Variational Formulation

Recap – Three Methods of formulating FEM

- Weighted Residual Method starting from governing diff eqn
- Variational formulation based on Stationarity of a functional
- Direct Method

FINITE ELEMENT FORMULATION BASED ON STATIONARITY OF A FUNCTIONAL

Functional is used to represent a function of a function e.g., an expression such as

$$I = \int \left\{ \left(\frac{1}{2}\right) AE \left(\frac{du}{dx}\right)^2 - (q)(u) \right\} dx$$

is a function of *u*, which itself is a function of *x*.

We state that the problem of finding *u*(*x*) that makes *I* stationary with respect to small, admissible variations in *u*(*x*) is equivalent to the problem of finding *u*(*x*) that satisfies the governing differential equation.

Principle of Stationary Total Potential

“*Among all admissible displacement fields, the equilibrium configuration makes the total potential of the system stationary with respect to small, admissible variations of displacement.*”

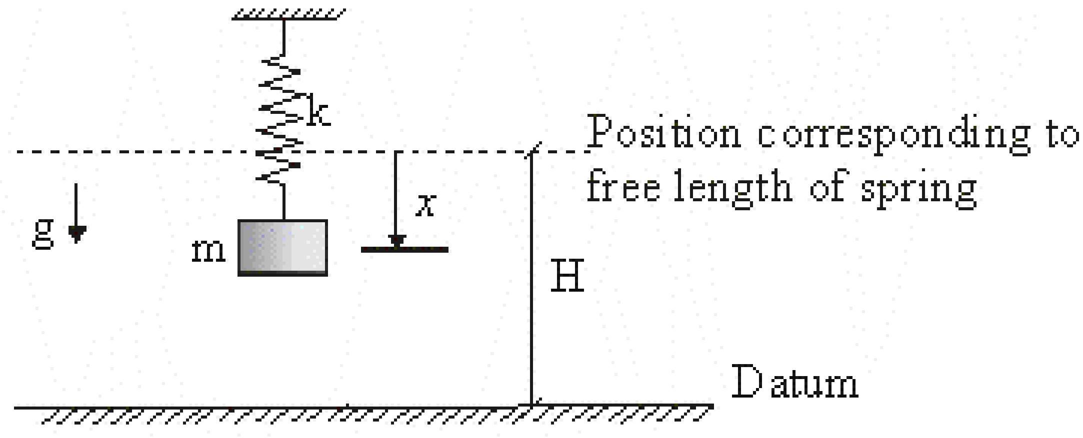
Principle of Stationary Total Potential (PSTP)

- In the spring-mass system

$$U = (1/2)(kx^2) \qquad \& \qquad V = mg(H-x)$$

$$\Pi_p = \left(\frac{1}{2}\right)(kx^2) + mg(H-x)$$

- PSTP → $\dfrac{d\Pi_p}{dx} = 0$ → $x_{eq} = (mg/k)$



- ✓ finite element equations can be derived systematically, with equal ease, starting from either form of representation of governing equations – differential or functional.
- ✓ Given a functional, we can always find the corresponding differential equation for the problem but some differential equation forms may not have a corresponding functional form.
- ✓ Unlike the total potential of a structure, in many cases the functional (even when it exists) may not have a direct physical meaning.
- ✓ While traditionally, the functional form (via PSTP for structures) is used to introduce the finite element method, it must be appreciated that the finite element formulation itself does not require the existence of a functional.

$$k\frac{d^2T}{dx^2} + q_0 = 0$$

subject to $T(0) = T_0$; $q\big|_{x=L} = h(T_L - T_\infty)$

is equivalent to minimising the functional

$$I = \int_0^L \frac{1}{2} k\left(\frac{dT}{dx}\right)^2 dx - \int_0^L q_0 T \, dx + \frac{1}{2} h(T - T_\infty)^2$$

subject to $T(0) = T_0$

$$k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + q_0 = 0$$

with the boundary conditions

$T = T_0$ on S_T , $q_n = \bar{q}$ on $S\underline{q}$,

$q_h = h\,(T - T\infty)$ on S_c

is equivalent to minimising the functional

$$\mathrm{I} = \frac{1}{2}\int\int_A \left(k\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 \right] - 2\,\mathrm{q}_0 T \right) dA + \int_{S_1} \bar{q}\; T\; ds + \int_{Sc} \frac{1}{2}\, h(T - T_\infty)^2\; ds$$

subject to $T = T_0$ on S_T

Rayleigh – Ritz (R–R) Method

Step 1– *Assume a displacement field*

$$\{\phi(x)+\sum c_i N_i\}, \qquad i = 1, 2, \ldots, n,$$

where N_i are the shape functions and c_i are the as yet undetermined coefficients. should satisfy the essential b.c. of the problem and also satisfy internal compatibility i.e. there should be no kinks, voids, etc.

Step 2 – *Evaluation of the total potential*

For the system under consideration, evaluate the total potential Πp

Step 3 – *Set-up and solve the system of equations*

By virtue of the PSTP,

$$\frac{\partial \Pi_p}{\partial c_i} = 0 \qquad i = 1, 2, \ldots, n$$

which will yield the necessary equations to be solved for the coefficients c_i.

