2-D Modeling in FEM

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Dimensionality of the problem

- Dimensionality - Variation of unknown field variables
- 1-D, 2-D and 3-D problems
- Real life problems – 3D →**computationally expensive**
- 1D problem – variation along y and z direction assumed
  - Eg. Cantilever beam – Euler-Bernoulli beam theory → relate deformations in y and z direction with that of x direction
- 2D problem – suitable assumption regarding variation of field variable in z-direction

Dimensionality of the problem

2D problems in structural Mechanics:

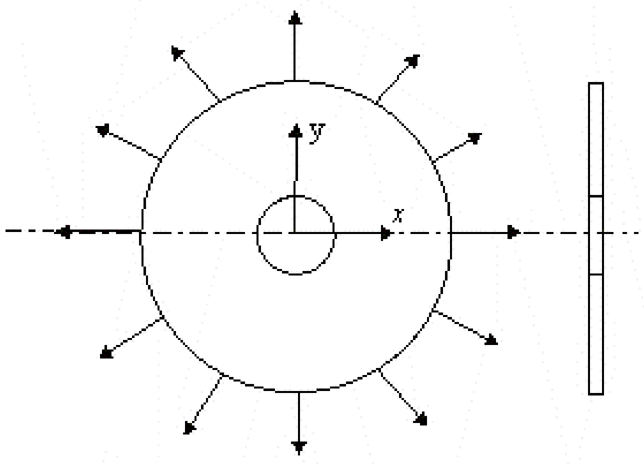
- Plane stress
- Plane strain
- Axi-symmetric

Plane stress $\quad \sigma_z = 0$

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

eg. Thin rotating disc

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu^2}{2} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \right)$$

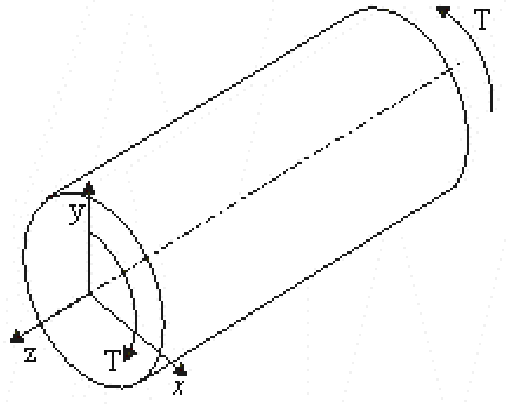


Plane strain ($\varepsilon_z = 0$)

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eg. Long slender shaft subjected to torsion

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)\,(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \right)$$



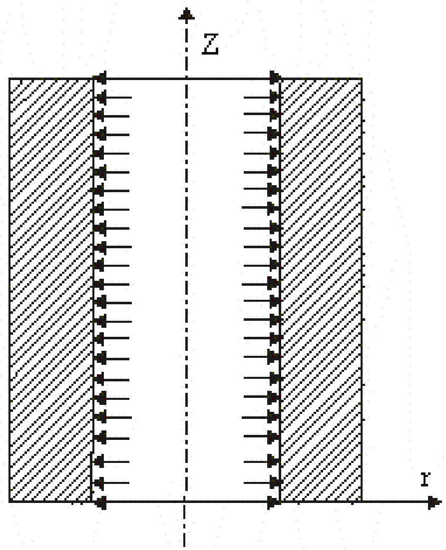
Axi-symmetric

$$\sigma = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_\theta & 0 \\ \tau_{zr} & 0 & \sigma_z \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_r & 0 & \varepsilon_{rx} \\ 0 & \varepsilon_\theta & 0 \\ \varepsilon_{zr} & 0 & \varepsilon_z \end{bmatrix}$$

- Geometry, material properties, support conditions and loading – need to be axis-symmetric

- no variation along θ direction

eg. Pressure vessel



$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 \\ & & 1 & 0 \\ Sym. & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} \right)$$

Modeling geometry/deformation

- For 2D and 3D problems we need to model physical geometry of structure and variation of disp./strain/stress

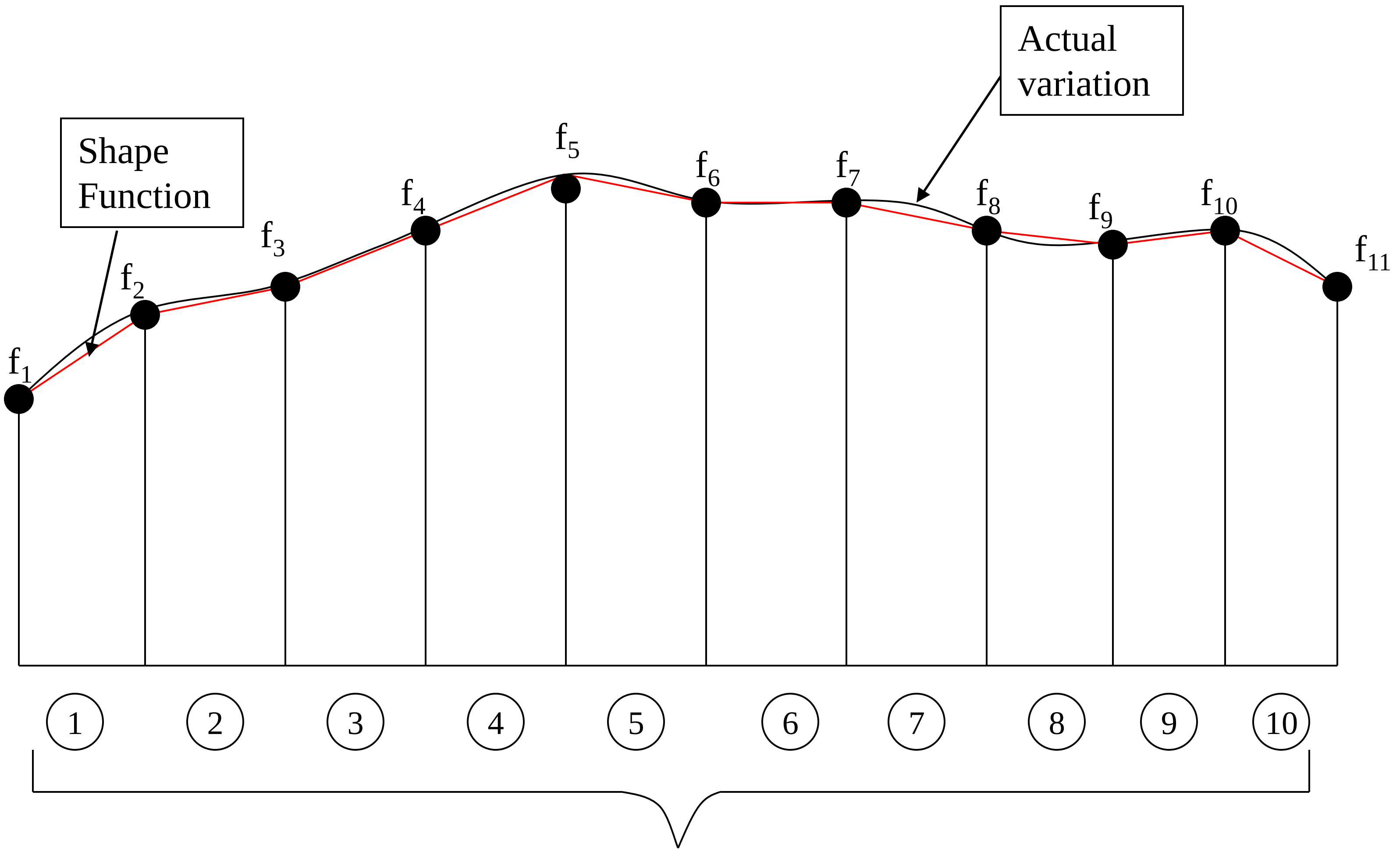
- Approximation of geometry of lower order than that of field variable → Sub-parametric formulation

