1/(1+\exp(-a_j))$
1
0.5
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
$a_j = \sum_{i=1..d} w_{ji}^{(1)} x_i + w_{j0}^{(1)}$
$\hat{Y}$
……
$x_1$
$x_2$
$x_{d\text{-}1}$
$x_d$

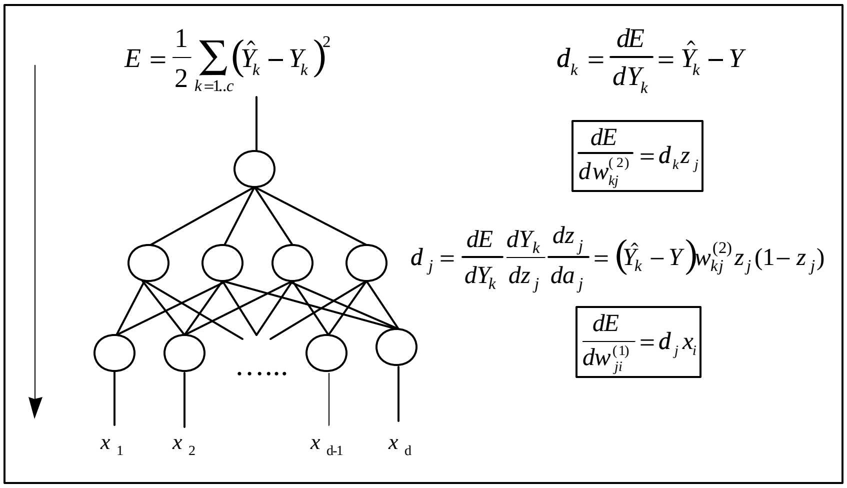
Figure 2.7: Generating a response from the neural network involves transmitting the input stimulus through a first "layer" of weights before being combined in a summation and passed through a nonlinear transform (generally sigmoidal) to obtain the activation levels of the "hidden units". These are then modulated by a second layer of weights and combined to provide the network response.

The first "layer" of weights consists of parameters which determine the locations, projection direction and slope of the basis functions. The second "layer" of weights consists of the parameters which determine the manner in which the basis functions are combined. The expression for the network response as a whole is given by the equation:

$$\hat{Y}_k = \tilde{g}\left( \sum_{j=1..M} w_{kj}^{(2)} g( \sum_{i=1..d} w_{ji}^{(1)} x_i + w_{j0}^{(1)} ) + w_{k0}^{(2)} \right) \tag{2.19}$$

The "hidden unit" transformation $g$ is typically sigmoidal. The output transformation $\tilde{g}$ is linear for real-valued regression and sigmoidal for 0-1 classification problems. The neural network models described in this thesis are intended for regression-type problems and all employ linear output units.

The parameters or "weights" of the network are obtained by minimising a measure of the network "error" over a given training sample. The key phase of the learning process is the "reverse pass" in which errors are attributed to the various components of the network. Figure 2.8 illustrates this process in the most common case, where the "error function" is Mean-Squared Error (MSE)[5] and is identical to the Ordinary Least Squares (OLS) of statistics.



$E = \frac{1}{2}\sum_{k=1..c}\left(\hat{Y}_k - Y_k\right)^2$
$d_k = \frac{dE}{dY_k} = \hat{Y}_k - Y$
$\frac{dE}{dw^{(2)}_{kj}} = d_k z_j$
$d_j = \frac{dE}{dY_k}\frac{dY_k}{dz_j}\frac{dz_j}{da_j} = \left(\hat{Y}_k - Y\right)w^{(2)}_{kj} z_j (1 - z_j)$
$\frac{dE}{dw^{(1)}_{ji}} = d_j x_i$
……
$x_1$
$x_2$
$x_{d\text{-}1}$
$x_d$

Figure 2.8: Calculating the error derivative for the gradient descent learning procedure involves a so-called "reverse pass". The chain rule of derivatives is used to propagate the error from the output of the network back through the various sets of parameters. The process can be viewed as one of allocating the "blame" for the error to the various components of the network.

The purpose of the reverse pass is to calculate the gradient of the error function with respect to the parameter set. After presenting each pattern in the sample, the average value of the error-derivative is calculated and the parameters updated according to an iterative gradient descent procedure[6]:

$$\Delta w = -h \sum_{t=1..T} \frac{dE_t}{dw} \tag{2.20}$$

[5] The use of MSE (or OLS) leads to the interpretation of the network response as a "conditional mean". The iterative nature of the learning procedure means that it is straightforward to replace the MSE cost function with any differentiable function. For instance in the case of an Absolute Deviation cost function, the network response is interpretable as a "conditional median" (see Burgess, 1995c)

[6] Clearly any nonlinear optimisation procedure can be used to find the optimal parameter values. The simple iterative gradient descent is illustrated here for historical reasons.

Where the step-size parameter $\eta$ is the "learning rate" of the network.

Such a network is essentially a non-linear non-parametric regression model which consists of a linear combination of flexible basis functions. Multi-layer networks of this type, sometimes referred to as "multi-layer perceptrons" (MLPs), have been shown to be universal approximators in that a sufficiently large neural network can approximate arbitrarily closely any deterministic relationship between the input and output variables (Hornik et al., 1989). Whilst this flexibility is generally considered advantageous, it was only gradually that the dangers of *overfitting* were recognised. Overfitting arises because with a finite learning set the neural network may correctly learn the examples from the training data but not necessarily *generalise* this performance to out-of-sample data. In the worst case, the network may have sufficient learning capacity to match the training examples in an arbitrary manner, leaving the response to previously unseen input patterns effectively random.

The solution to the problem of overfitting is to constrain the neural network in such a way that the fit to non-training points is "reasonable", in other words to optimise the tradeoff between restrictive assumptions on the one hand and excessive flexibility on the other. A variety of methods have been proposed for controlling overfitting and thus dealing with the "bias/variance dilemma" (Geman et al, 1992). Weight decay (Hinton, 1987, 1989) and similar "regularisation" techniques[7] add an additional term to the cost function which penalises high absolute values of the parameters and hence, indirectly, high curvature of the network response function. In this case the update procedure shown above is modified to:

$$\Delta w = \left( -h \sum_{t=1..T} \frac{dE_t}{dw} \right) - r.w \qquad (2.21)$$

The additional "weight decay" term penalises large connection weights and ensures that the network function is smooth. Weight decay can be considered as analogous to statistical "ridge regression" (Hoerl and Kennard, 1970a,b) and, as for many smoothing techniques (Titterington, 1985), can also be interpreted in a Bayesian context (MacKay, 1992) - in this case implying a gaussian prior distribution for the network weights. More recent weight decay

[7] For a review of regularization techniques see Bishop (1995), pp. 338-353

variants (Moody and Rognvaldsson, 1997) use a modified penalty term which is directly related to the curvature of the estimated network function.

An alternative approach to dealing with over-fitting is that of “early stopping” [e.g. Morgan and Bourlard (1990), Weigend *et al*, (1990)]. In this case the training data is divided into an estimation (training) set and a “validation” set. The parameters are estimated using only the training data but during this process the error on the validation data is also tracked. Whilst the validation error is decreasing, the network is deemed to be learning useful patterns and relationships within the data and the learning process is allowed to continue. When the validation error starts to increase the network is deemed to be over-fitting the specific characteristics of the training set and the learning process is terminated.

A final group of techniques for dealing with overfitting aim to limit the learning capacity of the network by means of model selection, i.e. identifying an appropriate network architecture. Architecture selection procedures can be divided into *pruning* algorithms, which start with a complex network and systematically remove unnecessary components, and *constructive* algorithms which start with simple networks and add components as necessary. Examples of pruning algorithms are two-stage pruning (Sietsma and Dow, 1991), Optimal Brain Damage (Le Cun, *et al*¸1990), and Optimal Brain Surgeon (Hassibi and Stork, 1993). Examples of constructive algorithms are the tiling algorithm (Mezard and Nadal, 1989), the upstart algorithm (Frean, 1990), Cascade Correlation (Fahlman and Lebiere, 1990) and the CLS procedure (Refenes and Vithlani, 1991). Recent surveys of constructive and pruning techniques for neural networks are presented by Kwok and Yeung (1995), Anders and Korn (1996) and Renner (1999).

Many “flavours” of neural network have been developed. By far the most commonly used is the multi-layer perceptron (MLP) based on the backpropagation algorithm of Rumelhart et al (1986); other influential developments have included the Self-Organising Feature Map of Kohonen (1982), Radial Basis Function networks [Broomhead and Lowe (1988), Moody and Darken (1989), Poggio and Girosi (1990)] and various forms of temporal or “recurrent” neural networks [Elman (1990), Werbos (1990), Connor et al, (1994)]. In recent years the statistical properties of neural networks have become increasingly well-understood with important reference works due to Bishop (1995), Amari (1995) and Ripley (1994, 1996).

**Decomposition of forecasting error**

In order to motivate the use of flexible modelling techniques such as neural networks, let us consider the task of modelling and forecasting a random variable $y_t = g(\mathbf{z}_t) + e(\mathbf{v}_t)_t$ which is the result of a data-generating process which is partially deterministic and partially stochastic as defined in Eqn. (2.4). The fundamental modelling task is to approximate the deterministic component of the time-series as closely as possible through the construction of a model:

$$\hat{y}_t = f(\mathbf{x}_t) \tag{2.22}$$

This model estimation problem can be broken down into two subproblems, namely *variable selection*, and *model specification*. Variable selection is the problem of identifying the optimal set of explanatory variables $\mathbf{x}_t$, whilst model specification is the problem of identifying the optimal function of those variables, $f$. In the context of neural network modelling, the model specification problem consists of defining the "architecture" of the neural network, and is referred to as *architecture selection*.

The advantages and disadvantages of flexible modelling techniques can be highlighted by considering the various steps of the modelling procedure in the light of the way in which they influence the forecasting accuracy of a model; in particular we now present a breakdown of the various components of the model error which are a result of the assumptions and restrictions imposed during the modelling process.

**Variable selection:** selection of the "information set" upon which the model is based introduces a "missing variables" component $\mathrm{m}(\mathbf{z}_t, \mathbf{x}_t)$ to the model error which corresponds to the difference between the true function $g(\mathbf{z}_t)$ and the estimator $g_R^*(\mathbf{x_t})$ which is "optimal", but only with respect to the restricted information set $\mathbf{x}_t$:

$$\mathrm{m}(\mathbf{z}_t, \mathbf{x}_t) = g(\mathbf{z}_t) - g_R^*(\mathbf{x}_t) \tag{2.23}$$

**Architecture selection:** imposing a particular parametrisation or functional form on a model necessarily introduces a second component to the model error. This so-called "model bias" can be defined as the difference between the optimal model $\mathrm{f}^*\left(\mathbf{x}_t, q^*\right)$ given the parametrisation, and the optimal model $g_R^*\left(\mathbf{x_t}\right)$ given the restricted information set :

$$\mathrm{b}\left(\mathrm{f}, \mathbf{x}_t\right) = \mathrm{g}_R^*\left(\mathbf{x}_t\right) - \mathrm{f}^*\left(\mathbf{x}_t, q^*\right) \tag{2.24}$$

**Sample selection**: estimating the parameters of the model with respect to a particular finite sample will create a third component of the model error. This “sampling error” is also referred to as “model variance” and can be defined as the difference between the asymptotically optimal estimator $\mathrm{f}^*\left(\mathbf{x}_t, q^*\right)$ and the empirically optimised estimator $\mathrm{f}\left(\mathbf{x_t}, q\right)$ :

$$v\left(\mathrm{f}, \mathbf{x}_t, q\right) = \mathrm{f}^*\left(\mathbf{x}_t, q^*\right) - \mathrm{f}\left(\mathbf{x}_t, q\right) \tag{2.25}$$

We can now relate these quantities to the overall expected loss (in terms of mean-squared error) of the model $\hat{y} = \mathrm{f}\left(\mathbf{x_t}, q\right)$ :

$$\begin{aligned} \mathrm{E}\left[(y_t - \hat{y}_t)^2\right] &= \mathrm{E}\left[(\mathrm{f}\left(\mathbf{x}_t, q\right) + \mathrm{v}(\mathrm{f}, \mathbf{x}_t, q) + \mathrm{b}(\mathrm{f}, \mathbf{x}_t) + \mathrm{m}(\mathbf{z}_t, \mathbf{x}_t) + e(\mathbf{v}_t)_t - \mathrm{f}\left(\mathbf{x}_t, q\right))^2\right] \\ &= \mathrm{E}\left[\mathrm{v}(\mathrm{f}, \mathbf{x}_t, q)^2\right] + \mathrm{E}\left[\mathrm{b}(\mathrm{f}, \mathbf{x}_t)^2\right] + \mathrm{E}\left[\mathrm{m}(\mathbf{z}_t, \mathbf{x}_t)^2\right] + \mathrm{E}\left[e(\mathbf{v}_t)_t^{\,2}\right] \end{aligned} \tag{2.26}$$

Where the result is obtained under the assumption that the four sources of error are all independent, thus allowing the cross-terms within the expectation to disappear. A more extensive discussion of this breakdown of model error and the connection to performance nonstationarity is presented in Burgess (1998).

The first two terms of Eqn. (2.26) comprise the well known “bias-variance tradeoff”: model variance is high for flexible functions and low for more restricted model families; conversely model bias is low for flexible functions and high for more-restricted function classes. We can perhaps summarise the potential, and the risks, of flexible computational modelling techniques in the form of the bias-variance trade-off curve in Figure 2.9.

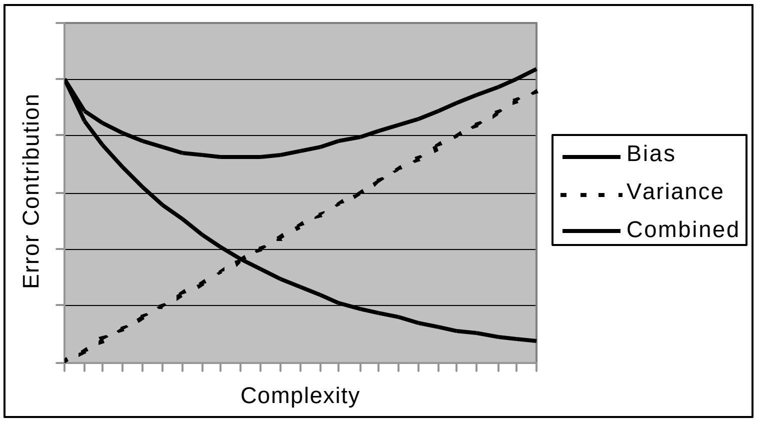


Figure 2.9: Illustration of the "bias variance trade-off" in idealised form. The chart illustrates how adding (appropriate) complexity to a model will reduce the error component which is due to model bias, but at the price of increasing the error component which is due to model variance. The minimal error, and hence optimal performance, will lie at a particular level of complexity which achieves the best balance of bias against variance.

The figure illustrates the fact that adding flexibility to a model will tend to reduce the error which is due to *model bias* (provided the additional flexibility is represented in a manner which is appropriate) but increase the error which is due to *model variance*. Due to the fact that there is typically a diminishing return from marginal increases in model complexity, the optimal model performance will lie at a complexity level which balances the marginal impact of model bias and model variance. In cases where linear models lie to the left of the optimal trade-off, the additional flexibility of techniques such as neural networks offers the opportunity of reducing the error and thus improving forecasting performance. On the other hand, the danger of using the emerging methods lies in the fact that beyond the optimal trade-off the reduction in model bias is more than offset by the increase in model variance, leading to an increase in the overall error and a decrease in forecasting performance.

With an understanding of the manner in which different factors, such as variable selection, model specification and the use of finite samples, contribute to the overall modelling error, there has grown an understanding that for many practical problems a single model, however powerful, may not be the optimal solution. Motivated from this perspective, the following section presents a review of methods for improving forecasting ability by *combining* computational models.

## 2.2.5 Model Combination

In this section we first review the early development of techniques for combining forecasts which were primarily motivated by the desire to exploit the availability of models and forecasts from different sources. Secondly we review the recent development of automated procedures for model combination which have been developed by the machine learning community in an attempt to improve the bias-variance trade-off and thus better exploit the opportunities presented by the use of flexible computational models. We then reconcile these two strands of development from the perspective of the break-down of model error which was presented in the previous section.

### Combining Time-series Models

The fact that forecasting performance can be improved by combining a number of models was first demonstrated in the forecasting literature by Barnard (1963) who compared the performance of Box-Jenkins and exponential smoothing models and found that, although Box-Jenkins out-performed the smoothing method, the best performance was actually obtained by taking an average of the forecasts provided by the two methods. This finding was seized upon by Bates and Granger (1969) who investigated weighted combinations of forecasts, an idea independently proposed by Nelson (1972), and shown to result in significant performance improvement by Newbold and Granger (1974) who applied the methodology to 80 different time-series. The original purely pragmatic motivation for combining forecasts was soon complemented by a Bayesian perspective [Zellner, (1971); Geisel, (1974)], leading to a framework for combining information from different sources (Bunn, 1975) motivated in part by methods for aggregating the opinions of different experts (Morris, 1974).

Amongst a large literature on this subject, Winkler and Makridakis (1983) provided further empirical evidence for the success of combining methods in a large scale study of 1001 time-series. Bunn (1989) noted the increasing popularity brought about by improvements in computer software, and Granger (1989) reviewed the state of the art, emphasizing in particular the fact that model combination will only result in superior forecasts either if the individual models are suboptimal, or if there is a difference in the information sets (explanatory variables) upon which they are based.

### Procedures for Model Combination

The merits of model combination approaches have also been discovered by the machine learning community, with a number of automated procedures being developed in the last decade. These algorithms for model combination can be divided into two broad groups: *ensemble* algorithms and *modular* algorithms.

Perhaps the simplest ensemble algorithm is “bootstrap aggregation” or *bagging* (Breiman, 1996) in which a set of models are generated from a single dataset by resampling the data using bootstrap techniques (Efron and Tibshirani, 1993) and estimating individual models for each sample thus obtained. Similar resampling concepts underlie the *ensemble learning* of Hansen and Salamon (1990) and Krogh and Vedelsby (1995) amongst others. A more sophisticated approach which is designed for classification problems is the *boosting* algorithm [Schapire, (1990), Freund (1990)] in which additional models are based on selected examples which are incorrectly classified by the existing set of models. Forecasts may be based either on voting or weighting methods.

Modular algorithms consist of a set of specialised models in combination with some form of method for selecting the appropriate model at a given point in time, generally in the form of combination weights for the individual forecasts. Examples of the modular approach are the “mixture of experts” model (Jacobs et al, 1991) and “stacked generalisation” (Wolpert, 1992). A mixture of experts model is illustrated in Figure 2.10.

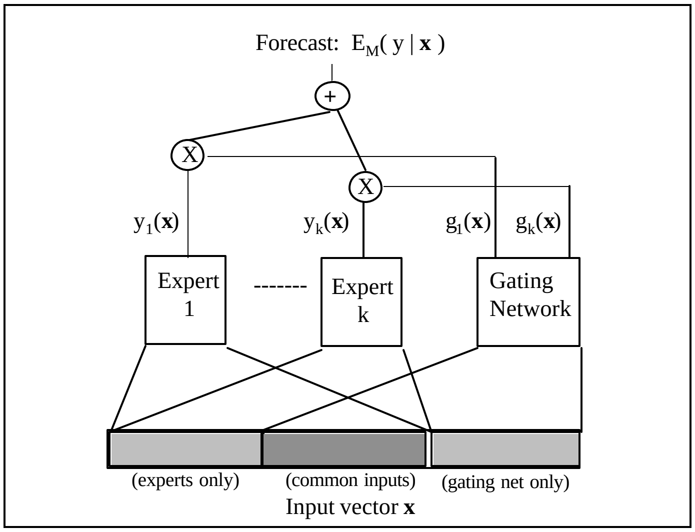


Figure 2.10: Illustration of a “mixture of experts” model. The “expert” models learn to specialise in different subspaces of the input domain. The “gating” network learns to identify conditions under which each expert performs best, generating combination weights for the forecasts of the individual modules.

A mixture-of-experts model is a combination of simpler models together with a “gating” network. The role of the gating network is to identify which of the simpler models, the “experts”, is most appropriate to respond to the current input pattern. Estimation of the parameters of the mixture of experts model as a whole is rather complex, involving a combination of “supervised” learning by the individual experts, together with “unsupervised” learning by the gating network in order to allocate the examples to different experts. The basis of the estimation procedure is the *Expectation Maximisation* (EM) algorithm of Dempster *et al.* (1977). Convergence of this algorithm for the mixture of experts framework has been shown by Jordan and Xu (1995).

The fact that model combination only adds value in the case of models whose errors are at least partially decorrelated was noted by Krogh and Vedelsby (1995) (a view widely understood by the forecasting community even before Granger’s review paper in 1989) with a significant impact upon subsequent developments. An extensive review of model combination approaches for neural networks in presented in the collection of papers edited by Sharkey (1999).

### Analysis of Model Combination

The two strands of development shown above can be reconciled with each other from the perspective of the break-down of forecasting error which is presented in Eqn. (2.26). Whilst, the machine learning literature is primarily concerned with optimising the trade-off between bias and variance, the forecasting literature makes it clear that it is **at least as important** to diversify the information set upon which a final forecast is made. Consider the simple example of combining two models.

$$\begin{aligned}
&\text{Model 1:}\quad y_t = \mathrm{f}_1\left(\mathbf{x}_t, q\right) + \mathrm{v}_1(\mathrm{f}, \mathbf{x}_t, q) + \mathrm{b}_1(\mathrm{f}, \mathbf{x}_t) + \mathrm{m}_1(\mathbf{z}_t, \mathbf{x}_t) + e_1(\mathbf{v}_t)_t \\
&\text{Model 2:}\quad y_t = \mathrm{f}_2\left(\mathbf{x}_t, q\right) + \mathrm{v}_2(\mathrm{f}, \mathbf{x}_t, q) + \mathrm{b}_2(\mathrm{f}, \mathbf{x}_t) + \mathrm{m}_2(\mathbf{z}_t, \mathbf{x}_t) + e_2(\mathbf{v}_t)_t
\end{aligned} \tag{2.27}$$

Then the expected loss of the combined model is given by:

$$\begin{aligned}
\mathrm{E}\left[\left( y_t - \hat{y}_t \right)^2\right] &= \mathrm{E}\left[\left( y_t - \left\{ w\,\mathrm{f}_1\left(\mathbf{x}_t, q\right) + (1-w)\,\mathrm{f}_2\left(\mathbf{x}_t, q\right)\right\}\right)^2\right] \\
&= \mathrm{E}\left[\begin{pmatrix} w\left\{ \mathrm{v}_1(\mathrm{f}, \mathbf{x}_t, q) + \mathrm{b}_1(\mathrm{f}, \mathbf{x}_t) + \mathrm{m}_1(\mathbf{z}_t, \mathbf{x}_t) + e_1(\mathbf{v}_t)_t \right\} \\ + (1-w)\left\{ \mathrm{v}_2(\mathrm{f}, \mathbf{x}_t, q) + \mathrm{b}_2(\mathrm{f}, \mathbf{x}_t) + \mathrm{m}_2(\mathbf{z}_t, \mathbf{x}_t) + e_2(\mathbf{v}_t)_t \right\} \end{pmatrix}^2\right]
\end{aligned} \tag{2.28}$$

Defining $s_{\mathrm{V},1}^{2} = \mathrm{E}\left[\mathrm{v}_1(\mathrm{f},\mathbf{x}_t,q)^2\right], s_{\mathrm{V},2}^{2} = \mathrm{E}\left[\mathrm{v}_2(\mathrm{f},\mathbf{x}_t,q)^2\right]$, etc. this simplifies to:

$$\begin{aligned}\mathrm{E}\left[\left(y_t-\hat{y}_t\right)^2\right] &= w^2 s_{V,1}^{2} + (1-w)^2 s_{V,2}^{2} + 2w(1-w)r_{\mathrm{v}}s_{V,1}s_{V,2} \\ &+ w^2 s_{B,1}^{2} + (1-w)^2 s_{B,2}^{2} + 2w(1-w)\,r_B s_{B,1}s_{B,2} \\ &+ w^2 s_{M,1}^{2} + (1-w)^2 s_{M,2}^{2} + 2w(1-w)\,r_M s_{M,1}s_{M,2} \\ &+ s_{E,1}^{2}\end{aligned} \qquad (2.29)$$

In the case where the two models are identical, then $r_V = r_B = r_M = 1$ and the expected loss (MSE) reduces to the single-model case shown in Eqn. (2.26). Combining resampled models within the same family reduces the average loss by the extent to which the model variances are uncorrelated ($r_V < 1$), but has no effect on errors due to model bias ($r_B = 1$), missing variables ($r_M = 1$), and the inherent stochastic term. Combining models from different functional families, but based on the same set of variables/representations, may reduce errors due to model bias ($r_B < 1$), but not those related to missing variables ($r_M = 1$).

In the context of traditional forecasting models, the model estimation procedures are based upon analytic solutions and thus insensitive to heuristic parameters. Additionally the use of linear models will tend to mean that the use of resampling techniques will add little value: the underlying bias of linearity is still present in each model and the effect of averaging across resampled data sets will be similar to estimating a single model from the entire dataset (no new data is created by resampling).

In the machine learning context, however, the combination algorithms are concerned with models which are both *nonlinear* and *subject to instability* in the estimation procedure. From this perspective the use of resampling techniques may add value by reducing unnecessary sources of additional bias and variance which are induced by the heuristic aspects of the model estimation procedures.

In the light of the review and analysis presented in this and the previous section, it is clear that recent developments in machine learning have a great deal of potential to offer in applications of time-series forecasting, particularly in the form of flexible modelling techniques which are not dependent on strong assumptions regarding the nature of the underlying data-generating processes of financial time-series. A certain amount of the danger which is posed by the potentially high variance of neural networks and related models, can perhaps be controlled both by the regularisation techniques discussed in the previous section and the model combination techniques discussed above. The next section presents a review of further exciting possibilities which are presented by developments in machine learning in the form of advanced methods for model *optimisation*.

### 2.2.6 Advanced Optimisation

In this section we review the development of recent techniques for advanced optimisation. These techniques have been developed largely within the machine learning community and, as yet, have been relatively little applied to problems in the financial domain. However the flexibility of the techniques and the ability which they offer to broaden the scope of the modelling process suggests that they offer exciting opportunities in this, as in other, fields. Firstly we review the emergence of evolutionary optimisation algorithms such as “Genetic Algorithms”. Secondly we review the development of techniques to perform “Reinforcement Learning”.

**Evolutionary Optimisation**

Evolutionary optimisation algorithms are search algorithms inspired by the mechanisms of natural selection and “survival of the fittest”. The first work in this field was performed during the mid-60’s with the introduction of the concept of “genetic algorithms” [Holland (1975), Goldberg (1989)]. More recent reference works are provided by Mitchell (1996) and Michalewicz (1996).

The basic operation of a genetic algorithm (GA) is illustrated in Figure 2.11 below. An initial population of candidate solutions is iteratively evolved through selective pressure over a number of cycles or “generations”. The stages of each cycle are (i) decoding of each candidate from the genetic representation, (ii) evaluation of the “fitness” of each candidate solution, (iii) selection of suitable “parent” solutions for the next generation, (iv) creation of the next generation by combining the genetic information contained in the parents through a process of “reproduction”.

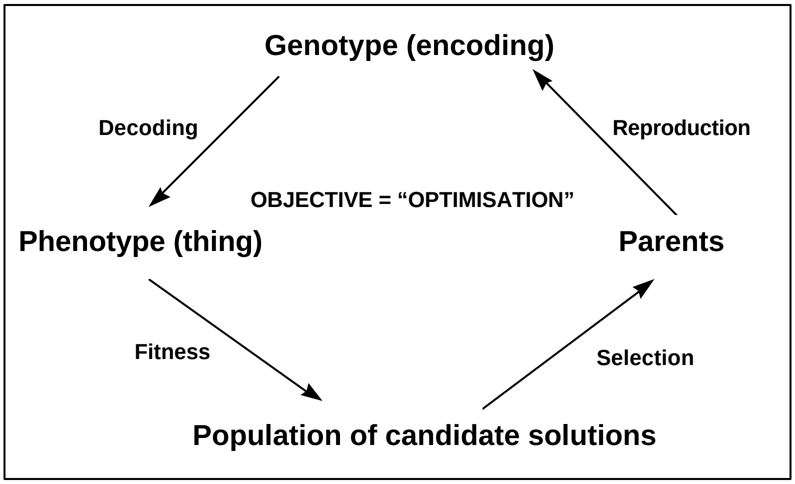


Figure 2.11: Schematic overview of a genetic algorithm. In the population cycle, successful genetic configurations create individuals which survive to contribute to the genetic mix of the subsequent generation.

Conceptually, the three novel aspects of GAs compared to more traditional algorithms can be thought of as the use of optimisation by iterative selection, the use of genetic search and other stochastic search mechanisms, and the maintenance of a population of candidate solutions. Although these three strands are usually intimately linked within a single algorithm, they nevertheless represent different motivations for the use of such algorithms and hence are reviewed independently below.

### Optimisation by iterative selection

Evolutionary algorithms such as GAs can be considered as optimisation algorithms which operate through the iterative application of selective pressure. In each generation, the "parents" are selected according to their "fitness"; the fittest solutions will thus have an increased representation in the new generation and advantageous traits will be propagated through the population.

Much of the motivation behind the early development of genetic algorithms in particular was the fact that they provide a "robust" or general purpose mechanism for optimisation (Goldberg, 1989). From this perspective the objective of the algorithm is to find a single "best" solution, with the fitness function playing the role of the traditional "objective" function which is to be optimised. In such cases, the attraction of evolutionary algorithms lies in the fact that, in principle, they impose no requirements regarding the availability of gradient information, continuity of the fitness function, uni-modality of the fitness function, convexity of the feasible region, and similar constraints imposed by traditional optimisation approaches. It is primarily in this context that GAs and other evolutionary optimisation procedures are increasingly being used to solve complex real-world problems [Davis (1991), Winter et al (1995)].

### Genetic and stochastic search

Genetic search can essentially be considered as a particular form of stochastic search procedure in which new solutions are derived from existing solutions by the use of "reproductive operators" such as "cross-over" and "mutation". *Mutation* involves introducing minor variations upon existing solutions, and, together with a selection procedure, provides a basis for "hillclimbing" towards improved solutions. *Cross-over* involves the combination of genetic material from two or more parents and provides a basis for more significant improvements through the concentration of independently-evolved advantageous traits. The efficiency of genetic search mechanisms is based upon the fact that evaluating an individual provides information about the fitness of the "building blocks" or subsets of parameters which it contains, and hence indirectly about the likely fitness of other individuals which contain these various building blocks. This is referred to by Holland (1975) as *implicit parallelism*.

For genetic algorithms with binary encodings, the incidence of a particular *schema* (pattern of 0s, 1s and "don't cares") in the population has been shown to be exponentially related to the relative fitness of individuals containing that particular pattern. More recent analyses, however, have shown that simple algorithms are strongly dependent upon the choice of a problem representation which contains <u>suitable</u> building blocks [Thierens and Goldberg, (1993), Thierens (1995)].

A significant problem with genetic search is that of "genetic drift" (De Jong, 1975) in which diversity in the population is lost prior to identification of the optimal solution. This tendency, sometimes known as "premature convergence", can be combated by methods which attempt to maintain diversity in the population; such methods include *crowding* (De Jong, 1975), *sharing* (Holland, 1975), *deterministic crowding* (Mahfoud, 1995) and *dynamic niche sharing* (Miller and Shaw, 1996).

As with neural networks, much of the recent work in the field has tended to abstract away from the biological inspiration of the earlier algorithms. In the use of an essentially stochastic search mechanism, evolutionary algorithms bear many similarities to another powerful search technique: simulated annealing, which are examined by Davis (1987). More recently, Baluja (1997) has presented evidence that modified algorithms which maintain explicit population-based search statistics will outperform standard GAs over a wide range of optimisation tasks. Similarly, Juels and Wattenberg (1996) show that for many simple optimisation tasks, the use of reproductive operators such as crossover and mutation is unnecessary and adds no performance improvement over simpler stochastic hill-climbing algorithms. In recent years, researchers in the field have started to focus more on the practical issues which arise from real-world applications, with interesting work on such issues as dealing with situations where the fitness function evaluation itself is contaminated by noise (Miller and Goldberg, 1996), hybrid combinations of genetic algorithms and simple hillclimbing techniques (Lobo and Goldberg, 1997) and identifying the optimal population size for a particular problem (Harik et al, 1997). An engineering-style methodology for the use of genetic algorithms in real-world problems has recently begun to emerge (Goldberg, 1998, 1999).

**Population-based algorithms**

The fact that evolutionary algorithms work on a population of candidate solutions in parallel, has a number of important implications which are often overlooked. Most often the generation of a *population* of solutions is viewed as "a means to an end", the "end" in this case being the optimisation of potentially multi-model functions for which traditional algorithms are likely to converge only to a local optimum (Deb, 1989). In this context, the "niching" strategies of Mahfoud (1995) and Miller and Shaw (1996) are primarily intended as means to subdivide the population and so help avoid *premature* convergence to a single fit individual which represents only a local optimum.

More excitingly, the generation of a population of solutions can be viewed as an end in itself. One important area where this is the case is in multi-objective optimisation where a variety of conflicting objectives must be balanced against each other. In this context evolutionary algorithms can be used to generate populations which represent sets of "Pareto-optimal" solutions [Horn, Nafpliotis and Goldberg (1994), Srinivas and Deb (1995)]. A "pareto optimal" or "non-dominated" solution is one which does not lose, on all criteria, to any single individual (see Section 13.1.2). The connection between *pareto-optimality* and *diversity* in natural evolution is illustrated by the diversity of traits which reflect the individual "ecological niche" of each species, as illustrated in Figure 2.12 below.

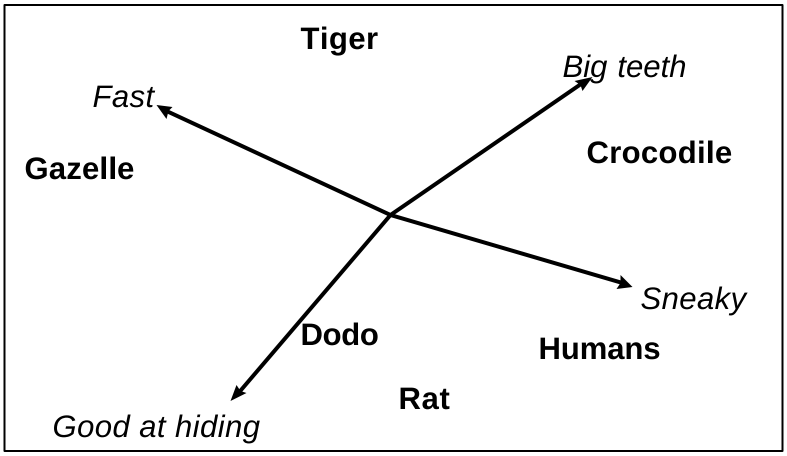


Figure 2.12: Pareto-optimality: natural species adapt to specialise in different ecological niches which can be considered to be particular combinations of ability, measured against a number of different "fitness" criteria. Species achieve the ultimate objective of survival, not necessarily by all trying to grow the largest teeth, but by competing for scarce resources according to their own abilities. The use of population-based algorithms offers similar possibilities to solve underlying problems by means of a diverse set of strategies.

In the case of multi-objective optimisation the generation of a set of Pareto optimal solutions avoids the need to prespecify a somewhat arbitrary tradeoff, thus leaving the final selection decision to a later stage, at which time other less quantitative features may be taken into account.

The generation of a population of candidate solutions also creates the potential for competitive or co-operative interaction between individuals within the population. This feature is a central aspect of the "classifier systems" first introduced by Holland and Reitman (1978) and more recently studied in the context of niching algorithms [Horn, Goldberg and Deb (1994), Horn and Goldberg (1996)]. Alternatively, from the perspective of optimising forecasting models or trading strategies, it opens up the possibility of avoiding model selection as such and instead exploiting an approach based on *combining* the population of candidate solutions [Burgess, (1997); Opitz and Shavlik (1999)].

From the review above it can be seen that evolutionary algorithms offer useful insights and exciting potential both for general purpose optimisation and also for optimisation of an entire population of complementary models. Another development from the field of machine learning which offers interesting possibilities for advanced optimisation in financial modelling is that of "reinforcement learning", a brief review of which is presented below.

**Reinforcement Learning**

Reinforcement learning (RL) is a machine learning approach to the optimisation of multi-period decision making. The foundations of RL are based upon the work of Bellman (1957) who developed the concept of a Markovian or state-dependent decision and formulated the "principle of optimality". The principle of optimality states that the value of taking a particular action in a given state is equal to the *immediate payoff* of the action plus the *expected future value* of the new state which is arrived at by taking the action. Mathematically this can be expressed as:

$$Q(a_t, s_t) = \max_{a_{t+1}} \mathrm{E}\left[r(a_t, s_t) + gQ(a_{t+1}, s_{t+1})\right] \tag{2.30}$$

where $a_t$ is the action taken at time *t*, $s_t$ is the current "state" at time *t*, $r(a_t, s_t)$ is the immediate payoff function of taking action $a_t$ when in state $s_t$, and $Q(a_t, s_t)$ is the overall value of taking action $a_t$ when in state $s_t$ when future consequences are also taken into account.

The recursive expression of the principle of optimality provided in Eqn. (2.30) forms the basis of the "dynamic programming" (DP) approach to solving multi-period decision problems (Bellman, 1957). Given the state-transition probabilities and a defined payoff function, DP methods solve the decision problem by working backwards from a known terminal date and recursively constructing the value function contingent on the agent following an optimal sequence of decisions from that point. The limitations of DP methods are that payoff function, state-transition probabilities and terminal date must all be known *a priori*.

Within the machine learning community, the limitations of exact DP methods inspired the development of approximate "reinforcement learning" (RL) methods for *estimating* the value function $Q(a_t, s_t)$ based on observed sequences of payoffs. Particularly important developments in this area included temporal difference learning (Sutton, 1988) and Q-learning (Watkins, 1989). Temporal difference learning is a learning method which iteratively updates an estimate of the reward function $r(a_t, s_t)$ based upon the differences between estimated and observed payoffs. Q-learning extends temporal-difference learning to the case of dynamic programming, introducing the “future consequences” component and thus estimating the total value function $Q(a_t, s_t)$.

Reinforcement learning techniques, largely based on Q-learning, have recently achieved success in fields as disparate as playing backgammon (Tesauro, 1992) and improving the performance of elevators (Crites and Barto, 1996). From the perspective of financial modelling, RL offers a systematic framework for optimising trading systems in terms of the actual trading performance or “payoff” which they produce, rather than in terms of statistical measures of performance accuracy which may not necessarily translate directly into trading performance [Moody *et al*, (1998), Towers and Burgess, (1999b)].

In this section we have reviewed recent developments in advanced optimisation techniques based upon the concepts of evolutionary optimisation and reinforcement learning. Together with previous sections this comprises an extensive review of a number of strands of developments in time-series and financial modelling upon which our methodology for statistical arbitrage is based. In the following section, we present a further review of the field of “Computational Finance” in which we consider specific developments and applications of computational modelling techniques to problems in investment finance.

## 2.3 Applications of Low-bias Modelling in Computational Finance

In the first section of this chapter we described the concept of statistical arbitrage, noting that the competitive activity of arbitrageurs will tend to ensure that market dynamics are almost entirely unpredictable, but there is no necessary reason that price regularities should not be discovered by looking at the market from new angles and using new tools. This perspective echoes that expressed by Lo and MacKinley (1999):

> “*…financial markets* **are** *predictable to some degree, but far from being a symptom of inefficiency or irrationality, predictability is the oil that lubricates the gears of capitalism.*”[8]

[8] Quote taken from the Introduction of “A Non-Random Walk down Wall Street”, Lo and MacKinley (1999), emphasis in original.

Many other researchers and practitioners share this perspective also. Recent years have seen the emergence and rapid growth of a multi-disciplinary field known as "Computational Finance", driven on the one hand by advances in computing power, data availability and computational modelling methods, and on the other by the competitive and continually evolving nature of the financial markets. Many of the more important recent advances in modelling methodology were reviewed in the previous section. In the remainder of this section we review recent attempts to apply these methods to meet the particular challenges posed by applications in investment finance.

The field of Computational Finance brings together researchers and practitioners with a wide range of interests, both in terms of the modelling methodologies employed and the application areas addressed. In the review below we particularly focus on the specific developments which provide the methodological basis for, and insights behind, the framework for statistical arbitrage modelling which is presented in the remainder of the thesis.

### Overview

The remainder of the thesis presents a methodological framework for exploiting recent advances in computational modelling as a means of identifying and forecasting predictable components in the dynamics of combinations of asset prices. From a practical perspective the thesis can be seen as an attempt to reconcile the *capabilities* of the modelling techniques with the *challenges* posed by the inherent characteristics of financial forecasting problems. In the remainder of this section we consider other attempts to achieve this reconciliation, reviewing firstly the issues raised in terms of methodological shortcomings and secondly the issues raised by the particular requirements of financial applications.

### Methodological Issues

In considering the most significant challenges which are posed by adapting flexible modelling techniques such as neural networks to forecasting in the financial and economic domains, Moody (1995), highlighted the two problems of noise and nonstationarity. The low signal to noise ratios of economic and financial time-series leads to two separate trade-offs. The "bias-variance" trade-off ensures that whilst traditional methods may tend to suffer from high model bias, sample size effects will lead to model variance problems if the underlying modelling assumptions are relaxed too far. The "noise/nonstationarity" trade-off entails that any attempt to reduce sample size effects by using more distant historical data will lead to "nonstationarity" in the general sense of performance degradation over time. The nonstationarity issue arises firstly due to evolution in the dynamics of the financial markets over time and secondly due to "missing variables" effects. Similarly, contrasting financial data to the more well-behaved nature of most problems in the engineering and scientific domains, Abu-Mostafa (1990, 1993, 1995) characterised financial forecasting as a domain where the success target was "50% + ε", rather than the "100% - ε" of engineering applications. Refenes, Burgess and Bentz (1997) review the development of neural network methodology in financial engineering, emphasising both the importance of methodological developments such as model and variable selection algorithms and also the importance of problem formulation and data representation in maximising the extent to which the potential of the modelling methodology is realised.

### Optimising the Bias-Variance trade-off through Low-bias Models

Inspired by the success of neural networks and other low-bias models in the domains of pattern recognition and engineering, many researchers felt that such techniques also presented the solution to the "apparent" unpredictability of financial markets.

In particular, early experiments to exploit this potential were inspired by the demonstrated ability of neural networks to model deterministically-chaotic time-series (Lapedes and Farber, 1987), i.e. time series with non-linear dynamics and apparently random or "chaotic" behaviour. This success led to unrealistically optimistic expectations that dynamics of financial markets, seemingly random according to traditional approaches, would turn out to be largely or wholly deterministic from a non-linear perspective. The results from early studies were mixed, with some showing positive levels of predictability [Dutta and Shashi (1988), Schoenenberg (1990), Bosarge (1991), Refenes (1992)] and others failing to detect evidence of significant market inefficiency (White, 1988). The assumption of deterministically non-linear dynamics led to a common tendency to use large networks, resulting in heavy overfitting of the data; a notable exception being the statistical/econometric approach of White (1988).

However, an increased understanding of the stochastic nature of financial time-series led to the development of a range of techniques for controlling model complexity and optimising the bias-variance trade-off. Weigend et al (1992) applied a weight-decay variant known as weight-elimination and achieved significant out-of-sample forecasting ability for foreign exchange markets. Moody and Utans (1992) adopted a model selection methodology based on variable pruning and found positive results in predicting corporate bond ratings. Recognising the challenges posed by the combination of noise and nonstationarity, Abu-Mostafa (1990, 1993, 1995) suggested the incorporation of prior knowledge as a model bias, in the form of "hints" or assumed invariances of the target domain. More appropriate performance metrics for financial forecasting, to replace the simple mean-squared-error criterion which had been commonly used previously, were presented in Chapter 5 of Refenes (1995).

The acknowledgement of a significant stochastic element in financial time-series led researchers to turn to traditional statistical and econometric models. Links between neural networks and econometrics were made by White (1991), and neural-network based statistical tests for neglected nonlinearity were presented by Lee et al (1993). Traditional ARMA modelling was generalised to the non-linear case to give so-called error-feedback [Burgess and Bunn, (1994), Burgess and Refenes, (1999)] or "NARMA" modelling (Connor *et al*.,1994); in addition Connor *et al* considered the effects of the heavy-tailed noise distributions which are common in financial time-series. Variable and architecture selection methods for neural networks, based on nonparametric statistical tests, were developed by Burgess (1995).

The primary justification for applying neural networks to financial forecasting tasks is the potential presence of non-linear relationships, be they "direct" non-linearities in the relationship between the dependent and a particular independent variable, or indirect non-linearities caused by non-independence in the effects of two or more variables, i.e. so-called "interaction" or "conditional" effects. In this context statistical tests of the significance of a neural network model, such as the "neglected non-linearity" test of White (1989), or performance comparisons between linear and neural network models [eg. Meier et al (1993), Steiner and Wittkemper (1995), Burgess (1996), Steurer and Hann (1996)] can be considered as indirect tests for such relationships. This approach, of using a neural network as a stronger test than a linear model, was made explicit by Bentz et al (1996) in the context of testing for cyclicality in stock prices when considered relative to the market index. Explicit tests for non-linear effects are generally of two kinds, firstly by visual analysis of low-dimensional cross-sections of the estimated network function [Bilge and Refenes (1993), Grudnitski and Do (1995), Refenes et al (1995)], secondly by non-parametric statistical tests for low dimensional non-linear relationships [Burgess (1995)]; the two approaches were combined in Bentz et al (1996).

The global approach of testing for non-linearities via the performance of the neural network model as a whole, and the local approach of testing for non-linear effects between specific variables can be combined in a model selection methodology which attempts to optimise overall performance by testing the statistical significance of individual relationships. Variable pruning algorithms operate by removing those *variables* which do not contribute usefully to the model; the best example is "sensitivity based pruning" [Utans and Moody (1991), Moody and Utans (1992)], with a development (Burgess and Refenes, 1995, 1996) which uses a statistical test based on the approximate number of degrees of freedom which are related to each variable. Weight pruning techniques operate more indirectly by eliminating insignificant *weights* from the network, although methods such as Optimal Brain Surgeon (Hassibi and Stork, 1993) have not been extensively used in financial forecasting; a similar effect may be obtained by using *regularisation* techniques such as the weight elimination of Weigend et al (1992) or the more-principled smoothing regularizers of Moody and Rognvaldsson (1997). An alternative to pruning is the more parsimonious "constructive" approach in which a minimal linear model is enhanced in a stepwise fashion in order to eliminate regularities in the residual error; an example of this approach is the "Neural Additive Model" of Burgess (1995b) which is inspired by the non-statistical "Cascade correlation" algorithm of Fahlman and Lebiere (1990). More recently, significant steps have been taken to generalise statistical modelling methodology to the nonlinear nonparametric models represented by neural networks and other low-bias modelling techniques. These include the development of neural estimation methods which are robust to non-gaussian noise distributions, outliers and influence points (Bolland, 1998) and neural model identification procedures (Zapranis and Refenes, 1999).

### Optimising the Noise/Nonstationarity trade-off through Adaptive Models

Nonstationary or time-varying relationships in financial time-series can be caused either by model misspecification or by true time-variation. This issue is studied extensively by Bentz (1997, 1999) in the context of factor models of asset price returns. The causes and implications of performance degradation due to underlying nonstationarities are studied by Burgess (1998, 1998b). Nonstationarity due to model misspecification arises primarily through the violation of modelling assumptions, such as assuming that a relationship is independent of exogenous factors when in fact there is a nonlinear dependency or interaction effect. The effects of such nonstationarity can best be mitigated through the use of flexible low-bias models in combination with appropriate model and variable selection procedures such as those referred to above.

The second main approach to dealing with model instability is the use of adaptive models. In this case the adaptive model is considered as a local linearisation in time of a more complex underlying relationship. The simplest adaptive method for estimating regression models is the “rolling window” method which has been employed by many authors, including Moody and Wu (1994), Burgess and Refenes (1995, 1996), and Rehfuss et al (1996).

A more elegant approach than the rolling window is the use of recursive estimation procedures based on the Kalman filter, which has the advantage of being equivalent to an exponentially decaying weighting of the observations. Bolland and Connor (1996), embedded a non-linear multivariate autoregressive (NMAR) model of exchange rates within a state-space representation which captured the no-arbitrage relationships between the different exchange rates; the neural network which embodied the NMAR model was used to forecast short-term price movements in the underlying “base” rates. Connor, Bolland and Lajbcygier (1997) used a Kalman filter to model intraday changes in the term structure of interest rates. Niranjan (1997) embedded a radial-basis-function neural network within an “Extended Kalman Filter” (Candy, 1986) in order to track the price of options contracts. Bentz and Connor (1997) used a Kalman filter to estimate conditional “factor sensitivities”, i.e. the relationships between asset price movements and fundamental economic factors, as they evolve over time. Burgess (1997) describes an “adaptive cointegration” framework in which the equilibrium relationship between a set of assets is allowed to evolve over time, with the cointegrating vector being estimated using a “time-varying parameter” model.

Recursive estimation techniques have the effect of implicitly discounting past observations using an exponential decay function over time and have the advantage of being computationally efficient. With the increased availability of computing power, researchers have started to consider alternative time-discounting functions. Refenes *et al* (1994, 1997) consider more general time-discounting or “credibility” functions which consist of parametrised decay terms; these are implemented by adding an explicit discount term to the cost function, as a means of optimising the “noise/nonstationarity trade-off” during model estimation.

### Application Issues

Many of the early attempts to apply neural network techniques to financial modelling failed to recognise either that financial time-series are generally nonstationary in mean (random walk) or that this causes complications in both the modelling process and the analysis of the model forecasts [Bergerson and Wunsch (1991), Refenes (1992), Tiam (1993)]. Since these early experiments, the majority of authors have transformed the target data series in order to obtain stationarity, the two most common approaches are to define a "trading signal" [Bosarge (1991), Refenes and Azema-Barac (1994), Refenes and Zaidi (1995)], typically a binary buy/sell signal, in which case the problem becomes essentially one of classification (up-moves versus down-moves) or to use the more traditional approach of *differencing* the time-series and hence trying to forecast asset *returns* rather than asset *prices* [White (1988), Weigend *et al* (1992), Steiner and Wittkemper (1995)].

In conjunction with these methodological developments, came a more subtle and intelligent **application** of the emerging techniques, largely brought about by cross-disciplinary interaction with statisticians, econometricians and financial economists. The common trend of these new applications was that they attempted to exploit *indirect* inefficiencies in the markets, rather than trying to forecast directly observed price movements as had the earlier studies.

One strand of this development consists of applications to option pricing, in which the objective is to forecast derived properties of a conditional probability distribution. Hutchison *et al* (1994) applied neural network techniques to the problem of pricing options through approximating the non-linear pricing function to exploit minor inaccuracies in the traditional "Black Scholes" model (Black and Scholes, 1973). This approach was extended by Lajbcygier et al. (1996) to cover the "modified Black" option-pricing model of Black (1976). In contrast, Gonzalez-Miranda (1993) adopted the indirect approach of using neural networks to forecast changes in the "implied volatility" of the Spanish stock market, which in turn drives changes in option prices; this approach was extended by Gonzalez-Miranda and Burgess, (1997).

A second strand of applications are concerned with forecasting returns of derived assets or combinations of assets, the motivation for this approach being that dynamics which are not directly observable by market participants are less likely to be completely efficient (i.e. unpredictable). As a topic of central importance to our statistical arbitrage methodology, this use of what may be considered "intelligent pre-processing" as a means of enhancing the predictable component in asset dynamics is reviewed further below.

### Enhancing Predictability through Intelligent Pre-processing

In many cases the volatility in asset returns is largely due to movements which are market-wide or even world-wide in nature rather than specific characteristics of the particular asset; consequently there is a risk that this "market noise" will overshadow any predictable component of asset returns. A number of authors have recently suggested approaches which attempt to reduce this effect by suitably transforming the financial time-series.

A number of authors have employed *multivariate* modelling techniques to identify statistically interesting *combinations* of assets. Lo and MacKinley (1995) used canonical correlation as a tool for identifying maximally predictable linear combinations of asset prices, conditioned upon a particular information set. Bentz *et al* (1996) employed a modelling framework in which prices and returns were viewed *relative* to the market as a whole; this "de-trending" was found to remove typically 80% of the volatility of asset returns, a finding consistent with the Capital Asset Pricing Model (CAPM) [Sharpe, 1964] of finance theory. This approach is extended to multi-variate "factor models" in Bentz (1997, 1999).

In contrast to the studies above, which included "exogenous" variables which were not in all cases tradable assets, other authors have used multivariate methods which are concerned solely with relationships within sets of asset prices. Burgess and Refenes (1995, 1996) used a cointegration framework in which FTSE returns were calculated relative to a portfolio of international equity indices, with the weightings of the portfolio given by the coefficients of a cointegrating regression. Steurer and Hann (1996) also adopted a cointegration framework, modelling exchange rates as short-term fluctuations around an "equilibrium" level dictated by monetary and financial fundamentals. This type of approach in general is characterised as "statistical arbitrage" in Burgess (1996) where a principle components analysis was used to create a eurodollar portfolio which was insulated from shifts and tilts in the yield curve and optimally exposed to the third, "flex" component; the returns of this portfolio were found to be partly predictable using neural network methodology but not by linear techniques. The "statistical arbitrage" perspective was applied to the options prices and implied volatilities of equity market indices by Bolland and Burgess (1997). Back and Weigend (1998) applied "Independent Component Analysis" to stock price returns in the Japanese equity market in an attempt to isolate the driving mechanisms behind market movements.

## Model Risk and other Implementation Issues

The use of a predictive model as the basis of trading decisions necessarily introduces an additional source of risk to the investment process as a whole. Nonstationarities, such as “regime shifts” in the underlying data generating process create a risk that model performance will degrade and entail that future model performance must of necessity include an element of uncertainty (Burgess, 1998, 1998b). This risk is exacerbated by the fact that sampling error in model selection criteria will create an upward bias in the expected performance of the selected model. This selection bias may lead to spuriously “significant” results and is sometimes referred to as “data snooping” [Lo and MacKinley, (1990), White (1997), Sullivan et al (1998)].

One response to potential regime-shifts in the data-generating process is to attempt to learn to *distinguish between* the different regimes. Weigend and Mangeas (1995, 1996) considered the case of “multi-stationary” time-series where the dynamics oscillate between a number of different regimes; the approach employed was based on the “mixture of experts” model of Jacobs *et al* (1991). An extension of this approach to the task of modelling probability distributions of asset returns was presented by Weigend  and Shi (1998).

Where the nonstationarity is more significant, for instance when the different regimes are not distinguishable and/or are changing through time, a more robust approach is to *combine* independent models. Refenes, Bentz and Burgess (1994) suggested that an appropriate framework for weighting independent trading models is provided by portfolio optimisation theory (Markowitz, 1952, 1959). Markovitz mean-variance optimisation takes into account the correlations between the profit and loss patterns of assets and thus encourages effective diversification of risk. Rehfuss *et al* (1996) motivated a “committee” approach where the model forecasts were combined according to a fixed weighting, noting that the most robust approach is to assign an equal weight to each model. In Burgess (1997) we suggested that ideally it is appropriate to *jointly* optimise the models to be combined, rather than doing so on an *individual* basis, and that this should be done in such a manner as to minimise the correlations between the profits and losses of the different models. This approach forms the motivation for the methodology which is described in Part III of the thesis.

A further body of recent work has acknowledged the importance of the decision making component of the investment strategy as of equal importance to the predictive modelling itself

(Towers and Burgess, 1998). Predictive information is of little use if it is inefficiently translated into trading decisions, or leads to profits which are no larger than the transaction costs which are incurred in the process of trading. One approach is to directly employ a (risk adjusted) profit-maximization criterion during the learning process, either in the context of neural network learning, [Choey and Weigend, (1997), Bengio (1997)] or in the more formal setting of reinforcement learning [Moody and Wu, (1997), Moody et al (1998), Moody and Saffell (1999)]. The optimisation of trading strategies for a given forecasting model has been investigated by Towers and Burgess (1998, 1999a) using a modified form of reinforcement learning. Joint optimisation of predictive model and decision policy has been studied by Towers and Burgess (1999b) and developed further by Towers (1999).

## 2.4 Summary

In this chapter we have presented the background to our methodology. We have discussed the concept of "statistical arbitrage" as a generalisation of traditional "riskless" arbitrage strategies and shown how it can be used to motivate the use of predictive modelling in investment finance. We have outlined the elements which comprise a statistical arbitrage strategy and noted the challenges which they pose to modelling methodology. We have reviewed the state of the art in computational modelling, examining relevant techniques from fields such as econometrics, machine learning and time-series modelling. Finally, we have presented an overview of recent developments in the application of flexible computational modelling procedures to problems involving financial forecasting and trading. In the following chapter, we consider these developments from the specific perspective of statistical arbitrage, outlining the opportunities which they provide and also the weaknesses which they expose in the form of "methodological gaps" which remain to be addressed as part of our methodology for statistical arbitrage.

# *3. Assessment of Existing Methods*

In this chapter, we assess the advances in computational modelling which were reviewed in the previous chapter, from the perspective of statistical arbitrage. In the first section we assess the strengths and weaknesses of the various techniques and identify the opportunity which they provide for the creation of a model-based approach to statistical arbitrage. In the second section we evaluate the requirements of such an approach, identifying the "methodological gaps" which are exposed in the current literature.

## 3.1 Potential for Model-based Statistical Arbitrage

A common theme which emerges from our review of Computational Finance is that the predictability of financial time-series is to some extent a function of the data-representation adopted. In particular a number of authors use multivariate techniques to create **combinations** of time-series, generally as a preliminary step to performing predictive modelling itself. Such methods include factor models (Jacobs and Levy, 1988), canonical correlation (Lo and MacKinley, 1995), relative prices (Bentz et al, 1996), principle-components analysis (Burgess, 1996), cointegration [Burgess and Refenes, (1995, 1996); Steurer and Hann (1996)] and independent components analysis (Back and Weigend, 1998).

Such methods can be motivated from two perspectives; on a purely practical basis the combinations thus created represent time-series which are not directly tradable in the marketplace, and hence are less likely to have had any predictable component "arbitraged away"; a second motivation is that the combinations can be seen as a means of improving the signal to noise ratio of the data, and thus enhancing the predictable component in the data. This second view can be seen as a consequence of modern asset pricing models such as the capital asset pricing model (CAPM) of Sharpe (1964) and the "arbitrage pricing theory" (APT)[†] of Ross (1976). Essentially these pricing models take the form:

[†] "Arbitrage" is used here with a different sense to that used in the rest of this thesis. In fact it may be more appropriate to refer to the Ross pricing model as the "no-arbitrage pricing theory" as the basis of the approach is that predictable components of asset returns should be "arbitraged-away" using a methodology such as our own, thus the residual price changes should be based only on fundamental sources of risk in the global economy.

$$\Delta y_{i,t} = a_i + b_{i,M}\Delta Mkt_t + b_{i,1}\Delta f_{1,t} + .... + b_{i,n}\Delta f_{n,t} + e_{i,t} \tag{3.1}$$

In the CAPM model, $n$=0 and the systematic component of asset price dynamics $\Delta y_t$ is solely attributed to the link with market movements $\Delta Mkt_t$. In the APT model, and so-called "multi-factor" versions of CAPM, there may be a number of additional market-wide risk factors $\Delta f_{j,t}$. The essence of modern portfolio theory is that in well-diversified portfolios, the net effect of the asset specific risks $e_{i,t}$ is small and thus the major drivers of portfolio performance are the underlying risk factors, $\Delta Mkt_t$ and $\Delta f_{j,t}$. From this perspective appropriately constructed combinations of assets will "hedge" most of the market-wide risks and thus enhance any predictable component which may exist in the asset specific components $e_{i,t}$.

As a powerful means of testing for deviations from random-walk behaviour, the variance ratio tests described in Section 2.2.2 provide a valuable tool for identifying predictable components in asset price dynamics. Figure 3.1 below shows the results of calculating the variance ratio functions for two equity market indices, namely the German Dax 30 index and the French Cac 40 index. In addition the figure shows the variance ratio of the "relative" price series constructed by dividing the level of the Cac 40 by the level of the Dax 30.

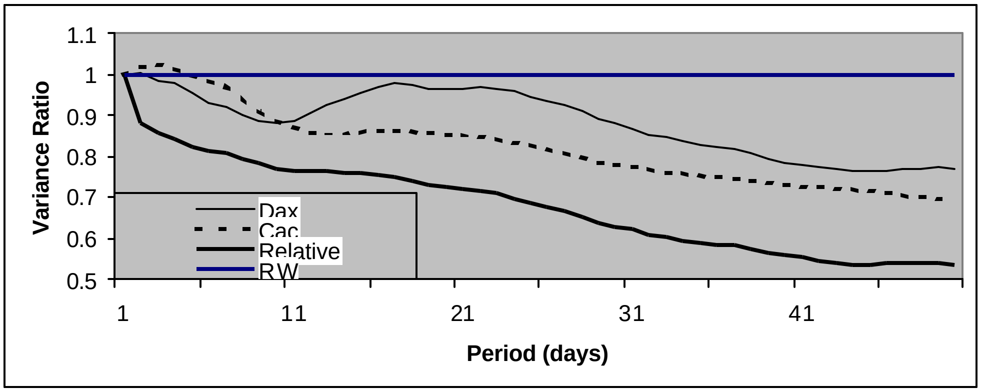


Figure 3.1: The Variance Ratio profile of the Dax and Cac indices individually and in relative terms. The x-axis is the period over which asset returns are calculated (in days), the y-axis is the normalised variance of the returns.

The figure shows that the returns calculated over $n$-day periods have lower variance than $n$ times the variance of one-day returns. This implies a certain degree of mean-reversion in all three of the price series (displayed as a tendency for short term volatility to cancel out over the longer term). Further motivation for our statistical arbitrage perspective is provided by the fact that the variance ratio for the *relative* price series deviates further from the random walk profile than do those of either of the individual series.

In viewing the French and German stock market indices from a global perspective, it seems plausible that the exposure of the two indices to “market wide” risk factors is quite similar and that the fluctuations in “relative price” will be driven by the country-specific aspects of the dynamics. Whilst the variance ratio profiles in Figure 3.1 provide *qualitative* evidence for a predictable component in the relative price dynamics, time-series identification tests such as those described in Section 2.2.2 can help to provide *quantitative* evidence. Table 3.1 presents the result of applying some predictability tests to the relative Cac/Dax price series. In addition to the DF unit-root test and two variance ratio statistics, the table presents a further “cyclicity” test based upon the performance of a simple “implicit statistical arbitrage” trading rule. Although the Dicky-Fuller test fails to reject the null hypothesis that the time-series is nonstationary (‘t’-statistic of 2.34 against a 10% critical value of  2.84), the more powerful “Variance Ratio” tests strongly reject the random walk hypothesis.

| Test | Test Statistic | p-value | Conclusion |
|---|---|---|---|
| Dicky-Fuller | 2.34 | > 0.10 | fail to reject random walk |
| 10-period VR | 0.754 | 0.04 | mean-reverting effect |
| 100-period VR | 0.295 | 0.01 | strong mean-reversion |
| Sharpe (Cyc50) | 1.15 | 0.009 | tradable statistical arbitrage |

Table 3.1: The Dicky-Fuller test fails to reject the null hypothesis that the Cac/Dax relative price is nonstationary; in contrast, the more general Variance Ratio tests indicate the presence of a mean-reverting component in the price dynamics, and the opportunities for statistical arbitrage are confirmed by the performance of a simple trading rule

As the table also indicates, the presence of mean-reverting or “cyclical” effects can be turned into profitable statistical arbitrage through the use of a simple technical trading rule (details of similar “implicit statistical arbitrage rules” are presented in Chapters  4 and 7).

As the “relative price” approach is limited to the bivariate case, multivariate techniques may offer a broader range of potential statistical arbitrage opportunities. From this perspective, techniques such as cointegration analysis offer a natural generalisation of the relative price approach. Figure 3.2 shows the results of estimating a cointegrating regression between the UK FTSE 100 index and a combination of international equity indices.

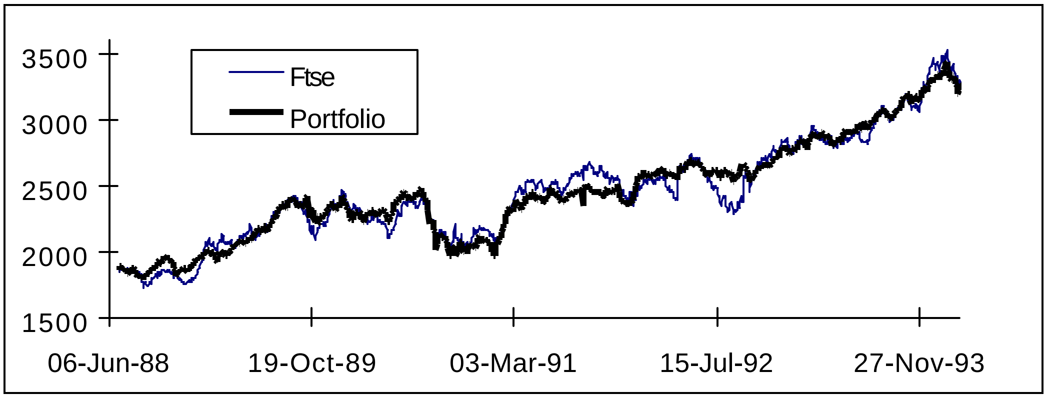


Figure 3.2: Equilibrium relationship between the level of the UK FTSE 100 index and a portfolio constructed from a weighted combination of international equity market indices, over the period 6th June 1988 to 17th November 1993. The combination weights are estimated by means of a cointegrating regression as described in Section 2.2.2.

The figure illustrates that the effect of estimating the cointegrating regression is to construct a combination of the other indices which can be considered as a “portfolio” which closely matches the movements of the FTSE over time. However whilst the price of the combined portfolio both begins and ends the estimation period close to the level of the FTSE, there are substantial short-term deviations between the two “assets”. In retrospect these deviations can be thought of as potential opportunities for statistical arbitrage.

The most straightforward way to exploit statistical arbitrage opportunities in the form of temporary deviations (or “mispricings”) between assets that are correlated (or “cointegrated”) in the long term would simply be to buy the underpriced asset and sell the overpriced asset whenever a sufficiently large mispricing occurs. However such an approach incurs unquantified risks in that short term dynamics and exogenous factors may be acting to increase the deviation, leading to short term loses or “draw downs” before the longer term relationship is restored. A more principled approach is to use the “mispricing” information as the basis of a predictive model of the relative returns on the two assets.

Using the traditional (linear) time-series models which were described in Section 2.2.3 would implicitly assume that the underlying data-generating process of the mispricing time-series is linear in nature. As there is no reason for making such an assumption it is preferable, in principle at least, to employ more flexible modelling procedures such as the low-bias neural estimation techniques reviewed in Section 2.2.4. A particular motivation for the use of low-bias models is that they can capture direct nonlinear dependencies such as threshold or saturation effects, as well as interaction effects in which the values of other factors may modulate the effect of the current statistical mispricing. From a statistical arbitrage perspective, the use of low-bias models can be motivated by a desire to identify opportunities which would be overlooked or misinterpreted by more traditional modelling techniques. Figure 3.3 illustrates this insight that opportunities which are overlooked by linear modelling techniques may be correctly identified by more flexible methods.

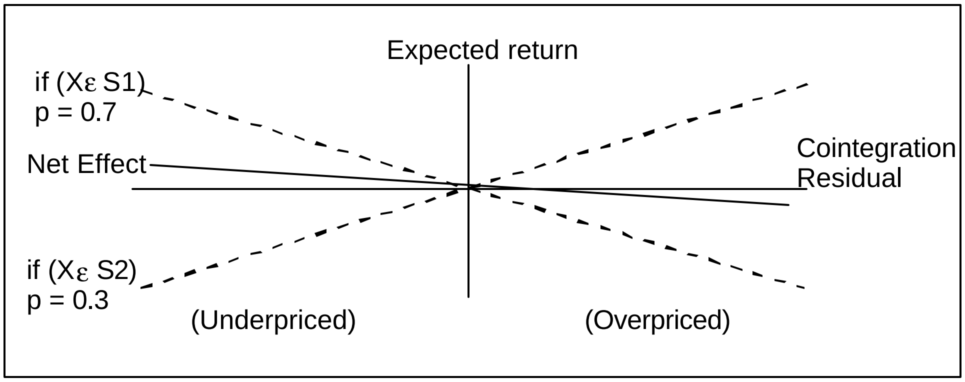


Figure 3.3: An example of a situation in which flexible nonlinear models would identify an opportunity which is invisible to linear modelling techniques. Whereas the linear model can only identify an average-case effect which is too weak to be exploited, the nonlinear model can in principle identify the significant “conditional effects” which prevail under the two different regimes S1 and S2.

In this particular case, evidence for the potential advantages to be gained from the use of low-bias modelling techniques is presented in Table 3.2. The table presents the performance of a simple buy-sell rule based upon predictive models of the relative performance of the FTSE index and the cointegrating portfolio illustrated in Figure 3.2. The general functional form of the models is:

$$\Delta(F-P)_t = f\left(d_t, \Delta(F-P)_{t-5}, \mathbf{z}_t ; \mathrm{q}\right) + e_{i,t} \tag{3.2}$$

where $\Delta(F-P)_t$ is the 5-day relative return of buying the FTSE and selling the cointegrating portfolio and $d_t$ is the statistical mispricing represented by the deviation between the two series. The lagged relative price change $\Delta(F-P)_{t-5}$ is included as a means of capturing short-term autoregressive dynamics in the mispricing series, $\mathbf{z}_t$ is a vector of exogenous variables and $\mathrm{q}$ is the vector of model parameters. The table presents results both for linear models estimated using OLS regression, and for neural “multi-layer perceptron” models (Eqn. 2.19) estimated using gradient descent. Further details of the models are provided in (Burgess and Refenes, 1995, 1996).

| Model Number | Model type | Variable Selection? | Predictive Correlation | Cumulative Return | Directional Ability | Sharpe Ratio |
|---|---|---|---|---|---|---|
| 1 | Linear | No | 0.15 | 8.28% | 54% | 0.36 |
| 2 | Nonlinear | No | 0.21 | 23.2% | 59% | 1.02 |
| 3 | Linear | Yes | 0.30 | 26.8% | 58% | 1.20 |
| 4 | Nonlinear | Yes | 0.34 | 46.1% | 63% | 2.16 |
| 5 | Nonlinear (MAD) | Yes | 0.30 | 45.4% | 65% | 2.05 |
| Combined | N/A | N/A | N/A | 45.5% | 72% (of 66%) | 3.74 |

Table 3.2: Specification and performance of predictive models for relative returns of FTSE index versus cointegrating portfolio of international indices. To account for in-sample biases, performance measures are reported for a separate out-of-sample period of 500 days, trading performance metrics are based upon a simple buy/sell rule.

In all cases the linear model is outperformed by the nonlinear model with otherwise identical specification, providing evidence for the presence of **nonlinear relationships** within the data. Models 1 and 2 contain 12 explanatory variables in total, based upon a set of economic factors recommended by expert advice. Models 3-5 were based upon a reduced set of 3 variables, of which the cointegration residual $d_t$ was one, identified by a nonparametric screening procedure described in Burgess and Refenes (1995, 1996). The improved results for the parsimonious models thus obtained highlight the fact that **variable selection** can have a more significant impact on the results even than the assumed functional form of the model.

Model 5 is a modification of Model 4 but with the standard OLS error-criterion replaced with a Mean-Absolute Deviation (MAD) measure (see Burgess 1995c for details), resulting in an estimator of conditional median returns rather than the usual conditional expectation (mean). The final row of the table refers to a simple combination of Models 4 and 5 where a modified trading rule is adopted which only takes a position in the market when the two models are in agreement as to the expected sign of the (relative) return. The improved reliability in this case is an example of the potential benefits which can accrue from the use of **model combination** techniques.

It should be noted that the performance of the models reported in Table 3.2 is based upon the use of a naïve trading rule which exploits the predictive information in a very crude manner. For instance the rule is sensitive only to the *sign* of the forecasted relative returns and does not take into account in any way the *magnitude* of the forecasts. On the one hand this may result in lost opportunities if particularly favourable circumstances are exploited no more aggressively than are marginal situations. On the other hand it may involve conducting unnecessary transactions. A more sophisticated, reinforcement learning system based on the techniques described in Section 2.2.6 may for instance learn to avoid incurring high transaction costs in the situation where a marginal forecast fluctuates between positive and negative without ever indicating any significant opportunities for statistical arbitrage.

Whilst the results in Table 3.2 represent only a particular set of assets during a particular period, they do serve to illustrate the potential benefits which can be obtained by combining both low-bias modelling and variable selection within a statistical arbitrage context. The final major observation which arises from the literature review concerns the potential benefits to be obtained by the use of model combination techniques such as those reviewed in Section 2.2.5.

Whilst the use of model combination techniques will not necessarily improve the **expected** level of performance, it does have the potential to improve the reliability with which this level of performance can be attained. This is particularly important in the case of statistical arbitrage models where the basic reliability of the models is low (or alternatively the “model risk” is high) due to inherently low degrees of predictability and also to modelling issues such as overfitting, nonstationarity and data snooping. In this context, Figure 3.4 illustrates the potential which model combination techniques offer for improving the risk-adjusted performance of the statistical arbitrage modelling strategy as a whole.

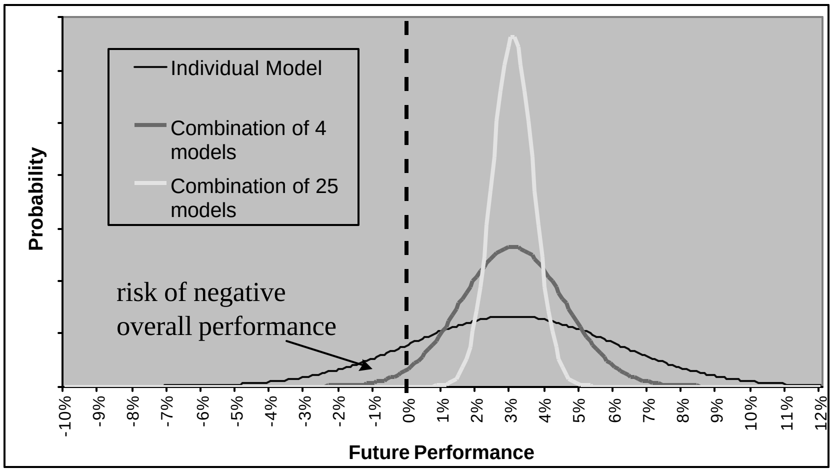


Figure 3.4: An illustration of the potential for improved reliability which is made possible by the use of model combination. Whilst the expected level of performance is not necessarily increased, the variability in performance may be much reduced and hence the **risk-adjusted** performance can be improved significantly.

It is perhaps in this area that the evolutionary optimisation procedures which were reviewed in Section 2.2.6 have the most to offer. Unlike traditional optimisation procedures, the population-based nature of the algorithms offers the possibility of optimising an entire set of models that could form the basis of a well-diversified portfolio and thus greatly reduce the risks that are inherent in relying on any single model. The nonstationary nature of trading strategy performance bears a strong analogy to the case of naturally evolved species which survive tenuously in narrow and fragile ecological niches, such as the case of the dodo illustrated in Figure 3.5.

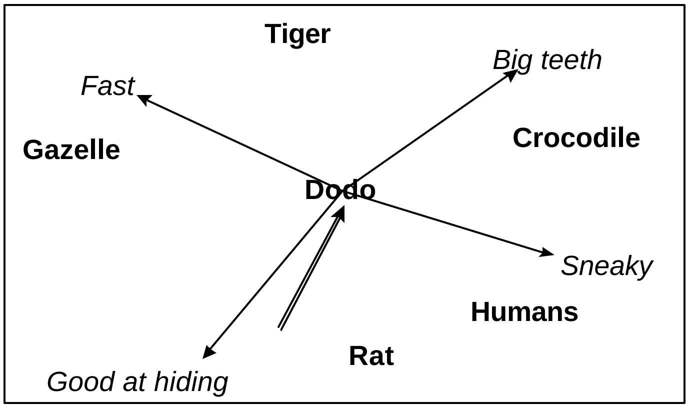


Figure 3.5: Illustration of an analogy between natural species living in fragile ecological niches, and trading strategies which are subject to “model breakdown”. The ecological niche (fitness) of the dodo was destroyed when its hidden island was discovered by man, compared to the earlier situation illustrated in Figure 2.12. A similar fragility may be expected in the case of market-beating trading strategies.

In this section, we have presented an overview of the opportunities for model-based statistical arbitrage which are offered by the current state of the art in computational modelling. In summary, a suitable framework should address the following three issues:

- multivariate analysis to identify combinations of assets which exhibit statistical fair-price relationships, deviations from which can be considered as *statistical mispricings;*

- low-bias predictive modelling to capture the unknown characteristics of the mispricing dynamics;

- model combination and trading rule implementation to exploit the *opportunities* offered by the predictive information whilst simultaneously controlling the level of *risk* which is incurred in so doing.

In the following section we consider the methodological requirements of such a framework, highlighting the particular areas in which existing methodology falls short of these requirements. The *methodological gaps* thus identified form the motivation for the novel developments which are described in the remaining chapters of the thesis.

## 3.2 Methodological Gaps

In this section we evaluate the state of the art in computational modelling in order to identify to what extent it provides the methodology capable of realising the statistical arbitrage opportunities outlined in Section 3.1. The objective of this section is to highlight the specific weaknesses of existing methodology which provided the motivation for developing the novel tools and techniques which are described in the remainder of the thesis.

We divide our analysis into four parts. The first part discusses the need for an overall framework for developing statistical arbitrage models. The remaining three parts address the three components within the framework, namely identifying statistical mispricings, modelling the mispricing dynamics and exploiting the predictive information.

### 3.2.1 Modelling Framework

From the perspective of statistical arbitrage, the primary weakness of existing methodology is the lack of an integrated modelling framework which addresses the three modelling stages of intelligent pre-processing, predictive modelling, and decision implementation. This is a crucial failing because the practical advantage to be gained from any one of these stages is highly dependent upon the other two aspects: “statistical mispricings” are only useful if they are in some sense predictable, and “predictive information” is only useful if it can be successfully exploited. Conversely, the *trading strategy* is dependent upon the quality of the predictive information and the *predictive model* is reliant upon the (continuing) properties of the mispricing dynamics.

A number of methodologies do address two of these modelling stages at the same time, at least to some extent, and some of the more recent techniques offer the potential to deal with all three simultaneously. Table 3.3 presents an overview of candidate methodologies to form the basis of an integrated statistical arbitrage framework, together with an assessment of the extent to which the different methodologies either currently do address, or have the potential to address, the issues of concern to each of the three stages.

| | **Stage 1:**<br>**Data representation** | **Stage 2:**<br>**Modelling the mispricing dynamics** | **Stage 3:**<br>**Exploiting predictive information** |
|---|---|---|---|
| Technical analysis (chartists) | Graphical | Implicit | Trading signals |
| ARIMA modelling<br>(Box-Jenkins) | Univariate, stationarity | Time-series dynamics | NO |
| Portfolio theory (Markowitz) | NO | NO | Asset Allocation |
| *Cointegration* | Multivariate, stationarity, mean-reversion | Error-correction models | NO |
| *Neural networks* | (PCA/ICA networks) | Nonlinear dynamics | Reinforcement learning |
| *Evolutionary algorithms*<br>*(Population-based)* | maybe<br>(hybrid model) | maybe<br>(hybrid model) | general purpose optimisation |

Table 3.3: Evaluation of different modelling methodologies in terms of suitability as the basis of a statistical arbitrage framework. Shadings are used to distinguish between beneficial and detrimental characteristics.

The first three rows of Table 3.3 represent what might be considered "established" techniques. Whilst the technical analysis methods used by market practitioners known as "chartists" can be considered to address all three stages of the modelling process, they have severe limitations from a statistical arbitrage perspective. The representation of financial data as time-series graphs is primarily suited to univariate analysis, the simple extension to bivariate "relative prices" is possible but clearly offers limited possibilities compared to true multivariate analysis. Similarly, technical analysis approaches are limited in that they contain no explicit forecasting models, instead relying on the generation of simple "buy/sell" signals.

Moving on to the other "established" methods, time-series modelling techniques such as Box-Jenkins address the issue of data-representation only as a subsidiary component of the modelling itself, and fail to address the decision making stage at all. Meanwhile portfolio theory will generate an asset allocation based upon forecasted risks and returns, but makes no suggestions how the estimates are obtained.

The bottom three rows of Table 3.3 represent what we consider "emerging" techniques, from the perspective of financial forecasting at least. Thus cointegration can be considered as an alternative method of dealing with pre-processing issues, and can also be used as the basis of error-correction models. In principle, neural network modelling has the potential to address all three stages of the modelling process. However, the majority of NN development has focused on predictive modelling through the use of so-called "supervised networks" of the type reviewed in Section 2.2.4. Evolutionary algorithms also provide a potential solution to all stages of the modelling process, particularly when used in combination with other techniques (so-called "hybrid models"). They are most naturally positioned, however, at the third stage of the methodology due to their ability to perform general-purpose optimisation.

The conclusion of this evaluation is that at the current state of the art no one methodology is inherently suited to dealing with all three stages of the modelling process, although both neural networks and evolutionary algorithms have perhaps the potential to do so in future. Thus the first "methodological gap" which we address in this thesis is the development of an integrated modelling procedure which, by combining suitable techniques within a suitable framework, will address all three stages of the modelling process from a statistical arbitrage perspective.

### 3.2.2 Identifying Statistical Mispricings

In considering the first stage of the modelling process, the main failing existing of current methodology is the lack of an approach which explicitly supports the identification of statistical mispricings within sets of asset prices. From the perspective described in Section 2.1, the overall objective of the first of the three components of a statistical arbitrage strategy was noted as "*construction of statistical fair-price relationships between assets such that the "mispricings" have a predictable component (through time-series analysis of historical asset price movements)*" with the corresponding modelling challenge being stated as: "*given an asset (or portfolio) $X_t$ to identify an appropriate combination of assets to form the corresponding statistical hedge, or "synthetic asset", $SA(X_t)$*" .

The primary objective of the methodology at this stage is to construct statistical mispricings which contain a potentially predictable component. A secondary requirement is the existence of tests to identify whether the deterministic component of the dynamics is statistically significant (rather than simply an artefact of the construction procedure). A final requirement is that the mispricings generated are amenable to statistical arbitrage trading. Table 3.4 presents an overview of candidate methodologies for constructing statistical mispricings:

| | **Induces stationarity** | **Enhances Predictability** | **Predictability tests** | **Known distribution** | **Statistical arbitrage interpretation** |
|---|---|---|---|---|---|
| ARIMA +ACF | YES | NO | short-term dynamics | YES | NO |
| Cointegration + unit root tests | YES | YES | stationarity => mean-reversion | YES | YES |
| *Cointegration + VR tests* | YES | YES | deviations from random walk | NO | YES |
| *ICA/PCA* | Some components | Not directly | Some | NO | Maybe |

Table 3.4: Evaluation of different modelling methodologies in terms of for constructing statistical mispricings and testing for potential predictability in the resulting time-series. Shadings are used to distinguish between beneficial and detrimental characteristics.

The disadvantage of standard time-series modelling approaches such as Box-Jenkins ARIMA modelling are that the pre-processing is univariate and hence not specifically designed to identify predictable components amongst combinations of assets. Cointegration *is* a multivariate technique and has the dual advantages of enhancing predictability by identifying mean-reverting components whilst providing a natural statistical arbitrage interpretation in the creation of balanced sets of “hedged” portfolios which can be traded against each other. A disadvantage, however, is the low-power of standard “unit root” cointegration tests at detecting predictable components, as illustrated in Table 3.1. This weakness of cointegration methodology may in principle be overcome by replacing the unit root tests with more powerful tests based on concepts such as the variance ratio, however the distribution of statistics for such tests is not known and certainly would be expected to be nonstandard due to biases induced by the cointegrating regression.