Example: A bar under uniform load

$$AE\,\frac{d^2u}{dx^2} + q = 0$$

with the boundary conditions $u(0) = 0$; $\left.\frac{du}{dx}\right|_{x=L} = 0$

Strain energy stored in the bar,

$U =$ $\int_0^L \left[\frac{1}{2}\,AE\left(\frac{du}{dx}\right)^2\right] dx$

Potential of the external forces,

$$V = -\int_0^L q_0 u\; dx$$

To find $u(x)$ that minimises the total potential of the system:

$$\Pi_p = \int_0^L \left[\frac{1}{2} AE \left(\frac{du}{dx} \right)^2 - q_0 u \right] dx$$

Step 1 – Let us assume that $u(x) \approx c_1 x + c_2 x^2$

This satisfies the essential boundary condition that $u(0) = 0$.

Step 2 – The total potential of the system is given as

$$\Pi_p = \int_0^L \left[\frac{AE}{2} (c_1 + 2c_2 x)^2 - q_0 (c_1 x + c_2 x^2) \right] dx$$

$$= \frac{AE}{2} \left[c_1^2 L + \frac{4c_2^2}{3L^3} + 2c_1 c_2 L^2 \right] - q_0 \frac{c_1 L^2}{2} - q_0 c_2 \frac{L^3}{3}$$

Step 3 – From the principle of stationary total potential (PSTP) , we have

$$\frac{\partial \Pi_p}{\partial c_i} = 0$$

$$\frac{\partial \Pi_p}{\partial c_1} = 0 \Rightarrow \frac{AE}{2}(2c_1 L + 2c_2 L^2) - \frac{q_0 L^2}{2} = 0$$

$$\frac{\partial \Pi_p}{\partial c_2} = 0 \Rightarrow \frac{AE}{2}\left(8c_2 L^3/3 + 2c_1 L^2\right) - \frac{q_0 L^3}{3} = 0$$

Bar Element formulated from the stationarity of a functional

$$u = \left(1 - \frac{x}{\ell}\right) u_1 + \left(\frac{x}{\ell}\right) u_2$$

$$U^e = \int_0^\ell \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx \qquad = \frac{AE}{2}\frac{(u_2 - u_1)^2}{\ell}$$

$$V^e = -\int_0^\ell q_0 u \, dx - F_1 u_1 - F_2 u_2 \qquad = -q_0 \frac{\ell}{2}(u_1 + \mathrm{u}_2) - F_1 u_1 - F_2 u_2$$

$$\Pi_p^e = \frac{AE}{2}\frac{(u_2 - u_1)^2}{\ell} - \frac{q_0 \ell}{2}(u_1 + u_2) - F_1 u_1 - F_2 u_2$$

$$\Pi_p = \sum_{k=1}^{n} \Pi_p^e \qquad\qquad \frac{\partial \Pi_p}{\partial u_i} = 0$$

$$\frac{\partial \Pi_p^e}{\partial u_1} = 0 \;\Rightarrow\; \frac{AE}{\ell}(u_1 - u_2) = \frac{q_0 \ell}{2} + F_1$$

$$\frac{\partial \Pi_p^e}{\partial u_2} = 0 \;\Rightarrow\; \frac{AE}{\ell}(u_2 - u_1) = \frac{q_0 \ell}{2} + F_2$$

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} q_0 \ell / 2 \\ q_0 \ell / 2 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

One Dimensional Heat Transfer Element based on the stationarity of a functional

$$\Pi = \frac{1}{2}\int_0^{\ell} k\left(\frac{dT}{dx}\right)^2 dx \;-\; \int q_0 T\, dx \;-\; Q_1 T_1 \;-\; Q_2 T_2$$

$$T(x) = \left(1 - \frac{x}{\ell}\right) T_1 \;+\; \left(\frac{x}{\ell}\right) T_2$$

$$\Pi^e = \frac{1}{2}\left(\frac{k}{\ell}\,(T_2 - T_1)^2\right) \;-\; \frac{q_0 \ell}{2}(T_1 + T_2) \;-\; Q_1 T_1 \;-\; Q_2 T_2$$

$$\Pi \;=\; \sum_{k=1}^{n} \Pi^e$$

$$\frac{\partial \Pi}{\partial T_i} = 0$$

$$\frac{k}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_0\ell/2 \\ q_0\ell/2 \end{Bmatrix} - \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

Generic FEM equations for Structures

- 3 types of forces – 1. Body force (eg gravity)

  2. Surface force (eg. Pressure)

  3. Point force

- Initial stresses/strains (eg preloading, thermal strains)
- Stress, $\sigma_x = E(\varepsilon_x - \varepsilon_{x0}) + \sigma_{x0}$
- Strain energy per unit volume

$$dU = [F]\,[d\alpha] = [(\sigma_x)(1)(1)]\,[(d\varepsilon_x)(1)] = \sigma_x \; d\varepsilon_x$$

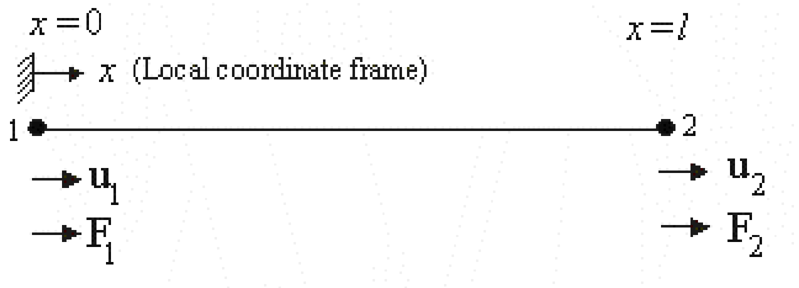
- Strain energy of the structure,

$$U = \int\left[\left(\frac{1}{2}\right)E\varepsilon_x^2\right]A\,dx - \int \varepsilon_{\mathrm{x}} E \varepsilon_{x0} A\,dx + \int \varepsilon_{\mathrm{x}} \sigma_{x0} A\,dx$$