  e.g. Euler-Bernoulli beam

- Approximation of geometry of equal order as that of field variable → Iso-parametric formulation

  e.g. an axial bar modeled with linear elements

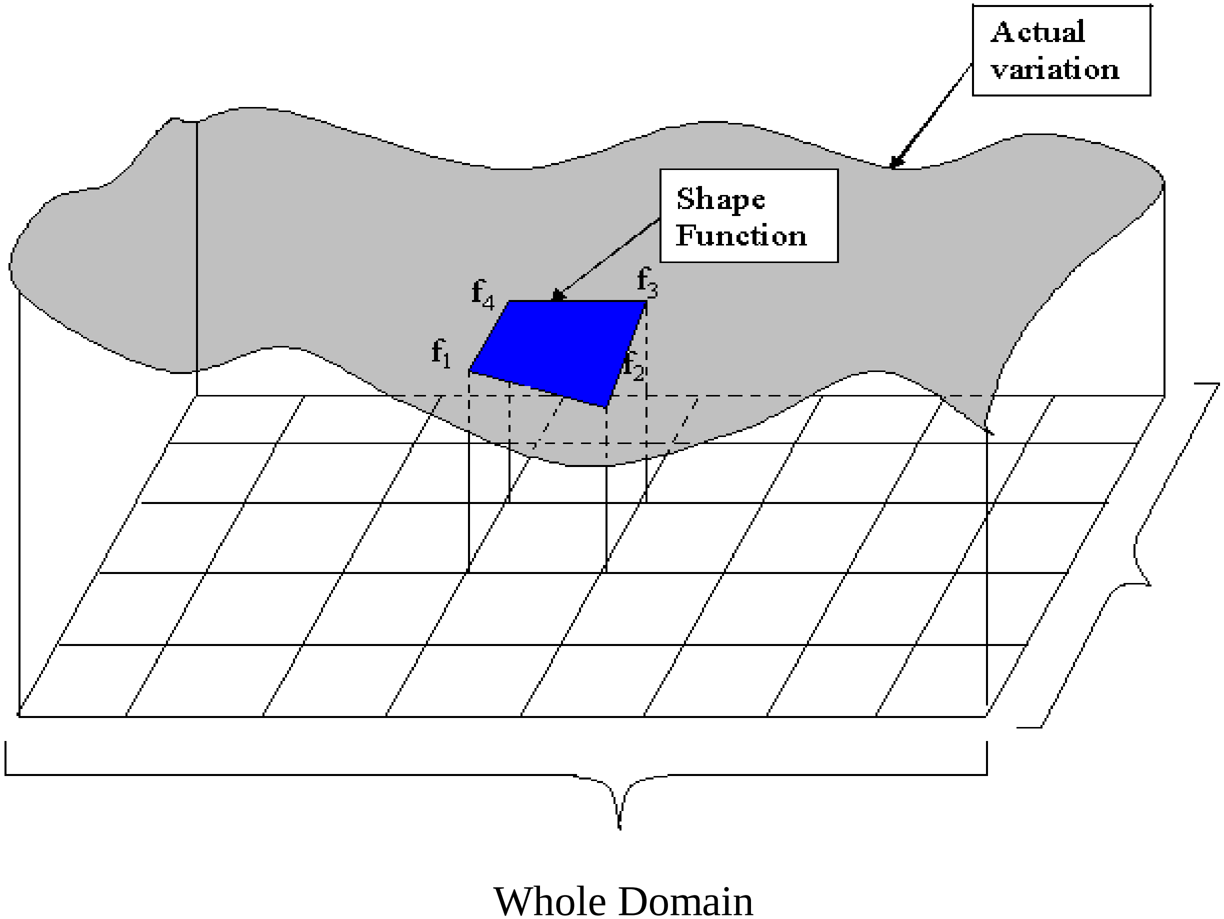
- Approximation of geometry of higher order than that of field variable → Super-parametric formulation