A final possibility would be to employ a methodology based upon multivariate methods such as principal components analysis (PCA) or independent components analysis (ICA). Whilst these methods do not explicitly induce either stationarity or predictability, a natural advantage of these approaches is that the components would be decorrelated and hence suitable candidates for a well-diversified statistical arbitrage strategy. Again, it is likely that suitable test statistics would suffer from nonstandard distributions.

Thus the second methodological gap identified in this analysis can be stated as the need to combine multivariate analysis techniques with predictability testing procedures in a manner which is aimed at explicitly creating and testing for predictable components in relative asset price dynamics.

### 3.2.3 Modelling the Mispricing Dynamics

The objective of the second, predictive modelling, stage of the modelling process was stated in Section 2.1 as: “*to create models capable of predicting the changes in the “statistical mispricing” between the two portfolios,* $\mathrm{E}\left[ payoff\left( X_t - SA\left( X_t \right)\right)\right]$”.

Thus the primary objective of the methodology at this stage is to maximise the extent to which the potentially predictable component in the mispricing dynamics can be captured within an appropriate predictive model. As discussed in Section 2.4, and reinforced by the results in Table 3.1, the main issues are to firstly identify the appropriate explanatory variables, secondly to identify the functional form of the model and thirdly to minimise the disruption to the model which is due to finite sample effects. Table 3.5 presents an overview of candidate modelling methodologies evaluated according to criteria derived from the need to address these underlying issues:

| | **Arbitrary function form** | **Model interaction effects** | **Variable selection methodology** | **Model Identification** | **Noise tolerance** |
|---|---|---|---|---|---|
| Regression | must be specified | must be specified | manual (t-tests) | manual (diagnostics) | Fair |
| Ridge regression | must be specified | must be specified | implicit | implicit | Good |
| Stepwise regression | must be specified | must be specified | F-tests | automatic | Fair/Good |
| *Neural Regression* | automatic | automatic | manual (“influential variables”) | cross-validation | **Weak** |
| *Regularised NN* | automatic | automatic | implicit | cross-validation | Fair |
| *Cascade Correlation* | automatic | automatic | manual | automatic | **Very Weak** |

Table 3.5: Evaluation of different modelling methodologies in terms of suitability for modelling and forecasting the mispricing dynamics. Shadings are used to distinguish between beneficial and detrimental characteristics.

Table 3.5 indicates that the main weaknesses of traditional regression based approaches to predictive modelling lie in the underlying assumption that the (usually linear) functional form has been correctly specified. This is a particularly significant drawback in the case of predicting the dynamics of financial markets because any predictable components may be missed entirely by such “high-bias” models. On the other hand, the strengths of traditional methods lie in the availability of diagnostic statistics and modelling procedures which support tasks such as model identification and variable selection. A further advantage of traditional statistical methods is the relatively low complexity of the resulting models and hence a tolerance to "sampling effects" or model variance.

Conversely, low-bias models such as neural regression techniques have the significant advantage of requiring very few *a priori* assumptions about the underlying functional form of the data-generating process. However this advantage is substantially negated by the lack of statistically principled procedures for variable selection and model identification, although some basis for the development of such techniques does exist in the form of regularisation techniques such as weight decay, and architecture selection procedures such as "cascade correlation".

Thus the conclusion of this analysis is that an important methodological gap currently exists between the "high-bias/low-variance" techniques of traditional statistical modelling, which are relatively resistant to small sample effects but heavily dependent upon correct model specification, and the "low-bias/high-variance" techniques of neural networks and other nonparametric techniques. The "semi-parametric" nature of neural networks offers an important opportunity to create smooth yet flexible models of multivariate relationships. However, for the limited sample sizes and high noise levels which are available in financial modelling this opportunity is only likely to be realised if suitable neural model estimation procedures are in place.

### 3.2.4 Exploiting the Predictive Information

It was noted in Sections 2.1 and 3.1 that the objective of the third stage of the modelling process is to exploit the opportunities offered by the predictive information which is the product of the previous two stages. This exploitation requires that the predictive information be used as the basis of a trading strategy, or more-generally an asset–allocation strategy[9], which returns significantly positive profits (after transaction costs) whilst simultaneously controlling the risks which are a consequence both of asset price dynamics in general and the specific

[9] We use the term "asset allocation" strategy to refer to the overall process by which capital is allocated to the various assets in the universe, this may be the net result of a number of different trading strategies.

statistical arbitrage models in particular. Table 3.6, below, presents an overview of candidate methodologies for implementing a model-based asset allocation strategy with these underlying objectives:

| | **Profit Maximisation** | **Control of Asset risk** | **Control of Model risk** | **Control of Transaction Costs** |
|---|---|---|---|---|
| Portfolio Theory | YES | YES | NO | NO |
| Model Selection/ Backtesting | YES | maybe | NO | indirectly |
| Model Combination | indirectly | indirectly | in principle | indirectly |
| *Population-based algorithms* | YES | in principle | in principle | in principle |
| *Reinforcement Learning* | YES | YES | maybe | in principle |

Table 3.6: Evaluation of different modelling methodologies in terms of suitability for exploiting predictive information within a statistical arbitrage strategy. Shadings are used to distinguish between beneficial and detrimental characteristics.

Table 3.6 indicates that in many ways this third stage of the modelling process is the one least well served by existing methodology. Portfolio theory is well suited to dealing with the aspect of asset allocation which is related to the risk/return characteristics of the market, but provides little support for dealing with model related issues.

Model selection techniques can identify the “best” model and hence provide an alternative method of profit maximisation. So-called “backtesting”, or simulating the historical trading performance of the strategy allows transaction costs and market risk to be accounted for as part of the selection criteria but provides no protection against “data snooping” effects or “model breakdown” caused by nonstationarities in the underlying relationships.

Model combination strategies are intended to control “model risk” in the form of forecasting errors, but are not naturally suitable for the dual task of maximising return whilst simultaneously minimising risk. Furthermore the analysis in Section 2.2.5 highlights the often-overlooked fact that model risk is reduced only in cases where the various models are decorrelated.

In contrast, the two emerging techniques of population-based algorithms and reinforcement learning offer important *potential* for the optimisation of model-based decision-making, but further developments are required before this potential can be fully realised. Reinforcement learning techniques perhaps form an ideal basis for optimising risk-adjusted return in a manner which also accounts for market imperfectations such as transaction cost. Finally population-based algorithms can be seen as general purpose optimisation techniques and hence in principle can account for all factors simultaneously; furthermore the joint optimisation of an entire population of models would in principle allow for explicit decorrelation of asset risk and model risk between different models, allowing both of these sources of risk to be effectively diversified.

In conclusion, sizeable weaknesses in existing methodology come to light when we consider the issue of optimising the *after-costs risk-adjusted return of model-based trading strategies*. The most basic requirement is the identification of suitable trading strategies to effectively exploit the predictable component of mispricing dynamics. A second objective would be a method of effectively diversifying risk across a number of statistical arbitrage models. A more ambitious objective would be to recognise the interdependence of the modelling stages, and develop methodology for actively encouraging the diversification of model risk.

## 3.3 Summary

In this chapter we have presented an overview of the opportunities for model-based statistical arbitrage which are offered by the current state of the art in computational modelling. In particular we have highlighted a number of "methodological gaps" or weaknesses in existing modelling techniques. Chapter 4 contains an outline of the modelling framework which forms our response to these issues, presenting an overview of our methodology for statistical arbitrage and providing a "route map" to the remaining chapters of the thesis.

# *4. Overview of Proposed Methodology*

In this chapter we present a brief overview of our statistical arbitrage methodology and provide a “route map” to the specific developments and experiments which are described in the remaining chapters of the thesis. In Section 4.1 we present a summary of the overall methodology, whilst in the following sections we summarise the contents of the remaining parts of the thesis. The methodology is based upon the background issues and techniques in Chapter 2 and the assessment of existing techniques in Chapter 3. The novel tools and techniques which are described in the thesis are intended to address the gaps in existing methodology which are identified in the analysis presented in the second part of Chapter 3.

## 4.1 Overview of Modelling Framework

The overall objective of our methodology is to provide a modelling framework for identifying and exploiting particular types of regularities in the dynamics of asset prices, namely opportunities for “statistical arbitrage”. By analogy with traditional riskless arbitrage strategies, our statistical arbitrage strategies involve three stages:

- identification of statistical fair-price relationships between assets such that the deviations from the relationships can be considered “statistical mispricings” which exhibit a small but consistent regularity (deterministic component) in their dynamics

- construction of predictive models which capture as much as possible of the deterministic component in the mispricing dynamics

- implementation of appropriate trading strategies which exploit the predictive information by buying assets (or combinations of assets) which are forecasted to *outperform*, and selling assets (or combinations of assets) which are forecasted to *underperform*.

From a methodological perspective, each of these three stages requires a focus on different aspects of the modelling process, pre-processing at stage 1, predictive modelling at stage 2, and model implementation or decision-making at stage 3. Although these tasks are intimately linked they can nevertheless be considered as raising different underlying modelling issues and our methodology is divided into three parts to address these three sets of issues. An overview of our methodology for statistical arbitrage modelling is shown in Figure 4.1 below:

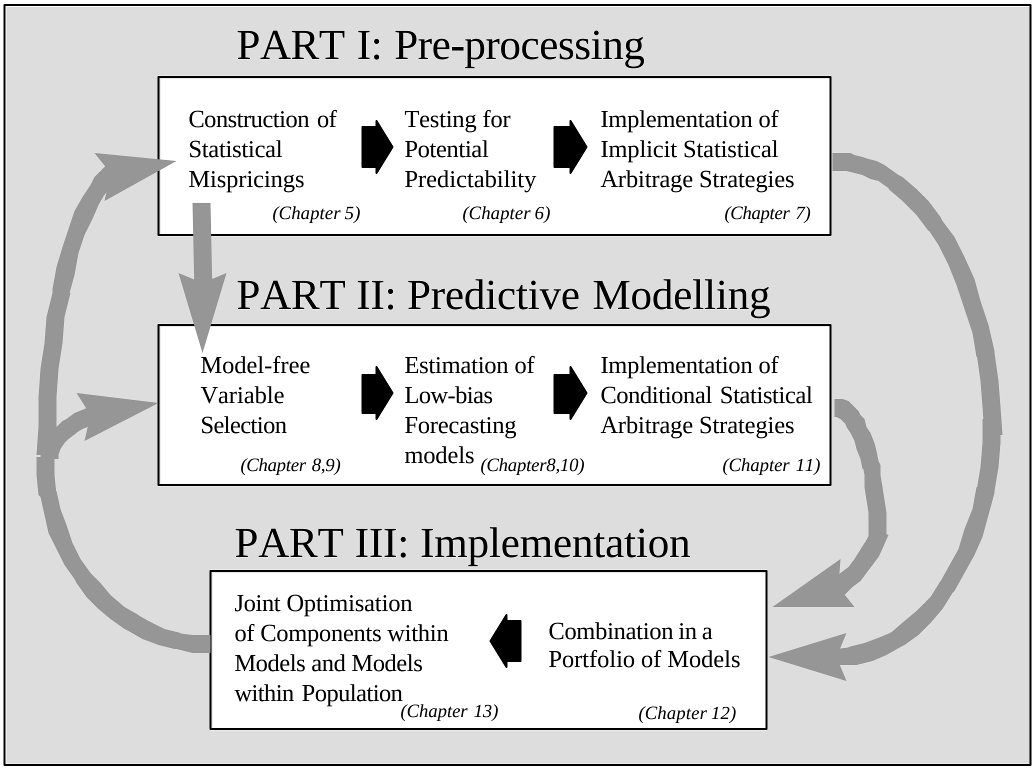

PART I: Pre-processing
Construction of Statistical Mispricings
(Chapter 5)
Testing for Potential Predictability
(Chapter 6)
Implementation of Implicit Statistical Arbitrage Strategies
(Chapter 7)
PART II: Predictive Modelling
Model-free Variable Selection
(Chapter 8,9)
Estimation of Low-bias Forecasting models
(Chapter8,10)
Implementation of Conditional Statistical Arbitrage Strategies
(Chapter 11)
PART III: Implementation
Joint Optimisation of Components within Models and Models within Population
(Chapter 13)
Combination in a Portfolio of Models
(Chapter 12)

Figure 4.1: An overview of the methodology for statistical arbitrage which is described and evaluated in the remainder of the thesis. The methdology is divided into three parts which address the issues involved in pre-processing, predictive modelling and model implementation.

Part I focuses on the issue of **pre-processing**, which in our case consists of constructing and identifying combinations of time-series which contain a predictable component which can be exploited in a statistical arbitrage context. This part of the methodology is divided into three components. The first component is concerned with constructing statistical mispricings (Chapter 5) and is based upon the econometric concept of *cointegration*, with extensions to the standard methodology to deal with time-varying parameters and high-dimensional datasets. The second component is concerned with testing for potentially predictable components (Chapter 6) in the mispricing dynamics and employs standard tests for autoregressive, mean-reverting and non-random walk dynamics as well as introducing novel tests based upon the shape of the *variance ratio* function of the time-series. The third component is concerned with the case of implicit statistical arbitrage strategies (Chapter 7) where the typically mean-reverting nature of the mispricing dynamics can be exploited directly, without the need for an explicit forecasting model.

Part II deals with the issues involved in **predictive modelling** of the mispricing dynamics in the situation where few assumptions can be made about the underlying data-generating process, the amount of data available is small, and the noise content of the data is very high (Chapter 8). This part of the methodology is also divided into three components. A methodology for model-free variable selection (Chapter 9) is intended to distinguish which variables from a set of candidates should be included in the modelling procedure proper, and is based upon methods from *non-parametric statistics*. The actual estimation of low-bias forecasting models (Chapter 10) is performed through novel algorithms which balance the flexibility of *neural networks* with the noise-tolerance and diagnostic procedures of *statistical regression*. The third component of the methodology consists of applying the modelling procedures in the form of conditional statistical arbitrage stategies (Chapter 11).

Part III addresses the **implementation** issues which arise in the context of applying predictive models to risk-averse decision making in general and statistical arbitrage trading in particular. This part of the methodology is divided into two components. A portfolio of models methodology (Chapter 12) is proposed as a means of diversifying both model risk and asset risk, and can be seen as a synthesis of *portfolio theory* and *model combination*. The final component is a *population-based* algorithm which performs joint optimisation (Chapter 13) of the models in a portfolio in order to promote explicit decorrelation between the different models and thus increase the advantages which accrue from risk diversification.

## 4.2 Part I: A Cointegration Framework for Statistical Arbitrage

The objective of the first part of our methodology is to support the pre-processing of financial time-series in a manner which allows the construction and identification of combinations of assets which contain a predictable component which can be exploited in a statistical arbitrage context. Such combinations are referred to throughout the thesis as (statistical) mispricings.

### Construction of Statistical Mispricings

The approach which we take is essentially an extension of the concept of *relative value* and is motivated by the view that asset prices are both more amenable to statistical arbitrage and potentially more predictable when viewed in combined rather than individual terms. Appropriately constructed combinations of prices will be largely independent of *market-wide sources of risk* and will instead highlight the *asset specific aspects of the price dynamics*. Such combinations of assets are amenable to statistical arbitrage because they represent opportunities to exploit predictable components in asset specific price dynamics in a manner which is (statistically) independent of changes in the level of the market as a whole, or other market-wide sources of risk. Furthermore, as the asset-specific component of the dynamics is not directly observable by market participants, but only in combination with market-wide effects, it is plausible that regularities in the dynamics may exist from this perspective which have not yet been “arbitraged away” by market participants.

The methodology which we describe in this thesis is a generalisation of relative value modelling which is inspired by the econometric concept of “cointegration” (see section 2.2.1-3). This represents a major advance over the standard pairs trading approach described in Section 2.1.3 because it allows the modelling of relationships between more than two assets. Where pairs trading relies on the existence of naturally similar pairs of assets, the basis of our approach is to **construct** synthetic pairs from a **combination** of assets.

The advantages of the multivariate approach can clearly be seen in the artificial example shown in Figure 4.2. In the left hand figure, the price of asset X fluctuates around a “fair price” which is determined neither by asset Y alone, nor asset Z alone, but rather by the linear combination 0.2Y+1.4Z. The implications of this relationship are shown in the relative value charts in the right hand figure. Pairs trading strategies based on X/Y and X/Z would result in losses due to persistent trends in the relative prices. In contrast, the multivariate relationship X/(0.2Y+1.4Z) remains stable and could be used as the basis of a profitable strategy.

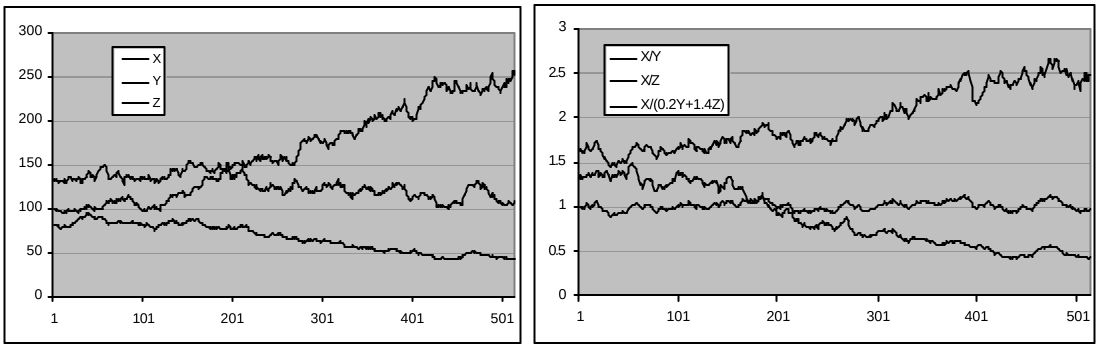


Figure 4.2: An illustration of the advantage of multivariate strategies over simple “pairs trading”. In this artificial example the price of asset X fluctuates around a stable “fair price” equal to 0.2Y + 1.4Z , offering opportunities for statistical arbitrage which are by no means apparent from either of the bivariate perspectives offered by X/Y and X/Z.

The relevance of *cointegration* lies in the fact that the parameters $b_i$ can be optimised from historical prices using a “cointegrating regression” (Granger, 1983). Using this approach the "fair price" relationship is constructed by regressing the historical price of $T$ on the historical prices of a set of "constituent" assets $C$:

$$\mathrm{E}[T_t] = SA(T)_t = \sum_{C_i \in C} b_i C_{i,t} \tag{4.1}$$

where the $b_i$ coefficients are estimated using the regression procedure. The synthetic asset can be considered an optimal statistical hedge in that the OLS procedure ensures that the deviation between the two price series is minimal in a mean-squared-error sense. Given an estimated fair price relationship of the form shown in Eqn (4.1) the "statistical mispricing" is defined as the deviation from the fair price relationship, i.e. the difference between the target and synthetic assets:

$$M_t = T_t - \sum_{C_i \in C} b_i C_{i,t} \tag{4.2}$$

In the simplest case where the asset universe is small and the optimal "constituent weights" $b_i$ are assumed constant over time then standard OLS regression can be used to estimate the fair price relationship. As these situations are more the exception than the norm we develop two extensions to the basic methodology. The first extension uses adaptive modelling techniques to allow the constituent weights $b_i$ to vary slowly over time, in this way allowing the model to capture any long term drift which may occur in the fair price relationship between the assets:

$$M_t = T_t - \sum_{C_i \in C} b_{i,t} C_{i,t} \tag{4.3}$$

The second extension is intended to deal with the case where the asset universe is large and employs stepwise regression techniques. Starting with the asset price itself, i.e. $C(0) = \varnothing,\ M_t^{(0)} = T_t$, the set of constituent assets is constructed step by step. At each step the set of constituents is extended by including the variable which is most highly correlated with the current mispricing:

$$C(k+1) = C(k) \cup \underset{C_j \in U_A - C(k)}{\arg\max}\ \mathrm{E}\left[ {M_t^{(k)}}^2 - \left( M_t^{(k)} - b_j C_j \right)^2 \right] \tag{4.4}$$

### Testing for Potential Predictability

The next component of the methodology involves testing for the presence of a potentially predictable component in the mispricing dynamics. Unfortunately it is not easy to find examples of strictly cointegrated relationships in financial asset prices, and even the relationships which are discovered may be unstable over time. Rather than imposing the strict requirement of stationarity, we instead employ a broader range of tests which are capable of identifying both trending and mean-reverting behaviour and which are robust to the presence of a nonstationary component in the time-series dynamics.

We introduce two methodological innovations as part of our predictability-testing procedures. The first of these are a new set of tests for deviations from random-walk behaviour based on the *shape* of the variance ratio profile. These tests are based upon the **joint** distribution of the set of VR statistics $\mathbf{VP}t = \left\{VR(1), VR(2), VR(3), \ldots . VR(t)\right\}$ where the individual VR statistics are defined by:

$$\mathrm{VR}(t) = \frac{\sum_t \left(\Delta^t y_t - \overline{\Delta^t y}\right)^2}{t \sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2} \qquad (4.5)$$

Viewing the vector of variance ratio statistics as a multivariate normal distribution with mean $\overline{\mathbf{VP}t}$ and covariance matrix $S_{\mathbf{VP}}$ then the first statistic $VPdist\left(t\right)$ represents the Mahalanobis distance of the observed vector $\mathbf{VP}t$ from the centre of the distribution:

$$VPdist\left(t\right) = \left(\mathbf{VP}t - \overline{\mathbf{VP}t}\right)' S_{\mathbf{VP}}{}^{-1}\left(\mathbf{VP}t - \overline{\mathbf{VP}t}\right) \qquad (4.6)$$

A second set of tests exploit the structure within the set of VR statistics. In particular we consider the projections of the vector $\mathbf{VP}t$ onto the directions which correspond to the eigenvectors $\mathbf{e}_i$ of the covariance matrix $S_{\mathbf{VP}}$ .

$$VRproj(t, i) = \left(\mathbf{VP} - \overline{\mathbf{VP}}\right)' \mathbf{e}_i \qquad (4.7)$$

The magnitude of these projections can be interpreted as the extent to which a given variance ratio profile matches a characteristic set of common “patterns”.

The motivation for these tests is to improve the power with which slight deviations from random walk behaviour can be detected. The flexibility of VR-profile statistics is illustrated in Figure 4.3 below.

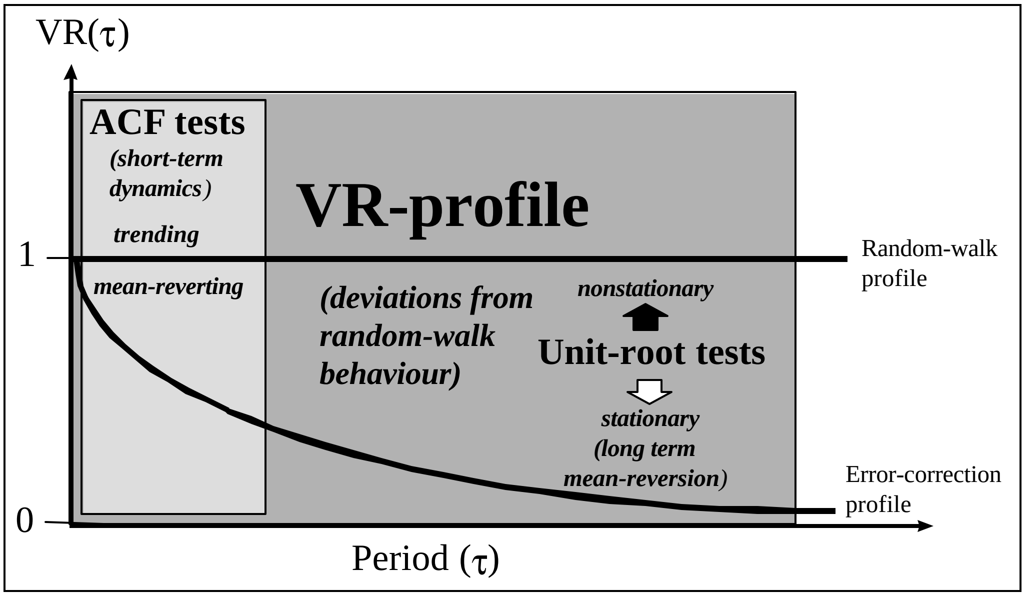


Figure 4.3: An illustration of the flexilbility of variance ratio profile statistics. Whilst tests based on the autocorrelation function (ACF) are mainly designed to identify short-term dynamics and unit-root tests are designed to distinguish between two specific types of long-term dynamics, tests based on the entire variance ratio profile are sensitive to general deviations from random-walk behaviour, whether short-term or long-term effects.

Tests based on the autocorrelation function (ACF) are most appropriate at detecting short-term effects such as such as momentum or short-term correction effects. In contrast, unit-root tests and standard variance ratio tests are most appropriate for observing long term effects, such as distinguishing between stationary and nonstationary behaviour. Tests based on the overall shape of the variance ratio profile are capable of recognising specific combinations of short-term and long-term dynamics which are particularly amenable to statistical arbitrage.

The second innovation in our methodology for predictability testing is the application of both the standard tests and the new tests in the context of multi-variate modelling. Except in the case of unit root tests, which are used in cointegration modelling, the other tests are generally used in a univariate setting rather than being applied to the results of a preliminary multivariate transformation. In order to apply the tests in a multi-variate setting it is necessary to correct for the **biases** which are induced by the construction procedure which is used to create the statistical mispricing time-series[10]. We achieve this by deriving appropriate empirical distributions of the test statistics by the use of Monte-Carlo simulations. As well as correcting for the biases induced by the algorithm itself, this approach has the additional advantage of automatically taking sample size effects into account, rather than relying on asymptotic behaviour.

**Implicit Statistical Arbitrage Strategies**

In cases where the mispricing dynamics contain a substantial mean-reverting component, we have devised a set of "implicit statistical arbitrage" (ISA) trading strategies which are designed to exploit the mean-reversion effect without the need for an explicit forecasting model.

The underlying assumption of the ISA strategies is that future price changes will be such as will tend (on average) to reduce the mispricing between the target asset and the combination of constituent assets. Thus over a period of time strategies based on selling the overpriced (combination of) asset(s) and buying the underpriced (set of) asset(s) should realise positive profits.

The ISA strategies operate by viewing the statistical mispricing $M_t$ as a single portfolio consisting of the assets $\left\{T_t, C_1, C_2, \ldots, C_{n_c}\right\}$ with weightings $\left\{1, -b_1, -b_2, \ldots, -b_{n_c}\right\}$ respectively, where $n_c$ is the number of constituent assets. Thus buying the target asset $T$ and selling the synthetic asset $SA(T)_t = \sum b_i C_{i,t}$ is thought of as the single action of buying the mispricing $M_t$. Similarly selling the target asset and buying the synthetic asset is thought of as selling the mispricing $M_t$.

The ISA strategies are implemented by means of parametrised trading rules which define a desired holding in the mispricing portfolio as a function of the current level of the mispricing:

[10] For instance, the use of regression will induce a certain amount of spurious mean-reversion in the residuals which may incorrectly lead to the rejection of random walk behaviour even when no true predictable component is present (see Section 6.3).

$$\text{posn}(M_t, k, h) = -\frac{\sum_{j=1..h} \text{sign}\left(M_{t-j}\right)\left|M_{t-j}\right|^k}{h} \tag{4.8}$$

Note the minus sign which indicates that the position taken should be *opposite* in sign to the current mispricing. The parameter $h$ can be thought of a as “holding period” which is used to smooth the output of the trading rule and reduce the number of transactions which are generated. The parameter $k$ defines the sensitivity of the trading rule to the actual *magnitude* of the mispricing as opposed to simply the *sign* of the mispricing, as shown in Figure 4.4 below:

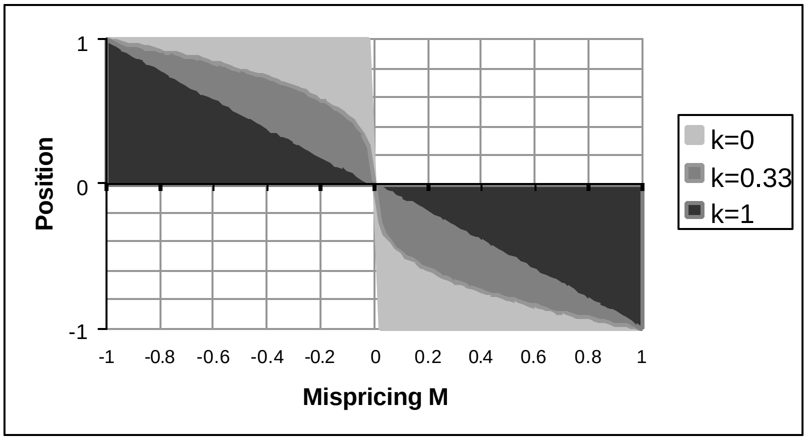


Figure 4.4: The relationship between the level of the statistical mispricing $M_t$ and the output of the implicit statistical arbitrage trading rule posn($M_t$, k, 1) for different values of the sensitivity parameter k.

The ability of ISA strategies to exploit mean-reverting behaviour is illustrated in Figure 4.5 which shows the results of applying the strategy to a synthetic time-series with a moderate degree of mean-reversion:

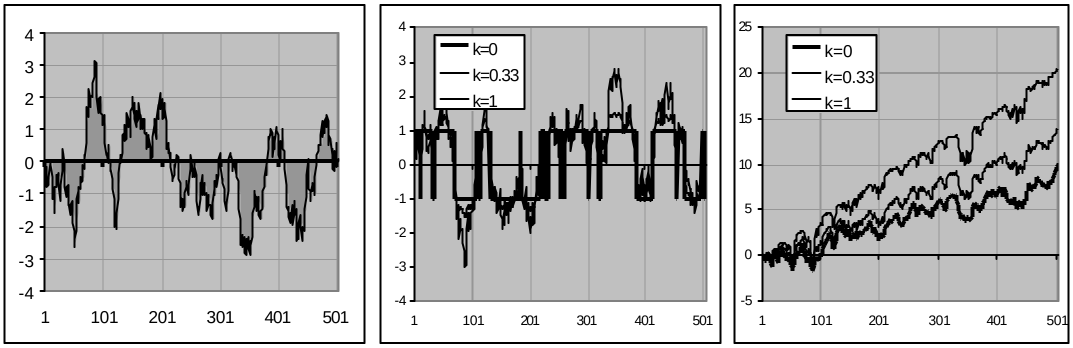


Figure 4.5: Results of applying the ISA strategy to an artifical time-series which exhibits moderate mean-reversion. The left hand chart shows the time-series itself. The centre chart shows the output of the ISA rule (with different levels of $k$) . The right hand chart shows the accumulated profit and loss, or “equity curves” for the three cases.

## 4.3 Part II: Forecasting the Mispricing Dynamics using Neural Networks

The objective of the second part of our methodology is to support the building of predictive models of the mispricing dynamics. The methodology aims to maximise the extent to which the deterministic, and hence *potentially* predictable, component of the mispricing dynamics can be *actually* captured in forecasting models and then exploited as the basis of practical statistical arbitrage trading strategies.

### Model-free Variable Selection

A key factor in the success of the modelling process is the accuracy of the information set upon which the model is conditioned. This fact is highlighted in the breakdown of model error in Section 2.2.4, where the “missing variables” term is shown to be one of the three components which inflate the model error over and above the inherent stochastic component of the data.

Thus the first aspect of our procedure for building forecasting models is a methodology for “model-free” variable selection. The term *model free* reflects the intention that the methodology should be sensitive to a wide range of relationships, rather than being predicated upon the assumption of a particular functional form. The variable selection methodology is intended for use in situations where the small size of the available data sample, together with the large stochastic component in the data, precludes the inclusion of all of the variables in the model building procedure proper. The flexibility of the variable selection procedure is vital in order to avoid imposing a hidden bias on the subsequent stages of modelling.

In order to achieve this flexibility, our variable selection methodology is based upon techniques from nonparametric statistics, and in particular *nonparametric regression*. The approach which we take is to regress the dependent variable upon nonlinear projections of the original variables to produce submodels of the form:

$$y = a + \sum_{i} q_i b_i\left(x_j\right) + e \qquad (4.9)$$

Where the basis functions $b_i\left(x_j\right)$ are chosen in such a way that in combination they can approximate a wide range of functional relationships. Under the null hypothesis that independent variable $x_j$ is unrelated to $y$, it is known that the F-ratio of the model should follow an F-distribution with $p$ degrees of freedom in the numerator and $n-(p+1)$ in the denominator, i.e.:

$$\frac{\frac{1}{p}\sum\left(\left[a + \sum_{i} q_i b_i\left(x_j\right)\right] - \bar{y}\right)^2}{\frac{1}{n-(p+1)}\sum\left(y - \left[a + \sum_{i} q_i b_i\left(x_j\right)\right]\right)^2} \sim F_{p,n-(p+1)} \qquad (4.10)$$

The tests can also be used to test for *interaction effects* between variables; to do so the basis functions act as a nonlinear projection of the subset of the original variables which is being tested. In the two dimensional case the submodels are of the form:

$$y = a + \sum_{i} q_i b_i\left(x_j, x_k\right) + e \qquad (4.11)$$

The variable selection algorithm consists of “screening” all the low-dimensional combinations of the candidate variables by constructing and testing appropriate submodels. Flexibility is achieved in the first place by using basis functions which are known to be capable of closely reproducing a wide range of relationships, and secondly by using a range of different basis functions. Suitable basis functions include polynomials in $x_j$, “bin smoothers” and “radial basis functions” as well as the linear (identity) case and the more-flexible models offered by neural estimation methods. An example of a “bin-smoother” applied to nonlinear relationship is shown in Figure 4.6 below.

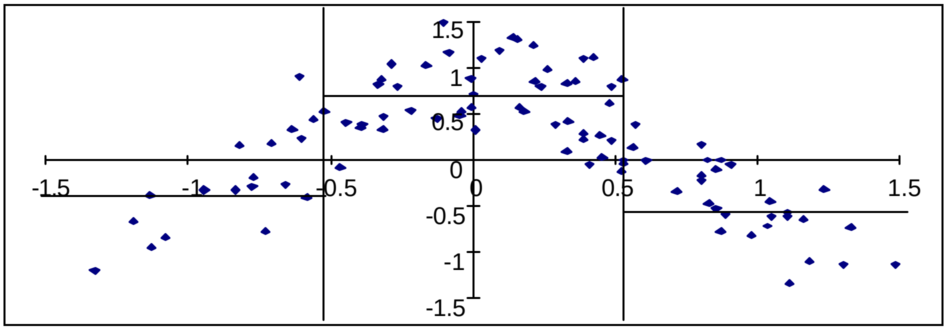


Figure 4.6: An example of a non-linear relationship which may be identified by means of a bin smoother (3 bins in the example) but which exhibits zero correlation and hence would be missed by a linear test.

The basis functions used in the bin-smoother consist of indicator values which take the value 1 if an observation is in a particular region of input space, or “bin”, and 0 otherwise. If the mean value of the dependent variable *y* varies substantially from one bin to another, the associated F-test will indicate the presence of a statistically significant relationship which may be missed by a linear test.

### Estimation of Low-bias Forecasting Models

Having performed any necessary variable selection, the next task is to approximate the deterministic component of the mispricing dynamics in the form of a predictive model. The estimation procedure should optimise the complexity trade-off between bias and variance, whilst at the same time incorporating underlying modelling assumptions which ensure that the complexity is allocated in an appropriate manner. The methodology which we have developed avoids the problem of excessive bias by using flexible neural network models and avoids excessive estimation variance by using statistical significance testing to control the specification of the neural network models.

As discussed above, perhaps the most important modelling assumptions relate to the choice of variables to include within the model. If the model is lacking important explanatory variables then it cannot capture the associated deterministic component of the mispricing dynamics, irrespective of the flexibility of the functional form of the model. Likewise the incorporation of irrelevant “noise variables” can only ever decrease model performance, irrespective of the methodology which is used to control model variance (see Section 10.1). For these reasons *variable selection*, in the sense of the allocation of model complexity to each of the explanatory variables, is treated as an integral part of our neural model estimation procedures.

Existing algorithms for automatic neural network specification can be considered as belonging to four main classes, namely: (i) early-stopping of learning through the use of (cross-) validation, (ii) regularisation approaches; (iii) constructive algorithms and (iv) deconstructive or "pruning" algorithms. For our purposes the first of these can be eliminated in that termination of learning when the network is not even in a local optimum invalidates the use of hessian-based statistical diagnostics such as the degrees of freedom absorbed by the model (the first order conditions used to derive the degrees of freedom measure no longer hold). Our methodology encompasses, to some extent, each of the other three classes of algorithm. It differs from existing methods in three main ways.

- our methodology is **statistical** in nature and provides diagnostics which avoid the need to lay aside large fractions of the dataset for use in determining hyperparameters such as training time, regularisation coefficients and number of hidden units.

- our methodology views the modelling process from a *data-centred* rather than a *parameter-centred* viewpoint. Rather than viewing the neural network in terms of the individual **weights** which it contains, we adopt a nonparametric perspective based upon the "equivalent kernels" which describe the sensitivity of the estimated model to each observation in the learning set. The estimated model as a whole is considered from the perspective of the degrees of freedom which it absorbs and the amount of variance which they explain.

- our methodology treats the *allocation* of model complexity as being of equal importance as the optimisation of the *level* of complexity; thus variable selection and architecture selection are considered as inter-related aspects of the overall model specification.

Within this general perspective our methodology supports three alternative model estimation algorithms. The three algorithms share a basic testing methodology in which the statistical significance of model components is calculated by comparing the degrees of freedom which they absorb with the additional explanatory power which they provide. The basic selection criterion which we use might be termed a “change of model” F-test[11] and is of the form:

$$F_i = \frac{\sum\left(y-\hat{y}_a\right)^2 - \sum\left(y-\hat{y}_b\right)^2 \Big/ \sum df_A - \sum df_B}{\sum\left(y-\bar{y}\right)^2 - \sum\left(y-\hat{y}_a\right)^2 \Big/ (n-1) - \sum df_B} = \frac{\Delta RSS/\Delta df}{RSS\ /\ n-df} \tag{4.12}$$

The first algorithm might be considered as a “benchmark” as it is simply based on the concept of controlling the degrees of freedom within a model by varying the degree of regularisation which is used during the estimation procedure. The “pseudo-constructive” form of the algorithm starts with a heavy regularisation term, and successively relaxes the regularisation whilst the additional degrees of freedom thus created are deemed to be statistically *significant.* The “pseudo-deconstructive” form of the algorithm starts with a low degree of regularisation which is successively increased whilst the degrees of freedom thus removed from the model are shown to be *insignificant.*

The second algorithm is based upon a “constructive” approach which starts with a null model which is successively enhanced by the addition of components to the model. The additional components are selected on the basis of partial F-tests for significant low-dimensional relationships in the residual errors of the current model. The tests for residual structure are based upon the nonparametric tests provided by the model-free variable selection methodology. The algorithm combines the tasks of variable and architecture selection in that the additional complexity which is added to the model is conditioned only upon the particular input subspace (set of independent variables) within which residual structure has been identified.

The third algorithm within our (partial-)F testing methodology is based upon a “deconstructive” approach. The algorithm starts with an overparametrised architecture which is successively refined by iteratively removing variables which are found to provide statistically insignificant contributions to the overall model. This approach can be seen as complementary to the constructive approach in that the two algorithms approach the bias-variance tradeoff from

[11] sometimes referred to as a “partial” F-test in that only a part of the entire model is being tested for statistical significance

opposite directions. As with the constructive algorithm, the deconstructive algorithm also provides an integrated approach to the two tasks of complexity control and variable selection.

**Conditional Statistical Arbitrage Strategies**

Our conditional statistical arbitrage strategies are a mechanism for exploiting the predictive information which is contained in forecasting models of asset price dynamics. Rather than being concerned with changes in individual asset prices, as would be the case in a standard ECM (see Section 2.2.3), our models are concerned with changes in the prices of the *combinations* of assets which define statistical mispricings. The general form of our "mispricing correction models" or MCMs is:

$$\Delta\left(T_t - \sum_{C_i \in C} b_{i,t} C_{i,t}\right) = \Delta M_t = f\left(M_t, \Delta M_{t-t}, Z_t\right) + e_t \tag{4.13}$$

where $M_t$ is the deviation from the statistical fair price relationship, or "statistical mispricing", $\Delta M_{t-t}$ represents a set of lagged changes in the mispricing, and $Z_t$ represents a selected set of exogenous variables which either directly influence the mispricing dynamics or serve to modulate the mispricing-correction effect.

The "conditional statistical arbitrage" (CSA) trading rules which are designed to exploit the predictions of the MCMs take a similar form to the implicit statistical arbitrage (ISA) rules from Part I of the methodology. As in the case of the implicit rules, the CSA rules define the position which should be taken in terms of the portfolio which corresponds to the statistical mispricing $M_t$, namely a portfolio with assets $\left\{T_t, C_1, C_2, \ldots, C_{n_c}\right\}$ and relative weightings $\left\{1, -b_1, -b_2, \ldots, -b_{n_c}\right\}$. The difference in this case is that the position to be taken is defined as a function of the MCM *forecast* $\mathrm{E}\left[\Delta M_t\right]$, rather than of the *level* of mispricing $M_t$:

$$\mathrm{posn}(\mathrm{E}\left[\Delta M_t\right], k, h) = \frac{\sum_{j=1..h} \mathrm{sign}\left(\mathrm{E}\left[\Delta M_{t-j}\right]\right) \left|\mathrm{E}\left[\Delta M_{t-j}\right]\right|^k}{h} \tag{4.14}$$

As in the case of the ISA rules, $h$ and $k$ represent smoothing and sensitivity parameters respectively. The relationship is a *positive* one because the implicitly negative relationship between the level of the mispricing and expected future changes in the mispricing is now absorbed in the forecasting model.

We can define a taxonomy of statistical arbitrage strategies based upon different functional forms of mispricing correction model and thus the types of behaviour that they are designed to accommodate. Such a taxonomy is presented in Table 4.1 below:

| **Strategy** | **Mispricing Correction Model**<br>**(*implicit* or explicit)** | **Properties** |
|---|---|---|
| *riskless* | $\mathrm{E}\left[\Delta M_t\right] = -M_t$ | *imposition of fair price at expiry* |
| *implicit* | $\mathrm{E}\left[\Delta M_t\right] \propto -M_t$ | *strong reversion* |
| *adaptive* | $\mathrm{E}\left[\Delta M_t\right] \propto L_t - M_t$ | *time-variation in reversion level* |
| time-series | $\mathrm{E}\left[\Delta M_t\right] = a + b_0\left(L_t - M_t\right) + \sum b_i \Delta M_{t-i}$ | momentum plus other dynamics |
| conditional | $\mathrm{E}\left[\Delta M_t\right] = f\left(L_t - M_t, \Delta M_{t-i}, Z_t\right)$ | threshold effects, exogenous variables |

Table 4.1: A taxonomy of statistical arbitrage strategies, based upon the assumed form of the mispricing correction model (MCM). The MCM may be implicit, as in the first three rows, or explicit, as in the time-series and conditional models.

The taxonomy highlights the conceptual progression from *riskless* strategies to *implicit* statistical arbitrage strategies and then to *conditional* statistical arbitrage strategies. Riskless strategies are based upon the expectation that the mispricing will be eliminated totally, at the expiration date if not sooner. Implicit statistical arbitrage strategies exploit mean-reversion tendencies in the mispricing dynamics, be they with respect to a fixed or time-varying reversion level. Conditional statistical arbitrage strategies involve the construction of an explicit model to exploit more general regularities in the mispricing dynamics.

The advantage of employing conditional statistical arbitrage strategies based upon flexible forecasting models is that other deterministic components of the dynamics can be accounted for, in addition to the basic tendency of mean-reversion. By capturing these additional components in the forecasting model it may be possible both to decrease risk and increase profitability. A decrease in risk may occur because otherwise unfavourable effects, such as a short-term momentum acting to temporarily *increase* the mispricing, can be accounted for in the model forecasts. An increase in profits may occur because the overall accuracy of the forecasting model will improve due to capturing a larger part of the deterministic component in the underlying dynamics.

The advantage of employing conditional statistical arbitrage strategies in cases where the mispricing dynamics are other than straightforwardly mean-reverting is illustrated in Figure 4.7. The figure compares the performance of a number of alternative statistical arbitrage strategies applied to an artificial time-series with the data-generating process:

$$\Delta M_t = a\Delta M_{t-1} - \left(b + gZ_t\right)M_t + e_t \tag{4.15}$$

The parameters $a, b,$ and $g$ were chosen such that the linear mean-reversion component, $-bM_t$, accounted for approximately 1% of the total variance of the time-series, the momentum term, $a\Delta M_{t-1}$, a further 1% and a modulated term, $-gZ_t M_t$, a further 2% of the total variance, in the form of an interaction effect with a (randomly generated) exogenous variable $Z_t$.

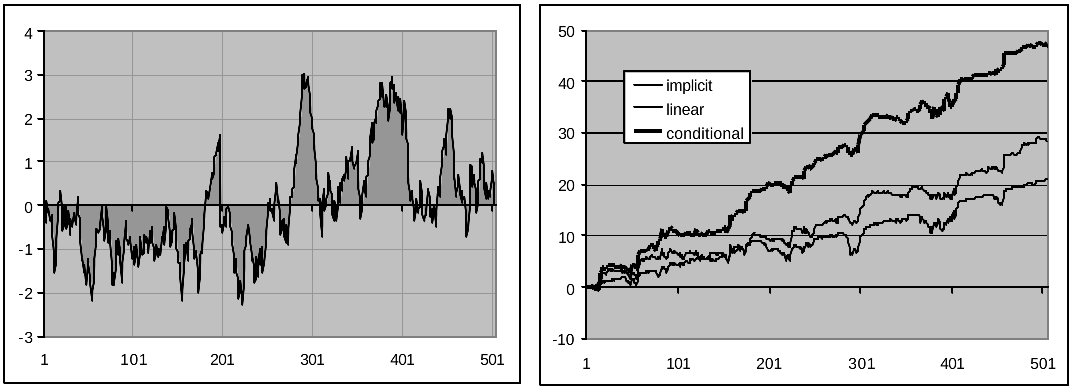


Figure 4.7: Comparison of the performance of different statistical arbitrage strategies when applied to an artificial time-series with the data-generating process in Eqn. (4.15). The left hand chart shows the time-series itself. The right hand side shows the cumulative profit and loss curves for three alternative models: an implicit model, a linear time-series model, and a full conditional model.

The figure clearly shows the performance improvement which is attained by the models which more-accurately capture the deterministic component of the time-series dynamics. The *implicit* model only captures the mean-reversion effect which accounts for 1% of the total variability in the time-series. Whilst generally profitable, the implicit strategy suffers from severe drawdowns during the periods where the time-series exhibits short term trends away from the equilibrium (for example around periods 300, 400 and 450). The linear *time-series* model captures both the linear component of the mean-reversion together with the short-term momentum effect, accounting in total for 2% of the total variability. The improved accuracy of the time-series model leads to higher profits and less severe drawdowns than are achieved by the implicit model. Finally, the *conditional* model also captures the modulation of the mean-reversion effect with respect to the exogenous variable, accounting for 4% of the variability in all and achieving substantially higher profits and lower drawdowns than either of the more limited models.

## 4.4 Part III: Diversifying Risk by Combining a Portfolio of Models

The objective of the third part of our methodology is to support the diversification of model risk. The methodology aims to reduce the risks which are inherent in the modelling process itself, thus increasing the likelihood of achieving successful statistical arbitrage strategies.

### Combination in a portfolio of models

Irrespective of the statistical rigour of the modelling process, the *future* performance of forecasting models in general, or statistical arbitrage models in particular, will necessarily be uncertain. The fact that model selection criteria are evaluated with respect to a particular finite sample will cause sampling error. This in turn means that the model which is apparently optimal, according to the model selection criteria, will not necessarily be optimal in future. Furthermore this risk is much increased when the noise content of the data is high and/or the performance may be unstable over time due to the presence of nonstationarities in the underlying data-generating process.

The effect of using small and/or noisy samples is that the sampling error on any insample performance estimate will necessarily be large. Similarly, the effect of any time-variation in the underlying data-generating process will be to increase the variability of future performance, given a particular level of insample performance. This serves to increase the model selection risk, and reduce the probability that the optimal insample model will also perform optimally during an out-of-sample period. The model selection risks which arise from the use of finite, noisy and possibly nonstationary datasets are illustrated in Figure 4.8 below:

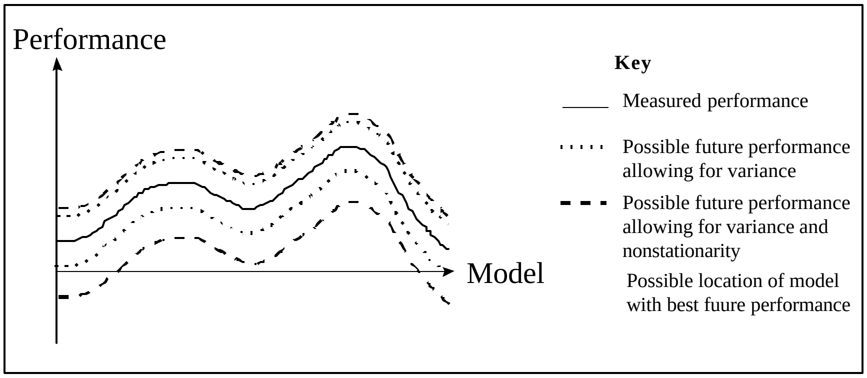


Figure 4.8: Distribution of the possible future performance of a set of models, allowing for both sampling error in the performance statistic and the possible effects of nonstationarity.

Note that, whilst the effects of variance are equally likely to be positive as negative, time-variation or "nonstationarity" is more likely to lead to performance degradation than to an improvement.

Additional complications to the model selection procedure are caused by "criterion risk" and "selection bias". Criterion risk arises in the case where the metric which is used for model selection is not the same as the metric by which future performance will be evaluated. Selection bias, or "data-snooping", arises because the measured performance of a *selected* model will be positively biased with respect to the true expectation of future performance, due to the fact that:

$$\max(m_i) + \mathrm{E}[e_i] \leq \mathrm{E}[\max(m_i + e_i)] \leq \max(m_i) + \max(e_i) \tag{4.16}$$

where $m_i$ is the true insample performance of model *i* and $e_i$ represents the sampling error of the performance measure. The selection bias is caused by the discrepancy between the first expression, which represents the true expectation of future performance, and the second expression, which is the measured insample performance (after correcting for any other biases). The third expression indicates that the maximum selection bias is determined by the sample variability of the performance metric.

The problems posed by model selection risk are not addressed by standard modelling procedures. The model combination techniques discussed in Section 2.2.5 are intended not to reduce model **risk** as such but rather to improve model **performance**, in the sense of minimising forecasting error. In finance however, there is an explicit requirement to take into account not only performance (or expected return) but also risk (or uncertainty in the level of return), with the ultimate measure of utility being some form of “risk-adjusted return”.

The methodology which we propose to deal with this issue is an adaptation of the model combination approach in the context of modern portfolio theory (Markowitz, 1952). The core inspiration of Markowitz portfolio theory is that a suitable combination (portfolio) of *assets* may achieve a higher level of risk-adjusted return than any individual asset, due to the ability to diversify risk through a combination of less than perfectly correlated assets. The basis of our model combination methodology is to adapt the Markowitz framework by substituting the profits and losses of statistical arbitrage *models* in place of the returns on individual assets, i.e. to create a “portfolio of models”.

Given a set of models $\mathbf{m} = \left\{ m_1\left(\mathrm{q}_1\right), m_2\left(\mathrm{q}_2\right), \ldots, m_{nm}\left(\mathrm{q}_{nm}\right) \right\}$, (uncertain) performance estimates $\mathbf{r}_M = \left[ \mathrm{E}\left[r_1\right] \quad \mathrm{E}\left[r_2\right] \quad \cdots \quad \mathrm{E}\left[r_2\right] \right]^T$ and covariance matrix $\mathbf{V}_M = \mathrm{E}\left\{ \left[ e_1 \quad e_2 \quad \cdots \quad e_{nm} \right]^T \left[ e_1 \quad e_2 \quad \cdots \quad e_{nm} \right] \right\}$. Then the portfolio of models is given by:

$$P_M = \sum_i w_{M,i} m_i \tag{4.17}$$

and the optimal vector of weights is that which maximises risk-adjusted performance:

$$\mathbf{w}_M{}^* = \arg\max_{\mathbf{w}} \left\{ \mathbf{w}_M{}^T \mathbf{r}_M - \frac{1}{2T_M} \mathbf{w}_M{}^T \mathbf{V}_M \mathbf{w}_M \right\} \tag{4.18}$$

As in the case of standard portfolio optimisation, the risk-tolerance parameter $T_M$ embodies the desired trade-off between increased expected performance and decreased levels of risk.

**Joint optimisation of components within models, and models within a population**

The final component of our methodology is designed to address two important issues which in fact are facets of the same underlying problem. The first issue is the so-called “forecasting bottleneck” which arises when the performance of a trading rule is conditioned upon the output of a forecasting model. The second issue is that of successful diversification of model risk within the context of the “portfolio of models” approach.

The underlying issue in both cases is the potential inefficiency which is caused by the use of indirect optimisation, by which we mean optimising a model, or a model component, with respect to criteria which are different to those by which the ultimate performance of a model will be judged. This risk is illustrated in Figure 4.9 below.

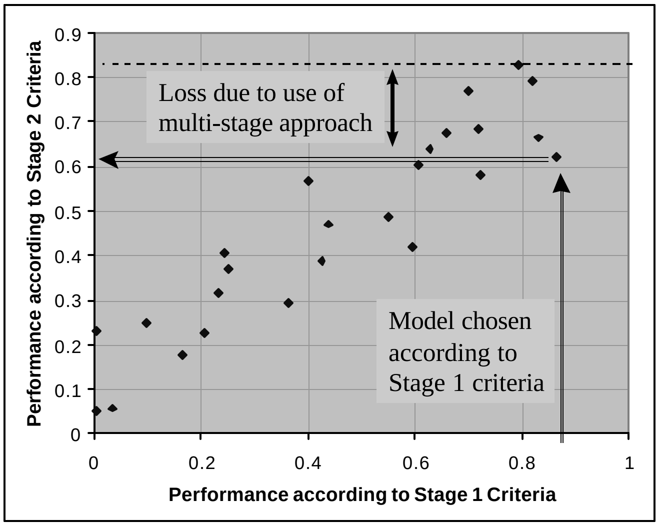


Figure 4.9: Illustration of the potential inefficiency which arises through the use of different criteria at different stages of the modelling process, or when the criteria used during model construction are different to those by which actual performance is measured.

In the figure, the model selected according to the criteria of Stage 1 is significantly less than optimal at Stage 2 even though the criteria at each stage are 90% correlated, clearly highlighting the potential inefficiency of indirect optimisation. In this thesis we refer to this potential inefficiency, and source of additional uncertainty, as “criterion risk”.

The criterion risk which arises in the context of the “forecasting bottleneck” is that the forecasting model which is optimal with respect to the statistical criteria used during the modelling procedure will not necessarily lead to the optimal performance when used in conjunction with a trading strategy which is optimised independently. In particular a model which has the most accurate forecasts may not lead to the optimal trading performance if the volatility of the forecasts leads to large numbers of transactions being performed and the profitability being eroded by transaction costs.

In the case of diversifying model risk using a portfolio of models approach, the criterion risk is that models which appear attractive in terms of their individual risk-adjusted performance may not necessarily add value to an existing portfolio of models. In particular, models which are highly correlated with other models will offer little opportunity for risk-diversification. This is a common problem because the use of model selection criteria which view models on an individual basis will tend to select models which have advantageous, yet similar, characteristics. In the context of genetic algorithms and similar population-based algorithms this phenomenon is sometimes referred to as "premature convergence".

The final part of our methodology is a **population-based algorithm** which is intended to mitigate the potential inefficiencies which are discussed above. The key elements of the specification of each of the three components of a statistical arbitrage model (mispricing model, forecasting model and trading rule) are grouped together as a set of "meta-parameters". The "fitness" of the set of meta-parameters is evaluated in terms of the actual trading performance of the overall model, thus serving to avoid the criterion risk which would arise from separate optimisation of the different model components. This process can be seen as avoiding the forecasting bottleneck by allowing feedback from the trading performance to be passed back to the other model components, thus closing the modelling "loop" (see Figure 4.1) and allowing *joint optimisation* of all model components with respect to a single set of criteria.

Criterion risk due to the evaluation of models on an individual basis is controlled by means of "conditional fitness" measures which evaluate models in the context of the broader set of models. The conditional fitness is defined as the *marginal utility* of the model, i.e. the improvement in the performance of the population as a whole which is caused by adding the model to an existing portfolio of models. An example of a conditional fitness measure is presented in Figure 4.10.

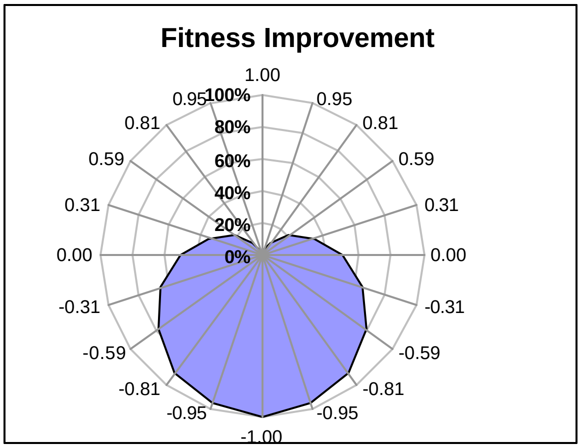


Figure 4.10: Illustration of the added value that a new model provides to an existing portfolio, according to a standard measure of risk-adjusted return. The correlation between the returns of the model and those of a pre-existing portfolio is displayed as a function of the angle (ρ=cos(θ)) with correlation of 1 at zero degrees from vertical, 0 correlation at +/- 90$^{o}$ and -1 correlation at 180$^{o}$. The expected risk and return of the new model are equal to those of the existing portfolio

The figure clearly shows demonstrates that, irrespective of the fact that the expected return and risk of the new model are identical to those of the existing portfolio, the *added-value* which the new model represents is heavily dependent on the extent to which it is decorrelated with the existing portfolio. This demonstrates how, in the context of trading a portfolio of statistical arbitrage models, the *conditional fitness* approach employed in our population-based algorithm explicitly accounts for the correlations between models and thus actively encourages the creation of a diversified set of models.

## 4.5 Summary

In this chapter we have outlined our modelling framework and summarised the three main components of our methodology. The following three parts of the thesis present the details of the methodology, together with evaluations of the modelling procedures with respect to both artificial simulations and real-world problems.

# Part I: A Cointegration Framework for Statistical Arbitrage

In this part of the thesis we describe the first of the three parts of our methodology for statistical arbitrage modelling. This consists of a framework for constructing combinations of time-series which contain a potentially predictable component. An overview of the methodology described in this part of the thesis is presented in Section 4.2.

Chapter 5 describes a framework for modelling combined asset prices, which is inspired by the econometric concept of cointegration. The cointegration framework is used to generate potential statistical mispricings, by performing stochastic detrending of asset prices with respect to other, related, asset prices. The objective of this pre-processing is to generate combinations of time-series which are largely immunised against market-wide uncertainties and which enhance the potentially predictable components of asset price dynamics. The basic methodology is extended to time-varying relationships and also to high-dimensional problems where the number of assets is numbered in the tens or hundreds.

Chapter 6 describes a range of tests which are designed to identify potential predictability in the mispricing time-series. The tests include standard autocorrelation tests, cointegration tests for stationarity, and variance ratio tests for deviations from random walk behaviour as well as novel tests based upon the shape of the variance ratio profile as a whole. The strengths and weaknesses of the various tests are examined under controlled circumstances using Monte-Carlo simulations. Modifications of the predictability tests are presented which correct for the bias induced by the construction procedure. The final section of the chapter contains the results of applying the tests to combinations of FTSE 100 equity prices.

Chapter 7 describes a set of "implicit statistical arbitrage" (ISA) trading strategies which are designed to directly exploit any mean-reverting component in the mispricing dynamics, bypassing the intermediate stage of constructing an explicit forecasting model. The underlying assumption of the ISA strategies is that future price changes will be such as will tend (on average) to reduce the mispricing between a given "target" asset and the associated "synthetic asset". The ISA rules are used to perform an empirical evaluation of the statistical mispricing methodology. The high-dimensional version of the mispricing construction methodology is evaluated with respect to models of the daily closing prices of FTSE 100 stocks; the adaptive version of the methodology is evaluated on the French CAC 40 and German DAX 30 stock market indices.

# *5. Methodology for Constructing Statistical Mispricings*

This chapter describes the methodology which we use to construct combinations of asset prices which represent statistical mispricings. Section 5.1 describes the basic cointegration framework which is used to generate the mispricing time-series. The methodology operates by performing stochastic detrending of related asset prices in order to partially immunise against market-wide sources of uncertainty, or “risk factors”, thus enhancing the potentially predictable asset specific components of the dynamics. The following two sections present extensions of the basic methodology. In Section 5.2, an adaptive filtering methodology is presented which is used to remove nonstationary components from mispricing time-series in order to qualify them for use as the basis of statistical arbitrage models. Section 5.3 describes a stepwise approach which extends the basic methodology to the case of high-dimensional problems where the number of assets is numbered in tens or hundreds, as is typically the case in modelling the constituents of indices such as the CAC, DAX, FTSE and Dow Jones.

## 5.1 Description of Cointegration Methodology

In this section we describe the basic cointegration framework which we use to generate combinations of time-series which represent statistical mispricings. Whilst the econometric methods used in cointegration modelling form the base upon which our methodology has been developed, we apply and further develop these tools according to a rather different perspective than the traditional one.

Cointegration *testing* (see Section 2.2.2) in particular is often seen as an end in itself, with the objective of testing an economic hypothesis regarding the presence or absence of an equilibrium relationship between a set of economic variables. This stage may or may not be followed by the construction of an Error-Correction Model (ECM, see Section 2.2.3) in order to establish the dynamics of the mechanism by which short-term deviations are corrected towards the long-term equilibrium.

In our methodology the use of tools from cointegration modelling is seen as a “means to an end”, with the “end” being the creation of successful statistical arbitrage trading strategies. From this perspective the cointegration framework is a useful starting point, but one in which the underlying modelling assumptions and restrictions are overly strict. Whilst the object of cointegration testing is to identify combinations of time-series which are *stationary*, our statistical arbitrage modelling is concerned with the more general case of combinations of time-series which simply contain *a predictable component.* This generalisation of the cointegration framework is both necessary and useful because the requirement for strict stationarity is over-restrictive and would cause many potential opportunities for statistical arbitrage to be overlooked.

Our use of cointegration methods to construct statistical mispricings can be thought of as a principled extension of the *relative value* strategies, such as “pairs trading”, which are used by some market practitioners. Before describing the details of our mispricing construction methodology we first provide a more formal motivation for the use of a relative value approach to modelling asset price dynamics.

### Relative Value Modelling

In order to motivate the use of modelling techniques in which prices are viewed in relative (or combined) rather than absolute (or individual) terms, we refer back to Section 2.2.1 and note that any random variable $y_t$ can be considered as being the result of a data-generating process which is partially deterministic and partially stochastic:

$$y_t = g(\mathbf{z}_t) + e(\mathbf{v}_t)_t \tag{5.1}$$

where $g(\mathbf{z}_t)$ is the deterministic component of the time-series and $e(\mathbf{v}_t)_t$ is the stochastic component. Considering that the purpose of modelling the data-generating process is to construct an estimator $\hat{y}_t = f(\mathbf{x}_t)$ we note that the maximum possible predictive ability (according to the $R^2$ metric of proportion of variance explained) is limited by the proportion of the variance of $y_t$ which is due to the deterministic component:

$$R^2(y_t, f(\mathbf{x_t})) \leq \frac{\mathrm{E}\left[\left(g(\mathbf{z}_t) - m_y\right)^2\right]}{\mathrm{E}\left[\left(g(\mathbf{z}_t) - m_y\right)^2\right] + \mathrm{E}\left[e(\mathbf{v}_t)^2\right]} \tag{5.2}$$

In many forecasting applications, this issue is overlooked, largely because the target series is often predetermined. However in an application such as statistical arbitrage there are many possible combinations of assets which could be used as target series, and thus the construction of time series with significant deterministic components is a key component of our methodology.

The primary motivation for adopting a "combined price" approach is based upon the recognition that much of the "noise" or stochastic component in asset returns is common to many assets. This viewpoint forms the basis of traditional asset pricing models such as CAPM (Capital Asset Pricing Model) and the APT (Arbitrage Pricing Theory). Essentially these pricing models take the form:

$$\Delta y_{i,t} = a_i + b_{i,Mkt}\Delta Mkt_t + b_{i,1}\Delta f_{1,t} + .... + b_{i,n}\Delta f_{n,t} + e_{i,t} \tag{5.3}$$

This general formulation, which is discussed in Section 3.1, relates changes in asset prices $\Delta y_t$ to sources of systematic risk (changes in the market, $\Delta Mkt_t$, and in other economic "risk factors", $\Delta f_{j,t}$) together with an idiosyncratic asset specific component $e_{i,t}$.

If markets are more efficient at discounting major sources of economic uncertainty than asset specific risks, then the proportion of the variance of market-wide risk factors which is due to a deterministic and hence potentially predictable component ($d_F$) will be lower than the deterministic component in idiosyncratic effects ($d_I$). The overall level of potential predictability of a time-series will lie somewhere between the two values depending upon the extent to which the dynamics are influenced by the two types of effect:

$$R^2\left(\Delta y_{i,t}, \hat{y}_t\right) \leq w_F d_F + \left(1 - w_F\right) d_I \tag{5.4}$$

where $\hat{y}_t$ represents the forecasting model, $R^2$ the proportion of variance correctly forecasted by the model, and $w_F$ the proportion of the dynamics which are caused by market-wide risk factors. For instance, if $d_F$ = 1% and $d_I$ = 5%, an asset with market-wide exposure of $w_F = 0.75$ would have a potentially predictable component equivalent to $0.75 \times 1\% + 0.25 \times 5\% = 2\%$ of its total variance. In contrast, a price series with lower exposure to market-wide factors, say $w_F = 0.25$, might contain a potentially predictable component equivalent to $0.25 \times 1\% + 0.75 \times 5\% = 4\%$ or a 100% increase on the first case.

This distinction between the dynamics which are due to market-wide risk factors and the asset specific component of price dynamics provides a strong motivation for believing that the returns of appropriate combinations of asset prices may be potentially more predictable than the raw (individual) returns. Consider a portfolio consisting of a long (bought) position in an asset $y_1$ and a short (sold) position in an asset $y_2$. If the asset price dynamics in each case follow a data-generating process of the form shown in Eqn. (5.3) then the *combined* returns $\Delta y_{1,t} - \Delta y_{2,t}$ are given by:

$$\begin{aligned} \Delta y_{1,t} - \Delta y_{2,t} &= \left(a_1 - a_2\right) \\ &\quad + \left(b_{1,Mkt} - b_{2,Mkt}\right)\Delta Mkt_t + \left(b_{1,1} - b_{2,1}\right)\Delta f_{1,t} + \ldots + \left(b_{1,n} - b_{2,n}\right)\Delta f_{n,t} \\ &\quad + \left(e_{1,t} - e_{2,t}\right) \end{aligned} \tag{5.5}$$

If the factor exposures are similar, i.e. $b_{1,j} \approx b_{2,j}$, then the proportion of variance which is caused by market-wide factors will be correspondingly reduced and the potential predictability correspondingly increased. This effect is illustrated in Figure 5.1 below.

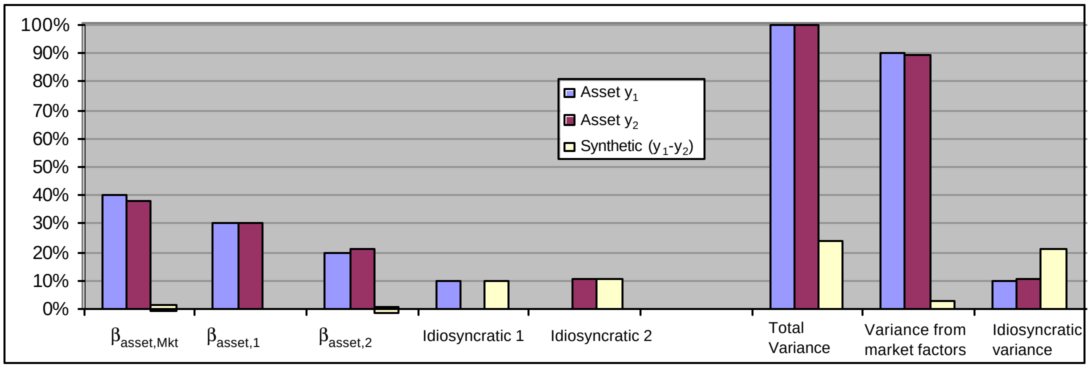


Figure 5.1: Illustration of advantages of modelling within a combined price framework: whilst the individual assets Y1 and Y2 are primarily influenced by changes in market-wide risk factors, the price-changes of the "synthetic asset" Y1-Y2 are largely immunised from such effects and magnify the effect of stock-specific factors which are more likely to contain systematic (and hence potentially predictable) components.

Thus the motivation for a relative value approach is that asset prices viewed in relative (i.e. combined) rather than absolute (i.e. individual) terms are both more amenable to statistical arbitrage and potentially more predictable. The analysis above demonstrates that, in principle, appropriately constructed combinations of prices can be largely immunised against market-wide sources of risk and will instead highlight the asset specific aspects of the price dynamics. Such combinations of assets are amenable to statistical arbitrage because they represent opportunities to exploit predictable components in asset specific price dynamics in a manner which is (statistically) independent of changes in the level of the market as a whole, or other market-wide sources of uncertainty. Furthermore, as the asset-specific component of the dynamics is not directly observable by market participants it is plausible that regularities in the dynamics may exist from this perspective which have not yet been "arbitraged away" by market participants.

This analysis helps to explain the popularity of relative value strategies amongst hedge funds, proprietary trading desks and other "risk arbitrageurs". In its simplest form this approach is called "pairs trading" and consists of trend and reversion analysis of a graph of relative prices, with the assets X and Y being selected either on the basis of intuition, economic fundamentals, long term correlations or simply past experience. Opportunities for pairs trading in this simple form, however, are reliant upon the existence of naturally similar pairs of assets and thus are very limited. We describe below the details of our methodology in which cointegration techniques are used to create a wider range of opportunities, by **constructing** synthetic "pairs" in the form of appropriate combinations of two **or more** assets.

### Construction of Statistical Mispricings

In an analogy to the no-arbitrage relationships upon which riskless arbitrage strategies are based, the objective of our methodology is to identify combinations of assets which represent statistical "fair price" relationships upon which to base "statistical arbitrage" strategies.

More specifically, given an asset universe $U_A$ and a particular "target asset", $T \in U_A$, our objective is to create a "synthetic asset" $SA(T)$ such that the value of the synthetic asset can be considered a statistical "fair price" for the target asset:

$$\mathrm{E}\left[T_t\right] = SA(T)_t \tag{5.6}$$

Furthermore the fair price relationship in Eqn (5.6) should be such that deviations from the relationship can be considered as "statistical mispricings":

$$M_t = T_t - SA(T)_t \tag{5.7}$$

where the dynamics of the mispricing time-series $M_t$ contain a predictable component which can be exploited as the basis of a statistical arbitrage trading strategy.

Our methodology for constructing statistical mispricings is based upon the use of cointegration techniques to estimate the fair price relationships. A "cointegrating regression" (Granger, 1983) is used to estimate a linear combination of assets which exhibits the maximum possible long-term correlation with the target asset $T$. The coefficients of the linear combination are estimated by regressing the historical price of $T$ on the historical prices of a set of "constituent" assets $C \subset U_A - T$:

$$SA(T)_t = \sum_{C_i \in C} b_i C_{i,t} \text{ s.t. } \{b_i\} = \arg\min \sum_{t=1..n} \left( T_t - \sum_{C_i \in C} b_i C_{i,t} \right)^2 \tag{5.8}$$

and the "cointegrating vector" $\mathrm{b} = \begin{bmatrix} b_1 & \cdots & b_{n_C} \end{bmatrix}^T$ of constituent weights is given by:

$$\mathrm{b}_{\mathrm{OLS}} = \left(\mathbf{C}^T \mathbf{C}\right)^{-1} \mathbf{Ct} \tag{5.9}$$

where $\mathbf{C}$ is the $n_c = |C|$ by $n$ matrix of historical prices of the constituents and $\mathbf{t} = \begin{bmatrix} T_1 & \cdots & T_n \end{bmatrix}^T$ is the vector of historical prices of the target asset.

The synthetic asset can be considered an optimal statistical hedge, conditioned upon the set of constituent assets $C$, in that the standard properties of the OLS procedure used in regression ensure both that the synthetic asset will be an unbiased estimator for the target asset, i.e. $\mathrm{E}\left[T_t\right] = SA(T)_t$, and also that the deviation between the two price series will be minimal in a mean-squared-error sense.

We formally define a "synthetic asset model" as a triple: $SA = \left\{T \in U_A; C \subset U_A - \{T\}; b \in \Re^{|C|}\right\}$ where $U_A$ is the asset universe, $T \in U_A$ is the "target asset", $C \subset U_A - \{T\}$ is the set of "constituent assets" and $b \in \Re^{|C|}$ is the vector of constituent weights. Given such a model, we can derive the time-series which represents the associated statistical mispricing:

$$M_t = T_t - \sum_{C_i \in C} b_i C_{i,t} \tag{5.10}$$

The statistical mispricing $M_t$ can be considered as a composite portfolio consisting of the assets $\left\{T_t, C_1, C_2, \ldots, C_{n_C}\right\}$ with weightings $\left\{1, -b_1, -b_2, \ldots, -b_{n_C}\right\}$ respectively. The price of this portfolio represents the excess value of the target asset $T$, relative to the linear combination of assets $SA(T)_t = \sum b_i C_{i,t}$ and can be thought of as a "stochastically detrended" version of the original asset price $T_t$ (i.e. detrended with respect to observed time-series which are generated by (at least partially) stochastic processes rather than with respect to a deterministic function of time) .

In this context the set of constituent assets $C$ can be considered as acting as proxies for the unobserved risk factors which act as common stochastic trends in market prices. In maximising the correlation between the target asset and the synthetic asset the construction procedure cannot (by definition) account for the “asset specific” components of price dynamics, but must instead indirectly maximise the sensitivities to common sources of economic risk. In the context of Eqn (5.5) the effect of the construction procedure is to artificially create a pair of assets (the target asset $T_t$ and the synthetic asset $SA(T)_t$) which have similar exposures to the underlying (but not necessarily directly observable) risk factors which drive the asset price dynamics.

**Example**

We now illustrate the manner in which the cointegrating regression acts to “cancel out” the exposure to underlying risk-factors and thus enhance the relative magnitude of the asset specific component of the price dynamics. Consider a set of three assets, each following a two-factor version of the data-generating process shown in Eqn (5.3). Price changes within the set of three assets are driven by a total of five factors, two common risk factors $f_1$ and $f_2$ and three asset specific components $e_1$, $e_2$, and $e_3$. Furthermore $f_1$ and $f_2$ are assumed to follow random walk processes whilst the dynamics of the asset specific factors contain a mean-reverting component.

The factor exposures of the three assets X, Y and Z are assumed to be as shown in Table 5.1:

| Asset | f1 | f2 | ε1 | ε2 | ε3 |
|---|---|---|---|---|---|
| X | 1 | 1 | 1 | 0 | 0 |
| Y | 1 | 0.5 | 0 | 1 | 0 |
| Z | 0.5 | 1 | 0 | 0 | 1 |

Table 5.1: Price sensitivity of three assets X, Y and Z to changes in common risk factors $f_1$ and $f_2$ and asset specific effects $e_1$, $e_2$, and $e_3$.

Thus the dynamics of the system are given by:

$$\begin{aligned}
\Delta f_{i,t} &= h_{i,t} & i &= 1,2 \quad h_{i,t} \sim N(0,1) \\
\Delta e_{j,t} &= -0.1e_{j,t} + e_{j,t} & j &= 1,2,3 \quad e_{\mathrm{j},t} \sim N(0,0.25) \\
\Delta X_t &= \Delta f_{1,t} + \Delta f_{2,t} + \Delta e_{1,t} \\
\Delta Y_t &= \Delta f_{1,t} + 0.5\Delta f_{2,t} + \Delta e_{2,t} \\
\Delta Z_t &= 0.5\Delta f_{1,t} + \Delta f_{2,t} + \Delta e_{3,t}
\end{aligned} \tag{5.11}$$

A realisation of the asset prices generated by the system is shown in Figure 5.2 below. Note the broad similarity between the price movements of the three assets.

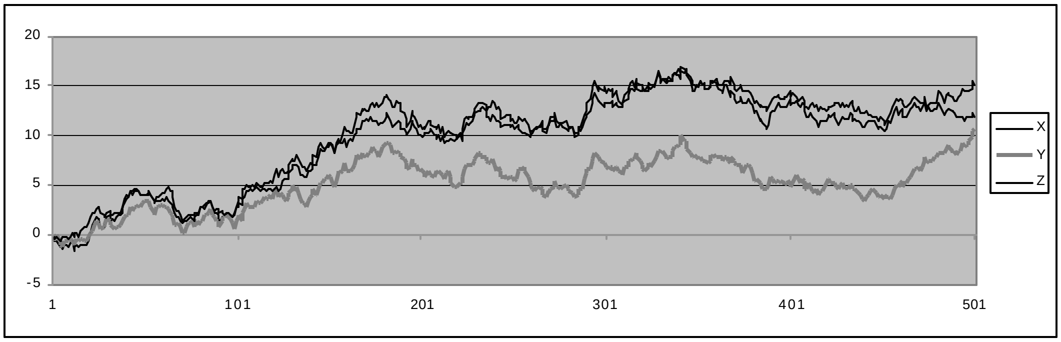


Figure 5.2: Realisation of three simulated asset price series which are driven by two underlying common factors in addition to asset-specific components.

In the case of the realisation shown in Figure 5.2, a cointegrating regression of X on Y and Z gives the estimated fair price relationship:

$$X_t = 0.632Y_t + 0.703Z_t + M_t \tag{5.12}$$

The estimated relationship differs slightly from the theoretically optimal $X_t = 2/3Y_t + 2/3Z_t + M_t$, due to sampling error. The resulting statistical mispricing time-series $M_t$ is shown in Figure 5.3 below. Unlike the nonstationary asset prices X, Y and Z, the mispricing can clearly be seen to be mean-reverting.

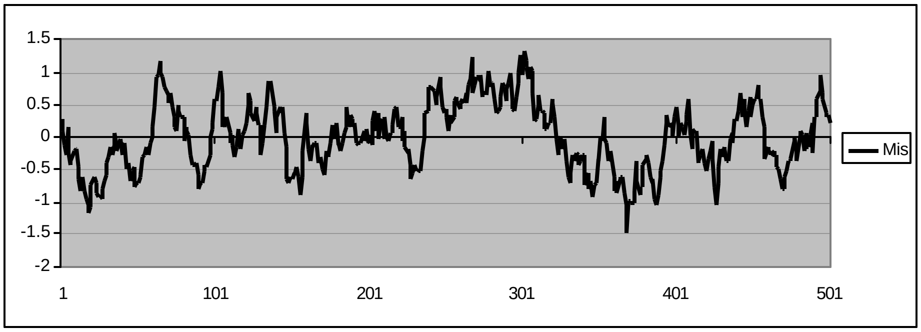


Figure 5.3: Mispricing time-series from the fair price relationship $X_t = 0.632Y_t + 0.703Z_t + M_t$ where the time-series for X, Y and Z are as shown in Figure 5.2.

The mean-reverting nature of the mispricing time-series, compared to the close to random walk behaviour of the original time-series X, Y and Z, is highlighted by the variance ratio profiles shown in Figure 5.3. Whilst the variance ratio for all three original assets remains close to unity in each case, the variance ratio of the *mispricing* falls substantially below one as the period over which the differences are calculated increases. This indicates that the volatility which is present in the short-term dynamics is not reflected in the long term volatility, thus providing evidence for a substantial mean-reverting component in the dynamics of the linear combination $X_t - \left(0.632Y_t + 0.703Z_t\right)$.

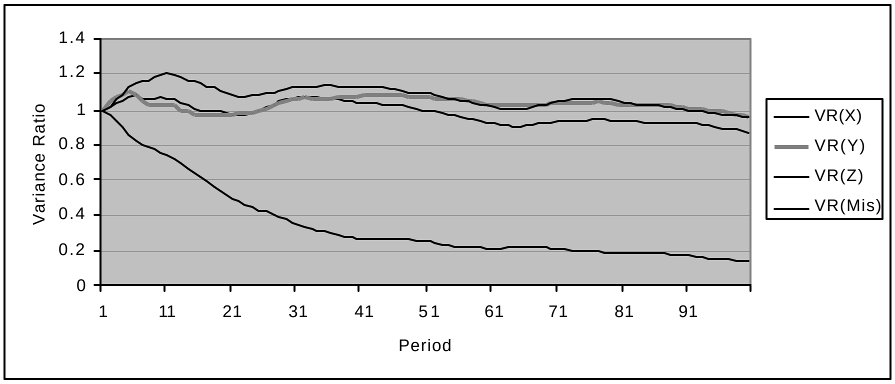


Figure 5.4: Variance ratio profiles for the three time-series X, Y and Z (depicted in Figure 5.2) and the mispricing time-series $Mis_t = X_t - \left(0.632Y_t + 0.703Z_t\right)$

In attempting to replicate the "target" time-series X the cointegrating regression procedure creates the "synthetic asset" $0.632Y + 0.703Z$ which has similar exposures to the common factors $f_1$ and $f_2$. Thus in the mispricing time-series $X_t - \left(0.632Y_t + 0.703Z_t\right)$ the *net* exposure to the common factors is close to zero, allowing the mean-reverting asset specific effects $e_1$, $e_2$, and $e_3$ to dominate the mispricing dynamics. This "statistical hedging" of the common risk factors is quantified in Table 5.2, below, which reports the proportion of the variance of each time-series which is associated with each of the sources of uncertainty:

| | X | Y | Z | Mis |
|---|---|---|---|---|
| $f_1$ | 46.1% | 64.2% | 16.8% | 0.2% |
| $f_2$ | 41.5% | 10.1% | 67.8% | 0.0% |
| $e_1$ | 11.5% | 0.2% | 0.0% | 54.5% |
| $e_2$ | 0.5% | 25.1% | 0.1% | 23.7% |
| $e_3$ | 0.4% | 0.4% | 15.3% | 21.6% |
| Total | 100.0% | 100.0% | 100.0% | 100.0% |

Table 5.2: Sensitivity of price changes of the original time-series X, Y and Z and the "mispricing" time-series $X_t - (0.632Y_t + 0.703Z_t)$. The table entries show the proportion of the variance of each time-series which is associated with changes in common risk factors $f_1$ and $f_2$ and asset specific effects $e_1$, $e_2$, and $e_3$.

This demonstrates the fact that the use of cointegrating regression can immunise against common stochastic trends which are not observed directly but instead proxied by the observed asset prices. Whilst the variance of changes in the original time-series X, Y and Z is primarily (70-90%) associated with the common risk factors $f_1$ and $f_2$, the effect of these factors on the "mispricing" is minimal (0.2%). Conversely the effect of the asset specific factors is greatly magnified in the mispricing time-series, growing from 10%-30% in the original time-series to 99.8% in the linear combination.