- Potential of external forces, $V = -(\int \alpha q\,dx + \sum \alpha i P i)$
- Total potential of structure, $\Pi p = U + V$

Generic FEM equations

- Displacement, $\alpha = [N]\{\delta\}^e$
- For a 1D bar element,



$$\delta^e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad [N] = \begin{bmatrix} \left(1 - \frac{x}{l}\right) & \frac{x}{l} \end{bmatrix}$$

- Strains, $\varepsilon x = [\partial](\alpha) = [\partial][N]\{\delta\}^e = [B]\{\delta\}^e$

  $[B] = [\partial][N] = [-1/l, 1/l]$

- Substituting in the expression for total potential,

$$\prod p = U + V$$

$$\prod_p^e = \int \frac{1}{2}\{\delta\}^{eT}[B]^T[B]\{\delta\}^e EA\ dx - \int \{\delta\}^{eT}[B]^T E\varepsilon_{x_0} A\ dx$$

$$+\int \{\delta\}^{eT}[B]^T \sigma_{x_0} A\, dx - \int \{\delta\}^{eT}[N]^T q\ dx - \sum \{\delta\}^{eT}[N]_i^T P_i$$

Generic FEM equations

• Define: Stiffness matrix - $[k]^e = \int [B]^T [B] EA \; dx$

Load vector - $\{f\}^e = \int [B]^T E\varepsilon_{x_0} A \; dx - \int [B]^T \sigma_{x_0} A \; dx + \int [N]^T q \; dx + \sum [N]_i^T P_i$

• Total potential of the element, $\Pi_p{}^e = \frac{1}{2}\{\delta\}^{eT}[k]^e\{\delta\}^e - \{\delta\}^{eT}\{f\}^e$

• Total potential of the system, $\Pi_p = \sum_{1}^{NOELEM} \Pi_p^e = \frac{1}{2}\{\delta\}^T[K]\{\delta\} - \{\delta\}^T\{F\}$

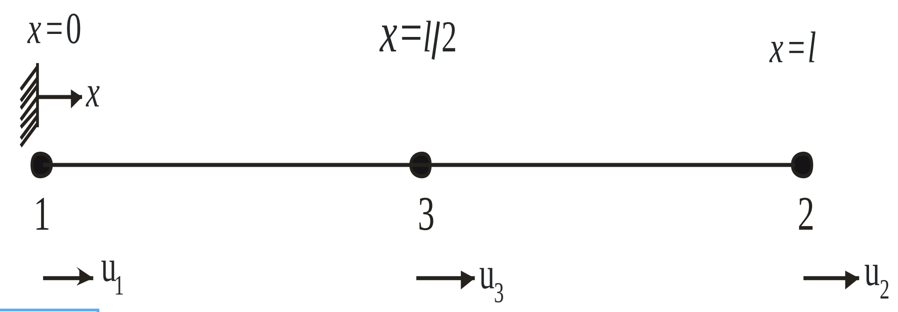
• According to PSTP, $\prod_p$ should be minimum

$\rightarrow \quad \frac{\partial \Pi_p}{\partial\{\delta\}^T} = 0 \quad \rightarrow \quad \boxed{[K]\{\delta\} = \{F\}}$

• Global stiffness matrix, $[K] = \sum_{n=1}^{NOELEM} [k]^e$

• Global load vector, $\{F\} = \sum_{1}^{NOELEM} \{f\}^e$

The Quadratic Bar Element



$$\boxed{u(x) = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3}$$

$$u(x) = c_0 + c_1x + c_2x^2$$

$$u(x) = u_1\left(1 - \frac{3x}{\ell} + \frac{2x^2}{\ell^2}\right) + u_2\left(-\frac{x}{\ell} + \frac{2x^2}{\ell^2}\right) + u_3\left(\frac{4x}{\ell} - \frac{4x^2}{\ell^2}\right)$$

$$\boxed{N_1 = 1 - \frac{3x}{\ell} + \frac{2x^2}{\ell^2}}$$

$$\boxed{N_2 = \frac{-x}{\ell} + \frac{2x^2}{\ell^2}}$$

$$\boxed{N_3 = \frac{4x}{\ell} - \frac{4x^2}{\ell^2}}$$

The Quadratic Bar Element

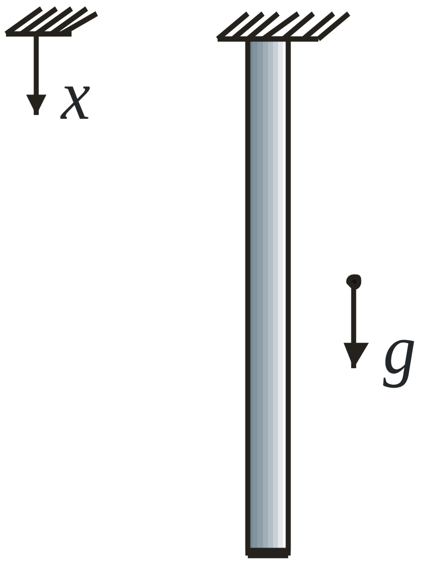
- Element Matrices

$$[k]^e = \int [B]^T [B] EA \; dx$$

$$\boxed{[k]^e = \frac{AE}{3\ell} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}}$$

A bar subjected to self weight

--Linear bar elements



Exact Solution:

Displacement $u(x) = \dfrac{\rho g}{E}\left(Lx - \dfrac{x^2}{2}\right)$

Stress $\sigma(x) = \rho g\left(L - x\right)$

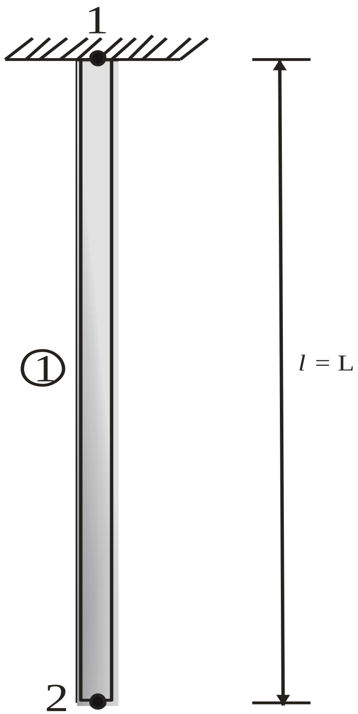
One element solution :

$$\boxed{\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}}$$

Using $l = L$ *and b.c.* $u_1 = 0$

$$u_2 = \left(\frac{\rho A g L}{2}\right)\frac{L}{AE} = \frac{\rho g L^2}{2E}$$

$$\boxed{u(x) = \left(\frac{\rho g L}{2E}\right)x}$$



Strain $\varepsilon_x = \dfrac{\rho g L}{2E}$

Stress $\sigma_x = E\varepsilon_x = \dfrac{\rho g L}{2}$

Two element solution :

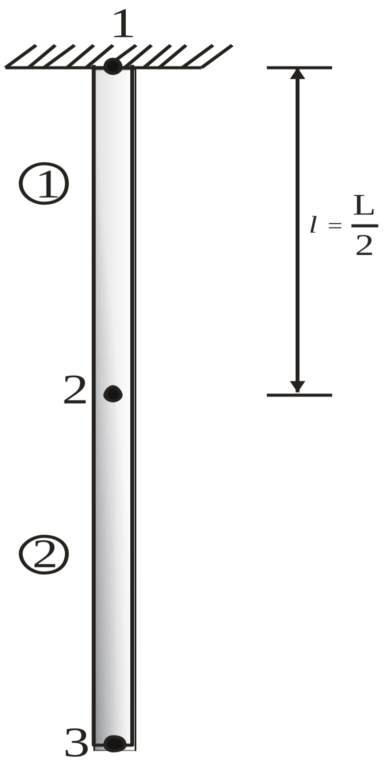
For the first element we have

$$\frac{AE}{(L/2)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}f_1^{(1)}\\ f_2^{(1)}\end{Bmatrix}$$

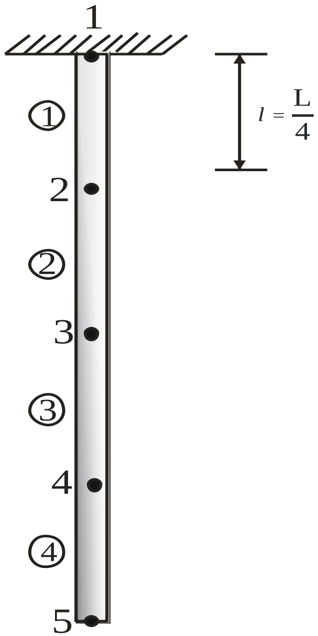
For the second element we have

$$\frac{AE}{(L/2)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}f_1^{(2)}\\ f_2^{(2)}\end{Bmatrix}$$

$$\boxed{\frac{AE}{(L/2)}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}f_1^{(1)}\\ f_2^{(1)}+f_1^{(2)}\\ f_2^{(2)}\end{Bmatrix}}$$