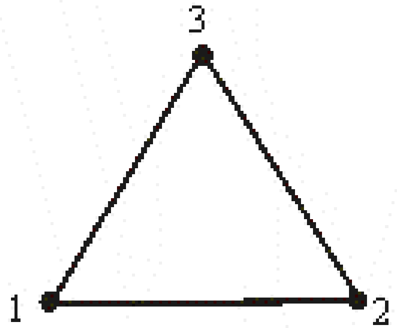
Whole Domain

Modeling geometry/deformation

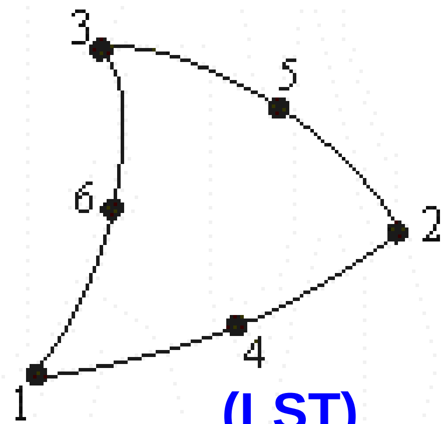
We need to model physical geometry of structure and variation of disp./strain/stress



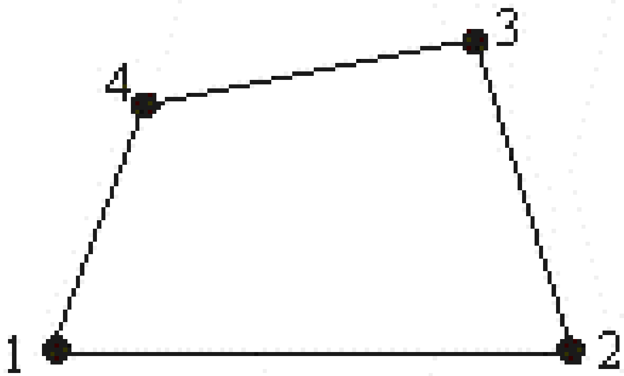
Two dimensional elements



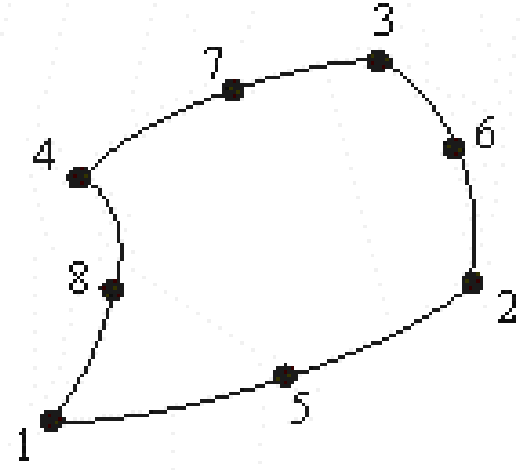
CST



(LST)