By magnifying the component of the dynamics which is associated with asset specific effects, we would expect to magnify the predictable component which is present in the asset specific effects but not in the common factors. This effect can be quantified by considering the Dickey-Fuller statistics obtained from simple error-correction models (ECMs) of the time-series dynamics:

$$DF(s_t) = \hat{b} \big/ s_{\hat{b}} \text{ from regression } \Delta s_t = a - \hat{b} s_t + e_t \tag{5.13}$$

The details of the estimated ECMs are presented in Table 5.3. The results confirm that the predictable component in the asset specific dynamics is masked by the common risk factors in the original time-series but is almost as strongly present in the mispricing time-series as in the underlying, but also unobserved, asset specific dynamics themselves. The magnitude of the deterministic component in the mispricing is 3.9%, which is comparable to the 4.8%, 6.2% and

4.0% in the true asset specific dynamics, and a negligible amount in the case of the original time-series X, Y and Z.

| Factor/Asset | $f_1$ | $f_2$ | $e_1$ | $e_2$ | $e_3$ | X | Y | Z | M |
|---|---|---|---|---|---|---|---|---|---|
| estimated $\hat{b}$ | *0.014* | *-0.001* | ***0.096*** | ***0.124*** | ***0.079*** | -0.001 | -0.000 | -0.001 | **0.077** |
| std. error $s_{\hat{b}}$ | *0.008* | *0.001* | *0.020* | *0.022* | *0.018* | 0.002 | 0.003 | 0.001 | 0.018 |
| $DF(s_t)$ | *1.631* | *-0.580* | *4.908* | *5.644* | *4.454* | -0.452 | -0.004 | -0.591 | 4.398 |
| R$^2$ | *0.5%* | *0%* | *4.8%* | *6.2%* | *4.0%* | 0% | 0% | 0% | 3.9% |

Table 5.3: Details of simple Error Correction Models estimated to quantify the mean-reverting component in both the unobserved factors and the observed time-series. Values in bold correspond to cases where the estimated mean-reversion coefficient $\hat{b}$ is significant at the 0.1% level. The rows in the table are: estimated reversion parameter $\hat{b}$ ; standard error of estimate; associated DF statistic (approximately equivalent to the 't'-statistic in a standard regression); proportion of variance explained by model (R$^2$)

### Discussion

The major objective of this stage of our methodology is to construct combinations of time-series which are both decorrelated with major sources of economic risk and contain a deterministic (and hence potentially predictable) component in the dynamics. The first objective, of decorrelation with major risk-factors, is desirable in order to aid diversification, both within a set of statistical arbitrage strategies, and also between the statistical arbitrage strategies as a whole and other sources of investment income. The second objective simply recognises the fact that the magnitude of the deterministic component represents the **maximum** possible performance of the statistical arbitrage strategies (which can only be **reduced** by imperfections in the subsequent stages of predictive modelling and decision rule implementation).

We have shown above that, under the assumption that the asset specific components of price dynamics are more predictable than those which are due to market-wide risk factors, these two objectives can be jointly achieved by using cointegrating regression to perform stochastic detrending of time-series. However other methods exist which could also be used for this purpose, and indeed which would also fit within our overall statistical arbitrage framework.

Given a particular asset $y_1$ there are two main methods for identifying appropriate assets $y_2$ with similar exposures to market-wide risk factors. The "indirect approach" of relative value modelling, has been described above; the other approach is to build explicit "factor models" of the form shown in Eqn (5.3). From a statistical arbitrage perspective, the price sensitivity or "exposure" $b_{i,j}$ of each asset $y_i$ to each "risk factor" $\{M, f_j\}$ can be used as the basis for creating combinations of assets which have low net exposure to the major risk factors whilst enhancing the asset specific components of the price dynamics.

The standard approach to estimating explicit factor models is to postulate a set of financial and economic variables as the risk factors and to use regression-based techniques to estimate the sensitivity coefficients $b_{i,j}$ (e.g. the sensitivity of a particular stock to, say, long term interest rates). Evidence for mean-reverting dynamics in the residuals of factor models was reported by Jacobs and Levy (1988). However, the task of constructing such models is a difficult problem in itself, and the interested reader is directed to a recent extensive study by Bentz (1999).

A second approach to factor-modelling is to perform multivariate analysis of asset price returns and reconstruct the unobserved risk factors as linear combinations of observed asset returns. The technique of "principal components analysis" (PCA) is a natural tool in this case as it generates sequences of linear combinations (factors) which account for the greatest possible amount of the total variance, subject to the constraint of being orthogonal to the previous factors in the sequence. In an earlier paper (Burgess, 1995) we described an application of this approach to modelling statistical arbitrage in the eurodollar money markets. In this case over 98% of the variability in 12 futures contracts was found to be due to two factors representing a "shift" and a "tilt" in the yield curve[†]. Further applications of PCA to statistical arbitrage modelling are described in Schreiner (1997); Towers (1998); Tjangdjaja *et al* (1998) and Towers and Burgess (1998).

A generalisation of PCA which has recently attracted much attention in engineering disciplines is so-called "independent components analysis" (ICA) using algorithms developed for the blind separation of signals [Bell and Sejnowski, (1995), Amari *et al* (1996)]. In ICA the orthogonality condition of PCA is strengthened to one of complete statistical *independence*,

[†] The "yield curve" is a term used to describe the market interest rate (or "yield") as a function of the time period; a shift in the yield curve can be interpreted as an across the board change in interest rates; a tilt in the yield curve reflects a relative change in long- versus short-term interest rates.

thus taking into account higher moments. Whilst ICA techniques offer exciting potential in computational finance, particularly in that they account not only for expectations ("returns") but also for variances ("risk"), applications in this domain are still rare [Moody and Wu (1997b), Back and Weigend (1998)].

Whilst these factor modelling approaches offer interesting possibilities for statistical arbitrage, the relative value approach based upon cointegration has important practical advantages. The most important of these is that the cointegration models can be conditioned upon the choice of a particular target asset and (relatively small) set of constituent assets, whereas in PCA for example, every asset will be included in every factor. Thus the cointegration approach is both more manageable, in terms of the number of assets which are traded in a given model, and more intuitively understandable. For instance the "statistical mispricings" generated by the cointegration methodology can be associated with the particular target asset upon which they are conditioned; representing a "sophisticated relative value" perspective on the current price of the target asset. This perspective forms the basis of a set of advanced analytics products which we are currently developing in conjunction with Reuters.

In terms of the relative value approach which is implicit in our cointegration framework, a number of recent developments are also of interest, in particular the development of techniques for estimating *nonlinear* cointegration relationships [Granger and Hallman (1991), Markellos(1997), Breitung (1998)]. However the case where a nonlinear combination of financial assets exhibits stationary behaviour is difficult both to motivate and to exploit. The move away from linearity acts to undermine the use of relative value modelling as "implicit" factor modelling. Bringing nonlinearity into the statistical hedge portfolio would complicate the process of trading the mispricing because the relative portfolio weights would themselves become *conditional* on the price levels of the various time-series. This would require additional trades to reweight the portfolio even at times where no change has occurred in the forecasted mispricing dynamics. It is because of these complications that, in the lack of strong evidence for nonlinear cointegration in financial time-series, we choose to develop our methodology on the basis of linear rather than nonlinear cointegration[12].

[12] These reservations do not apply in the case of the mispricing **dynamics**, where Part II of our methodology supports the use of low-bias (neural network) forecasting models which are able to capture smooth nonlinear relationships

## Practical Issues

In order to highlight the difficulties which are involved in estimating statistical fair price relationships in practice, we consider the example of the UK FTSE index and its relationship with a basket of other international equity market indices, comprising the US S&P, German DAX, French CAC, Dutch EOE and Swiss SMI. The data are daily closing prices for all the indices from 6 June 1988 to 17 Nov. 1993. The “fair price” relationship estimated by cointegrating regression is:

$$FTSE_t = 68.4 + 0.055\,DAX_t + 0.122\,CAC_t - 0.060\,EoE_t + 0.367\,SMI_t + 3.78\,S\&P_t + M_{FTSE,t} \qquad (5.14)$$

The R$^2$ statistic for the cointegrating regression is 95.6% indicating that a very large proportion of the volatility of the FTSE index is associated with changes in global risk-factors which are reflected in the other market indices. The time-series for the FTSE 100 index and the synthetic asset portfolio are shown in Figure 3.2, whilst the statistical mispricing between the two series is presented in Figure 5.5 below.

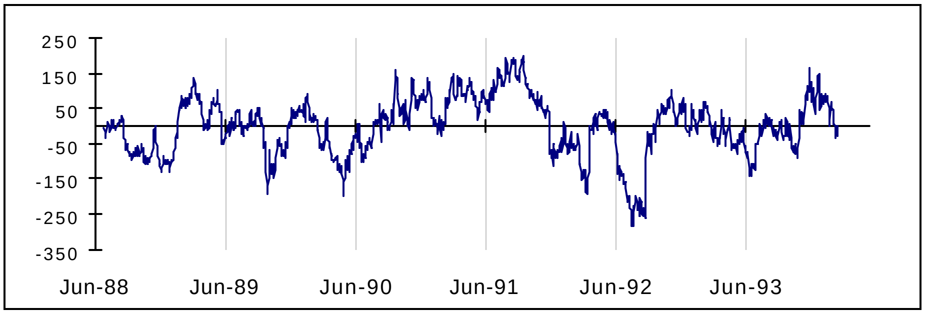


Figure 5.5: Statistical mispricing between the FTSE 100 index and a weighted portfolio of international equity market indices, over the period 6th June 1988 to 17th November 1993.

The chart shows that the mispricing oscillates around a mean value of zero with an apparently high degree of mean-reversion. The largest mispricing is that which follows the UK currency crisis in the late summer of 1992. During this period, the exit of the UK from the Exchange Rate Mechanism (ERM) led initially to a heavy fall in the FTSE relative to the other indices; however, after a few weeks the FTSE had returned approximately to its fair price level relative to the other markets. This sequence of events clearly represented a substantial opportunity for statistical arbitrage had it been predicted in advance. Furthermore, the DF statistic for the mispricing series is 3.5 which is significant at the 0.01 level, indicating that the mean-reversion is not merely an artefact of the estimation procedure but instead represents a significant cointegration between the set of market indices. The error-correction effect is relatively weak in this case, with the mean-reversion component found to account for only around 1% of the variance of the mispricing dynamics.

The major practical complication which arises when applying the mispricing construction methodology is that the estimated fair price relationship may be unstable over time. Such instability may be due either to true time-variation in the cointegrating vector, or by errors in the estimation procedure. The implication is that in-sample significance of the cointegrating relationship is not sufficient to guarantee that the **future** mispricing will be mean-reverting.

Whilst it is not generally possible to *predict* changes in the fair price relationship, it is possible to *adapt* to the changes after they become manifest in the observed data. Thus in section 5.2 we describe an extension of the basic methodology in which an adaptive modelling procedure is used to track changes in the fair price relationship between a set of assets.

Instability in the estimation procedure itself is most likely to be caused by the presence of near-collinearity in the set of constituent asset prices. This danger is heightened when the number of regressors is large because the number of cross-correlations is O($N^2$) in the number of regressors. In order to address this issue, Section 5.3 describes a stepwise procedure to select appropriate subsets of the asset universe as constituent assets, thus extending our basic methodology to the case where the size of the asset universe is numbered in tens or hundreds.

A further precaution against instability would fall into the category of biased and/or constrained estimation procedures, and would consist of replacing the standard OLS regression with a regularised or “ridge” regression. In this case a diagonal term is added to the covariance matrix in order to artificially deflate the covariances and reduce the estimation variance which is caused by near-collinearity. In this “regularised least squares” (RLS) case the cointegrating vector is no longer given by Eqn (5.9) but instead by the modified expression:

$$b_{RLS} = \left(\mathbf{C}'\mathbf{C} + l\mathbf{I}\right)^{-1}\mathbf{C}\mathbf{t} \qquad (5.15)$$

As noted above, other biases or constraints could be imposed upon the mispricing construction procedure, for example a natural constraint may be to disallow negative coefficients in the cointegrating vector as these would correspond to short positions in the synthetic asset. This may serve to further reduce instability, but is actually more likely to be imposed for practical reasons if it is felt that it eases interpretation of the fair price relationships. Further discussion of these modifications is beyond the scope of this thesis, but preliminary developments and results are presented elsewhere (Burgess, 1999).

## 5.2 Extension to Time-varying Relationships

The basic methodology for constructing statistical mispricings assumes that the underlying cointegrating relationship is itself stable through time. There is, however, every reason to believe that the market’s opinion as to what constitutes a “fair price” of one asset relative to another will change over time, with changes in the surrounding economic and political climate, and in market sentiment. For instance, in the Intertemporal Capital Asset Pricing Model (ICAPM) of Merton (1973), shifts in the risk premia attached to the underlying sources of financial and economic uncertainty will cause shifts in asset prices. In cases where the fair price relationship between a set of assets describes a less than perfect hedge against the underlying risk factors such price shifts will act as shocks to the statistical mispricing and thus lead to time-variation in the coefficients of the underlying fair price relationship.

The implication of this is that even models which have worked well in the past are liable to “break down”, i.e. suffer performance degradation due to nonstationarity in the model parameters. In modelling terms, such nonstationarity could arise either because of unobserved components which although present in the true equilibrium relationship are not included in the model or alternatively because the true constituent weightings are time-varying (perhaps due to the influence of conditioning factors).

In this section we describe an extension to the methodology in which parameter nonstationarity can be controlled by means of adaptive modelling techniques. The methodology is based on exponential smoothing techniques in the simplest case and on a state-space implementation of time-varying regression in the more general case. The purpose of these extensions is to allow the constituent weightings to evolve slowly according to a “random walk parameter” model.

The random walk time-varying parameter model which is analogous to the cointegrating regression given in Eqn. (5.8) is:

$$T_t = \sum_{C_i \in C} b_{i,t} C_{i,t} + M_t$$

where (5.16)

$$b_{i,t} = b_{i,t-1} + h_{b_i} \qquad h_{b_i} \sim \mathrm{N}(0,\ s^2_{b_i})$$

The important difference between Eqn (5.16) and Eqn (5.8) is that the parameter vector $\mathrm{b}_{\mathbf{t}} = \left[ b_{1,t} \quad b_{2,t} \quad \dots \quad b_{n_C,t} \right]^T$ is now conditioned upon time to reflect the fact that the parameters are no longer assumed constant but instead follow a random walk process.

In this extended model, deviations from the fair price relationship are considered as being only partially due to statistical mispricings. Instead some of the deviation will be considered as being due to errors in the parameter estimates, $\mathrm{b}_{\mathbf{t}}$, which will be updated accordingly. The extent to which deviations are associated with the mispricing $M_t$ and parameter errors $h_{b_i}$ will be determined by the assumed variance of each of these two sources of innovation in the dynamics.

### Exponential Smoothing of the Reversion Level

The simplest implementation of our adaptive modelling can be considered as a restriction of the more general model in which only the offset term $a_t$ is allowed to have non-zero variance. In this case the model can be simplified to:

$$
\begin{aligned}
T_t &= \sum_{C_i \in C} b_i C_{i,t} + a_t + M_t \\
a_t &= a_{t-1} + \frac{s_a}{s_M} M_{t-1}
\end{aligned}
\tag{5.17}
$$

The constituent weightings are estimated in a one-off regression procedure as in the basic methodology, but a certain amount of the deviation is absorbed in the time-varying “constant” term $a_t$. If the assumed variance of the offset term is small relative to that of the mispricing, i.e. $s_a / s_M << 1$, then $a_t$ will be fairly stable, and will only adjust if the deviations remain consistently of one sign for a long period. The value of $a_t$ can be considered as a time-varying reversion level which reflects changes in outside factors which are not included in the fair price relationship. In particular this “exponential smoothing” approach can be viewed as a means of gradually absorbing any nonstationarities which are asset specific rather than caused by common stochastic trends in the set of asset prices.

This ability to absorb nonstationary components in the mispricing dynamics is illustrated in Figure 5.6 below. The figure illustrates a case where the observed deviation from the fair price relationship is affected by two asset specific events, a step change at period 100 and a gradual decline between period 200 and period 300.

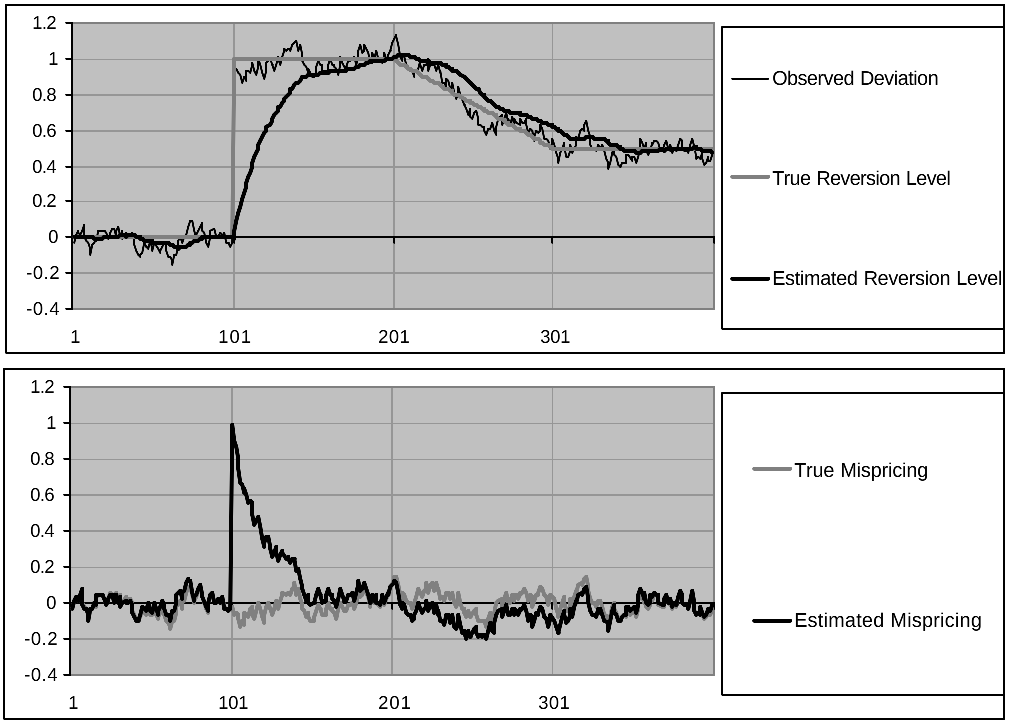


Figure 5.6: Illustration of the absorbtion of nonstationarity in the mispricing dynamics by a model with time-varying reversion level $a_t$ as defined in Eqn (5.15). In the case illustrated, the sensitivity ratio $s_a / s_M = 0.05$.

The upper figure clearly illustrates the manner in which the adaptive reversion level serves to compensate for the nonstationarity in the observed deviation series. Although the step change at period 100 is initially treated as a component of the mispricing, its persistence leads to it being gradually absorbed within the time-varying reversion level. After this adjustment, the estimated mispricing again becomes close to the true mispricing (lower figure). Similarly, the gradual decay between period 200 and period 300 causes a temporary tendency for the estimated mispricing to over-state the true case but again, with time, this nonstationarity is absorbed and the estimated mispricing regains its accuracy.

### State-space Formulation of Adaptive Regression

The more general case of our adaptive modelling methodology is where the fair price relationship is estimated using time-varying parameter regression in which not only the constant term $a_t$ but also the constituent weights $b_{i,t}$ are allowed to vary over time. This formulation, which corresponds to that shown in Eqn. (5.14), is more appropriate than the simpler “exponential smoothing” case when it is the factor exposures to the common stochastic trends which vary with time, as opposed to asset specific price shocks.

The time-varying parameter model can be expressed in state space form and optimised using Kalman filter methodology. In this case the “state” of the model consists of the parameter vector $\mathrm{b}_{\mathbf{t}} = \begin{bmatrix} b_{1,t} & b_{2,t} & \ldots & b_{n_C,t} \end{bmatrix}^T$ (which may or may not be supplemented by a constant term). The current parameter estimates are compared to the observed value of the target asset by means of the so-called “measurement equation”:

$$T_{\mathbf{t}} = \mathbf{C}_{\mathbf{t}}\mathrm{b}_{\mathrm{t}} + e_t \qquad e_t \sim N(0, s_M^2) \tag{5.18}$$

The parameter estimates (states) are then updated according to the “transition equation”:

$$\mathrm{b}_t = \mathbf{T}_{\mathbf{t}}\mathrm{b}_{t-1} + \mathrm{h}_t \qquad \mathrm{h}_t \sim N\left(\mathbf{0}, \mathbf{R}_{\mathbf{t}}\right) \tag{5.19}$$

In the simple random walk parameter model the state transition matrix $\mathbf{T}$ is the identity matrix $\mathbf{I}$, the state (parameter) covariance matrix is assumed to be diagonal $\mathbf{R} = \begin{bmatrix} s_{b_1}^2 & s_{b_2}^2 & \ldots & s_{b_{nC}}^2 \end{bmatrix}\mathbf{I}$, and the model can be seen to correspond to the formulation in Eqn. (5.16). Generalisations of this model can be constructed which allow for trends, reversion and other deterministic components in the parameter values. The detailed formulation of these more general models can be found in any advanced time-series text, such as Harvey (1993). Such models are beyond the scope of this thesis but are discussed extensively by Connor and Bentz (1997) in the context of estimating explicit factor models of asset price dynamics.

**Example**

We demonstrate the filtering methodology using a simulated cointegration relationship with time-varying parameters and the following data-generating process:

$$\begin{aligned} &y_t = \begin{bmatrix} 1 & x_t \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix} + d_t \\ &\text{where } a_{t+1} = a_t + \mathrm{N}\left(m_a, s_a^2\right),\ \ b_{t+1} = b_t + \mathrm{N}\left(m_b, s_b^2\right) \end{aligned} \tag{5.20}$$

The dynamics of the “target” series $y_t$ are governed by a reversion towards the current “equilibrium value” $y_t^E = a_t + b_t x_t$, whilst the “constituent asset” $x_{\mathrm{t}}$ follows a random walk:

$$\begin{aligned}\Delta y_t &= -r d_t + N\left(m_y, s_y^2\right) \\ \Delta x_t &= N\left(m_x, s_x^2\right)\end{aligned} \tag{5.21}$$

A realisation of the system is illustrated in Figure 5.7 below. The parameters of the system are as follows: $x_0=y_0=1$, $m_x = 0.05\%$, $m_y=0.03\%$, $s_x=s_y=0.7\%$, $a_0=0$, $b_0=1$, $m_a=0.01\%$, $m_b=0$, $s_a=0.1\%$, $s_b=0.1\%$. Finally the mean reversion parameter $r = 0.1$ (i.e. the stationary component has an autoregressive root of 0.9). Knowing the values for $a_t$ and $b_t$ we also plot $y_t$ against the true $y_t^E$ .

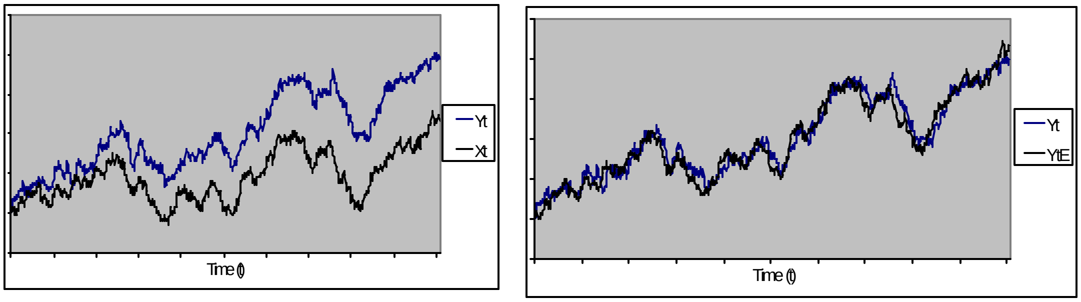


Figure 5.7: Realisation of a system exhibiting dynamic cointegration; the left hand chart shows that the deviation between the series is nonstationary, leading us to fail to reject the null hypothesis of no cointegration. The right hand chart shows, however, that the series $y_t$ actually exhibits a strong cointegration with $x_t$ when the **time-varying** cointegrating vector is taken into account.

The state-space form of the adaptive modelling methodology was employed to estimate the time-varying cointegration vector. We measure the success of the filter in two ways: firstly by the correlation between deviations from the *true* "equilibrium" $y_t^E = \begin{bmatrix}1 & x_t\end{bmatrix}\begin{bmatrix}a_t \\ b_t\end{bmatrix}$ and those from the *estimated* filter $\hat{y}_t^E = \begin{bmatrix}1 & x_t\end{bmatrix}\begin{bmatrix}\hat{a}_t \\ \hat{b}_t\end{bmatrix}$, and secondly by the significance of the reversion effect which is implied by the filtered deviation series. This can be tested by examining a modified DF-statistic based upon the regression:

$$\Delta^* d_t = \hat{g} d_t + e_t \tag{5.22}$$

Our "Dynamic Dickey-Fuller" (DDF) test then consists of examining the strength of the reversion effect: the *significance* of the mean-reverting effect is indicated by the degree to which the 't' statistic of the $\gamma$ coefficient is significantly negative; the *magnitude* of the mean-reverting component (as a proportion of the total variance) is quantified by the $R^2$ statistic of the regression.

However, we cannot apply the difference operator directly to the deviation time-series compute the dependent variable for the regression in Eqn. (5.22). This is because the time-varying parameters will absorb some of the nonstationarity in $d_t$ and create spurious mean-reversion in the deviation time series. Instead we use a difference operator in which the cointegration vector is held constant for the period over which the difference is computed:

$$\Delta^* d_t = d^*{}_{t+1} - d_t = \left[ y_{t+1} - (\hat{a}_t + \hat{b}_t x_{t+1}) \right] - \left[ y_t - (\hat{a}_t + \hat{b}_t x_t) \right] \tag{5.23}$$

A benchmark level of performance was established by applying the DDF test to the true mispricing series. The regression equation in this case was found to be $\Delta^* d_t = -0.07 d_t + N\left(0,\ 0.0196^2\right)$ with a ‘t’ statistic of -12.4. The effect of the assumed balance between parameter (state) noise and observation noise was then evaluated by varying the sensitivity ratio $S = {s^2_{a,b}} \big/ {s^2_y}$ . Sample results for the case $S = 10^{-5}$ are shown in Figure 5.8:

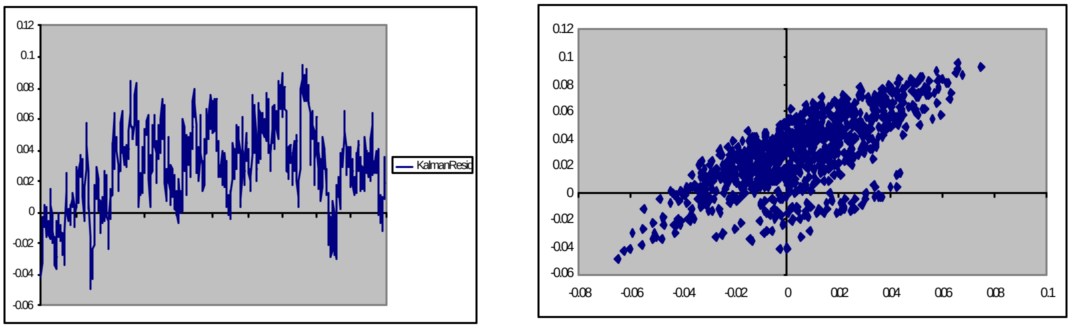


Figure 5.8: The results recovered by the adaptive methodology for the case of sensitivity ratio $S=10^{-5}$; the left hand chart shows the estimated series of deviations which is clearly stationary and represents a suitable candidate as a predictive input variable to a forecasting model. The right-hand chart shows the scatterplot between the true deviation and that recovered by the adaptive methodology, in this case the correlation between the two series was found to be 0.696

Table 5.4 below reports the results of varying the sensitivity ratio of the filter from $10^{-10}$ and $10^{1}$. Also shown in the table are comparable figures for the ideal (true) model and for a nonadaptive standard regression.

| Log10(S) | Correl | DDF |
|---|---|---|
| *(regression)* | *0.238* | *-5.24* |
| -10 | 0.238 | -5.24 |
| -5 | 0.696 | -10.71 |
| -4 | 0.758 | -12.44 |
| -3 | 0.663 | -12.42 |
| -2 | 0.507 | -11.13 |
| -1 | 0.309 | -7.35 |
| 0 | 0.150 | -4.03 |
| 1 | 0.097 | -2.99 |
| *Ideal* | *1.000* | *-12.41* |

Table 5.4: Performance of adaptive methodology at extracting the deviations from dynamic relative equilibrium of the time series in Figure 5.7. Performance figures are reported for models with different levels of sensitivity *S* as well as for the *true* deviations and a nonadaptive model. **Correl** is correlation coefficient between estimated and true deviations; **DDF** is the Dynamic Dickey Fuller statistic.

The results demonstrate that the filtering methodology is capable of extracting a good approximation to the true deviations. The maximum correlation between estimated and true deviations is just over 75% for a model with a sensitivity $S = 10^{-4}$. This model also has a DDF statistic which is very close to that obtained using the true deviations. The table also indicates that although a certain level of adaptation can improve over the static model, too much adaptation becomes counterproductive. In fact, in this instance, both the correlation and DDF figures for over-sensitive models are worse than the comparable figures for the nonadaptive model.

## 5.3 Extension to High-dimensional Problems

Although the concepts which underlie cointegration analysis are equally valid for systems with tens or hundreds of variables as they are for systems with a handful of variables, in practice cointegration methods are generally used for low dimensional problems. For instance the review and analysis of cointegration methods in Hargreaves (1994) includes a Monte-Carlo analysis of performance for a four dimensional problem, whilst noting that this is an improvement on previous studies in which the problem dimension was only two. In contrast, the asset universe under consideration for statistical arbitrage opportunities is generally of the order of tens (40 constituents of the French Cac 40 index, 30 constituents of the German Dax or Italian Mib indices) or even hundreds (100 constituents of the UK FTSE, 300 of the Eurotop, 500 of the US S&P index etc.).

In this section we describe a practical extension of the mispricing construction methodology to the high-dimensional case where the size of the asset universe is of the order of tens or hundreds. In order to reduce the dimensionality of the problem we identify relationships between relatively small subsets of the data. There remains the problem of identifying the most appropriate subsets to form the basis of the statistical arbitrage models. In order to ensure a reasonable span of the entire space, we take each asset in turn as the dependent variable of a cointegrating regression. To identify the most appropriate subspace for the cointegrating vector we replace the standard "enter all variables" regression procedure with a "stepwise" regression. In this approach, cointegrating variables are added sequentially to the model. At each stage the variable added to the model is that which is most significantly correlated to the current mispricing and hence, when added to the model, will result in the largest possible reduction in the residual variance.

An economic interpretation of the stepwise methodology is that the first asset selected to cointegrate with the target asset will be the one which has the most similar exposure to the market-wide risk factors which dominate the price variance of the target asset. However the second selected asset will be the one which is most correlated with the *residual* exposures which remain after the target asset has already been stochastically detrended with respect to the first selected asset.

Starting with the asset price itself, i.e. $C(0) = \varnothing, M_t^{(0)} = T_t$, the set of constituent assets is constructed step by step. The form of the synthetic asset models which are generated by the stepwise procedure is as follows. Let $M_t^{(k)}$ be the mispricing generated by the model at stage $k$, i.e. where the target asset is cointegrated with the first $k$ variables selected from the asset universe $C(k) \subset U_A$:

$$M_t^{(k)} = \left( T_t - \sum_{C_i \in C(k)} b_i{}^{(k)} C_{i,t} \right) \qquad (5.24)$$

Then the model for stage $k + 1$ is generated by first identifying the currently-excluded variable which accounts for the greatest amount of the variance of the current mispricing:

$$C(k+1) = C(k) \cup \underset{C_j \in U_A - C(k)}{\arg\max} \, \mathrm{E}\left[ M_t^{(k)^2} - \left( M_t^{(k)} - b_j C_j \right)^2 \right] \tag{5.25}$$

Having extended the set of constituents, the full set of beta parameters are re-estimated in order to account for the effects of cross-correlations between the explanatory variables.

The procedure will continue until either a prespecified maximum number of cointegrating assets have been added to the model, or there are no statistically significant relationships between the current residual (synthetic asset) and the variables remaining outside the model. As in standard stepwise regression modelling, the significance test which is used is a partial-F test of variable significance which relates the reduction in the variance to the additional degree of freedom which is added to the model[13].

As noted in Section 5.1, our cointegrating-regression approach shares with explicit factor methods the general idea of detrending the asset prices relative to common stochastic trends. A key difference in our approach, and particularly in this stepwise methodology, is that the risk factors are implicitly determined with respect to each target asset on an *individual* basis. For instance, in the case of a bank stock which is primarily sensitive to economic growth, market risk, interest rate risk and investor sentiment, the selected group of cointegrating assets will consist of individual assets which, in a suitable linear combination, have similar exposure to these underlying factors. In the case of a major retailer with significant export business the selected assets may include other retailers, stocks which are sensitive to exchange rates, and stocks of companies which have similar export markets. Thus rather than trying to identify a single set of factors which represent all stocks approximately equally, the stepwise methodology implicitly allows the risk factors to be adjusted according to the particular characteristics of the target asset under consideration.

[13] Similar F-tests of statistical significance underly the variable selection and neural estimation procedures in Part II of the thesis, and are discussed in detail in Chapters 9 and 10.

**Example**

We now illustrate the operation of the stepwise methodology for synthetic asset construction with respect to the particular example of daily closing prices of the constituents of the FTSE 100 index.

The objective of the methodology is to construct the synthetic assets which represent the mispricings of the individual equities. This is performed by taking each asset in turn and using the stepwise regression procedure to sequentially identify the five most suitable assets[14] in each case. The general form of the resulting mispricing models is:

$$M_{\mathrm{s,t}} = P_{s,t} - \left(\sum_{i=1}^{5} w_{s,i} P_{c(i,s),t} + c\right) \tag{5.26}$$

where $M_{\mathrm{s,t}}$ is the statistical mispricing (synthetic asset) for stock $s$ at time $t$; $P_{s,t}$ is the actual stock price of asset *s* at time *t*; $P_{c(i,s),t}$ is the price of the *i*'th constituent asset selected for target asset *s* and $w_{s,i}$ is the associated weighting parameter.

The data consists of daily closing prices from 13th June 1996 to 13th May 1998 and is divided into an insample period of 400 days and an out-of-sample period of 100 days. After eliminating assets for which a continuous history was unavailable (mainly due to mergers) we were left with 94 assets including the index itself. The table below shows the specification of the first 10 synthetic assets, listing first the target asset and then the constituents, together with their weightings, in reverse order of selection.

[14] Note that these will not necessarily (and in fact very rarely) be the five assets which are most correlated with the target asset. After the first stage it is the correlation with the *residual mispricing* which is important, and only at the first stage will this be equivalent the target asset itself.

| Number | Target | Constit1 | Wt1 | Constit2 | Wt2 | Constit3 | Wt3 | Constit4 | Wt4 | Constit5 | Wt5 | Constant |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | FTSE | SCTN | 0.916 | RTR | 0.414 | REED | 0.82 | III | 1.878 | LLOY | 2.72 | 633.711 |
| 2 | ABF | ICI | -0.084 | CW | 0.315 | GUS | -0.16 | WLMS | -0.362 | PRU | 0.527 | 334.975 |
| 3 | ALLD | PSON | 0.197 | STAN | -0.043 | SFW | -0.407 | WLY | 0.121 | SBRY | 0.382 | 290.694 |
| 4 | ANL | SFW | 0.199 | BT | 0.564 | BSY | -0.092 | WLMS | -0.27 | LGEN | 1.497 | 13.321 |
| 5 | ASSD | REED | -0.034 | ICI | 0.031 | BAY | -0.05 | BSY | -0.05 | TSCO | 0.246 | 87.192 |
| 6 | AVZ | RR | -0.375 | CPG | 0.177 | ORA | 0.272 | BASS | 0.149 | ULVR | 0.944 | -253.124 |
| 7 | BA | SEBE | 0.16 | PSON | 0.175 | RTR | -0.203 | WLY | 0.303 | BARC | 0.101 | -87.8 |
| 8 | BAA | BASS | -0.085 | WLMS | 0.287 | CCM | 0.252 | RR | -0.515 | HSBAx | 0.1 | 341.542 |
| 9 | BARC | ICI | 0.208 | ORA | -1.154 | REED | -0.624 | SMIN | 0.746 | BSCT | 2.429 | 117.822 |
| 10 | BASS | SFW | 0.241 | EMI | 0.148 | GAA | 0.225 | RIO | -0.289 | SCTN | 0.846 | 151.271 |

Table 5.5: A selection of synthetic assets which represent statistical mispricings of FTSE 100 constituent stocks. In each case the constituent assets are listed in **reverse** order of selection and the parameter values are those which apply when all five constituent assets are included in the model

As an example, consider the 10th listed asset, BASS, representing Bass Plc., the brewery group. The associated synthetic asset model is given by:

$$M_{\mathrm{BASS,t}} = P_{BASS,t} - \begin{pmatrix} 0.241P_{SFW,t} + 0.148P_{EMI,t} \\ +0.225P_{GAA,t} - 0.289P_{RIO,t} \\ +0.846P_{SCTN,t} + 151.271 \end{pmatrix} \tag{5.27}$$

In the order selected, the constituent assets are SCTN (Scottish and Newcastle - another brewery group), RIO (Rio Tinto Plc - Mining and Natural Resources), GAA (Granada - TV and entertainment), EMI (music), and SFW (Safeway supermarket group). The choice of a brewery as the first selected constituent is unsurprising as we would expect the two stocks are sensitive to similar economic factors. The final three stocks have probably been selected on the basis of a shared sensitivity to the UK economy as a whole and the luxury/entertainment side in particular. Perhaps the only surprising choice, in this case, is that of Rio Tinto. However, note that the coefficient in this case is negative (-0.289) indicating that (net of the effect of the other constituents) factors to which the Bass share price responds positively tend to have a negative impact on Rio Tinto, and vice versa.

Whilst we have singled out one model in particular, a similar explanation could be produced for each of the other models. These relationships are not always obvious: they may for instance relate to a company's financial setup rather than its industrial sector, and may also be complicated, as in the case of Bass/Rio Tinto, by the presence of negative coefficients. However, in all cases there is at least an implicit relationship in the fact that the share price movements of the companies have been found to be sufficiently related as to select out the five particular stocks from a choice of 93 possible assets.

To further illustrate the operation of the synthetic asset modelling procedure, we now present additional details of this one arbitrarily selected model. Firstly, the table below describes the successive refinements of the synthetic asset model, starting from the basic price of Bass stock and then describing the parameters as the cointegrating assets are incorporated into the model. At each stage the variance of the residual statistical mispricing is reported, together with the percentage reduction relative to the original asset price.

| Model | Constant | SCTN | RIO | GAA | EMI | SFW | Variance | Reduction |
|---|---|---|---|---|---|---|---|---|
| BASS | | | | | | | 1129.0 | |
| SA0 | 822.9 | | | | | | 1129.0 | 0% |
| SA1 | 383.3 | 0.645 | | | | | 725.9 | 36.64% |
| SA2 | 560.4 | 0.693 | -0.220 | | | | 441.3 | 61.35% |
| SA3 | 344.1 | 0.764 | -0.230 | 0.206 | | | 370.8 | 67.35% |
| SA4 | 215.5 | 0.889 | -0.277 | 0.214 | 0.134 | | 315.9 | 72.47% |
| SA5 | 151.2 | 0.846 | -0.289 | 0.225 | 0.148 | 0.241 | 287.5 | 74.87% |

Table 5.6: Synthetic asset models for the mispricing of the equity price of Bass Plc., one of the FTSE 100 constituents. The table reports the parameter values as additional assets are incorporated into the model, together with the variance of the resulting mispricing time-series and the *reduction* in variance when compared to the original Bass share price.

In this particular case we note that the variance of the Bass price is 1129. Adjusting the prices to allow for a constant term results in a new series with zero mean but has no effect on the variance. The first selected constituent asset is SCTN (Scottish and Newcastle) and the variance of the Bass price around the “fair price” of 383.3 + 0.645 SCTN is only 725.9, i.e. a 36.64% reduction in the overall variance. The addition of a second constituent asset to the cointegrating regression results in a new relationship: 560.4 + 0.693 SCTN - 0.220 RIO. The variance of the Bass price around this new fair price relationship is only 441.3, eliminating almost a further 25% of the initial variance to give a total variance reduction of 61.35%. In other words, 61.35% of the variance in the price of Bass shares can be accounted for by factors which are also reflected in the share prices of Scottish and Newcastle and Rio Tinto. By the time that all five assets have been added into the model, the total variance reduction is 74.87% - as expected the marginal contribution of the latterly selected variables is much lower than for the first variables.

A comparison of the actual time-series is presented in Figure 5.9 below, which compares the original Bass share price to that of the statistical mispricing relative to the final synthetic asset.

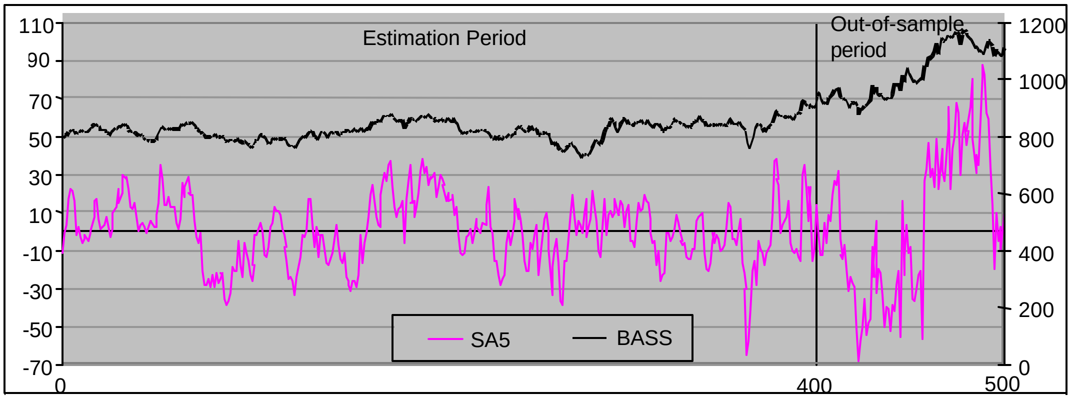


Figure 5.9: Time-series of the share price of Bass Plc (right hand axis) and the associated statistical mispricing series (left hand axis) which represents the deviation relative to a "synthetic asset" composed of a combination of five other share prices. The first 400 observations were used to estimate the cointegration relationship. During this period the synthetic asset has a 74.87% lower variance than the original Bass share price. The final 100 observations are "out of sample".

The mispricing is clearly much more stable than the underlying share price. This fact is emphasised during the "out of sample" period, during which the Bass share price trends up to new levels whereas the statistical mispricing remains relatively stable and maintains a high degree of mean-reversion. This supports the hypothesis that the mispricing is largely immunised from the unpredictable sources of market risk which dominate the changes in the original share price, and is instead more representative of stock specific factors which are lower in variance and yet may contain a substantial component which is potentially predictable.

## 5.4 Summary

In this chapter we have described our cointegration framework for **constructing** synthetic assets in the form of statistical mispricings of underlying asset prices relative to appropriately constructed "synthetic assets". The synthetic assets consist of linear combinations of other asset price series and are constructed by methodology based on the concept of a "cointegrating regression", with extensions to the time-varying and high-dimensional cases. The intention behind this pre-processing is to generate combinations of time-series which are largely immunised against market-wide uncertainties and which enhance the potentially predictable components of asset price dynamics.

In the following chapter we discuss the next component of the methodology, namely a broad range of tests which are designed to identify the presence of a potentially predictable component in the mispricing dynamics.

# *6. Testing for Potential Predictability*

This chapter describes the tests which we have developed for the purpose of identifying potentially predictable components in the dynamics of the statistical mispricing time-series. These tests include tests for *short-term effects*, which are based on the autocorrelation function of the time-series, tests for *stationarity* in the form of cointegration "unit root" tests, and tests for general *deviations from random walk behaviour* which are based upon the variance ratio function of the time-series. We also introduce novel tests based upon the shape of the variance ratio profile as a whole. The tests themselves are described in Section 6.1. The strengths and weaknesses of the various tests can be defined in terms of their relative *size* and *power* in distinguishing between the null hypothesis of purely stochastic dynamics (random walk behaviour) and various alternative hypotheses which correspond to different types of partially deterministic dynamics. Section 6.2 presents the results of an examination of the properties of the various tests. The analysis was conducted under controlled circumstances using Monte-Carlo simulations of a range of data-generating processes which represent plausible alternative hypotheses. Section 6.3 presents modifications of the predictability tests which correct for the bias induced by the construction procedure. Section 6.4 presents the results of applying the tests to statistical mispricings generated from combinations of FTSE 100 equity prices.

## 6.1 Description of Predictability Tests

Having created a time-series which represents a statistical mispricing between a set of assets, the next stage of our methodology involves testing the time-series for the presence of a potentially predictable component in the mispricing dynamics. Given that our mispricing construction procedure is based upon *cointegration* modelling techniques, the natural choice would be to use cointegration tests such as unit root tests (see Section 2.2.2) in order to detect the presence of a mean-reverting "error-correction" effect. Unfortunately it is not easy to find examples of statistically significant cointegration relationships in financial asset prices. Typically, even combinations of asset prices which contain a stationary component will be contaminated by the presence of a nonstationary component. In such cases, the cointegration tests will tend to fail to reject the null hypothesis of nonstationarity in spite of the fact that a potentially predictable mean-reverting component is present in the dynamics of the time-series.

The reason why standard cointegration tests will in some cases fail to identify potential opportunities for statistical arbitrage is simple: it is not the *purpose* for which the tests are designed. Whilst the object of cointegration testing is to identify combinations of time-series which are *stationary*, statistical arbitrage modelling is concerned with the more general case of combinations of time-series which contain a stationary *component*, or indeed any other deterministic component. Rather than imposing the strict requirement of stationarity, we instead employ a broader range of tests which are capable of identifying both trending and mean-reverting behaviour and which are also **robust** to the presence of a nonstationary component in the time-series dynamics.

The following subsections define the set of tests which we employ to identify predictable components in noisy time-series. The tests include autocorrelation tests for short-term dynamics, unit root tests for stationarity, and variance ratio tests for the presence of mean-reversion. We introduce a fourth set of novel tests which are based upon the shape of the variance ratio profile as a whole and which are intended to identify general deviations from unpredictable (random walk) behaviour. The power of these tests against a range of alternative hypotheses will be tested in the following section using a Monte-Carlo simulation.

### 6.1.1 Autocorrelation tests for short-term dynamics

The individual autocorrelation statistics which collectively comprise the autocorrelation function (ACF) of a time-series are very sensitive to noise. To compensate for this weakness, it is common to instead employ “portmanteau” statistics which are joint tests of a set of individual correlation coefficients. In particular, we base our benchmark test for the presence of autocorrelation effects in the mispricing time-series on the Box-Lyung test (see also Section 2.2.2) which quantifies the *absolute magnitudes* of the relationships in the short-term dynamics of the time-series.

#### Q-BL statistic

The Box-Pierce $Q$ statistic consists of the (unweighted) sum of the squares of the first $p$ components of the sample correlogram. The $Q$ statistic is commonly used to test whether a series is white noise and is asymptotically distributed as chi-squared with $p$ degrees of freedom. The Box-Lyung statistic (Q-BL) is a modification which is known to have better finite sample properties, and is defined by:

$$Q' = T(T+2)\sum_{k=1}^{p} \frac{r_k^2}{T-k} \tag{6.1}$$

Where $T$ is the sample size, or length of the time-series and $r_k$ is the k'th autocorrelation coefficient as defined in Eqn. (2.8). The exact specification of the Q-BL statistic will depend on the parameter $p$ which determines the number of autocorrelation coefficients to combine in the summation.

### 6.1.2 Unit root tests for Stationarity

To represent the stationarity tests of cointegration analysis (see Section 2.2.2), we take the two standard tests which are recommended by Engle and Granger (1987), namely the Dickey-Fuller (DF) test and the Cointegrating Regression Durbin-Watson (CRDW). These are defined as follows:

**DF statistic**

Calculating the DF statistic involves regressing changes in the series $\Delta y_t = y_{t+1} - y_t$ on the actual level of $y_t$ itself. The test statistic is the 't' statistic of the regression beta:

$$\Delta y_t = a + by_t + e_t \tag{6.2}$$

If the 't' statistic indicates a beta value which is significantly less than zero then the series is stationary. Under the null hypothesis of nonstationarity the DF statistic follows a non-standard distribution which is tabulated in Fuller (1976).

**CRDW statistic**

The CRDW statistic is calculated as for the standard Durbin-Watson test for autocorrelated regression residuals:

$$CRDW = \frac{\sum_t \left(y_t - y_{t-1}\right)^2}{\sum_t y_t^2} \tag{6.3}$$

If $y_t$ and $y_{t-1}$ are independent and zero mean then the variance of their sum will equal the sum of their variances and the $DW$ or $CRDW$ statistic will be within sampling error of the value two. Perfect negative correlation gives a statistic close to 4 and perfect positive correlation a value close to zero. Under the null hypothesis of a random walk the $CRDW$ follows a nonstandard distribution with values very close to zero (i.e. high positive autocorrelation).

### 6.1.3 Variance Ratio tests for mean-reversion

The variance ratio function is defined as the normalised ratio of long term variance (calculated over period $\tau$) to single-period variance and is thus:

$$\mathrm{VR}(t) = \frac{\sum_t \left(\Delta^t y_t - \overline{\Delta^t y}\right)^2}{t \sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2} \tag{6.4}$$

If the increments to the series are uncorrelated then the ratio will be close to 1. It can be shown that suitably expressed VR statistics are equivalent to weighted sums of *actual* (rather than squared - as in Section 6.1.1) autocorrelation coefficients (see Eqn. (2.13)). $\mathrm{VR}(t) > 1$ indicates higher long-term variance than would be expected given the short-term variance, and hence that the **net** effect over timescale $\tau$ is of positive autocorrelation or trending behaviour. $\mathrm{VR}(t) < 1$ indicates lower long-term variance than would be expected, given the short term variance, and hence that the **net** effect over the period is of negative autocorrelation or mean-reverting behaviour.

The common use of the VR statistic is to test for the presence of a significant mean-reverting component in the dynamics of the time-series when viewed over a particular time-scale, i.e. $\mathrm{VR}(t) < 1 - d$ (where $d$ represents a critical value based upon the required significance level).

### 6.1.4 Variance Profile tests for deviations from random walk behaviour

As noted above, the standard form of the VR statistics requires the specification of a timescale $\tau$. Furthermore, such statistics can be adversely affected by short term effects in the dynamics. For instance a short-term <u>positive</u> autocorrelation (trending behaviour) combined with longer-term <u>negative</u> autocorrelation (mean-reversion) may lead to a VR statistic close to 1, and hence failure to reject the null hypothesis of no mean-reversion. In order to overcome this limitation of *individual* VR statistics we use two types of tests which are derived from the *joint* distribution of the set of VR statistics $\mathbf{VP}t = \{VR(1), VR(2), VR(3), \ldots VR(t)\}$. Such statistics can be viewed as summarising the overall "shape" of the variance ratio profile.

Viewing the vector of variance ratio statistics as a multivariate normal distribution with mean $\overline{\mathbf{VP}t}$ and covariance matrix $S_{\mathbf{VP}}$ then the first type of variance profile statistic which we investigate, denoted $VPdist(t)$, represents the Mahalanobis distance of the observed vector $\mathbf{VP}t$ from the centre of the distribution:

$$VPdist(t) = \left(\mathbf{VP}t - \overline{\mathbf{VP}t}\right)^T S_{\mathbf{VP}}{}^{-1} \left(\mathbf{VP}t - \overline{\mathbf{VP}t}\right) \qquad (6.5)$$

This statistic was first used by Eckbo and Liu (1993), who also demonstrate that the distribution converges asymptotically to a chi-squared distribution with $t$ degrees of freedom.

Our second set of variance profile statistics are a novel set of tests based upon the correlation structure within the set of VR statistics. In particular, we consider the projections of the vector $\mathbf{VP}t$ onto the directions which correspond to the eigenvectors $\mathbf{e}_i$ of the covariance matrix $S_{\mathbf{VP}}$.

$$VRproj(t,i) = \left(\mathbf{VP} - \overline{\mathbf{VP}}\right)^T \mathbf{e}_i \qquad (6.6)$$

where the eigenvectors $\mathbf{e}_i$ are the set of projections such that $S_{\mathbf{VP}}\mathbf{e}_i = l\,\mathbf{e}_i$ and can be identified using the standard technique of principal components analysis (PCA), for which a good reference text is Jolliffe (1986).

The magnitude of the projections $VRproj(t,i)$ can be interpreted as the extent to which the variance profile as a whole matches the characteristic "pattern" which corresponds to the eigenvector $\mathbf{e}_i$ - and thus the extent to the time-series dynamics correspond to those which generate variance profiles with a shape similar to $\mathbf{e}_i$.

As an illustration, a Monte-Carlo simulation was conducted in order to generate the variance profiles $\mathbf{VP}_{100} = \{VR(1), VR(2), VR(3), \ldots VR(100)\}$ of 1000 random-walk time-series. Figure 6.1 below, presents the first three eigenvectors obtained by PCA of the resulting correlation matrix.

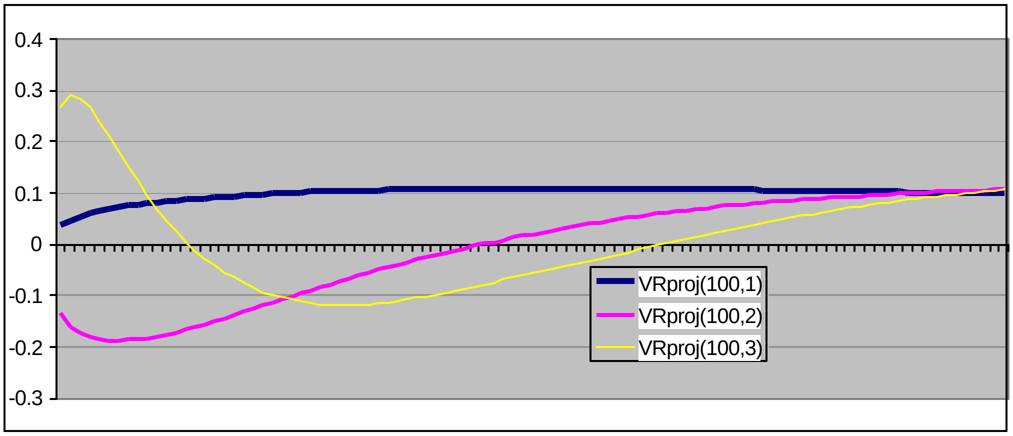


Figure 6.1: The first three eigenvectors (principal components) of the correlation matrix of the 100-period variance profile, estimated under the null hypothesis of no predictable component (random walk behaviour). The eigenvectors represent the common structure which is found in deviations from the average variance profile.

For instance, the magnitude of the projection on the first eigenvector represents the extent to which a variance profile exhibits a global shift upwards. Positive projections on this eigenvector will correspond to positive autocorrelation (increasing VP) in the relatively short-term dynamics, followed by zero autocorrelation (flat VP) in the medium and long-term dynamics. Conversely, negative projections on this eigenvector will correspond to negative autocorrelation in the short-term dynamics, and zero autocorrelation in the medium and long term dynamics.

Interpreting the second component in a similar manner, we note that positive projections correspond to short term reversion (low VR) and longer-term trending behaviour (increasing VR). The third component represents a very short term trend, followed by a weaker medium term reversion and a weaker still long term trend.

The PCA of the 100-period variance ratio profile $\mathbf{VP}_{100}$ also indicated that for random walk series approximately 80% of the total variability is captured by the first principal component (corresponding to projections onto the first eigenvector). The fact that such a high proportion of the variance in a set of 100 metrics can be represented by a single statistic indicates a high degree of structure within the correlation matrix of the different $\mathrm{VR}(t)$ statistics. Figure 6.2 below shows the scree plot of variance explained by each principal component, together with cumulative variance for the first 'n' components.

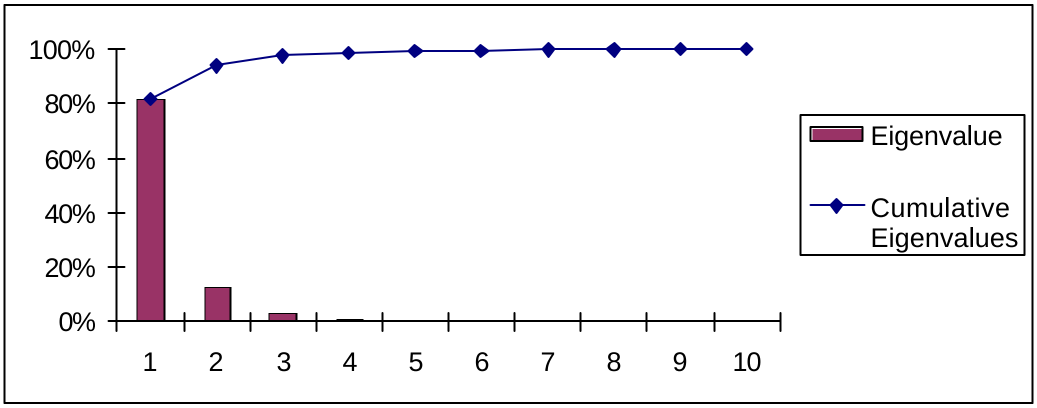


Figure 6.2: Scree plot showing the variance accounted for by the first ten principal components of the variance ratio profile $\mathbf{VP}_{100}$ evaluated over 1000 realisations of random walk time-series. The fact that the first few components account for almost all the variance indicates a high degree of correlation between the various VR(τ) statistics.

Table 6.1, below, reports the actual and cumulative figures for the proportion of variance explained by each principal component:

| Principal Component Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Variance Explained | 81.6% | 12.4% | 3.6% | 1.2% | 0.5% | 0.3% | 0.1% | 0.1% | 0.1% | 0.0% |
| Cumulative Variance | 81.6% | 94.0% | 97.6% | 98.8% | 99.3% | 99.6% | 99.7% | 99.8% | 99.9% | 99.9% |

Table 6.1: Variance Explained by the principal components of the Variance Ratio profile $\mathbf{VP}_{100}$

The first five components account for over 99% of the total variability in the 100-period profile. A similar analysis of the 25-period variance ratio profile was found to produce similar results, except that the overall time-scale is shorter and the "frequency" of the components is correspondingly higher. Given that five components are sufficient in each case to account for 99% of the total variability in the VR profile, and that the different length profiles may be sensitive to different dynamic effects, it was decided to include test statistics derived from each of the first 5 components of both the 25-period and 100-period variance profiles within our subsequent Monte-Carlo experiments.

## 6.2 Monte-Carlo Evaluation of Predictability Test Power

This section presents an analysis of the various predictability tests which were described in the previous section. The usefulness of the predictability tests lies in their ability to distinguish between different classes of time-series dynamics. In particular, this ability can be quantified by analysing the *size* and *power* of the tests in differentiating between the *null hypothesis* of random-walk dynamics and different *alternative hypotheses* which correspond to different types of partially deterministic dynamics.

The set of statistics included in our analysis are listed in Table 6.2 below:

| Statistic | Type | Parameter | Equation |
|---|---|---|---|
| Q5 | Q (Box Lyung) | 5 | $T(T+2)\sum_{k=1}^{5}\frac{r_k^2}{T-k}$ |
| Q10 | Q (Box Lyung) | 10 | $T(T+2)\sum_{k=1}^{10}\frac{r_k^2}{T-k}$ |
| DF | Dickey-Fuller | none | $b \big/ se_b$ from regression $\Delta y_t = a + by_t + e_t$ |
| CRDW | Durbin-Watson | none | $CRDW = \frac{\sum_t (y_t - y_{t-1})^2}{\sum_t y_t^2}$ |
| VR10 | Variance Ratio | 10 | $\sum_t \left([y_t - y_{t-10}] - \overline{\Delta y}\right)^2 \Big/ 10\sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2$ |
| VR25 | Variance Ratio | 25 | $\sum_t \left([y_t - y_{t-25}] - \overline{\Delta y}\right)^2 \Big/ 25\sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2$ |
| VR50 | Variance Ratio | 50 | $\sum_t \left([y_t - y_{t-50}] - \overline{\Delta y}\right)^2 \Big/ 50\sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2$ |
| VR75 | Variance Ratio | 75 | $\sum_t \left([y_t - y_{t-75}] - \overline{\Delta y}\right)^2 \Big/ 75\sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2$ |
| VR100 | Variance Ratio | 100 | $\sum_t \left([y_t - y_{t-100}] - \overline{\Delta y}\right)^2 \Big/ 100\sum_t \left(\Delta y_t - \overline{\Delta y}\right)^2$ |
| VP25d | Variance Profile (distance) | 25 | $\left(\mathbf{VP}_{25} - \overline{\mathbf{VP}_{25}}\right)' \mathrm{S}_{\mathbf{VP}25}^{-1}\left(\mathbf{VP}_{25} - \overline{\mathbf{VP}_{25}}\right)$ |
| VP100d | Variance Profile (distance) | 100 | $\left(\mathbf{VP}_{100} - \overline{\mathbf{VP}_{100}}\right)' \mathrm{S}_{\mathbf{VP}100}^{-1}\left(\mathbf{VP}_{100} - \overline{\mathbf{VP}_{100}}\right)$ |
| VP25c1..5 | Variance Profile (projection) | 25, i=1...5 | $\left(\mathbf{VP}_{25} - \overline{\mathbf{VP}_{25}}\right)' \mathbf{e}_i$ |
| VP100c1..5 | Variance Profile (projection) | 100, i=1..5 | $\left(\mathbf{VP}_{100} - \overline{\mathbf{VP}_{100}}\right)' \mathbf{e}_i$ |

Table 6.2: Predictability tests selected for analysis in the Monte Carlo simulations

The approach which we take to compare the various tests is to fix the *size* of the tests (false positive rate, $\alpha$) and then compare their *power* with respect to different alternative hypotheses (1-$\beta$, the false negative rate). This tradeoff between the size and power of a test is illustrated in Figure 6.3 below.

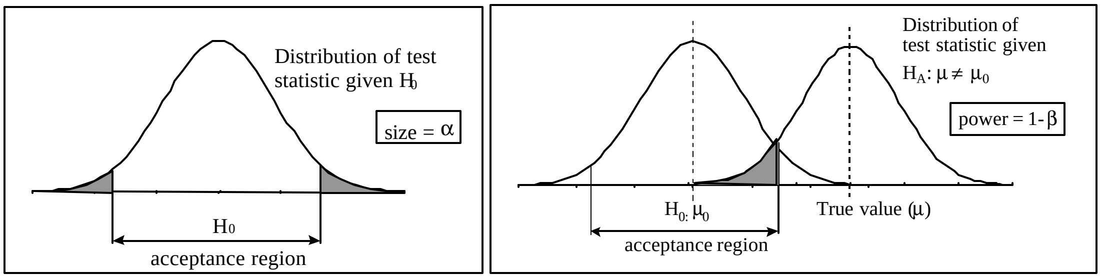

Distribution of test statistic given H0
size = α
H0
acceptance region
Distribution of test statistic given HA: μ ≠ μ0
power = 1-β
H0: μ0
True value (μ)
acceptance region

Figure 6.3: Tradeoff between size and power of a test. The size of the test is α, the probability of a Type I error or “false positive” (the alternative is falsely accepted). The power of the test is (1-β), where β is the probability of a Type II error or “false negative” (the alternative is falsely rejected).

The first stage of the process is to generate the empirical null hypothesis distribution of the various test statistics, i.e. in the case where the dynamics are completely stochastic:

$$y_t = y_{t-1} + e_t \qquad\qquad e_t \sim \mathrm{N}\left(0, s^2\right)$$

or alternatively: (6.7)

$$\Delta y_t = e_t$$

For this purpose, 1000 simulations of random-walk time-series were performed, each of length 1000 observations. The “acceptance region” for each of the test statistics was identified by selecting critical values such that the false positive rate on the empirical sample corresponded to the required *size* of the test. These ranges were then validated with respect to a second simulation of 1000 random walk series in order to allow a comparison to be made between the nominal and actual sizes (false positive rates) of the tests, as presented in Table 6.3 below.

| Nominal | DF | CRDW | Q5 | Q10 | VR10 | VR25 | VR50 | VR75 | VR100 | VP100d | VP25d |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1% | 1.6% | 1.3% | 0.8% | 0.9% | 0.9% | 0.9% | 1.2% | 1.0% | 1.6% | 5.9% | 1.3% |
| 5% | 4.5% | 3.9% | 4.0% | 5.2% | 3.6% | 6.0% | 4.7% | 4.4% | 5.6% | 18.8% | 4.5% |
| 10% | 9.1% | 8.0% | 9.8% | 8.8% | 9.9% | 12.2% | 9.1% | 9.2% | 10.4% | 26.5% | 10.6% |

| Nominal | VP100c1 | VP100c2 | VP100c3 | VP100c4 | VP100c5 | VP25c1 | VP25c2 | VP25c3 | VP25c4 | VP25c5 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1% | 1.2% | 1.0% | 1.1% | 1.6% | 2.0% | 0.8% | 1.3% | 1.4% | 1.8% | 1.0% |
| 5% | 4.4% | 4.7% | 5.3% | 6.0% | 7.3% | 3.7% | 6.6% | 4.9% | 5.8% | 4.9% |
| 10% | 8.7% | 9.4% | 9.9% | 12.4% | 12.4% | 9.8% | 10.4% | 10.4% | 10.9% | 12.3% |

Table 6.3: Nominal size of predictability tests versus actual size on a calibration set of 1000 random walk time-series

On the whole the validation results confirm that the actual size of the tests is close to the 1%, 5% and 10% nominal levels selected for the experiment. The only significant deviation from the nominal levels was found for the VP100d statistic with false positive rates of {5.9%, 18.8% and 26.5%} instead of the nominal {1%, 5% and 10%}[15].

### 6.2.1 Specification of Alternative Hypotheses for Simulation Experiments

The *power* of the tests was then evaluated against a range of alternative hypotheses in which the corresponding time-series dynamics contain a deterministic and hence potentially predictable component in combination with a (much larger) stochastic component. Two simple forms of deterministic component are considered, namely reversion and momentum. To generalise the reversion component, however, from the case of standard “mean reversion” or stationarity, we also allow for a degree of random-walk drift in the reversion level (which can

[15] Whilst the VP100d statistic suffers from a certain degree of instability, the corresponding eigenvector projections VP100c1..5 are close to the nominal size. Hence the source of the instability appears to be the later projections which only account for a tiny proportion of the overall variance of the VR profile. As the Mahalanobis distance metric effectively attaches an equal weight to each projection it would appear to be a dangerous metric to use in unadjusted form.

be compensated for adaptive modelling methods such as that in Section 5.2). Thus the data-generating processes are of the form:

$$\Delta y_t = (1 - w)\Delta s_t + w\Delta n_t + \Delta m_t$$

where:

$$\begin{aligned} \Delta s_t &= -rs_{t-1} + e_t & \qquad e_t &\sim \mathrm{N}\left(0, s^{2}\right) \\ \Delta n_t &= h_t & \qquad h_t &\sim \mathrm{N}\left(0, s^{2}\right) \\ \Delta m_t &= a\left(y_{t-1} - y_{t-(p+1)}\right) & & \end{aligned} \tag{6.8}$$

In all, simulation experiments were conducted to test the *power* of the predictability tests against 10 different parametrisations of the system defined in Eqn. (6.8). The parameters of the different experiments are specified in the Table 6.4 below.

| Experiment | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Reversion | *r* | 0.01 | 0.02 | 0.05 | 0.25 | 0.02 | 0.02 | 0.02 | 0.05 | 0.05 | 0.05 |
| Nonstationarity | *w* | 0 | 0.25 | 0.5 | 0.6 | 0.25 | 0.25 | 0.25 | 0.5 | 0.5 | 0.5 |
| Momentum | *a* | | | | | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 |
| Momentum period | *p* | | | | | 1 | 3 | 10 | 1 | 3 | 10 |

Table 6.4: Parameters of the dynamics for the predictability test experiments; each set of parameters represents a different alternative hypothesis to the null hypothesis of unpredictable (random-walk) behaviour.

Each experiment involved the generation of 1000 realisations of the process defined in Eqn. (6.8), using the appropriate set of parameters from Table 6.4. The experiments break down into three groups depending upon the types of predictable components which are present in the dynamics: the first set (Expt 1) consists of stationary time-series, which contain a mean-reverting component and a noise term, but no nonstationary component; the second set (Expts. 2-4) involve time-series which contain a mean-reverting component and a noise term, but which are also "contaminated" by a nonstationary (random-walk) component in the dynamics; the final set of experiments (Expts. 5-10) combine mean-reversion with both momentum and nonstationary effects, in addition to the noise term. The results of the three sets of experiments are presented in turn below.

## 6.2.2 Power against Mean-reversion (Expt 1)

Experiment 1 is an extremely weak case of a mean-reversion term $-rs_{t-1}$ which, because the other terms have weighting parameters *w=a=0*, in fact represents a standard mean-reversion $-ry_{t-1}$. In the case of applying this test to the mispricing between a set of assets this experiment would correspond to testing for *cointegration* between the assets.

Before moving on to the results of the predictability tests, let us consider the nature of the alternative hypothesis being tested. Figure 6.4 presents sample realisations of the alternative hypothesis time-series, together with the 100-period variance ratio profile averaged over all 1000 realisations.

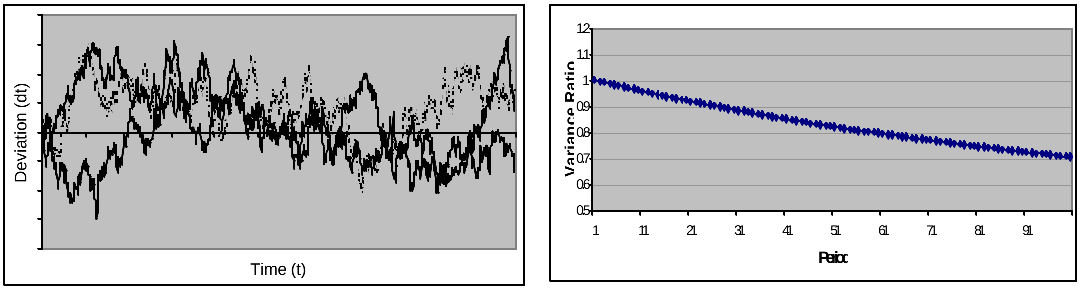


Figure 6.4: Realisations of time-series which exhibit mean-reversion (left) and the variance ratio profile averaged over 1000 such series (right). For a truly stationary process the variance ratio will tend to zero as the period over which the increments are calculated increases.

The mean-reversion term can clearly be seen to induce a smooth downward slope in the variance ratio profile. In fact the variance ratio of a stationary process will asymptotically trend towards zero as the length of the period $t$ approaches infinity. One interpretation of this type of variance ratio profile is that the mean-reversion causes a tendency for short-term volatility in the series to partly cancel out, leading to longer term volatility being lower than would be expected from the short-term volatility.

The power of the various tests is presented in Figure 6.5 and Table 6.5 below for the three cases where the nominal size (expected false positive rate) of each test is 1%, 5% and 10%.

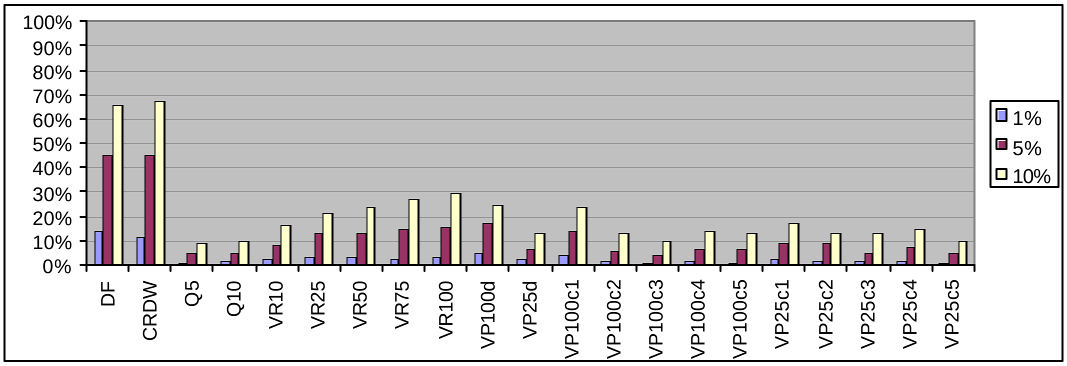


Figure 6.5: Illustration of the power of the predictability tests against the alternative hypothesis of uncontaminated mean-reversion. The sets of bars represent false positive rates of 1%, 5% and 10%.

| Size | DF | CRDW | Q5 | Q10 | VR10 | VR25 | VR50 | VR75 | VR100 | VP100d | VP25d |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1% | **14.3%** | 11.4% | 0.7% | 1.3% | 2.5% | 3.4% | 2.9% | 2.7% | 3.3% | 4.5% | 2.4% |
| 5% | **45.3%** | 45.1% | 4.5% | 5.1% | 8.1% | 12.8% | 13.3% | 15.0% | 15.8% | 17.0% | 6.2% |
| 10% | 65.5% | **67.1%** | 9.3% | 10.1% | 16.4% | 21.2% | 24.1% | 27.5% | 29.4% | 24.7% | 13.1% |

| Size | VP100c1 | VP100c2 | VP100c3 | VP100c4 | VP100c5 | VP25c1 | VP25c2 | VP25c3 | VP25c4 | VP25c5 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1% | 3.7% | 1.3% | 0.8% | 1.2% | 1.0% | 2.1% | 2.0% | 1.1% | 1.8% | 0.9% |
| 5% | 13.6% | 5.5% | 4.2% | 6.7% | 6.2% | 8.9% | 9.0% | 5.1% | 7.1% | 5.1% |
| 10% | 23.8% | 12.9% | 9.9% | 14.0% | 12.9% | 17.6% | 13.2% | 12.9% | 14.6% | 10.0% |

Table 6.5: Power of the different predictability tests to detect deviations from random walk behaviour in the form of pure mean-reversion, uncontaminated by either nonstationary drift term or momentum.

Cases in which the *power* of a test is greater than the *size* of the test indicate that the test has a positive ability to identify the deviation from random walk behaviour. Clearly, the standard stationarity tests are the most successful in this first experiment which corresponds to the standard cointegration hypothesis for which these tests are designed. Next in power come the longer-period variance ratio tests VR75 and VR100 which detect the lower than expected long term variance which is due to the mean-reverting nature of the series. None of the VR projections are particularly powerful but the highest power is shown by the first principal components (VP100c1, VP25c1) reflecting the fact that the mean-reversion induces a global downward shift in the variance ratio profile.

### 6.2.3 Power against Nonstationary Reversion (Expts. 2-4)

Experiments 2 to 4 represent a contaminated form of mean-reversion in which the random walk term $w\Delta n_t$ is assigned a nonzero weight and hence introduces a nonstationary drift in the reversion level of the series. As this term degrades the effect of the reversion term, a compensating increase in the reversion level $r$ is employed to maintain an approximately equivalent level of overall predictability.

The average variance ratio profiles for the three experiments are presented in Figure 6.6; the reversion term induces a negative slope in the VR profile as in Experiment 1. The nonstationary component, however, serves to maintain the long-term variance and hence induce a curvature in the VR profile. In the case of Experiment 4, the reversion is at its strongest, giving a high negative slope at the beginning of the profile, but so is the degree of nonstationarity, causing the profile to flatten out fairly rapidly so that the slope is almost zero beyond the 30-period VR.

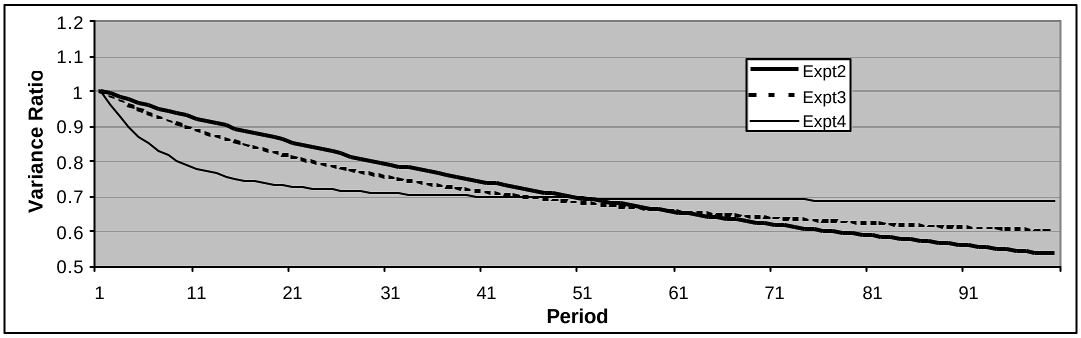


Figure 6.6: The average variance ratio profiles of the simulated time-series used in experiments 2, 3 and 4: mean-reversion contaminated by a nonstationary drift component. Whilst the mean-reverting component causes a decay in the short-term variance, the addition of a nonstationary (random walk) term prevents the longer term variance ratio from approaching zero.

Realisations of the data-generating processes for experiments 2 and 4 are presented in Figure 6.7. The data-generating process for experiment 3 has properties midway between the two.

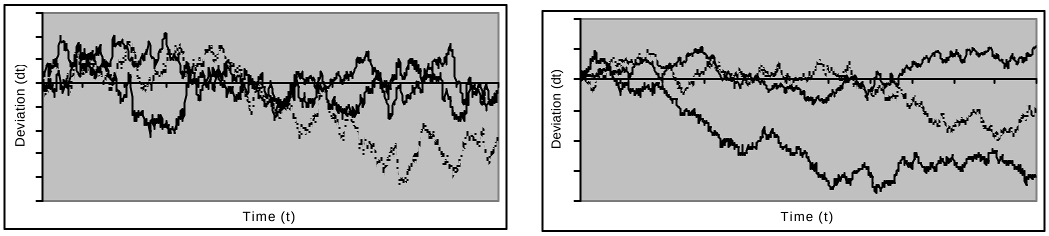


Figure 6.7: Realisations of the data-generating process used in the Monte-Carlo simulations for predictability test experiments 2 (left) and 4 (right)

The power of the different tests against the contaminated reversion processes is shown in the figure below. For the sake of brevity, only the figures for the test of size 10% are presented.

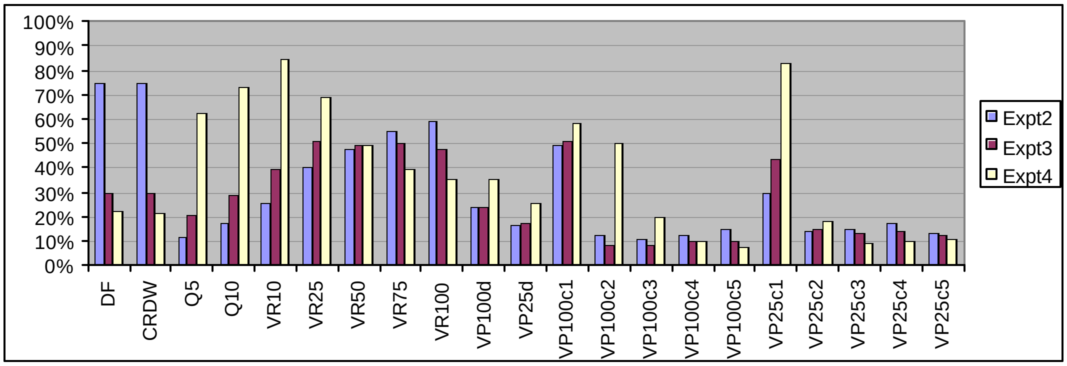


Figure 6.8: Illustration of the power of the predictability tests against the alternative hypotheses of experiments 2, 3 and 4 - mean-reversion which is contaminated by a low level of nonstationary drift. The size (false positive rate) of the tests is 10%.

The overall picture is already very different to the simple case of the pure stationary time-series in Experiment 1. A number of the other tests are now comparable or even superior in power to the standard DF and CRDW tests. The detailed results are presented in Table 6.6 below.

| | DF | CRDW | Q5 | Q10 | VR10 | VR25 | VR50 | VR75 | VR100 | VP100d | VP25d |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Expt2 | **75.0%** | 74.9% | 11.2% | 17.1% | 25.6% | 40.3% | 47.6% | 54.9% | 59.5% | 23.9% | 16.1% |
| Expt3 | 29.8% | 29.4% | 20.8% | 29.3% | 39.5% | **51.1%** | 49.4% | 49.9% | 47.9% | 23.5% | 17.4% |
| Expt4 | 22.2% | 21.9% | 62.5% | 73.3% | **84.5%** | 69.3% | 49.2% | 39.7% | 35.5% | 35.1% | 25.8% |

| | VP100c1 | VP100c2 | VP100c3 | VP100c4 | VP100c5 | VP25c1 | VP25c2 | VP25c3 | VP25c4 | VP25c5 |
|---|---|---|---|---|---|---|---|---|---|---|
| Expt2 | 49.5% | 12.2% | 11.1% | 12.4% | 15.1% | 29.9% | 13.8% | 15.1% | 17.7% | 13.1% |
| Expt3 | 50.8% | 8.5% | 8.5% | 10.0% | 9.5% | 43.5% | 15.1% | 13.0% | 14.3% | 12.2% |
| Expt4 | 58.8% | 50.0% | 19.8% | 9.9% | 7.5% | 82.8% | 17.9% | 9.0% | 10.4% | 11.1% |

Table 6.6: Power of the different predictability tests to detect deviations from random walk behaviour in the form of mean-reversion which is contaminated by a nonstationary drift term. For the sake of brevity, the power statistics are presented only for the case where the nominal size of the tests is 10%

Of the various tests, the stationarity tests (DF and CRDW) are most severely degraded by the presence of a nonstationary component. In Expt. 2, this component only has weight 0.25 (accounting for only 1/16th of the short-term variance) and these tests remain the most powerful of the set of tests under consideration, but by a much reduced margin than in the purely stationary case of Expt 1. The other tests which perform relatively well in this case are the long-period VR statistics and the first components of the VR profiles.

As the amount of nonstationarity increases ($w=0.5$ in Expt. 3 and $w=0.6$ in Expt 4) the stationarity tests and long-period VR tests lose power at the expense of the Q tests and the short-period VR tests. Whilst the nonstationary component degrades the power of the long-period tests, the increased level of mean-reversion (which is included to maintain the overall predictability at approximately the same level as the previous experiments) has the effect of inducing stronger short-term correlations which are detected by the Q and short-period VR statistics. The most consistent performance is shown by the medium term VR50 statistic and the first component of the 100-period profile, VP100c1.

### 6.2.4 Power against a combination of Momentum and Reversion (Expts. 5-10)

Experiments 5 to 10 represent a more complex, yet extremely plausible, form of dynamics in which a relatively short-term trend or momentum term is combined with an underlying reversion. Three different strengths of momentum are considered over different periods: a relatively strong momentum based on single period changes, a weaker momentum over 3-period changes and a weaker still momentum over 10-period returns. The adjustment of the strength of the effect to the length of the period is to compensate for the higher variance over longer periods and maintain the overall level of potential predictability approximately constant. This avoids the two extremes of very weak predictability which is almost impossible to detect and, on the other hand, very strong predictability which is almost impossible to *fail* to detect - neither of which cases would provide much useful information about the relative strengths of the various predictability tests. The three momentum specifications are explored in the context of two combinations of reversion with nonstationary drift.

Firstly, the set of experiments 5-7 consider a combination of reversion with different period momentum effects when the level of nonstationarity is relatively low ($w$=0.25). The variance ratio profiles are shown in Figure 6.9.