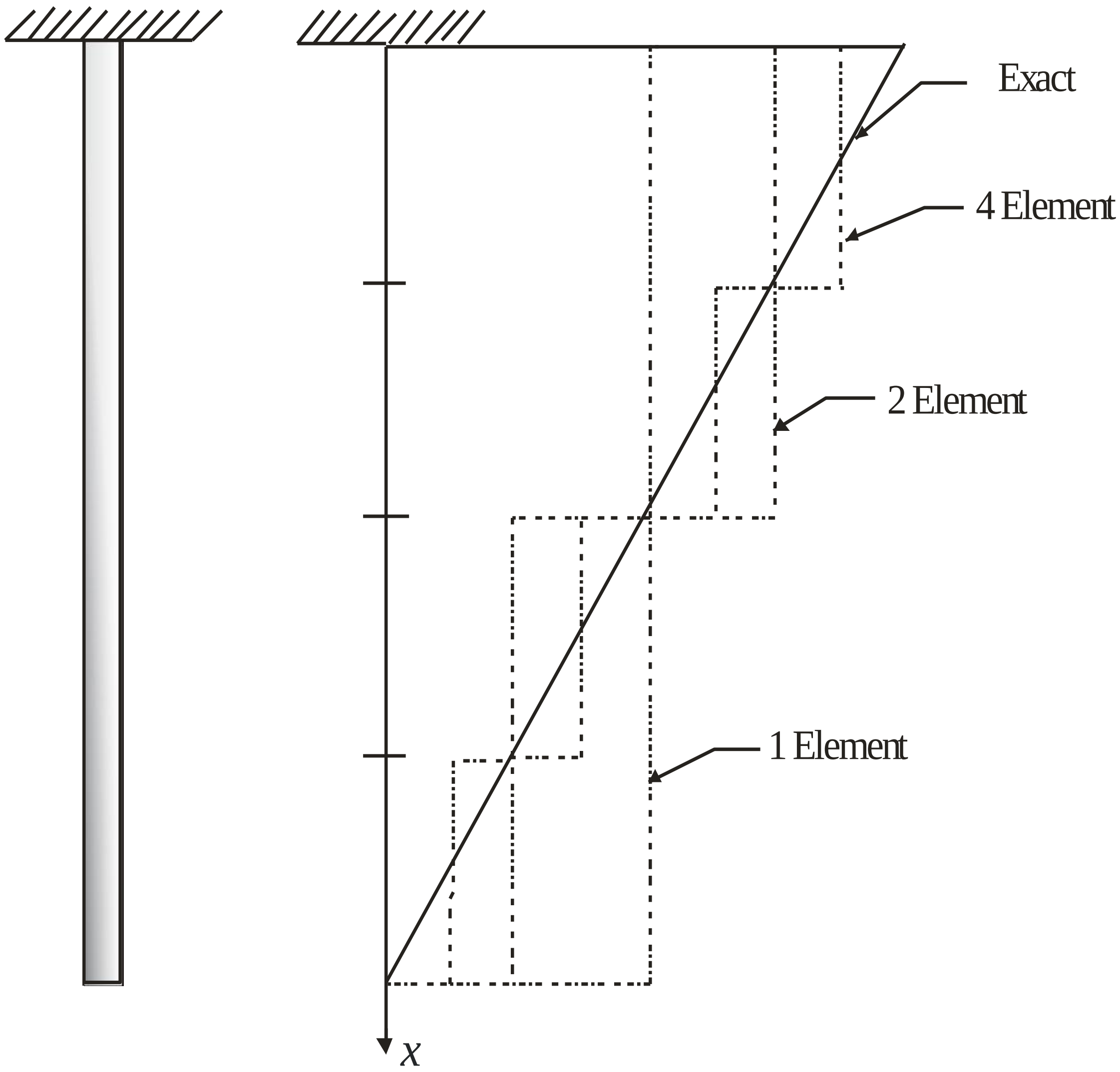


Four element solution :

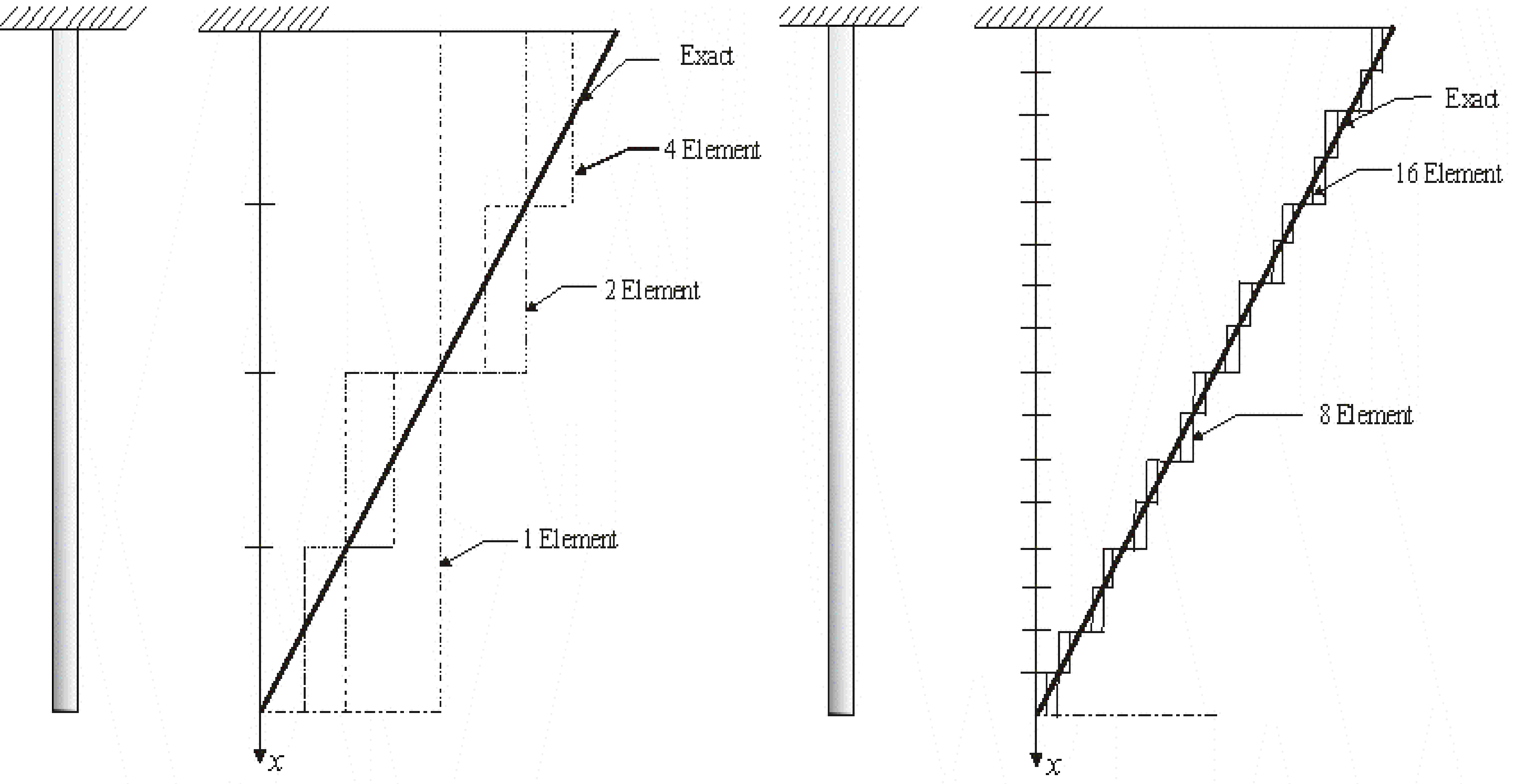


$$\frac{AE}{(L/4)}\begin{bmatrix} 1+1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1/2+1/2 \\ 1/2+1/2 \\ 1/2+1/2 \\ 1/2 \end{Bmatrix}\frac{\rho ALg}{4}$$