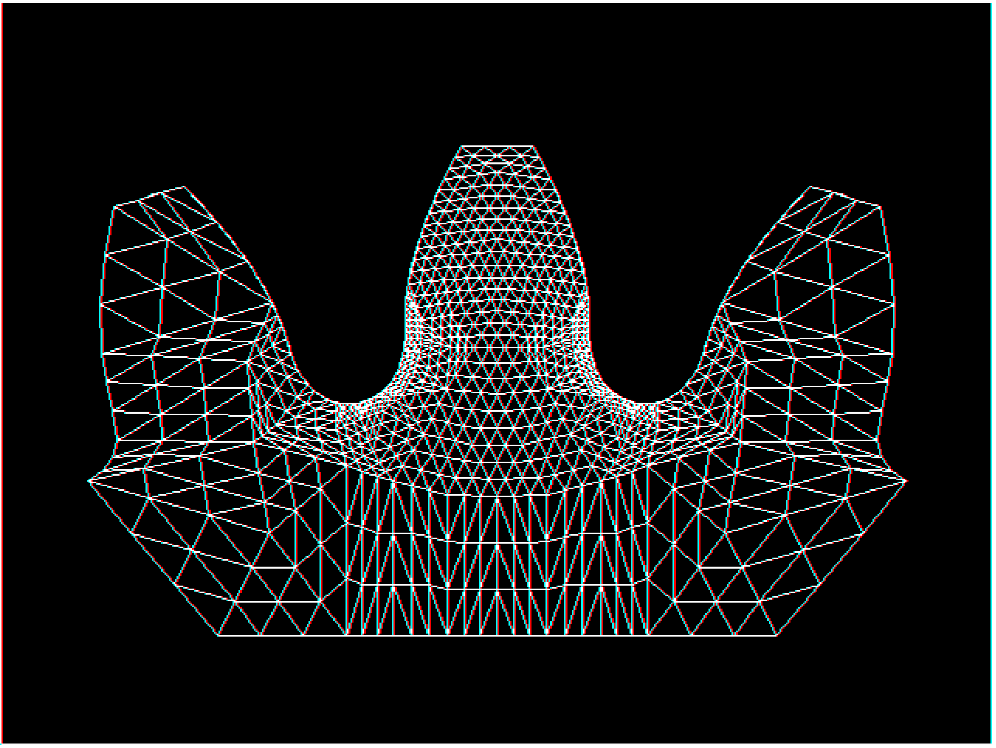
Quad 4



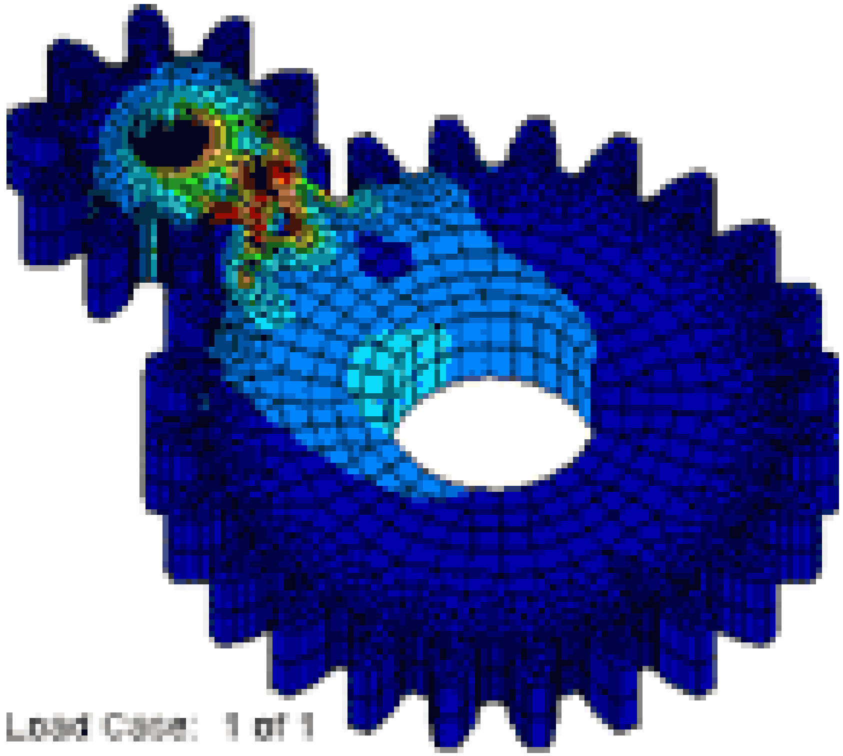
Quad 8

Gear Modeling

2D Model

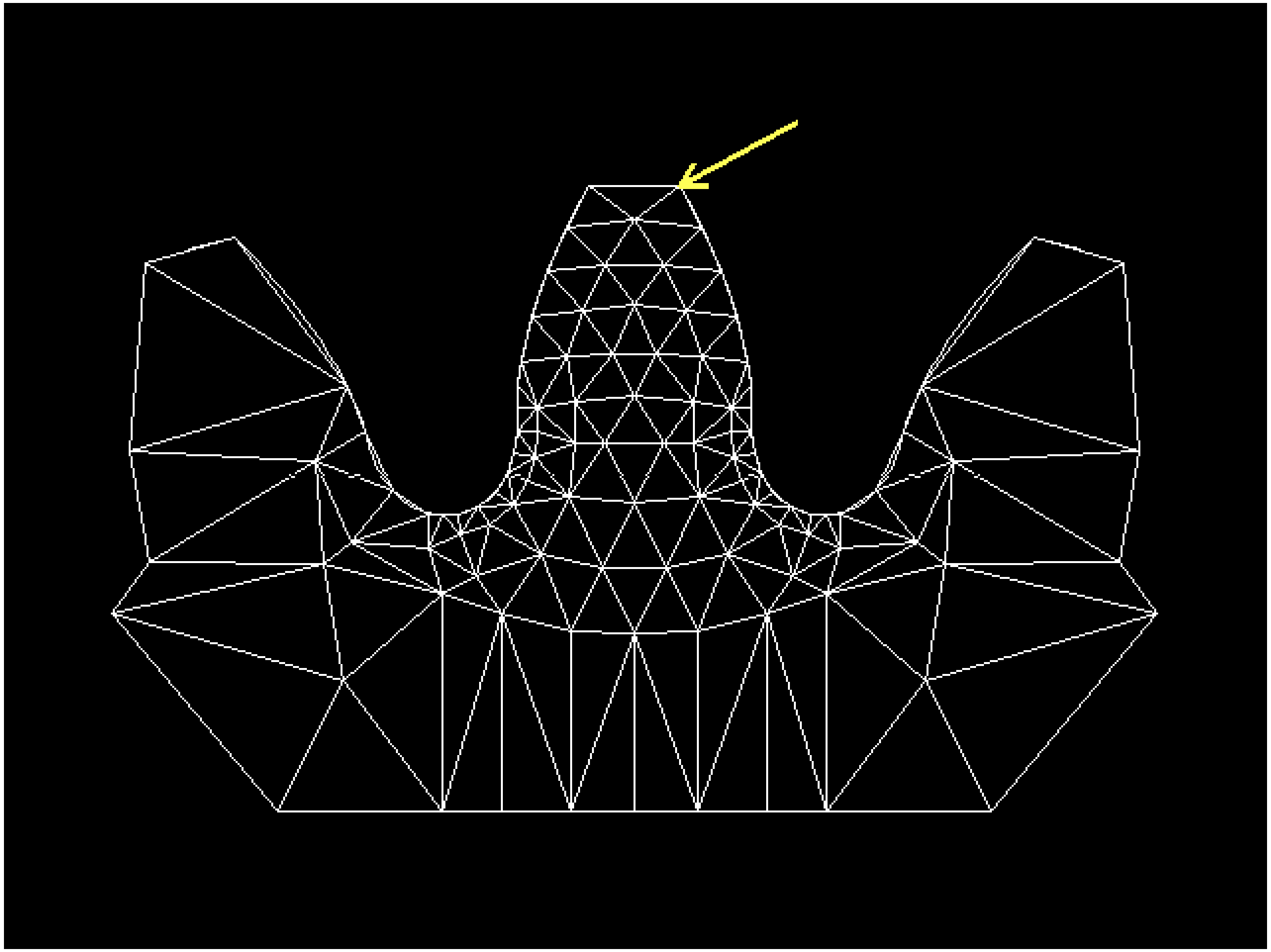


3D Model



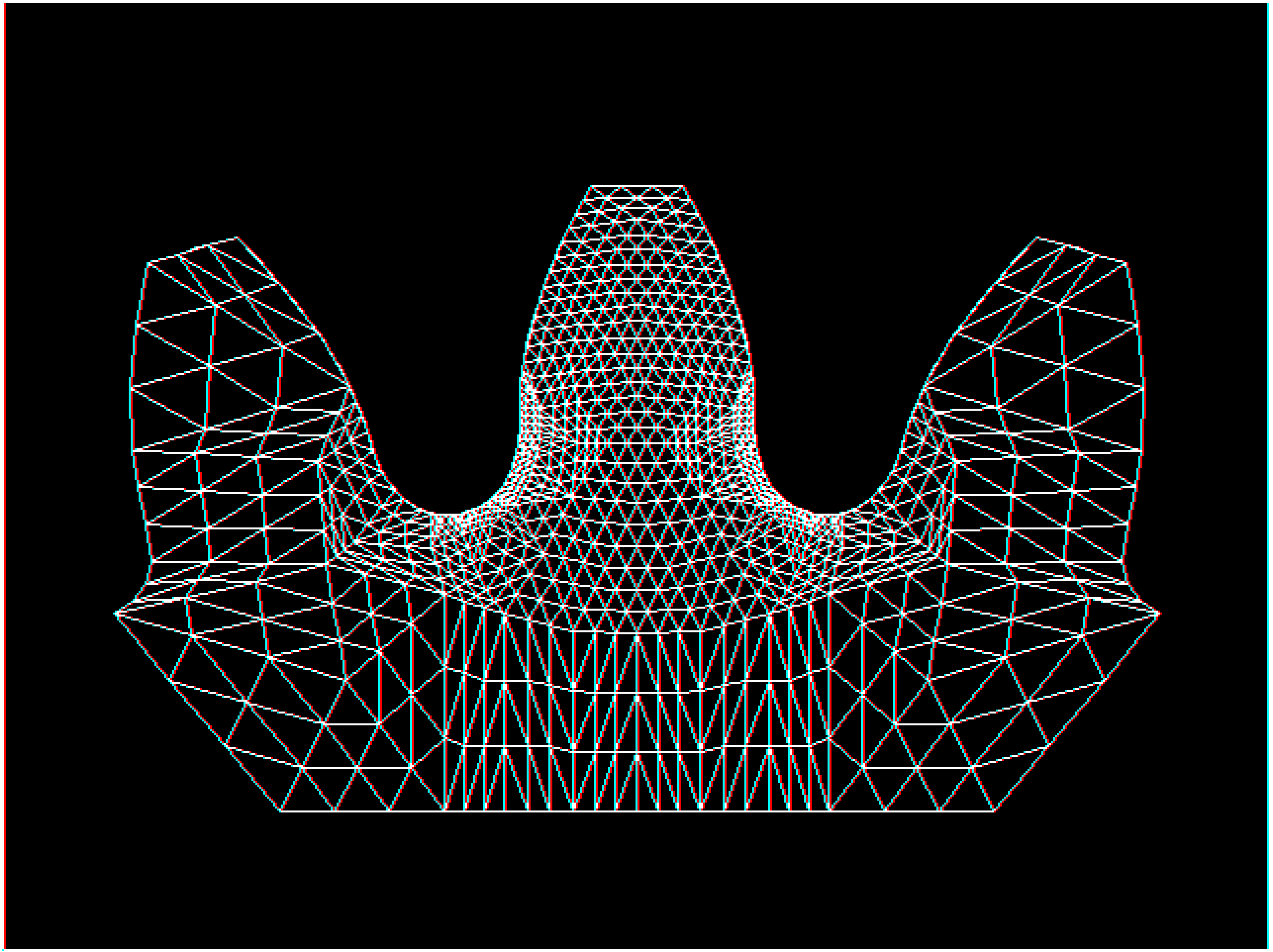
DOF reduction- Cyclic symmetry

Approximately 150 triangular elements



Source: http://members.aol.com/gearLab/FEM.htm

Approximately 1500 triangular elements

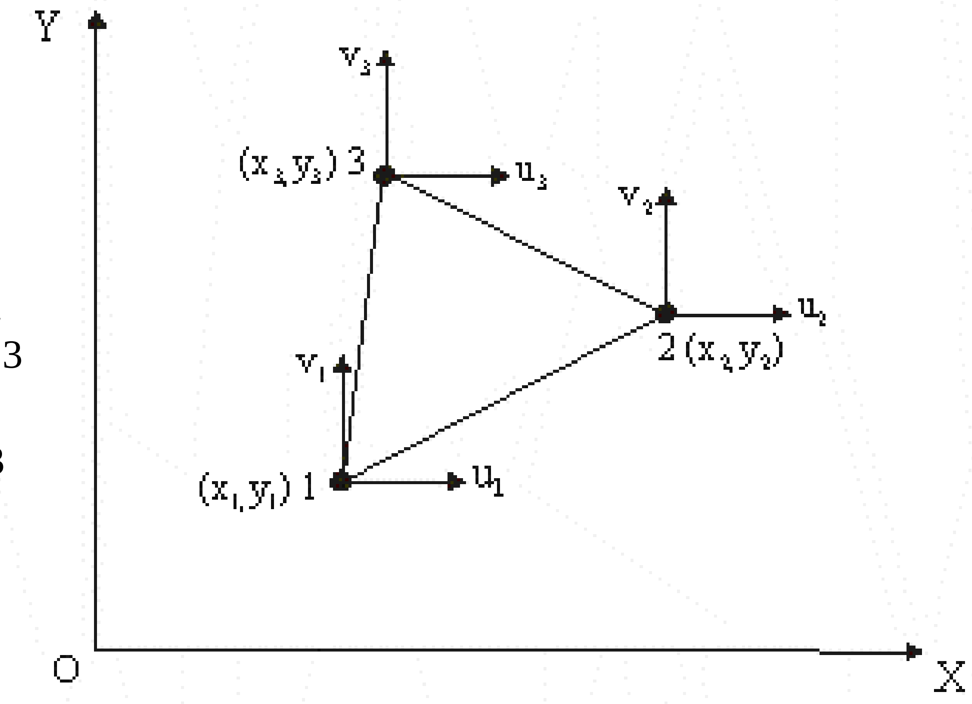


Source: http://members.aol.com/gearLab/FEM.htm

3-noded triangular element

Scalar Field - temperature

Vector Field – displacement



$$u(x,y) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x,y) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$T(x,y) = N_1 T_1 + N_2 T_2 + N_3 T_3 = [N]\{T\}^e$$

N_1, N_2, N_3 - shape functions

Simple three noded triangular element

$$T(x,y) = c_0 + c_1 x + c_2 y$$

$$T_1 = c_0 + c_1 x_1 + c_2 y_1$$

$$T_2 = c_0 + c_1 x_2 + c_2 y_2$$

$$T_3 = c_0 + c_1 x_3 + c_2 y_3$$