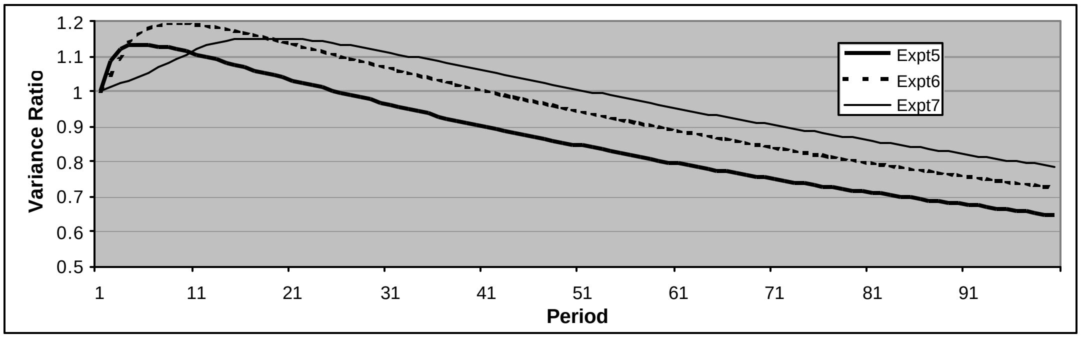


Figure 6.9: The variance ratio profiles for experiments 5, 6 and 7: longer-term mean-reversion combined with short-term momentum and contaminated by a low-level of nonstationary drift, for short periods the variance ratio is above 1 reflecting the positive autocorrelation or momentum effect; for longer time-periods the ratio drops below 1 reflecting the effect of the mean reversion component.

The figure clearly indicates the increased short-term variance which is caused by the momentum term. The point at which the profile attains a maximum can be seen to depend on the period over which the momentum term operates. The power of the predictability tests in the case of these alternative hypotheses is summarised in Figure 6.10 below.

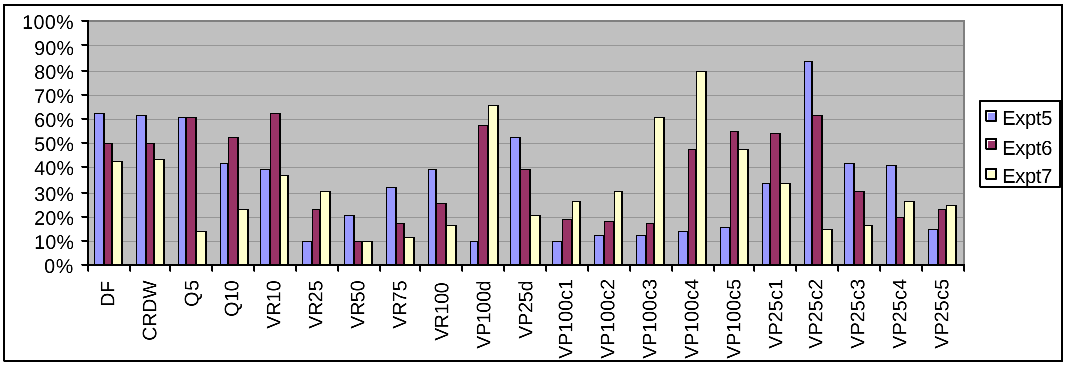


Figure 6.10: Illustration of the power of the predictability tests against the alternative hypotheses of experiments 5, 6 and 7 - mean-reversion which is contaminated by a moderate level of nonstationary drift. The size (false positive rate) of the tests is 10%, figures above 10% indicate a positive ability to detect the predictable component in the data-generating process above and beyond that which would be due to chance alone.

Again the picture is a mixed one; although the stationarity tests perform reasonably well, they are in no case the best performing tests, and in Expt. 7 in particular they are only half as powerful as the best of the tests. The detailed performance figures are shown in Table 6.7 below.

|  | DF | CRDW | Q5 | Q10 | VR10 | VR25 | VR50 | VR75 | VR100 | VP100d | VP25d | VP100c1 | VP100c2 | VP100c3 | VP100c4 | VP100c5 | VP25c1 | VP25c2 | VP25c3 | VP25c4 | VP25c5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Expt5 | 62.2% | 62.1% | 60.4% | 42.3% | 39.3% | 9.6% | 21.1% | 32.1% | 39.4% | 9.8% | 52.8% | 10.3% | 12.0% | 12.4% | 14.5% | 15.9% | 33.8% | **84.0%** | 41.7% | 40.8% | 15.1% |
| Expt6 | 50.4% | 50.6% | 60.6% | 52.7% | **62.9%** | 23.0% | 10.3% | 16.9% | 25.7% | 57.6% | 39.8% | 18.7% | 18.2% | 17.4% | 47.5% | 55.0% | 53.8% | 61.3% | 30.2% | 19.9% | 22.7% |
| Expt7 | 43.1% | 43.9% | 14.4% | 22.8% | 37.1% | 30.3% | 10.0% | 11.5% | 16.3% | 65.5% | 20.9% | 26.1% | 30.7% | 60.6% | **79.3%** | 47.9% | 33.4% | 15.2% | 16.7% | 26.6% | 24.6% |

Table 6.7: Power of the different predictability tests to detect deviations from random walk behaviour in the form of a combination of mean-reversion with short-term momentum effects, contaminated by a low level of nonstationary drift.

The majority of the information appears to be contained in the relatively short term relationships, as reflected by the fact that, on balance, the Q5 test outperforms the Q10 and the VR10 outperforms the longer-period measures. Amongst the variance profile projections, the second principal component of the 25-period VR profile is the highest performing test in Experiment 5 and the second best test in Experiment 6. The third and fourth components of the longer 100-period VR profile are the most powerful tests against the lower frequency momentum exhibited by the time-series in Experiment 8.

Finally, the last set of experiments (numbers 8-10) consider the case where reversion and momentum are combined with a moderately large degree of nonstationarity ($w$=0.5). The average case VR profiles are illustrated in Figure 6.11.

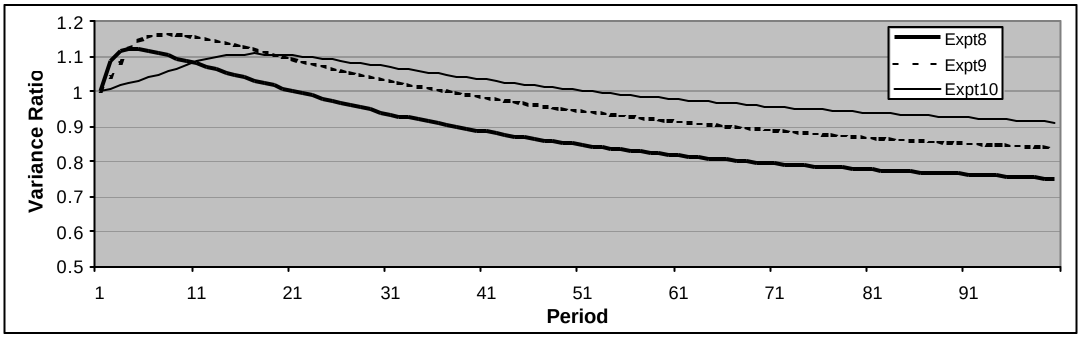


Figure 6.11: Averaged variance ratio profiles from experiments 8, 9 and 10: longer-term mean-reversion combined with short-term momentum and contaminated by a moderate level of nonstationary drift, note that the higher level of nonstationarity means the longer-term variance ratios remain higher than is the case in Figure 6.9.

In each case the profiles indicate the three characteristic features: short-term variance increased by momentum effect, medium term variance decreased by reversion and long term variance stabilised by a nonstationary component. The power of the tests is presented graphically in Figure 6.12.

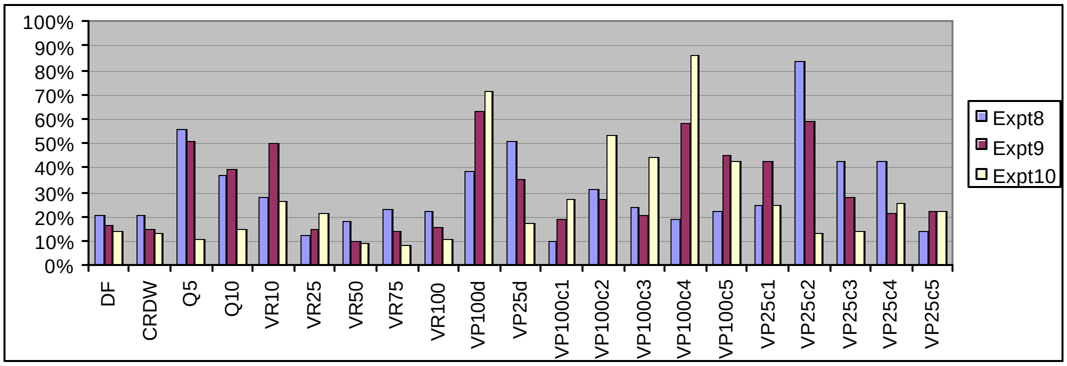


Figure 6.12: Illustration of the power of the predictability tests against the alternative hypotheses of experiments 8, 9 and 10 - mean-reversion which is contaminated by a moderate level of nonstationary drift. The corresponding size (false positive rate) of the tests is 10%.

These experiments depart from the assumptions of stationarity in *two* ways: firstly in the presence of a momentum effect and secondly because the reversion level itself suffers from a moderate degree of nonstationarity. It is clear that the more traditional tests (of either the stationarity, variance ratio or correlation type) are not very well suited to identifying predictable components under these circumstances, in spite of the fact that the underlying dynamics are simple enough to be represented by only 4 parameters. It is here that the various projections of the VR profile come into their own - each capable of detecting a different set of deviations from random walk behaviour. The actual performance figures are presented in Table 6.8.

| | DF | CRDW | Q5 | Q10 | VR10 | VR25 | VR50 | VR75 | VR100 | VP100d | VP25d |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Expt8 | 20.7% | 21.0% | 55.9% | 37.0% | 28.0% | 12.3% | 17.9% | 23.4% | 22.1% | 38.8% | 51.4% |
| Expt9 | 16.7% | 15.2% | 50.7% | 39.6% | 50.0% | 14.7% | 10.3% | 14.2% | 16.0% | **63.6%** | 35.2% |
| Expt10 | 14.1% | 13.4% | 11.0% | 14.8% | 26.7% | 21.6% | 9.3% | 8.6% | 10.8% | 71.5% | 17.0% |

| | VP100c1 | VP100c2 | VP100c3 | VP100c4 | VP100c5 | VP25c1 | VP25c2 | VP25c3 | VP25c4 | VP25c5 |
|---|---|---|---|---|---|---|---|---|---|---|
| Expt8 | 9.6% | 31.0% | 24.1% | 19.3% | 22.4% | 24.8% | **84.2%** | 43.3% | 42.7% | 14.0% |
| Expt9 | 19.1% | 26.8% | 21.0% | 58.5% | 45.1% | 42.5% | 59.5% | 28.5% | 21.7% | 22.3% |
| Expt10 | 27.4% | 53.7% | 44.4% | **86.2%** | 42.7% | 25.1% | 13.1% | 14.2% | 25.4% | 22.6% |

Table 6.8: Power of the different predictability tests to detect deviations from random walk behaviour in the form of a combination of mean-reversion with short-term momentum effects, contaminated by a moderate level of nonstationary drift

The performance of the stationarity tests is very weak, only slightly above the 10% of false positives which would be expected from random-walk time-series. On the other hand some of the VR projections appear to be very well suited to identifying this type of potentially predictable behaviour. For instance, in Experiment 10, VR100c4 has a power of 86.2% compared to only 14.1% for DF, 14.8% for Q10 and 26.7% for VR10. Even in the case of Experiment 8 where Q5 performs moderately well with a power of 55.9%, this is significantly outperformed by the 84.2% of the variance profile projection VP25c2.

### 6.2.5 Discussion

On the whole, the results presented above highlight the dangers of relying on standard tests for predictability in time-series when the underlying dynamics do not match the implicit assumptions of the tests being used. On the other hand, the projections of the variance ratio profile provide a powerful method of identifying a wide variety of deviations from random-walk behaviour.

Table 6.9 presents the most significant correlations between the test statistics, averaged across the whole set of experiments.

|  | DF | CRDW | Qstat5 | Qstat10 | VR10 | VR50 | VP100d | VP100c1 | VP25d | VP25c1 |
|---|---|---|---|---|---|---|---|---|---|---|
| DF | 1 |  |  |  |  |  |  |  |  |  |
| CRDW | -0.974 | 1 |  |  |  |  |  |  |  |  |
| Qstat5 | 0.008 | 0.000 | 1 |  |  |  |  |  |  |  |
| Qstat10 | -0.013 | 0.012 | 0.890 | 1 |  |  |  |  |  |  |
| VR10 | 0.130 | -0.110 | 0.302 | 0.112 | 1 |  |  |  |  |  |
| VR50 | 0.372 | -0.374 | 0.050 | -0.011 | 0.588 | 1 |  |  |  |  |
| VP100d | 0.089 | -0.094 | 0.086 | 0.117 | 0.180 | 0.192 | 1 |  |  |  |
| VP100c1 | 0.354 | -0.351 | 0.154 | 0.058 | 0.746 | 0.941 | 0.279 | 1 |  |  |
| VP25d | -0.064 | 0.070 | 0.258 | 0.207 | 0.121 | -0.044 | 0.175 | 0.035 | 1 |  |
| VP25c1 | 0.093 | -0.075 | 0.012 | -0.166 | 0.700 | 0.446 | 0.165 | 0.562 | 0.222 | 1 |

Table 6.9: Correlations between the predictability test statistics, calculated across all simulated time-series from the whole set of ten experiments.

The two stationarity/cointegration tests, the DF and CRDW statistics, are (negatively) correlated to a very high degree (0.974) as might indeed be expected given that they are designed for the same purpose. The two portmanteau statistics (Qstat5, Qstat10) are also relatively highly correlated (coefficient of 0.890) which suggests that the addition of squared autocorrelation statistics 6 through 10 adds relatively little information to the first five. The variance ratio tests VR10 and VR50 are also correlated, although at a lower level of 0.588. Interestingly, the variance ratio statistics are not highly correlated with either the stationarity tests (DF and CRDW) or the autocorrelation tests (Q5 and Q10), confirming that they do contain information which cannot be provided by either of these more commonly used families of tests.

The variance ratio **distance** statistics (VP25d, VP100d) are largely uncorrelated both with each other and with the other tests. This is likely to be due to the fact that by weighting deviations by the inverse of the associated eigenvalue, they are highly sensitive to any deviation at all on the higher order principal axes. Combined with the uninspiring performance of these tests in the various Monte-Carlo experiments, this suggests that these particular tests are perhaps the least useful of all of those considered.

Of the variance ratio **projections**, only the first principal components turn out to be substantially correlated with any of the standard tests. Not only are they correlated with each other (0.562), but also with the basic variance ratio statistics and, in the case of VP100c1, the stationarity tests DF and CRDW. From this we can take two main messages: the first principal components of the variance ratio profiles are perhaps the best tests overall, and the subsequent principal components can be used to identify classes of potentially predictable dynamics which are likely to be completely missed by standard predictability tests - be they stationarity tests, autocorrelation tests or even variance ratio tests.

## 6.3 Bias-Correction of Predictability Tests

In addition to the introduction of novel predictability tests based upon the joint distribution of the variance ratio profile, the second major innovation in our methodology for predictability testing is the application of both the standard tests and the new tests in the context of multivariate modelling.

Except in the case of unit root tests, which are used in cointegration modelling, the other tests are generally used in a univariate setting rather than being applied to the results of a preliminary multivariate transformation. In order to apply the tests in a multivariate setting it is necessary to correct for the **biases** which are induced by the construction procedure that is used to create the statistical mispricing time-series. In particular, the problem with using the tests in conjunction with a cointegration methodology is that the residuals of a cointegrating regression (even when the variables are random walks) will not behave entirely as a random walk – for instance, they are forced, by construction, to be zero mean. More importantly, the regression induces a certain amount of spurious "mean-reversion" in the residuals and the impact of this on the distribution of the test statistics must be corrected if they are to detect truly deterministic components rather than spurious artefacts of the construction methodology itself. A further complication arises in the case of the extended stepwise methodology for use in high-dimensional problems, because the "selection bias" inherent in choosing the best *m* out of $n > m$ regressors must also be accounted for.

This bias in the tests is effectively due to the fact that the acceptance regions and critical values for the test statistics are based implicitly on the assumption of a null hypothesis of random-walk behaviour. In our case the test statistics are applied to time-series which are the result of the preliminary mispricing construction procedure and so the random walk null hypothesis is no longer appropriate. In order to account for the biases introduced by the construction procedure, the appropriate null hypothesis would be that the series is the result of regressing one random walk upon a set of random walks. The distribution of the test statistics may be quite different under this corrected null hypotheses, and using the incorrect null hypothesis may result in false rejections of the null in favour of an assumed alternative that the time-series contains a significant predictable component.

Our solution to this problem is to derive the empirical distribution of the test statistics under the corrected null hypothesis, by the use of Monte-Carlo simulations. Given the experimental parameters such as the length of the time-series, the number of regressors, etc. the empirical distribution of the test statistic is generated by creating many realisations of random walk processes and calculating the value of the test statistic in each case. As well as correcting for the biases induced by the algorithm itself, this approach has the additional advantage of automatically taking into account the appropriate sample size, rather than relying on asymptotic behaviour.

In particular, we focus on the bias induced in the variance ratio profile when considering time-series that are the result of regressing random walks upon each other, rather than in the case where the time-series is itself a simple random walk. The size of the induced bias is a function of the number of regressors used to construct the synthetic asset, the size of the asset universe from which the regressors were selected, and the length (sample size) of the time-series.

### Number of Regressors

To illustrate the sensitivity of the variance ratio profile we present one dimensional slices through the design space by varying each of the parameters in turn. Figure 6.13 illustrates effect of varying the *number of regressors*.

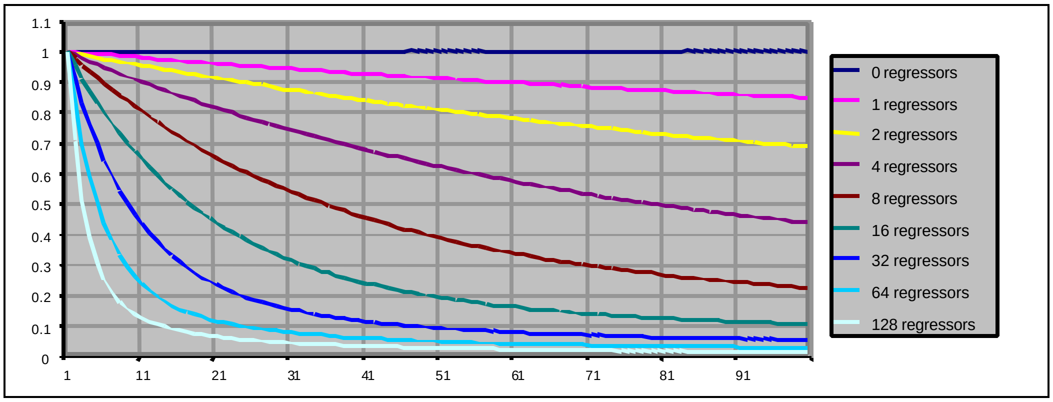


Figure 6.13: Illustration of the bias induced in the variance ratio profile of the residuals obtained by regressing a random walk on a varying number of other random walk series. The variance ratio declines substantially below 1, which might erroneously be taken to indicate the presence of mean-reversion but is in fact merely a bias induced by the regression procedure. Results presented are calculated over 10000 Monte-Carlo simulations with a sample size of 500 in each case.

In the case of 0 regressors, the behaviour of the series is as we would expect from a random walk - namely an average variance ratio profile with a constant value of one. However, this result is only obtained in the case where 0 regressors are used. In all other cases we notice a distinct bias induced in the variance ratio profile. This bias takes the form of a decline in the variance ratio profile which has a similar appearance to the profile produced by time-series which contain a mean-reverting component. However in these cases, the apparent mean-reversion is a completely spurious artefact of the regression procedure and an even more extreme deviation from the random walk profile would be required to indicate the presence of a true mean-reverting component.

## Selection of Regressors

A further bias is caused by the use of a stepwise procedure, as the candidate variables which are selected from the pool of available regressors will tend to be precisely those which induce the greatest bias in the variance ratio profile. The effect of keeping the number of selected regressors constant, but varying the *size of the pool of variables* from which they are selected, is illustrated in Figure 6.14.

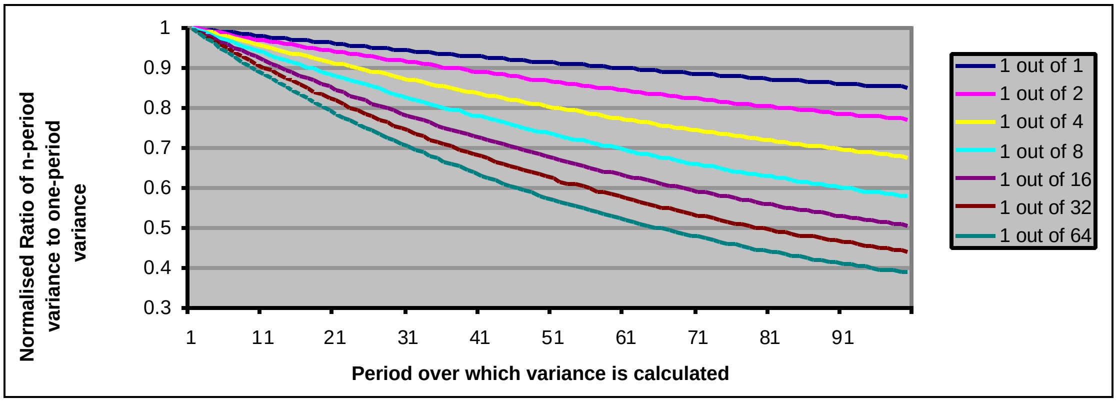


Figure 6.14: Illustration of the induced bias as a function of the size of the pool of variables from which the regressors are selected. The profiles correspond to selecting a single regressor from a pool of 'n' candidate variables, with 'n' ranging from 1 to 64. The sample size is 500 observations and each curve is calculated as an average over 10000 Monte-Carlo simulations.

The figure clearly illustrates that an additional bias is induced by the fact that the regressors were selected from a larger set of candidate variables. This is a prime example of the dangers posed by selection bias, or "data snooping" (see Section 12.1.2).

In this particular case, where the sample size is 500 and the number of selected regressors is 1, the number of candidate regressors has a large impact on the size of the induced bias. Where only a single potential regressor is available the average value for the 100-period variance ratio is around 0.86 - a relatively small deviation from the 1.00 expected if the series were a pure random walk. On the other hand, where the single regressor is selected from a pool of 64 candidates, the 100-period variance ratio is only 0.4 - a much more substantial bias. These results clearly indicate that the size of the asset universe should be taken into account when testing for potential predictability in the dynamics of mispricing time-series which were constructed using the stepwise version of our methodology.

### Sample Size

The final factor which determines the size of the bias which is induced by the construction procedure is the *sample size*. Figure 6.15 illustrates the average variance ratio profiles for the residuals of regressing a random walk on a set of 8 regressors, where each regressor is also a random walk.

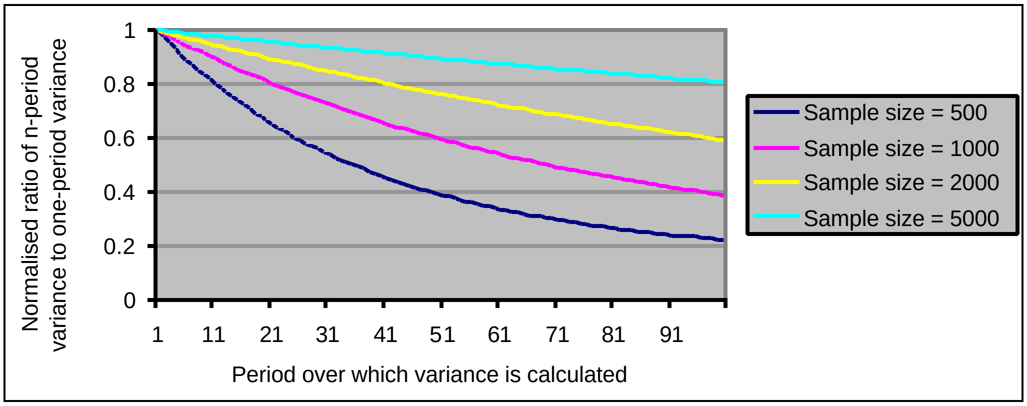


Figure 6.15: Illustration of the induced bias as a function of sample size. The profiles correspond to using 8 regressors in sample sizes ranging from 500 to 5000 observations. Each curve is calculated as an average over 10000 Monte-Carlo simulations.

These results clearly demonstrate that the magnitude of the bias is inversely related to the sample size, thus indicating a greater ability of the preliminary regression procedure to "overfit" the observed data when the sample size is small. The bias induced in the large sample case (sample size = 5000) is relatively small, with the 10-period variance ratio taking an average value of 0.81, compared to the small-sample case (sample size = 500) where the bias induced is such that the 100-period variance ratio takes an average value of only 0.23.

### Two-dimensional sensitivity: Joint effect of Sample Size and Number of Regressors

The results presented above should be considered as one-dimensional slices through the three-dimensional space defined by {number of regressors, size of regressor pool, sample size}. By performing an appropriate simulation, we can measure (and correct) the biases which are introduced by two or more of these factors jointly. Table 6.10 presents the average VR(50) statistics obtained by jointly varying both the *number of regressors* and the *sample size*.

|  | *500* | *1000* | *2000* | *5000* |
|---|---|---|---|---|
| 0 | 1.00 | 1.00 | 1.00 | 1.00 |
| 1 | 0.91 | 0.96 | 0.97 | 0.99 |
| 2 | 0.81 | 0.89 | 0.95 | 0.98 |
| 3 | 0.71 | 0.84 | 0.91 | 0.96 |
| 4 | 0.62 | 0.78 | 0.88 | 0.95 |
| 8 | 0.39 | 0.59 | 0.76 | 0.89 |
| 16 | 0.19 | 0.37 | 0.58 | 0.80 |
| 32 | 0.10 | 0.18 | 0.36 | 0.64 |
| 64 | 0.05 | 0.09 | 0.18 | 0.43 |
| 128 | 0.03 | 0.05 | 0.09 | 0.22 |

Table 6.10: The average value of the VR(50) statistic as a function of the sample size (columns) and number of regressors used in the mispricing construction procedure (rows).

These results clearly demonstrate the fact that the bias induced by the regression procedure *increases* with the number of regressors and *decreases* with the length of the time-series (sample size).

**Summary**

The results reported above clearly demonstrate that variance ratio profiles which are generated from regression residuals behave very differently from those of true random walks. The induced bias grows with the number of regressors and also, in the stepwise procedure, with the number of candidate variables from which the variables were selected. The size of the bias is less when the sample size is large and greater when the sample size is small. When testing for predictability in mispricing dynamics it is important to use test statistics which correct for the induced bias. In particular, this correction, will reduce the risk of concluding that a time-series contains a predictable component when the deviation from the random walk profile is caused by the construction procedure itself rather than a systematic component in the actual dynamics.

Our solution to this problem is to generate the appropriate empirical distributions of our test statistics prior to performing the modelling procedure itself, and such that the experimental parameters (number of regressors, size of the pool form which the regressors are chosen, and sample size) match those which will be used in the modelling procedure. This approach is rather computationally intensive, given that the number of simulations required is of the order of 1000s or 10000s depending on the accuracy required, but is quite within the capabilities of standard PCs. In principle, the use of such empirical distributions is very flexible as it allows for further modification of the null hypothesis data-generating process should it be so desired. An example of such a modification would be to take into account conditional volatility effects through the use of the GARCH processes discussed in Section 2.2.3 rather than the standard homoskedastic (constant variance) random walks discussed above.

## 6.4 Evidence for Potential Predictability in Statistical Mispricings

In this section we illustrate the use of the bias-corrected predictability tests, and the variance profile tests in particular, with respect to the models described in Section 5.3. These models represent statistical mispricings of the daily closing prices of the equities which constitute the FTSE 100 index and, as described in Section 5.3, were constructed using the stepwise regression methodology with a maximum of five constituent assets in each model.

In order to calculate the appropriate distributions for the variance ratio profile statistics we apply the bias correction procedure described in Section 6.3. This consists of performing a Monte Carlo simulation under the null hypothesis of regressing random walk variables on other random walks (i.e. no predictable component). This procedure automatically accounts for the impact of (a) the mean-reversion induced by the regression itself, and (b) the selection bias introduced by the use of the stepwise procedure.

The empirical distribution was calculated from 1000 simulations, in each case the parameters of the simulation match those of the subsequent statistical arbitrage modelling: that is, a 400 period realisation of a random walk is regressed upon 5 similarly generated series, selected sequentially from a set of 93 such series, using a forward stepwise selection procedure, and the variance ratio profile calculated from the residuals of the regression[16]. The variance ratio is calculated for returns varying from one period up to fifty periods. Note however, that by construction the value of VR(1) can only take the value 1.

Given the average profile and covariance matrix of the profile under the null hypothesis of random walk behaviour, we then tested the residuals of the *actual statistical arbitrage*

[16] Clearly it would be straightforward to repeat the procedure for other experimental parameters: sample size, number of variables etc, but as noted in section 6.3 the huge number of possible combinations leads towards recalibrating for particular experiments rather than attempting to tabulate the entire distribution.

*models* for significant deviations from these profiles. Some examples of the variance profiles for the mispricing models are presented in Figure 6.16.

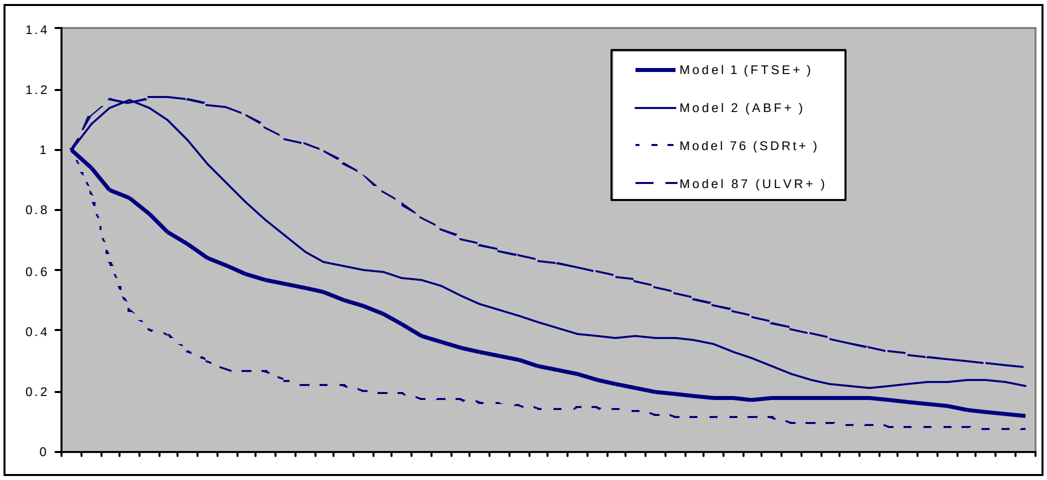


Figure 6.16: Selected variance ratio profiles for statistical mispricings obtained through stepwise regression of each target asset on five related assets from the FTSE 100 universe

The tests employed were the two types of variance ratio profile statistics described in Section 6.1 and analysed in Section 6.2, namely the Mahalanobis distance of the observed profile from the average profile under the null hypothesis (VPd) and the projection of this difference onto the first five eigenvectors VPc1..VPc5.

The results of the bias-corrected predictability tests are shown in Tables 6.11-6.13 below. The nominal size of the tests is 1% in Table 6.11, 5% in Table 6.12 and 10% in Table 6.13. The nominal test sizes are based upon the assumption that the tests statistics follow normal distributions, in order to account for deviations from this assumption we present results both for the calibration sample itself, for a second similar but independent "Test" sample, and for the mispricing models. For the variance profile projections, we use eigenvectors derived from both the correlation and the covariance matrix of the variance profile; statistics derived from the covariance matrix are indicated by a (v).

| | VPd | VPc(v)1 | VPc(v)2 | VPc(v)3 | VPc(v)4 | VPc(v)5 | VPc1 | VPc2 | VPc3 | VPc4 | VPc5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Cal | 1.8% | 1.6% | 1.4% | 1.4% | 0.9% | 1.4% | 1.7% | 1.1% | 1.7% | 1.5% | 1.2% |
| Test | 4.3% | 1.2% | 0.9% | 1.8% | 1.3% | 1.2% | 1.3% | 0.9% | 1.3% | 1.3% | 1.6% |
| Model | 36.2% | 8.5% | 1.1% | 2.1% | 3.2% | 3.2% | 8.5% | 4.3% | 3.2% | 4.3% | 8.5% |

Table 6.11: Comparison of VR tests for random-walk simulations and actual mispricings, nominal size of test = 1%

| | VPd | VPc(v)1 | VPc(v)2 | VPc(v)3 | VPc(v)4 | VPc(v)5 | VPc1 | VPc2 | VPc3 | VPc4 | VPc5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Cal | 6.6% | 4.5% | 5.1% | 4.8% | 5.2% | 6.0% | 4.7% | 5.8% | 4.0% | 4.1% | 4.8% |
| Test | 9.9% | 3.9% | 5.5% | 4.6% | 4.8% | 5.4% | 4.1% | 4.2% | 4.3% | 5.6% | 6.2% |
| Model | 53.2% | 11.7% | 8.5% | 7.4% | 12.8% | 11.7% | 11.7% | 9.6% | 8.5% | 14.9% | 13.8% |

Table 6.12: Comparison of VR tests for random-walk simulations and actual mispricings, nominal size of test = 5%

| | VPd | VPc(v)1 | VPc(v)2 | VPc(v)3 | VPc(v)4 | VPc(v)5 | VPc1 | VPc2 | VPc3 | VPc4 | VPc5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Cal | 11.7% | 8.7% | 9.8% | 9.5% | 10.6% | 10.5% | 8.3% | 10.0% | 8.4% | 9.4% | 9.7% |
| Test | 14.5% | 8.2% | 10.5% | 9.3% | 10.9% | 10.7% | 7.5% | 10.1% | 8.5% | 12.1% | 10.4% |
| Model | 59.6% | 20.2% | 13.8% | 14.9% | 18.1% | 18.1% | 19.1% | 14.9% | 16.0% | 19.1% | 23.4% |

Table 6.13: Comparison of VR tests for random-walk simulations and actual mispricings, nominal size of test = 10%

The tests indicate that the mispricings of the statistical arbitrage models deviate significantly from the behaviour of the random data – providing evidence of potentially predictable deviations from randomness. Table 6.14 below shows ‘z’ tests for significant deviations in the mean scores of the observed mispricings when compared to the random validation sample:

| | VPd | VPc(v)1 | VPc(v)2 | VPc(v)3 | VPc(v)4 | VPc(v)5 | VPc1 | VPc2 | VPc3 | VPc4 | VPc5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| AveTest | 50.99 | -0.01 | -0.01 | 0.00 | 0.01 | 0.00 | -0.15 | 0.03 | 0.10 | 0.08 | -0.07 |
| VarTest | 135.19 | 0.22 | 0.04 | 0.01 | 0.01 | 0.00 | 33.57 | 5.56 | 2.72 | 1.57 | 0.83 |
| AveModel | 70.79 | -0.23 | 0.01 | 0.07 | -0.03 | -0.03 | -2.34 | 0.32 | 1.02 | -0.65 | 0.61 |
| VarModel | 676.34 | 0.41 | 0.04 | 0.02 | 0.01 | 0.00 | 62.67 | 7.45 | 3.83 | 1.86 | 1.06 |
| z' stat | 7.3 | -3.2 | 0.5 | 4.0 | -4.1 | -5.0 | -2.6 | 1.0 | 4.4 | -5.0 | 6.3 |
| p-value | 0.0000 | 0.0012 | 0.6116 | 0.0001 | 0.0000 | 0.0000 | 0.0089 | 0.3282 | 0.0000 | 0.0000 | 0.0000 |

Table 6.14: Comparison of mean values of the various VR tests for random-walk simulations and actual mispricings

The presence of significant deviations in the mean test statistics of the mispricing models, compared to those of the simulated random walk series, reinforces the findings that the actual mispricings deviate significantly from random behaviour and supports the hypothesis that there may be potentially predictable components in the dynamics of the statistical mispricings.

## 6.5 Summary

In this chapter we have described the tests which we have developed for the purpose of identifying potentially predictable components in the dynamics of the statistical mispricing time-series, including both standard predictability tests and novel tests based upon the joint distribution of the variance ratio profile of the time-series. Extensive simulations against a range of alternative hypotheses provided evidence that these tests can outperform standard tests in cases where the dynamics combine short-term trending behaviour (momentum) with longer term reversion effects and suffer from a degree of nonstationary contamination. The biases which are induced by applying both the standard and the novel tests in a multivariate setting have been identified, and quantified by means of  Monte Carlo simulations. Finally the bias-corrected statistics have been applied to a set of mispricing models for FTSE 100 equity prices and found to provide statistically significant evidence that the average mispricing dynamics contain statistically significant deviations from random behaviour.

In the following chapter we present a set of specialised trading rules which are designed to directly exploit any mean-reverting component in the mispricing dynamics, without the need for an explicit forecasting model of the mispricing dynamics.

# 7. Empirical Evaluation of Implicit Statistical Arbitrage Models

In this chapter we describe a set of "implicit statistical arbitrage" (ISA) trading strategies which are designed to directly exploit any mean-reverting component in the mispricing dynamics, bypassing the intermediate stage of constructing an explicit forecasting model. The ISA trading rules are defined in Section 7.1. In the following two sections, the ISA rules are used to perform an empirical evaluation of the statistical mispricing methodology. The high-dimensional version of the mispricing construction methodology is evaluated on FTSE 100 equity prices in Section 7.2 and the adaptive version of the methodology is evaluated on a pair of European equity indices in Section 7.3.

## 7.1 Implicit Statistical Arbitrage Strategies and Trading Rules

In this section we describe a set of "implicit statistical arbitrage" (ISA) trading strategies, so-called because the trading rules upon which they are based rely *implicitly* on the mean-reverting behaviour of the mispricing time-series. An example of such a trading rule was given in Eqn. (4.8). Given a mispricing model of the form shown in Eqn. (5.10), the underlying assumption of the ISA strategies is that future price changes will be such as will tend (on average) to reduce the mispricing between the target asset and the combination of constituent assets. Thus the ISA strategies act by exploiting what might be considered a "mispricing correction effect", and will realise profits in cases where the mean-reversion component in the mispricing dynamics is sufficiently large to overcome the transaction costs which are associated with following the strategy.

The ISA strategies are implemented by parametrised trading rules which define the sign and magnitude of the currently desired holding in the mispricing portfolio, which consists of the set of assets $\left\{T_t, C_1, C_2, \ldots, C_{n_c}\right\}$ in the proportions $\left\{1, -b_1, -b_2, \ldots, -b_{n_c}\right\}$ respectively. The basic strategy defines the desired holding according to the trading rule:

$$ISA(M_t, k)_t = -\operatorname{sign}\left(M_{t-j}\right)\left|M_{t-j}\right|^k \tag{7.1}$$

The negative sign indicates that the mispricing should be bought when it is negative, and sold when it is positive. The sensitivity parameter $k$ allows the magnitude of the holding to vary as a function of the size of the current mispricing. With $k=0$ the function is a step function, and the entire holding is always invested in the mispricing portfolio (whether bought "long", or sold "short"). With $k>0$ the size of the portfolio will be increased if the magnitude of the mispricing increases, and decreased if the magnitude of the mispricing decreases; the exact nature of this scaling will depend upon the value $k$ and is illustrated for selected values in Figure 4.4.

As discussed in Towers and Burgess (1998a) and Towers (1999), the performance of the basic ISA rule in Eqn. (7.1) can be highly sensitive to the level of transaction costs. Fluctuations in the mispricing can lead to similar fluctuations in the desired holding, thus generating frequent transactions and potentially undermining otherwise profitable strategies by incurring high levels of transaction costs. Thus we define generalisations of the basic rule which are designed to reduce the trading activity by smoothing the trading signal:

$$ISA(M_t, k, h)_t = \frac{1}{h} \sum_{j=0..h-1} ISA(M_t, k)_{t-j} \tag{7.2a}$$

$$ISA(M_t, k, q)_t = (1-q) ISA(M_t, k)_t + q\, ISA(M_t, k, q)_{t-1} \tag{7.2b}$$

The modified rules offer some control over the performance of the trading strategy **net** of costs. This can be achieved by choosing smoothing parameters which optimise the tradeoff between exploiting the predictive information (which is implicit in the level of the mispricing) on the one hand, and incurring transaction costs through over-frequent trading on the other. Thus increasing either the moving-average parameter $h$ (in Eqn. 7.2a) or the exponential smoothing parameter $q$ will tend to reduce the level of transaction costs, but also the accuracy of the smoothed trading signal. In a manner analogous to the bias-variance tradeoff of model estimation, the imposition of a low level of smoothing ("regularisation term") may increase the overall performance due to a more-than-offsetting reduction in transaction costs ("model variance").

The "return" made by the ISA rule during a given period $t$ to $t+1$ is deemed to be:

$$ISARET(M_t, T_t, SA(T)_t, k)_t = ISA(M_t, k)_t \frac{\Delta M_t}{T_t + SA(T)_t} - c\left|\Delta ISA(M_t, k)_t\right| \tag{7.3}$$

i.e. the current portfolio holding, multiplied by the change in the portfolio value, scaled by the total (rather than net) value of the holdings, and then adjusted to account for transaction costs incurred as a result of changes in the trading signal. Thus the first term on the right hand side of Eqn (7.3) corresponds to the proportional change in the mispricing relative to the total size of the absolute holdings in the mispricing portfolio, i.e. the value of the target asset plus the value of the corresponding synthetic asset. The transaction cost is approximated as a proportion $c$ of the change in the desired holding, note that $c$ represents one half of the "bid-ask spread" in that changing from a position of +1 to one of –1 will incur costs of 2$c$.

The cumulative profit from period 1 to period $n$ is simply given by:

$$ISAPROF(M_t, T_t, SA(T)_t, k)_t = \sum_{t=1..n} ISARET(M_t, T_t, SA(T)_t, k)_t \tag{7.4}$$

i.e. no compounding of profits and losses is generally taken into account (although such a measure can clearly easily be defined).

An important performance metric which accounts not only for the level of profitability, but also the associated level of variability in profits (risk), is the Sharpe Ratio (Sharpe, 1966). This measure of *risk-adjusted return* is the ratio of the mean return to the standard deviation of returns. The Sharpe Ratio is commonly used by market practitioners as a means of quantifying the amount of profitability per unit of risk[17]. The Sharpe Ratios for our statistical arbitrage strategies are calculated as the ratio of the mean profitability of a trading strategy to the standard deviation of the profits of the strategy:

$$ISASR(M_t, T_t, SA(T)_t, k)_t = \frac{\frac{1}{n}\sum_{t=1..n} ISARET(M_t, T_t, SA(T)_t, k)_t}{\sqrt{\frac{1}{n-1}\left[\sum_{t=1..n} ISARET(M_t, T_t, SA(T)_t, k)_t - \frac{1}{n-1}\sum_{t=1..n} ISARET(M_t, T_t, SA(T)_t, k)_t\right]^2}} \tag{7.5}$$

The ability of the ISA trading rules to effectively exploit a mean-reverting component in mispricing dynamics is illustrated in Figure 7.1. which shows the cumulative profit curves (ISAPROF as defined in Eqn. (7.4)) for the statistical mispricing model of Bass Plc. which served as an example of the stepwise construction methodology in Section 5.3.

[17] The Sharpe Ratio scales with the square root of the number of periods $n$. To aid comparison between strategies it is often *annualised* by multiplying by $\sqrt{ppy/n}$ where *ppy* is the number of trading periods in a year.

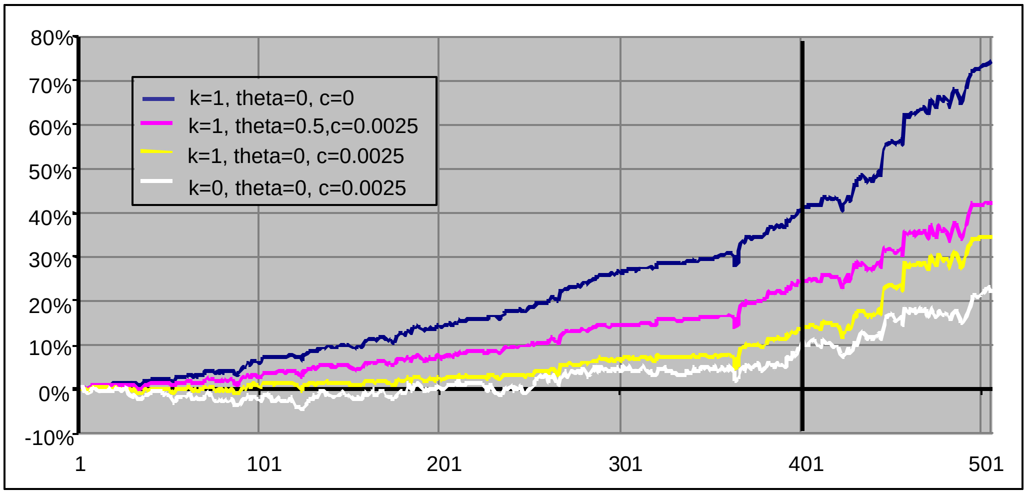


Figure 7.1: Cumulative equity curves for ISA strategies applied to statistical mispricing of Bass Plc. The first 400 observations correspond to the estimation period, whilst the final 100 observations represent a subsequent "out of sample" period. The corresponding mispricing time-series is illustrated in Figure 5.9.

The equity curves clearly demonstrate the sensitivity of the performance not only to the assumed level of transaction costs, $c$, but also to the parameters of the trading rule, which in this case are the sensitivity parameter $k$ and the exponential smoothing parameter $q$ of the rule in Eqn (7.2b).

In the following section we apply the ISA trading rules to a more extensive set of statistical mispricing models of FTSE 100 stocks.

## 7.2 Empirical Results for FTSE 100 Equity Prices

In this section we present an empirical evaluation of the implicit statistical arbitrage strategies applied to statistical mispricings between the equities which comprise the FTSE 100 index. The data consists of daily closing prices between 13th June 1996 and 5th October 1998. The 600 observations are divided into a 400 day "insample" period which is used to estimate the statistical mispricing models, and a subsequent 200 day "out-of-sample" period which is used to present an unbiased estimate of the generalisation performance of the models. After removing the assets for which continuous data samples were not available, the number of assets in the sample was 89 constituents plus the index itself.

### Construction of the synthetic assets using the stepwise cointegration methodology

Due to the large size of the asset universe, the *stepwise* variant of the construction methodology (Section 5.3) was used to generate the synthetic asset models. Each asset in turn was taken as the “target” asset, and the number of constituent assets *n* varied from 0 to 4 (the case *n*=0 corresponding to the raw asset price). The mispricing models are thus of the form:

$$M_{\mathrm{s,t}} = P_{s,t} - \left( \sum\nolimits_{i=1}^{n} w_{s,i} P_{c(i,s),t} + c \right) \tag{7.6}$$

In total, this procedure resulted in the creation of 90*5 = 450 synthetic asset models, each representing a candidate for trading through the use of an implicit statistical arbitrage trading rule.

### Correlation of trading performance with predictability test statistics

The first experiment was to investigate the *correlation* between the various predictability tests described in Chapter 6 and the performance of the ISA trading rules. A summary of the correlations which are exhibited between the **insample profitability** (ISAPROF defined by Eqn. (7.4)) and the predictability test statistics of the 450 mispricing time-series (also evaluated only on the insample period) is presented in Table 7.1. The ISA rule used was the basic rule (Eqn. (7.1) with $k$ = 1.

| *n* | DF | DW | Q5 | Q10 | VR5 | VR10 | VR15 | VR20 | VR25 | VP25d | VP25c1 | VP25c2 | VP25c3 | ISASR(IS) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -73% | 69% | -11% | -3% | -24% | -30% | -29% | -27% | -25% | -17% | -28% | 20% | -13% | 95% |
| 1 | -13% | 8% | -4% | 4% | -35% | -28% | -29% | -30% | -29% | 0% | -30% | 26% | 23% | 60% |
| 2 | -16% | 11% | 4% | 11% | -24% | -23% | -28% | -31% | -32% | 11% | -29% | 12% | 9% | 60% |
| 3 | -7% | 6% | 9% | 18% | -39% | -35% | -34% | -34% | -32% | 17% | -36% | 32% | 15% | 42% |
| 4 | -11% | 9% | 12% | 12% | -38% | -33% | -30% | -32% | -35% | 10% | -34% | 30% | 21% | 34% |
| Overall | -16% | 14% | 2% | 6% | -53% | -55% | -60% | -62% | -63% | 4% | -62% | 25% | 11% | 82% |

Table 7.1: Correlation between in-sample predictability test statistics applied to the statistical mispricings, and the in-sample profitability (ISAPROF) of the associated ISA trading rules.

The statistics most correlated with the trading profitability are the variance ratio statistics (negative correlation with profits indicates that low VR statistics tend to indicate high profits) and the first variance profile projection, VP25c1. The *risk-adjusted* measure of insample profits, ISASR as defined in Eqn. (7.5), is found to be highly, but not perfectly, correlated with the *cumulative* profits ISAPROF indicating that the strategies vary in both return and risk.

It is interesting to note the extent to which the correlations between the test statistics and trading performance will generalise to future trading performance. Table 7.2 presents the correlations between the insample predictability test statistics and the **out-of-sample** profitability:

| *n* | DF | DW | Q5 | Q10 | VR5 | VR10 | VR15 | VR20 | VR25 | VP25d | VP25c | VP25c | VP25c | ISAPROF | ISASR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | (IS) | (IS) |
| 0 | -13% | 13% | 11% | 12% | 12% | 7% | 1% | -5% | -8% | -3% | -1% | 3% | -26% | 5% | 5% |
| 1 | 11% | -10% | 4% | 13% | -12% | -20% | -26% | -27% | -26% | 10% | -25% | -22% | -1% | 3% | 24% |
| 2 | 11% | -11% | 0% | 9% | -7% | -15% | -21% | -20% | -18% | 19% | -19% | -16% | -1% | 8% | 28% |
| 3 | 6% | -2% | -3% | -1% | -1% | -5% | -10% | -10% | -8% | 12% | -8% | -5% | -9% | 25% | 34% |
| 4 | 3% | -1% | -4% | -7% | -3% | -3% | -4% | -3% | -3% | 9% | -3% | -2% | -5% | 3% | 20% |
| Overall | 3% | -2% | 1% | 4% | -16% | -22% | -27% | -29% | -30% | 9% | -27% | -25% | -8% | 28% | 38% |

Table 7.2: Correlation between in-sample predictability test statistics applied to the FTSE mispricings, and the out-of-sample profitability. The rightmost columns show the correlation between the insample and out-of-sample profitability of the ISA rules themselves.

The correlations here show a similar pattern to those in Table 7.1, but are approximately halved in magnitude. This degradation suggests that the profitability of the ISA strategies is unstable over time – a fact which is confirmed in the rightmost columns by the relatively low level of correlation between **insample** and **out-of-sample** profits. Note also that the raw insample profit, ISAPROF(IS), generalises less well to out-of-sample profit (correlation of 28%) than does the *risk-adjusted* insample profit, ISASR(IS) (correlation to out-of-sample profit = 38%) suggesting that the best indicator of continued profitability is the *consistency* of the in-sample performance rather than the mere magnitude of profits.

### Trading performance and model selection

The second experiment was to investigate the trading performance of the models, and in particular the effect of *model selection*. From each set of 90 models, the 10 best models were selected according to each of four criteria: the VR20 and VP25c1 statistics and the insample profitability in both raw and risk-adjusted form (ISAPROF and ISASR). The average out-of-sample profitability of the selected sets of models is shown in Table 7.3.

| *n* | All models | ISAPROF(IS) | ISASR(IS) | VR20 | VP25c1 |
|---|---|---|---|---|---|
| *0* | *-1%* | *4%* | *3%* | *-1%* | *-3%* |
| 1 | 5% | 10% | 10% | 16% | 15% |
| 2 | 6% | 6% | 10% | 12% | 11% |
| 3 | 8% | 11% | 16% | 13% | 16% |
| 4 | 8% | 8% | 11% | 10% | 8% |

Table 7.3: Trading performance of the FTSE ISA models. The first column corresponds to the number of constituent assets included in the specification of the synthetic asset models. The second column corresponds to the average profitability of all 90 models (with no selection criterion) during the 200-day out-of-sample period. The remaining four columns show the performance of the top ten models of each specification, selected by the different criteria.

The first main result is that the statistical mispricing models ($n$>0), which are based on *transformed* prices, significantly outperform the models which are based on *raw* asset prices. For the 90 models with $n$=0, the mean performance is -0.9% and the standard deviation of performance is 8.9%; for the 360 models with $n$>0, the mean performance is 6.8% and the standard deviation is 10.0%. The $z$-statistic for the difference between the two sample means evaluates to 7.104, with a p-value of 6.11E-13, indicating that the out-of-sample performance gain from using the stochastic detrending procedure is highly statistically significant.

The second main result is that, in almost all cases, the subsets of models identified by the selection criteria outperform the baseline for each specification. Table 7.4 presents an analysis of the performance increases from using model selection:

| | Profits | | | | Sharpe Ratio | | | |
|---|---|---|---|---|---|---|---|---|
| | ISAPROF | ISASR | VR20 | VP25c1 | ISAPROF | ISASR | VR20 | VP25c1 |
| Max | 34.8% | 32.6% | 32.6% | 32.6% | 3.69 | 3.69 | 3.69 | 3.69 |
| Min | -25.6% | -11.2% | -1.9% | -25.6% | -0.93 | -0.93 | -0.18 | -0.93 |
| Median | 7.1% | 11.5% | 10.6% | 11.6% | 0.65 | 1.22 | 1.10 | 1.30 |
| | | | | | | | | |
| Average | 8.7% | 11.8% | 12.9% | 12.5% | 0.84 | 1.20 | 1.35 | 1.36 |
| Gain | 2.0% | 5.0% | 6.2% | 5.8% | 0.07 | 0.43 | 0.58 | 0.60 |
| z'-stat | 0.95 | 3.17 | 4.68 | 3.45 | 0.41 | 2.69 | 4.26 | 3.98 |

Table 7.4: Out-of-sample performance analysis of the sets of implicit statistical arbitrage models selected by different criteria.

In all four cases the model performance is increased above the baseline of 6.8% (average across all 360 synthetic assets with number of constituents $n>0$). In all cases except that of using the insample profit as the selection criterion, the increase in average performance is indicated as being statistically significant. A similar improvement in shown in the risk-adjusted figures which are presented in the form of annualised Sharpe ratios.

### Diversification across the set of models

Whilst the above results show that positive profits can be achieved from the application of the ISA rules to the FTSE mispricings, they do not appear to be very impressive, particularly in risk-adjusted terms. A general rule of thumb amongst practitioners when evaluating Sharpe Ratio statistics for trading models, is that (annualised) values of below 1 are uninteresting, between 1 and 2 indicates that a model shows promise, but only values of 2 or more are considered noteworthy. In contrast, the average performance of the selected sets of ISA models is only just over 1, and even then this is **before** transaction costs are taken into account.

However, this rather pessimistic view overlooks the advantages which can be obtained through *diversification*. Just as traditional portfolio optimisation can benefit from risk-diversification across a number of different underlying **assets**, so model-based trading can benefit through diversification across a number of **models**. This issue is in fact central to Part III of our methodology, which presents the "portfolio of models" approach to risk diversification amongst trading models.

As a preliminary indication of the advantages of model combination in a trading context, we arbitrarily select the set of models chosen by the VR20 criteria and divide the nominal capital equally across all 40 models. Table 7.5 presents the performance of the combined set of models.

|  | Overall Return | Profitable Periods | Sharpe Ratio |
|---|---|---|---|
| Part1 | 7.6% | 61% | 4.64 |
| Part2 | 5.3% | 57% | 2.67 |
| Overall | 12.9% | 59% | 3.56 |

Table 7.5: Out-of-sample performance analysis of the combined set of 40 models which were selected by the VR20 criterion. (with zero transaction costs). The results are presented both for the 200-day out-of-sample period as a whole and also broken down into two equal parts.

The overall Sharpe Ratio is improved by almost a factor of three through diversifying across the set of 40 models. As the mean return is unchanged, this improvement must be due to a reduction in the risk (standard deviation of returns), relative to that of the individual models. Figure 7.2 presents the equity curve for the combined set of models:

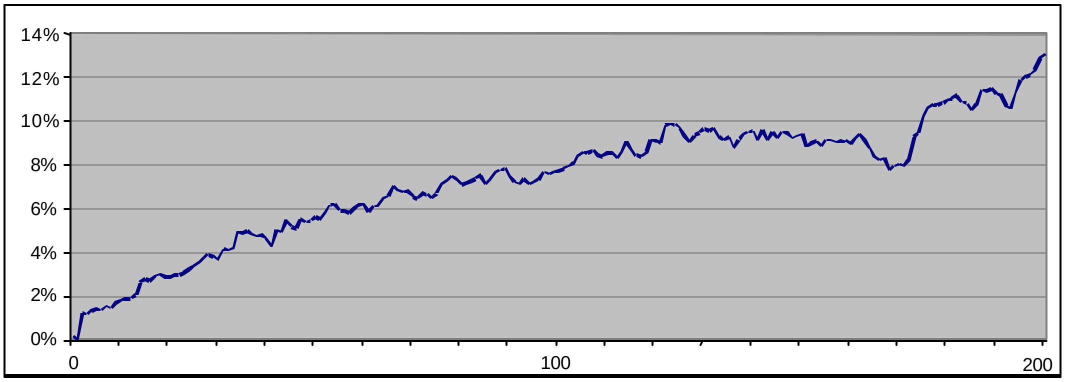


Figure 7.2: Equity curve of cumulative profits and losses for combined portfolio of ISA models of the FTSE 100 constituents, over the 200-day out-of-sample period.

The equity curve is quite smooth, reflecting a low level of risk in the combined set of models. The period of relatively weak performance corresponded to a period of excessive market volatility (May-October 1998) but is also in part due to a certain degree of instability in the performance of the statistical arbitrage models. Within our methodology we propose two main ways of controlling the effects of such performance nonstationarity, the first is to deal with the nonstationarity at the parameter level by means of the adaptive cointegration methodology which was described in Section 5.2 and which is illustrated in Section 7.3; the second method is to deal with the nonstationarity at the model level through the population-based model selection and combination methodology which is described in Part III of the thesis.

From Sections 2.2.5 and 2.2.6 we note that the potential gain through risk diversification is crucially dependent upon the *correlations* between the models which are being combined. Figure 7.3 illustrates the effective diversification within the set of models, in the form of a PCA of the model returns:

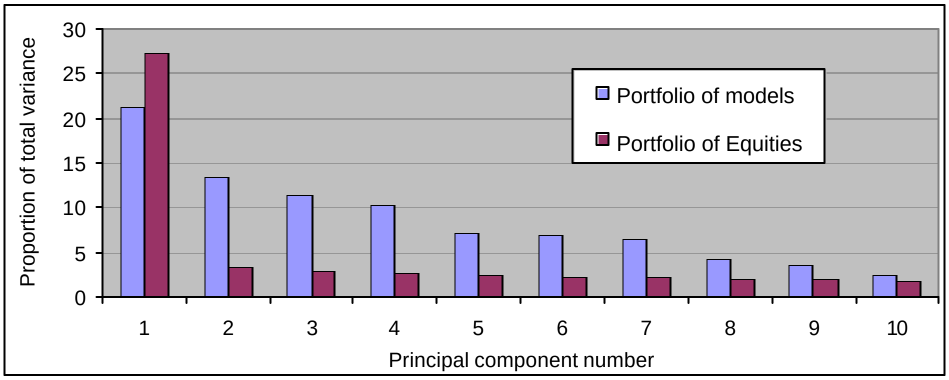


Figure 7.3: Illustration of diversification within a set of 40 implicit statistical arbitrage models. The chart shows the percentage of the total variance which is explained by the principal components of the profits and losses of the 40 models, in comparison to an equivalent analysis of the *raw* equity prices.

The set of statistical arbitrage models is substantially more diversified than the set of underlying equity prices. In the original asset prices, there is a clear break after the first principal component, which accounts for over 25% of the variance in the entire set of 90 models. Within the set of models, the strongest component is much less dominant and there are a further 6 factors which each account for over 5% of the total variance. This result provides support for our approach of inducing a certain degree of diversity in the pool of models, by using each asset in turn as the “target” of the mispricing construction procedure. Nevertheless, the diversification is less complete than might be desired. This is a necessary consequence of using a model selection criterion which considers models in *isolation*, an issue which serves as a major motivation for the population-based methodology which is described in Part III of the thesis.

### Trading Rule Implementation and Transaction Costs

In this section we discuss the impact of transactions costs and other trading parameters. Table 7.6 extends the results in Table 7.5 to take into account the impact of *transaction costs* on the performance of the combined set of models. The first two levels of $c$ (0.1% and 0.25%)

represent plausible costs for large institutions whereas the third level (0.5%) is the lowest figure that could be expected for an individual investor.

| | Overall Return | Profitable Periods | Sharpe Ratio |
|---|---|---|---|
| *c* = 0.1% | | | |
| Part1 | 6.2% | 59% | 3.77 |
| Part2 | 4.4% | 57% | 2.24 |
| Overall | 10.6% | 58% | 2.93 |
| *c* = 0.25% | | | |
| Part1 | 4.0% | 57% | 2.46 |
| Part2 | 3.1% | 55% | 1.59 |
| Overall | 7.1% | 56% | 1.98 |
| *c* = 0.5% | | | |
| Part1 | 0.4% | 47% | 0.24 |
| Part2 | 1.0% | 54% | 0.51 |
| Overall | 1.4% | 51% | 0.39 |

Table 7.6: Out-of-sample analysis of the impact of transaction costs on the performance of the combined set of models selected by the VR20 criterion; these results extend the zero-cost case for which results were presented in Table 7.5.

The results show that the effect of transaction costs on model profitability is very significant, both in terms of the magnitude of the profits and on the annualised Sharpe Ratio. It is clear that the market participants best placed to benefit from statistical arbitrage models of this type are market-makers and other large players who benefit from relatively low transaction costs. Even in these cases the absolute levels of profitability may appear low. However, in principle, these strategies can either be leveraged, in which case profitability is *multiplied* by the leverage factor, and/or overlayed on an underlying market long position, in which case the performance would be *added* to the market return. As such it is the *consistency* of the profits which is of paramount importance, and according to the Sharpe Ratio benchmark the results above remain promising, even after moderate levels of transaction costs.

The final aspect of this evaluation concerns the effect of the *trading parameters* in the ISA rules. In general, the implementation of the trading strategy is of at least comparable importance to the informational advantage upon which the strategy is based [see for instance Towers and Burgess (1998)]. This is an issue to which we return in Part III of the thesis, as it provides a motivation for the *joint* optimisation of mispricing/predictive model and trading strategy. Table 7.7 presents a simple sensitivity analysis of the average performance of the individual models with respect to the parameters of the modified trading rule in Eqn. (7.2a).

| | Rule1 | Rule2 | Rule3 | Rule4 | Rule5 |
|---|---|---|---|---|---|
| Power (*k*) | 1 | 1 | 1 | 0.5 | 2 |
| Period (*h*) | 1 | 1 | 5 | 1 | 1 |
| Cost (*c*) | 0 | 0.10% | 0.10% | 0.10% | 0.10% |
| Total Return | 12.9% | 10.6% | 10.9% | 11.1% | 9.8% |
| Sharpe Ratio | 1.35 | 1.10 | 1.11 | 0.96 | 1.27 |
| Profitable Periods | 51.7% | 49.9% | 50.1% | 50.7% | 47.8% |
| Predictive Correlation | 0.142 | 0.142 | 0.139 | 0.124 | 0.156 |

Table 7.7: Sensitivity analysis of the average performance of the individual models to the parameters of the implicit statistical arbitrage trading rule ISA($M_t$, *k*, *h*)

Firstly, note that the analysis is conducted as an average across the 40 individual models and thus the advantages of diversification are not reflected in the proportion of profitable periods or the risk-adjusted return.

Of the five cases presented in Table 7.7, **Rule 1** was used previously to generate the results in Table 7.5, whilst **Rule 2** corresponds to the first case in Table 7.6 and forms the basis of the other three parameterisations. In **Rule 3** the holding period parameter *h* is increased from 1 to 5. The resulting smoothing of the trading signal causes fewer transaction costs to be incurred, but at the price of a slight reduction in the average predictive correlation between the trading signal and the price changes in the mispricing. In this case the net effect is a marginal increase in profitability. In **Rule 4** the trading signal is set to vary with the square root of the magnitude of the mispricing, which increases profitability but disproportionally increases the risk, as indicated in the reduced average Sharpe Ratio of 0.96. In **Rule 5** the trading signal varies with the square of the mispricing, leading to an increase in the average Sharpe Ratio. The increased predictive correlation between the signals generated by this nonlinear rule and the returns on the mispricing portfolios suggests that the mispricing dynamics may contain a *nonlinear* component.

**Summary**

In this section we have presented an empirical evaluation of the stepwise extension of the mispricing construction methodology in the high-dimensional case of a set of assets consisting of the FTSE index and 89 of its constituents. The mispricing models were evaluated in the context of implicit statistical arbitrage strategies which aim to exploit the presence of a mean-reverting component in the mispricing dynamics.

The predictability tests which were found to be most highly correlated with insample model profitability were the variance ratio statistics and the first component of the variance profile. Out-of-sample profitability was also correlated with these statistics but most of all with the insample risk-adjusted profitability in the form of the Sharpe Ratio metric.

Using either of these measures as the basis of a model selection process was found to significantly increase the average profitability of the models. Combining a set of models was found to result in an almost three-fold improvement in risk-adjusted performance, achieved through the diversification of model risk.

Transaction cost levels were found to be very important, with meaningful levels of profitability obtained at institutional cost levels (0.1%, 0.25% one way) but negligible profitability obtained at cost levels closer to those charged to individual investors (0.5% one way). The profitability is also sensitive to the form of the ISA trading rules, with small changes in the parameters leading to performance increases or decreases of up to around 10% of total profitability.

In the following section we present a further empirical evaluation of the mispricing construction methodology where the adaptive modelling techniques of Section 5.2 are applied to the relationship between the German DAX 30 and French CAC 40 equity market indices.

## 7.3 Empirical Results for Equity Market Indices

In this section we present an empirical evaluation of the *adaptive* version of the mispricing construction methodology which was described in Section 5.2. We apply the methodology to the task of identifying statistical arbitrage opportunities in the joint price dynamics of two European equity market indices. The specific data upon which the evaluation is based consists of the daily closing levels of the French CAC 40 index and the German DAX 30 index between August 1988 and August 1996. A simple relative value analysis of these two series was presented in Section 3.1.

In this case we first motivate the use of the *adaptive* modelling approach by demonstrating evidence of parameter instability in the basic cointegration model between the two indices. The statistical mispricing between the two indices can be constructed by means of a cointegrating regression as described in Section 5.1. The fair price relationship obtained by regressing the level of the CAC index on the level of the DAX between August 1988 and August 1990 is given by:

$$CAC_t = 0.888\, DAX_t + 388 + M_t \tag{7.7}$$

This statistical mispricing of the CAC, relative to the DAX is then given by the deviation from the fair price relationship:

$$M_t = CAC_t - 0.888\, \left(DAX_t + 388\right) \tag{7.8}$$

We can plot the price of the CAC against the "fair price" of 0.888 times the price of the DAX plus 388 points, as shown in Figure 7.4 below:

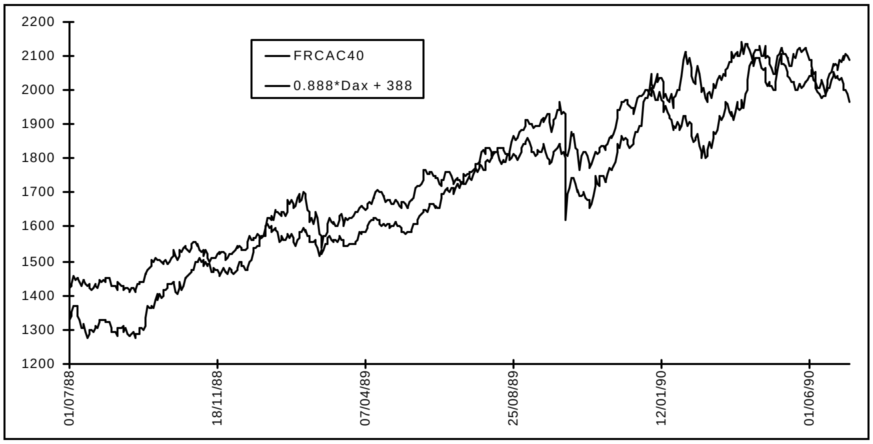


Figure 7.4: The French Cac index and its "fair price" versus the German Dax over the period August 1988 to August 1990

As discussed in Section 6.3, the cointegrating regression will induce a bias in the mispricing dynamics during the insample period. This bias manifests itself in the form of a spurious mean-reversion component that in turn will induce a bias in the apparent profitability of implicit statistical arbitrage strategies which are applied to the mispricing time-series. In order to obtain an unbiased evaluation of the effectiveness of the statistical arbitrage model between the indices, we calculate the performance of a simple ISA trading rule (with $k$=0), when applied to subsequent, **out-of-sample** data. Table 7.8 summarises this trading performance; the table consists of six entries representing yearly periods from August to August together with a final entry for the period as a whole.

| Period | 90-91 | 91-92 | 92-93 | 93-94 | 94-95 | 95-96 | 90-96 |
|---|---|---|---|---|---|---|---|
| Total Return | 11.3% | 22.4% | 17.7% | 7.1% | -4.2% | -7.2% | 47.2% |
| Sharpe Ratio | 0.72 | 1.78 | 1.18 | 0.53 | 0.25 | -0.56 | 1.30 |

Table 7.8: Out-of-sample performance of a basic ISA($k$=0) model applied to the mispricing between the CAC and DAX indices. Total Return is calculated according to Eqn. (7.4), and the Sharpe Ratio using the annualised form of Eqn (7.5)

In this simple analysis the model continues to perform well for the first few years but then exhibits a substantial degradation in performance. A possible cause of this degradation is that the model is based on the assumption that the parameters of the fair price relationship are stable. There is, however, every reason to believe that the market's opinion as to what constitutes a "fair price" of one asset to another will change over time. This risk is even greater in the case where the assets are stock market indices from different countries, where there may be many "hidden variables" which influence the perceived fair price relationship but which are not explicitly included in the model. Thus even statistical arbitrage models which have worked well in the past are liable to "break down". The fact that this has indeed happened in this case can be seen from the plot of the extended price series which is presented in Figure 7.5.

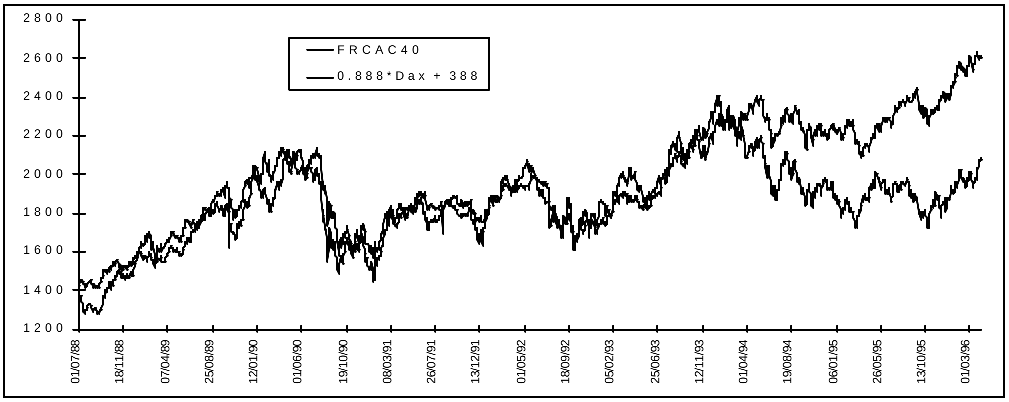


Figure 7.5: Extended price series for the CAC and (rescaled) DAX indices over both the insample and out-of-sample periods; the figure demonstrates the presence of a fair price relationship which subsequently breaks down as the two markets drift apart.

From early 1994 the two series begin to diverge and, contrary to the cointegration hypothesis, do not move back into line with each other. Whatever relationship was present truly has "broken down". The effect of this nonstationarity on the statistical mispricing and the cumulative profits of the ISA trading rule is illustrated in Figure 7.6.

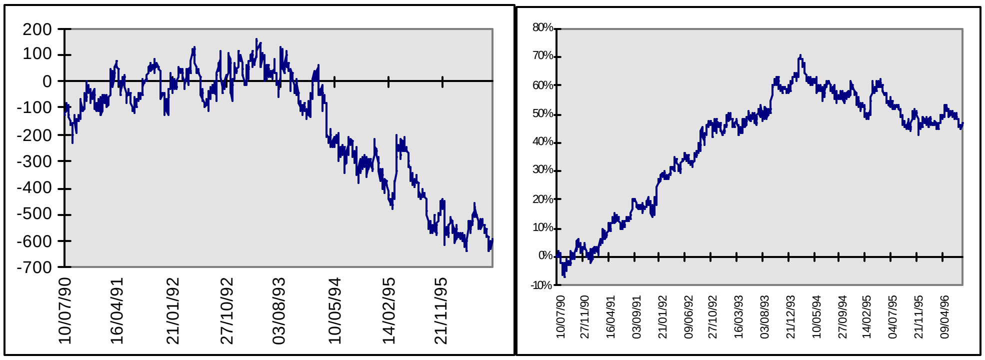


Figure 7.6: The statistical mispricing (left) between the CAC and DAX indices remains stable for a number of years but then begins to "drift"; the degradation in model performance is seen in the equity curve (right) for an ISA trading rule which implicitly relies upon mean reversion in the mispricing.

The breakdown of the original relationship between the two series appears to occur somewhere in early 1994 and is manifested both in the gradual downward drift of the mispricing and the losses incurred by the associated ISA trading rule. However, an examination of the mispricing series in Figure 7.6 also suggests that the relationship has not disappeared *completely* but instead persists in a *modified* form. There appears to be a regularity in the mispricing dynamics in the form of a mean-reversion around the downward drift, a type of "channel" effect. In fact the persistence of a mean-reverting component can easily be confirmed by analysing the variance ratio of the mispricing time-series (which in this case is very similar to that used as an example in Figure 3.1). As discussed in Section 5.2, the motivation for using an adaptive modelling technique in such a case is that it enables at least part of the nonstationary component of the dynamics to be captured within the model parameters, thus maximising the predictive information contained in the filtered mispricing.

The "model nonstationarity" phenomenon has been noted for other financial time series. For instance, in modelling the FTSE against a basket of international equity indices (Burgess and Refenes (1996)) we adjusted for a changing relationship by periodically recalculating the coefficients of the cointegrating regression using a "rolling window" approach. Bruce, Connor and Martin (1996) use a more sophisticated approach which is robust to both level and trend shifts and find that it significantly improves the stability of a cointegrating relationship between the FTSE and the U.S. S&P index. Bentz (1999) provides an extensive study of similar effects in *explicit* factor models.

Both the “rolling window” and the “trend/level shift” approaches create *discontinuities* in the beta parameters of the fair price relationship. These discontinuities might affect the accuracy of any subsequent modelling procedure, by changing the semantics of the underlying synthetic asset price. They may also require additional transactions to be performed in order to rebalance the portfolio to reflect the new weightings. To avoid these complications, we use the adaptive modelling methodology of Section 5.2 to allow the cointegration to evolve smoothly according to a “random walk parameter” model of the form shown below:

$$T_t = \sum_{C_i \in C} b_{i,t} C_{i,t} + M_t$$

where (7.9)

$$b_{i,t} = b_{i,t-1} + h_{b_i} \qquad\qquad h_{b_i} \sim \mathrm{N}(0,\ s^2_{b_i})$$

This is a state-space model in which the states are the cointegration coefficients $b_{i,t}$ and which can easily be implemented within the framework of a standard Kalman filter. The scale parameters $s^2_{b_i}$ determine the rate at which the coefficients may evolve through time; in the special case of $\forall_i : s^2_{b_i} = 0$ the model collapses to a (fixed coefficient) regression model. Further discussion of the random walk parameter model and related models can be found in (Harvey, 1993).

In this case the CAC is the “target asset” , $T_t$ , and the two constituent assets are the DAX and the constant term. The adaptivity of the model is specified through a single scale parameter $q = \dfrac{s^2_a}{s^2_M / s^2_{Cac}} = \dfrac{s^2_b / s^2_{Dax}}{s^2_M / s^2_{Cac}}$ which controls the rate at which the parameters are allowed to vary, simplifying the more general case presented in Eqn. (5.19).

There is a slight complication in calculating the returns of the ISA trading strategies in the case of a mispricing model which is itself adaptive. Changes in the mispricing will partially be due to changes in the **weightings** of the mispricing portfolio rather than being purely attributable to changes in the prices of the assets. To compensate for this effect it is necessary to apply a similar correction to that used for calculating the Dynamic Dickey-Fuller (DDF) statistics in Section 5.2. As changes in the portfolio are implemented only at the end of the trading period, the actual return is determined by taking into account changes in the values of the assets but not changes in the estimated fair price coefficients; this corrected measure is given by:

$$\Delta^* M_t = M^*{}_{t+1} - M_t = \left[Cac_{t+1} - (\hat{a}_t + \hat{b}_t Dax_{t+1})\right] - \left[Cac_t - (\hat{a}_t + \hat{b}_t Dax_t)\right] \tag{7.10}$$

The (corrected) performance of the ISA strategy applied to the adaptive mispricing model is presented in Table 7.9; in order to illustrate the effect of varying the degree of adaptivity of the model, the performance is reported over a range of different values of the scale parameter *q*.

| Period | 90-91 | 91-92 | 92-93 | 93-94 | 94-95 | 95-96 | 90-96 |
|---|---|---|---|---|---|---|---|
| *q=0* | *11.3%* | *22.4%* | *17.7%* | *7.1%* | *-4.2%* | *-7.2%* | *47.2%* |
| q=2.5e-6 | 12.9% | 11.3% | 20.9% | 11.1% | 0.3% | 3.2% | 59.8% |
| q=1.6e-4 | 15.4% | 7.9% | 19.4% | 5.4% | 13.6% | 9.9% | 71.5% |
| q=0.01024 | 23.3% | -5.5% | -16.9% | 9.7% | 14.6% | 13.7% | 38.9% |

Table 7.9: Performance of implicit statistical arbitrage model for Dax-Cac indices as a function of the parameter sensitivity *q* (Note: *q* = 0 is equivalent to fixed model)

Even with a **low** level of sensitivity (*q*=2.5e-6) the model adjusts sufficiently to be marginally profitable even when the original relationship breaks down in early 1994. With a **moderate** level of sensitivity (*q*=1.6e-4) the system performance is consistent across all periods, with the least profitable period being the one in which the major adjustment is made. When the sensitivity is **high** (*q*=0.01024) the system becomes unstable and performs less well than even the fixed parameter model.

The performance of the system can be confirmed statistically by referring to the DDF statistics for different levels of sensitivity *q* which are presented in Table 7.10.

| Sensitivity (q) | DDF-statistic |
|---|---|
| 0 | -1.022 |
| 2.5e-06 | -2.454 |
| 1.6e-04 | -3.170 |
| 0.01024 | -2.695 |

Table 7.10: Significance tests for reversion in the residuals of the adaptive mispricing models between the Dax and Cac indices. The table reports the DDF statistic as a function of the degree of adaptivity (*q*)

Note the similar pattern of results to the *simulated* example in Section 5.2, albeit with a smaller underlying degree of mean reversion. In this case, the $R^2$ figures of the DDF regressions are between 0.5% and 1%, corresponding to a predictive correlation in the range of 0.07 to 0.10. With this low degree of predictability, it is clear that many other factors influence the German and French stock markets in addition to the mispricing correction effect which exists between the two.

The specific effect of the adaptive modelling can be seen in the model parameters, the cointegration residual, and the performance of the "implicit" trading models. Noting that $q$=0 is the fixed parameter case which has already been discussed, the results for the low sensitivity case of $q$=2.5e-6 are shown in Figure 7.7a and Figure 7.7b:

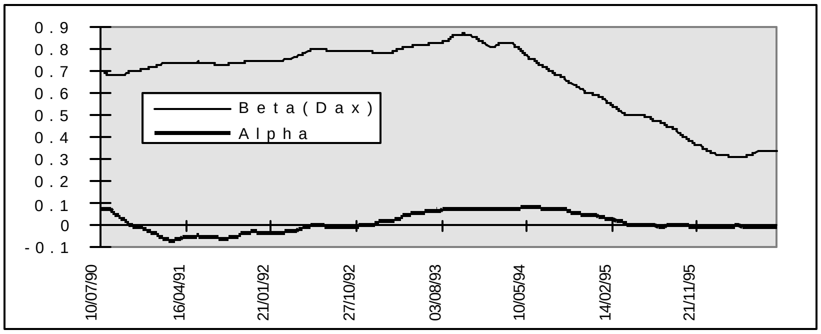


Figure 7.7a: Scaled parameters for the adaptive Dax-Cac relationship with low sensitivity (q=2.5e-6)

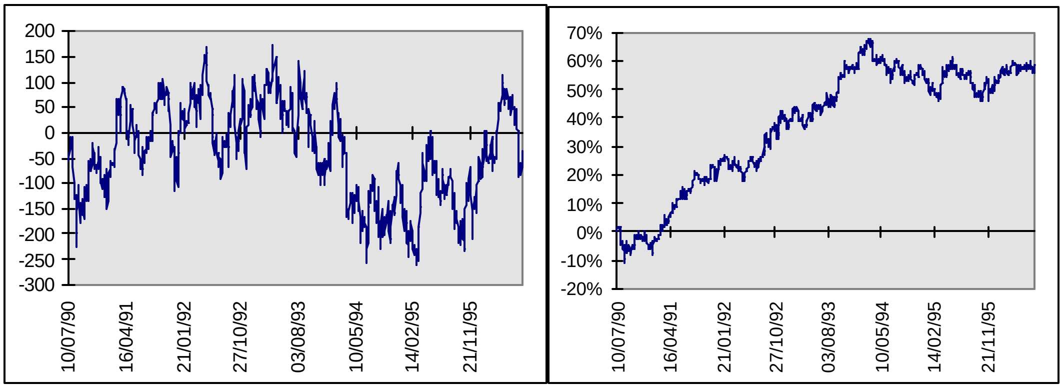


Figure 7.7b: Filtered mispricing (left) and associated equity curve (right) for Dax-Cac model with low sensitivity (q=2.5e-6)

Figure 7.7a shows that the drifting apart of the two indices is reflected by a gradual change in the beta parameter which represents the equilibrium ratio between the two indices. Figure 7.7b shows that this adjustment is reflected in the statistical mispricing, which also initially drifts downwards but eventually compensates for the change in the system and reverts to a fluctuation around zero. In comparison to the fixed parameter model, the right hand chart shows that adaptive model suffers greatly reduced losses during the period in which the nonstationarity occurs and, in terms of the year-on-year results presented in Table 7.9, is at least marginally profitable in each period.

With an increased sensitivity, *q*=1.6e-4, more of the nonstationary component is absorbed by the parameters of the cointegration (synthetic asset) model. Once the changing parameters are taken into account, the mispricing no longer drifts away from zero and the system is consistently profitable, even during the periods in which the nonstationarity is strongest. The associated curves are presented in Figure 7.8.