Comparison of stresses



Bar subjected to self-weight



Bar subjected to self-weight

- Many elements my be required to model the field accurately
- Solution for displacement at nodes is exact
- The error in the derivatives (e.g. stress here) could be considerable
- Stress averaging will be necessary to get a more realistic value of stress at a node
- There exist certain points within a finite element where the error in the estimation of stresses is the least
- Quadratic element required for modeling quadratic variation of field variable

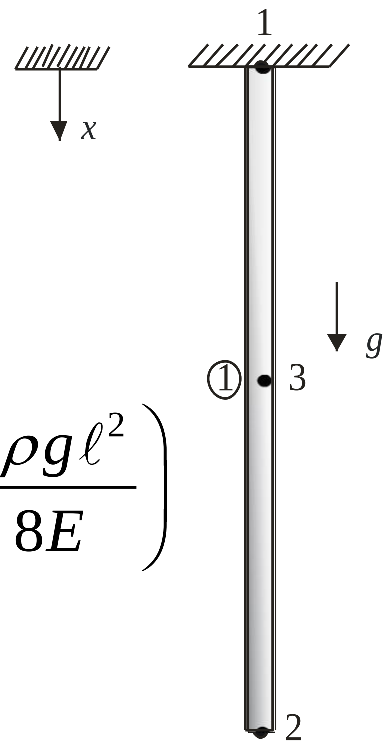
A bar subjected to self weight --Quadratic bar elements

$$\frac{AE}{3\ell}\begin{bmatrix} 7 & -8 \\ -8 & 16 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1/6 \\ 4/6 \end{Bmatrix}\rho A g \ell$$

Solving

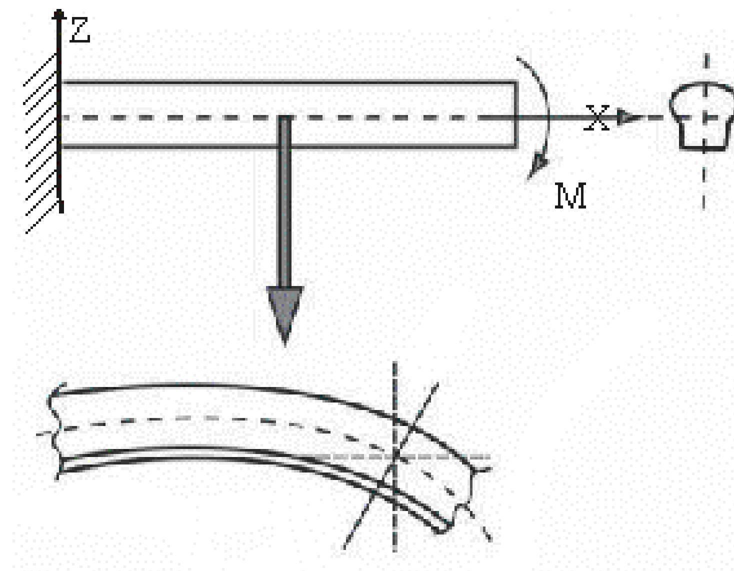
$$u(x) = \left(\frac{-x}{\ell} + \frac{2x^2}{\ell^2}\right)\frac{\rho g \ell^2}{2E} + \left(\frac{4x}{\ell} - \frac{4x^2}{\ell^2}\right)\left(\frac{3\rho g \ell^2}{8E}\right)$$

$$= \frac{\rho g \ell^2}{2E}\left(\frac{2x}{\ell} - \frac{x^2}{\ell^2}\right) = \frac{\rho g}{E}\left(Lx - \frac{x^2}{2}\right)$$



A few higher order elements are far superior to several lower order elements

Beam Element



• Euler-Bernoulli beam theory

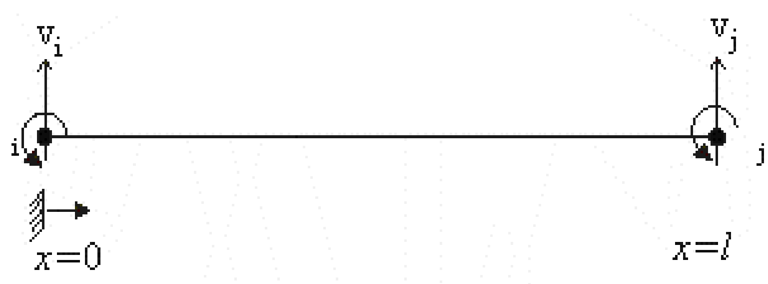
• c/s has the same transverse deflection as the neutral axis

• sections perpendicular to the neutral axis remain so after bending

• axial deformation, $uP = -(z)\ (dv/dx)$

• Beam element – line element representing neutral axis

• To ensure continuity of deformation at any point – use v and dv/dx as the nodal dof