$$\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$T(x,y) = \left(\frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2\Delta}\right) T_1 + \left(\frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2\Delta}\right) T_2 + \left(\frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2\Delta}\right) T_3$$

$$T(x,y) = N_1 T_1 + N_2 T_2 + N_3 T_3 = [N]\,\{T\}^e$$

$$\alpha_1 = x_2 y_3 - x_3 y_2$$

$$\beta_1 = y_2 - y_3$$

$$\gamma_1 = x_3 - x_2$$

Other coefficients (α_2, β_2, γ_2) and (α_3, β_3, γ_3) can be obtained by a simple cyclic permutation of subscripts 1,2,3.

$$2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2\ (\text{Area of triangle } 123)$$

In our standard notation

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

- Linear variation of unknown field variable

$$T(x, y) = c_0 + c_1 x + c_2 y$$

- Heat flux / Strains and stresses →constant over element
- Referred as **Constant Strain Triangle (CST) element**

Six Node Triangle Element

$$T(x,y) = c_0 + c_1 x + c_2 y$$

$$+c_4 x^2 + c_5 xy + c_6 y^2$$

$$T_1 = T(x_1, y_1)$$

$$T_2 = T(x_2, y_2)$$

$$T_3 = T(x_3, y_3)$$

$$T_4 = T(x_4, y_4)$$

$$T_5 = T(x_5, y_5)$$

$$T_6 = T(x_6, y_6)$$

TEDIOUS TO FIND SHAPE FN.