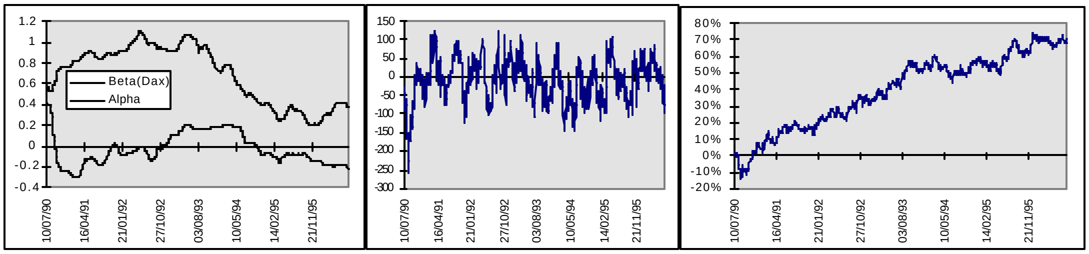


Figure 7.8: Cointegration parameters (left), filtered mispricing (centre), and ISA equity curve (right) for Dax-Cac model with moderate sensitivity (q=1.6e-4)

This level of sensitivity represents the optimal bias-variance tradeoff for this particular case. The parameters adapt rapidly enough to capture the drift in the underlying relationship, without being so sensitive as to become overly contaminated by the underlying noise in the time-series. The effect of this is that the ISA model remains profitable even during the period where the parameters of the underlying fair price relationship are changing most rapidly. Both the profitability (Table 7.9) and statistical significance of the mean-reversion effect (Table 7.10) are maximum at this moderate level of adaptability.

At higher levels of sensitivity the system becomes unstable due to the effect of noise. Too much of the relative volatility of the two series is absorbed by the parameters of the fair price model. The residual “mispricing” thus becomes less meaningful - statistically the DDF statistic becomes less significant - and the performance falls away. The corresponding curves are shown in Figure 7.9.

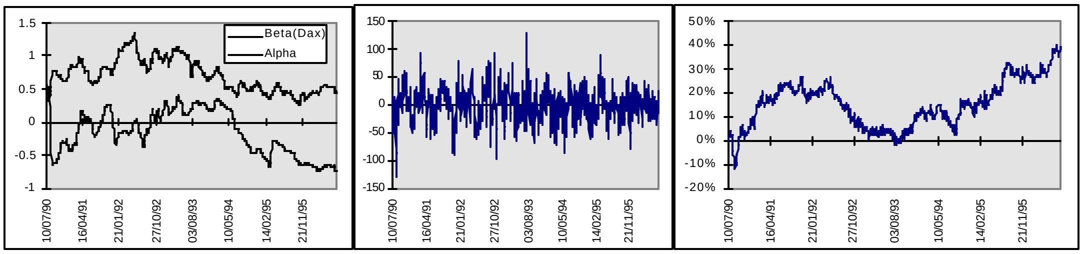


Figure 7.9: Cointegration parameters (left), filtered mispricing (centre), and ISA equity curve (right) for Dax-Cac model with moderate sensitivity (q=0.01024)

### Summary

In this section we have conducted an empirical evaluation of the adaptive variant of the mispricing construction methodology. The consistent profitability of an associated implicit statistical arbitrage (ISA) trading rule demonstrates the ability of the methodology to compensate for a smooth evolution (weak nonstationarity) in the parameters of the underlying fair price relationship.

The results indicate that a statistically significant but time-varying relationship exists between the French CAC 40 and the German DAX 30 equity market indices, and that this relationship can be exploited by a similar implicit statistical arbitrage approach as was applied successfully to the mispricing models of FTSE 100 constituents in the previous section. Furthermore the general pattern of the results is similar in nature to that which was exhibited by the controlled simulation experiment presented in Section 5.2, with the results improving along with increased adaptivity up to a certain point, but then degrading beyond that point.

The main difference between the actual model and the earlier simulation is that the relationship between the equity indices is relatively weak - only accounting for between 0.5% and 1% of the variability in the relative levels of the two markets. In spite of the fact that the mean-reversion is relatively slight, with a moderate degree of adaptivity the model is able to generate consistent profits over a number of years and hence appears to offer real opportunities for statistical arbitrage.

## 7.4 Summary

In this chapter we have described a set of "implicit statistical arbitrage" (ISA) trading strategies which are designed to exploit any mean-reverting component in the dynamics of the mispricing time-series which are constructed using the cointegration framework of Chapter 5. Two sets of results are presented in order to support the economic significance of the opportunities provided by the ISA strategies .

The first set of results concern mispricing models of the daily closing prices of FTSE 100 constituents and were created using the **stepwise** extension to the basic mispricing construction methodology. Whilst the results were found to be sensitive to model selection criteria, transaction costs and trading rules parameters, a set of 40 combined models, selected on the basis of their variance ratio statistics, produced an out-of-sample, market-neutral, unleveraged return of 12.9% (before costs) over a 200 day out-of-sample period.

The second set of results are based on a time-varying fair price relationship between the French CAC and German DAX stock market indices, which was estimated using the **adaptive** variant of our mispricing construction methodology. A model with a moderate degree of adaptivity was found to be able to capture the time-variation of the relationship in the parameters of the fair price model and produce consistently positive returns over a six year out-of-sample period.

The implicit statistical arbitrage models are designed to exploit simple mean-reverting components in mispricing dynamics. In cases where the true mispricing dynamics are more complex, the ISA strategies will be at best inefficient at exploiting the deterministic component of the dynamics. Part II of the thesis describes the second part of our methodology, which aims to support the construction of explicit forecasting models which are capable of exploiting a wide range of potentially predictable behaviour in the dynamics of statistical mispricing time-series.

# Part II: Forecasting the Mispricing Dynamics using Neural Networks

In this part of the thesis we describe the second of the three parts of our methodology for statistical arbitrage modelling. This consists of algorithms, tools and procedures for supporting the construction of predictive models of the dynamics of statistical mispricing time-series. The methodology is designed to address the particularly hard problems that arise in the context of building predictive models in investment finance. An overview of the methodology described in this part of the thesis is contained in Section 4.3.

Chapter 8 provides the motivation and general framework for our predictive modelling methodology. It firstly presents a general formulation of the model estimation process before moving on to discuss the particularly hard nature of the problems which arise in the case of building predictive models of asset price dynamics. These problems include high noise, low degree of prior knowledge, small sample sizes and potential time-variation (nonstationarity) in the underlying data-generating processes. An "equivalent kernels" perspective is used to highlight the similarities between neural modelling and recent developments in non-parametric statistics. This in turn motivates the use of neural estimation methods in conjunction with statistical testing procedures as a means of achieving both flexibility and parsimony in an attempt to overcome the "bias-variance dilemma".

Chapter 9 describes our methodology for model-free variable selection, which is based upon methods from non-parametric statistics and is intended to distinguish which variables from the information set should be included in the modelling procedure proper. The purpose of this "pre-selection" stage is to reduce the complexity, and hence **variance**, of the modelling process as a whole. The flexibility of the tests provides an important role in retaining the largest possible amount of *relevant* information upon which to condition the forecasting models. In particular, the tests are capable of identifying both nonlinear dependencies and interaction effects and thus avoid discarding variables which would be passed over by standard linear methods.

Chapter 10 describes our methodology for the actual estimation of low-bias forecasting models. This task is performed through novel algorithms which balance the flexibility of *neural networks* with the noise-tolerance and diagnostic procedures of *statistical regression*. Statistical testing and selection procedures are employed within a rigorous modelling framework which automatically optimises the specification of the neural network. Our integrated approach to the model estimation problem combines the two aspects of variable selection and architecture selection, within a common framework based upon the statistical significance tests which are developed in Chapter 9. Within this common framework we describe three alternative algorithms which aim to optimise both variable selection and model complexity using the constructive, deconstructive and regularisation-based approaches to model building.

Chapter 11 describes an empirical evaluation of the methodology from Part I and Part II of the thesis used in combination. The model-free variable selection procedures and neural model estimation algorithms are applied to the problem of forecasting the dynamics of the statistical mispricings generated by the first part of our methodology. The objective of this exercise is to generate *conditional statistical arbitrage* (CSA) models in which nonlinearities, interaction effects and time-series effects can all be captured and exploited without being explicitly prespecified by the modeller. An empirical evaluation is presented of a set of CSA models which are based upon statistical mispricings between the constituent stocks of the FTSE 100 index.

# *8. Low-bias forecasting in highly stochastic environments*

This chapter describes our methodology for building predictive models of the mispricing dynamics. Section 8.1 describes a general formulation of the model estimation process in terms of the different elements which are involved, namely *variable selection*, *model specification*, *parameter estimation* and *model selection*. Section 8.2 then highlights the particularly hard nature of the problem posed by building predictive models of asset price dynamics, which is due to a combination of the highly stochastic nature of financial time-series, the lack of a theoretical basis upon which to formulate modelling assumptions, the high model variance caused by small sample sizes and the performance instabilities caused by time-variation in the underlying data-generating processes. Section 8.3 describes an "equivalent kernels" perspective which is used to highlight the similarities between neural modelling and recent developments in non-parametric statistics. This perspective emphasises the "data driven" nature of neural estimation and provides the basic statistical tools which underpin the model-free variable selection procedure and neural estimation procedures described in the subsequent chapters.

## 8.1 General Formulation of the Model Estimation Problem

The second stage of our methodology is concerned with modelling the predictable component of mispricing dynamics. Where the *implicit* statistical arbitrage models of Chapter 7 can exploit only the purely mean-reverting component of mispricing dynamics, the objective of the methodology described in this section is to capture and exploit as much as possible of the deterministic component in the mispricing dynamics. This task is essentially one of identifying models of the form:

$$\mathrm{E}\left[\Delta M_t\right] = f\left(M_t, \Delta M_{t-i}, Z_t\right) \tag{8.1}$$

These *explicit* models of the mispricing dynamics produce estimates of the expected (mean) innovation in the mispricing, conditioned upon the information contained in the level of the mispricing ($M_t$), past innovations in the mispricing time-series ($\Delta M_{t-i}$), and other variables ($Z_t$) which are either predictive of the mispricing innovations directly, or serve to modulate the information contained in other variables. Forecasting models of the form shown in Eqn.

(8.1) comprise the basis of the *conditional statistical arbitrage* (CSA) strategies which are described and evaluated in Chapter 11.

The task of building such forecasting models can be considered as an instance of the general model estimation problem, which consists of constructing estimators of the form:

$$\mathrm{E}\left[ y_t \middle| I_t \right] = \hat{y}_t = m\left\{ S\left( X \subset I, f \right); q\left(D, L\right); \mathbf{x}_t \right\} \tag{8.2}$$

where the forecast $\hat{y}_t$ is the value produced by the model $m$ with specification $S$, parameters $q$, and input vector $\mathbf{x}_t$.

The model estimation problem can be decomposed into the following elements:

1) selection of the information set $I$

2) model specification

   a) variable selection ( $X \subset I$ )

   b) specification of parameterised functional form $f\left(q\right)$

3) parameter estimation

   a) selection of data sample ( $D$ )

   b) application of learning algorithm, to estimate parameters $q = L(D)$

4) diagnostic testing (reformulate model and return to (2))

5) model selection (repeat process from (2) onwards and then select the best model)

It is important to recognise that **every single one** of these stages can make a crucial impact on the success or failure of the modelling process as a whole. In a general sense, the *assumptions* and *restrictions* made at each stage will jointly comprise the **bias** of the model and the effect to which they are jointly affected by the particular *data sample* $D$ will comprise the **variance** of the model.

Both the assumptions which comprise model bias and the sampling effects which comprise model variance may be either *explicit* (as in the case of the specification of a particular functional form), or *implicit* (such as choosing the set of variables after first examining the data itself). Section 2.2.4 contains a discussion of the *explicit* manner in which the different components of the modelling process introduce sources of forecasting error over and above the inherent stochastic component of the system. Furthermore, the fact that these decisions are not generally considered jointly, but rather sequentially, will introduce *additional* sources of bias and variance into the modelling process as a whole (the recognition of this fact plays an important motivation for the methodology described in Part III of the thesis, and Chapter 13 in particular).

Whilst both bias and variance are unavoidable consequences of imperfect knowledge combined with empirical inference, and hence are issues common to a wide range of computational modelling problems, their importance is vastly magnified in the case of building predictive models of asset price dynamics. The properties of asset price dynamics which cause the modelling process to so often fail in the case of financial forecasting are discussed in the following section.

## 8.2 Estimating Predictive Models in Investment Finance

This section highlights the particularly difficult nature of the problems which are posed by model bias and model variance in the case of financial forecasting. A simple illustration serves to demonstrate the severity of the test which the task of estimating predictive models in investment finance poses to a modelling methodology.

Consider an idealised view of the performance of a forecasting model, in which the expected level of model performance, $P_{EXP}$, is given by:

$$P_{EXP} = \mathrm{E}\left[R^2\right] = d(1-b) - v \tag{8.3}$$

where we define $\mathrm{E}\left[R^2\right]$ as the expected performance, measured in terms of the proportion of the variance in the dependent variable which is correctly forecasted by the model, $d \geq 0$ as the deterministic proportion of the dynamics of the dependent variable (and hence the performance of a perfect model), $b \geq 0$ as the idealised model bias (which represents the general effect of incorrect modelling assumptions leading to a possibly imperfect ability to capture the underlying deterministic component) and $v \geq 0$ as the idealised model variance (which represents any degradation in performance which is caused by estimating model parameters using finite and noisy samples). For real problems, the model *variance* can be quantified in terms of the “degrees of freedom” in a model (Hastie and Tibshirani, 1990) which can be interpreted as the number of “free” parameters which are estimated from the data (see

Sections 8.3 and 9.1). The *bias* of a real model is dependent on the specific modelling assumptions (e.g. a particular parametrisation of the model) and the true nature of the underlying relationships - which in real-world problems is generally unknown. An empirical example of quantifying model bias in controlled experiments (where the underlying relationships are pre-specified) is presented in Section 9.4.

Returning to the idealised case, let us consider three broadly representative problems in the form of statistical regression analysis applied to a *business modelling problem*, a neural network applied to a *pattern recognition problem*, and an unknown methodology applied to a *financial forecasting problem.* As a perspective which synthesises the approaches of Akaike (1973), Haerdle (1990), Hastie and Tibshirani (1990), Moody (1992) and Amari and Murata (1993) for a wide range of model classes, we define the idealised model variance as being equal to the product of the noise content $(1-d)$ of the dependent variable and the ratio of an idealised measure of model complexity (following Moody (1992) we refer to this as the "effective number of parameters" ( $p_{eff}$ )) to the sample size $n$ :

$$v = (1-d)\,\frac{p_{eff}}{n} \tag{8.4}$$

In general, the effect of using flexible modelling techniques such as neural networks is to reduce the level of model bias $b$ but at the price of an increase in model complexity and hence the "effective number of parameters" $p_{eff}$. In order to illustrate the sensitivity of the expected performance to the tradeoff between these two quantities, Table 8.1 below, presents a number of hypothetical scenarios in which assumed values of the problem and the model characteristics are used to calculate the expected performance by substitution in Eqns. (8.3) and (8.4):

| Problem | Deterministic Component ($d$) | Sample Size ($n$) | Model Type | Complexity ($p_{eff}$) | Model Bias (b) | Expected Performance $d(1-b)-\frac{(1-d)p_{eff}}{n}$ |
|---|---:|---:|---|---:|---:|---:|
| Business Modelling | 70% | 100 | linear model | 5 | 20% | 54.5% |
|  | 70% | 100 | neural network | 20 | 5% | 60.5% |
|  | 70% | 100 | parametric | 7 | 10% | 60.9% |
| Pattern Recognition | 95% | 1 million | linear model | 10 | 20% | 76.0% |
|  | 95% | 1 million | neural network | 100000 | 5% | 89.8% |
|  | 95% | 1 million | parametric | 20 | 15% | 80.7% |
| Financial Forecasting | 5% | 400 | linear model | 5 | 80% | -0.2% |
|  | 5% | 400 | NN/naïve | 20 | 5% | 0.0% |
|  | 5% | 400 | NN/possible? | 16 | 5% | 1.0% |
|  | 5% | 400 | parametric/naïve | 10 | 53% | 0.0% |
|  | 5% | 400 | parametric/possible? | 10 | 33% | 1.0% |

Table 8.1: Expected performance as a function of hypothetical model and problem characteristics for a number of scenarios which are intended to be representative of applying different modelling approaches to problems in three different problem domains.

The examples in the table offer an interesting perspective on the historical development of computational modelling techniques.

For the moderate noise levels and small samples typical of business modelling problems, the parsimonious nature (low $p_{eff}$) of traditional regression modelling is seen to be very useful in controlling the performance degradation which is due to model *variance*. In this context, the appeal of flexible modelling techniques (such as neural networks) can be seen to be relatively limited.

For instance, the table illustrates a case in which a nonlinear relationship accounts for 20% of the deterministic component in the data (which is taken to be $d$=70% of total variance). In this case, the standard regression assumptions of linearity might lead a linear model to fail to capture the nonlinear component, representing a model bias of $b$=20%. In this first scenario the combined effect of bias and variance is to reduce the expected performance level to 54.5% (compared to the 70% of the variance which is due to a deterministic and hence theoretically predictable component.) In this context, the second scenario illustrates that the use of a neural network may allow the majority of the nonlinear component to be absorbed within the model, reducing the bias to say 5% at the price of employing a total of perhaps 20 effective parameters, and giving an improved expected performance of the model of 60.5%. On the other hand, the third scenario indicates that a similar performance improvement may perhaps be achieved by means of the more traditional approach in which judicious examination of suitable diagnostic statistics might lead an originally linear parametric model to be *reformulated*, leading to model bias being (in this scenario) halved by the addition of only two new parameters. These scenarios indicate that for problems with these types of characteristics, the advantages of neural networks over more established techniques will tend to be marginal, and may be as much due to a reduction in the labour intensity of the modelling process as to an actual improvement in the ultimate level of performance.

The second set of three scenarios in Table 8.1 correspond to an idealisation of problems in the field of pattern recognition. The representative characteristics indicate that the relationships may be less stochastic (higher degree of determinism *d*), more complex (tendency for naïve models to be more highly biased), and more data-rich (large sample sizes *n*) than the business modelling problem discussed above. The corresponding analyses in Table 8.1 appear to suggest that problems of this type provide strong motivation for the development and application of **low-bias** modelling techniques such as neural networks. The first scenario in this context illustrates that a 20% bias (in a linear model) degrades what may in principle be a 95% performance level (based on deterministic content) down to an expected level of only 76%. In this context, the following scenario indicates that the additional flexibility of a neural network model (reduction in bias from *b*=20% for the linear case to *b*=5%) compensates many times over for additional model complexity ($p_{eff}$=100000), giving an improved performance of 89.8%. The relative importance of bias-reduction over variance-control in this context is illustrated in the third scenario, which indicates that the diagnostic testing/reparametrisation approach may also lead to performance improvements over an initial model, but typically not of quite the same scale as those achieved by the truly low-bias neural network approach.

However, whilst both regression modelling and neural networks represent natural and useful solutions to the bias-variance tradeoff given the types of problems that they were designed to solve, the final set of scenarios indicate that the situation is very different for problems involving financial forecasting – in today's markets at least. Under the assumption that easily detectable linear components of asset price dynamics have already been largely "arbitraged away" it is plausible that naively parametrised linear models can be considered highly-biased with respect to any deterministic component in asset price dynamics. Meanwhile the increased number of parameters in even relatively parsimonious neural networks will tend to negate the effect of any bias reduction through an associated increase in variance. The performance improvement between the two scenarios, neural network ($p_{eff}$=20) and neural network

($p_{eff}$=16), suggests that the specification of the neural model may play a key role in determining whether a positive level of performance can be achieved. The final two scenarios likewise suggest that for a problem with characteristics typical of the financial forecasting domain, a parametric model will only be expected to achieve positive performance if the underlying modelling assumptions are not greatly dissimilar to the actual nature of the underlying relationships.

Furthermore the discussion above neglects the effect of model selection bias or "data snooping" (see Section 12.1.2). This is the bias which is introduced by repeatedly "tweaking" or reformulating a model, selecting the formulation which leads to the greatest (apparent) level of performance, and taking this as a measure of expected future performance. The magnitude of the upward bias depends on the number of models investigated, the model variance, and the correlation between the models (choosing between 20 identical models will induce no selection bias). To illustrate this effect, we performed a simple Monte Carlo analysis based on the expected value of the **maximum** of a set of *n* trials, and discovered that choosing between 10 uncorrelated values will induce a bias of approximately 1.6 times the individual variance, 100 models 2.5 times variance and 1000 models 3.1 times variance.

In our idealised business modelling example, model selection bias can be considered relatively insignificant because the "performance signal to noise" ratio (of expected performance to model variance) is approximately 45. In the pattern recognition case the equivalent ratio is almost 200 and the deleterious effects of selection bias are again likely to be more than offset by the potential for bias reduction which is achieved by considering alternative parameterisations or formulations of the estimator. In contrast, the situation in financial forecasting is that model variance is approximately comparable in magnitude to the deterministic component in the data and hence "data snooping" represents a very real risk - typically leading to the creation of models with an apparent, but in fact totally spurious, predictive ability.

Thus neither traditional statistical methodology nor the emerging flexible modelling techniques of machine learning are ideally suited to the particular nature of problems in investment finance. The inherent weaknesses of the two "families" of modelling tools, in the context of financial forecasting, forms the primary motivation for all three parts of our modelling methodology, but this second, predictive modelling, part in particular.

The pre-processing methods in **Part I** are intended to look at prices from the novel perspective of statistical mispricing, aiming to maximise the predictable component $d$ and to enable the possibility of forecasting the dynamics of the mispricing time-series. The model combination methods in **Part III** are intended to control the risks which are posed by selection bias and model instability and thus enable exploitation of the potential informational advantage provided by the predictive models. However, the modelling procedures described in the subsequent chapters of **this** part of the thesis form the core of our methodology as they are used to actually construct the forecasting models themselves.

From a methodological perspective, the objective of the modelling algorithms is to capture the maximum possible amount of the predictable component in the mispricing dynamics. This involves minimising model bias by building only *appropriate* assumptions into the models, whilst simultaneously controlling the effects of model variance, by keeping the effective number of parameters in the models sufficiently low. The basis of our attempted solution to this problem is a synthesis of statistical and neural modelling techniques, which aims to exploit the flexibility of neural models whilst ensuring that this flexibility is employed in a controlled and appropriate manner. The medium through which this synthesis has been achieved is a particular perspective, from which neural networks and traditional statistical models are viewed as being based on a similar, fundamentally data-centred, approach to modelling. The "equivalent kernels" perspective views both families of models as ways of predicting the outcome of future situations through the construction of a weighted combination of "similar" observations from the past. This perspective is derived from the field of nonparametric statistics [e.g. Hastie and Tibshirani (1990); Haerdle (1990), Wahba (1990)] and its application to neural regression modelling is described in the section below.

## 8.3 An Equivalent Kernels perspective on Neural Network learning

In this section we describe a perspective which places neural networks in the context of other forms of *regression models*. A key feature of this perspective is that it allows us to identify, and start to quantify, both the similarities and the distinctions between neural networks and other types of regression estimator - thus highlighting the relative strengths and weaknesses of the various techniques and identifying the conditions under which they can used to best effect.

In particular, the equivalent kernels perspective allows us to identify the effective number of parameters or “degrees of freedom” which are contained in a neural network model; this information is used as the basis of the model-free variable selection methodology of Chapter 9, the neural estimation algorithms of Chapter 10 and the statistical arbitrage forecasting models which are presented in Chapter 11.

### 8.3.1 Smoother matrix representation of nonparametric regression models

Many non-parametric regression techniques, such as kernel smoothing, local regression and nearest neighbour regression (Haerdle, 1990), can all be expressed in the form:

$$\hat{\mathrm{y}}(\mathrm{z}) \;=\; \int_{\mathrm{x}=-\infty}^{\infty} j\,(z,x)y(x)p(x)\;dx \tag{8.5}$$

where $\hat{\mathrm{y}}(\mathrm{z})$ is the response at the query point z, $j\,(z,x)$ is the weighting function, or **kernel**, which is “centred” at z, $y(x)$ is the observed value and $p(x)$ the input density at point $x$. In finite samples, this integration can be approximated by the summation:

$$\hat{\mathrm{y}}(x_i) \;=\; \sum_{j=1}^{n} \varphi(x_i, x_j)\, y_j \tag{8.6}$$

Thus the response at each point $x_i$ is a weighted average of the observed target values across the entire dataset. The complete set of weighting kernels comprise the *smoother matrix* **S**:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} \varphi(x_1,x_1) & \varphi(x_1,x_2) & \cdots & \varphi(x_1,x_n) \\ \varphi(x_2,x_1) & \varphi(x_2,x_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ \varphi(x_n,x_1) & \cdots & \cdots & \varphi(x_n,x_n) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{8.7}$$

Which can be expressed in matrix notation as:

$$\hat{\mathbf{y}} = \mathbf{S}\mathbf{y} \tag{8.8}$$

Smoother matrices provide a natural representation for non-parametric statistical models. The simplest example of a smoother matrix is that of a “look up table”:

$$\mathbf{S}_{LUT} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{8.9}$$

In this case the smoother matrix is the identity matrix and the "fitted value" or response at point $x_i$ is simply the corresponding observed value $y_i$. Another model with a simple smoother-matrix representation is the sample mean:

$$\mathbf{S}_{MEAN} = \begin{bmatrix} 1/\mathrm{n} & 1/\mathrm{n} & \cdots & 1/\mathrm{n} \\ 1/\mathrm{n} & 1/\mathrm{n} & & \vdots \\ \vdots & & \ddots & \vdots \\ 1/\mathrm{n} & \cdots & \cdots & 1/\mathrm{n} \end{bmatrix} \tag{8.10}$$

The smoother matrix in this case reflects the fact that the fitted value is an equally weighted combination of all of the observed values $y_i$, irrespective of the corresponding input values $x_i$.

The smoother matrix **S** can be used as the basis of many useful statistical diagnostics. For instance, because the model is expressed in terms of the *influence* of each observation on the response at each sample point, it is possible to identify those observations which dominate the model prediction at a particular point. An in-depth study of this, and related, issues regarding extreme observations in both input-space (influence points) and output space (outliers) is presented in (Bolland, 1998). It is also possible to calculate the model bias and variance at each sample point [see (Haerdle, 1990) for details of this process in the context of nonparametric regression].

The key measure in which we are interested is the "**degrees of freedom**" which are absorbed by the model. The degrees of freedom measure summarises the extent to which the reduction in *insample* mean-squared error (MSE) is merely due to the fact that the observed values are used both to estimate the model fit and to calculate the MSE, as opposed to the capture of true structure within the data.

Consider the two examples of the look-up table and the sample mean. In the case of the look-up table, the fitted value is identical to the observed value and so the (insample) residuals are all zero. However this does not (necessarily) reflect any generalisation ability in the model, but merely the fact that all of the degrees of freedom in the original data are absorbed by the model. In contrast, the residual errors around the sample mean will typically be non-zero, reflecting the fact that only one degree of freedom has been absorbed.

Depending upon the assumptions which are made about the nature of the model, a number of slightly different expressions can be used to relate the structure of the smoother matrix $\mathbf{S}$ to the degrees of freedom $dof_{\mathbf{S}}$; however in the case of fitting using ordinary least squares (OLS) they all turn out [see pp 52-55 of (Hastie and Tibshirani, 1990)] to be equivalent to:

$$dof_{\mathbf{S}} = \sum_{i=1..n} \mathbf{S}_{ii} = \text{trace}(\mathbf{S}) \tag{8.11}$$

It is straightforward to confirm that in the case of the look-up table $\mathbf{S}_{LUT}$, this measure equals the number of observations 'n', and that for the sample mean $\mathbf{S}_{MEAN}$ absorbs a single degree of freedom.

Perhaps the initial lack of a similar degrees of freedom measure for neural networks has been one of the largest obstacles to the development of statistical significance tests and diagnostics equivalent to those used in regression modelling. Somewhat paradoxically, this weakness has had the advantage of stimulating the development of alternative approaches to model validation and selection, such as cross-validation, bootstrapping and out-of-sample testing. However, it has probably also reduced the credibility of the emerging modelling techniques and hindered their uptake within the statistical modelling community. More importantly, the lack of statistical significance measures that can be applied on an *insample* basis reduces the effectiveness of neural modelling in situations where only small samples of noisy data are available. In such cases, leaving data aside for purposes of selection or validation may seriously degrade the quality of the model fit itself.

However, in recent years the nonparametric approach to estimating the degrees of freedom in a model has been generalised to include both simple parametric models and neural networks. The following two sections review these extensions and discuss their implicaitons, firstly in the case of standard "linear in the parameters" approaches such as OLS regression, and secondly to more flexible neural network models.

### 8.3.2 Equivalent kernels for parametric models

If the smoother matrix representation and degrees of freedom measure were only applicable to nonparametric models then they would still be very useful and important tools. However, the same concepts can be extended to traditional parametric regression models through the perspective of “equivalent kernels”. Whereas in the nonparametric case the “kernels” which comprise the smoother matrix are explicitly specified, in the case of parametric models the smoother matrix is *implicitly* defined by the combination of parameterisation and estimation procedure.

Consider a model of the form:

$$\hat{y}(\mathbf{x}_i) = \sum_{k=1..m} w_k^* \phi_k(\mathbf{x}_i) \tag{8.12}$$

i.e. a weighted combination of some arbitrary transformations of the input variables. Note that the vast majority of statistical models, including many neural network formulations, can be represented in such an “additive” form. The crucial feature of the model is that although the basis functions $\phi_k$ are described with respect to the vector of independent variables $\mathbf{x}$, the parameters $w_k^*$ are determined, through the estimation procedure, with respect to the set of observed values $y_i$. In particular, the effect on the model estimates $\hat{y}(\mathbf{x}_i)$ of a perturbation or contamination of a particular observation $y_j$ is given by the chain rule of derivatives:

$$\frac{\P\hat{y}(\mathbf{x}_i)}{\P y_j} = \sum_{k=1..m} \frac{\P w_k^*}{\P y_j}\phi_k(\mathbf{x}_i) = \sum_{k=1..m} \frac{\P w_k^*}{\P y_j}\frac{\P\hat{y}(\mathbf{x}_i)}{\P w_k^*} = \frac{\P\mathbf{w}}{\P y_j}\nabla_{\mathbf{w}}\hat{y}(\mathbf{x}_i) \tag{8.13}$$

i.e. the effect on the fitted value $\hat{y}(\mathbf{x}_i)$ of perturbing the observed values $y_j$ can be defined in terms of the changes which are caused in the weights by which the (predefined) basis functions are combined. By comparison with Eqn. (8.6) we note that this partial derivative is directly equivalent to the elements which comprise the kernel functions $\varphi(x_i, x_j)$. Hence we obtain the so-called “equivalent kernels” and the smoother matrix for the parametric model is defined by:

$$\mathbf{S} = \begin{bmatrix} \frac{\partial \hat{y}(\mathbf{x}_1)}{\partial y_1} & \frac{\partial \hat{y}(\mathbf{x}_1)}{\partial y_2} & \cdots & \frac{\partial \hat{y}(\mathbf{x}_1)}{\partial y_n} \\ \frac{\partial \hat{y}(\mathbf{x}_2)}{\partial y_1} & \frac{\partial \hat{y}(\mathbf{x}_2)}{\partial y_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial \hat{y}(\mathbf{x}_n)}{\partial y_1} & \cdots & \cdots & \frac{\partial \hat{y}(\mathbf{x}_n)}{\partial y_n} \end{bmatrix} \tag{8.14}$$

This formulation allows us to compare on equal terms models with different parametric representations and even parametric models with models which are not explicitly parameterised (i.e. non-parametric models). Examples of equivalent kernels for different classes of parametric and non-parametric models are given by (Hastie and Tibshirani, 1990) whilst a treatment for Radial Basis Function (RBF) networks is presented in (Lowe, 1995).

For models with low-dimensional input-space, the equivalent kernels can be visualised by plotting the kernel weightings as a function of the input vector $\mathbf{x}$. Figure 8.1 illustrates equivalent kernels from simple fixed-basis regression models with different types of basis function:

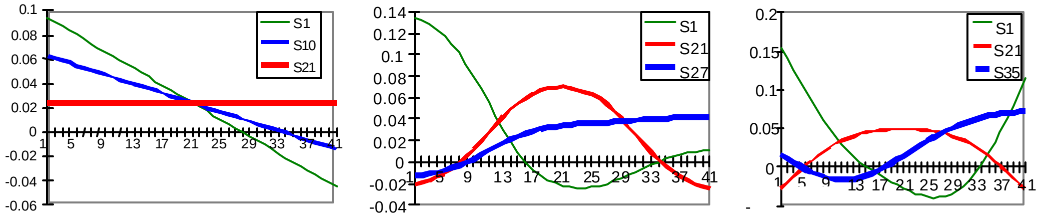


Figure 8.1: Examples of *equivalent kernels* for simple regression models. The kernels show the “influence” functions of the models, sampled at different points in input space. The models are, from left to right: linear regression, superposition of two sigmoid functions, and superposition of two gaussians. Note the rigid structure of the linear model as opposed to the localised and more flexible kernels from the non-linear models.

The equivalent kernels illustrate the *sensitivity* of the model $\hat{y}(\mathbf{x}_i)$ at a given point $\mathbf{x}_i$ to the observed values $y_j$ at all other points in the estimation set. For the linear case (on the left) the kernel labelled S21 indicates that the response at the mean *x*-value will be an equally weighted combination of all observations, i.e. the sample mean. In contrast, equivalent kernel S1 corresponds to the mimimum *x*-value, the kernel weights are greatest at nearby observations and decline linearly as a function of distance. In fact, a defining feature of linear models is that the equivalent kernel is always a hyper-plane in input-space, with the orientation and slope of the kernel depending on the direction and distance from the sampled point to the centre of the input distribution.

The equivalent kernels for the two nonlinear models appear very different to the linear case, in that the kernels are more highly localised and vary in *shape* depending upon the point in input space at which they are sampled. This effect is most noticeable for the equivalent kernels sampled at the centre of the input space (labelled S21 in each case). Unlike the linear case, the response at the mean $x$-value will not necessarily equal the (global) sample mean.

This difference between the linear and nonlinear models is actually slightly misleading, as all three models are of the additive form defined in Eqn. (8.12). The transformation by the nonlinear basis functions changes the notions of "distance", and "locality" to reflect the similarity of the *transformed* values as opposed to the original $x$-values, causing the equivalent kernels to appear nonlinear when viewed in the original space

If the transformed values better reflect the inherent "similarity" of two observations, then the use of a transformation should be expected to improve the accuracy of the estimated model. The estimation method itself, however, is blind to the origin of the variables and thus, for a predefined set of transformations, the smoother matrix follows the properties of linear regression in that it is both symmetric and idempotent. An important consequence of this is that the number of degrees of freedom absorbed by such an additive model is equal to the number of transformed variables *m*.

### 8.3.3 Equivalent kernels for neural networks

The ability to estimate equivalent kernels, and to calculate the number of degrees of freedom which are absorbed by the model, depends on the use of both fixed basis functions and on standard Ordinary Least Squares (OLS) estimation. This is typically the case in parametric models and even for some neural networks (such as some forms of RBF networks). However, for a neural network with more than one layer of adjustable weights, the basis functions themselves are *parameterised* (rather than fixed) and thus become dependent on the training data. Consequently the equivalent kernels are also data-dependent, and the problem of finding the equivalent kernels becomes non-linear. Similar complications arise in the case where the estimation procedure itself is biased by the use of a regularisation procedure such as ridge regression or the equivalent “weight decay” approach to neural estimation (see Section 2.2.4).

The solution to this problem is found in the work of Bolland (1998), following earlier work by Moody (1992) and Burgess (1995). Moody (1992) was the first to note that not all parameters in a neural network should be viewed as absorbing a full “degree of freedom”. In his terminology, the number of *effective parameters*, $p_{\mathrm{eff}}$, may be **less** than the actual number of parameters contained in the neural network model. The equivalent kernels perspective on neural networks was first investigated in Burgess (1995) using an empirical perturbation-based methodology. Bolland (1998) built on both the equivalent kernels perspective of Burgess (1995) and the analytic approach of Moody (1992), presenting an analytic approximation to the degrees of freedom in a neural network and extending this perspective in the context of general loss functions other than OLS, as reviewed below.

### Analytic approximation of the smoother matrix for neural networks

From Eqn (8.14) we note that the elements of **S** correspond to the partial derivatives of the fitted model at each point with respect to each observed value. Expanding the derivatives in terms of the model parameters, in this case the vector of neural network weights $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}^T$ we obtain:

$$\frac{\partial \hat{y}_i}{\partial y_j} = \sum_{k=1..m} \frac{\partial \hat{y}_i}{\partial w_k} \frac{\partial w_k}{\partial y_j} = \left(\nabla_{\mathbf{w}} \hat{y}_i\right)^T \frac{\partial \mathbf{w}}{\partial y_j} \tag{8.15}$$

which corresponds to Eqn. (8.13) for the fixed basis function models discussed above. Taking a first-order Taylor expansion of the loss function of the perturbed model about the original model we obtain:

$$\begin{aligned} \nabla_w L\left(\mathbf{y} + \mathrm{d}\mathbf{y}, \mathbf{X}, \mathbf{w}^+\right) = {} & \nabla_w L\left(\mathbf{y}, \mathbf{X}, \mathbf{w}\right) \\ & + \nabla_w \nabla_{y'} L\left(\mathbf{y}, \mathbf{X}, \mathbf{w}\right)\mathrm{d}\mathbf{y} + \nabla_w \nabla_{w'} L\left(\mathbf{y}, \mathbf{X}, \mathbf{w}\right)\mathrm{d}\mathbf{w} \\ & + O(\ldots) \end{aligned} \tag{8.16}$$

Assuming that both the original set, $\mathbf{w}$, and perturbed set, $\mathbf{w} \rightarrow \mathbf{w}^+$, of weights minimise the empirical loss function *L*, then $\nabla_w L\left(\mathbf{y}, \mathbf{X}, \mathbf{w}\right) = 0$, and $\nabla_w L\left(\mathbf{y} + \mathrm{d}\mathbf{y}, \mathbf{X}, \mathbf{w}^+\right) = 0$. Thus, neglecting higher order terms, we obtain the *change* in the network weights which is due to a perturbation in the observed values:

$$\mathrm{d}\mathbf{w} = \left(\nabla_w \nabla_{w'} L(\mathbf{y},\mathbf{X},\mathbf{w})\right)^{-1} \nabla_w \nabla_{y'} L(\mathbf{y},\mathbf{X},\mathbf{w})\mathrm{d}\mathbf{y} \tag{8.17}$$

The first term on the right-hand side, the second derivative of the loss function with respect to the model parameters, is known as the Hessian of the model. Substituting for d**w** into Eqn (8.15) and expressing the result in matrix form gives:

$$\mathbf{S} = \left(\nabla_{\mathbf{w}} \hat{y}\right)^T \left(\nabla_w \nabla_{w'} L(\mathbf{y},\mathbf{X},\mathbf{w})\right)^{-1} \nabla_w \nabla_{y'} L(\mathbf{y},\mathbf{X},\mathbf{w}) \tag{8.18}$$

However for loss functions where the error component is quadratic, $L = 1/2\left(\mathbf{y} - \hat{\mathbf{y}}\right)^2$, then even in the presence of a regularisation term, we have $\nabla_w \nabla_{y'} L(\mathbf{y},\mathbf{X},\mathbf{w}) = -\nabla_w \hat{y}$, and so the smoother matrix is:

$$\mathbf{S} = \left(\nabla_{\mathbf{w}} \hat{y}\right)^T \left(\nabla_w \nabla_{w'} L(\mathbf{y},\mathbf{X},\mathbf{w})\right)^{-1} \left(\nabla_{\mathbf{w}} \hat{y}\right) \tag{8.19}$$

A final simplification is sometimes used in which the full Hessian $\nabla_w \nabla_{w'} L(\mathbf{y},\mathbf{X},\mathbf{w})$ is replaced by a more-stable linearised approximation $\left(\nabla_{\mathbf{w}} \hat{y}\right)\left(\nabla_{\mathbf{w}} \hat{y}\right)^T$ giving the expression:

$$\mathbf{S} = \left(\nabla_{\mathbf{w}} \hat{y}\right)^T \left[\left(\nabla_{\mathbf{w}} \hat{y}\right)\left(\nabla_{\mathbf{w}} \hat{y}\right)^T\right]^{-1} \left(\nabla_{\mathbf{w}} \hat{y}\right) \tag{8.20}$$

As before the number of degrees of freedom in the network are given by $dof_{\mathbf{S}} = \sum_{i=1..n} \mathbf{S}_{ii} = \mathrm{trace}(\mathbf{S})$.

### Simulation results

Figure 8.2 shows examples of equivalent kernels from a univariate neural network trained to reproduce the first two periods of a sine-wave, sampled at 41 points evenly spaced between 0 and 4π.

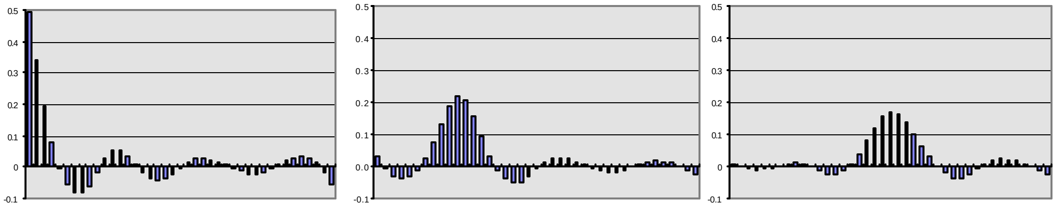


Figure 8.2: Equivalent Kernels for a univariate neural network trained to reproduce Sin(x); from left to right, the three figures represent the kernels for x = 0, π, and 2π. The neural network consisted of a single hidden-layer of four sigmoid units, a shortcut connection from input to output, and a linear output unit, trained using the standard backpropagation learning algorithm (see Section 2.2.4)

The trained network was found to absorb 8.2 degrees of freedom, compared to the 14 *potential* degrees of freedom represented by the parameters, and the 6 degrees of freedom for an equivalent model with fixed transfer functions. This confirms (a) that the network is not completely utilising its potential capacity, as suggested by Moody (1992); and (b) that perturbations in the training data are partially accommodated by adjustments to the basis function themselves.

We can also investigate the effects of model regularisation on the ability of the network to reproduce the target function, the number of degrees of freedom absorbed by the network, and the kernel functions themselves. Using standard quadratic weight decay, we minimise a loss function of the form:

$$L = \frac{1}{n}\sum_{i=1..n}\left(y_i - \hat{y}\left(\mathbf{x}_i\right)\right)^2 + r\sum_{j=1..m} w_j^2 \qquad (8.21)$$

Selected results from this analysis are presented in Figure 8.3, below.

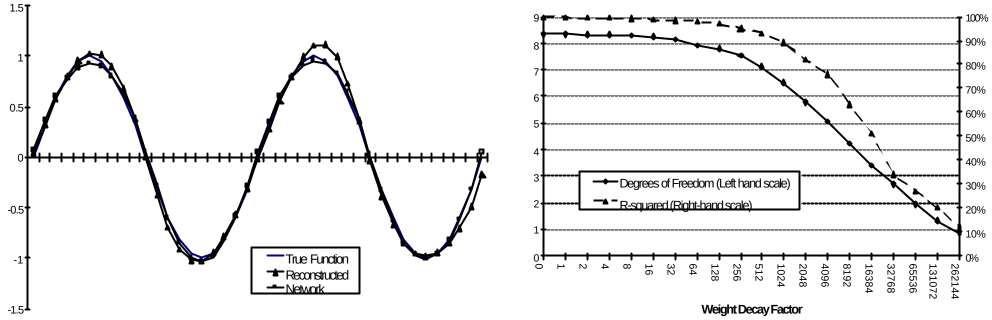


Figure 8.3: Analysis of univariate network trained to reproduce a sign wave. The left-hand chart presents a comparison of the network function (with *r*=0) and the function reconstructed from the estimated kernels; the right-hand chart demonstrates the effect of the weight decay factor, *r*, on both network performance ($R^2$) and capacity (degrees of freedom).

The results in the figure indicate that the estimated kernel function reproduces the actual network function very closely, indicating that the estimated equivalent kernels are also highly accurate. Increasing the amount of weight decay serves to reduce the degrees of freedom of the model, and in this noise-free example, the performance of the model is seen to degrade accordingly.

It is in multidimensional problems, however, that the real power of neural networks as parsimonious yet flexible function estimators is demonstrated. This power lies in the fact that the *shape* of the kernels can vary in different regions of the input space. As an illustration, kernels from a 2-dimensional model are shown in Figure 8.4 below.

The figure clearly illustrates the changing shape of the kernel functions in different parts of the input space. The first kernel, generated at (-4/7, -4/7), exhibits a ridge-like structure, albeit with some curvature. The second kernel, generated at (0,0), is a broad hummock with negative sensitivity to points near the corners. The third kernel, generated for (2/7, 6/7), shows a distinct curvature which echoes the ring shape of the original function.

These results demonstrate that the equivalent kernel functions of the neural network vary significantly in different regions of the input space, not just in terms of "spread" but also regarding the overall *shape* of the kernels. This supports the view that the representational power of neural networks arises from an ability to parsimoniously exploit the degrees of freedom within the model, in a way which is in some sense optimally related to *local* properties of the data itself, rather than according to some external *global* parameterisation such as the “bandwidth” parameters used in the kernel smoothing techniques of nonparametric statistics.

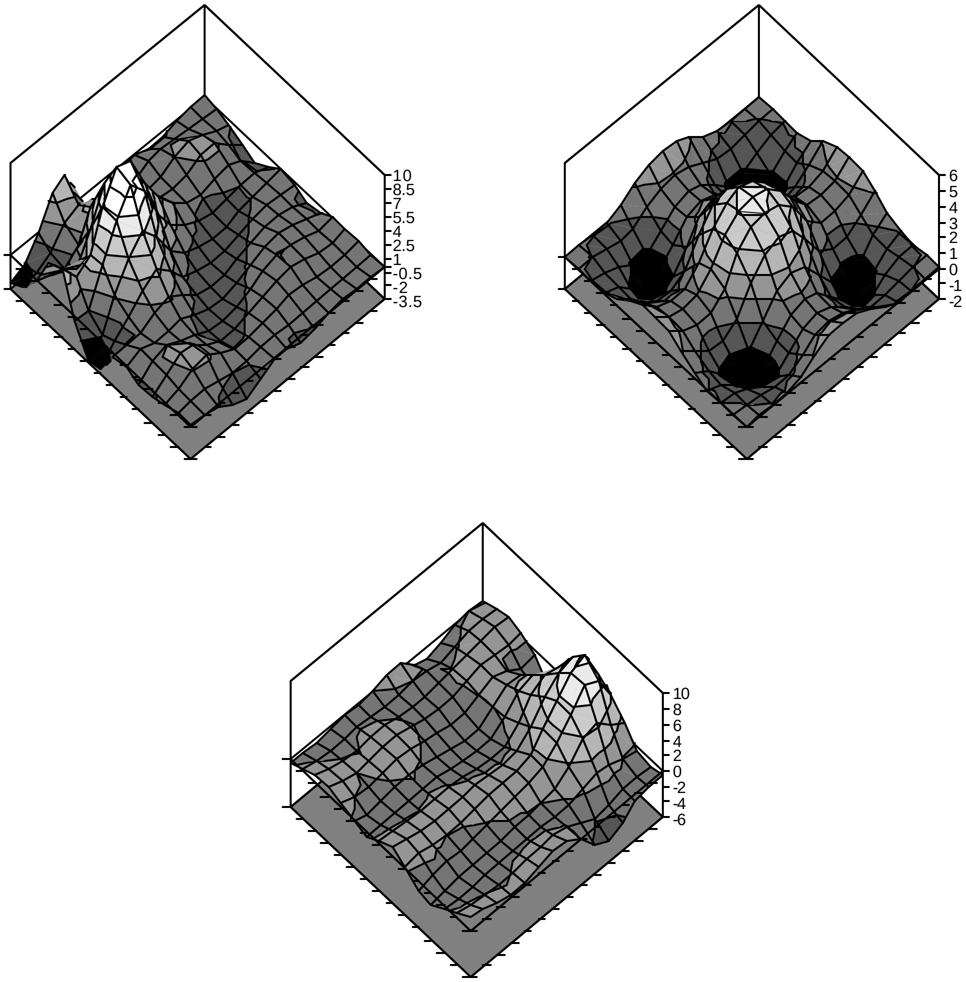


Figure 8.4: Equivalent Kernels for a neural network estimator of the function $z = 1 \Big/ \left[1 + 30\left(x^2 + y^2 - 0.5\right)^2\right]$ sampled on a regular 15 by 15 grid running between plus and minus one. This function was approximated using a 2-8-1 network with sigmoidal hidden units and a linear output unit.

## 8.4 Summary

In this chapter we have noted that estimating forecasting models of the mispricing dynamics can be seen as a particular case of the general model estimation problem. A simple analysis of the effect of model bias and variance on the expected level of forecasting ability highlights the particularly difficult nature of financial forecasting problems. We suggest that these difficulties may be alleviated by developing methodology which represents a *synthesis* of statistical and neural modelling techniques, aiming to exploit the flexibility of neural models whilst ensuring that this flexibility is employed in a controlled and appropriate manner.

The ability to transform neural network regression models into an equivalent kernel representation raises the possibility of harnessing the whole battery of statistical methods which have been developed for non-parametric techniques: model selection procedures, prediction interval estimation, calculation of degrees of freedom, and statistical significance testing amongst others. The recent work of a number of authors has a similar underlying theme. Reference has already been made to the work of Moody (1992), Amari (1995) and Bolland (1998), in addition Williams *et al* (1995) describe an equivalent kernel approach to estimating error bars for RBF networks with a single layer of adjustable weights.

Most importantly for our purposes, the equivalent kernel perspective allows the modelling process to be viewed as an *explicit* trade-off between the variance which is explained and the number of degrees of freedom which are absorbed by a model. In the following chapters we develop model-free variable selection and neural estimation procedures which are based upon the tradeoff between these two fundamental properties of regression estimators.

# *9. Model-free variable selection*

This chapter describes our methodology for *model-free* variable selection, by which we indicate the intention that the methodology should be sensitive to a wide range of relationships, rather than being predicated upon the assumption of a particular functional form. Section 9.1 describes the statistical significance tests which we use as the basis of the methodology. These tests exploit the *equivalent kernels* perspective, in which all regression estimators can be seen as ways of predicting the outcome of future situations through the implicit construction of a weighted combination of "similar" observations from the past. In particular, the significance testing methods are based upon the "F-ratio" of the amount of variance which is absorbed by a model, and the degrees of freedom which are employed in so doing. Section 9.2 describes the implementation of our methodology, in which a diverse set of basis functions are employed in order to screen data for a wide range of potentially predictable components. Section 9.3 describes a Monte Carlo evaluation of the statistical **power** of the variable selection tests against alternative hypotheses which contain only a *small* deterministic component and hence are close to the null hypothesis of *no* predictable component. The power of the tests is evaluated in two contexts, firstly in terms of the ability to detect *any* predictable component in the data, secondly to detect explicitly *nonlinear* relationships. Section 9.4 contains the results of a second Monte Carlo experiment to quantify the model **bias** which is imposed by the use of each particular family of basis functions.

## 9.1 Statistical methodology for model-free variable selection

In this section we describe the basis of our methodology for model-free variable selection. The objective of this stage of the modelling process is to identify the subset of variables $X \subset I$ which should be used as the basis of the model estimation process. The reason for this pre-selection may simply be that it is <u>infeasible</u> to include all of the variables in the model estimation proper, perhaps because the resulting model would have too many parameters to estimate from the available data. Alternatively, the preliminary filtering process may be intended simply to *reduce* the number of variables with the intention of reducing the complexity of the estimated model and hence the error which is due to model variance.

The aim of the methodology is to be "model free" in the sense that it should be sensitive to a wide range of relationships, rather than only those which follow a particular functional form.

The flexibility of the variable selection procedure is vital in order to avoid imposing a hidden bias on the subsequent stages of modelling by removing deterministic components in the data simply because they are invisible to the variable selection procedure. In order to achieve this flexibility, our variable selection methodology is based upon techniques from nonparametric statistics, and in particular *nonparametric regression*. Specifically, we regress the dependent variable upon a wide range of transformations of the variables and test for significant relationships by analysing the so-called "F-ratio" of the variance explained by the model to the degrees of freedom that it contains.

**Linear Relationships**

In order to motivate the use of a flexible variable selection methodology, let us first consider the problems which would arise by using standard linear methodology at this stage of the modelling process.

The standard linear regression model takes the form:

$$y_i = a + b_1 x_{1,i} + b_2 x_{2,i} + \ldots. + b_m x_{m,i} + e_i \tag{9.1}$$

Variable selection in this case is relatively straightforward, because the effect of each variable $x_j$ is associated with a single parameter $b_j$. Under the null hypothesis that $x_j$ is unrelated to $y$ then the so-called 't'-statistic $b_j \big/ se_{b_j}$ is known to follow a standard 't' distribution with $n-(m+1)$ degrees of freedom, where $n$ is the number of observations and $m$ is the number of variables in the model. A less precise measure of (linear) variable significance is based on the (Pearson) correlation coefficient $r_{xy}$:

$$r_{xy} = \frac{\frac{1}{n}\sum_{i=1..n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n}\sum_{i=1..n}(x_i - \bar{x})^2}\sqrt{\frac{1}{n}\sum_{i=1..n}(y_i - \bar{y})^2}} = \frac{s_{xy}}{s_x s_y} \tag{9.2}$$

Unlike the measures based on the regression coefficients $b_j$, the correlation coefficient ignores any *cross-correlations* between the various $x_j$'s. However, because it has the advantage of being easy to compute for large numbers of variables, it is often used as a "prescreening" tool to reduce the number of variables to a manageable level before moving on to the more computationally and data intensive phase of model estimation proper.

The additional complexity of **nonlinear** models entails that the variable selection process is both more *important* and more *difficult* to perform – at least without imposing substantial biases to the modelling procedure as a whole. A general nonlinear model with $m$ regressors can be broken down into submodels which relate to subspaces ranging from one-dimensional up to the entire space:

$$y = \sum_{i=1..m} f_i(x_i) + \sum_{i=1..m}\sum_{j=1..i} f_{i,j}(x_i, x_j) + \sum_{i=1..m}\sum_{j=1..i}\sum_{k=1..j} f_{i,j,k}(x_i, x_j, x_k) + ..... + f(x_1, \ldots x_m) + e \qquad (9.3)$$

In the linear case the higher terms do not exist, as can easily be seen in the case of the second-order term:

$$\begin{aligned} f_{i,j}(x_i, x_j) &= f_i(x_i) + f_j(x_j) \\ &= b_i x_i + b_j x_j \end{aligned} \qquad (9.4)$$

Therefore, the properties of *independence* and *linearity* vastly simplify the nature of the variable selection problem. Unfortunately, applying a <u>linear</u> test, such as the correlation coefficient $r_{x_j y}$, in a situation where <u>nonlinear</u> relationships may exist is highly dangerous as it will fail to detect either higher order "interaction" effects such as $f_{i,j}(x_i, x_j)$ or univariate relationships $f_i(x_i)$ which do not contain a significant linear component. Thus, in order to avoid discarding potentially invaluable information, we have developed an alternative "model free" variable selection procedure which relaxes the bias towards purely linear relationships and is capable of identifying a wide range of nonlinear and interaction effects.

### Basic Methodology

The approach which we take is to regress the dependent variable upon nonlinear projections of the original variables to produce submodels of the form:

$$y = a + \sum_{j=1..m} q_j b_j\left(x_i\right) + e \qquad (9.5)$$

Where the *basis functions* $b_j\left(x_i\right)$ are chosen in such a way that in combination they can approximate a wide range of functional relationships, with properties similar to those which are believed to exist in the true mispricing dynamics (i.e. smooth functions of low-dimension and high noise content). The fundamental test for a statistically significant relationship is that the amount of variance explained by the model should be large, relative to the number of free parameters, *p*, which were fitted to the data. Under the null hypothesis that $x_i$ is unrelated to $y$, it is known that the F-ratio of the model should follow an F-distribution with $p$ degrees of freedom in the numerator and *n-(p+1)* in the denominator, i.e.:

$$\frac{\frac{1}{p}\sum_{i=1..n}\left(\left[a+\sum_{j=1..m}q_j b_j\left(x_i\right)\right]-\bar{y}\right)^2}{\frac{1}{n-(p+1)}\sum_{i=1..n}\left(y-\left[a+\sum_{j=1..m}q_j b_j\left(x_i\right)\right]\right)^2} \sim F_{p,n-(p+1)} \tag{9.6}$$

The tests can also be used to test for *interaction effects* between variables; to do so the basis functions act as a nonlinear projection of the subset of the original variables which is being tested. In the two dimensional case the submodels are of the form:

$$y = a + \sum_{k=1..m} q_k b_k\left(x_i, x_j\right) + e \tag{9.7}$$

The variable selection algorithm consists of “screening” all the low-dimensional combinations of the candidate variables, constructing appropriate submodels, and testing for evidence of a significant deterministic component in the target series when it is conditioned on the particular submodel. As the number of combinations grows exponentially with the dimensionality, the maximum feasible cardinality of the subspace will typically be either 2 or 3. Clearly this imposes a bias on the modelling procedure as a whole. Given the underlying assumption, however, that the predictable component is primarily contained in low-dimensional subspaces, it is hoped that the bias will be more than offset by the simultaneous reduction in model-variance which results from reducing the input space used by the model estimation procedure.

**OLS procedure**

In the simplest case the model parameters $q_j$ are estimated using ordinary-least-squares (OLS) regression, where the cost function with respect to which the parameters are optimised is simply the mean-squared error:

$$E_{OLS} = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \left[a + \sum_{j=1}^{m} q_j b_j\left(\mathbf{x}_i\right)\right]\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2 \tag{9.8}$$

If we define the basis-transformed set of observations:

$$\mathbf{B} = \begin{bmatrix} b_1(\mathbf{x_1}) & b_2(\mathbf{x_1}) & \cdots & b_m(\mathbf{x_1}) \\ b_1(\mathbf{x_2}) & b_2(\mathbf{x_2}) & & \\ \vdots & & \ddots & \\ b_1(\mathbf{x}_m) & & & b_m(\mathbf{x}_m) \end{bmatrix} \tag{9.9}$$

Then the optimal OLS estimator, with parameters $\mathrm{q} = \left[q_1 \quad \cdots \quad q_m\right]^T$, is given by:

$$\hat{\mathbf{y}} = \mathbf{B}\mathrm{q}_{\mathrm{OLS}} = \mathbf{B}\left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}\mathbf{y} = \mathbf{S}_{\mathrm{OLS}}\mathbf{y} \tag{9.10}$$

As the smoother matrix $\mathbf{S}_{\mathrm{OLS}}$ is both symmetric and idempotent (Bolland, 1998), it follows that:

$$dof\left(\mathbf{S}_{\mathrm{OLS}}\right) = Trace\left(\mathbf{S}_{\mathrm{OLS}}\right) = Rank\left(\mathbf{S}_{\mathrm{OLS}}\right) = m \tag{9.11}$$

The total variance of the dependent variable can then be broken down into two components: one which is explained by the *model*, and the unexplained *residual* variance:

$$tss = \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n}\left(\hat{y}_i - \bar{y}\right)^2 + \sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2 = mss + rss \tag{9.12}$$

The test for statistical significance is based on the fact that under the null hypothesis, in which $\mathbf{y}$ is linearly independent from $\mathbf{B}$, the ratio of the average variance absorbed by the model degrees of freedom to the residual error variance follows an F distribution:

$$\left(\frac{mss}{m}\right)\Big/\left(\frac{rss}{n-(m+1)}\right) \sim F_{m,n-(m+1)} \tag{9.13}$$

### Adjustments for using Regularised Models

However, our prior assumption that the underlying relationships in mispricing dynamics are relatively smooth implies that standard OLS is not the most appropriate parameter estimation technique. The reason for this is that the expected curvature of the estimated function grows monotonically with the magnitude of the parameters. Thus a prior view that function complexity, and hence curvature, is low implies that parameter values of high magnitude are less likely than parameter values of small magnitude. In contrast, OLS treats all parameter values as being equally likely and hence ignores the “prior” information.

The solution to this discrepancy is to use an appropriate regularisation technique, which biases the estimation procedure in such a way as to reflect the prior assumptions. In this case we use “ridge regression” (Hoerl and Kennard, 1970) which is equivalent to assuming a prior distribution for the parameters $q_j$ which is gaussian and centred on zero. The optimal regularised parameters will depend on the degree of regularisation $r$, producing the estimator:

$$\hat{\mathbf{y}} = \mathbf{B}\mathrm{q}_{\mathrm{REG}} = \mathbf{B}\left(\mathbf{B}^T\mathbf{B} + r\mathbf{I}\right)^{-1}\mathbf{B}\mathbf{y} = \mathbf{S}_{\mathrm{REG}}\mathbf{y} \tag{9.14}$$

An important feature of ridge regression is that, relative to standard OLS, the number of *degrees of freedom* absorbed by the model is reduced:

$$dof\left(\mathbf{S}_{\mathrm{REG}}\right) = Trace\left(\mathbf{S}_{\mathrm{REG}}\right) \leq m \tag{9.15}$$

and the F-test for statistical significance is adjusted appropriately:

$$\left(\frac{mss}{dof\left(\mathbf{S}_{\mathrm{REG}}\right)}\right) \Bigg/ \left(\frac{rss}{n - \left[dof\left(\mathbf{S}_{\mathrm{REG}}\right) + 1\right]}\right) \sim F_{dof\left(\mathbf{S}_{\mathrm{REG}}\right), n-\left(dof\left(\mathbf{S}_{\mathrm{REG}}\right)+1\right)} \tag{9.16}$$

### Adjustments for using Neural Networks with adaptive basis functions

As discussed in Section 8.3, the basis functions, or rather the “equivalent kernels”, for neural network models are not predetermined, but instead are themselves estimated from the data. The purpose of the neural network learning algorithm is to determine the parameters which minimise the “cost function” $E_{NN}$ :

$$\mathrm{q}_{\mathrm{NN}} = \min_{q} E_{NN} = \min_{q}\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i - \mathrm{NN}(\mathbf{x}_i, \mathrm{q})\right)^2 + r\sum_{j=1}^{m} q_j^2\right] \tag{9.17}$$

Where the *r* term represents a “weight decay” or regularisation term which controls model smoothness in a manner similar to the ridge regression used for the fixed basis-function models. From Section 8.3.3, the degrees of freedom absorbed by the neural network model are given by:

$$\mathbf{S}_{\mathrm{NN}} = \nabla_q\ \mathrm{NN}(\mathbf{X},\mathrm{q})^T\left[\nabla_q \nabla_q E_{NN}\right]^{-1}\nabla_q\ \mathrm{NN}(\mathbf{X},\mathrm{q}) \tag{9.18}$$

and the test for significance is based on the degrees of freedom obtained from the smoother matrix:

$$\left(\frac{mss}{trace(\mathbf{S}_{\mathrm{NN}})}\right)\Bigg/\left(\frac{rss}{n-\left[trace(\mathbf{S}_{\mathrm{NN}})+1\right]}\right) \sim F_{trace(\mathbf{S}_{\mathrm{NN}}),n-(trace(\mathbf{S}_{\mathrm{NN}})+1)} \tag{9.19}$$

## 9.2 Implementation of Variable Selection Methodology

The main issue in implementing the variable selection methodology is the specification of the basis functions $b_j\left(x_i\right)$. In avoid to avoid imposing any strong biases at this preliminary stage of the modelling procedure we employ a range of different model types, from untransformed linear models, through fixed basis-function models to neural network models. The particular parameterisations which we implement within our procedure are defined in Table 9.1 below.

| Model Type | Specification | Number of Parameters |
|---|---|---|
| Linear | $\mathrm{L}(x_1, x_2, \ldots, x_{nd}) = a + \sum_{i=1}^{nd} b_i x_i$ | $nd+1$ |
| Polynomial | $\mathrm{P}(x_1, x_2, \ldots, x_{nd}) = \sum_{i_1=0}^{c} \ldots \sum_{i_{nd}=0}^{c-(i_1+..+i_{nd-1})} b_{i_1,..,i_{nd}} x_1^{i_1} x_2^{i_2} \ldots x_{nd}^{i_{nd}}$ | $\sum_{i=0}^{c} {}^{nd}_{\ i}C$ |
| Bin-smoother | $\mathrm{B}(x_1, x_2, \ldots, x_{nd}) =$<br>$\sum_{i_1=1}^{c} \ldots \sum_{i_{nd}=1}^{c} b_{i_1,..,i_{nd}} \prod_{d=1}^{nd} \begin{cases} 1 & \text{if } (i_d - 1) \leq \frac{\mathrm{rank}(x_d)}{n} c < i_d \\ 0 & \text{otherwise} \end{cases}$ | $c^{nd}$ |

| | | |
|---|---|---|
| RBF Network | $\mathrm{R}\left(x_1, x_2, \ldots, x_{nd}\right) = a + \sum_{i_1=1}^{c} \ldots \sum_{i_{nd}=1}^{c} b_{i_1,\ldots,i_{nd}} k e^{-\sum_{i=1}^{nd}\left(\frac{x_i - m}{s_i}\ {}_{i}\right)}$ | $c^{nd} + 1$ |
| MLP Network | $\mathrm{M}\left(x_1, x_2, \ldots, x_{nd}\right) = a + \sum_{i=1}^{c} b_i \tanh\left(b_i + \sum_{j=1}^{nd} w_{i,j} x_j\right)$ | $(nd + 2)c + 1$ |

Table 9.1: Specification of the different families of basis function which are implemented within the variable selection methodology; *nd* is the dimensionality of the input subspace; *c* indicates the relative level of complexity of the model.

Figure 9.1 presents examples of the basis functions generated by the five families of model.

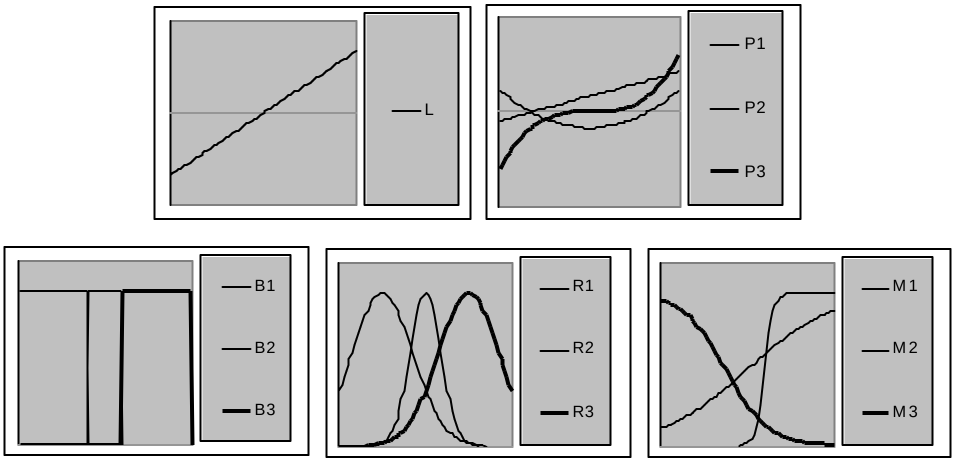


Figure 9.1: Univariate examples of the basis functions generated by each of the model families; from top-left to bottom-right the models are linear (L), polynomial (P), bin-smoother (B), Radial Basis Function (R) and Multi-layer perceptron (M).

The **linear models** (model type L) are of the standard linear regression form and are based on the assumptions of linearity and independence between variables.

The **polynomial models** (model type P) can be considered a generalisation of the linear model to allow for particular types of *nonlinearity* and *interaction* effects. Note that the $b_{00\ldots0}$ term is equivalent to the constant term $a$ of the other formulations and that the polynomial model with complexity $c$=1 is equivalent to the linear model.

The **bin-smoother models** (or regressograms (Haerdle, 1990)) represent *localised* sets of basis functions which are piece-wise constant indicator variables. The input space is divided into a set of *nd*-dimensional boxes or "bins" and the indicator variable for the bin in which an observation falls is set to 1, whilst the other indicator variables are set to 0. The bin-smoother model is capable of representing both nonlinear and interaction effects but unfortunately the number of parameters scales exponentially in the number of input-dimensions. An additional problem of the bin-smoother is the lack of smoothness of the estimated function, which is discontinuous at the boundaries of the bins.

The **Radial Basis Function** models (model type R) are similar to the bin-smoothers in that the basis functions are *centred* at particular points in the input space, but differ in that they have a smooth (gaussian) response which decays gradually with the distance from the centre. The number of parameters again scales exponentially with the input-space dimensionality.

The **MLP Network** models (model type M) are of the type of neural network known as "multi-layer perceptrons", and can be considered as summations of *projective* basis functions which have sigmoidal responses along arbitrary projections of the original input space. The particular flexibility of these models derives from the fact that the basis functions themselves are parameterised (by the projection "weights" $w$) and estimated from the data itself, rather than being prespecified as in the cases of the other models. In neural network terminology the number of basis functions $c$ is often referred to as the number of "hidden neurons" in the network. A further important advantage of the neural network models is the parsimonious scaling behaviour, with the number of parameters being only linear in the input-space dimensionality $nd$.

### Statistical variable significance tests

Within the general framework of the statistical significance tests described in Section 9.1, we have implemented four specific types of variable significance tests. These differ in the manner in which the estimators are applied and in the component of the deterministic relationship which is under scrutiny.

The first two types of test are simply aimed at detecting *any* deterministic component in the data. The basic test in this case is a **single-estimator test** in which a given type of estimator is applied to the data and tested for statistical significance. The second type of test recognises that in a preliminary "filtering" stage it is preferable to err on the side of caution before concluding that a variable should be eliminated from the analysis.  Thus we define a **joint test** in which all five types of estimator are applied to the data, with the most significant of the findings taken into account before deciding to eliminate a particular variable.

Note that by explicitly taking the maximum of a set of statistics, this approach introduces a "data snooping" effect in which the relationships will tend to seem more significant than they actually are, and in this case our results indicate that the false positive rate is approximately double the nominal size of the test[18]. If necessary, this bias could be explicitly corrected by constructing a Monte-Carlo distribution for the test statistic, as described for the predictability tests in Part I of the thesis.

The remaining tests reflect the fact that, in some cases, the presence of *nonlinear* structure may be of particular importance, either because it may influence the type of estimator which will be employed in the modelling process proper, or, as in the case of many financial time-series, because there is an underlying assumption that the linear component of any relationships will already have been "arbitraged away" by market participants. Thus whilst the basic tests treat linear and nonlinear components alike, we define modified test procedures in order to explicitly identify nonlinear structure over and above any linear component. The test then becomes a partial F-test in which the variance explained and degrees of freedom absorbed by the nonlinear models are compared to the equivalent figures for the linear estimator. The F-test is then conducted in terms of the *additional* variance which is explained by the nonlinear model, and the *additional* degrees of freedom which are required in order to do so.

### Empirical evaluation of variable selection properties through Monte Carlo simulation

The empirical properties of the variable selection methodology were tested by means of extensive Monte-Carlo simulations. The objective of the evaluation was twofold: firstly to quantify the ability to *detect* significant deterministic components and secondly to quantify the **bias** of the different model types in terms of the extent to which they fail to *represent* the underlying relationship in a manner which generalises to additional out-of-sample data.

In order to evaluate the flexibility of the variable selection procedure, a range of alternative hypotheses are investigated, corresponding to the functional forms of the five different model

[18] The exact magnitude of the effect will depend upon the correlation between the individual p-values derived from the five different types of estimator (e.g. taking the maximum of 5 identical tests would not produce a data snooping effect)

families specified in Table 9.1. Thus, in conducting a given simulation run, the deterministic component of the target data is constructed by taking a model of the appropriate class, $g \in \{L, P, B, R, M\}$, drawing the parameters at random from a (normal) distribution, and applying the model thus generated to random input-vectors $\left(x_{i,1}, x_{i,2}, \ldots, x_{i,nd}\right)$ drawn from a multivariate normal distribution.

The target data is then constructed from a weighted combination of the (normalised) deterministic component $g\left(x_{i,1}, x_{i,2}, \ldots, x_{i,nd}\right)$ and a stochastic component drawn from a standard normal distribution. The resulting generating equation is given by:

$$y\left(x_{i,1},\ldots, x_{i,nd}\right) = \sqrt{d}\ \frac{g\left(x_{i,1},\ldots, x_{i,nd}\right) - n^{-1} \sum\limits_{i=1\ldots n} g\left(x_{i,1},\ldots, x_{i,nd}\right)}{n^{-1} \sum\limits_{i=1\ldots n} \left[ g\left(x_{i,1},\ldots, x_{i,nd}\right) - n^{-1} \sum\limits_{i=1\ldots n} g\left(x_{i,1},\ldots, x_{i,nd}\right) \right]^2} + \sqrt{(1-d)}\ N(0,1) \tag{9.20}$$

The parameter $d$ controls the degree of determinism: with $d$=1 the relationship is purely deterministic; with $d=0$ the relationship is purely stochastic; for 0 < $d$ < 1 the relationship is such that a fraction $d$ of the variance of $y$ is deterministic and fraction (1-$d$) is stochastic.

The results of the Monte Carlo experiments are described in the following two sections. Section 9.3 describes the results of the evaluation of the statistical *power* of the variable selection tests. Section 9.4 describes the result of the evaluation of the *bias* of the families of estimator.

## 9.3 Evaluation of Variable Selection Test Power by Monte Carlo Simulation

The effectiveness of the variable selection methodology was evaluated in terms of the *power* of the tests, i.e. the probability that they will correctly identify the presence of a deterministic component in the data. The flexibility of the tests was investigated by considering the five different “alternative hypotheses” which are represented by the five families of model which are presented in Table 9.1. In order to investigate the sensitivity to noise, three cases were considered in which the magnitude of the predictable component was set to $d$=33%, 10% and 3.3%. The input dimensionality ($nd$) was set to two, reflecting both the prior assumption of low dimensional relationships and a recognition of the poor scaling properties of some of the model families (R, B and P).

For each predictability level, 100 realisations of each of the five model types were used to generate a target time-series, and for each such time-series an estimator was constructed from each of the different model families. The total number of models estimated was thus 3*5*5*100 = 7500. The length of each time-series was chosen to be 400 observations as being representative of the typical sample-size available for modelling asset prices on a daily basis.

### Single-estimator Tests

Table 9.2 presents a summary of the results of the basic form of the variable selection methodology, i.e. *single-estimator* tests in which a given type of estimator is applied to the data and the F-ratio of the model is tested for statistical significance .

| | | *Estimator Model Type* | | | | |
|---|---|---|---|---|---|---|
| *Target Model Type* | *d* | L | B | P | R | M |
| Linear (L) | 0.33 | 100% | 100% | 100% | 100% | 100% |
| | 0.1 | 99% | 95% | 96% | 99% | 99% |
| | 0.033 | 55% | 22% | 36% | 48% | 54% |
| Bin-smoother (B) | 0.33 | 67% | 100% | 96% | 99% | 100% |
| | 0.1 | 22% | 99% | 46% | 64% | 56% |
| | 0.033 | 4% | 46% | 6% | 14% | 11% |
| Polynomial (P) | 0.33 | 96% | 89% | 100% | 100% | 100% |
| | 0.1 | 79% | 45% | 93% | 86% | 92% |
| | 0.033 | 25% | 8% | 42% | 33% | 36% |
| RBF (R) | 0.33 | 95% | 100% | 100% | 100% | 100% |
| | 0.1 | 83% | 81% | 99% | 100% | 96% |
| | 0.033 | 38% | 29% | 44% | 59% | 50% |
| MLP (M) | 0.33 | 98% | 98% | 100% | 100% | 100% |
| | 0.1 | 82% | 82% | 94% | 95% | 96% |
| | 0.033 | 29% | 29% | 23% | 36% | 38% |
| Average | 0.33 | 91% | 97% | 99% | 100% | 100% |
| | 0.1 | 73% | 80% | 86% | 89% | 88% |
| | 0.033 | 30% | 27% | 30% | 38% | 38% |
| Overall | | 65% | 68% | 72% | 76% | 75% |

Table 9.2: Power of the variable significance tests at different levels of predictability '*d*', broken down by type of basis function used in the estimator, and also the nature of the underlying relationship. The nominal size or "false positive rate" of the F-test for model significance is 1%, indicating that in the absence of any true detection capability the test would be expected to indicate a significant result in 1 case out of 100.

When the deterministic component represents 33% of the total variance (*d*=0.33) the majority of the tests are 100% effective. The only substantial deviation from this pattern is the 67% obtained by the linear estimator when the deterministic relationship is generated by a bin-smoother model – highlighting the differences in the underlying assumptions of these two families of model.

When the potential predictability level falls to 10% the power of the tests is degraded somewhat and a clustering of the results begins to appear. The relatively inflexible linear estimator is the most affected, with an average power of only 73%, being sensitive only to the *linear* component in any deterministic relationship. The smooth nonlinear estimators {P, R, M} exhibit power of 85%+ with respect to each other but are much less efficient at detecting the relationships generated by bin-smoother models. Similarly, the bin-smoother is 99% effective at detecting relationships of the corresponding type, but only 45-82% effective at detecting *smooth* nonlinear relationships.

In the third scenario, where the potentially predictable component represents only 0.033 of the total variance of the target series, the power of the tests drops off noticeably. The three clusters {L}, {P, R, M} and {B} are still apparent, but the *average* power of the estimators across all types of relationship is roughly comparable. The most powerful tests overall are the **neural network** families R and M.

On a general level the results indicate that, for sample sizes in the order of a few hundred observations, it is relatively easy to *detect* significant relationships when they account for as much as 0.33 of the total variance of the target time-series. When the magnitude of the predictable component is of the order of 0.10 it is still possible to identify relationships with 95% plus effectiveness, but typically only when the nature of the estimator closely reflects that of the underlying relationship. In cases where the predictable component is of the order of 0.033 the situation is much more difficult, with an average identification rate of only around 30%.

### Joint Tests and the effect of the Significance Level

Table 9.3 presents the results of the *joint tests* which are conducted by applying all five types of estimator to the data, and taking the most significant statistic as an indicator of the presence of a deterministic relationship. The table also presents the results of increasing the test sensitivity in another manner, by increasing the significance level (p-value) which represents the acceptable “false positive” rate of the test.

| d | p-value | L | B | P | R | M |
|---|---|---|---|---|---|---|
| 0.33 | 0.01 | 100% | 100% | 100% | 100% | 100% |
| 0.1 | 0.01 | 99% | 99% | 95% | 100% | 97% |
|  | 0.05 | 100% | 100% | 100% | 100% | 100% |
| 0.033 | 0.01 | 61% | 48% | 51% | 67% | 49% |
|  | 0.05 | 82% | 75% | 74% | 86% | 73% |
|  | 0.1 | 91% | 82% | 85% | 92% | 86% |

Table 9.3: Power of the joint variable significance test in which all five estimators are used and the **most significant** result taken. Each column corresponds to a different type of underlying relationship. (alternative hypothesis) The table also shows the effect of relaxing the significance requirement by increasing the size (false positive rate) of the test.

As expected, the joint test is substantially more powerful than the single-estimator tests, particularly for the lower levels of predictability. In the case $d$=0.33 there is no need to increase the p-value as all relationships are detected successfully even at p = 0.01. At $d$=0.10, relaxing the p-value from 0.01 to 0.05 marginally improves the results from the already quite impressive 95% plus to the maximum possible 100%. In the high noise case, $d$=0.033, relaxing the p-value from 0.01 to 0.05 increases the detection rate from the range 48%-67% depending on the nature of the relationship, to a range of 73% to 86%. A further relaxation to p=0.10 improves the detection rate slightly, to a range of 82%-92%, but at this point the marginal improvements appear unjustified by the increase in false positives which they would cause.

### Explicit detection of nonlinear structure

Table 9.4 presents the power of the tests when applied in the context of detecting explicitly nonlinear structure which is over and above any linear component. As noted in Section 9.2, this test is in the form of a partial F-test which relates the *additional* variance which is explained by the nonlinear model to the *additional* degrees of freedom which it contains.

| Nature of relationship | d | B | P | R | M | Joint |
|---|---|---|---|---|---|---|
| Linear (L) | 0.33 | 0% | 0% | 1% | 2% | 2% |
|  | 0.1 | 0% | 2% | 3% | 6% | 7% |
|  | 0.033 | 0% | 2% | 4% | 7% | 11% |
| Bin-smoother (B) | 0.33 | 100% | 89% | 99% | 97% | 100% |
|  | 0.1 | 97% | 38% | 68% | 53% | 97% |
|  | 0.033 | 54% | 13% | 26% | 17% | 58% |
| Polynomial (P) | 0.33 | 7% | 100% | 60% | 90% | 100% |
|  | 0.1 | 7% | 80% | 42% | 59% | 82% |
|  | 0.033 | 5% | 31% | 21% | 24% | 41% |
| RBF (R) | 0.33 | 26% | 72% | 83% | 76% | 84% |
|  | 0.1 | 12% | 52% | 64% | 59% | 70% |
|  | 0.033 | 5% | 27% | 40% | 31% | 47% |
| MLP (M) | 0.33 | 41% | 89% | 94% | 97% | 99% |
|  | 0.1 | 20% | 45% | 68% | 74% | 81% |
|  | 0.033 | 10% | 13% | 30% | 22% | 38% |

Table 9.4: Power of the variable significance tests to explicitly identify *additional* nonlinear structure over and above any linear component. Column ‘d’ indicates the magnitude of the deterministic component of the data. Columns B, P, R and M indicate the power of the single-estimator tests. The final column indicates the power of the **joint** test based upon the most significant of the four estimators.

The results indicate that the ability to detect nonlinear structure varies widely depending on (a) the nature of the underlying relationship (b) the type of estimator used and (c) the magnitude of the predictable component.

When the underlying relationship is linear, any indication of nonlinearity is a “false positive” and gives an indication of the true **size** of the test. In practice the figures range from 0 to 11 out of the sample of 100, thus demonstrating that the *actual* size of the test can be much larger than the *nominal* size of 1%. In the worst case, for the joint statistic, the size of the test is inflated by the “data snooping” effect. A secondary pattern is the general trend for the false positive rate to increase as the magnitude of the predictable component *d* decreases, indicating that at low levels of predictability it is particularly difficult to distinguish between linear and nonlinear effects.

When the underlying relationship is generated by a bin-smoother model, the detection of nonlinear components is particularly high. The most effective case is the correctly specified one in which the estimator is also a bin-smoother, with detection rates of 100%, 97% and 54%

for *d* values of 0.33, 0.1 and 0.033 respectively. The neural network model types R and M perform only slightly worse and the polynomial model is least effective at detecting this type of relationship. For the polynomial relationship the well-specified model performs best with rates of 100%, 80% and 31%, the neural network models perform moderately, and the bin-smoother is particularly ineffective. These results suggest that the underlying biases of the polynomial and bin-smoother models are particularly incompatible, and that where they do coincide in detecting structure the most likely reason is that they have both identified a significant *linear* relationship in the data.

Finally, in the case of general, smooth, nonlinear relationships the two types of neural network model perform similarly well, and substantially better than both the polynomial and, least effective of all, the bin-smoother. Finally the overall detection rate is highest for the “Joint” statistic; as mentioned above, however, this improved performance does come at the price of a slight increase in the corresponding false positive rate.

### Conclusion

The conclusion of this analysis is that the F-test methodology can serve as a suitable tool for detecting both general relationships and also explicitly nonlinear relationships. The “detection rate” varies greatly as a function of the magnitude of the deterministic component, from generally close to 100% when the potentially predictable component accounts for a third of the total variance ($d$=0.33), reducing to around 90% at $d$=0.1 and around 40% at $d$=0.033. This suggests that for datasets of a typical size of a few hundred observations then the practical limit of effectiveness for forecasting models is when the potentially predictable component accounts for somewhere around this figure of 0.033 of the total variance, i.e. a signal to noise ratio of approximately 1/33. Furthermore, in cases close to this borderline it may be preferable to increase the p-value in acknowledgement of the uncertainty in the variable selection procedure, and to err on the side of caution even at the price of risking the inclusion of insignificant variables in the model estimation procedure proper.