Meaning of finite element equations

• element level equations, $[K]\,\{u\} = \{F\}$

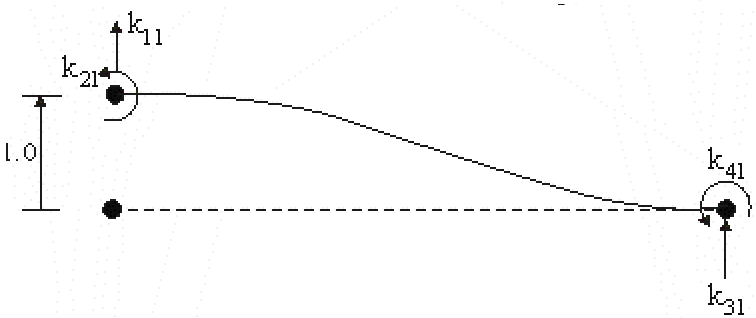
$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

• "*elements of each column of a stiffness matrix actually represent the forces required to cause a certain deformation pattern*"

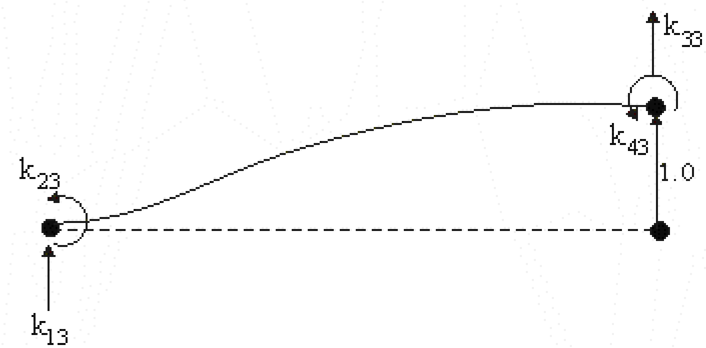
• *i*th column of the stiffness matrix shows a deformation pattern wherein the *i*th d.o.f. is given unit displacement (translational or rotational) and all other d.o.f. are held zero

• This is called the direct method of formulation for FE equations

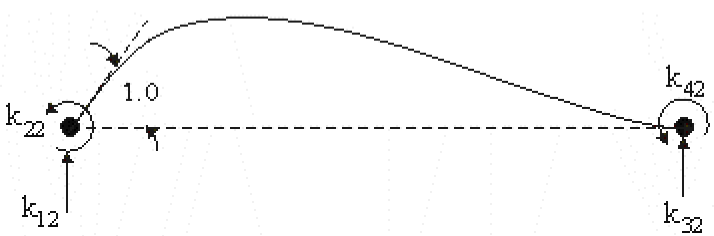
Direct method for beam element



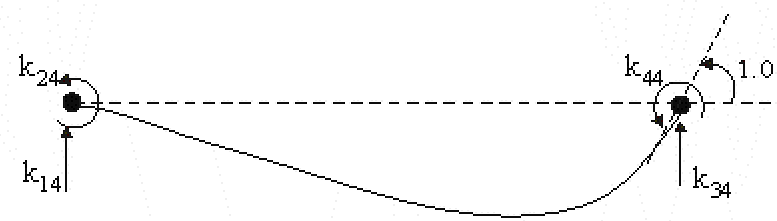
$v_1=1,\ v_2 = \theta_1 = \theta_2 = 0$



$v_1=0,\ v_2 =1,\ \theta_1 = \theta_2 = 0$



$v_1= v_2=0,\ \theta_1 = 1,\ \theta_2 = 0$



$v_1= v_2=0,\ \theta_1 = 0,\ \theta_2 = 1$

Direct method for beam element

- element level equations,

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

- Using direct method,

$$[\mathrm{k}] = \begin{bmatrix} \frac{12EI}{\ell^3} & & & Sym. \\ \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & & \\ \frac{-12EI}{\ell^3} & \frac{-6EI}{\ell^2} & \frac{12EI}{\ell^3} & \\ \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & \frac{-6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix}$$