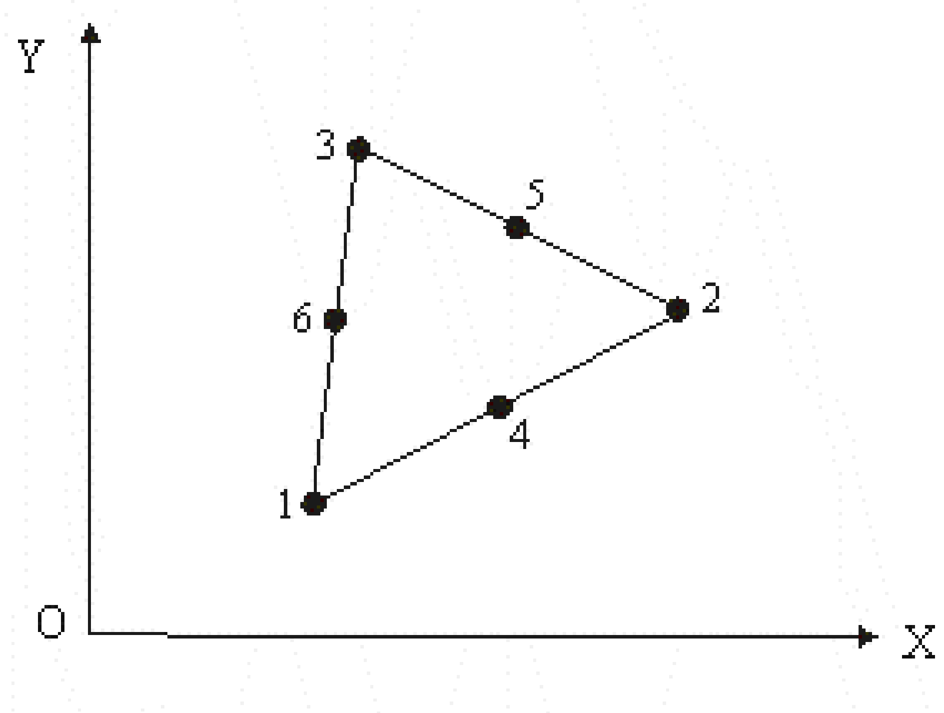
6-noded triangular element

- Scalar temperature field

$$T(x,y) = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2$$

- Pascal triangle

$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad xy^2 \quad x^2y \quad y^3$$
$$x^4 \quad xy^3 \quad x^2y^2 \quad x^3y \quad y^4$$



- Linear variation of stresses & strains / Heat flux
- Referred as **Linear Strain Triangle (LST) element**

Four Node Quadrilateral Element

$$T(x,y) = c_0 + c_1 x + c_2 y + c_3 x^2$$
$$T(x,y) = c_0 + c_1 x + c_2 y + c_4 xy$$
$$T(x,y) = c_0 + c_1 x + c_2 y + c_5 y^2$$

What to choose? Why?

$$T_1 = T(x_1, y_1)$$
$$T_2 = T(x_2, y_2)$$
$$T_3 = T(x_3, y_3)$$
$$T_4 = T(x_4, y_4)$$

Strain - Displacement Relation in 2-D

Nodal displacement δ^e

For example, CST $\delta^e = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{bmatrix}^T$

Strains:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\{\varepsilon\} = [B]\,\{\delta\}^e$$

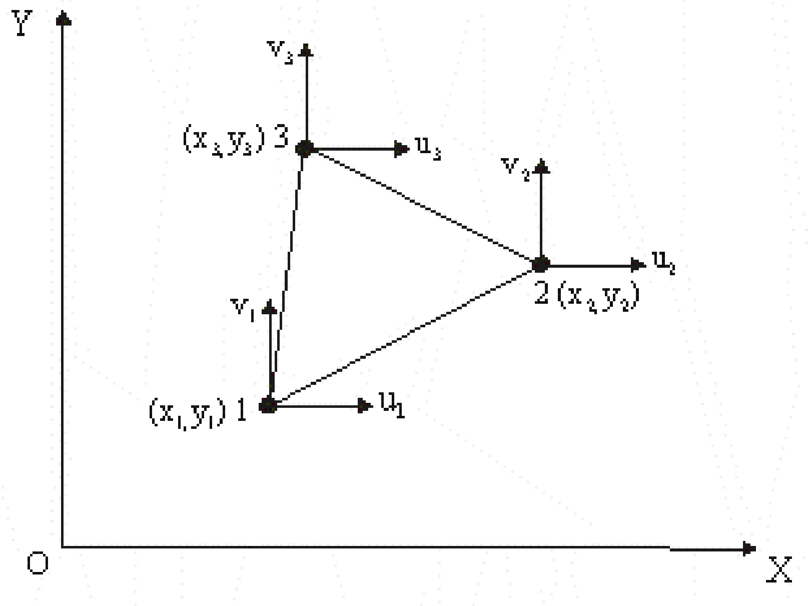
$$\varepsilon_y = \frac{\partial v}{\partial y}$$

[B] contains derivatives of the shape function

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

3-noded triangular element

- Iso-parametric formulation



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$u(x,y) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x,y) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$N_1 = \frac{1}{2\Delta}(\alpha_1 + \beta_1 x + \gamma_1 y) \qquad N_2 = \frac{1}{2\Delta}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2\Delta}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