## 9.4 Investigation of Model Bias using Monte Carlo Simulation

The results of the simulation experiments described in the previous section, to evaluate the **power** of the variable selection methodology, can also be used to provide valuable information regarding the relative **biases** of the different types of estimator.

The results in the previous section are primarily concerned with the *detection* of statistically significant relationships, rather than the extent to which the estimator which is produced as a means to this end accurately *represents* the underlying relationship. While this issue is more appropriate to the model estimation stage proper, as described in Chapter 10, it is in any case useful to understand the properties of the different families of estimator. In particular the ability of the different types of estimator to accurately capture low-dimensional components of the underlying relationship may have particular relevance to the design and implementation of a **constructive** approach to neural model estimation. The reason for this is that the objective of a constructive algorithm is to represent the broader relationship as an additive combination of a number of simpler submodels, and clearly the properties of the individual submodels will influence the overall performance of the algorithm.

The basis of our approach to predictive modelling is to jointly optimise the bias-variance tradeoff by the use of appropriate assumptions and modelling procedures. Furthermore, we have shown that the model variance is intimately linked to the concept of model complexity, in particular as represented by the degrees of freedom which are absorbed by the model. There is no general method of estimating model bias, however, because this is conditional both on the form of the estimator and on the form of the (unknown) generating process (see also Section 8.2). From this perspective the Monte-Carlo experiments, across a range of both underlying generators (alternative hypotheses) and types of estimator, present a valuable source of information in assessing the bias properties of the estimators under modelling conditions which are close to those which might be expected in practice.

However, given that in these controlled experiments we either know or can calculate both the magnitude of the deterministic component and the performance degradation due to model variance, we can define the **effective model bias** as the value of $b$ which equates the observed performance with the expected performance (as defined by the relationship in Eqn. (8.3). The effective model bias is the average proportion of the predictable variance which is left uncaptured due to discrepancies between the true model and the assumed form of the estimator.

$$
\begin{aligned}
bias_{eff} &= \mathrm{E}\left[1-\frac{R^2{}_{actual}}{R^2{}_{theoretical}}\right] = \mathrm{E}\left[1-\frac{1-{}^{rss}\!/_{tss}}{d-{}^{m}\!/_{n}}\right] \\
&= \mathrm{E}\left[1-\frac{1-\sum\limits_{outsample}\left(y-\hat{y}\right)\Big/\sum\limits_{outsample}\left(y-\bar{y}\right)}{d-{}^{m}\!/_{n}}\right]
\end{aligned}
\qquad (9.21)
$$

For instance a model with an empirical out-of-sample R$^2$ of 5%, in a case where the target data is known to contain a potentially predictable component equivalent to 10% of the total data variance, would have a bias of 1-0.05/0.1 = 50%. Note from the equation that the magnitude of the deterministic component must first be adjusted for model variance effects based on the model degrees of freedom $m$ relative to the sample size $n$, before the effective model bias can be computed.

Table 9.5 below presents a summary of the results of the analysis of effective bias for the five different model types which are listed in Table 9.1. The table shows the average degrees of freedom of the estimated models, the generalisation performance in terms of out-of-sample R$^2$, and the resulting bias. Target time-series are generated according to the procedure described in Section 9.2, with two input variables ($nd$ = 2). In all cases except the linear model, the complexity parameter $c$ is chosen in order to produce models with approximately 10 degrees of freedom.

The results are particularly interesting when compared with Table 9.2. For instance, for generators of type R and $d$=0.33, Table 9.2 reports a 95% detection rate for the linear estimator (L), however the corresponding generalisation performance is only 20.1% compared to the ideal benchmark of 33%. This behaviour is not as paradoxical as it might at first appear. The interpretation of this result is that in 95% of cases, the underlying relationship contains a significant linear component but that this linear component only accounts for approximately two-thirds of the total predictable component. In essence, the underlying biases of the linear estimator may mean that although it can *detect* the presence of a predictable component, it is not, in general, able to *represent* or model this component with any great accuracy. In this particular case we can consult the table to note that in the case of a linear estimator versus RBF generator, the **effective bias** turns out to be 37%.

|  | Estimator D.o.f. |  |  |  |  | Estimator Generalisation |  |  |  |  | Estimator Bias |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | L | B | P | R | M | L | B | P | R | M | L | B | P | R | M |
| d=0.33 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| L | 2.97 | 7.68 | 9.78 | 6.22 | 4.80 | 32.8% | 22.9% | 31.5% | 31.1% | 32.5% | 0% | 25% | 0% | 2% | 0% |
| B | 2.97 | 7.68 | 9.77 | 6.22 | 10.69 | 5.2% | 31.5% | 9.1% | 11.9% | 16.3% | 82% | 0% | 65% | 59% | 42% |
| P | 2.97 | 7.68 | 9.78 | 6.22 | 8.58 | 14.5% | 7.6% | 31.1% | 17.2% | 24.2% | 53% | 71% | 0% | 43% | 20% |
| R | 2.97 | 7.68 | 9.77 | 6.22 | 8.64 | 20.1% | 18.5% | 31.0% | 32.1% | 29.6% | 37% | 38% | 0% | 0% | 3% |
| M | 2.97 | 7.68 | 9.77 | 6.22 | 7.72 | 18.3% | 19.7% | 23.0% | 24.3% | 27.7% | 42% | 34% | 22% | 21% | 9% |
| d=0.10 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| L | 2.97 | 7.68 | 9.78 | 6.22 | 4.64 | 9.6% | 6.0% | 7.9% | 8.8% | 9.4% | 0% | 22% | 0% | 0% | 0% |
| B | 2.97 | 7.68 | 9.77 | 6.22 | 8.41 | 1.2% | 8.5% | 0.7% | 2.8% | 3.2% | 80% | 0% | 68% | 56% | 47% |
| P | 2.97 | 7.68 | 9.78 | 6.22 | 6.64 | 4.0% | 1.2% | 7.6% | 4.3% | 5.8% | 51% | 68% | 0% | 40% | 24% |
| R | 2.97 | 7.68 | 9.77 | 6.22 | 6.92 | 5.8% | 4.5% | 7.6% | 9.0% | 8.0% | 34% | 36% | 0% | 0% | 2% |
| M | 2.97 | 7.68 | 9.77 | 6.22 | 6.61 | 5.1% | 4.7% | 4.9% | 6.4% | 7.0% | 40% | 32% | 25% | 18% | 11% |
| d=0.033 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| L | 2.97 | 7.68 | 9.78 | 6.22 | 4.13 | 2.9% | 1.1% | 1.0% | 2.2% | 2.7% | 0% | 12% | 0% | 0% | 0% |
| B | 2.97 | 7.68 | 9.77 | 6.22 | 5.49 | 0.1% | 1.8% | -1.7% | 0.2% | 0.3% | 75% | 0% | 77% | 48% | 49% |
| P | 2.97 | 7.68 | 9.78 | 6.22 | 5.18 | 1.0% | -0.7% | 0.8% | 0.6% | 1.2% | 46% | 60% | 1% | 32% | 24% |
| R | 2.97 | 7.68 | 9.77 | 6.22 | 5.69 | 1.6% | 0.4% | 0.8% | 2.2% | 2.1% | 27% | 30% | 0% | 0% | 0% |
| M | 2.97 | 7.68 | 9.77 | 6.22 | 5.68 | 1.3% | 0.4% | -0.3% | 1.3% | 1.5% | 35% | 26% | 31% | 11% | 8% |
| Average |  |  |  |  |  |  |  |  |  |  | 40% | 30% | 19% | 22% | 16% |

Table 9.5: Analysis of degrees of freedom, out-of-sample **generalisation**, and effective model bias. Values are calculated as averages over 100 Monte Carlo runs, each with sample sizes of 400 and 1000 observations for estimation and generalisation set respectively. The parameter *d* indicates the magnitude of the deterministic component.

Remaining with the same underlying generator, we can further see that the bin-smoother model (B) has a very similar level of effective bias (38%) to the linear model, corresponding to an average generalisation ability of only 18.5% and a correction for the slight increase in the average degrees of freedom, and therefore model variance, which are associated with the bin-smoother model. Meanwhile, the three smooth nonlinear estimators of types P (polynomial), R (RBF) and M (MLP) achieve close-to-optimal generalisation performance of 31.0%, 32.1% and 29.6%, indicating biases of 0%, 0% and 3% respectively.

In general, the effective bias results are very similar at all three levels of predictability, a finding which is in contrast to the variable significance tests which are adversely affected by higher levels of noise. As expected, the linear model (L) tends to suffer from quite high bias.

This supports the view that whilst many of the models may contain a linear component, they also contain nonlinear and interaction effects which cannot be adequately captured by the highly-biased model. The bin-smoother (B) also suffers from relatively high bias, but in this case the effect is symmetric in that the other model types suffer from large biases when the underlying model type is a bin-smoother. This result emphasises the different underlying assumptions of smoothly changing functions on the one-hand and piecewise-constant but globally-discontinuous functions on the other hand.

The three model types {P, R, M} which represent smooth nonlinear relationships suffer from only mild bias with respect to each other. Amongst the three, the MLP model appears to be most flexible, even in the worst case suffering only a 24% bias. On the whole it appears that all three of these estimators can perform relatively well across a wide range of underlying generators of the deterministic component.

The bias figures for the MLP estimator, when correctly specified (i.e. applied to models generated from the same family), tend to be higher than might have been expected. The reason for this is the use of a regularisation term in our model estimation procedure, which introduces an additional bias in the estimator. In all models except type M the effect of the regularisation is independent of the degree of predictability and can thus be fine-tuned in advance to give models of the desired complexity. The adaptive basis functions used by the MLP neural network cause the degrees of freedom to vary substantially such that neural network model tends to employ more degrees of freedom when the deterministic component is larger and/or more complex. This adaptability means that the regularisation parameter is harder to fine-tune than for the fixed basis function models, creating the risk of a minor bias being introduced even when the model is otherwise well-specified.

Finally, the relatively low $R^2$ figures achieved in the $d$=0.033 case emphasise that at this low level of predictability, model variance becomes a significant influence on the results. A simple calculation suggests that in this scenario, even a completely unbiased model would have zero expected out-of-sample $R^2$ if the model complexity was as low as 0.033/400 = 13 parameters. This reinforces the fact that at low levels of potential predictability even the most principled of modelling approaches will be at the very limit of its effectiveness.

**Conclusion**

In comparison to the analysis in the previous section, the ability of the models to accurately *represent* the underlying relationship is clearly seen to be lower than their ability simply to *detect* the presence of a relationship, particularly in the case where the magnitude of the predictable component is low relative to that of the noise component. For small data sets of a few hundred observations, the effect of model variance alone will negate the predictive ability of moderately complex models (10-20 parameters) if the magnitude of the predictable component is much less than 5% of the total variance.

Defining "effective bias" as the average percentage of the predictable variance which is left uncaptured due to the assumed model being of a different family to the true relationship, we note that across the five model classes, both the linear model (L) and the bin-smoother (B) have biases of around 50% whilst the three smooth nonlinear model classes (P, R, M) are closer to 25%. This supports the view that in a case where the underlying relationship is of unknown form the best performance is likely to be achieved by using a smooth nonlinear estimator.

## 9.5 Summary

In this chapter we have described our statistical methodology for model-free variable selection. The empirical evaluation of the methodology suggests that under realistic modelling conditions the methodology is capable of identifying the presence of a wide range of potentially predictable components in the underlying data-generating process. However at the low signal-to-noise ratios which we expect to see in financial data it is easier to detect the presence of a deterministic component than it is to capture the form of the corresponding functional relationship. Both model bias and model variance will have a major impact on the average generalisation ability of a model and thus both flexibility and parsimony are important properties in an estimator.

In the following chapter we describe a development of the statistical testing methodology of this chapter to the problem of low-bias model estimation in the context of small and highly noisy data samples. The methodology is based upon the flexibility of neural network estimation, coupled with the parsimony which is provided by the use of statistical significance tests.

# *10. Statistical Methodology for Neural Network Model Estimation*

This chapter describes our methodology for the estimation of low-bias forecasting models. This task is performed through novel algorithms which balance the flexibility of *neural networks* with the noise-tolerance and diagnostic procedures of *statistical modelling*. Section 10.1 extends the statistical perspective of the previous chapters to form the basis of a rigorous modelling framework within which to automatically optimise the specification of neural network models. Within this framework we develop an integrated approach to the model estimation problem which combines the two tasks of variable selection and architecture selection. Section 10.2 describes three specific algorithms which share the underlying statistical methodology but approach the model estimation task from the different perspectives afforded by the constructive, deconstructive and regularisation-based approaches. Section 10.3 presents an experimental validation of the properties of the three algorithms under controlled circumstances. The results demonstrate the ability of the algorithms to estimate neural models with significant generalisation ability even when operating under conditions of high noise, small sample size and the presence of spurious variables.

## 10.1 Framework for Optimising the Bias-Variance Tradeoff in Neural Networks

Having performed any necessary variable selection, the next task is to approximate the deterministic component of the data in the form of a *predictive model*. Within this section we describe a statistical framework for neural network model estimation. The objective of our estimation procedures is to generate models which are neither too low in complexity (overly biased) nor too high in complexity (overly susceptible to variance) whilst simultaneously ensuring that the complexity is exploited in an appropriate manner. The essence of our methodology is to avoid the problem of excessive **bias** by using flexible neural network models and avoid excessive estimation **variance** by using statistical significance testing to control the specification of the neural network models.

Within the general formulation of the model estimation process which was presented in Section 8.1, the first stage, that of preselecting the information set upon which to condition the model, is addressed by the model-free variable selection methodology described in Chapter 9.

The methodology described in this chapter is designed to jointly address the second, third and fourth stages of the process, namely model specification, parameter estimation and diagnostic testing. Whilst certain aspects of the fifth stage (model selection) are also addressed in this chapter, a more general discussion and proposed solution to the issues involved in model selection is presented in Part III of the thesis.

**Bias-variance tradeoff**

In Section 8.2 we presented an idealised discussion of the particular difficulties which arise in the case of estimating financial forecasting models. We can consider the performance of a model in terms of the proportion of variance of the data which is captured in the model in a manner which will *generalise* to future observations. The maximum expected performance of a forecasting model was defined (Eqn (8.3)) in terms of the proportional magnitude $(d)$ of the deterministic component of the data, the *bias* $(b)$ which is imposed upon the modelling procedure in the form of incorrect modelling assumptions, and the *variance* $(v)$ which is caused by sampling error.

$$P_{EXP} = \mathrm{E}\left[R^2\right] = d(1-b) - v \tag{10.1}$$

The equivalent kernels perspective of section 8.3 allows us to quantify the model variance which is induced by the parameter estimation process in terms of the “effective number of parameters” $p_{eff}$ in the model as a ratio of the sample size $n$.

$$v = (1-d)\frac{p_{eff}}{n} \tag{10.2}$$

This definition of model variance allows us to define the “effective bias” of the estimator as the value $b$ which equates the *actual* generalisation ability of the model to the *expected* performance given the known level of predictability $d$, “degrees of freedom” $p_{eff}$, and sample size $n$. The simulation experiments of Section 9.4 indicate that under realistic circumstances, and averaging over a wide range of data-generating processes, the average level of “effective bias” is likely to be of the order of 25% for smooth nonlinear models and around 50% for less flexible models. The results in Section 8.4 also highlight the difficulty of the task of achieving positive generalisation ability in cases where the sample size is small (in this case 400 observations) and the data is highly noisy (deterministic component of the data accounting for 33%, 10% and 3.3% of the data variance).

The basis of our model estimation framework is to explicitly optimise the tradeoff between model bias and model variance by varying the complexity of the model. Increasing the effective number of parameters will increase model variance in a *linear* manner, but reduce model bias in a *nonlinear* manner with an decreasing marginal effect as the effective number of parameters grows. This phenomenon is illustrated in Figure 10.1 below, which illustrates in idealised form, the growth in both model variance and the ability to capture the structural component of the data, as the degrees of freedom in the model increase:

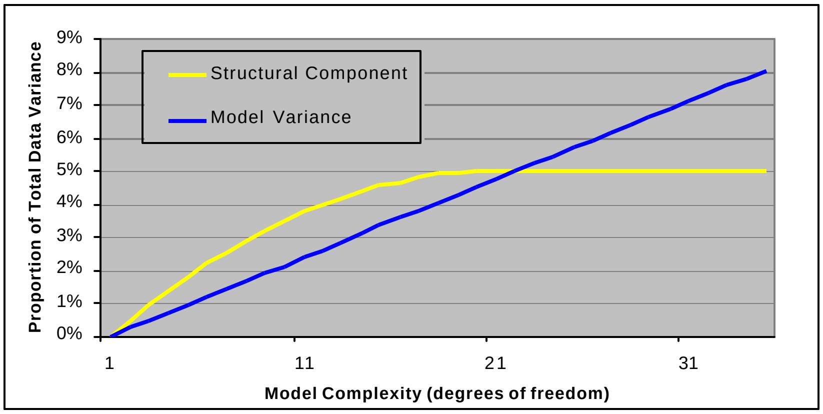


Figure 10.1: Idealised illustration of the manner in which additional model degrees of freedom are allocated between the structural component in the data and idiosyncratic noise effects (model variance).

Initially the degrees of freedom are used to capture the major structural components of the data. Additional degrees of freedom will be of decreasing marginal benefit up to the point where the modelled structural component equates to the entire deterministic component of the data (5% in the example).

The decreasing marginal effect of additional model flexibility has important practical consequences for model estimation in high-noise problem domains such as financial forecasting. Whilst the additional degrees of freedom will always improve the model fit to the particular estimation sample, the same is **not** true of the generalisation ability of the model when applied to additional observations. The reason for this is that the extent to which the model flexibility is used to capture idiosyncratic noise within the data sample will necessarily *reduce* the apparent error of the model on the sample itself, but will actually serve to *increase* the model error with respect to another sample. A well-known result in nonparametric statistics tells us that whilst the insample errors are reduced by an amount equal to the model variance, the expected out-of-sample error is increased by the same amount, i.e.

$$\mathrm{E}_{INSAMPLE}\left[(y-\hat{y})^2\right] = s_{TRUE}{}^2\left(1-\frac{p_{eff}}{n}\right) \tag{10.3a}$$

$$\mathrm{E}_{OUTSAMPLE}\left[(y-\hat{y})^2\right] = s_{TRUE}{}^2\left(1+\frac{p_{eff}}{n}\right) \tag{10.3b}$$

where $s_{TRUE}{}^2$ is the actual noise content of the data. A detailed derivation and discussion of these results is contained in Chapters 5 and 6 of Hastie and Tibshirani (1990); for the equivalent analysis for neural network modelling see for example Amari (1995).

The tradeoff between bias and variance is the basis of almost all statistical model selection criteria. For instance the purpose of the "adjusted $R^2$" measure in regression analysis (e.g. Weisberg, 1985) is to subtract the component of the insample performance which is simply due to model variance, thus allowing models to be compared in terms of an *unbiased* measure of insample performance. When the objective of the modelling procedure is to optimise the expected generalisation performance, it can be seen from Eqns. (10.3a,b) that it is appropriate to penalise model variance *twice,* which is the approach implicit in measures such as the Akaike Information Criterion (AIC) [Akaike, 1973, 1974b], the Generalised Prediction Error (GPE) [Moody, 1992], and the Network Information Criterion (NIC) [Murata et al, 1993]. The motivation for this double penalisation is illustrated in the comparison of insample model fit and generalisation performance which is shown in Figure 10.2 below.

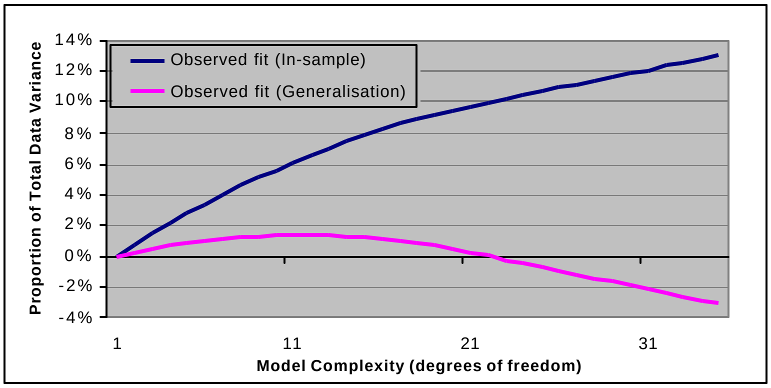


Figure 10.2: Illustration of the comparison between insample model fit and generalisation ability for a model with bias/variance properties corresponding to those shown in Figure 10.1. The insample fit equals the structural component captured by the model *plus* the model variance; the out-of-sample generalisation ability equals the structural component *minus* the model variance.

However, these "variance correction" criteria are not strictly suitable for use within the modelling process because they fail to take into account the effect of sampling error on the selection criterion itself. As discussed in Section 8.2 this drawback is not critical in cases where the sample size is large and/or noise content is low, and hence has almost entirely been overlooked in the neural network literature – perhaps because in such cases the added-value of the variance correction itself is probably questionable. In contrast, standard statistical procedures such as stepwise regression tend to adopt an approach which at least partially accounts for the unsuitability of model selection criteria for use *within* the model estimation process.

The solution to the problem of sampling error in the model selection criteria is simply to take into account the probability distribution of the test statistics, and to test not only for a "high enough" value but instead for a **statistically significant** value. Thus whilst the adjusted-$R^2$ correction is equivalent to requiring a performance increase over and above model variance ($\Delta fit > s^2 \Delta p$), and AIC is equivalent to requiring the increase in the structural component to be greater than the increase in model variance ($\Delta fit > 2s^2 \Delta p$), the partial F-testing methodology of stepwise regression requires that the increase in fit be statistically significant.

$$\frac{\Delta fit}{s^2 \Delta p_{eff}} > F^{crit(a)}_{\Delta p, n-p_{eff}} \qquad (10.4)$$

From this perspective the F-ratios of 1 and 2 required by adjusted $R^2$ and AIC respectively correspond to low levels of statistical significance. For example, with 1 additional degree of freedom in the model and 100 residual degrees of freedom, the adjusted $R^2$ threshold of 1 corresponds to a significance level for the standard $F_{1,100}$ distribution of 0.32, the AIC threshold of 2 corresponds to a significance level of 0.16 and the actual threshold required for significance at the 0.01 level would be 6.89.

Thus whilst the standard model selection criteria remain appropriate in the case of one-off *selection*, they are **not** suitable for use in the repeated testing involved in the model estimation process, which consists of an iterative process of specification/parameter estimation/diagnostic testing/reformulation. The low significance levels implicit in such methods will tend to exacerbate any "data snooping"/selection biases which are inherent in the model construction process and tend to result in spuriously overparameterised models. This recognition may well account for the fact that the more-parsimonious Bayesian or Schartz Information Criterion (B/SIC) has been found to produce superior empirical performance to the AIC (see Diebold, (1998), page 91), in spite of the fact that AIC is theoretically an asymptotically efficient criterion.

In the context of the small sample sizes and high noise levels of financial forecasting the effects of model variance can be particularly debilitating, particularly when exacerbated by data-snooping effects. For this reason we base our methodology for neural model estimation upon a similar partial-F testing approach as that used in stepwise regression and in the "neglected nonlinearity" tests of Lee et al (1993). This methodology can also be viewed as a natural extension of the F-ratio approach which we use as the basis of the model-free variable significance tests which are described in Chapter 9. Preliminary versions of this methodology have been reported in Burgess (1995, 1998) and Burgess and Pandelidaki (1996, 1998).

### Variable Selection and allocation of model complexity

It is rarely recognised that the *allocation* of complexity within a model plays an equally important role as the optimisation of the overall *level* of complexity. In particular, the level of performance achievable by a neural network or any other model will be heavily dependent upon the choice of explanatory (input) variables within the model and the allocation of degrees of freedom to the different variables.

The complexity of the model is determined by a combination of two factors, namely the parameterisation of the model, referred to as the neural network "architecture", and the parameter estimation process. In particular, we focus on projective basis function networks of the form:

$$\mathrm{M}\left(x_1, x_2, \ldots, x_{nd}\right) = a + \sum_{i=1}^{c} b_i \tanh\left(b_i + \sum_{j=1}^{nd} w_{i,j} x_j\right) \qquad (10.5)$$

The **maximum** degrees of freedom within such a model is equal to the number of free parameters $(nd+2)c+1$, however the **effective** number of parameters is reduced from this maximum by the inclusion of a "weight decay" or regularisation term in the parameter estimation procedure, which modifies the basic mean-squared error criterion to give a loss function of the form:

$$L = \frac{1}{n} \sum_{i=1..n} \left(y_i - \hat{y}\left(\mathbf{x}_i\right)\right)^2 + r \sum_{j=1..m} w_j^2 \qquad (10.6)$$

An illustration of the effect of varying the weight decay parameter, $r$ , on the effective degrees of freedom in a neural network is presented in Figure 8.3. However, this approach is a global approach to optimising model complexity which fails to take into account any local variations in the reliability or complexity of the relationships between the input variables and the dependent variable. The allocation of model complexity can be thought of as a "second dimension" to the model optimisation process, as illustrated in Figure 10.3.

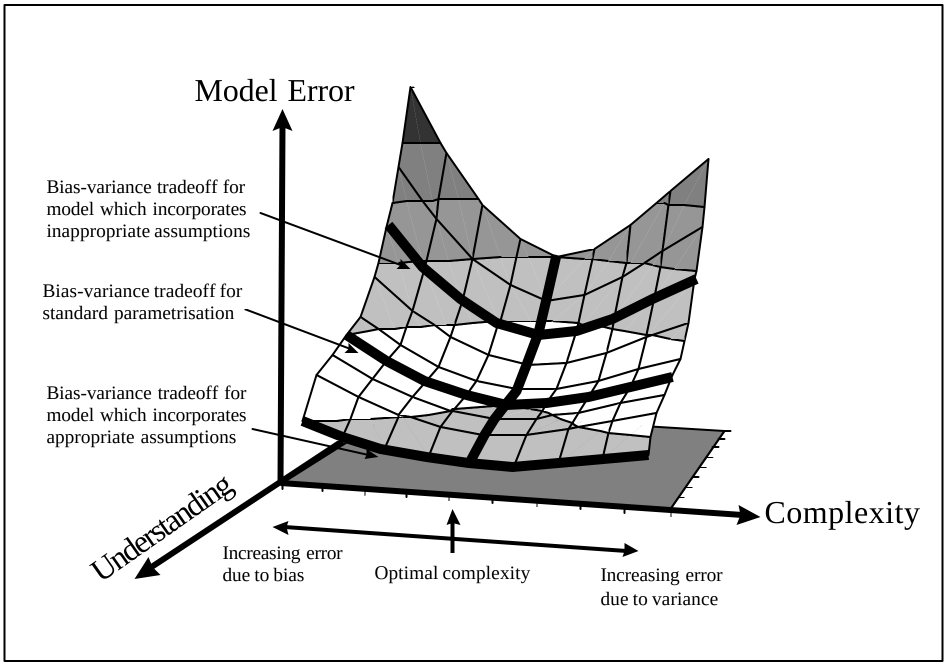

Model Error
Bias-variance tradeoff for model which incorporates inappropriate assumptions
Bias-variance tradeoff for standard parametrisation
Bias-variance tradeoff for model which incorporates appropriate assumptions
Understanding
Complexity
Increasing error due to bias
Optimal complexity
Increasing error due to variance

Figure 10.3: Illustration of the multi-dimensional nature of the model optimisation problem. Whilst optimising the bias-variance tradeoff is a central element in model estimation, the assumptions which are embodied within the modelling process are equally important in that they determine whether the degrees of freedom are efficiently allocated within the model.

Figure 10.3 illustrates that the overall “baseline” performance of a model can be degraded by imposing *inappropriate* assumptions, such as that all variables are equally important. Similarly the overall performance can be improved (model error decreased) by imposing *appropriate* assumptions, such as that relationships are both smooth and relatively low dimensional.

Within our methodology we treat **variable selection** as an important element of the modelling process which is placed on an equal footing with the optimisation of model complexity. It is clear that incorrectly omitting variables will entail that a model will be unable to capture that part of the predictable component of the target series dynamics which is conditioned on the omitted variable. What is less obvious is that the inclusion of spurious or nuisance variables, which are unrelated to the target time-series, can also have a significant effect on model performance by increasing the amount of model variance and thus degrading model performance. As such nuisance variables contribute nothing to the representational ability of the model, their elimination represents a “free lunch” (Wolpert, 1992) in the sense of reducing model variance at no cost in terms of model bias.

In particular, the regularisation-based approach to optimising model complexity is found to be an inefficient method of allocating degrees of freedom in the case where some of the input variables are merely spurious "noise" variables which have no relationship to the dependent variable. Figure 10.4 illustrates the results of a Monte-Carlo simulation experiment in which a varying number of spurious variables are included in the modelling process. The underlying data generating process is 90% stochastic noise and 10% deterministic, being generated from a univariate neural network with three hidden units.

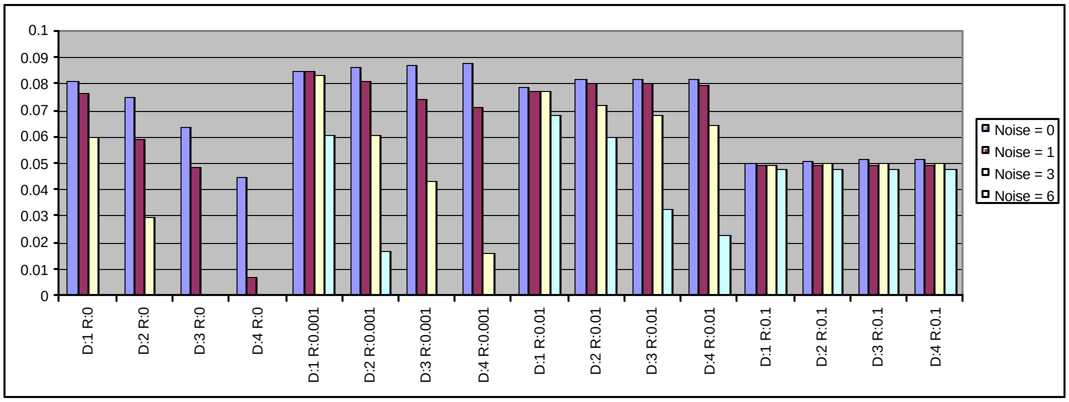


Figure 10.4: Generalisation ability as a function of the number of noise variables, neural architecture specification and the degree of regularisation. The underlying model is a univariate neural network with three hidden units and accounts for 10% of the total variance. The notation D:x R:y corresponds to the *median* performance of the neural network with x hidden units, and regularisation level y, averaged over 100 Monte-Carlo realisations, each consisting of 400 observations insample, and 1000 out-of-sample.

In these results, the absence of regularisation (cases where R=0) can be considered as a failure to incorporate the appropriate assumption that the estimated relationships should be smooth. In these cases the model performance is heavily degraded by model variance even in the case where there are no noise variables. With R=0.001 the variance is controlled and the optimal performance is achieved by the correctly specified model with 3 hidden units, however the performance is still heavily degraded when spurious variables are present. It appears that in order to compensate for the presence of noise variables on this *global* basis (i.e. not through explicit variable selection) it is necessary to impose substantial bias to the model both in the form of regularisation and in the form of underparameterisation. For instance, the optimal generalisation performance in the presence of 6 noise variables is achieved by models with only one hidden unit instead of the correct specification of three. Very high levels of regularisation can be seen to over-penalise complexity with the associated models becoming insensitive to the number of hidden units D and clearly capturing only the linear component of the relationship.

These results motivate the use of algorithms which can perform the joint tasks of variable selection and complexity optimisation in a localised manner, with an explicit allocation of complexity to different input variables. In the following section we describe three alternative algorithms for neural network estimation which are designed for use in situations where spurious variables may be present, sample sizes are small and the magnitude of the predictable component is low.

## 10.2 Algorithms for Neural Network Model Estimation

In this section we present three algorithms which provide alternative methods for optimising the bias-variance tradeoff in neural estimation. Although equally applicable to cases where the deterministic component dominates the noise term, the algorithms are specifically designed for the context where spurious variables may be present and the magnitude of the predictable component in the data is relatively low (accounting for between 0 and 25% of the variance of the target variable).

Within this general perspective our methodology supports three alternative model estimation algorithms. The three algorithms share a basic testing methodology in which the statistical significance of model components is calculated by comparing the degrees of freedom which they absorb with the additional explanatory power which they provide. The basic selection criterion which we use is a partial F-test of the form:

$$F_i = \frac{\left(\sum\left(y-\hat{y}_a\right)^2 - \sum\left(y-\hat{y}_b\right)^2\right) \Big/ \left(\sum df_A - \sum df_B\right)}{\left(\sum\left(y-\overline{y}\right)^2 - \sum\left(y-\hat{y}_a\right)^2\right) \Big/ \left((n-1) - \sum df_B\right)} = \frac{\Delta RSS/\Delta df}{RSS\ / n - df} \tag{10.7}$$

where $\hat{y}_a$ is the estimator which consists of the set of components $A$ and $\hat{y}_b$ is the estimator which consists of the set of components $B$. The test compares the ratio of the *variance per degree of freedom* which is explained by the set of components $\left\{f_i : f_i \in A - B\right\}$ to the average residual variance (adjusted for both variance and degrees of freedom which are absorbed by the model). Under the null hypothesis that component $f_i$ is irrelevant then the statistic $F_i$ follows an F-distribution with $df_i$ degrees of freedom on the numerator and $n - \sum_k df_k$ degrees of freedom on the denominator.

This F-testing approach is the basis of common statistical tools such as “stepwise regression” (e.g. Weisberg, 1985). It has a similar motivation to the econometric tests for neglected nonlinearity used by Lee *et al* (1993) for selecting the number of hidden units in a neural network. The partial F-testing approach was first applied to neural network variable selection by (Burgess, 1995) where a heuristic method was used to estimate the number of degrees of freedom associated with each input variable. In the algorithms described below the degrees of freedom for a neural network model are calculated from the smoother matrix according to the methods described in Chapter 8.

The partial F-testing methodology can be used to optimise the bias-variance tradeoff by approaching the optimal level of complexity either from below or from above. Methods which approach the optimal complexity from below are referred to the neural network literature as “constructive” approaches in that they can be viewed as gradually building-up or “constructing” a model. By analogy, we choose to refer to the alternative approach, of searching for the optimal complexity from above, as the “deconstructive” approach. The constructive and deconstructive approaches to the optimisation of model complexity are illustrated in Figure 10.5 below.

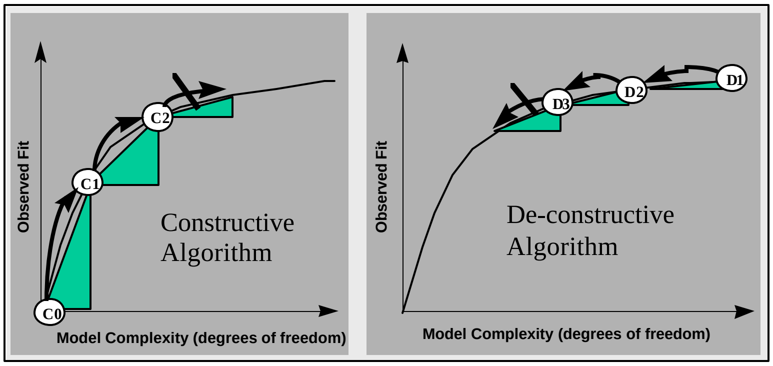

Observed Fit
C0
C1
C2
Constructive Algorithm
Model Complexity (degrees of freedom)
Observed Fit
D1
D2
D3
De-constructive Algorithm
Model Complexity (degrees of freedom)

Figure 10.5: The constructive and deconstructive approaches to optimising model complexity. The **constructive** approach starts with a null model which is successively *enhanced* whilst the additional components are found to be statistically *significant*. The **deconstructive** approach starts with an over-complex model which is successively refined by *removing components* which are found to be statistically *insignificant*.

We now describe our three algorithms for automated neural model estimation. The first algorithm is based upon the use of a regularisation term to control global model complexity.

The algorithm can be used in a "pseudo constructive" manner by using a high initial level of regularisation which is then relaxed in order to add model complexity. The "pseudo deconstructive" version of the algorithm uses an initially low degree of regularisation which is increased in order to remove complexity from the model. We refer to the algorithm as *pseudo* (de)constructive because although the model complexity will vary in terms of the degrees of freedom in the model, the actual parameterisation of the model remains constant throughout.

The other two algorithms combine the use of regularisation with the explicit addition and subtraction of model components. In particular these algorithms combine the tasks of complexity optimisation and variable selection. The first is a **constructive** algorithm which operates by *adding* model components, whilst the second is a **deconstructive** algorithm which operates by *removing* model components. In all cases the model estimation procedure is guided by the use of statistical significance tests based upon the partial-F test of Eqn. (10.7).

Detailed descriptions and discussions of the three algorithms are presented below.

### Algorithm 1: Complexity control using regularisation

The first algorithm is simply based on the concept of controlling the degrees of freedom within a model by varying the degree of **regularisation** which is used during the estimation procedure. This approach derives its inspiration from early work on "weight decay" (Hinton, 1987) and "weight elimination" (Weigend *et al*, 1990) in neural networks and the vast body of statistical literature on smoothing and regularisation (Titterington, 1985).

The algorithm combines the use of regularisation, in order to control the model complexity, with the partial-F testing methodology to determine whether the increased model complexity is exploited in a statistically significant manner. The "pseudo-constructive" form of the algorithm starts with a heavy regularisation term, and successively relaxes the regularisation whilst the additional degrees of freedom thus created are deemed to be statistically *significant*. The "pseudo-deconstructive" form of the algorithm starts with a low degree of regularisation which is successively increased whilst the degrees of freedom thus removed from the model are shown to be *insignificant*.

The algorithm might be considered a "global" algorithm in that all input variables are considered on an equal basis and any "variable selection" which might occur will be on an implicit basis, through competition for the degrees of freedom which are available to the model as a whole.

The model architecture is fixed, but the flexibility of the model is varied by changing the regularisation term. In the "pseudo-constructive" version of this algorithm, illustrated in Figure 10.6 below, the regularisation term is initially very strong, giving a model with low complexity. The regularisation term is then successively relaxed and the model re-estimated. This process is continued until an insignificant value for the change of model F-test indicates that the additional complexity is no longer justified.

```
initialise:  $bestmss = 0, bestmdof = 0, rss = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2, rdof = n-1$

For ( $r = r_{max}$ ; $r > r_{min}$ ; $r = r/ratio$ )

        train network: $q_{NN} = \min_q E_{NN} = \min_q\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i - \mathrm{NN}(\mathbf{x}_i, q)\right)^2 + r\sum_{j=1}^{m} q_j^2\right]$

        calculate smoother matrix: $\mathbf{S}_{NN} = \nabla_q \mathrm{NN}(\mathbf{X}, q)' \left[\nabla_q \nabla_q E_{NN}\right]^{-1} \nabla_q \mathrm{NN}(\mathbf{X}, q)$

        calculate degrees of freedom: $mdof = trace(\mathbf{S}_{NN}) - bestmdof$

        calculate variance explained: $mss = \left[\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 - \sum_{i=1}^{n}\left(y_i - \mathrm{NN}(\mathbf{x}_i, q)\right)^2\right] - bestmss$

        calculate partial-F:   $F = \frac{mss/mdof}{rss/rdof}$

        if (partial-F > $F_{thresh}$) then update model:
                $bestmss = mss, bestmdof = mdof$
                $rss = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 - mss, rdof = n - (mss + 1)$
Next
```

Figure 10.6: Details of the **regularisation-based** algorithm for neural model estimation.

A typical value for the acceptance threshold $F_{thresh}$ would be 4.0 - equating to a significance level (false positive rate) of approximately 1%.

The alternative "pseudo-deconstructive" version of this algorithm is obtained by reversing the direction of the search in "complexity space", i.e. the regularisation term is initially very small, giving a model with maximal complexity for the specified architecture, and then gradually *increased* until the change of model F-test indicates a statistically significant *reduction* in performance.

The key feature of this algorithm is that it allows the complexity of the model to be (approximately) matched to the complexity of the underlying data-generating process. Additional complexity is only accepted in the model if it is justified by a *statistically significant* improvement in the amount of variance explained by the model, with the statistical test used being a partial F-test. The major disadvantage of this approach is the lack of explicit *variable selection*; in cases where spurious explanatory variables are included in the model then at least some of the degrees of freedom will be allocated to these variables, increasing model variance without improving the representational ability of the model, and thus leading to unnecessary performance degradation. A discussion of this phenomenon was contained in the previous section following the results of the simulation experiment which are shown in Figure 10.4.

**Algorithm 2: Constructive Algorithm using Residual Testing: Neural Additive Model**

Our second algorithm employs a "**constructive**" approach to optimising both model complexity and variable selection. The algorithm starts with a null model which is successively enhanced by the addition of components to the model. This approach is based upon the "cascade correlation" algorithm of Fahlman and Lebiere (1990) but improves upon the earlier method by firstly using the principled methodology of the previous section to identify whether the residual structure is *statistically significant*, and secondly by only adding model complexity on a *local* rather than on a global basis. In other words, when additional complexity is included within the model, it is conditioned only upon the particular input subspace (set of independent variables) within which residual structure has been identified. In this way the algorithm combines the two tasks of variable selection and model specification.

The constructive algorithm starts with a null model which is successively enhanced by the addition of potentially-nonlinear components to the model. The additional components are selected on the basis of partial F-tests for significant low-dimensional relationships in the residual errors of the current model, as described in the model-free variable selection methodology in Chapter 9. The details of the constructive algorithm are presented in Figure 10.7.

```
Initialise null model: $\mathrm{q}_{\mathrm{NN}} = c, \forall : \hat{y}_i = \bar{y}, \quad rss = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2, rdof = n-1$
Repeat:
        $F_{max} = 0$
        For each subspace: $\mathbf{x}_{test} = x_i\left(\cup x_j\left(\cup x_j(\ldots)\right)\right)$
                estimate submodel of residual errors:
                    $\mathrm{q}_{\mathrm{TEST}} = \min_q E_{TEST} = \min_q\left[\frac{1}{2n}\sum_{i=1}^{n}\left((y_i - \hat{y}_i) - \mathrm{NN}(\mathbf{x}_{test}, \mathrm{q})\right)^2 + r\sum_{j=1}^{m} q_j^2\right]$
                calculate partial F:
                    $mdof = trace(\mathbf{S}_{\mathrm{TEST}})$
                    $mss = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 - \sum_{i=1}^{n}\left((y_i - \hat{y}_i) - \mathrm{NN}(\mathbf{x}_{test}, \mathrm{q}_{\mathbf{TEST}})\right)^2$
                    $F_{test} = {}^{mss/mdof}\!\Big/_{rss/rdof}$
                if ( $F_{test} > F_{max}$ ) then $F_{max} = F_{test}$ , $\mathrm{q}_{\mathbf{REF}} = \mathrm{q}_{\mathbf{TEST}}$
        Next
        if ($F_{max} > \mathrm{F}_{\mathrm{thresh}}$) then supplement model and continue:
            $\mathrm{q}_{\mathbf{NN}}^{(n+1)} = \min_q E_{NN} = \min_q\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i - \mathrm{NN}(\mathbf{x}_i, \mathrm{q}_{\mathbf{NN}}^{(n)} \cup \mathrm{q}_{\mathrm{REF}})\right)^2 + r\sum_{j=1}^{m} q_j^2\right]$
            $\forall_i : \hat{y}_i = \mathrm{NN}(\mathbf{x}_i, \mathrm{q}_{\mathbf{NN}}^{(\mathbf{n+1})}), rss = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2, rdof = n - \left[trace(\mathbf{S}_{NN}^{(n+1)}) + 1\right]$

        else terminate: $finished = TRUE$

Until ( $finished$ = TRUE )
```

Figure 10.7: Details of the **constructive** algorithm for neural model estimation

The dual advantages of this algorithm are that additional complexity is only included in the model when explicit structure is identified in the residual errors of the existing model, and that the allocation of this additional complexity is targeted at the particular input subspace in which the structure has been identified. This may be considered a "local" form of variable selection in that variables are included in the model not on a global basis, but on the basis of particular relationships which may exist only within the particular regions of the subspace in which they have been identified.

The disadvantage of the algorithm lies in the fact that computational issues result in its limitation to relatively low-order interaction effects. The residual testing algorithm requires that the variable selection methodology be applied to all the low-dimensional combinations of the candidate variables. As the number of combinations grows exponentially with the dimensionality, the maximum feasible cardinality of the subspaces tested will typically be only either 2 or 3. In cases where the predictable components are primarily contained in low-dimensional subspaces, the resulting bias on the modelling procedure as a whole will be more than offset by a reduction in model variance. This is achieved by *targeting* the allocation of model complexity, rather than simply spreading degrees of freedom over the entire input space. In cases where higher-order effects occur, however, the constructive algorithm will fail to accurately model these relationships (although it *will* be able to model their low-dimensional projections) resulting in a performance degradation due to excessive model bias.

### Algorithm 3: Deconstructive Algorithm using Variable Elimination

The third algorithm within our (partial-)F testing methodology is based upon a “deconstructive” approach. The algorithm starts with an overparametrised architecture which is successively refined by iteratively *removing* variables which are found to provide statistically *insignificant* contributions to the overall model. This approach can be seen as complementary to the constructive approach in that the two algorithms approach the bias-variance tradeoff from the opposite extremes. As with the constructive algorithm, the deconstructive algorithm also provides an integrated approach to the two tasks of complexity control and variable selection. The details of the deconstructive algorithm for neural model estimation are presented in Figure 10.8 below.

```
Initialise:   $\mathbf{x}_{excluded} = \varnothing$
Repeat:
        train network:   $q_{NN} = \min_{q} E_{NN} = \min_{q}\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i - NN\left(\mathbf{x} - \mathbf{x}_{excluded}, q\right)\right)^2 + r\sum_{j=1}^{m} q_j^2\right]$
                $rss = \sum_{i=1}^{n}\left(y_i - NN\left(\mathbf{x} - \mathbf{x}_{excluded}, q\right)\right)^2, rdof = n - \left[trace\left(\mathbf{S}_{NN}\right) + 1\right], F_{\min} = F_{thresh}$
        For each variable:   $x_{test} \in \mathbf{x} - \mathbf{x}_{excluded}$

                train reduced network:
                        $q_{NNR} = \min_{q} E_{NNR} = \min_{q}\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i - NN\left(\mathbf{x} - \left[\mathbf{x}_{excluded} \cup x_{test}\right], q\right)\right)^2 + r\sum_{j=1}^{m} q_j^2\right]$

                calculate reduction in degrees of freedom:   $mdof = trace\left(\mathbf{S}_{NN}\right) - trace\left(\mathbf{S}_{NNR}\right)$

                calculate reduction in variance explained:
                        $mss = \left[\sum_{i=1}^{n}\left(y_i - NN\left(\mathbf{x}_i - \left[\mathbf{x}_{excluded} \cup x_{test}\right], q\right)\right)^2 - \sum_{i=1}^{n}\left(y_i - NN\left(\mathbf{x}_i - \mathbf{x}_{excluded}, q\right)\right)^2\right]$
                calculate partial-F:   $F_{test} = {}^{mss/mdof}\!\Big/_{rss/rdof}$

                if ( $F_{test} < F_{min}$ ) then $F_{min} = F_{test}$ , $x_{ref} = x_{test}$
        Next
        if ($F_{min}$ < $F_{thresh}$) then remove variable and continue:   $\mathbf{x}_{excluded} = \mathbf{x}_{excluded} \cup x_{ref}$ , $finished = FALSE$

        else terminate:   $finished = TRUE$

Until ( $finished$ = TRUE )
```

Figure 10.8: Details of the **deconstructive** algorithm for neural model estimation

The deconstructive algorithm starts with an *overparametrised* architecture which is capable of modelling arbitrary nonlinear and interaction effects on a global basis. This architecture is successively refined by removing variables which are found to provide statistically insignificant contributions to the overall model. This process continues until the removal of any further variables is found to result in a significant performance degradation

The methodology used can be considered as a refinement of the variable selection procedure of Moody and Utans (1992), Burgess (1995) and Refenes *et al* (1997). For computational reasons, the previous approaches used an approximation of the reduced model by simply retaining the enhanced model but holding the "removed" variable to a constant average value. This approximation relies upon an assumption of independence between the input variables in that it does not allow for redundant information in the removed variable which is also provided by one or more other variables in the model. In contrast our new methodology involves re-estimating each reduced model in order to fairly evaluate the "added value" which is provided by each variable, conditioned upon the other variables contained in the model.

Additionally, in Moody and Utans (1992) and Refenes *et al* (1997) the quantity obtained is better considered a “metric” of model sensitivity rather than a “statistic” which has a known distribution and hence can be tested for statistical significance. The approach of Burgess (1995) represents an advance in that it relates the variance explained to a (heuristic) estimate of degrees of freedom in an approximation of a (partial-) F statistic. In contrast the F-test used in our new methodology can be considered a true F-test in that both the estimates of variance explained and degrees of freedom absorbed are obtained using the methods which are described in Chapter 8 and which are motivated by the “equivalent kernel” perspective of neural networks as a form of nonparametric statistical estimator.

The underlying biases of the deconstructive algorithm differ from those of the constructive algorithm in that variables are selected on a *global* rather than *local* basis. Rather than decomposing the input space into merely additive components, the deconstructive algorithm estimates a *single* component which allows high-dimensional interactions between any and all selected variables. Thus the algorithm approaches the bias-variance tradeoff from the opposite direction to the constructive algorithm not only in terms of the quantitative degrees of freedom in the model but also in terms of the conceptual “model space” which is under consideration. The computational complexity of the algorithm is lower than in the case of the constructive algorithm, as at each stage only $v$ reduced models need be estimated, where $v$ is the remaining number of variables in the model. The approach is still computationally rather expensive however, as the **total** number of models which may be estimated during the procedure is $O(v^2)$.

### Summary

In this section we have described three alternative algorithms for neural model estimation. The following section presents an empirical evaluation of the capabilities of the algorithms, paying particular attention to their ability to control the effect of model degradation due to the presence of spurious variables.

## 10.3 Experimental Validation of Neural Model Estimation Procedures

In this section we perform an experimental validation of the neural estimation algorithms of the previous section. Two experiments are described, the first being to investigate the performance degradation caused by the *inclusion of spurious variables* in the set of explanatory variables, and in particular the extent to which the three algorithms provide protection against this degradation. The aim of the second experiment was to hold the number of noise variables constant, whilst *varying the dimensionality* of the underlying data-generating process in order to investigate the ability of the three algorithms to model relationships which consist, at least in part, of high dimensional interaction effects.

### Experiment 1: Effect of spurious variables

The results are obtained through Monte-Carlo simulations with 100 realisations of each particular scenario. The deterministic component of the data is generated by a univariate neural network model with three hidden units and the data is constructed such that this deterministic component accounts for 10% of the variance of the target variable. The size of the insample (estimation) dataset is 400 observations - sufficiently small that *model variance* will be expected to have a significant effect on performance. Table 10.1 presents the average generalisation ability of the models constructed using each of our three neural estimation algorithms; benchmark performance figures are provided in terms of a neural network with three-hidden units and no regularisation.

| | Noise = 0 | Noise = 1 | Noise = 3 | Noise = 6 |
|---|---|---|---|---|
| Benchmark | 6.38% | 4.86% | -0.16% | -6.53% |
| Regularisation-based | 8.83% | 7.68% | 4.74% | 2.81% |
| | (0.22%) | (0.22%) | (0.22%) | (0.24%) |
| Constructive | 8.95% | 9.00% | 8.93% | 8.65% |
| | (0.23%) | (0.26%) | (0.31%) | (0.32%) |
| Deconstructive | 8.32% | 8.29% | 7.37% | 7.27% |
| | (0.22%) | (0.22%) | (0.28%) | (0.38%) |

Table 10.1: Generalisation performance of the automated neural selection algorithms as a function of the number of noise variables included in the estimation procedure. Results represent out-of-sample $R^2$ over 100 Monte-Carlo realisations of the data; standard errors are presented in brackets below the corresponding performance statistics.

The results are also presented graphically in Figure 10.9 below:

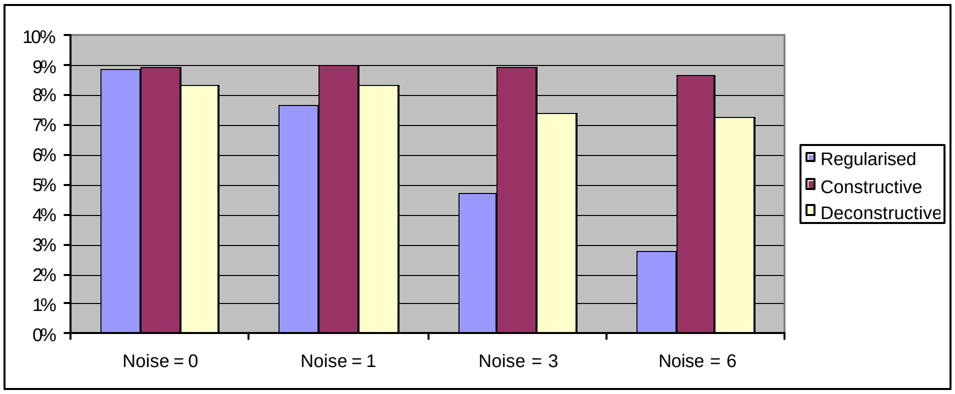


Figure 10.9: Performance of the automated neural specification algorithms, in terms of out-of-sample $R^2$, as a function of the number of noise variables included in the estimation procedure. The magnitude of the underlying deterministic component is 10%.

The results clearly indicate the effectiveness of the neural estimation algorithms at minimising the effect of the spurious variables. Only marginal degradation is seen in the case of the constructive algorithm, demonstrating the low size and high power of the F-tests for residual structure upon which the models are based. The deconstructive algorithm suffers a slightly greater degradation due to fact that the underlying bias of this approach is to favour more complex models. In contrast to the constructive and deconstructive algorithms, which are based on the statistical significance testing methodology of Chapter 9, the regularisation-based algorithm performs noticeably less well in the presence of spurious noise variables. This is because the underlying modelling bias of the regularisation approach is that all parameters and hence variables are to be treated equally, which in this case is incorrect. The algorithm does at least prevent the catastrophic performance degradation which is seen in the benchmark unregularised models, and performs comparably to the other two approaches in the case where only 0 or 1 noise variables are present.

### Experiment 2: Effect of increasing the model dimensionality

The objective of the second experiment is to compare the ability of the different algorithms to model relationships which at least partly consist of interaction effects between two or more variables. The experiment consists of a set of Monte-Carlo simulations in which the deterministic component is generated by an arbitrary non-linear relationship in the form of a neural network with 3 hidden units. The dimensionality of the relationship (number of non-noise variables) is varied between 1 and 5, with the number of noise variables being held constant at one.

The results of the experiment are presented in Table 10.2 below. The figures reported consist of average generalisation ability over 100 realisations of the data; the size of the estimation and generalisation sets are 400 observations and 1000 observations respectively.

| Number of Variables | 1+1noise | 2+1noise | 3+1noise | 4+1noise |
|---|---|---|---|---|
| Theoretical Max | 8.75% | 7.50% | 6.25% | 5.00% |
| Regularisation-based | 7.68% | 6.45% | 5.21% | 3.55% |
| | (0.22%) | (0.44%) | (0.46%) | (0.65%) |
| Constructive | 9.00% | 5.85% | 4.50% | 2.41% |
| | (0.26%) | (0.59%) | (0.57%) | (0.60%) |
| Deconstructive | 8.29% | 6.56% | 5.43% | 4.80% |
| | (0.22%) | (0.43%) | (0.49%) | (0.37%) |

Table 10.2: *Generalisation* (out-of-sample) performance of the automated neural estimation algorithms, in terms of out-of-sample $R^2$, as a function of the underlying **dimensionality** of the deterministic component (number of non-noise variables). The number of noise variables is held constant at one. The magnitude of the underlying deterministic component is 10%, the size of the estimation set is 400 observations and the size of the out-of-sample set is 1000 observations.

The results illustrate that model performance *reduces* as the dimensionality (and complexity) of the data generating process *increases*. A significant component of this performance degradation is due to unavoidable effects of model variance. This is illustrated in the table in the form of the "Theoretical Max" figures which represent the theoretically achievable (average case) performance given the deterministic component of the data (10%) and the model variance (sampling error) which is due to the underlying complexity of the data-generating process (DGP).

The results reflect the *different* biases of the three algorithms, and in particular they highlight the effect of increasing the dimensionality of the underlying DGP. The constructive algorithm performs well whilst the underlying dimensionality corresponds to that of the residual tests (1 in this case) but degrades rapidly beyond that point. In contrast, the deconstructive algorithm is comparatively robust to the increasing dimensionality because its variable selection procedure is capable of identifying structure involving arbitrarily high-order interaction effects between the variables. The regularisation-based algorithm performs relatively inefficiently in most cases but is better able to model the high-dimensional structure than is the constructive algorithm.

### Conclusion

The experiments in this section have demonstrated the key properties of the three algorithms. The results of the first experiment demonstrate that all three algorithms are capable of extracting significant deterministic components even in the situation where a high proportion of the input variables are nuisance variables, i.e. completely unrelated to the target variable. This is in contrast to the benchmark performance using a simple OLS cost function and no variable/architecture selection where the presence of noise variables greatly degrades, or even eliminates, the ability to extract a predictable component from the data. The regularisation-based algorithm employs a global approach to complexity optimisation which provides a limited degree of robustness to noise. In contrast, both the constructive and deconstructive algorithms are very robust to the presence of noise variables which are almost always excluded by the explicit variable selection procedures which they employ.

The second experiment highlights the effect of increasing the dimensionality of the deterministic component. The constructive algorithm is demonstrated to perform best in situations where the deterministic component is of *low dimensionality*, commensurate with the cardinality of the subspaces which are searched by the residual testing procedure. The deconstructive algorithm avoids this bias and is capable of modelling the high-order interaction effects within the data, outperforming the constructive algorithm in high-dimensional cases. The regularisation-control algorithm can be considered a “poor relation” in that its inability to explicitly differentiate between noise and non-noise variables creates a general tendency to underperform both the constructive and deconstructive algorithms.

## 10.4 Summary

In this chapter we have described our methodology for neural model estimation. The statistical perspective of the previous chapters is extended to form the basis of a rigorous modelling framework within which to automatically optimise the specification of neural network models. Three neural estimation algorithms have been described which in each case integrate the two tasks of variable selection and model specification. The three algorithms approach the model estimation task from the different perspectives afforded by the constructive, deconstructive and regularisation-based approaches. An experimental validation of the properties of the algorithms demonstrates the ability to estimate neural models with significant generalisation ability even when operating under conditions of high noise, small sample size and the presence of spurious variables.

It should be emphasised that the scenarios under which the algorithms have been evaluated represent almost the extreme limits of feasibility for predictive models, in which model variance alone will tend to almost negate any predictive capability. The results in Table 10.1 demonstrate that the use of naïve models can lead to catastrophic generalisation performance in these circumstances, thus motivating the use of statistically well-principled estimation procedures. Under more benign circumstances it might be expected that the performance improvements over simpler “non-statistical” approaches would be smaller, although potentially very valuable depending on the nature of the application and the manner in which the predictive information is exploited.

In the following chapter we describe the application of the low-bias neural modelling methodology to the task of estimating forecasting models for innovations in statistical mispricing time-series. These models are used to implement advanced statistical arbitrage strategies in which trading decisions are conditioned upon the level of mispricing, the recent dynamics of the mispricing time-series and other variables which contain predictive information concerning future changes in the mispricing.

# 11. Empirical Evaluation of Conditional Statistical Arbitrage Models

In this chapter we describe an empirical evaluation of the first two stages of our methodology used in combination. The model-free variable selection procedures and neural model estimation algorithms are applied to the problem of forecasting the dynamics of the statistical mispricings generated by the first part of our methodology. Section 11.1 provides a brief description of the Conditional Statistical Arbitrage (CSA) trading strategies which provide a mechanism for exploiting the predictive information contained in the neural network forecasting models. The remainder of this chapter then describes an empirical evaluation of the methodology in the context of statistical mispricing models between the stocks which constitute the FTSE 100 index. Section 11.2 describes the results of the variable selection procedure for the FTSE 100 models. Section 11.3 presents an empirical evaluation of a set of CSA strategies which are based upon neural network forecasting models of the mispricing dynamics.

## 11.1 Conditional Statistical Arbitrage Models

In this section we define a set of “conditional statistical arbitrage” (CSA) strategies which generate trading signals conditioned upon the output of forecasting models. The objective of the CSA strategies is to efficiently exploit any deterministic component of the mispricing dynamics which has been captured in an appropriate forecasting model. Unlike the implicit (ISA) strategies of Chapter 7, the CSA strategies make no assumptions about the nature of the mispricing dynamics, but rather are based upon the assumption that the future changes in the mispricing will be correlated with the output of the forecasting model.

Given a statistical mispricing model $M_t = T_t - \sum_{C_i \in C} b_i C_{i,t}$ (Eqn. (5.10)) we first use the low-bias modelling methodology of Chapters 9 and 10 to estimate “mispricing correction models” (MCMs) which are of the general form:

$$\frac{\Delta\left( T_t - \sum_{C_i \in C} b_{i,t} C_{i,t} \right)}{T_t + \sum_{C_i \in C} b_{i,t} C_{i,t}} = \Delta M_t = f\left( M_t, \Delta M_{t-t}, Z_t \right) + e_t \qquad (11.1)$$

The MCM in Eqn. (11.1) can be considered as a novel modification of the standard error-correction models (ECMs) which were described in Section 2.2.3. Rather than being concerned with changes in *individual* asset prices, as would be the case of a standard ECM, our models are concerned with changes in the *combined* prices of the sets of assets which define statistical mispricings. The use of the general notation $f(\ )$, as opposed to a particular functional form, reflects the fact that the MCMs are estimated using the low-bias **neural model estimation** procedures which are described in Chapter 10. The specific choice of lagged terms $\Delta M_{t-t}$ and exogenous variables $Z_t$ is performed by means of the **model-free variable selection** methodology which is described in Chapter 9. Conditional strategies based on MCMs of the form in Eqn. (11.1) can be related to riskless arbitrage and implicit statistical arbitrage by means of the taxonomy of arbitrage strategies which is presented in Table 4.1.

Our CSA trading rules for exploiting the predictions of the MCMs take a similar form to the implicit statistical arbitrage (ISA) rules from Part I of the methodology. As in the case of the implicit rules, the CSA rules define the position which should be taken in terms of the portfolio which corresponds to the statistical mispricing $M_t$. The basic CSA treading rule defines the desired position in the mispricing portfolio as a function of the MCM forecast:

$$CSA(\mathrm{E}[\Delta M_t], k)_t = \mathrm{sign}\left(\mathrm{E}[\Delta M_t]\right)\left|\mathrm{E}[\Delta M_t]\right|^k \tag{11.2}$$

where $k$ is a parameter which determines the sensitivity of the trading position to the magnitude of the forecasted return, $\mathrm{E}[\Delta M_t]$. In contrast to the equivalent ISA rule (Eqn. (7.1)) the implicitly negative relationship between the *level* of the mispricing and expected future *changes* in the mispricing is now absorbed in the forecasting model. As in the case of the ISA strategies, we can define generalisations of the basic CSA rule which are designed to reduce the trading activity (and hence trading costs). This is achieved by *smoothing* the trading signal using either a moving-average or an exponential moving average with parameters $h$ and $q$ respectively:

$$CSA(\mathrm{E}[\Delta M_t], k, h)_t = \frac{1}{h}\sum_{j=0..h-1} CSA(\mathrm{E}[\Delta M_t], k)_{t-j} \tag{11.3a}$$

$$CSA(\mathrm{E}[M_t], k, q)_t = (1-q)CSA(\mathrm{E}[M_t], k)_t + q\ CSA(\mathrm{E}[M_{t-1}], k, q)_{t-1} \tag{11.3b}$$

The advantage of employing conditional statistical arbitrage strategies based upon flexible, low-bias, forecasting models is that, in addition to the basic tendency of mean-reversion, more general deterministic components of the dynamics can be captured in the forecasting model and thus exploited by the trading strategy. In particular, the use of **neural network** methodology makes it possible to automatically model both direct *nonlinearities* and also *interaction effects* without having to specify such effects in advance. Whilst it is of course possible to capture nonlinear effects using parametric statistical techniques, this is only the case if either an examination of the data, or some associated theory, suggests an appropriate parameterisation which contains the desired effect as a component of the model. In practice, the high level of noise in asset returns means that predictable components are very difficult to detect without the use of principled methodology such as that described in the previous chapters.

Figures 11.1 and 11.2 illustrate the types of relationships which we would hope to be able to capture through the use of flexible modelling techniques, but not necessarily through parametric methods. Figure 11.1 presents two smooth nonlinear relationships which contain negligible linear components. In each case a nonlinear model can capture a significant component of the overall variance of the data which the linear technique fails to detect.

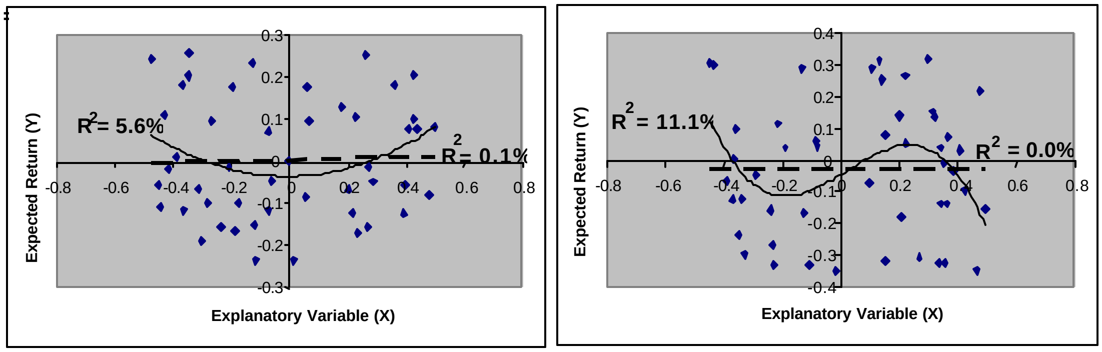


Figure 11.1: Illustration of smooth nonlinear relationships which can be captured by low-bias neural forecasting models

Figure 11.2 contains an example of a multivariate interaction effect in which the influence of independent variable $x$ on dependent variable $y$ is modulated by the value of a third variable $z$.

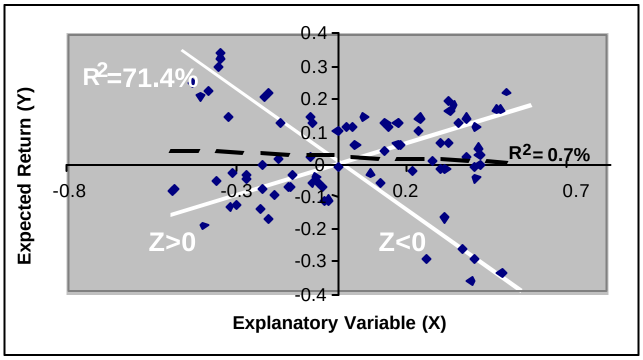


Figure 11.2: Illustration of a multivariate interaction effect.

By capturing these additional components in the forecasting model it may be possible both to decrease risk and increase profitability. A decrease in risk may occur because otherwise unfavourable effects such as a short-term momentum acting to temporarily *increase* the mispricing are now accounted for in the model forecasts. An increase in profits may occur because the overall accuracy of the forecasting model will improve due to capturing a larger part of the deterministic component in the underlying dynamics. The advantage of employing conditional statistical arbitrage strategies in cases where the mispricing dynamics are other than straightforwardly mean-reverting was illustrated in Figure 4.8 which compared the performance of a number of alternative statistical arbitrage models applied to an artificial time-series containing a momentum term and a conditional or modulated mean-reversion effect.

The CSA strategies are ultimately dependent upon the predictive ability of the associated forecasting models. This sensitivity is illustrated in Table 11.1 below. The table presents a summary of the performance of the basic CSA strategy (with $k=1$) as a function of the *predictive correlation* between the actual returns $y$ and the forecasted returns $\hat{y}$.

| $\text{correl}\left(y,\hat{y}\right)$ | $R^2\left(y,\hat{y}\right)$ | Directional Correctness | Annualised profit | Annualised risk | Sharpe Ratio |
|---|---|---|---|---|---|
| 0.05 | 0.25% | 51.5% | 11.8% | 16.1% | 0.738 |
| | | (0.2%) | (0.7%) | (0.1%) | (0.047) |
| 0.1 | 1% | 53.1% | 26.0% | 16.2% | 1.605 |
| | | (0.1%) | (0.8%) | (0.1%) | (0.048) |
| 0.15 | 2.25% | 54.8% | 38.9% | 16.3% | 2.393 |
| | | (0.1%) | (0.7%) | (0.1%) | (0.045) |
| 0.2 | 4% | 56.4% | 51.4% | 16.3% | 3.142 |
| | | (0.2%) | (0.8%) | (0.1%) | (0.050) |
| 0.25 | 6.25% | 58.0% | 64.4% | 16.5% | 3.904 |
| | | (0.2%) | (0.8%) | (0.1%) | (0.047) |
| 0.3 | 9% | 59.6% | 76.4% | 16.7% | 4.579 |
| | | (0.1%) | (0.6%) | (0.1%) | (0.035) |
| 0.35 | 12.25% | 61.2% | 89.3% | 16.9% | 5.282 |
| | | (0.1%) | (0.7%) | (0.1%) | (0.039) |
| 0.4 | 16% | 63.0% | 101.4% | 17.1% | 5.915 |
| | | (0.1%) | (0.7%) | (0.1%) | (0.038) |
| 0.45 | 20.25% | 64.7% | 114.1% | 17.4% | 6.570 |
| | | (0.1%) | (0.6%) | (0.1%) | (0.032) |
| 0.5 | 25% | 66.5% | 127.4% | 17.6% | 7.231 |
| | | (0.1%) | (0.6%) | (0.1%) | (0.036) |

Table 11.1: Sensitivity of the Conditional Statistical Arbitrage (CSA) strategy performance to the predictive correlation between the actual returns $y$ and the forecasted returns $\hat{y}$. Standard errors are presented in brackets underneath the associated performance metrics. The data in constructed such that the standard deviation of the underlying time-series is 1% per day; the trading signal is normalised to have an average magnitude of 1. Transaction costs are *not* included.

The table demonstrates that only relatively low levels of predictive ability are required in order to achieve economically meaningful performance. For instance, a predictive correlation of 0.2 will produce on average a strategy with an annualised Sharpe Ratio of approximately 3.1. As this corresponds to capturing a deterministic component equivalent to only 4% of the total variance in asset returns, it is quite plausible that such levels of performance might be achievable in reality.

In the remainder of this chapter we present an empirical evaluation of the first two parts of our methodology used in combination, in the form of a set of CSA strategies which are based upon neural network forecasting models. The forecasting models are trained to predict the changes in the statistical mispricings between the daily closing prices of the FTSE 100 constituents.

## 11.2 Application of Variable Selection Methodology to FTSE 100 Data

In this section, we describe an application of the model free variable-selection methodology of Chapter 9, to the statistical mispricings between the constituent stocks of the FTSE 100 index. An evaluation of these mispricing models, in the context of *implicit* statistical arbitrage strategies was presented in Chapter 7. The data consists of daily closing prices between 13th June 1996 and 5th October 1998. The 600 observations are divided into a 400 day "insample" period which is used to estimate the statistical mispricing models, and a subsequent 200 day "out-of-sample" period which is used to present an unbiased estimate of the generalisation performance of the models. After removing the assets for which continuous data samples were not available, the number of assets in the sample was 89 constituents plus the index itself.

The mispricing models of FTSE stocks which were presented in Chapter 7 were of the form:

$$M_{\mathrm{s,t}} = P_{s,t} - \left( \sum_{i=1}^{n} w_{s,i} P_{c(i,s),t} + c \right) \tag{11.4}$$

where $M_{\mathrm{s,t}}$ is the statistical mispricing for stock *s* at time *t*; $P_{s,t}$ is price of asset *s* at time *t*; $P_{c(i,s),t}$ is the price of the *i*'th constituent asset selected for target asset *s* and $w_{s,i}$ is the associated weighting parameter. In the remainder of this chapter we evaluate the more sophisticated "Conditional Statistical Arbitrage" strategies on the set of 90 mispricings which result from each asset in the universe being compared to an associated synthetic asset consisting of **three** constituent assets (i.e. *n*=3 in Eqn (11.4)).

The set of candidate variables for the forecasting models can be divided into two subgroups: **time-series** variables which are derived from the mispricing itself, and candidate **exogenous** variables. The pre-processing involved in creating these two sets of variables is described in turn below, before moving on to the testing methodology and the empirical results of the variable selection procedure.

### Representation of the mispricing time-series

In order to correct the mispricing time-series for any nonstationary components which may arise due to asset specific factors we apply the adaptive modelling methodology of Section 5.2. Whilst the full time-varying regression version of the filtering methodology *could* be applied in this case we choose instead to use the simpler exponential smoothing version of the methodology. In the exponential smoothing approach the relative weightings of the assets within the mispricing portfolio remain constant, thus simplifying the evaluation of the trading performance of the CSA strategies.

The effect of the exponential smoothing version of the adaptive modelling methodology is to create transformed versions of the mispricing time-series which represent the differences between the original (static) mispricing and an exponentially-smoothed "reversion level" which absorbs the nonstationary component in the mispricing dynamics. The resulting transformation of the mispricing depends upon the choice of smoothing constant α and is given by:

$$\begin{aligned}
&\Delta_a^E M_t = M_t - EMA(M, a)_{t-1} \\
&EMA(M, a)_t = aM_t + (1 - a)EMA(M, a)_{t-1} \\
&\text{where } EMA(M, a)_0 = 0
\end{aligned} \tag{11.5}$$

This representation allows the mispricing innovations to be sampled at different frequencies; special cases include the mispricing itself $\Delta_0^E M_t = M_t$ and the first-difference $\Delta_1^E M_t = M_t - M_{t-1} = \Delta M_t$. Exponential decay is a natural representation of (near) random walk time-series as it enjoys certain optimality properties, see for example Brown (1963). In this case the exponentially-smoothed transformation represents an optimal estimation of the location of the "fair price" of the mispricing time-series, under the assumption of different levels of contamination by a non-stationary drift which is caused by asset specific shocks to the mispricing time-series. With a high-value of the smoothing parameter $a$ the transformation can also be viewed as a measure of the **momentum** of the series. Optimising the smoothing parameter $a$ at this stage would impose an additional bias on the subsequent forecasting stage of the methodology. Therefore we choose instead to include three transformations of each series as candidate variables for our model-free variable selection methodology, with $a$ = 0.01, 0.3 and 0.9 corresponding to low, moderate and high frequency indicators respectively. An example of the result of the exponential smoothing is shown in

Figure 11.3 for the case $\Delta^{E}_{0.3} M_{FTSE,t}$, i.e. the mispricing of the index itself with a smoothing parameter of 0.3.

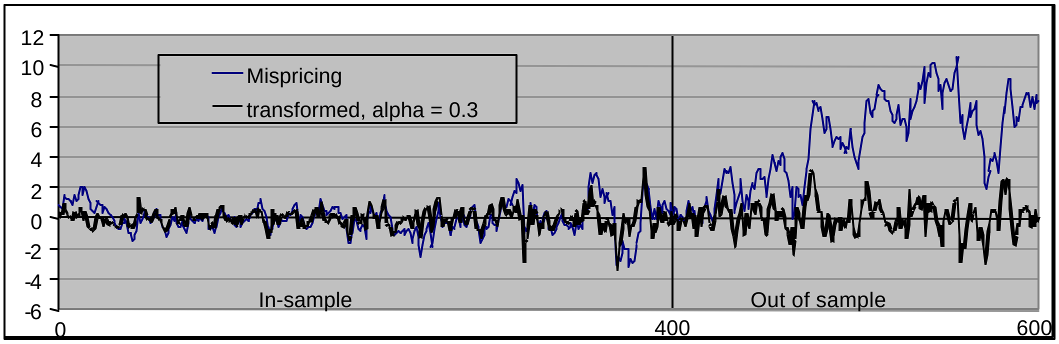


Figure 11.3: Illustration of the adaptive filtering methodology. The figure shows the mispricing $M_{FTSE,t} = FTSE_t - (1961 + 3.033\, LLOY_t + 1.100\, WLY_t + 0.377\, ICI_t)$ and the transformed version of the series with smoothing parameter $a$ =0. 3. The scale on the x-axis is measured in days.

### Candidate Exogenous Variables and Transformations

In addition to the (transformed) mispricing itself, it is plausible that other, exogenous, factors may influence the mispricing dynamics and as such represent predictive variables which will improve model performance. A set of candidate exogenous variables were transformed into **changes** in the time-series, which may differentially affect the prices of underlying assets and thus account in part for the dynamics of the mispricings also, and measures of the recent **volatility** of these same exogenous factors, which may be expected to modulate the relationships which drive the mispricing dynamics.

The set of exogenous variables, which were considered as candidate variables by the model-free variable selection methodology, comprised a mixture of equity indices, bond indices, commodity indices and exchange rates as listed in Table 11.2 below. The variables represent both "direct" factors, which may differently affect the prices of different assets, and "context" factors which may interact with or modulate the mispricing correction effect in some way.

| | | |
|---|---|---|
| Equity indices: | FTSE 100 (UK) | FTSE |
| | S&P 500 (US) | S&P |
| | Dax 30 (Germany) | Dax |
| Bond price indices: | Datastream 10-year bond price index (UK) | UK10yr |
| | Datastream 2-year bond-price index (UK) | UK2yr |
| Commodities: | Goldman Sachs Precious metals index | GSPM |
| | GS Industrial metals index | GSIN |
| | Oil price (Brent) | LCR |
| Exchange rates: | Sterling index | STER |
| | UK pound to US dollar | USD |
| | UK pound to German mark | DEM |

Table 11.2: Exogenous variables included in the variable selection procedure for the low-bias forecasting models of the mispricing dynamics

Each variable was subjected to the five transformations listed in Table 11.3 to give 55 variables in all.

| | |
|---|---|
| Differences | Daily change |
| | 5-day change |
| | 20-day change |
| Volatilities | 5 day variance in log(price) |
| | 20 day variance in log(price) |

Table 11.3: List of transformations which were applied to the exogenous variables prior to use in the variable selection procedure for the low-bias forecasting models of the mispricing dynamics

### Methodology

The variable selection methodology of Chapter 9 was employed to identify the candidate variables which may act as predictive indicators for each individual mispricing time-series.

The polynomial class of models (specification P from Table 9.1) were selected for the simple reason that the degrees of freedom can be trivially partitioned into direct (univariate) and interaction (multivariate) effects on the basis of the number of variables which are assigned non-zero powers within each polynomial term. Reflecting our underlying modelling bias towards relatively low-order and low-dimensional relationships the maximum degree of polynomial was taken as 3, and the maximum dimensionality of the model-free variable selection test (of the form given in Eqn. (9.13)) was 2, giving nine degrees of freedom in all. In each case, the dependent variable which is regressed onto the (transformed) candidate variables is the *one-day-ahead percentage change in the mispricing* defined by the left-hand-side of Eqn. (11.1).

Partly for practical reasons and partly to control the risks of "data snooping", a filtering procedure was used in order to determine which combinations of variables should be tested: firstly, all 58 candidate variables were tested on a univariate basis, and then bivariate tests were conducted between each *individually significant* variable and all other variables. Thus a (fairly typical) case with 4 variables being identified as univariately significant would result in 4*57 = 228 bivariate tests being performed.

The critical values chosen for the F-tests were 4.0 in the univariate case and 7.0 in the bivariate case. After correcting for data-snooping/repeated sampling effects (but not for redundancy of information across multiple variables) these critical values represent a significance level of approximately 1%.

### Empirical Results

A summary of the results of the variable selection procedure is presented below. Firstly, with respect to the filtered mispricing series themselves, the number of times each variable was indicated as significant is shown in Table 11.4:

| | $\Delta^E_{0.01} M_t$ | $\Delta^E_{0.3} M_t$ | $\Delta^E_{0.9} M_t$ |
|---|---|---|---|
| Direct | 82 | 17 | 15 |
| Indirect | 0 | 3 | 4 |
| Total | 82 | 20 | 19 |

Table 11.4: Frequency of selection of the mispricing transformations, either as *direct* relationships with future changes in the mispricing or as *indirect* or interaction effects involving a third variable. Indirect relationships are only reported in cases where no direct relationship is indicated.

Amongst the transformed mispricings, the low-frequency measure $\Delta^E_{0.01} M_t$, which is closely related to the untransformed variable, is indicated as significant in 82 out of 90 cases. This suggests a common feature of mean-reversion amongst the mispricing time-series. A significant part of these apparent relationships, however, will be a spurious artefact of the mispricing construction procedure. In the case of this indicator the only truly valid test will be the out-of-sample performance of the forecasting models. In contrast, the two higher-frequency transformations are each only selected with respect to about one-quarter of the synthetic assets. These variables will tend to correspond to relatively short short-term (momentum) effects in the mispricing time-series.

With regard to the *exogenous* variables, the number of times which each variable was selected on the basis of a significant *univariate* relationship is presented in Table 11.5:

| | Daily Change | 5-day change | 20-day change | 5-day vol. | 20-day vol. | Total |
|---|---|---|---|---|---|---|
| FTSE | 8 | 7 | 1 | 2 | 0 | 18 |
| S&P | 26 | 4 | 0 | 0 | 0 | 30 |
| Dax | 7 | 7 | 1 | 1 | 0 | 16 |
| UK10yr | 0 | 1 | 0 | 0 | 0 | 1 |
| UK2yr | 0 | 1 | 0 | 0 | 0 | 1 |
| GSPM | 3 | 1 | 2 | 0 | 0 | 6 |
| GSIN | 1 | 0 | 0 | 2 | 0 | 3 |
| LCR | 0 | 0 | 0 | 2 | 1 | 3 |
| STER | 7 | 2 | 0 | 1 | 0 | 10 |
| USD | 4 | 0 | 0 | 0 | 0 | 4 |
| DEM | 6 | 1 | 0 | 1 | 0 | 8 |
| Total | 62 | 24 | 4 | 9 | 1 | 100 |

Table 11.5: Number of times which each exogenous variable was selected on the basis of containing a significant *univariate* relationship with one-day-ahead changes in the mispricing

Unsurprisingly, the most commonly selected subset of variables are the recent changes in equity markets. Perhaps more surprising is that the US stock market, represented here by the broad S&P 500 index, is more commonly identified as a predictive indicator of changes in mispricings than is the UK market, in the form of the FTSE 100 index. Amongst the other variables, the commodities and exchange rates are selected on an infrequent basis and the two bond-price indices only once each. Amongst the different transformations, the daily changes are selected almost twice as often as all four other transformations combined.

Table 11.6 presents the number of times which each variable was selected on the basis of a significant indication of an *interaction effect*, when tested jointly with a variable previously indicated significant on a univariate basis. The figures **do not** include variables which also tested as significant during the univariate analysis.

| | Daily Change | 5-day change | 20-day change | 5-day vol. | 20-day vol. | Total |
|---|---|---|---|---|---|---|
| FTSE | 3 | 12 | 9 | 6 | 7 | 37 |
| S&P | 2 | 4 | 7 | 7 | 3 | 23 |
| Dax | 3 | 2 | 9 | 4 | 6 | 24 |
| UK10yr | 5 | 4 | 5 | 4 | 2 | 20 |
| UK2yr | 2 | 2 | 2 | 3 | 0 | 9 |
| GSPM | 1 | 7 | 3 | 6 | 4 | 21 |
| GSIN | 5 | 4 | 2 | 4 | 4 | 19 |
| LCR | 1 | 2 | 3 | 3 | 2 | 11 |
| STER | 2 | 1 | 4 | 2 | 3 | 12 |
| USD | 2 | 3 | 4 | 4 | 3 | 16 |
| DEM | 2 | 1 | 2 | 3 | 3 | 11 |
| Total | 28 | 42 | 50 | 46 | 37 | 203 |

Table 11.6: Number of times which each exogenous variable was selected on the basis of containing a significant *indirect* or interaction *effect* with respect to one-day-ahead changes in the mispricing

In this case the picture is much less clear than with the direct relationships. Some preference is shown for equity market variables and, amongst the transformations, 20-day changes and 5-day volatilities. Over the set of 90 synthetic assets, the total number of selected variables was thus 121 (mispricings) + 100 (direct exogenous) + 203 (indirect exogenous) = 424, an average of 4.7 per model.