- displacement field, $v(x) = N_1 v_i + N_2 \theta_i + N_3 v_j + N_4 \theta_j$

Consistent vs lumped load

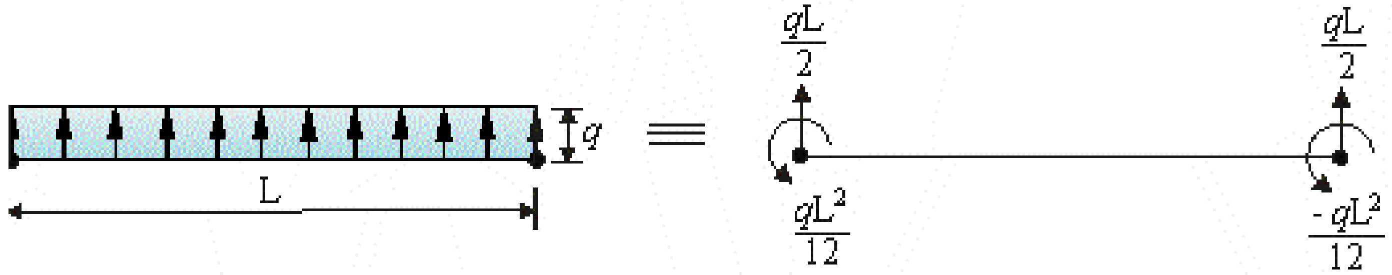
- uniform load q_0 on beam element,

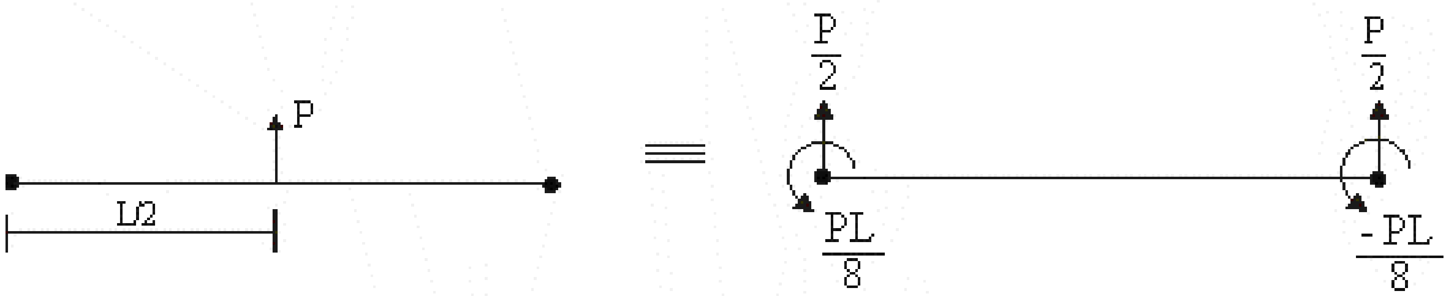
$$\{f\}^e = \int_0^{\ell} [N]^T q_0 \; dx = \begin{Bmatrix} q_0\ell/2 \\ q_0\ell^2/12 \\ q_0\ell/2 \\ -q_0\ell^2/12 \end{Bmatrix}$$

- **consistent load** – equivalent to the distributed force
  – nodal forces acting through the nodal displacements do the same amount of work as the distributed force
- **lumped load** – lump half of the total load on each node
  – do not consider moment load

$$\{f\}^e = \begin{Bmatrix} q_0\ell/2 \\ 0 \\ q_0\ell/2 \\ 0 \end{Bmatrix}$$

Consistent loads

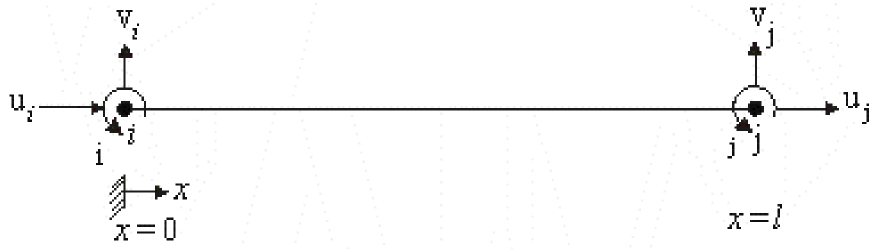




Frame element

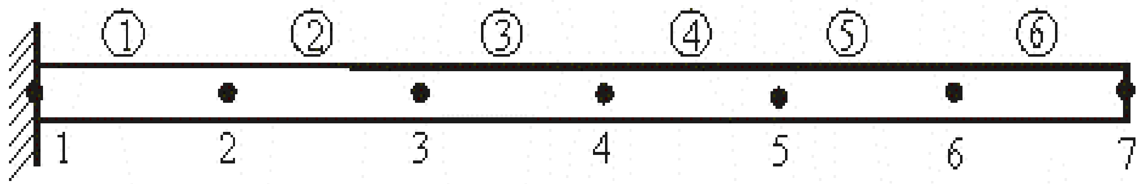
- combination of bar and beam element

$$\{\delta\}^e = \left\{ u_i \;\; v_i \;\; \theta_i \;\; u_j \;\; v_i \;\; \theta_j \right\}^T$$

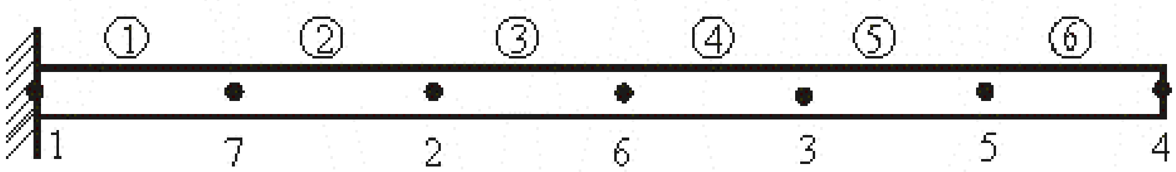


$$[k]^e = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12AE/L^3 & 6AE/L^2 & 0 & -12AE/L^3 & 6AE/L^2 \\ 0 & 6AE/L^2 & 4AE/L & 0 & -6AE/L^2 & 2AE/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12AE/L^3 & -6AE/L^2 & 0 & 12AE/L^3 & -6AE/L^2 \\ 0 & 6AE/L^2 & 2AE/L & 0 & -6AE/L^2 & 4AE/L \end{bmatrix}$$

Effect of node numbering on assembled matrix equation



(a) Mesh 1



(b) Mesh 2

- Element stiffness matrix

$$[k]^e = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Effect of node numbering on assembled matrix equation

- Assembled stiffness matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Mesh 1 (banded matrix) Mesh 1 (scattered matrix)

- Stiffness matrix is symmetric
- Efficient node numbering scheme reduces computational expense
- No need to store non-zero matrix coefficients
- Commercial softwares automatically apply efficient node numbering scheme and matrix storage schemes