α, β, γ - are constants

3-noded triangular element

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[B] = \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x \end{bmatrix}$$

- Since N_1, N_2, N_3 are linear, B matrix contains constant terms – Constant Strain Triangle (CST) Element

Stress – Strain Relation in 2-D

For example, plane stress:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu^2}{2} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \{\varepsilon\}^0 \right)$$

$$\{\sigma\} = [D]\,(\{\varepsilon\} - \{\varepsilon\}^0) + \{\sigma\}^0$$

Material Behavior affects these relations

Energy

Elastic strain Energy

$$U^e = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\}\ dv = \frac{1}{2}\int_v \{\varepsilon\}^T \left([D]\,(\{\varepsilon\} - \{\varepsilon\}^0) + \{\sigma^0\}\right) dv$$

$$= \int_v \left(\frac{1}{2}\{\varepsilon\}^T [D]\ \{\varepsilon\} - \{\varepsilon\}^T [D]\{\varepsilon\}^0 + \{\varepsilon\}^T \{\sigma^0\}\right) dv$$

Potential of External Forces

$$V^e = \int_v \frac{1}{2}\{\delta\}^T \{q_v\}\ dv - \int_s \{\delta\}^T \{q_s\}\ ds - \sum \{\delta_i\}^T \{P_i\}$$

Kinetic Energy to be taken for dynamic problems

Total Potential for an Element

$$\Pi^e = U^e + V^e$$

$$\Pi_p^e = \int \frac{1}{2} \{\delta\}^{eT} [B]^T [D] [B] \{\delta\}^e \; dv - \int \{\delta\}^{eT} [B]^T [D] \{\varepsilon\}^0 \; dv$$

$$+ \int \{\delta\}^{eT} [B]^T \{\sigma\}^0 dv - \int \{\delta\}^{eT} [N]^T \{q\} \, dv - \sum \{\delta\}^{eT} [N]_i^T \{P_i\}$$

Total Potential for an Element

$$\Pi_p{}^e = \frac{1}{2}\{\delta\}^{eT}[k]^e\ \{\delta\}^e - \{\delta\}^{eT}\{f\}^e$$

where,

$$[k]^e = \int_{\mathrm{v}} [B]^T[D][B]\,dv$$

element stiffness matrix

$$\{f\}^e = \int_v [B]^T[D]\{\varepsilon\}^0 dv - \int_v [B]^T\{\sigma\}^0 dv + \int_{\mathrm{v}} [N]^T\{q\}\,dv + \sum [N]_i^T\{P_i\}$$

Element load vector (consistent load vector)

Total Potential for the Structure

$$\Pi_p = \sum \Pi_p^e = \frac{1}{2}\{\delta\}^T[K]\{\delta\} - \{\delta\}^T\{F\}$$

$$[K] = \sum_{n=1}^{NOELEM} [k]^e$$

Global Stiffness Matrix

$$\{F\} = \sum_{1}^{NOELEM} \{f\}^e$$

Global Load Vector

Principal of Stationary Total Potential (PSTP)

Total potential of the system must be minimum

$$\frac{\partial \Pi_p}{\partial \{\delta\}^T} = \quad 0$$

$[K]\{\ \delta\ \} = \{F\}$ - static equilibrium relation in matrix form

$[K]$ – Global stiffness matrix, $\{F\}$ – Global load vector

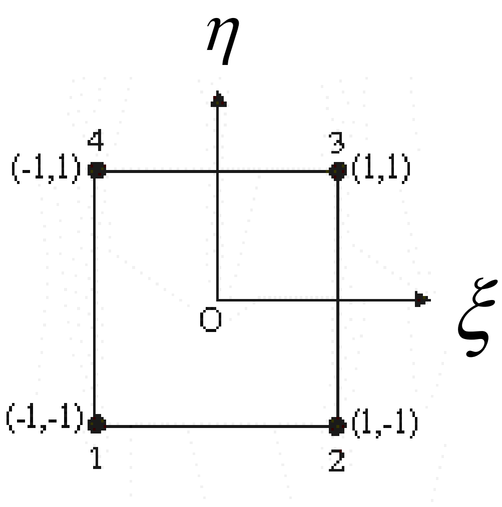
Natural Coordinates

- To help achieve curved edge elements
- To help ease derivation of shape functions

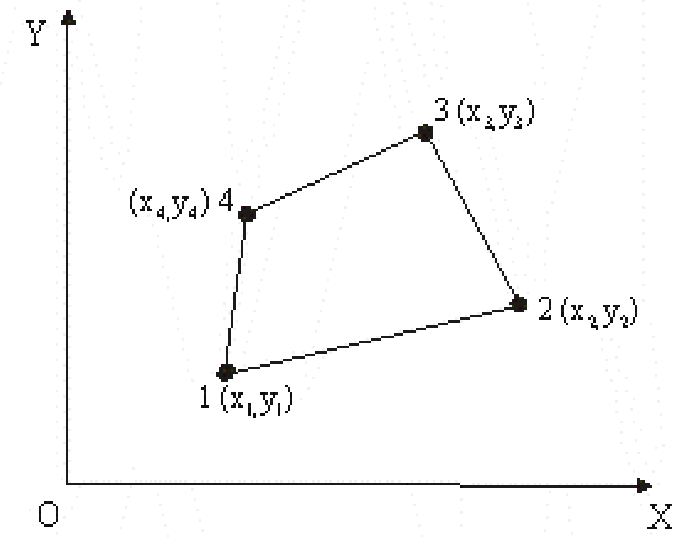
- Natural coordinates for one dimensional case is taken as ξ and in 2-D, they are taken as ξ and η.
- NC always vary from 0 to 1 or -1 to 1.