## 11.3 Empirical Results of Conditional Statistical Arbitrage Models

In this section we describe the results of applying the three neural estimation algorithms of Chapter 10 to each of the 90 statistical mispricings, a total of 270 models in all. The performance of the models is analysed below from a number of perspectives. Firstly the **average** performance of the individual models is evaluated; secondly a number of specific models with representative properties are analysed in more detail; thirdly the **collective** performance of the models is considered, as a function firstly of the different algorithms which were used to generate the models and secondly in the context of applying a number of different model selection criteria; finally the paper-trading performance of the models is evaluated, with particular emphasis being placed on the effect of transactions costs.

The data used for the analysis is the same as was used for the *implicit* statistical arbitrage models of Section 7.2 and consists of an in-sample set of 400 daily observations, covering the period 22nd May 1996 to 16th December 1997, and an out-of-sample set of 200 observations, covering the period 17th December 1997 to 5th October 1998; to provide a context for the analysis, the performance of the FTSE 100 index during this period is illustrated in Figure 11.4

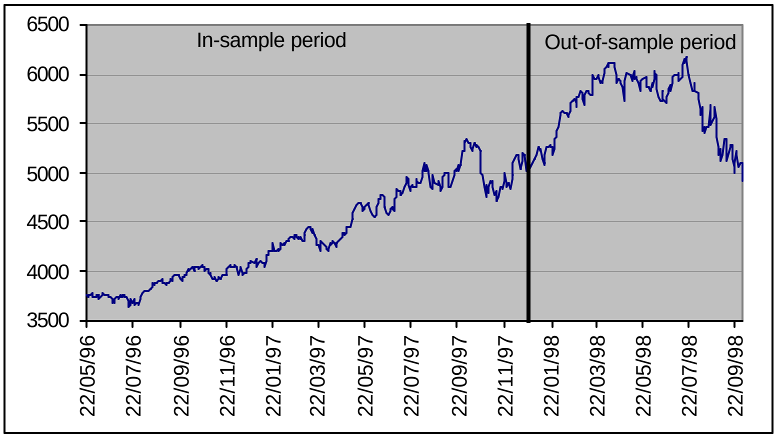


Figure 11.4: Performance of the FTSE 100 index during the period covered by the data used for the modelling and evaluation of the Conditional Statistical Arbitrage (CSA) models.

### Average Performance of the individual models

The average performance of the individual models, from a statistical perspective, is presented in Table 11.7, below. The first set of statistics correspond to in-sample measures of model significance which are used as the basis of the model selection criteria discussed in more detail below. The second set of measures correspond to the out-of-sample properties of the forecasting models.

| | VP25c1 | MSS | MDOF | F | $R^2_{adj}$ | Direction | Correl | $R^2$ | NonStat |
|---|---|---|---|---|---|---|---|---|---|
| Maximum | -0.04 | 140.99 | 35.37 | 14.41 | 0.29 | 59.2% | 0.27 | 0.06 | 3.43 |
| Median | -2.06 | 20.50 | 2.57 | 6.63 | 0.04 | 51.3% | 0.10 | -0.04 | 0.06 |
| Minimum | -3.02 | 0.00 | 0.00 | 0.00 | 0.00 | 43.3% | -0.13 | -3.41 | 0.00 |
| Average | -2.00 | 25.88 | 4.41 | 6.62 | 0.05 | 51.3% | 0.11 | -0.09 | 0.10 |

Table 11.7: Summary of the insample characteristics and out-of-sample performance across the set of 90*3=270 individual models. The statistics reported are VP25c1: projection onto the first principle component of the 25-period variance profile; MSS = (insample) variance explained by forecasting model; MDOF = degrees of freedom absorbed by forecasting model; F = F-ratio of forecasting model; $R^2_{adj}$ = insample variance explained, adjusted for degrees of freedom; Direction = proportion of out-of-sample periods in which sign(forecast)=sign(actual); Correl = out-of-sample correlation between forecasted and actual returns; $R^2$= 1-MSE/VAR = variance explained (out-of-sample); and a measure of model degradation due to nonstationarity, NonStat = $(\text{Correl})^2$-$R^2$ .

From this perspective, the performance of the models is not particularly promising. During the **in-sample period** the average proportion of variance explained ($R^2$) is only 5%. Model complexity varies between zero (in cases where the architecture selection algorithms fail to identify any significant structure) and 35.37, although the figures for average model degrees of freedom (MDOF) of 4.41 and median value of 2.57 suggest that the majority of models are relatively simple. The average F-statistic of 6.62 is highly significant, although it is not entirely clear how much of this is due to the spurious mean-reversion which is an artifact induced by the mispricing construction procedure.

The **out-of-sample** statistics appear to be equally uninspiring, with the average directional ability only 51.3% and predictive correlation of 0.11. Furthermore, the average out-of-sample $R^2$ of -0.09 indicates that the average mean-squared error of the models is *larger* than the variance of the target variable - suggesting a *negative* forecasting ability. Any deviation between $R^2$ (variance explained) and the squared correlation coefficient is indicative of nonstationarity in either the mean forecasted return or the mean actual return (or both). From the table we see that, on average, this effect corresponds to 10% of the variance of the target series, more than eliminating the forecasting ability of the models from the perspective of “variance explained”.

However, the true value of the forecasting models lies in the **profitability** (or otherwise) of the trading systems derived from them. Table 11.8 presents the out-of-sample performance of the basic CSA rule in Eqn (11.1) averaged across all 270 models.

|  | Overall | Q1 | Q2 | H1 | H2 |
|---|---|---|---|---|---|
| Start obs: | 1 | 1 | 51 | 1 | 101 |
| End obs: | 200 | 50 | 100 | 100 | 200 |
| Proportion of profitable models: | 80.4% | 72.2% | 69.3% | 76.7% | 71.1% |
| Maximum SR: | 3.93 | 5.88 | 5.07 | 5.25 | 4.52 |
| Median SR: | 0.86 | 1.29 | 1.03 | 0.97 | 0.58 |
| Minimum SR: | -1.68 | -3.29 | -3.49 | -2.92 | -3.25 |
| Average SR: | 0.85 | 1.37 | 0.93 | 1.02 | 0.76 |

Table 11.8: Summary of the average paper-trading performance across the set of 90*3=270 individual models. The out-of-sample period of 200 observations is broken down into two halves, H1 and H2, with the first half being further subdivided into two quarters Q1 and Q2. The performance metric reported in each case is a version of risk-adjusted return in the form of a Sharpe Ratio and represents the (annualised) mean return divided by the standard deviation of return.

From this perspective, the forecasting model performance is much more promising. Although the performance of most models is not significant on an individual basis (average SR = 0.85), an approximate[19] ‘t’-test for the **mean** performance indicates that the overall results are highly significant (‘t’-stat = 0.85*sqrt(270) = 13.97). Whilst the risk-adjusted performance, SR=0.85 is not particularly high (an approximate benchmark amongst practitioners is that a Sharpe Ratio above 2 is considered “good” and 3 or 4 is excellent) the results are at least evidence of positive performance by the forecasting models.

Having established a baseline level of performance, a number of important questions follow. On the positive side are the extent to which the performance may be improved by employing both model selection and model combination, whilst on the negative side are the potential performance degradation caused by nonstationarity and the impact of transaction costs. Furthermore, there is the technical issue of explaining the fact that the mean level of profitability is positive even though the mean out-of-sample $R^2$ is negative.

Before going on to consider these various issues in detail, we can make some initial observations based on Table 11.8. The superior performance of Q1 to Q2 and H1 to H2 provides evidence that the average model quality is degraded over time. This suggests that

[19] This test is only approximate because it assumes independence between the models; however a significance level of 0.1% (’t’-stat=3) would only require $(3/0.85)^2$=12 independent models from the entire set of 270.

performance improvements of almost a factor of 2 (Q1/H2=1.8) may be achievable simply by controlling the effects of *nonstationarity*, for instance by periodically re-estimating the models. Secondly, the wide spread of performance across the individual models (from 5.88 to -3.49) suggests the potential for further improvements, if the performance is in some way correlated with a suitable *model selection* criterion.

### Effect of mean-nonstationarity on trading performance

An important corollary of the use of the CSA trading rules is that they provide a certain degree of robustness to nonstationarity in the mean of the target variable. Whilst the mispricing construction process guarantees that the **insample** drift in the mispricing will be very close to zero, no such guarantee applies during the **out-of-sample** period. In fact, nonstationarity in the mean value of the mispricing innovations can be caused by a number of factors, including cases where the mean-reversion in the mispricing is completely spurious, is subject to a degree of nonstationary contamination, or suffers from a breakdown or "structural change" in the underlying relationship. This risk is confirmed by the performance statistics in Table 11.7 above, indicating that the average magnitude of this effect is equivalent to 10% of the variance of the mispricing.

The source of this robustness can be shown (Burgess, 1999b) to be due to the fact that the **mean** trading strategy return is actually unaffected by such a nonstationarity (intuitively this is due to the fact that the out-of-sample bias will sometimes have a negative effect on trading performance, and sometimes a compensating positive effect). Provided that the predictive **correlation** remains positive, then the expected trading performance will also be positive. The nonstationarity will, however, increase the *variability* of returns and hence degrade the risk-adjusted performance, as well as the measured $R^2$ during the out-of-sample period.

This phenomenon underlies the fact that the trading models, which from table 6 have an average out-of-sample correlation of 0.11, achieve *positive* trading performance in spite of the fact that the average out-of-sample $R^2$ is *negative*. This finding has important practical implications for building forecasting models within a statistical arbitrage context[20] and is illustrated in the Figure 11.5 below:

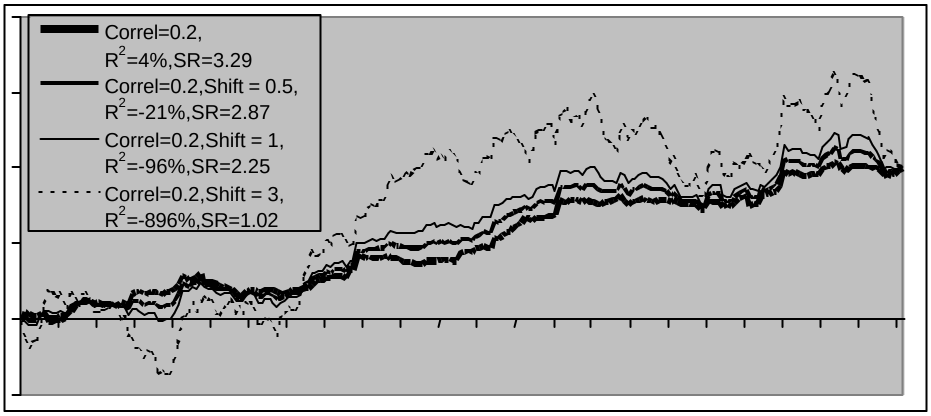


Figure 11.5: Robustness of the CSA strategy to mean-nonstationarity in the mispricing innovations. An out-of-sample **bias** in the mean value of the mispricing innovations may make the out-of-sample $R^2$ figure *negative* but the trading performance will remain *positive* provided the predictive correlation does so also. The increased variability (risk) results in a corresponding reduction in the Sharpe Ratio.

### Illustrative Examples of Forecasting Model Performance

Before moving on to consider the collective performance of the forecasting models in a portfolio content, it is interesting to consider **particular examples** of models whose performance in some way typifies certain properties of the set of models as a whole. In particular we present below examples of models with consistently positive performance, with nonstationary performance, with performance which is robust to nonstationarity in the target mean, and with performance which is highly sensitive to the level of transaction costs.

*Consistently profitable model*

Figure 11.6 illustrates the out-of-sample equity curve of a consistently profitable model, created using the *deconstructive* version of the neural estimation procedure, and based on the mispricing of Smiths Industries against Rolls Royce, Rank Group and BSKyB. The characteristics of the model are reported in Table 11.9 (model ref = 228). The exogenous

[20] Primarily, it suggests that the value of optimising "variance explained" (or squared error) is appropriate more because of the monotonic relationship between in-sample correlation and insample $R^2$ than because "variance explained" is important in itself.

variables selected for the model (by the procedure described in Section 11.2) were daily changes and 20-day volatility of the FTSE index.

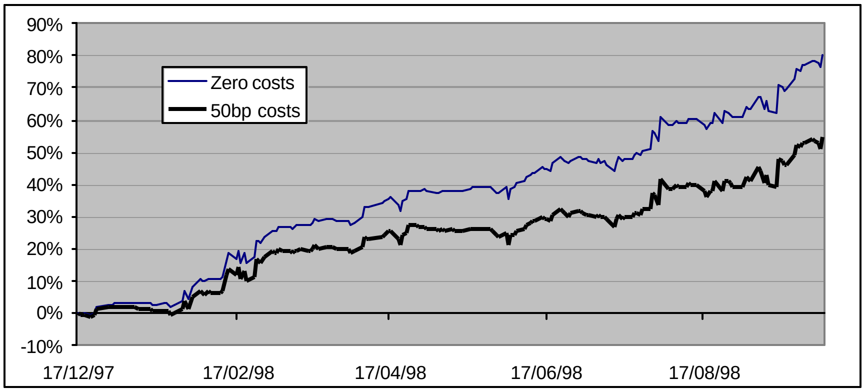


Figure 11.6: Example of a consistently profitable model, created using the *deconstructive* neural estimation algorithm and based upon the mispricing between Smiths Industries, and a combination of Rolls Royce, Rank Group and BSkyB.

### *Model with nonstationary performance*

Figure 11.7 presents the out-of-sample equity curve of Model 127 (see Table 11.9) whose performance undergoes a breakdown due to nonstationarity in the underlying relationships. The model was created using the *constructive* algorithm and based on the statistical mispricing between Lloyds TSB Group and a combination of the FTSE index, Allied Domecq, and Rentokil. No exogenous variables were selected for this particular forecasting model which was thus solely based upon the mispricing dynamics.

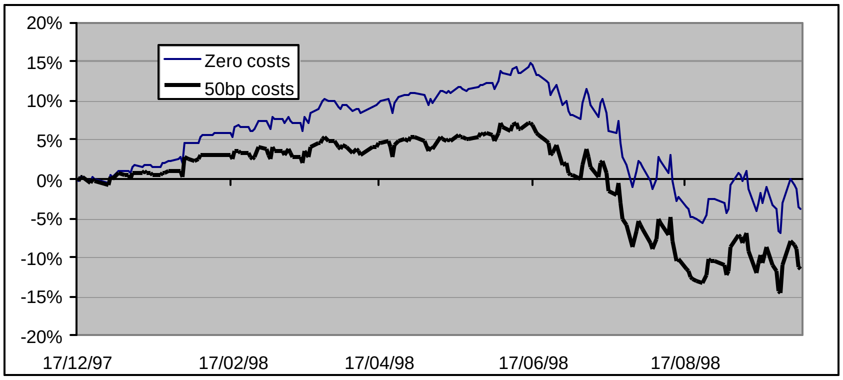


Figure 11.7: Example of a model which exhibits performance breakdown, created using the *constructive* neural estimation algorithm and based upon the mispricing of Lloyds TSB against the FTSE index, Allied Domecq and Rentokil.

### *Model which is profitable in spite of a negative $R^2$*

Figure 11.8 shows the out-of-sample equity curve for Model 83 which is consistently profitable in spite of a negative $R^2$ during the out-of-sample period. The model was created using the *regularised* version of the neural estimation methodology and is based upon the mispricing of Diageo against Next, Granada Group and Safeway. The exogenous variables selected for inclusion in the forecasting model were 5-day changes in the FTSE index plus 5-day volatilities in both the DEM exchange rate and the S&P 500 index.

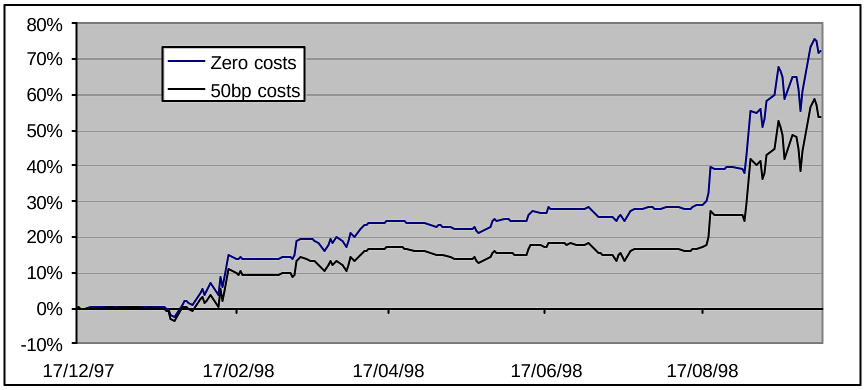


Figure 11.8: The performance of the *regularised* forecasting model for the mispricing of Diageo against Next, Granada and Safeway. The model is consistently profitable in spite of a negative $R^2$ of -69% during the out-of-sample period. The correlation between the forecasts and the actual relative returns, however, is 0.15 - demonstrating that profitability is determined by correlation rather than $R^2$.

### *Model which is highly sensitive to transaction costs*

The final example, Model 235, which is highly sensitive to the assumed level of transaction costs, is illustrated in Figure 11.9, and is based on the mispricing of Severn and Trent Water against the National Grid, Orange and P&O, with the forecasting model created using the *constructive* algorithm. Exogenous factors selected for inclusion in the model were 5-day changes in the FTSE and Dax indices, and 1-day changes in the S&P and Dax indices.

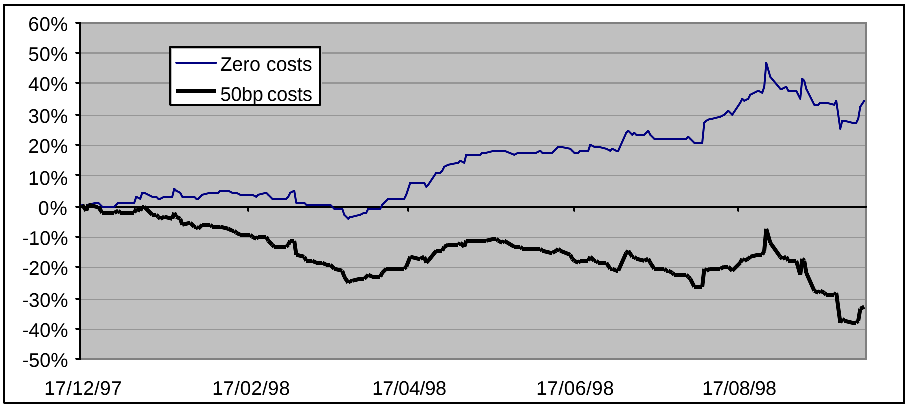


Figure 11.9: The performance of the forecasting model for the mispricing of Severn and Trent Water against the National Grid Group, Orange and P&O. The model is profitable on a zero cost basis but consistently loses money when the transaction cost spread of 50 basis points (0.5%) is accounted for. This suggests that the magnitude of the mispricings with respect to this particular fair price relationship is too low to exploit profitably - at least using a trading rule of this type.

The performance statistics and other characteristics of the four models are described in Table 11.9.

| Model | Algorithm | Mispricing | MDOF | F | H1/0bp | H2/0bp | H1/50bp | H2/50bp | Direction | Correl | $R^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 228 | D | 76 | 12.04 | 5.31 | 4.38 | 3.39 | 2.96 | 2.31 | 53.8% | 0.236 | 3.5% |
| 127 | C | 43 | 2.55 | 9.82 | 2.78 | -1.64 | 1.22 | -1.84 | 45.4% | 0.030 | -7.8% |
| 83 | R | 28 | 18.84 | 4.83 | 2.64 | 2.92 | 1.66 | 2.31 | 50.8% | 0.153 | -68.8% |
| 235 | C | 79 | 2.09 | 6.63 | 2.61 | 1.35 | -1.71 | -1.63 | 58.3% | 0.112 | 1.2% |

Table 11.9: Summary of the characteristics of the four models presented above. The first set of table entries describe the forecasting model and are ‘Model’: model reference number; ‘Algorithm’: D=deconstructive, C=constructive, R=regularised; ‘Mispricing’: reference to model used to generate the mispricings; degrees of freedom (MDOF) and F-ratio (F); the second set of figures present the Sharpe Ratio of the out-of-sample trading performance, divided between H1 and H2 and with costs at 0 and 50 basis points; the final set of values are the out-of-sample statistical performance metrics: ‘Direction’ = proportion of periods in which sign(forecast)=sign(actual); ‘Correl’ = correlation between actual and forecasted change in mispricing; ‘$R^2$’ = 1 - MSE/VAR(actual)

### Collective Performance of models created using different algorithms

When considering a set of models, for instance the set of forecasting models generated using a particular algorithm, measures of the average performance of *individual* models are of limited interest because they do not account for the relative correlations between the performance of the models within the set. Instead, the appropriate method for evaluating a set of models is to evaluate the combined performance of the entire set, thus taking into account the extent to which the strengths and weaknesses of the individual models will compensate for each other. In this section we employ a very simple approach to model combination, in which the notional capital is simply divided equally amongst the set of models. This "equal weighting" approach is the one which is most robust to sampling errors in the estimation of the cross-correlations of the various models. More sophisticated approaches to model combination, and the various issues involved in jointly optimising a set of models, are discussed in Part 3 of the thesis.

Table 11.10 presents the **collective** performance statistics of the sets of 90 models generated by each of the three neural estimation algorithms; the performance of a set of *linear* forecasting models is provided as a benchmark.

| Algorithm | Sharpe Ratio | H1 | H2 | Profitable | H1 | H2 | Return | H1 | H2 | StDev | H1 | H2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Linear | 3.26 | 3.79 | 3.53 | 58% | 60% | 56% | 28.2% | 8.0% | 20.2% | 8.7% | 2.1% | 5.7% |
| Constructive | 4.63 | 4.32 | 5.03 | 60% | 60% | 61% | 19.4% | 6.8% | 12.6% | 4.2% | 1.6% | 2.5% |
| Regularised | 4.68 | 5.21 | 4.72 | 63% | 67% | 58% | 19.9% | 7.4% | 12.5% | 4.3% | 1.4% | 2.6% |
| Deconstructive | 5.14 | 5.52 | 5.24 | 63% | 65% | 61% | 22.5% | 8.4% | 14.1% | 4.4% | 1.5% | 2.7% |

Table 11.10: Summary of the *collective* performance of the sets of 90 models generated by each of the three different neural network forecasting algorithms, together with benchmark figures for linear regression forecasting models. The performance statistics reported are annualised 'Sharpe ratio'; the proportion of trading periods which generated positive return across the portfolio as a whole; and first and second moments of the portfolio returns. Figures are presented across the out-of-sample period as a whole, and also broken down into two equal halves H1 and H2.

On the whole the results are broadly comparable across the different algorithms, with the deconstructive algorithm slightly outperforming in risk-adjusted terms. The linear models achieve the highest total return but this is mainly due to the second period only and at a high cost in terms of increased risk. On a risk-adjusted basis the low-bias models outperform the linear models by a ratio of approximately 1.5, a result which is consistent across both sub-periods.

The annualised Sharpe Ratios of the collective *sets* of models represent a substantial improvement over the performance of the *individual* models (e.g. SR=5.14 for the set of models created using the Deconstructive algorithm, as opposed to SR=0.85 for the average across the individual models), this improvement reflects the advantages of diversifying across a large set of models. The combined sets of models achieve profits of approximately 20% - although this is before transactions costs have been accounted for. The equity curves for the first half of the out-of-sample period are shown in Figure 11.10, below:

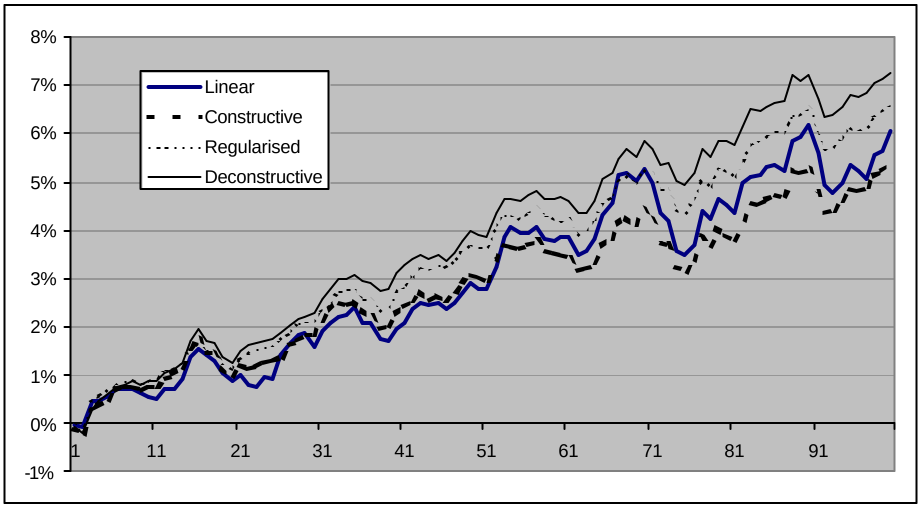


Figure 11.10: Equity curves for the *collective* performance of the sets of 90 models generated by each of the three different neural network forecasting algorithms, and linear regression forecasting models. The chart shows only the first half of the out-of-sample period; during the second half the performance of the linear models in particular becomes substantially more volatile.

The chart illustrates the strong correlation between the performance of the different sets of models. This suggests that, to a large extent, the models are exploiting the **same** information in the mispricing time-series and that the added-value of the more sophisticated low-bias models is relatively small - at least in non-risk-adjusted terms. This result may be considered a positive one in that it highlights the value of our modelling framework as a whole, and also a negative one in that it reinforces the view that, in the case of modelling highly noisy time-series at least, the use of neural network methods cannot be considered a “magic bullet” that achieves significant results in cases where other modelling approaches fail dismally.

In spite of these provisos, the deconstructive methodology in particular appears to add significant value in a risk-adjusted sense - achieving higher returns, and in a smoother manner than the linear model in particular but also the two other low-bias modelling algorithms. This result is consistent with the simulation results of Section 4.4 in which the deconstructive algorithm was found most robustly able to model nonlinear and interaction effects whilst being substantially resistant to the presence of spurious variables.

### The effect of applying different model selection criteria

Whilst the performance of the different algorithms is broadly similar, we have already seen that the performance varies widely across the individual models within each set. This motivates the use of **model selection criteria** to attempt to distinguish between the good and bad models and to improve performance by including only a subset of the models within the final portfolio.

Table 11.11 presents the results of applying a range of model selection criteria. The first is simply the projection of the (25-period) variance ratio profile of the mispricing onto the first principal component - i.e. an indication of the strength of the mean-reversion in the mispricing time-series. The other four model selection criteria are based upon the in-sample fit of the model, penalised according to the degrees of freedom. In order of increasing penalty they are: adjusted $R^2$, Akaike Information Criterion (AIC), Bayesian (or Swartz) Information Criterion (BIC/SIC) and F-ratio. In each case the model subset was chosen by ordering the models according to the selection criterion and then choosing a cutoff point at an appropriate discontinuity. In each case the resulting set of models consists of approximately 50 models chosen from the entire set of 270.

| Criterion | Sharpe Ratio | H1 | H2 | Profitable | H1 | H2 | Return | H1 | H2 | StDev | H1 | H2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Var Profile | 2.89 | 4.42 | 1.92 | 59% | 63% | 55% | 18.3% | 11.2% | 7.2% | 6.3% | 2.5% | 3.7% |
| Adjusted R | 3.27 | 4.89 | 2.57 | 57% | 63% | 50% | 17.3% | 8.9% | 8.4% | 5.3% | 1.8% | 3.3% |
| AIC | 3.15 | 4.51 | 2.59 | 59% | 62% | 55% | 16.0% | 7.8% | 8.2% | 5.1% | 1.7% | 3.2% |
| BIC/SIC | 3.59 | 5.49 | 2.57 | 63% | 68% | 58% | 17.8% | 10.1% | 7.7% | 5.0% | 1.8% | 3.0% |
| F-ratio | 3.88 | 6.23 | 2.96 | 58% | 64% | 52% | 18.1% | 9.4% | 8.7% | 4.7% | 1.5% | 2.9% |
| None | 3.92 | 5.23 | 3.45 | 60% | 64% | 55% | 14.2% | 6.4% | 7.8% | 3.6% | 1.2% | 2.3% |

Table 11.11: Summary of the collective performance of the *subsets* of the 270 models which were selected by different model selection criteria. The performance statistics reported are 'Sharpe ratio' of expected return to std. dev. of returns; the proportion of profitable trading periods; cumulative portfolio return and standard deviation of returns

Compared to the benchmark of simply including all 270 models, it appears that little is to be gained by making use of a model selection criterion. The variance ratio projection performs particularly badly, suggesting that it is not so much the degree of mean-reversion in the mispricing which determines model performance, but rather how well this is captured in the forecasting models themselves, possibly in conjunction with exogenous variables. Secondly, for all of the criteria the performance is lower during the second half of the out-of-sample period than during the first half, providing further evidence of performance nonstationarity amongst the models.

Amongst the complexity-penalising measures of insample model fit, there is a marked relationship between portfolio performance and the **severity** of the complexity penalty. The BIC penalises each degree of freedom by $\log(n)$ observations - in this case 5.99, whilst for the F-ratio the penalty for each additional degree of freedom is the F-ratio of the current model (Table 11.7 indicates that on average F=6.62). The selected portfolios for these two criteria significantly outperform the less-heavily penalised Adjusted $R^2$ and AIC, with the (most heavily penalised of all) F-ratio achieving the best performance. The implication of this is that in the content of highly noisy data with potentially unstable relationships, it is appropriate to penalise model complexity very highly when selecting amongst alternative models.

A final remark regarding these results is that although the BIC/SIC and F-ratio significantly outperform the benchmark ("None") criterion during the first half of the out-of-sample period, an increased level of performance degradation results in a relative underperformance during the second half of the period. This can also be seen in the equity curves for selected sets of models, shown in Figure 11.11 below:

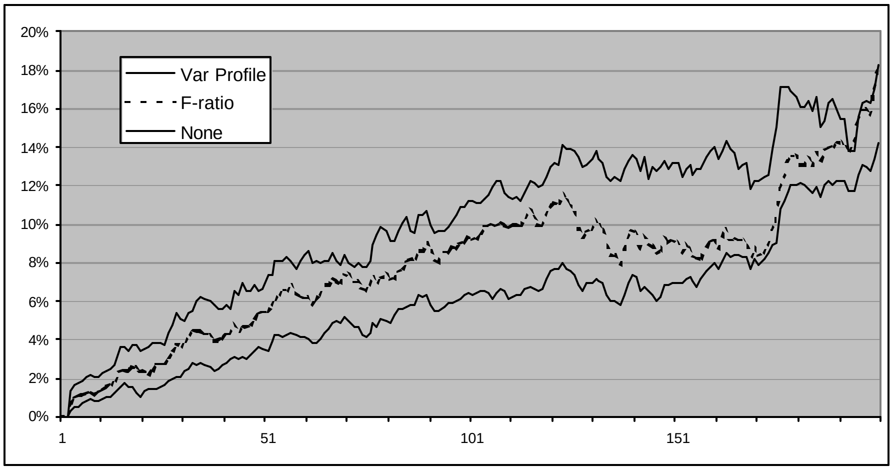


Figure 11.11: Equity curves for the collective out-of-sample performance of the models subsets selected by the variance ratio profile and F-ratio criteria, compared to the benchmark set of all 270 models.

The degradation in performance during the third quarter corresponds to the Russia/Brazil crisis of August 1998. Except for this period, the smoothest performance is shown by the set of models selected using the F-ratio criterion. This result indicates that there is a tradeoff between the short-term improvement in performance which can be achieved by an appropriate model selection criterion and the reduced robustness to performance nonstationarity due to the partial loss of diversification which is incurred by rejecting some of the models. In practical terms, it suggests that model selection should not be a "one-off" event, but be an ongoing process which is capable of responding to either temporary or permanent model breakdown. This and related issues regarding the **joint** optimisation of models within a portfolio are discussed in more detail in Part III of the thesis.

### The impact of transaction costs

The example in Figure 11.9 demonstrates that a model may have significant predictive ability in a statistical sense yet achieve consistently negative profits once transaction costs are accounted for. Table 11.12 presents the proportion of the 270 models which achieve positive out-of-sample profits for different assumed levels of transaction costs.

| | Full period | Q1 | Q2 | H1 | H2 |
|---|---|---|---|---|---|
| 0bp | 80.4% | 72.2% | 69.3% | 76.7% | 71.1% |
| 10bp | 75.6% | 66.7% | 65.9% | 73.3% | 68.5% |
| 20bp | 71.5% | 62.2% | 64.1% | 69.6% | 65.6% |
| 30bp | 67.4% | 60.4% | 62.6% | 64.4% | 59.3% |
| 40bp | 64.1% | 56.3% | 60.0% | 60.7% | 53.3% |
| **50bp** | **59.3%** | **54.4%** | **56.7%** | **57.4%** | **50.0%** |
| 75bp | 48.9% | 46.3% | 49.6% | 47.0% | 43.3% |
| 100bp | 38.1% | 39.6% | 35.9% | 31.1% | 37.8% |

Table 11.12: Proportion of individual models which achieve positive out-of-sample profits, as a function of the assumed level of transaction costs

The level of transaction costs is thus a major determinant of model profitability. The highlighted row of figures corresponds to bid-ask costs of 50 basis points (0.5%) which is a slightly higher value than would apply to a typical institutional trader (who is not a market maker). At this level the proportion of profitable models (over the entire out-of-sample period) has fallen from 80.4% at zero costs to only 59.3%; during the second half of the period the degradation is proportionally worse: from 71.1% to only 50.0%.

This result may suggest that the models are of little value to anyone except a market maker, with the inherent informational advantage being negated by market frictions. However, the impact of transaction costs may be reduced by the use of appropriate trading rules and model selection criteria. The trading rule issue concerns the trade-off between exploiting the available information whilst minimising losses due to transaction costs and is studied in more detail by Towers and Burgess (1998, 1999) and Towers (1999). The model selection issue relates to the consistency, or more correctly "persistence", in model performance and is explored below.

The examples in Figure 11.6-11.9 suggest that the profitability (or otherwise) of models contains a certain degree of persistence, and that this applies almost equally strongly even after transaction costs are accounted for. If this property of persistence in model performance generalises to the model set as a whole then we might expect to be able to identify subsets of models which are consistently profitable even after costs. In fact rather than identifying models with almost-certain *good* performance, it is more likely that we might identify the models with almost-certain *bad* performance, and improve the overall performance by **excluding** these from the model set. This is because while model performance may *degrade* through nonstationarity it is much less likely to suddenly *improve*.

In order to select models with positive performance *after transaction costs* it is necessary to set aside the first part of the out-of-sample period as a "selection period". Based on the (post transaction cost) **performance** during this period, it is possible to estimate the statistical significance of the model performance against the null hypothesis of zero profitability:

$$t_{\cos t=c} = \frac{\frac{1}{n}\sum_{i=1..n} p_i}{\sqrt{\frac{1}{n}\sum_{i=1..n}\left(p_i - \overline{p}\right)^2}} = \frac{\sqrt{n}\,\overline{p}}{s_{\,p}} \tag{11.6}$$

where $p_i$ is the (post-cost) profit achieved during period $i$. The results of applying this performance related selection criterion to the set of 270 forecasting models are presented in Table 11.13 below. The selected model set contains all models where the confidence level of positive performance is 85% or more, based on the performance during the first quarter (50 observations) of the out-of-sample period. The resulting portfolio consisted of 26 models.

| | Sharpe Ratio | | | | | Profitable | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Q1 | Q2 | Q3 | Q4 | **Q2-4** | Q1 | Q2 | Q3 | Q4 | **Q2-4** |
| Cost = 0bp | | | | | | | | | | |
| Selected models | *13.65* | 4.82 | 2.71 | 5.24 | **4.25** | *80%* | 60% | 56% | 64% | **60%** |
| All models | 7.42 | 3.96 | 0.62 | 5.60 | **3.52** | 72% | 56% | 50% | 60% | **55%** |
| Cost = 50bp | | | | | | | | | | |
| Selected models | *10.02* | 2.26 | 0.85 | 3.70 | **2.45** | *76%* | 56% | 48% | 62% | **55%** |
| All models | -0.26 | -0.49 | -2.76 | 2.52 | **0.13** | 52% | 48% | 44% | 52% | **48%** |

Table 11.13: Performance (in terms of both Sharpe Ratio and the proportion of profitable periods) of model portfolio selected on the basis of Q1 performance after accounting for **transaction costs** of 50bp (0.5%). First quarter figures for selected models are in italics to reflect the fact that they are in a sense insample results and so cannot be fairly compared to the benchmark during this period

Whilst the transaction costs almost eliminate the profitability of the **all-model** benchmark during quarters 2-4 of the out-of-sample period, the **selected** portfolio remains consistently profitable during all three sub-periods. This indicates that the (post-transaction-cost) performance of the individual models is sufficiently persistent to allow a meaningful filtering of the models. Note that the reduced performance during quarter three (Q3) corresponds to the period of abnormal market dynamics caused by the Russia/Brazil crises and that the improved performance in Q4 corresponds to the period during which the markets returned somewhat to normality. In spite of the turbulence the portfolio of selected models remains profitable during this period (after costs) albeit with a reduced risk-adjusted performance which is reflected by the lower Sharpe Ratio. The equity curves for the selected models against the benchmark all-model set, are presented in Figure 11.12 below.

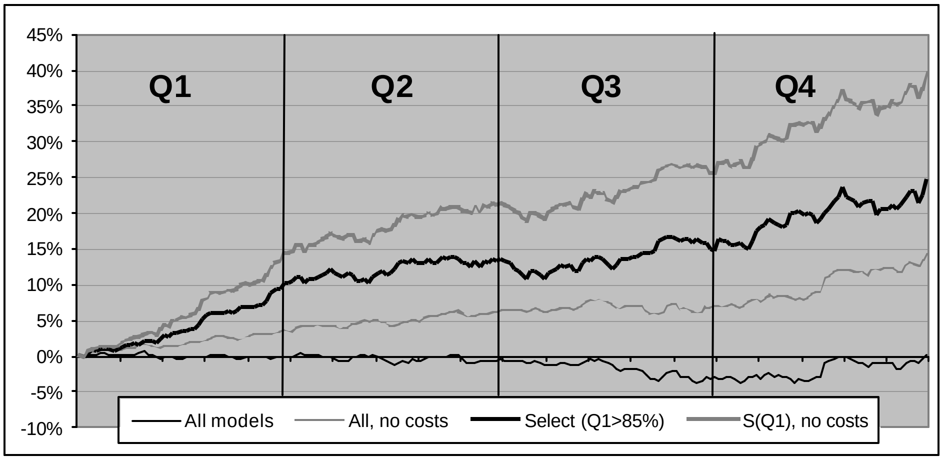


Figure 11.12: Cumulative profits of the **selected-set** of models, using the performance-based criterion, against the **all-model** benchmark. Profits are shown both gross, and net of round-trip costs of 50 basis points (0.5%). Note that Q1 performance contains an upward bias in the case of the selected set of models as it was used as the evaluation period on the basis of which the model selection was performed.

## Summary

In this section we have described the results of applying the model-free variable selection methodology of Chapter 9 and the low-bias modelling methodology of Chapter 10 to the problem of forecasting the price-dynamics of statistical mispricings amongst constituents of the FTSE 100 index. On an individual basis, the models are marginally profitable with an average annualised Sharpe Ratio of only 0.85

In contrast to the individual performance of the models, the *collective* performance of the entire portfolio of models benefits greatly through risk diversification, with a Sharpe Ratio of 4.88 for the portfolio consisting of all 270 low-bias forecasting models. The deconstructive version of the neural estimation methodology produces the most consistent risk-adjusted performance, corresponding to a 50% improvement over a set of linear models. The application of various statistical model selection criteria produced some evidence of short-term improvement in performance, for those criteria which most heavily penalise additional model complexity, but at the price of reduced diversification.

In a realistic evaluation, it was found that the positive information content in the models is largely negated by the cost of the transactions which are generated. However by setting aside the first part of the out-of-sample period to use as the basis of a *performance related* selection criterion, a subset of models was identified which achieved consistently profitable performance, even after costs, during the remainder of the out-of-sample period.

## 11.4 Summary

In this chapter we have described an empirical evaluation of the first two stages of our methodology used in combination. The model-free variable selection procedures of Chapter 9 and neural model estimation algorithms of Chapter 10 are used to forecast innovations in the statistical mispricings between FTSE 100 constituents. These forecasts are used as the basis of a set of “conditional statistical arbitrage” (CSA) strategies and found to be capable of generating consistently positive out-of-sample profits, even after realistic levels of transaction costs are accounted for.

The results obtained in this chapter demonstrate that the methodology of previous chapters can be successfully applied to the problem of forecasting the dynamics of statistical arbitrage. They also highlight the importance of a number of issues related to model selection and combination, particularly in the presence of transaction costs and where the ultimate performance metric is different to that used during the model estimation process. These issues are developed in Part III of the thesis.

# Part III: Diversifying Risk by Combining a Portfolio of Statistical Arbitrage Models

In this part of the thesis we describe the third part of our statistical arbitrage methodology. This addresses the implementation issues which arise in the context of applying predictive models to risk-averse decision-making in general and statistical arbitrage in particular. The methodology aims to reduce the risks which are inherent in the modelling process itself, thus increasing the extent to which the predictive information is efficiently exploited and increasing the likelihood of achieving successful statistical arbitrage strategies.

Chapter 12 describes our methodology for diversifying model risk. This is achieved through the use of model combination techniques which take into account the two equally important objectives of maximising return and minimising risk. Furthermore, the methodology emphasises the importance of using selection criteria which are as similar as possible to the ultimate performance measure (i.e. after-costs trading performance) rather than the traditional statistical criteria based upon forecasting accuracy alone. The traditional approach to model combination is used in conjunction with the risk-averse optimisation techniques of modern portfolio theory in order to achieve a "portfolio of models" approach. This approach is evaluated with respect to the conditional statistical arbitrage models described in the previous part of the thesis.

Chapter 13 represents a less developed solution to a much more ambitious task, namely that of integrating all stages of the modelling process in a single optimisation procedure. The objective of this approach is to reduce, and ultimately eliminate, the various inefficiencies which arise through the use of "multi-stage" approaches to modelling; for instance, when pre-processing, predictive modelling, and trading rule implementation as treated as separate rather than inter-dependent tasks. This chapter describes a population-based algorithm in which an entire set of models is generated in the context of a "joint" optimisation procedure. Through the use of optimisation criteria in which individual models are evaluated in terms of the *added-value* they provide to an existing portfolio, the algorithm actively encourages diversification and hence maximises the consequent opportunities for risk-reduction. The approach is evaluated with respect to both artificial and real-world problems.

# *12. A "Portfolio of Models" Approach to Statistical Arbitrage*

This chapter describes our "portfolio of models" methodology for diversifying the individual risks of the statistical arbitrage models. The basis of the methodology is a synthesis of the model combination approach with modern portfolio theory (Markowitz, 1952, 1959), thus creating a methodology which simultaneously optimises both risk and return within the context of trading a set of statistical arbitrage models. Section 12.1 discusses the risks which are involved in the use of model selection techniques which are aimed at identifying a single "best" model, and also the manner in which these risks are much increased by the noisy and nonstationary nature of asset price dynamics. Section 12.2 discusses how these risks can be much reduced by the use of model combination techniques and describes our "portfolio of models" algorithm for optimising the after-costs risk-adjusted return of a set of statistical arbitrage models. Section 12.3 describes the application of the portfolio of models methodology to the set of conditional statistical arbitrage models which were described in Chapter 11.

## 12.1 Model Selection in Noisy and Nonstationary Environments

Irrespective of the statistical rigour of the modelling process, the *future* performance of forecasting models in general, or statistical arbitrage models in particular, will necessarily be uncertain. The fact that model selection criteria are evaluated with respect to a particular finite sample will cause *sampling error*. This in turn means that the model which is apparently optimal, according to the model selection criteria, will not necessarily be optimal in future. Furthermore this risk is much increased when the noise content of the data is high and/or the performance may be unstable over time due to the presence of nonstationarities in the underlying data-generating process.

In this section we consider the issue of **model selection** and present an analysis of the risks which are induced by the model selection process.

### 12.1.1 Model Selection

Given a set of forecasting models $F = \left\{ f_1(\mathbf{x}_1, \mathrm{q}_1), f_2(\mathbf{x}_2, \mathrm{q}_2), \ldots, f_{nm}(\mathbf{x}_{nm}, \mathrm{q}_{nm}) \right\}$ and performance metric $m_i = M\left( f_i(\mathbf{x}_i, \mathrm{q}_i), y_i, \mathbf{z}_i \right)$ then the problem of model selection is simply to select the model $f^*$ which achieves the highest value of the performance metric:

$$f^* = f_i \ \text{s.t.}\ m_i = \max\left(m_1, m_2, \ldots, m_{nm}\right) \tag{12.1}$$

Performance criteria which are commonly used for model selection are referred to as *model selection* criteria and fall into three main categories: measures of (insample) model fit, measures of out-of-sample (statistical) performance and application-specific performance measures. Before discussing the complications which can arise during the model selection process, we briefly review the most common model-selection criteria.

**Selection criteria based upon Model fit**

A natural measure of model quality is the extent to which the model correctly represents the available (insample) data. Criteria based upon this approach (see Section 2.2.3) vary primarily in the extent to which they penalise the number of degrees of freedom which are required in order to achieve the observed fit; the motivation for penalising complexity is to optimise the bias-variance tradeoff by avoiding “overfitting”. In order of the severity of the complexity penalty, model selection criteria within this class include $R^2$, adjusted $R^2$, $C_p$ or Akaike Information Criterion (AIC), Bayesian or Schwarz Information Criterion (BIC or SIC) and F-ratio. These criteria can all be expressed as functions of two measures: the amount of insample variance which is captured by the model, and the number of degrees of freedom absorbed by the model. The evaluation in Chapter 11 suggests that statistical arbitrage models selected on the basis of criteria which heavily penalise additional complexity (degrees of freedom), such as BIC or F-ratio, tend to be superior to those selected using the less severe penalties.

**Selection criteria based upon out-of-sample (statistical) performance**

The second “family” of model selection approaches is that which relies in some sense on *out-of-sample* testing. The simplest form of this approach is simple “validation” in which the model is selected which performs best on a particular out-of-sample set. This method is commonly used in the “early stopping” approach to neural network learning (see Section 2.2.4). A more computationally intensive form of this approach is “k-fold cross validation”, in which the data set is divided into $k$ subsamples and the performance on each subsample is estimated with respect to the model optimised on all other subsamples. The extreme of this approach is full “leave one out” cross-validation (Wahba and Wold, 1975). A related but distinct approach is the use of “bootstrap” resampling (Efron and Tibshirani, 1993) of the data to obtain unbiased estimates of prediction error.

**Selection criteria based upon application-related criteria**

In general, statistical measures of model fit or forecasting performance will only be indirectly related to the ultimate objective of the wider decision making problem as a whole. This suggests the possibility of performing model selection based upon *application-specific* measures of performance. In the context of model-based trading systems, such criteria are concerned with the profitability of the trading signals which are generated by the system. The simplest such measure is the cumulative trading profit, measured over a particular, preferably out-of-sample, period. More commonly, the performance of the system is evaluated in terms of a metric which takes into account both the profitability, or expected **return**, and the variability of the returns, or **risk**. As noted in Section 7.1, a measure which is commonly used by practitioners is the Sharpe Ratio (SR) of average (excess) return divided by standard deviation of returns:

$$SR\left(f, y\right) = \frac{\frac{N}{n}\sum_{t=1..n} s(f_t)\Delta y_t}{\sqrt{\frac{N}{n}\left[\sum_{t=1..n} s(f_t)\Delta y_t - \left(\frac{1}{n}\sum_{t=1..n} s(f_t)\Delta y_t\right)\right]^2}} = \sqrt{N}\,\frac{\bar{r}}{s_r} \qquad (12.2)$$

Where the “trading signal” $s(f_t)$ indicates the exposure or “position taken” in asset $y$ at time *t* given the forecasted price change $f_t$. The Sharpe Ratio may be expressed in single-period form, in which case *N*=1, *or “annualised”* by setting *N* equal to the number of trading periods per year (approximately 250 for daily observations). Another common measure is the Risk-Adjusted Return (RAR) from Markovitz (1952, 1959) mean-variance portfolio theory:

$$RAR(f, y, T) = N\left(\bar{r} - \frac{s_r^2}{T}\right) \tag{12.3}$$

Where *T* can be viewed as a "risk appetite" or tolerance parameter. A survey of performance metrics for trading systems is presented in (Refenes, 1995).

### 12.1.2 Model Selection Risk

In situations where the performance metrics are both perfectly accurate and perfectly stable over time, then model selection poses relatively few risks and the main issue is simply one of choosing the appropriate selection criterion. From our perspective, however, all of the criteria listed in the previous section have two features in common which serve to make the distinctions between them relatively unimportant. Firstly, given a finite data set, they may select the wrong model; secondly, this risk is much increased when the noise content is high and/or model performance may be unstable over time due to the presence of nonstationarities in the underlying data-generating process.

**Model selection risk due to sampling variance**

In order to appreciate the primary source of model selection risk it is sufficient to consider that the use of finite, noisy datasets will induce sampling error not only in the estimated model, but also in the performance metric itself. Thus the "ideal" model selection task of Eqn. (12.1) is transformed into the finite sample case:

$$\begin{aligned} f^* &= f_i \text{ s.t. } m_i + e_{I,i} = \max\left(m_1 + e_{I,1}, m_2 + e_{I,2}, \ldots, m_{nm} + e_{I,nm}\right) \\ &= f_i \text{ s.t. } r_i = \max\left(r_1, r_2, \ldots, r_{nm}\right) \end{aligned} \tag{12.4}$$

where $e_{I,i}$ is the insample variation of performance metric *M* applied to model $i$. Where such sampling error is present, there is a possibility that a suboptimal model will be selected, i.e. $r_i = \max\left(r_1, r_2, \ldots, r_{nm}\right)$ but $m_i < \max\left(m_1, m_2, \ldots, m_{nm}\right)$. Figure 12.1 illustrates that when the base level of performance is relatively low, this risk is greatly exacerbated:

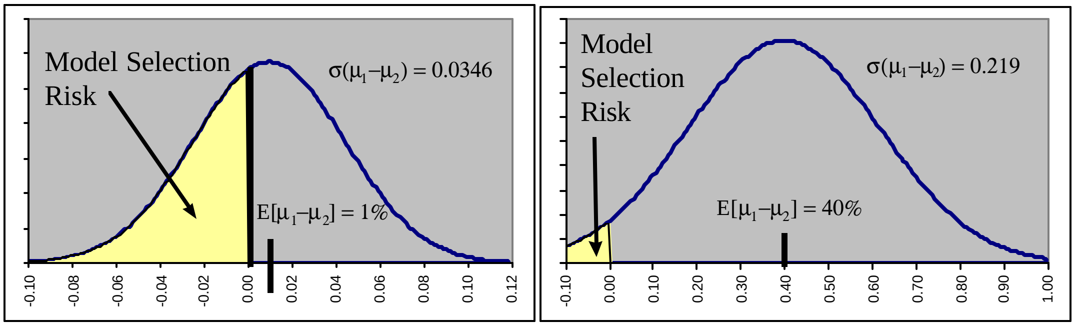


Figure 12.1: Illustration of the link between model selection risk and the base level of performance; in the first case model 1 has an expected performance ($R^2$) of 2% and model 2 1%; based on a sample of size 100 the chance of incorrectly selecting model 2 is 0.385; in the second case, model 1 has expected performance of 80% and model 2 of 40%, the risk of selecting the wrong model is only 0.033

The figure highlights the fact that "model selection risk" is related to the sampling error of the performance metric, and is thus greatest either when the sample size is small and/or the noise content of the data is high. Whilst the use of finite samples of noisy data is a sufficient cause of model selection risk, a number of other factors may increase the risk by inducing additional bias and/or variance into the model selection procedure. A number of these contributory factors are considered below.

### Use of finite out-of-sample period

Model selection risk is increased when the objective is to identify the model which will generalise best, not in an asymptotic sense, but during a *particular* finite out-of-sample period. In this case the out-of-sample performance, as well as the insample performance, is subject to sampling error. This will approximately double the risk that the selected model will perform suboptimally during a particular out-of-sample period.

### The effect of performance nonstationarity

The effect of any time-variation in the underlying data-generating process will be to contribute to the increase the variability of **future** performance, given a particular level of insample performance. This exacerbates the effect described above, further reducing the probability that the optimal insample model will also perform optimally during an out-of-sample period. Performance nonstationarity will also tend to induce a **bias** in the value of the model selection criterion being more likely to lead to a reduction in performance than to an improvement. The model selection risks which arise from the use of finite, noisy and nonstationary datasets were illustrated in Figure 4.9.

### Selection bias or "Data snooping"

Even assuming that no other biases are present in the model selection criterion, it is easy to demonstrate that the *measured* performance of the selected model will be positively biased with respect to the *true* expectation of future performance.

If $e_i$ represents the sampling variation of performance metric $M$ applied to model $i$ and the optimal model $f^*$ is selected according to Eqn. (12.4) then it is clear that:

$$\max(m_i) + \mathrm{E}[e_i] \leq \mathrm{E}\left[\max(m_i + e_i)\right] \leq \max(m_i) + \max(e_i) \tag{12.5}$$

As noted in Section 4.4, whilst the expected future performance of model $i$ is given by the first expression in Eqn. (12.5), the measured performance of the selected model is given by the second expression and will generally represent an inflated expectation of future performance. This discrepancy has been referred to throughout the thesis by the terms "selection bias" and "data snooping".

### Criterion Risk

We also noted in Section 4.4. that a final source of risk during model selection is similar to the effect of performance nonstationarity, in that it may represent either an additional bias, or an additional source of variability, or both. This is the choice of selection criterion.

In the case where the metric $M_S\left(f_i(\mathbf{x}_i, \mathrm{q}_i), y_i, \mathbf{z}_i\right)$ which is used for model *selection* is not the same metric $M_E\left(f_i(\mathbf{x}_i, \mathrm{q}_i), y_i, \mathbf{z}_i\right)$ which will be used for the ultimate *evaluation*, then there is no guarantee that the model which is selected according to criterion $M_S$ will be the same as that which would have been selected according to the true criterion $M_E$. This can occur when, typically for convenience, a *statistical* criterion such as $R^2$ is used to select between models, but an *application-related* performance measure such as Sharpe Ratio is used to evaluate the final models.

A more general form of this **criterion risk** is when the selection procedure is being applied to only a single component of a more complex system, such as selecting between synthetic assets before constructing appropriate forecasting models, or selecting between forecasting models independently of their associated trading rules. As with the use of statistical model selection criteria, simplifications of this type are typically a consequence of methodological limitations and will be discussed in more detail in Chapter 13.

### Summary

Irrespective of the statistical rigour of the modelling process, the *future* performance of forecasting models in general, or statistical arbitrage models in particular, will necessarily be uncertain. The fact that model selection criteria are evaluated with respect to a particular finite sample will cause sampling error. This in turn means that a model which is apparently optimal, according to the model selection criteria, will not necessarily be optimal in future. Furthermore this risk is much increased when the noise content of the data is high and/or the performance may be unstable over time due to the presence of nonstationarities in the underlying data-generating process.

In the following section we describe a portfolio of models approach which reduces model selection risk through the use of appropriate model combination techniques.

## 12.2 Controlling Model Selection Risk by Combining Models

From the discussion in the previous section we can see that performance statistics should be considered to be samples from noisy distributions - i.e. as being *indicative* rather than *definitive* of future performance. The implication of this is that unless we are in the fortunate position of being able to define a small set of models of which one is known to be correct, then model **selection**, as such, is both a highly biased and a highly risky methodology to adopt. Our solution to this problem is use model **combination** as a means of avoiding the risks which are inherent in selecting any single "best" model.

Consider the simple "risk averse" objective of maximising the *probability* of achieving positive performance. The *expected* performance of a combination $C = wM_1 + (1-w)M_2$ is given by:

$$m_C = \mathrm{E}\left[\mathrm{R}^2(C)\right] = wm_1 + (1-w)m_2 \qquad (12.6)$$

However the *standard deviation* of the combined performance depends not only on the individual standard deviations $s_i$ but also on the *correlation* $r$ between the performance of the two models.

$$s_C = SD\left[\mathrm{R}^2(C)\right] = \sqrt{w^2 s_1^2 + w^2 s_2^2 + 2w(1-w)r\ s_1 s_2} \qquad (12.7)$$

Figure 12.2 illustrates that the probability of achieving positive combined performance is a nonlinear function of *w* with an optimal value which tends to lie between the two extremes. In general, the advantages of model combination will depend upon the **expected** performance of the individual models, the **risk** which is associated with each performance estimate, and the **correlation** structure of the performance risks across the set of models. This recognition forms the basis of the "portfolio of models" approach which is described below.

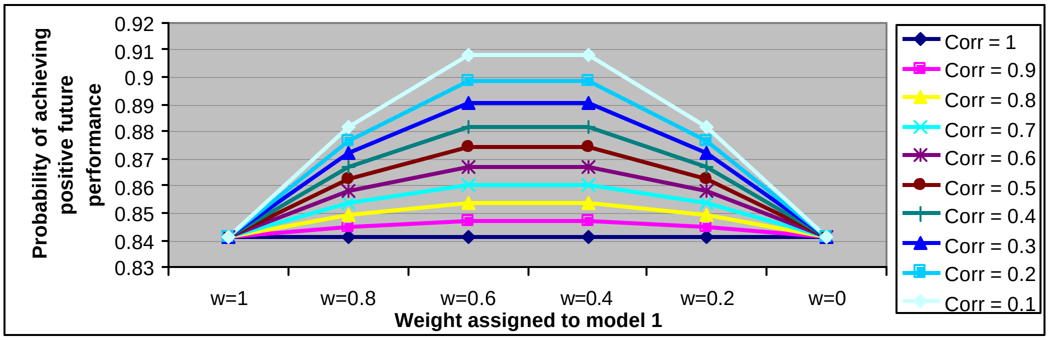


Figure 12.2: Figure illustrating the *probability* that a combination of two models will achieve positive future performance; the sampled $\mathrm{R}^2$ of both models is 1%, estimated over a sample of size 100; combining the two models reduces the variability in the overall performance and increases the probability that future performance will be positive.

## The Portfolio of models approach

Given a pre-existing set of forecasting models $F = \left\{f_1(\mathbf{x}_1, q_1), f_2(\mathbf{x}_2, q_2), \ldots, f_{nm}(\mathbf{x}_{nm}, q_{nm})\right\}$, and (uncertain) performance estimates $r_i = M\left(f_i(\mathbf{x}_i, q_i), y_i, \mathbf{z}_i\right) + e_i$, the task of model combination can be viewed as one of identifying suitable combination weights $w^*$ which optimise the expected (risk-adjusted) performance of the combined set of models:

$$w^{*} = \arg\max_{w} \ \mathrm{E}\left[ M\left( w_1 r_1 + w_2 r_2 + \ldots + w_{nm} r_{mn} \right) \right] \tag{12.8}$$

Typical constraints are that the weights should be non-negative and sum to one. The optimal combination of weights for a given set of models will depend upon the nature of the performance metric (which is now considered the "objective function" of the optimisation procedure), the set of performance estimates and the covariance structure of the performance uncertainties.

Fortunately, whilst it is not in general possible to optimise Eqn. (12.8) for arbitrary risk-adjusted objective functions, an ideal starting point for the case of trading strategies already exists in the form of the mean-variance approach to portfolio optimisation (Markowitz, 1952).

The core inspiration of Markowitz portfolio theory is that an investor's risk-adjusted performance can be increased by combining a set of investments in a suitable **portfolio**:

$$P = \sum_{i} w_i X_i \tag{12.9}$$

where the $X_i$'s represent the individual assets within the portfolio and the $w_i$ are the "portfolio weights". Assuming the returns on each individual asset $X_i$ are normally distributed with mean $r_i$ and standard deviation $s_i$, the vector of optimal portfolio weights is given by:

$$\begin{aligned} \mathbf{w}^{*} &= \arg\max_{\mathbf{w}} \left\{ \sum_{i} w_i r_i - \frac{1}{2T} \sum_{i} \sum_{j} w_i w_j s_i s_j r_{ij} \right\} \\ &= \arg\max_{\mathbf{w}} \ \left\{ \mathbf{w}^{T}\mathbf{r} - \frac{1}{2T} \mathbf{w}^{T}\mathbf{V}\mathbf{w} \right\} \end{aligned} \tag{12.10}$$

where $T$ is a "risk appetite" parameter, $\mathbf{w} = \left\{ w_1, w_2, \ldots, w_n \right\}^{T}$ is the vector of portfolio weights, $\mathbf{r} = \left\{ r_1, r_2, \ldots, r_n \right\}^{T}$ the vector of expected returns and $\mathbf{V}$ the covariance matrix of the returns. In practice, the elements of $\mathbf{r}$ and $\mathbf{V}$ are estimated from the first and second moments of historical returns over a particular time window $T-n.....T-1$:

$$\hat{r}_i = \frac{1}{n} \sum_{t=T-n}^{T-1} r_{i,t} \tag{12.11a}$$

$$\hat{s}_{i,j} = \frac{1}{n-1}\sum_{t=T-n}^{T-1}\left(r_{i,t} - \hat{r}_i\right)\left(r_{j,t} - \hat{r}_j\right) \qquad (12.11b)$$

The basis of our model combination methodology is to adapt the Markowitz framework by substituting **models** in place of individual assets, i.e. to create a "portfolio of models".

Given a set of models $\mathbf{m} = \left\{m_1\left(q_1\right), m_2\left(q_2\right), \ldots, m_{nm}\left(q_{nm}\right)\right\}$, performance estimates $\mathbf{r}_M = \left[\mathrm{E}[r_1] \quad \mathrm{E}[r_2] \quad \cdots \quad \mathrm{E}[r_2]\right]^T$ and covariance matrix $\mathbf{V}_M = \mathrm{E}\left\{\left[e_1 \quad e_2 \quad \cdots \quad e_{nm}\right]^T\left[e_1 \quad e_2 \quad \cdots \quad e_{nm}\right]\right\}$, the portfolio of models is given by a weighted linear combination of the individual models:

$$P_M = \sum_i w_{M,i}\, m_i \qquad (12.12)$$

and the optimal vector of weights is that which maximises *risk-adjusted* performance:

$$\mathbf{w}_M{}^* = \arg\max_{\mathbf{w}} \left\{\mathbf{w}_M{}^T\mathbf{r} - \frac{1}{2T_M}\mathbf{w}_M{}^T\mathbf{V}_M\mathbf{w}_M\right\} \qquad (12.13)$$

as with standard portfolio optimisation, the risk-tolerance parameter $T_M$ embodies the desired trade-off between increased expected performance and increased levels of risk.

The general "portfolio of models" framework could in principle be applied to situations where the concepts of "risk" and "return" are very different from those involved in portfolio investment. In this case the interpretation of the trade-off in Eqn. (12.13) is simply one between the *expected* level of performance and the *variability* in performance, with performance being defined in an application-specific manner. In the case of our **statistical arbitrage** methodology, however, there is a natural mapping between the traditional concepts of risk and return, as they relate to individual **assets**, and the risks and returns generated by the trading strategies which are based upon forecasting **models** of mispricing dynamics.

In the case of creating a portfolio of statistical arbitrage models, each model consists of three sub-models or components. The first component, which is constructed using the methodology described in Part I of the thesis and forms the basis of the model, is the specification of the statistical mispricing between the value $P_{T(i)}$ of the target asset *T(i)* and a weighted combination of "constituent" assets *C(i,j)*:

$$M_{i,t} = P_{T(i),t} - \sum_{j} b_{i,j} P_{C(i,j),t} \tag{12.14}$$

The second component of each statistical arbitrage model is the mispricing correction model which is created using the tools described in Part II of the thesis and which is used to forecast the expected innovation in the mispricing time-series:

$$\mathrm{E}\left[\Delta M_{i,t}\right] = \hat{f}_i\left(M_{i,t}, \Delta M_{i,t-k}, \mathbf{z_{i,t}}, \mathrm{q}_{\mathbf{i}}\right) \tag{12.15}$$

The third and final component of the model is the decision rule which transforms the forecasts into trading signals:

$$s_{i,t} = d_i\left(\hat{f}_i\left(M_{i,t}, \Delta M_{i,t-k}, \mathbf{z_{i,t}}, \mathrm{q}_{\mathbf{i}}\right), \mathrm{j}_{\mathbf{i}}\right) \tag{12.16}$$

A particular class of trading rules for statistical arbitrage models have been described in Chapter 11. These "conditional statistical arbitrage" (CSA) rules transform the forecasted innovation in the mispricing time-series into a trading signal (here denoted $s_{i,t}$) which indicates the desired holding in the mispricing portfolio. In general, the choice of a suitable trading rule with which to *exploit* predictive information is of comparable importance to the methodology which is used to *generate* the information (Towers and Burgess, 1998). A particular motivation for the portfolio of models approach is that it allows the performance of all three model components (statistical mispricing, predictive model and decision rule) to be jointly evaluated. This forms the basis of the population-based algorithm which is described in Chapter 13 and which is one approach to overcoming the "forecasting bottleneck". An alternative "reinforcement learning" methodology for performing joint optimisation of predictive models and trading rules is described in Towers and Burgess (1999b) and Towers (1999).

An important advantage of the portfolio of models approach is that the performance of the individual models can be represented in a manner which accounts for transaction costs. For instance, the performance estimates can allow for a proportional level of transaction costs *tc*:

$$r_{i,t}^{M} = s_{i,t}\Delta M_{i,t} - tc\left|s_{i,t} - s_{i,t-1}\right| \tag{12.17}$$

The relative weights of the models within the combined portfolio are then obtained by constructing the vector of expected model returns $\mathbf{r}_M$ and covariance matrix of model returns $\mathbf{V}_M$ and solving for the maximum risk-adjusted return of the portfolio as a whole:

$$RAR\left(P_M\right) = \max_{\mathbf{w}} \mathbf{r}_M \mathbf{w} - \frac{1}{T}\mathbf{w}^T \mathbf{V}_M \mathbf{w} \qquad \text{subject to} \quad \sum_j w_j = 1; \quad w_j \geq 0 \tag{12.18}$$

The final section in this chapter describes an empirical evaluation of the portfolio of models methodology, applied to the CSA models between FTSE 100 constituents which were described in Chapter 11.

## 12.3 Empirical Results for FTSE 100 models

In this section, we perform an empirical evaluation of the portfolio of models methodology which is described above. We compare the different approaches of model selection, simple model combination and "portfolio of models" combination in the context of the CSA models of FTSE 100 mispricings which were described in Chapter 11.

### 12.3.1 Evaluation of Model Selection Risk

In this section we present results, based upon the statistical arbitrage models of FTSE constituents, which illustrate the dangers of a "hard" model selection approach in a context where noise, nonstationarity and finite data impose significant bias and variance in the model selection criteria.

**Selection of "incorrect" models**

From amongst the set of 270 CSA models which were evaluated in Chapter 11, Table 12.1 presents an analysis of the performance of the models that are identified as the "best" models according to a range of different model selection criteria.

| | Model | SR | SR(Q1) | SR(Q2) | SRwc | SRwc(Q1) | SRwc(Q2) | Direction | Correl | $R^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| *Selection Criterion* | | | | | | | | | | |
| VRprof | 89 | 137 | 148 | 13 | 109 | 127 | 8 | 97 | 203 | 204 |

|  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|
| F-ratio | 89 | 137 | 148 | 13 | 109 | 127 | 8 | 97 | 203 | 204 |
| SIC | 239 | 73 | 72 | 175 | 75 | 72 | 156 | 66 | 86 | 237 |
| AIC | 240 | 79 | 91 | 120 | 71 | 77 | 97 | 66 | 74 | 249 |
| $R^2$(adj) | 240 | 79 | 91 | 120 | 71 | 77 | 97 | 66 | 74 | 249 |
| SR(Q1) | 37 | 95 |  | 59 | 79 |  | 43 | 74 | 138 | 63 |
| SRwc(Q1) | 37 | 95 |  | 59 | 79 |  | 43 | 74 | 138 | 63 |

Table 12.1: Performance of the "best" models which were identified by different model selection criteria. Each column indicates the *performance* rankings (out of 270) of the models which rank highest according to a range of different *selection* criteria. The model selection criteria are Variance ratio projection, F-ratio, SIC, AIC and adjusted $R^2$ of the forecasting model, Sharpe Ratio of trading performance during first 50 out-of-sample observations (wc = with costs at 50bps). Performance measures are Sharpe Ratio (overall, first 50, next 50) at zero and 50bps costs, Directional Correctness, predictive correlation, and (out-of-sample) $R^2$

The results emphasise the danger of employing a hard model selection approach. Out of all seven criteria, only the variance ratio profile and F-ratio select a model whose *performance* is ranked even in the top 20, and then only according to 2 of the 9 measures. The average ranking of the selected models, across all selection criteria and all performance measures is 102/270 - only slightly better than a purely random selection.

### Correlation of rankings between Model Selection criteria and out-of-sample performance

Table 12.2 presents a broader view of the accuracy of the rankings generated by the model selection criteria, in the form of the **rank-correlations** between the in-sample selection criteria and the out-of-sample performance metrics.

| | SR | SR(Q1) | SR(Q2) | SRwc | SRwc(Q1) | SRwc(Q2) | Direction | Correl | $R^2$ |
|---|---|---|---|---|---|---|---|---|---|
| *Selection Criterion* | | | | | | | | | |
| Vrprof | 0.064 | 0.141 | 0.102 | 0.117 | 0.173 | 0.134 | 0.074 | 0.235 | -0.129 |
| F-ratio | 0.084 | 0.146 | 0.079 | 0.121 | 0.154 | 0.143 | 0.050 | 0.199 | -0.114 |
| SIC | 0.064 | 0.113 | 0.046 | 0.074 | 0.096 | 0.108 | 0.043 | 0.175 | -0.173 |
| AIC | -0.027 | 0.053 | 0.026 | -0.156 | -0.071 | -0.020 | -0.050 | -0.018 | -0.473 |
| $R^2$(adj) | -0.032 | 0.037 | 0.021 | -0.189 | -0.097 | -0.040 | -0.033 | -0.041 | -0.471 |
| SR(Q1) | *0.446* | *1.000* | 0.187 | *0.415* | *0.903* | 0.132 | *0.198* | *0.395* | *0.293* |
| SRwc(Q1) | *0.395* | *0.903* | 0.193 | *0.565* | *1.000* | 0.244 | *0.170* | *0.360* | *0.298* |

Table 12.2: Rank correlations between the model selection criteria (rows), and the different out-of-sample performance measures (columns). Values in italics are biased due to overlapping evaluation periods.

The low levels of correlation in Table 12.2 emphasise the difficulty of performing model selection. The variance ratio profile, F-ratio and SIC are more positively correlated with the performance measures than are the AIC and adjusted $R^2$ criteria. The actual trading performance during the first quarter of the out-of-sample period is the most highly correlated criterion with respect to subsequent trading performance, especially in the "with costs" form, however note that only the Q2 correlations are meaningful in this case because the overlap with Q1 induces a bias in both the Q1 and overall figures.

### 12.3.2 Evaluation of Model Combination

The advantages to be gained from model **combination** are highlighted in the results shown in Table 12.3, which compares the average individual performance of the models to the collective performance of two combined sets of models. The first set consists of an equally weighted combination of all 270 models; the second set contains only those selected models whose performance during Q1 is statistically significant at the 0.15 level. (i.e. the final set of models evaluated in Chapter 11).

| | Cost | Sharpe ratio | | | | | Profitable | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Q1 | Q2 | Q3 | Q4 | **Q2-4** | Q1 | Q2 | Q3 | Q4 | **Q2-4** |
| Average of individual models | 0bp | 1.37 | 0.93 | 0.39 | 1.21 | **0.69** | 52% | 51% | 50% | 52% | **51%** |
| Combination (All models) | 0bp | 7.42 | 3.96 | 0.62 | 5.60 | **3.52** | 72% | 56% | 50% | 60% | **55%** |
| Selected models | 0bp | *13.65* | 4.82 | 2.71 | 5.24 | **4.25** | *80%* | 60% | 56% | 64% | **60%** |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Average of Individual models | 50bp | 0.10 | -0.06 | -0.59 | 0.47 | **-0.05** | 44% | 45% | 45% | 47% | **46%** |
| Combination (All models) | 50bp | -0.26 | -0.49 | -2.76 | 2.52 | **0.13** | 52% | 48% | 44% | 52% | **48%** |
| Selected models | 50bp | *10.02* | 2.26 | 0.85 | 3.7 | **2.45** | *76%* | 56% | 48% | 62% | **55%** |

Table 12.3: Comparison of performance of models on an *individual* basis, in a *fully-inclusive* combination and in a *selected* combination of models. The performance statistics quoted are the annualised Sharpe Ratio and the percentage of profitable periods. Q1 figures for “selected models” are upwardly biased through being used as part of the selection process itself.

The results represent an extension of those presented in Table 11.13, and clearly indicate the performance improvement which is achieved through diversifying the performance risk across the entire set of models. The inclusion of transactions costs at 50 basis points (0.5%) creates a sufficient downward bias in performance that the simple (equally weighted) model combination strategy is no longer successful. However, a subset selection criterion based on performance during the first subperiod achieves consistently profitable performance. This criterion exploits the persistence in risk-adjusted performance which was noted in the results of Table 12.2 above.

Additional results which demonstrate the improvement in performance which can be achieved through simple model combination methods are presented in Sections 7.2 and 11.3, concerning the ISA and CSA models respectively.

### 12.3.3 Evaluation of Portfolio of Models Approach

The results in this section were obtained by using the “**portfolio of models**” approach of Section 12.2 to optimise the combination weights of the FTSE 100 statistical arbitrage models. The portfolio of models approach can be thought of as a more sophisticated approach to model combination which is specifically designed for the case of combining trading models.

The methodology employed is as described in Section 12.2. The models to be combined consist of the 270 models which were generated in the experiments described in Section 11.3. For reasons of computational tractability the portfolio was constructed from a subset of these models, consisting of the 100 models which achieved the highest (risk-adjusted) performance during the first out-of-sample subperiod of 50 days. The portfolio of models was optimised with respect to risk-adjusted return on an annualised basis; the risk-tolerance parameter T was set to a realistic level of 20 (thus equating a risky expected performance of 20% p.a. with standard deviation 10% to a risk-free return of 20-(1/20)*$10^2$=15%).

The portfolio optimisation was performed on the basis of the first 50 out-of-sample observations and the resulting portfolio of models evaluated for a further 50 observations. After this period the portfolio was re-optimised to take into account systematic changes in the performance of individual models over time. The performance of the resulting portfolio of models is summarised in Table 12.4 below.

| | Costs = 0 bp | | | | Costs = 50bp | | | |
|---|---|---|---|---|---|---|---|---|
| | Q2 | Q3 | Q4 | **Q2-4** | Q2 | Q3 | Q4 | **Q2-4** |
| Return | 40.3% | 30.6% | 97.8% | **56.2%** | 12.2% | 6.5% | 64.9% | **27.9%** |
| Risk | 6.3% | 7.6% | 12.6% | **9.4%** | 6.3% | 7.6% | 12.6% | **9.3%** |
| RAR (T=20) | 38.3% | 27.8% | 89.9% | **51.9%** | 10.2% | 3.6% | 57.0% | **23.5%** |
| S.R. | 6.424 | 4.033 | 7.767 | **6.009** | 1.946 | 0.856 | 5.168 | **2.993** |

Table 12.4: Performance of **portfolio of models**; the statistics reported are (annualised) return and risk (of the portfolio as a whole), mean-variance Risk-Adjusted Return (with risk tolerance parameter T=20) and Sharpe Ratio. Portfolio is optimised with respect to zero transactions costs, and evaluated at both zero and 50 basis points transactions costs.

In comparison to the equivalent figures in Table 12.3, these results indicate that significant performance improvements, in risk-adjusted terms, can be achieved through combining the models within a portfolio. This improvement extends even to the selected subset of models and is due to the fact that the portfolio of models takes into account the covariance structure of the model returns whereas this was neglected in the previous equally-weighted approach.

Figure 12.3 presents the equity curves for the cumulative return of the portfolio of models under two different levels of assumed transaction cost.

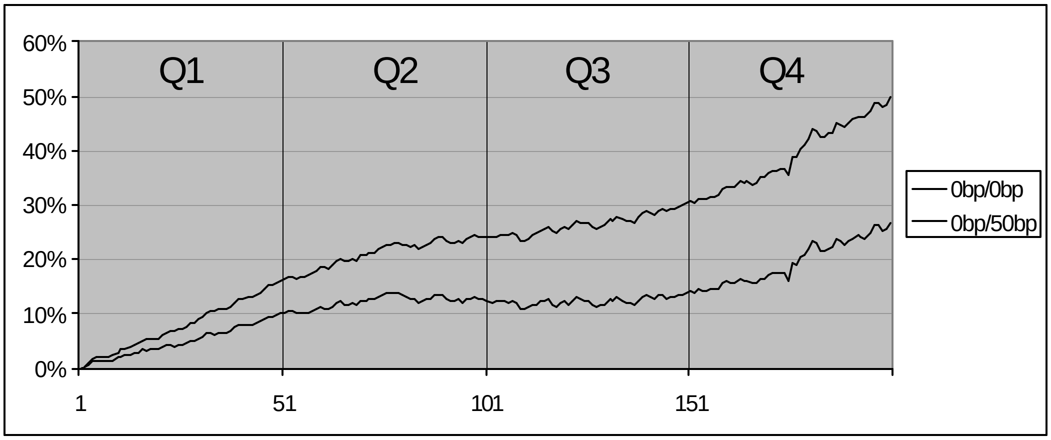


Figure 12.3: Cumulative profit and loss for portfolio of models. Portfolio optimisation is performed assuming zero transaction costs, and performance evaluation is conducted considering the two "extreme" scenarios of bid-ask spread = 0 and 50 basis points. Note that Q1 performance is effectively insample as it is used to estimate the expected returns and covariances of model returns. Performance estimates and portfolio weights are then updated after 100 and 150 periods.

Although the performance is very satisfactory when evaluated in the zero transaction cost case in which the portfolio of models was optimised, there is a substantial degradation when transactions costs are accounted for at the (rather high) level of 50 basis points. Particularly during the central part of the out-of-sample period, which corresponds to the Russia/Emerging markets crisis of late Summer 1998, the incorporation of these costs is seen to almost completely eliminate the overall profitability of the models. Note however that even during this period the portfolio of models remains marginally profitable and that performance improves substantially when the markets returned to more normal conditions towards the end of the out-of-sample period.

This negative impact of transactions costs may partly be caused by **criterion risk**, namely that the model weightings were selected under the assumption of zero transaction costs and may not necessarily remain optimal in the case of non-zero costs. In order to investigate this possibility the portfolio of models was re-evaluated in the context where the estimated returns upon which the portfolio weightings are based are pre-adjusted to take into account the effect of transaction costs at the 50 basis point level. The revised results in this case are presented in Table 12.5 and Figure 12.4.

|  | Costs = 0bp |  |  |  | Costs = 50bp |  |  |  |
|---|---|---|---|---|---|---|---|---|
|  | Q2 | Q3 | Q4 | **Q2-4** | Q2 | Q3 | Q4 | **Q2-4** |
| Return | 38.7% | 25.2% | 95.6% | **53.1%** | 16.0% | 6.5% | 69.6% | **30.7%** |
| Risk | 6.3% | 6.9% | 12.3% | **9.1%** | 6.3% | 7.0% | 12.3% | **9.0%** |
| RAR (T=20) | 36.7% | 22.8% | 88.1% | **49.1%** | 14.1% | 4.0% | 62.1% | **26.6%** |
| S.R. | 6.177 | 3.630 | 7.794 | **5.871** | 2.564 | 0.923 | 5.670 | **3.398** |

Table 12.5: Performance of portfolio of models optimised after transaction costs are accounted for; risk and return measures are all annualised; the portfolio of models is optimised with respect to transactions costs at 50 basis points, and evaluated at both zero and 50bp transaction cost levels.

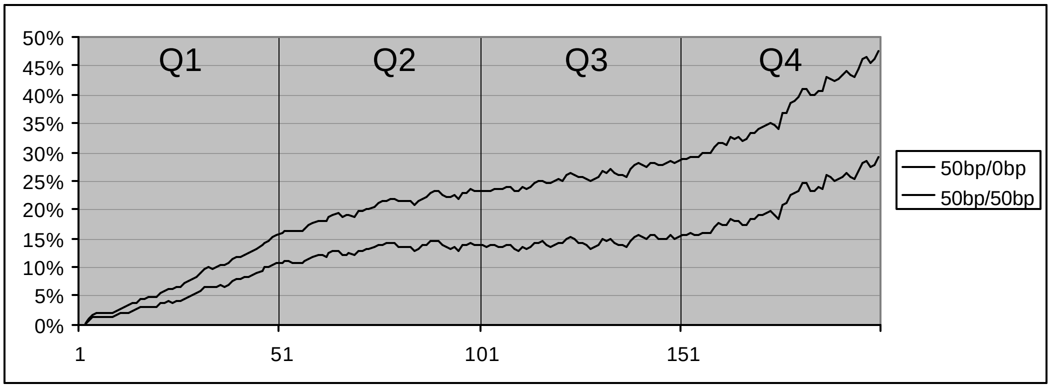


Figure 12.4: Cumulative profit and loss for portfolio of models. Portfolio optimisation is performed assuming transaction costs of 50 basis points, and performance evaluation is conducted considering the two scenarios of bid-ask spread = 0 and 50 basis points. The performance estimates and portfolio weights are updated after 100 and 150 periods.

Although slightly degraded in the zero-costs case, the revised Q2-4 results are significantly improved, by a factor of approximately 10% across both measures of risk-adjusted return, under the 50 basis point scenario. This provides evidence that criterion risk can have a significant influence on the overall performance and that it is best to optimise the portfolio of models under circumstances similar to those in which they will be evaluated.

## 12.4 Summary

The use of model selection is extremely hazardous in the case of statistical arbitrage trading strategies due to the high level of uncertainty which is attached to future trading performance. An empirical evaluation of a set of FTSE 100 statistical arbitrage models has been used to highlight the fact that model risk can be much reduced through the use of even simple model combination strategies. The more sophisticated “portfolio of models” approach improves performance further by allowing the correlation structure of the model performances to be exploited, and has the additional advantage of being applicable on an “after costs” basis.

# *13. A Population-based Algorithm for Joint Optimisation of Trading Models*

The previous chapter describes a framework for employing statistical arbitrage models within a "portfolio of models" context. In this chapter we investigate the mechanisms by which the risk-adjusted performance of a set of models is improved through the process of **risk-diversification**. Section 13.1 examines the circumstances which allow for successful risk diversification amongst a set of models and describes a population-based algorithm which has the objective of improving risk-diversification by performing **joint** optimisation of sets of models, rather than by optimising the models on an individual basis and leaving the diversification opportunities largely to chance. Section 13.2 illustrates the advantages of our methodology by a set of controlled simulations. Finally, Section 13.3 presents empirical results of applying the algorithm to generate diverse sets of statistical arbitrage models.

## 13.1 Risk-averse Optimisation

Given that a set of statistical arbitrage models will be *applied* within the context of a "portfolio of models", it is also preferable to *generate*, *optimise* and *select* models within this context. The motivation for such an approach is to minimise the effect of what we have referred to as "criterion risk": the fact that a model, set of models, trading system or whatever, which is optimal with respect to a *particular* selection or performance criterion will not necessarily be optimal with respect to a *different* criterion. Therefore under ideal circumstances, to minimise this risk, all parameters of a model would be optimised with respect to the ultimate performance criterion.

