**Assignment No**: 1

**Topic**: Formulation of simple 4-elements FEM for wave equation (Steady State)

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We have the wave equation as: $\nabla^2\varphi = \frac{1}{c^2}\frac{d^2\varphi}{dt^2}$ ....................( 1 )

Here φ is the displacement and "c" is the velocity of the medium.

For the steady state we assume time derivative part as "f", so the eqn. becomes $\nabla^2\varphi = f$ .....( 2)

Now our problem can be formulated as following:

$\int_0^L W(\nabla^2\varphi - f)\, dx = 0$ "W" is the weight function ................( 3 )

In one dimension above problem can be written as

$\int_0^L W(\frac{d^2\varphi}{dx^2} - f)\, dx = 0$ ................ ( 4 )

Subjected to the rigid boundary conditions (BC's) as $\varphi = Q_0\ at\ x = 0\ and\ \varphi = Q_L\ at\ x = L$

Now above equation 4 can be written as following: $\int_0^L \left[\frac{d}{dx}\left(w\frac{d\varphi}{dx}\right) - \frac{dW}{dx}\frac{d\varphi}{dx} - f\right] dx = 0$ ...........(5)

Which can be written as:

$$\int_0^L \left[\frac{dW}{dx}\frac{d\varphi}{dx}\right] dx = -\int_0^L Wf dx + \left(w\frac{d\varphi}{dx}\right)\Big|_0^L \qquad ....................(6)$$

Now this eqn is to discretized in n element so we can write :

$$\sum_{n=1}^{N}\int_0^l \left[\frac{dW}{dx}\frac{d\varphi}{dx}\right] dx = \sum_{n=1}^{N}\left[-\int_0^l Wf dx + \left(w\frac{d\varphi}{dx}\right)\Big|_0^l\right] \qquad ..................... (7)$$

Now we calculate the interpolating function, weights and corresponding derivatives, so

$$\left.\begin{array}{llll} \text{For the linear interpolation: } \varphi = \left(1-\frac{x}{l}\right)\varphi_k + \left(\frac{x}{l}\right)\varphi_{k+1} & \text{and} & \frac{d\varphi}{dx} = \frac{\varphi_{k+1}-\varphi_k}{l} \\ \text{And we can choose the weight as: } W_1 = \left(1-\frac{x}{l}\right) & \text{so} & \frac{dW_1}{dx} = \frac{-1}{l} \\ \text{And} \quad W_2 = \frac{x}{l} & \text{so} & \frac{dW_2}{dx} = \frac{1}{l} \end{array}\right\} \qquad .............(8)$$

On substituting the values into equation 7 from equation 8.

We place W=W1 and W=W2 one by one in equation 8 which yields two equations.

$$\left.\begin{array}{l} \int_0^l \frac{-1}{l}\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] dx = \int_0^l f\left(1-\frac{x}{l}\right) dx + \left(1-\frac{x}{l}\right)\frac{d\varphi}{dx}\Big|_0^l \\ \int_0^l \frac{1}{l}\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] dx = \int_0^l f\left(\frac{x}{l}\right) dx + \left(\frac{x}{l}\right)\frac{d\varphi}{dx}\Big|_0^l \end{array}\right\}$$

........................................ (9)

Which can be written as following

$$\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] = f\frac{l}{2} - \frac{d\varphi}{dx}\bigg|_{x=0}$$

$$\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] = f\frac{l}{2} + \frac{d\varphi}{dx}\bigg|_{x=l}$$

...................... (10)

Which can be written as

$$\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] = f\frac{l}{2} - Q_0$$

$$\left[\frac{\varphi_{k+1}-\varphi_k}{l}\right] = f\frac{l}{2} + Q_l$$

..........................................(11)

In Matrix form it can be written as

$$\frac{1}{l}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_k\\ \varphi_{k+1}\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-Q_0\\ Q_l\end{bmatrix}$$

..............................(12)

Now suppose we have four elements then we can write them in following manner:

Element 1 : $\frac{1}{l}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_0\\ \varphi_1\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-Q_0\\ Q_l\end{bmatrix}$ ......................(13A)

Element 2: $\frac{1}{l}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-Q_l\\ Q_{2l}\end{bmatrix}$ ......................(13B)

Element 3: $\frac{1}{l}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_2\\ \varphi_3\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-Q_{2l}\\ Q_{3l}\end{bmatrix}$ ......................(13C)

Element 4: $\frac{1}{l}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_3\\ \varphi_4\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-Q_{3l}\\ Q_{4l}\end{bmatrix}$ ......................(13D)

On merging above four equation we get:

$$\frac{1}{l}\begin{bmatrix}-1 & 1 & 0 & & & \\ 1 & -2 & 1 & \cdots & 0 & \\ 0 & 1 & -2 & & & \\ & \vdots & & \ddots & \vdots & \\ & 0 & & \cdots & -2 & 1\\ & & & & 1 & -1\end{bmatrix}\begin{bmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix} = \frac{fl}{2}\begin{bmatrix}1\\2\\2\\2\\1\end{bmatrix} + \begin{bmatrix}-Q_0\\ 0\\ 0\\ 0\\ Q_L\end{bmatrix}$$

................(14)

Which is the required equation.

QED.