### 13.1.1 The Potential Inefficiency of Indirect Optimisation

While this ideal is not generally feasible to achieve, it is nevertheless true that standard modelling approaches tend to deviate further from it than is strictly necessary. The algorithm described later in this section attempts to reduce this discrepancy in the case of generating sets of statistical arbitrage trading models which will be employed within a portfolio of models context.

Section 12.2.1 briefly touched on the issue of “criterion risk”, noting that the model and parameters $f_i(q_i)$ which maximise a particular criterion $M_1$ will not necessarily maximise an alternative criterion $M_2$. This **criterion risk** can be quantified as:

$$
\begin{aligned}
&CR(M_1, M_2, f, q) = \max_{f,q} M_2[f(q)] - M_2\left[f^{M1*}(q^{M1*})\right] \geq 0 \\
&\text{where } f^{M1*}(q^{M1*}) = \arg\max_{f,q} M_1[f(q)]
\end{aligned}
\tag{13.1}
$$

In general, the construction and use of predictive models is a multi-stage process. Although the details may differ depending on the precise methodology adopted, the general phases might be considered to be: selection of the target series, variable selection, specification and estimation of the forecasting model, model selection and combination, decision rule implementation. Each of these stages involves a number of decisions, choices, inferences or optimisations regarding issues such as: the underlying assumptions, parameter specification/estimation, selection between alternatives etc.

Whilst ultimately based upon its practical utility, this standard “divide and conquer” approach is nevertheless dangerous in that it introduces a number of inefficiencies to the model search process. In particular the selections and optimisations performed at previous stages will impose limitations on the options available at later stages. The modelling process can be thought of as a multi-stage filter with each stage reducing the range of possible models, until ultimately a single model or combination of models has been defined in its entirety. This perspective is illustrated schematically in Figure 13.1.

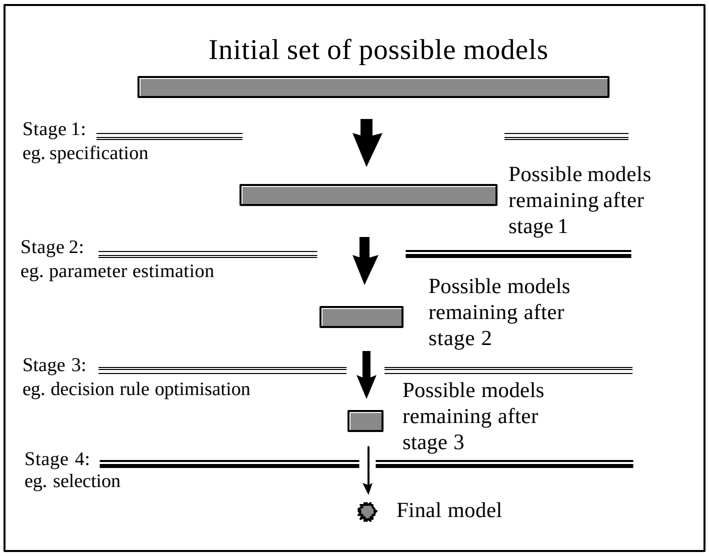


Figure 13.1: Schematic of the model-building process as a multi-stage filter; the wide range of possible models is reduced at each stage until the complete model emerges from the final selection stage.

Due to the nature of model development, it is generally infeasible to use the same criteria at all stages of the model-building process. For instance, the ultimate objective may be trading performance, but this can only be evaluated in the context of both a predictive model and a decision rule. Thus, whilst trading performance can be used as a criterion at the later stage of optimising the decision rule itself, estimation of the model parameters, and particularly selection of the target variable, are tasks which must be performed before the decision rule is known and hence must be performed using alternative criteria. Figure 13.2 illustrates the "forecasting bottleneck" (Moody *et al*, 1998) which can arise in the context of combining a predictive model with a trading rule:

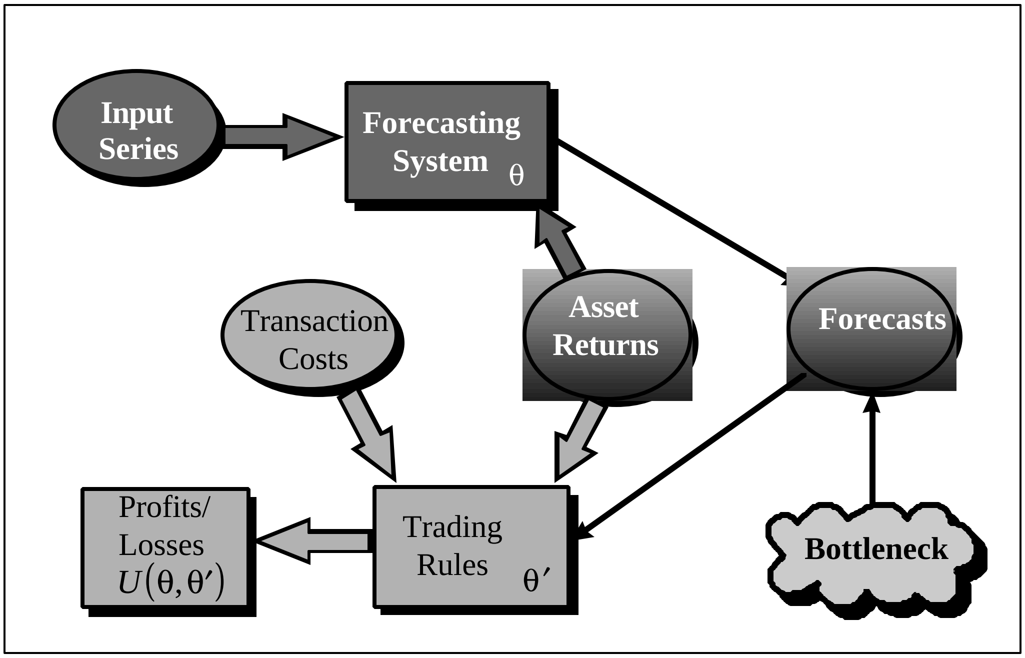


Figure 13.2: Illustration of the "forecasting bottleneck"; separate optimisation of the forecasting system and trading rule means that potentially useful information is ignored in each case; in particular the forecasts generated by the model are optimised by a different criterion than the trading rule and will for instance fail to take into account transaction costs and other market frictions; similarly the information in the input series which represent the current environment are only transmitted to the trading rule in the form of a single forecast, thus losing much information which is potentially important at later stages.

The population-based algorithm which we develop in this chapter represents an attempt to overcome this limitation of multi-stage approaches by effectively "closing the loop" in Figure 13.2. This perspective is reflected in Figure 4.1 which prevents an overview of our methodology, the basis of our approach is to "jointly optimise" the model components by both evaluating and optimising the entire set of components within a model in terms of the ultimate performance criterion.

Our statistical arbitrage models consist of three components which are generated sequentially: the "fair price model" $M_{i,t} = P_{T(i),t} - \sum_j b_{i,j} P_{C(i,j),t}$ (see Chapter 5) which generates the time-series of statistical mispricings; the "forecasting model" $\mathrm{E}\left[\Delta M_{i,t}\right] = \hat{f}_i\left(M_{i,t}, \Delta M_{i,t-k}, \mathbf{z_{i,t}}, \mathrm{q}_{\mathbf{i}}\right)$ (see Chapter 11) which predicts the future dynamics of the mispricing; and the "statistical arbitrage trading rule" $s_{i,t} = d_i\left(\hat{f}_i\left(M_{i,t}, \Delta M_{i,t-k}, \mathbf{z_{i,t}}, \mathrm{q}_{\mathbf{i}}\right), \mathrm{j}_{\mathbf{i}}\right)$ (see Chapters 7 and 11) which translates the predictions into a trading signal. The modelling methodology as a whole also contains a fourth stage, namely the combination of the individual models within a portfolio. Thus we potentially suffer from three major sources of criterion risk.

- the criterion for optimising the model weightings within the portfolio

- the criterion for optimising/selecting the individual models within the portfolio

- the "intermediate" criteria for optimising the various *components* of each model

The general weaknesses of multi-step and individual optimisation of models as opposed to the joint optimisation of the set of models as a whole are discussed in Burgess (1999c). In the particular context of combining a set of statistical arbitrage trading models, the crucial difference between the two approaches is that by optimising models on an individual basis we may undermine the potential advantages which are due to risk diversification across the population of models.

Intriguingly, due to the discrepancy between the criteria used at different stages, the danger of *inefficiency* at the portfolio level is exacerbated by the degree of *efficiency* with which individual models are optimised. The primary danger is that if the set of models "converge" towards the single model which is optimal on an *individual* basis then the opportunities for risk-diversification within the *portfolio* are correspondingly restricted. In the limit, where the "convergence" is complete, and all models are identical, then there is no possibility at all of benefiting from risk-diversification across the set of models.

### 13.1.2 Pareto Optimality, Diversity and Decorrelation

Having discussed the particular dangers which are posed by the use of multi-stage modelling procedures and identified the need for a framework for performing joint optimisation, we now consider the issues which are subsumed in the concept of "diversification" and the circumstances under which a diversification strategy can be successfully applied to the task of risk-averse optimisation.

In particular we consider below the issues of "Pareto Optimality", in which different candidate models or solutions may be optimal under different conditions, "Diversity" as represented by a set of such Pareto-optimal solutions, and "decorrelation" as a fundamental aspect of the time-series analogue of Pareto-optimality. From this perspective we note that successful diversification amongst a portfolio of trading models is dependent upon the models being at least partially decorrelated from each other.

#### Optimisation

The problem of optimisation can generally be considered one of selecting the parameters $q$ which maximise (or minimise) the value of some optimisation criterion, often referred to as a "fitness function" $F$ or "utility function" $U$.

Given a set of candidate solutions $C = \{C_1, C_2, \ldots, C_n\}$, the problem of optimisation can be considered equivalent to that of producing a preference relationship or *ordering* $\succ$ amongst the candidates such that $U(C_{o1}) \succ U(C_{o2}) \succ \ldots . U(C_{on})$. The optimal solution is then simply the candidate which is first in the ordering $C_{o1}$. Particularly in the case of multi-objective optimisation ($nf$ > 1), however, it may not be possible to produce such a total ordering because there may be candidates for which neither $U(C_i) \succ U(C_j)$ nor $U(C_j) \succ U(C_i)$. In such a situation it may be the case that no single optimal solution exists and the concept of optimality has to be broadened to that of "Pareto optimality" as discussed below.

#### Pareto Optimality

In order to illustrate the concept of Pareto-optimality, consider utility functions of the form:

$$U\left(C_k:q=q_k\right)=\sum_{i=1}^{nf} f_i\left(x_i\left(q_k\right),y\right) \tag{13.2}$$

In this formulation, the fitness of a candidate solution $C_k$ with parameters $q=q_k$ is defined in terms of the partial fitness $f_i$ which is conferred by possession of each attribute $i$ and which will in general depend on both the degree $x_i\left(q_k\right)$ to which the solution possesses the attribute and the influence of external circumstances $y$ which determine the “value” $f_i$ of the attribute.

In the case of single-objective optimisation (i.e. where $nf$ = 1) the task of ordering the candidate solutions is relatively straightforward, and in many cases can be performed even when the true fitness or “utility” function $U$ is unknown. Provided that the utility is both *monotonic* in the attribute, $d_1>d_2 \Rightarrow U\left(d_1\right)\succ U\left(d_2\right)$, and *independent* of external circumstances, $\forall_y: U\left(d,y\right)=U(d)$, then candidates can be ordered purely on the basis of the extent to which they possess the single desired attribute. In fact, even in the case where the utility function contains a stochastic or unknown component which is due to exogenous (external) factors $y$, only the weak assumption of linear separability $\left[U\left(d,y\right)=U(d)+U\left(y\right)\right]$ is required in order to still allow the candidates to be ranked with respect to $d$ alone, as the monotonicity assumption implies that $d_1>d_2 \Rightarrow U\left(d_1\right)+U\left(y\right)\succ U\left(d_2\right)+U\left(y\right)$.

In contrast, the problem of choosing between candidate solutions which represent alternative trade-offs between more than one desirable attributes is much less tractable. This difficulty is illustrated in Table 13.1 which illustrates the possible situations which arise when comparing candidate solutions on the basis of **two** desirable properties $x_1$ and $x_2$.

| | $x_1\left(q_A\right)>x_1\left(q_B\right)$ | $x_1\left(q_A\right)=x_1\left(q_B\right)$ | $x_1\left(q_A\right)<x_1\left(q_B\right)$ |
|---|---|---|---|
| $x_2\left(q_A\right)>x_2\left(q_B\right)$ | $M_A$ | $M_A$ | ? |
| $x_2\left(q_A\right)=x_2\left(q_B\right)$ | $M_A$ | indifferent | $M_B$ |
| $x_2\left(q_A\right)<x_2\left(q_B\right)$ | ? | $M_B$ | $M_B$ |

Table 13.1: Choosing the optimal model from two alternatives in the case of dual-objective optimisation; unless the relative values of the properties are known in advance, with certainty, then it may not be possible to choose between the two alternatives.

In situations (indicated by '?') in which circumstances exist in which either candidate may be superior, the candidates are said to be "non-dominated" by each other. When extended to the full set of candidate solutions, any candidate which is not dominated by any other candidate is said to be a "**Pareto Optimal**" solution. Formally, (again under the assumptions of monotonicity and independence) a candidate solution $C_A\!:\!q = q_A$ qualifies as Pareto-Optimal iff:

$$\neg\exists_B\!:\!\forall_j\!:\!x_j(q_B) \geq x_j(q_A) \tag{13.3}$$

i.e. there is no other solution which is at least as good on all attributes, and strictly superior on at least one[†] .

Figure 13.3 provides an illustration of the concept of Pareto-optimality in a 2-attribute setting:

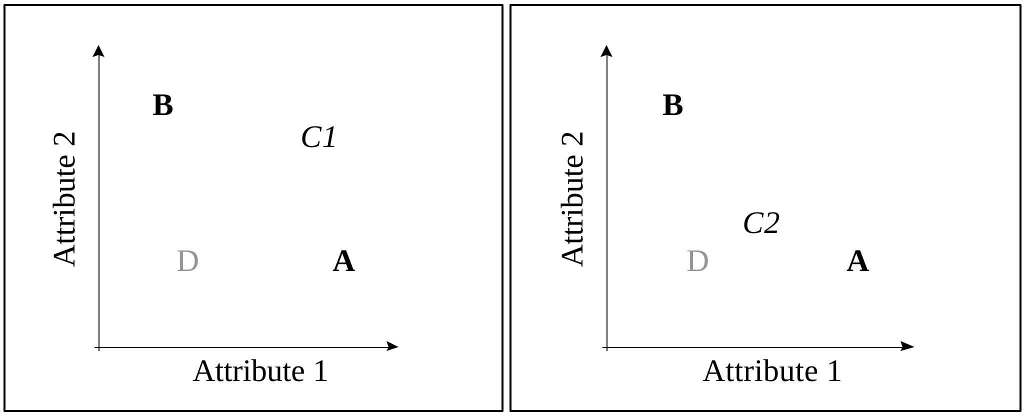


Figure 13.3: Illustration of Pareto-optimality in a 2-attribute optimisation problem. On the left the Pareto-optimal set is {A, B, C1}, and on the right {A, B, C2}.

On the left, candidate A is optimal with respect to the first attribute, and candidate B optimal with respect to the second attribute. However, model C1 is also Pareto-optimal because neither A nor B **dominates** C1 by being simultaneously superior on all attributes. Model D is

[†] or equivalently $\forall_B\!:\exists_j\!:x_j(q_A) > x_j(q_B)$, the candidate outperforms all others on at least one attribute

dominated by both A and C1 and under the monotonicity assumption cannot be optimal under any circumstances. In the similar example on the right the Pareto-optimal set is {A, B, C2}, although candidate C2 could only be optimal if the fitness function were nonlinear in the attribute values.

### Diversification

A principled motivation for the strategy of *diversification* arises from a conjunction of two factors. The first requirement is that the mapping from **parameters** to **attributes** must impose restrictions on the combinations of attributes which are realisable within an individual solution; i.e. even when the search within “parameter space” is unrestricted, the search within “attribute space” may be subject to unavoidable constraints. The second consideration is that the utility of the achieved solution must be subject to an element of uncertainty or risk, i.e. the function which relates the attributes of the solution to actual utility must contain either unknown parameters or a random component due to outside influences.

In order to illustrate these issues, and thereby the potential advantages which accrue from the use of diversification as a strategy, consider the following example. In this example the candidate models are represented by two real-valued parameters: i.e. $q = \{q_1, q_2\} \in \Re^2$ which map onto two attributes $x_1(q_1, q_2)$, $x_2(q_1, q_2)$ such that the utility of a solution is a linear combination of the attributes:

$$\begin{aligned} F(q_1, q_2) &= f_1 x_1(q_1, q_2) + f_2 x_2(q_1, q_2) \\ &= f_1 \Big/ \left(1 + 3[1 - q_1]^2 + 5[1 - q_2]^2\right) + f_2 \Big/ \left(1 + 5[2 - q_1]^2 + 3[2 - q_2]^2\right) \end{aligned} \tag{13.4}$$

The utility of a given solution will depend on the unknown attribute values $f_1$ and $f_2$. Figure 13.4 illustrates the “fitness landscape” as a function of the parameter values, in the case where $f_1 = f_2 = 1$.

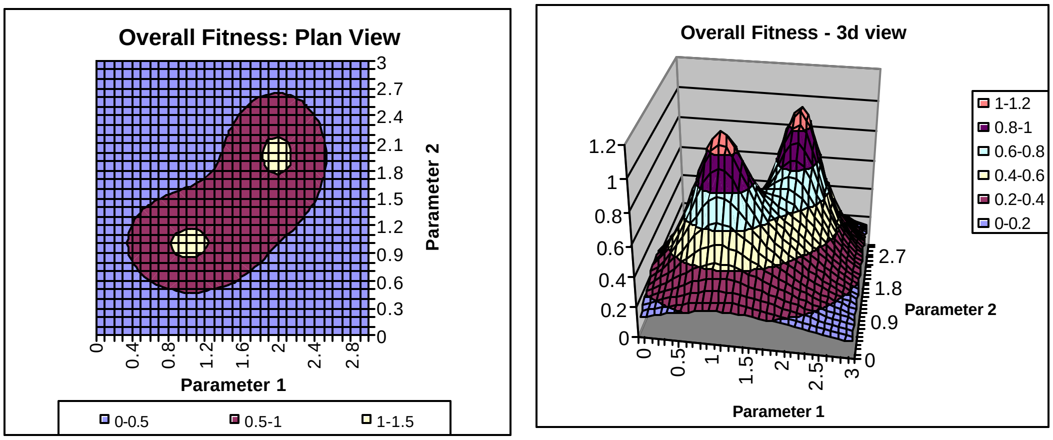


Figure 13.4: Mapping between parameters and overall fitness in the case where $f_1 = f_2 = 1$. Note that due to the nonlinear mapping between parameters and attributes, the fitness function (which is linear in the attributes) is also nonlinear in the parameters.

In general, the constants $f_1$ and $f_2$ which represent the relative desirability between the different attributes, are not simply "unknown" but instead are dependent on exogenous variables. In the context of natural selection and the evolution of life on earth, this dependence on external factors goes some way towards explaining the existence of "ecological niches" in which species evolve so as to exploit different sets of environmental conditions. In the context of risk-averse optimisation, the variability is typically of a probabilistic rather than geographical nature and the attribute values are seen as being drawn from probability distributions $p(f_1)$ and $p(f_2)$ respectively. It is this uncertainty in the value of a solution that we refer to as "risk", and it is within this context that we now motivate the use of diversification as a strategy for the control and reduction of risk.

In the context of risk-averse optimisation, the fitness function which is used to choose between alternative candidates will be a function both of the expected utility $\mathrm{E}[U(\mathbf{x}(q))]$ and the utility risk $U(\mathbf{x}(q), y) - \mathrm{E}[U(\mathbf{x}(q))]$. In the simplest case of quadratic optimisation, for instance, the objective is to maximise the expected utility whilst simultaneously minimising the variance of the utility risk:

$$F(\mathbf{x}(q), y) = \mathrm{E}[U(\mathbf{x}(q))] - \frac{1}{T}\mathrm{E}\left\{\left[U(\mathbf{x}(q), y) - \mathrm{E}[U(\mathbf{x}(q))]\right]^2\right\} \qquad (13.5)$$

where $T$ is a “risk tolerance” parameter which controls the trade-off between the two quantities. With minor modifications, the quadratic formulation encompasses a wide range of problems, from statistical parameter estimation (in which the objective is to minimise squared prediction risk), to portfolio construction (where the objective is to maximise the expected asset returns whilst penalising risk or volatility of returns).

Within this setting, the advantage of **model combination** strategies is that they allow diversification across different desirable attributes, thus reducing the overall level of uncertainty, whilst maintaining the expected level of performance. Furthermore, by performing optimisation at the combination or “population” level the range of achievable attribute combinations is guaranteed to increase, because:

$$\mathbf{x}\left(\sum_{q_i \in P} w_i q_i\right) \subseteq \mathbf{x}(q) \subseteq \sum_{j=1.n} w_j \mathbf{x}(q_j) \qquad (13.6)$$

and therefore the optimal fitness of the population will be at least as great as that of the optimal single model, which in turn will be at least as great as that of parameter-level combinations of individual models:

$$\max_{w} F\left[\mathbf{x}\left(\sum_{q_i \in P} w_i q_i\right)\right] \leq \max_{q} F[\mathbf{x}(q)] \leq \max_{w,q} F\left[\sum_{j=1.n} w_j \mathbf{x}(q_j)\right] \qquad (13.7)$$

An everyday analogy would be to say that whilst the best generalist may perform better than a team of generalists, the best results of all are likely to be achieved by a team of complementary specialists.

From a modelling perspective, the advantages of model combination arise from the fact that it may not be possible to construct a single set of **parameters** which represents a close to optimal tradeoff between the different **attributes**. By combining models, the tradeoff between attributes can avoid the constraints which are otherwise imposed by the parameter-attribute mapping. For this reason the risk-sensitive fitness of a population as a whole may greatly exceed that of the optimal individual within the population. Furthermore, the models which are most likely to add value at population level are most likely to be the non dominated or Pareto-optimal models, as these represent the boundaries of the attribute values which can be achieved by individual models and hence open up the space of possible solutions which can be achieved by attribute-level interpolation. This effect is illustrated in Figure 13.5.

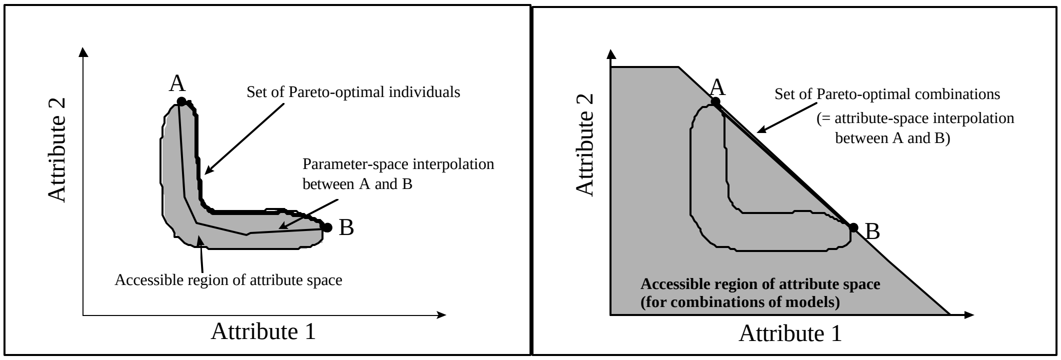

A
Set of Pareto-optimal individuals
Parameter-space interpolation between A and B
B
Accessible region of attribute space
Attribute 2
Attribute 1
A
Set of Pareto-optimal combinations
(= attribute-space interpolation between A and B)
B
Accessible region of attribute space (for combinations of models)
Attribute 2
Attribute 1

Figure 13.5: The advantage of model combination: through allowing interpolation between models at the attribute level, the range of potential solutions is larger than for individual models.

### Time series Diversification: Decorrelation

Unfortunately, in time-series problems, the underlying "attributes" themselves are typically not observed directly, but rather indirectly through the impact they have upon the observed time-series. The desirable attributes of statistical arbitrage models are conceptually of the form: "ability to exploit mean-reversion/trending behaviour between assets/markets X/Y and Z", "reliance on normal market conditions", "sensitivity to emerging markets crisis", and hence are difficult, if not impossible, to measure directly.

The solution to this dilemma is simply to recognise that although models cannot be compared directly at the attribute level, the net effect of the model attributes can be evaluated indirectly through the observed time-series of profits and losses. In particular the similarity of two models can be assessed through the **correlation** of the returns which they generate. Consider a general fitness function in which the profits and losses of a statistical arbitrage model are determined by the sensitivity to an arbitrary number of unknown factors:

$$r(q)_t = \sum_{jeF} \left( f_j + e_{j,t} \right) x_j(q) \qquad (13.8)$$

Where $r(q)_t$ is the return of the model with parameters $q$; $x_j(q)$ is the extent to which the model is sensitive to factor *j*; $f_j$ is the expected value of factor *j* and $e_{j,t}$ is the stochastic component associated with factor *j* at time t.

Assuming that the factors are independent of each other, then the correlation between the returns of two models A and B is given by:

$$r(A,B)=\frac{\operatorname{cov}\left(r(q_A)_t,r(q_B)_t\right)}{sd\left(r(q_A)_t\right)sd\left(r(q_B)_t\right)}=\frac{\sum_{j\in F}\mathrm{E}\left[e_j{}^2x_j(q_A)x_j(q_B)\right]}{sd\left(r(q_A)_t\right)sd\left(r(q_B)_t\right)}=\frac{\sum_{j\in F}s_j^2x_j(q_A)x_j(q_B)}{\sqrt{\sum_{j\in F}s_j^2x_j(q_A)^2}\sqrt{\sum_{j\in F}s_j^2x_j(q_B)^2}} \qquad (13.9)$$

where $s_j^2=\mathrm{E}\left[e_j^2\right]$ is the variance or risk associated with each factor $j$.

Thus the correlation will depend upon the similarity of the attributes $x_j(q_A)$ and $x_j(q_B)$ of the two models. In the extreme case where the sensitivities of one model are identical (within a scaling constant) to those of the other model, then $x_j(q_B)=kx_j(q_A)$ and the returns of the models will be perfectly correlated.

$$r(A,B)=\frac{\sum_{j\in F}s_j^2x_j(q_A)kx_j(q_A)}{\sqrt{\sum_{j\in F}s_j^2x_j(q_A)^2}\sqrt{\sum_{j\in F}s_j^2\left(kx_j(q_A)\right)^2}}=\frac{k\sum_{j\in F}s_j^2x_j(q_A)^2}{\sqrt{\sum_{j\in F}s_j^2x_j(q_A)^2}\,|k|\sqrt{\sum_{j\in F}s_j^2x_j(q_A)^2}}=\frac{k}{|k|}=\begin{cases}1 \text{ if k} > 0\\ -1 \text{ if k} < 0\end{cases} \qquad (13.10)$$

Thus, the time-series equivalent of preferring to combine solutions which are individually Pareto-optimal is to create a combination of models which differ in their (hidden) sensitivities to the underlying factors which generate the profitability of the models. The degree to which two models A and B differ is identified by the extent to which the correlation of their historical profits and losses is less than perfect, i.e. $d(A,B)=1-r(A,B)$.

In this section we have examined the underlying issues which determine the extent to which a model combination strategy can successfully be used to diversify risk. In the following subsection we use the insights gained from this analysis to develop a population-based algorithm for joint optimisation of a complementary set of models.

### 13.1.3 A Population-based Algorithm

In this section we describe the population-based algorithm for joint-optimisation of a set of statistical arbitrage models. The key features of the algorithm are:

- joint optimisation of components within each model through the use of “meta parameters”

- forced decorrelation of models through conditional fitness measures
- joint optimisation of the risk-adjusted performance of the “portfolio of models” as a whole

### Joint Optimisation

In Section 13.1.1 we highlighted the dangers posed by so-called “criterion risk”, namely that by optimising model parameters with respect to a “fitness” criterion which is different to the true utility function, an additional source of bias is introduced to the modelling procedure. This risk is exacerbated when a number of components within a model are optimised *sequentially*.

If we denote a model and its components as $m = \left(c_1\left(q_1\right), c_2\left(q_2\right), \ldots, c_{nc}\left(q_{nc}\right)\right)$, the optimisation criteria at each stage $i$ as $M_i$ and the true “utility” or ultimate performance metric as $U$, then the optimal model according to the multi-stage approach is given by:

$$m^{MS} = \left( \max_{q_1} M_1\left(q_1\right), \max_{q_2} M_2\left(q_2 \middle| q_1\right), \ldots., \max_{q_{nc}} M_{nc}\left(q_{nc} \middle| q_1, \ldots., q_{nc-1}\right)\right) \qquad (13.11)$$

The key danger of this approach is that a set of models will converge to the single optimal solution, thus reducing diversity and unnecessarily restricting the parameter space which is explored by the search procedure. In the evolutionary optimisation community this problem is known as “premature convergence”. Furthermore, a model which appears optimal when viewed in isolation will not necessarily be that which has the greatest marginal utility when added to an existing portfolio of models.

Within our population-based algorithm we reduce the “convergence” tendency by means of a hierarchical approach in which a set of meta-parameters $f = \left(f_1 \subset q_1\right) \cup \cdots \cup \left(f_{nc} \subset q_{nc}\right)$ are not conditioned on the previous modelling stages but rather form part of a single high-level model specification. The remaining parameters $q \cap \overline{f}$ are then optimised according to the standard sequential procedures, but conditioned upon the values of the meta-parameters $f$:

$$m^H\left(f\right) = \left( \max_{q_1 \cap \overline{f}_1} M_1\left(q_1 \middle| f_1\right), \max_{q_2 \cap \overline{f}_2} M_2\left(q_2 \middle| q_1, f_2\right), \ldots, \max_{q_{nc} \cap \overline{f}_{\mathrm{nc}}} M_{nc}\left(q_{nc} \middle| q_1, \ldots., q_{nc-1}, f_{\mathrm{nc}}\right)\right) \qquad (13.12)$$

By partitioning the parameter space in this manner, we reduce the dependency both on the indirect optimisation criteria $M_i \ldots M_{ns}$ and on the parameter estimation at previous stages, which is replaced by a joint dependency upon the meta-parameters $\mathrm{f}$. The key advantage of this approach is that the utility of the model specification at meta-parameter level can be evaluated as a whole using "conditional" fitness measures which take into account not only the performance of the individual model but also its relationship to the performance of the existing portfolio of models.

### Conditional Fitness

Given a candidate model $m^H\left(\mathrm{f}_\mathrm{k}\right)$, the standard approach to optimisation would be to compare the fitness of the model to that of the current optimal model, supplanting the existing model in the case where the fitness of the new model exceeds that of the existing solution. The weakness of this approach is that it overlooks any potential advantages which may be obtained through model combination. In the context of a population-based approach, the true choice is not **between** the new and a single old model, but rather whether the new model can **add value** to the existing population of models $P = \bigcup_{i=1..np} m^H\left(\mathrm{f}_\mathrm{i}\right)$. In other words, does the new model have a positive marginal utility:

$$mu\left[m^H\left(\mathrm{f}_\mathrm{k}\right)\right] = U\left[\left(\bigcup_{i \in P} m^H\left(\mathrm{f}_\mathrm{i}\right)\right) \cup m^H\left(\mathrm{f}_\mathrm{k}\right)\right] - U\left[\bigcup_{i \in P} m^H\left(\mathrm{f}_\mathrm{i}\right)\right] \tag{13.13}$$

This expression forms the basis for our solution to the problem of **jointly** optimising a population of models. In the particular case of optimising the risk-adjusted performance of a weighted combination of statistical arbitrage models, the marginal utility of model *m* is given by:

$$mu\left[m^H\left(\mathrm{f}_\mathrm{m}\right)\right] = \frac{d}{dw}\left\{\left[wr_m + (1-w)r_P\right] - \frac{1}{T}\left[ws_m^2 + (1-w)^2 s_P^2 + 2w(1-w)s_m s_P r_{mP}\right]\right\} \tag{13.14}$$

where $r_m, s_m^2$ are the expected return and risk of model *m*; $r_P, s_P^2$ are the expected return and risk of the existing portfolio of models and $r_{mP}$ is the correlation of returns between the new model and the existing portfolio. Solving for *w* in Eqn. (13.14) gives the optimal combined portfolio with:

$$w = \frac{\frac{T}{2}\left(r_m - r_P\right) + \left(\sigma_P^2 - \sigma_m \sigma_P \rho_{mP}\right)}{\sigma_m^2 + \sigma_P^2 - 2\sigma_m \sigma_P \rho_{mP}} \tag{13.15}$$

A positive value for $w$ in Eqn (13.15) indicates that the risk-adjusted performance of the portfolio as a whole should be improved by the addition of the new model *m*. All other factors being equal, a new model can be assigned a positive weight for any of three reasons: (a) a higher expected return than the current portfolio $\left(r_m - r_P > 0\right)$; (b) a lower level of risk than the current portfolio $\left(\sigma_P - \sigma_m > 0\right)$, or (c) a less than perfect correlation with the existing portfolio $\left(\rho_{mP} < 1\right)$.

This analysis highlights the fact that the value of a model cannot be correctly evaluated independently of the existing set of portfolios: whilst $r_m$ and $\sigma_m^2$ **can** be computed independently, the correlation $\rho_{mP}$ can only be calculated in the context of the existing portfolio *P*. Whilst it may seem that correlation plays a less important role than the individual levels of risk and return, this is often far from being the case. Figure 13.6 illustrates the size of the "niche" *w* and the marginal utility *mu* in the case where a new model has equal levels of expected return and risk to the existing portfolio.

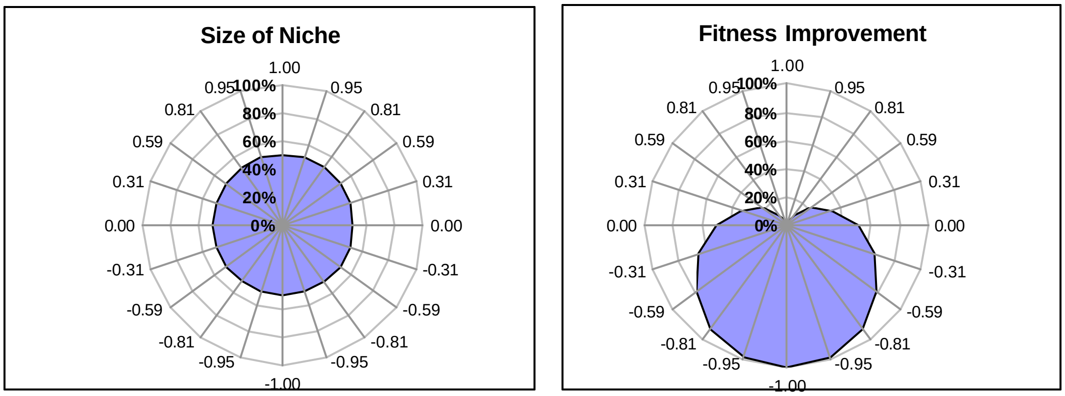


Figure 13.6: The marginal utility of a new model as a function of the **correlation** of its returns with those of a pre-existing portfolio. The correlation is displayed as a function of the angle (ρ=cos(θ)) with correlation of 1 at zero degrees from vertical, 0 correlation at +/- 90$^o$ and -1 correlation at 180$^o$. The figure on the left indicates the optimal weight $w$ for the new model, whilst the figure on the right indicates the proportional increase in overall utility relative to the old portfolio. The other parameters are $r_m = r_P = 10\%, \sigma_m = \sigma_P = 10\%.$ The assumed level of risk tolerance, T, is 20.

The figure on the left displays the optimal weighting assigned to the new model, which may be considered as the size of the "ecological niche" which is available for the new model. In this case the fact that the levels of risk and return are equal to those of the existing portfolio creates a symmetry in which the optimal weighting is always 50%. However the actual *added value* of the new model varies widely as a function of the correlation. At a correlation of +1 the new combination is completely equivalent to the old portfolio and there is zero added value, at the other extreme, a correlation of -1 allows the risk of the old portfolio to be completely cancelled out with a consequent fitness improvement from $10-(10^2)/20) = 5$ for the old portfolio to 10 for the new combination, or a 100% improvement in relative terms.

**Algorithm**

We now describe a population-based optimisation algorithm for maximising the combined utility of a set of models. In our case the set of models will represent a portfolio of statistical arbitrage models, although the methodology is equally applicable to other optimisation problems in which the fitness function is risk-averse or contains trade-offs with unknown parameters.

The key feature of the algorithm is that the maximisation is performed jointly on the entire set of models, and is performed with respect to the fitness of the population $P = \bigcup_{i=1..np} m^H\left(\mathrm{f}_\mathrm{i}\right)$ as a whole rather than in terms of the fitness of individual models within the population. The key innovation which allows the optimisation to be performed in this manner is the combination of a meta-parameter representation with the use of conditional fitness functions. The meta-parameters $\mathrm{f}_\mathrm{i}$ capture the key features of the individual models in a reduced search-space, whilst the conditional fitness measures $cf\left[m^H\left(\mathrm{f}_\mathrm{k}, P\right)\right] = U\left[P \cup m^H\left(\mathrm{f}_\mathrm{k}\right)\right] - U[P]$ ensure that the evaluation of new models is performed in a manner which accurately represents the added value which they provide to the existing portfolio. The complete algorithm is presented below:

```
Initialise population:    $\forall_{i=1..np}: \mathrm{f}_i = 0, f_i = 0$

For each population cycle
    Generate candidate models
    For each candidate
        Generate meta-parameters
        $\forall_{i=1..nc}: \mathrm{f}_{np+i} = h\,\mathrm{g}(P) + (1-h)randvec(\ )$
        For each model component $k$ = 1..$nc$
            Optimise parameters
            $\mathrm{q}_k = \mathrm{f}_{\mathrm{i,k}} \cup \max_{\mathrm{q}_{i,k} \cap \mathrm{f}_{\mathrm{i,k}}} M_k\left(\mathrm{q}_{i,k} \middle| \mathrm{q}_{i,1},....,\mathrm{q}_{i,k-1},\mathrm{f}_{\mathrm{i,k}}\right)$
        Calculate conditional fitness of model
        $cf\left[m\left(\mathrm{q}_{\mathrm{i}}, P\right)\right] = U\left[P \cup m\left(\mathrm{q}_{\mathrm{i}}\right)\right] - U[P]$
    Next model
    Update Population
    $P = s\left(P, C = \bigcup_{i=np+1..np+nc} m\left(\mathrm{q}_i\right), cf_{1..np+nc}\right)$
Next cycle
```

Figure 13.7: Details of the population-based algorithm for joint optimisation of a set of models

The application independent details of the algorithm consist of the "heredity" factor '*h*', the generation function 'g', and the survival function 's'. The heredity factor *h* determines the extent to which new candidates are derived from existing members of the population as opposed to random exploration of the meta-parameter space. For instance, the generation function for new meta-parameters, 'g', may act by combining the parameters of a number of "parent" models, in which case the algorithm may be considered a "genetic algorithm" and the term (1-*h*) is the "mutation rate". The survival function 's' governs the manner in which the population base at cycle *t+1* is derived from the previous population at time *t* together with the new candidates. One method is to hold the population size *np* constant, in which case 's' will act by replacing old members of the population with new candidates which have higher marginal utility with respect to the remainder of the population. Alternatively, we can allow the population to grow and shrink naturally as new candidates with positive marginal utility are discovered, incorporated into the population, and in some cases render existing models obsolete.

The application dependent aspects of the algorithm will include the utility function $U$, the model parametrisation $m = \left(c_1\left(\mathrm{q}_1\right), c_2\left(\mathrm{q}_2\right), \ldots, c_{nc}\left(\mathrm{q}_{nc}\right)\right)$, the set of meta-parameters $\mathrm{f}$, and the optimisation criteria for the individual model components $M_k$.

**Summary**

In this section we have described the framework for a population-based algorithm for risk-averse optimisation. In the following sections we investigate the properties of the algorithm, firstly with respect to a controlled experiment using simulated data, and secondly as a means of generating a diverse portfolio of statistical arbitrage models.

## 13.2 Controlled Evaluation of Population-based Algorithm

In this section we describe an experiment to verify the properties of the population-based algorithm. The experiment is based upon controlled simulations in which the model parameters determine the sensitivity to a number of underlying "factors". In turn, the factor sensitivities determine a time-series of "returns" which are a linear combination of the returns of the underlying factors. Some factors have a positive expectation and hence represent systematic *opportunities* for generating profits, other factors have zero expectation and merely represent additional sources of model-dependent *risk*.

An important aspect of the experiment is that there is no **single** combination of parameters which results in an optimal set of factor sensitivities. The performance of the population-based algorithm is compared both to optimising a single model via stochastic hill-climbing, and to a "naïve" population-based algorithm in which fitness is evaluated on an individual rather than conditional basis.

### 13.2.1 Description of controlled experiment

The basis of the controlled simulations is an abstracted form of the general problem of optimising systems for "model based" trading. The objective is to maximise the quadratic utility of risky returns which are computed as a linear combination of model attributes $x_j(q)$ and factor returns $f_j\left(y\right)$:

$$r(q,y)_t = \sum_{j=1..i} x_j(q) f_j(y)_t \qquad f_j(y)_t \sim N\left(m_j, s_j^2\right) \tag{13.16}$$

We partition the "risk factors" $f_j(y)$ into a set which represent potential sources of excess returns, i.e. $m_{j=1..i} > 0$, and a second set which merely represent sources of additional uncertainty, i.e. $m_{j=i+1..i+n} = 0$. Attributes $x_{j=1..i}(q)$ represent the extent to which the model has successfully captured potential sources of excess return, whilst attributes $x_{j=i+1..i+n}(q)$ represent the exposure to the additional risks which are associated with trading a particular set of assets, estimating a particular model, etc. The final link in the chain between parameter values $q$ and model fitness $F(q)$ is the parameter attribute mapping. For ease of visualisation we use an abstracted representation in which attributes are represented by two-dimensional gaussians with "centre" $(c_{1,j},c_{2,j})$ and "spread" $s_j$.

### Experimental Parameters

The parameters used for the simulation experiments are reported in Table 13.2.

| Factor | Return | Risk | Co-ordinates | Spread |
|---|---|---|---|---|
| $j$ | $\mu_j$ | $\sigma_j$ | $(c_{1,j},c_{2,j})$ | $s_j$ |
| 1 | 1.0 | 0.5 | (0.25, 0.25) | 0.25 |
| 2 | 0.8 | 0.5 | (0.25, 0.75) | 0.25 |
| 3 | 0.7 | 0.5 | (0.75, 0.75) | 0.25 |
| 4 | 0.6 | 0.5 | (0.75, 0.25) | 0.25 |
| 5..29 | 0.0 | 1.0 | ({0.1, 0.3, 0.5, 0.7, 0.9},<br>{0.1, 0.3, 0.5, 0.7, 0.9}) | 0.2 |

Table 13.2: The parameters of the risky factors which determine model profits in the simulation experiments

The risk tolerance parameter $T$ was set to 2. The overall risk and return of a model depend on the combination of attributes (factor exposures) and the risk and return associated with each individual factor. The set of attributes for model (0.5, 0.5) is shown in Figure 13.8.

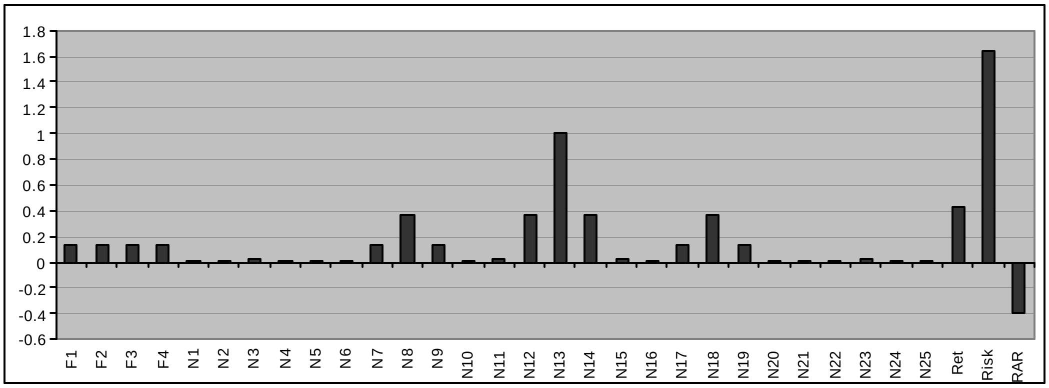


Figure 13.8: Attributes for model (0.5, 0.5). The first four columns represent the extent to which the model embodies the four factors which are responsible for generating positive expected returns. The following 25 columns represent the extent to which the model is influenced by each of the 25 noise factors, which are arranged in an even grid over the parameter space. The final three columns show the combined effect of the attributes in the form of expected return, risk, and risk-adjusted return.

The model with parameters (0.5, 0.5) lies mid-way between all four sources of positive returns and at the centre of noise factor N13; it has an expected return of 0.42, total risk of 1.62, and a risk-adjusted return of 0.42-(1/2)1.62 = -0.40.

Even in this relatively simple example, the nonlinearity in the parameter-attribute mapping creates a situation where, viewed from parameter space, the fitness function is both highly nonlinear and multimodal. The mapping between model parameters and risk-adjusted return is presented in Figure 13.9 below. A grid search indicates that the optimal (individual) model coincides with the centre of the strongest positive return factor at location (0.25, 0.25) and has an expected return of 1.03, risk of 1.82 and risk-adjusted return of 0.12.

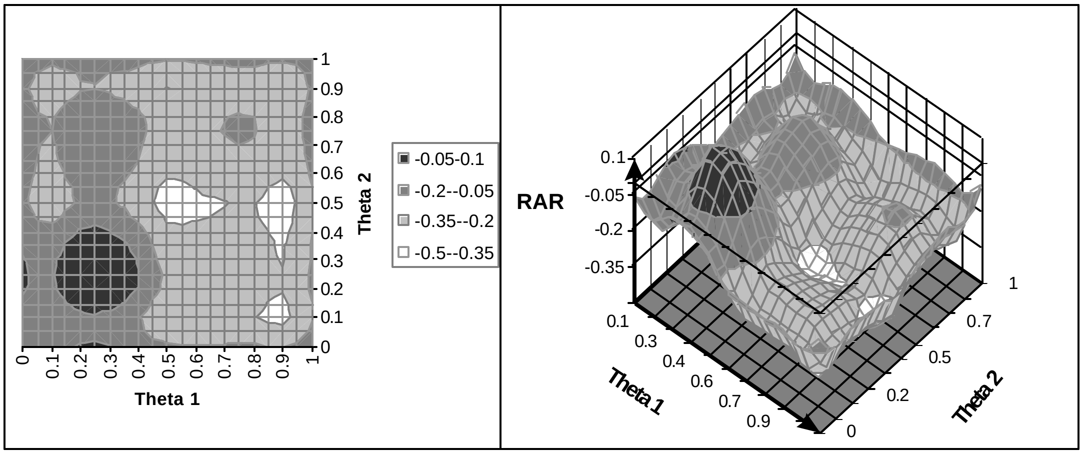


Figure 13.9: Risk-adjusted return as a function of the two model parameters; superior models lie in regions around the centre of each return-generating factor.

We verify the advantages of the portfolio of models approach by comparing its results to those achieved by a standard “stochastic hill climbing” algorithm used to optimise a single model. The importance of using conditional fitness measures is highlighted by comparing the results of the population based algorithm in two cases.

In each case, the general form of the population-based algorithm is as described in Figure 13.7. In the first case, **P0**, the fitness measure used is an individual measure of risk-adjusted return and the inner-loop of the algorithm operates as shown in Figure 13.10:

Generate candidate model: $q_1^{(cand)} = rand();q_2^{(cand)} = rand()$

Calculate factor exposures: $x_j^{(cand)} = e^{-\left(\frac{\left(q_1^{(cand)}-c_{1,j}\right)^2+\left(q_2^{(cand)}-c_{2,j}\right)^2}{s_j^2}\right)}$

Calculate individual fitness: $f(cand) = \sum_f m_f x_f^{(cand)} - \frac{1}{T}\sum_f s_f^2 x_f^{(cand)2}$

Update population: if $\exists_{m(k)\in P} : f(cand) > f(k)$ then replace model k with new model

Figure 13.10: Details of the population-based algorithm P0, in which fitness is evaluated on an individual basis.

In contrast, the second version of the population-based algorithm, **P1**, uses conditional fitness measures which are based upon the marginal increase in the utility of the population as a whole. The fitness of the population is defined by the solution to the following quadratic programming problem:

$$F(P) = \max_{\mathbf{w}} \sum_f m_f \left( \sum_{m(j)\in P} w_j x_f^{(j)} \right) - \frac{1}{T}\sum_f s_f^2 \left( \sum_{m(j)\in P} w_j x_f^{(j)} \right)^2 \tag{13.17}$$
$$\text{subject to } \sum_j w_j = 1; \quad w_j \geq 0$$

and the inner loop of algorithm **P1** operates as shown in Figure 13.11:

Generate candidate model: $q_1^{(cand)} = rand();q_2^{(cand)} = rand()$

Calculate factor exposures: $x_j^{(cand)} = e^{-\left(\frac{\left(q_1^{(cand)} - c_{1,j}\right)^2 + \left(q_2^{(cand)} - c_{2,j}\right)^2}{s_j^{\,2}}\right)}$

Calculate conditional fitness: $cf^{(cand)} = \max_{k \in P} F\left(P - \{m(k)\} \cup \{cand\}\right) - F(P)$

Update population: if $cf^{(cand)} > 0$ then replace model k with new model

Figure 13.11: Details of population-based algorithm **P1**, where fitness is evaluated in the context of the current portfolio.

The following subsection presents the results of the simulation experiment.

### 13.2.2 Discussion of Empirical Results

Table 13.3 presents a summary of the performance of the final portfolio of models produced by each of the two algorithms, as a function of the population size.

| Population Size | Return (indiv) | Risk (indiv) | RAR (indiv) | Port Return | Port Risk | Port RAR | OS Port Return | OS Port Risk | OS Port RAR |
|---|---|---|---|---|---|---|---|---|---|
| Algorithm P0 (individual fitness) | | | | | | | | | |
| 1 | 0.96 | 1.77 | 0.08 | 0.96 | 1.77 | 0.08 | 0.91 | 1.83 | 0.00 |
| 4 | 0.96 | 1.77 | 0.08 | 0.97 | 1.77 | 0.08 | 0.91 | 1.83 | 0.00 |
| 9 | 0.96 | 1.77 | 0.07 | 0.94 | 1.69 | 0.10 | 0.89 | 1.72 | 0.03 |
| 16 | 0.94 | 1.75 | 0.06 | 0.93 | 1.65 | 0.10 | 0.87 | 1.68 | 0.03 |
| 25 | 0.94 | 1.76 | 0.05 | 0.91 | 1.61 | 0.11 | 0.86 | 1.62 | 0.05 |
| Algorithm P1 (conditional fitness) | | | | | | | | | |
| 1 | 0.96 | 1.77 | 0.08 | 0.96 | 1.77 | 0.08 | 0.91 | 1.83 | 0.00 |
| 4 | 0.56 | 1.40 | -0.14 | 0.75 | 0.44 | 0.52 | 0.73 | 0.62 | 0.42 |
| 9 | 0.62 | 1.59 | -0.18 | 0.82 | 0.41 | 0.61 | 0.80 | 0.54 | 0.53 |
| 16 | 0.65 | 1.59 | -0.14 | 0.82 | 0.36 | 0.64 | 0.80 | 0.54 | 0.53 |
| 25 | 0.67 | 1.55 | -0.11 | 0.84 | 0.38 | 0.65 | 0.82 | 0.66 | 0.49 |

Table 13.3: Performance Analysis of the portfolios of models produced by the two algorithms as a function of population size. Algorithm P0 uses a fitness measure based solely on the performance of an individual model. Algorithm P1 uses a fitness measure which is based on "value added" to the existing portfolio.

The optimisation was conducted with respect to a 1000 observation “in-sample” realisation of the time-series of returns $r(q,y)_t$ and evaluated with respect to a further 1000 observations in an “out-of-sample” set. The same sequence of 1000 candidate models was used for each experiment, and in each case the rate of improvement had slowed almost to zero by this time. Note that the performance with population size 1 is equivalent to that of a traditional single-solution “hill climbing” approach.

With algorithm P0, there is little benefit from diversification, due to the population **converging** towards the optimal individual model. The results for algorithm P1 are very different. Viewed on an **individual** basis the quality of the models apparently *deteriorates* with larger population size, but the *improvement* in **portfolio** performance indicates that the reduced individual return is simply the price of achieving a diversified portfolio of models, with the net result being to optimise the risk-adjusted return of the portfolio as a whole. Note that in this case the bulk of the diversification can be achieved with only four models, although larger populations create some additional opportunities for diversifying away the “noise” factors.

### Learning trajectories

Figure 13.12 compares the trajectories of portfolio return, risk and risk-adjusted return for the two algorithms P0 and P1 (population size=16).

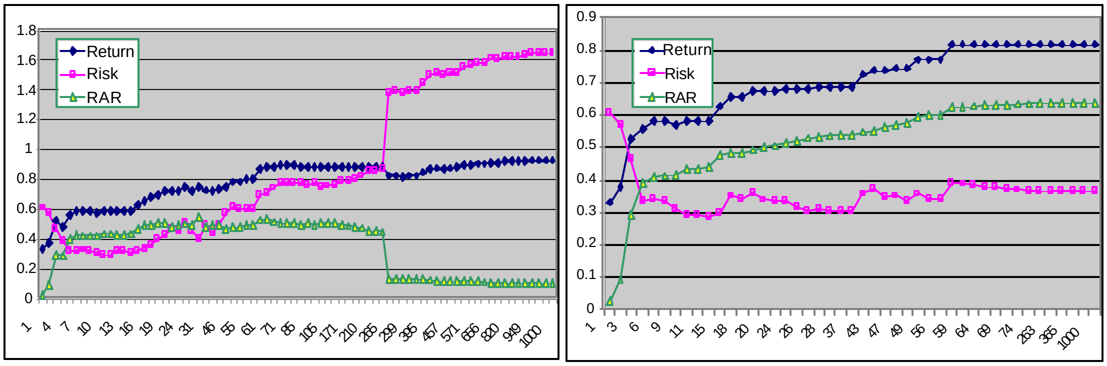


Figure 13.12: Evolution of portfolio risk and return in the case of algorithm P0 on the left (individual fitness) and P1 on the right (conditional fitness) The nonlinear time-axis indicates the decreasing rate at which interesting new models are discovered by the random search procedure.

In the early stages of the procedure, new models are discovered which increase the portfolio **return** through higher individual returns and also reduce the **risk** through improved diversification. However, a by-product of the convergence of algorithm P0 towards the individually optimal model, is an eventual loss of population diversity and an associated increase in portfolio risk. The “catastrophe” around time 285 corresponds to the loss of the last model which is not in the “cluster” around the *individual* optimum. In contrast, the conditional fitness measure used in the joint optimisation algorithm P1 recognises the value of diversity and is seen to ensure a monotonic improvement in the true performance measure of *portfolio* risk-adjusted return.

## Portfolio composition

Figure 13.13 presents the composition of the population which is maintained by the different algorithms at various stages of the optimisation procedure.

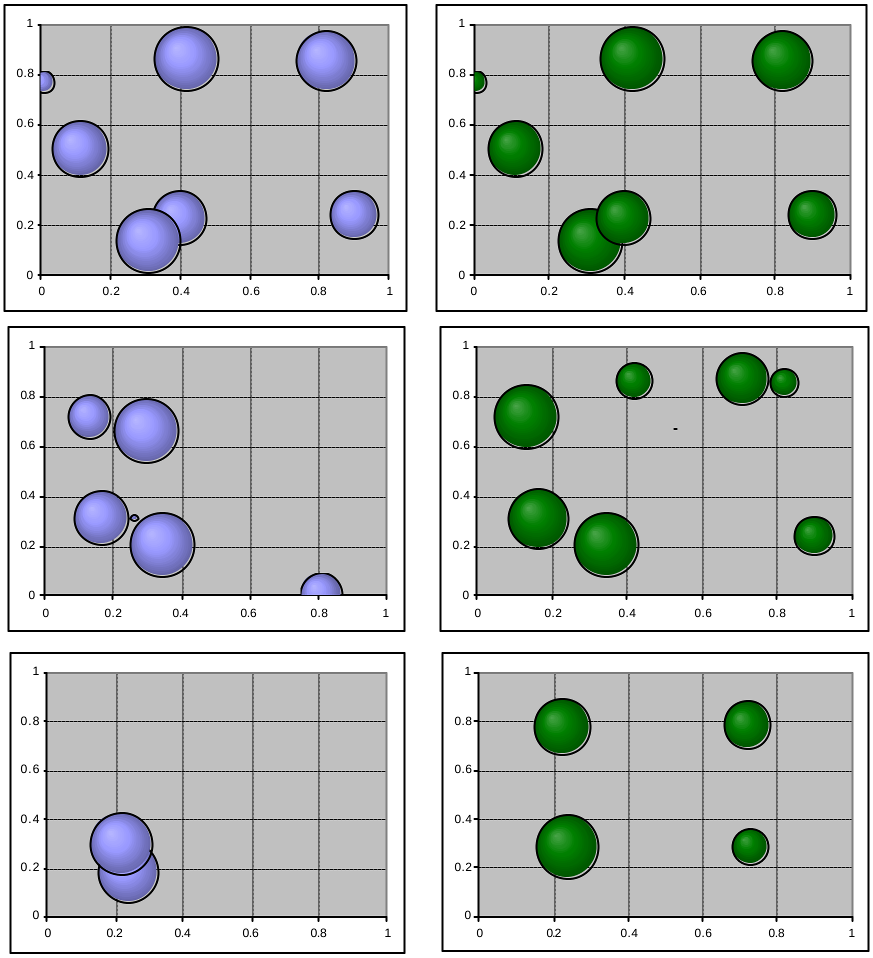


Figure 13.13: Evolution of **portfolio** composition for the two algorithms. Figures on the left hand side correspond to Algorithm P0 and on the right Algorithm P1. The top row is after 10 candidates have been evaluated; the second row is after 30 candidates; the final row is on termination of the algorithms after 1000 candidates. For each sphere, the x-coordinate equals the value of the first model parameter $\theta_1$, the y-coordinate is the second parameter $\theta_2$, and the size of the sphere is the weight allocated to the model within the portfolio.

Initially the composition of the portfolios is similar. In the case of algorithm P0 there is a clear convergence to a single optimal model, whilst in algorithm P1 the conditional fitness measure leads to only a localised convergence towards each of the sources of (risk-bearing) return.

**Summary**

In this section we have demonstrated that the use of a population-based approach is not sufficient in itself to guarantee that the potential advantages of diversification are realised in practice. In contrast the joint optimisation algorithm actively encourages diversity by evaluating models in terms of the **added-value** which they provide to the existing portfolio. The following section demonstrates the application of the algorithm to the real-world problem of jointly optimising a portfolio of statistical arbitrage models.

## 13.3 Empirical Evaluation on Equity Index Data

In this section we describe the empirical results which were obtained as a result of applying the population-based optimisation algorithm of Section 13.2 to the task of generating a portfolio of statistical arbitrage models based on mispricings between a set of international equity indices.

### 13.3.1 Joint Optimisation of a portfolio of Statistical Arbitrage Models

The purpose of the joint optimisation is to optimise the **risk-adjusted return** of the entire portfolio. If $np = |P|$ is the size of the portfolio, $\mathbf{r} = \left[\mathrm{E}(\mathrm{r}_1) \quad \mathrm{E}(\mathrm{r}_2) \quad \cdots \quad \mathrm{E}(\mathrm{r}_{np})\right]$ the vector of expected returns, $\mathbf{V}$ the ($np$ x $np$) covariance matrix of model returns, $\mathbf{w} = \left[w_1 \quad w_2 \quad \cdots \quad w_{np}\right]^T$ the portfolio weights and $T$ the risk-appetite parameter, then the optimal weights and expected risk-adjusted return of the portfolio are given by:

$$RAR(P) = \max_{\mathbf{w}} \mathbf{rw} - \frac{1}{T}\mathbf{w}^T\mathbf{V}\mathbf{w} \qquad \text{subject to} \quad \sum_j w_j = 1; \quad w_j \geq 0 \qquad (13.18)$$

and the "conditional fitness" or marginal utility of a candidate model is simply the increase in risk-adjusted return which is obtained by adding the model to the current portfolio:

$$\mathrm{mu}(cand) = RAR(P \cup cand) - RAR(P) \qquad (13.19)$$

In this manner the "fitness" of a candidate model takes into account not only the **expected return** (in $\mathbf{r}$), and **risk** of the model (diagonal elements of $\mathbf{V}$) but also the **correlations** between the various models (off-diagonal elements of $\mathbf{V}$). Straightforward modifications of the algorithm can be used to optimise alternative measures of risk-adjusted return such as the Sharpe Ratio.

### Generation of Candidate Models

The components within each model are jointly optimised with respect to the conditional fitness measure (marginal utility) at the level of a set of "meta-parameters". The meta-parameters are considered as a high-level "joint specification" of the model, from which the remaining parameters then follow as a natural consequence of the estimation procedure and the sample data. The objective of this approach is to reduce the dimensionality of the joint optimisation process whilst minimising the **criterion risk** which would otherwise arise through the use of "multi-stage" optimisation procedures.

Table 13.4 describes the general form of the three components which comprise a statistical arbitrage model, defining both the meta-parameters and (standard) parameters which apply at each stage:

| Component | General Form | Meta-Parameters | Parameters |
|---|---|---|---|
| Statistical Mispricing<br>(Chapter 5) | $M_{i,t} = P_{T(i),t} - \sum_j b_{i,j,t} P_{C(i,j),t}$<br>$b_{i,j,t} = b_{i,j,t-1} + h_t$ | $T(i), C(i,j)$<br>$q = s_B^2 / s_M^2$ | $b_{i,j,t}$ |
| Forecasting Model<br>(Chapter 10) | $\mathrm{E}\left[\Delta M_{i,t}\right] = \hat{f}_i\left(M_{i,t}, \Delta M_{i,t-j}, \mathbf{z_{i,t}}, \mathrm{q}_{\mathbf{i}}\right)$ | $\hat{f}_i, j, \mathbf{z_{i,t}}$ | $\mathrm{q}_{\mathbf{i}}$ |

| | | |
|---|---|---|
| Trading Strategy<br>(Chapters 7, 11) | $ISA(M_t, k)_t = -\text{sign}\left(M_{t-j}\right)\left\|M_{t-j}\right\|^k$<br>$CSA(\text{E}[\Delta M_t], k)_t = \text{sign}\left(\text{E}[\Delta M_t]\right)\left\|\text{E}[\Delta M_t]\right\|^k$ | $k$<br>(+smoothing parameters $h, q$ ) |

Table 13.4: Composition of an individual statistical arbitrage model: the meta-parameters are optimised at portfolio level and determine the high-level specification of the model components, in contrast the remaining parameters are estimated from the data and conditioned upon the values of the meta-parameters.

### 13.3.2 Results for Portfolio of Equity Index Models

In this section, we describe the results of applying the population-based algorithm to the problem of jointly optimising a portfolio of statistical arbitrage models based on mispricings between the following set of international equity indices:

Dow Jones Industrial Average, Standard and Poors 500 Index (US)

FTSE 100 (UK)

DAX 30 (Germany)

CAC 40 (France)

SMI (Switzerland)

IBEX (Spain)

OMX (Sweden)

Hang Seng (Hong Kong)

Nikkei 225 (Japan)

The data used in the experiment were daily closing prices for the period 1st August 1988 to 8th October 1997. From the total of 2419 observations the first 1419 were used for estimation of the parameters of the statistical arbitrage models, the following 500 observations (9th December 1993 to 9th November 1995) were used to perform model selection within the context of the population based optimisation procedure, and the final 500 observations (10th November 1995 to 8th October 1997) were withheld for an out-of-sample evaluation.

The population-based algorithm was applied to jointly optimise a set of 10 models, each consisting of three components. For the first component, the adaptive form of the statistical mispricing methodology was used, with meta-parameters specifying the target asset, the set of constituent assets, and the degree of adaptivity. The second component consisted of either a linear or neural forecasting model with meta-parameters specifying the functional form of the model together with the explanatory variables (statistical mispricing together with selected lagged innovations in the mispricing). The third group of meta-parameters specify the sensitivity parameter $k$ and the holding-period $h$ for the moving-average form of the CSA trading rule.

Table 13.5 summarises the out-of-sample performance of the individual models within the population, together with two portfolios, one formed from a weighted (WtAv) and the other an unweighted average (SimAv) of the individual models.

| | **Out-sample (1-250)** | | | **Out-sample (251-500)** | | |
|---|---|---|---|---|---|---|
| | Directional Ability | Sharpe Ratio | Total Profit | Directional Ability | Sharpe Ratio | Total Profit |
| **WtAv** | **51.2%** | **1.79** | **13.3** | **52.0%** | **0.58** | **8.8** |
| **SimAv** | **49.2%** | **1.60** | **11.8** | **49.6%** | **0.29** | **4.7** |
| Model1 | 46.4% | -0.31 | -4.5 | 47.6% | -0.51 | -8.4 |
| Model2 | 52.4% | 3.18 | 20.4 | 50.8% | 2.17 | 41.4 |
| Model3 | 50.4% | 0.57 | 6.8 | 55.6% | 2.86 | 39.6 |
| Model4 | 50.8% | 0.79 | 21.2 | 49.6% | 0.28 | 11.6 |
| Model5 | 54.8% | 1.04 | 19.8 | 47.6% | -0.52 | -33.8 |
| Model6 | 55.2% | 2.08 | 38.8 | 51.2% | 0.35 | 15.9 |
| Model7 | 51.6% | -0.01 | -0.1 | 53.6% | 0.00 | 0.0 |
| Model8 | 46.4% | -1.11 | -26.3 | 48.0% | -0.30 | -10.1 |
| Model9 | 45.6% | -0.06 | -0.6 | 43.6% | -1.53 | -59.5 |
| Model10 | 53.2% | 2.15 | 42.8 | 55.2% | 1.96 | 50.3 |

Table 13.5: Performance of a portfolio of statistical arbitrage models of international equity indices created using the population-based algorithm; the results are subdivided into two equal periods.

The wide diversity of the performance of the individual models highlights the high level of risk which is inherent in attempting to choose a single “best” model. The two portfolios constructed by combining the individual models, whether by the weighted average resulting from mean-variance optimisation or a simple unweighted average, both have a similar level of performance - less profitable than the best models, but a long way from making the losses of the worst models, and significantly positive over the out-of-sample period as a whole. The equity curves for the weighted combination of models, together with the individually best- and worst-performing models are presented in Figure 13.14.

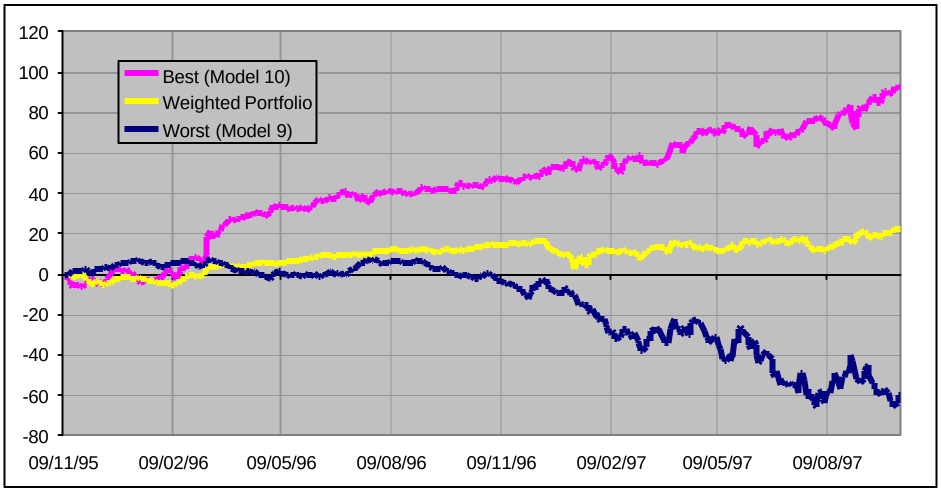


Figure 13.14: Performance of statistical arbitrage models for international equity indices

Notice the increased level of risk exhibited by the combined portfolio in the second half of the out-of-sample period. In practice however, it is unlikely that a single set of models would be traded in unchanged form for a period of two years. Thus whilst the experiment serves to demonstrate the feasibility, in principle, of using the population-based algorithm to generate a set of trading models, it is a rather crude analysis which neglects the potential for leveraged trading, de-selection of underperforming models, periodic refreshing of the portfolio and many other enhancements that would be open to a potential trader of such a system.

In practice statistical arbitrage between international equity markets is conducted not on the **spot** market index levels themselves, but through the **futures** markets. Moreover, the different opening and closing times of the different markets mean that the most meaningful way of analysing index futures strategies is through the use of synchronised intra-day observations. The following subsection presents results of applying the population-based algorithm in such a context.

### 13.3.3 Results for Portfolio of Intraday Equity Index Future Models

In this subsection we present the results from an application of the population-based methodology to the task of jointly optimising a portfolio of **intraday** statistical arbitrage models between an international set of equity index futures. In particular, the index futures were the September 1998 contracts on the FTSE 100, Dax 30, Cac 40, S&P 500 and Mib 30 and the July 1998 contract on the Swiss market index. The data consists of hourly samples taken live from a Reuters data-feed from 9am to 5pm daily during the period 15/5/98 through to 20/7/97, giving a total of 364 observations.

A population of 10 models was jointly optimised using the first 264 observations and evaluated on the remaining 100 out-of-sample observations. Due to the relatively small sample size, the models used were of the implicit statistical arbitrage type which aims to exploit the mean-reverting component of the mispricing dynamics. The meta-parameters which define each model thus consist of the target asset, the set of constituent assets, the adaptivity parameter for the weights of the fair price relationship, and the trading rule parameters. The performance of both the individual models and unweighted and weighted combinations is shown in Table 13.6., net of transaction costs of 10 basis points.

| | Weighting | Profit | S.R. | Direction |
|---|---|---|---|---|
| Model1 | 0.004 | -0.84% | -2.91 | 47.5% |
| Model2 | 0.055 | -0.18% | -0.61 | 46.5% |
| Model3 | 0.177 | 1.14% | 4.11 | 55.4% |
| Model4 | 0.084 | 2.06% | 6.71 | 58.4% |
| Model5 | 0.177 | 1.76% | 5.63 | 53.5% |
| Model6 | 0.114 | -0.57% | -2.39 | 45.5% |
| Model7 | 0.219 | 0.90% | 3.73 | 51.5% |
| Model8 | 0.038 | 1.22% | 5.00 | 54.5% |
| Model9 | 0.110 | 1.78% | 6.10 | 53.5% |
| Model10 | 0.021 | 0.69% | 2.55 | 51.5% |
| Best | - | 2.06% | 6.71 | 58.4% |
| Worst | - | -0.84% | -2.91 | 45.5% |
| Comb | - | 0.80% | 3.60 | 53.5% |
| CombWt | - | 1.06% | 4.57 | 53.5% |

Table 13.6: Performance analysis of a portfolio of intraday statistical arbitrage models based on a set of international equity index futures. Transaction costs of 10 basis points are included. "Comb" is the performance of an equally weighted combination and "CombWt" a weighted combination of the models.

The table demonstrates a wide spread of performance figures across the individual models. Note also that the model which is allocated the highest weight based on **in-sample** performance (Model 7) is far from the best during the **out-of-sample** evaluation. This indicates the danger of using a weighted combination scheme rather than the simpler equally-weighted combination, even though in this case the weighted combination outperforms overall.

Even this relatively simple set of models show useful promise, with the annualised Sharpe Ratio of 4.57 for the weighted portfolio suggesting that the performance is consistent enough to justify a moderate degree of leverage.

## 13.4 Summary

In this chapter we have described an ambitious approach to the task of creating model-based trading strategies. This approach aims to avoid the risks and inefficiencies which result from the separate optimisation of components within a model, and models within a population.

In our population-based approach, models are jointly optimised by the use of conditional fitness measures which quantify the marginal utility or *added value* which a model provides to the current population. As opposed to the traditional method of evaluating models on an individual basis, this approach actively encourages the generation of a set of well-diversified and hence complementary models. Furthermore, in an attempt to overcome the problem of criterion risk, or the “forecasting bottleneck”, the components within each model are also jointly optimised with respect to the marginal utility at the level of a set of “meta-parameters” which can be considered as a high-level specification of the model.

The properties of the algorithm have been verified using controlled simulation, in which the joint optimisation approach was demonstrated to outperform a similar population-based algorithm using a more traditional fitness measure. The algorithm has been applied to the real-world task of optimising a set of statistical arbitrage models within the “portfolio of models” context described in Chapter 12. Whilst many further improvements are envisaged, even these preliminary results demonstrate the potential to generate profitable strategies at levels of risk which are made tolerable through the strategy of diversification across a portfolio of models.

This chapter concludes the detailed exposition of our methodology for statistical arbitrage modelling which has formed the subject of Parts I , II and III of the thesis, and which follows the overview that was presented in the introductory part of the thesis. The remainder of the thesis presents a summary of our main conclusions, including an evaluation of the contribution which our current methodology makes to the state of the art, suggestions for future extensions and refinements of the methodology, and the scope of possible practical applications of our work both in statistical arbitrage specifically and in the broader context of investment finance in general.

# Conclusions and Bibliography

In this final part of the thesis we present a summary of our main conclusions and a bibliography. Chapter 14 presents the main conclusions of the thesis and discusses directions for further work; it contains an evaluation of the contribution which our current methodology makes to the state of the art, suggestions for future extensions and refinements of the methodology, and a discussion of the scope of possible practical applications of our work both in statistical arbitrage specifically and in the broader context of investment finance in general.

# 14. Conclusions

In this thesis we have proposed and developed an integrated framework which enables the use of recent advances in computational modelling as a means of exploiting small but consistent regularities in asset price dynamics. We adopt a holistic perspective in which our methodology is based on an extensive analysis of the obstacles which arise in financial forecasting and the manner in which they influence the effectiveness of the modelling process as a whole. We have addressed the weaknesses of existing methodology by combining computational modelling techniques from a number of different fields. Within our methodological framework we apply the different techniques only to the parts of the modelling process for which they are inherently suitable, thus maximising the strengths of the various techniques whilst minimising the effect of their weaknesses.

Within our integrated modelling framework, we have developed specific tools and techniques which represent significant advances upon the existing state of the art. We have consistently exploited the *flexibility* which is offered by emerging modelling techniques such as neural networks and genetic algorithms whilst ensuring that this flexibility is employed in an *appropriate* manner. This has been achieved by placing the emerging modelling techniques in the context of, and in partnership with, the methodological rigour and diagnostic techniques which are provided by established modelling tools from the fields of statistics, econometrics and time-series forecasting.

In particular, we have developed extensions of the econometric methodology of cointegration which are suitable for use in cases where the parameters of the underlying relationship are time-varying and for cases where the number of time-series involved in the analysis in numbered is tens or hundreds rather than units. We have developed novel tests for identifying deterministic components in time-series behaviour which are based upon the joint distribution of a set of variance ratio statistics. We have demonstrated, through controlled simulations, that our new tests are sensitive to a *wider range* of deviations from random behaviour than are standard predictability tests. By means of a computationally intensive approach based upon Monte-Carlo simulation, we have generalised the applicability of both our new tests and existing predictability tests to the case where the time-series under analysis represents the result of a cointegration-based pre-processing procedure. The advantage of the simulation-based approach is that the actual empirical distribution of the test statistics can be determined under equivalent experimental parameters to those which are present in a given analysis, thus accounting for any artefacts which are induced by the pre-processing stage, automatically adjusting for sample size effects and avoiding any inefficiencies which would be incurred through the use of incorrect theoretical assumptions.

Through the use of an “equivalent kernels” perspective taken from nonparametric statistics, we have achieved a synthesis which includes both traditional parametric regression modelling and neural network learning. In particular, this approach allows us to compute the “degrees of freedom” which are contained in a neural network model and use this as the basis of a variety of statistical significance tests for neural network models and components within such models. We have developed three variant algorithms for neural model estimation which combine the low-bias of neural modelling techniques with the low-variance of statistical modelling procedures. Through controlled simulations, we have verified the properties of the algorithms and demonstrated that such a combined approach is vital in the case of modelling highly noisy time-series in which few *a priori* assumptions can be made about the nature of the underlying data-generating process.

We have generalised the model combination approaches of statistical forecasting in order to achieve diversification, and hence reduction, of the model risk which applies in the case of trading model-based strategies. Our “portfolio of models” approach extends the ensemble approach to forecasting with ideas from portfolio theory in order to provide a means of maximising the expected returns whilst simultaneously minimising the level of risk of a combined set of trading strategies as a whole. We have built upon this approach in order to develop a population-based algorithm, which exploits the particular strengths of genetic (and evolutionary optimisation) algorithms as a means of jointly optimising the whole set of models within a population. We have demonstrated, through the use of controlled simulations, that this approach can overcome the *criterion* risk which arises in cases where complex models are optimised in multiple stages and on an *individual* rather than *collective* basis. In particular, the population-based algorithm can be used to generate a portfolio of complementary models by actively encouraging diversification within the population and thus maximising the benefits which can be achieved through the portfolio approach.

We have applied these various methodological developments from a particular perspective which we refer to as “statistical arbitrage”. We consider statistical arbitrage as a generalisation of traditional “riskless” arbitrage strategies which are based on predefined relationships between financial assets, typically between derivative instruments such as options and futures contracts and the “underlying” assets upon which the derivatives are based. From our statistical arbitrage perspective, we apply our extended cointegration methodology to identify statistical “fair price” relationships between sets of related asset prices. Just as deviations from theoretical no-arbitrage relationships are considered “mispricings” which represent potential opportunities for riskless arbitrage, we likewise consider deviations from the analogous statistical fair-price relationships as potential opportunities for statistical arbitrage, and refer to them as “statistical mispricings”.

We have demonstrated that our methodology is applicable to real-world problems by performing extensive experimental evaluations from the statistical arbitrage perspective. This approach can, perhaps, be considered the purest method of evaluating the added-value which is provided by a computational modelling approach to investment finance. This is because the profits and losses of the resulting models are almost entirely independent of the underlying movements in the market as a whole and instead reflect only the informational advantage, if any, which is provided by the models themselves. Furthermore, the significance of the resulting performance can be evaluated not only from a statistical perspective but also from a practical perspective in which the economic advantages of the models can be assessed after factors such as transaction costs have been taken into account. In principle, the risk and return of our strategies can either be multiplied through leverage (up or down) and/or overlayed as a market-timing component on top of a more traditional trading strategy. Thus the benefits of our approach are potentially of value to active fund managers in general, as well as arbitrageurs and hedge funds in particular.

From this perspective, the results of our empirical evaluations can be taken as being highly promising whilst at the same time not conclusive. The results are highly promising because they indicate that significant levels of profitability can be achieved, at acceptable levels of risk, even after transaction costs have been taken into account. At the same time we believe that the results are not conclusive because the true tests of a trading methodology cannot be evaluated in a research environment using historical data but must ultimately be performed in a true trading environment using real prices, real trading costs and real trading infrastructure. Having made this caveat, we do believe that our experimental results demonstrate significant potential. Our first set of extensive experiments were based upon an implicit assumption of mean-reverting behaviour in the time-series dynamics of statistical mispricings between daily closing prices of FTSE 100 constituents. During a 200 day out-of-sample period, a set of these “implicit statistical arbitrage” strategies produced a backtested performance of between 7 and

10%, assuming typical institutional levels of transaction costs, with the corresponding Sharpe Ratios of between 2 and 3 demonstrating a high degree of consistency within this performance. More advanced “conditional statistical arbitrage” strategies based upon low-bias neural models of the mispricing dynamics achieved a collective annualised out-of-sample performance of 21% return and 2.45 Sharpe Ratio even at a moderately high level of transaction costs (0.5%). This annualised out-of-sample performance was further improved to 26.6% return and 3.40 Sharpe Ratio by means of the “portfolio of models” approach. Additional experimental results of statistical arbitrage models between international equity market indices indicate that the methodology has real potential in these cases also.

Whilst beyond the scope of this thesis itself, certain additional evaluations and developments of the methodology have been made in the commercial world itself. The original inspiration for our methodology arose from collaborative projects between the Computational Finance Group of the Decision Technology Centre (formerly Neuroforecasting Unit) at London Business School and a number of financial institutions. An earlier version of the methodology formed the basis of the modelling work conducted in the ESPRIT research project “High frequency Arbitrage Trading” (HAT), with favourable live performance evaluations carried out by the two banks in the consortium, one bank evaluating models within the equity and equity derivatives markets and the other evaluating models within the fixed-income (and derivatives) markets. Additional developments of the methodology currently form the basis of commercial negotiations between an LBS spin-off company and a major financial information services company and data vendor.

In terms of further methodological developments we believe that our work raises many avenues for future research. In particular we have highlighted the important role played by the different sources of potential error during the modelling process, especially in the context of the high noise content and temporal instability (nonstationarity) of predictive relationships between asset prices. We believe that further analysis of the issues raised in the thesis will lead to developments concerning the identification of appropriate modelling biases for financial markets, and methods for controlling the sources of model error which are represented by model variance, data snooping, performance nonstationarity and criterion risk.

We believe that there is significant potential to develop the specific methodology described in this thesis, both in the context of statistical arbitrage modelling and also extensions to other modelling domains. Furthermore, our modelling framework as a whole should be considered as inclusive rather than exclusive. We have referred within the body of the thesis to the fact that our cointegration based approach for constructing statistical mispricings could be replaced or combined with other multivariate approaches such as principal components analysis, factor analysis and independent components analysis. Furthermore the linear fair price relationships, to which we restrict ourselves for reasons of implementational convenience, could nevertheless be extended to the nonlinear cases enabled by recent advances in these modelling techniques.

Similarly, the low-bias neural modelling methodology in the second stage of our framework could be either extended to include, or indeed be replaced by, related low-bias approaches from nonparametric statistics or machine learning. It is our belief that given appropriate underlying modelling assumptions the achievable level of performance will be limited more by the informational content of the data itself than by the differences between alternative modelling techniques. However, it would certainly be interesting to quantify the extent to which this is true, and also the circumstances under which techniques such as projection pursuit, smoothing splines and support vector machines will indeed achieve similar results to our synthesis of neural and statistical techniques.

It is at the final stage of our methodology that perhaps the largest questions remain, and the greatest potential for future development. Assuming that cases exist where financial asset price data *can* be preprocessed into a form which contains significant deterministic components, and predictive models *can* be estimated of the resulting time-series dynamics, then the most interesting and important question arises in the form of "how can this advantage best be exploited ?" We believe that we have made important first steps in this direction, in particular though our recognition of the inter-related nature of the various stages of the modelling process and our integration of these stages in the joint optimisation procedure of our population-based algorithm.

Perhaps a productive route towards future advances may be to adopt a similar philosophy to that which underlies this thesis, namely to identify the important problems which established techniques have been developed to solve, and then also to identify the manner in which overly restrictive assumptions of these techniques can be relaxed, through the appropriate application of the continual developments in computational hardware, software, data availability and modelling methodology. Another fruitful route may be to identify the important but previously unanswerable (and hence generally unasked) *questions* which these developments now make it possible to answer. If the financial markets can be thought of as an artificial ecology, then we believe that computational analogies with natural evolution, neural recognition and reinforcement learning, which have achieved such amazing results in the natural ecology, have an equivalent potential in finance which we are only just beginning to realise.

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