Natural co-ordinates

- To simplify element-level equation formulation
- To facilitate numerical-integration



Parent element

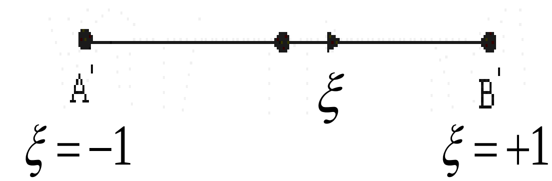


Child (physical) element

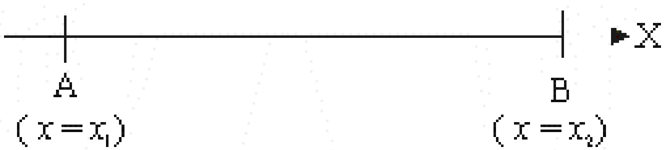
- ξ, η- Natural co-ordinates - vary from -1 to +1
- Nodal dof – along physical (x,y) co-ordinates

Natural co-ordinates – 1D case

- For linear 1D element



Natural co-ordinates



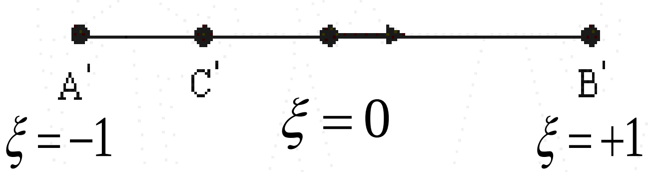
Physical co-ordinates

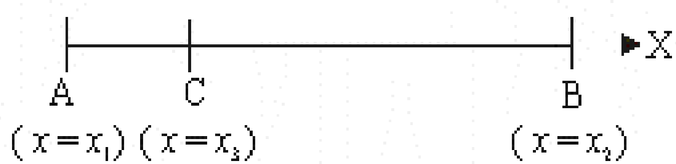
$$x(\xi) = \underbrace{\left(\frac{1-\xi}{2}\right)}_{N_1} x_1 + \underbrace{\left(\frac{1+\xi}{2}\right)}_{N_2} x_2$$

$$= N_1 x_1 + N_2 x_2$$

- For Quadratic 1D element

$$x(\xi) = N_i \; x_i \; = N_1 x_1 + N_2 x_2 + N_3 x_3$$





Methods for deriving shape functions

- Serendipity approach
  - $N_1 = 0$ at nodes 2 & 3
  - $N_1 = (SF)\,(\xi - 0)\,(\xi - 1)$
  - $N_1 = 1$ at node 1 – $SF = 0.5$
  - Similarly,

$$N2 = 0.5\,(\xi - 0)\,(\xi + 1)$$

& $$N3 = -\,(\xi + 1)\,(\xi - 1)$$

- Lagrange approach
  - $$N_1 = \frac{(\xi_2 - \xi)\,(\xi_3 - \xi)}{(\xi_2 - \xi_1)\,(\xi_3 - \xi_1)} = \frac{(1 - \xi)\,(0 - \xi)}{(1 + 1)\,(0 + 1)} = \frac{\xi\,(\xi - 1)}{2}$$
  - Similarly,

$$N2 = 0.5\,(\xi - 0)\,(\xi + 1)$$

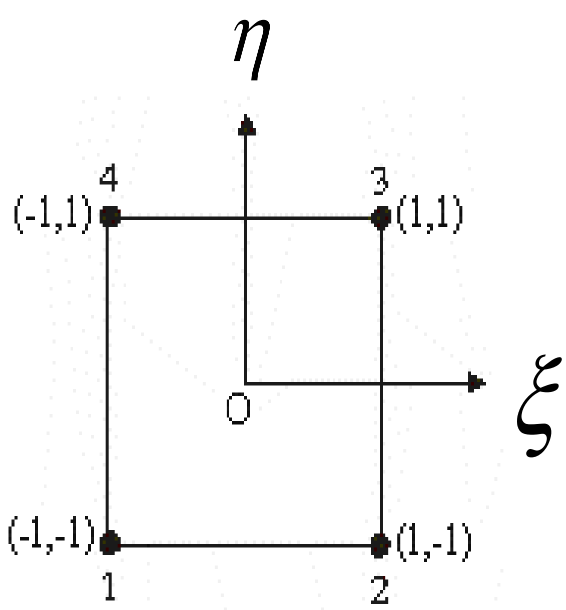
& $$N3 = -\,(\xi + 1)\,(\xi - 1)$$

$$x(\xi) = N_i\,x_i = N_1x_1 + N_2x_2 + N_3x_3$$

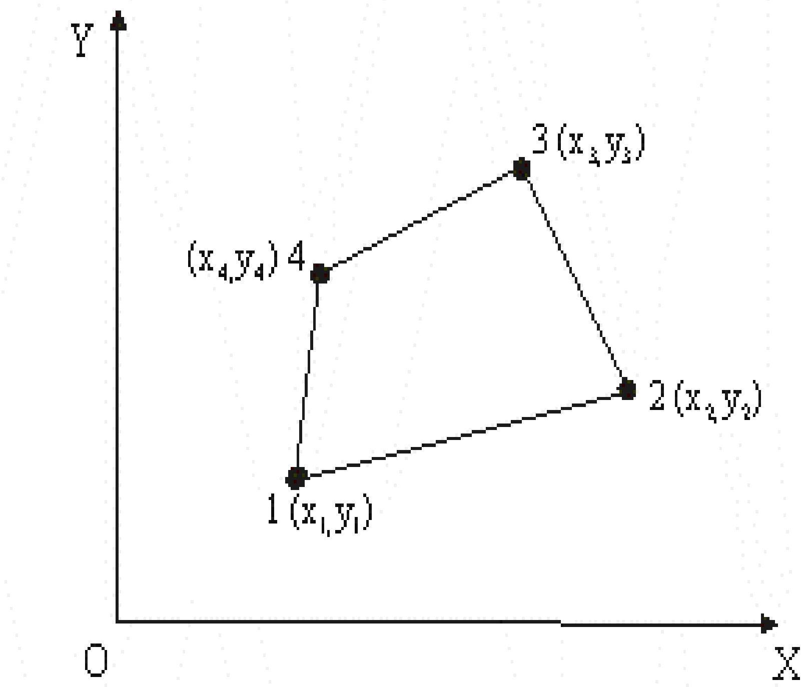
- Shape functions derived using the above 2 approaches may be different

Natural co-ordinates – Quadrilateral Elements (Quad 4)

- Linear variation of shape functions
- $(x, y) \Leftrightarrow f(\xi, \eta)$
- $x_P = \sum N_i x_i \qquad y_P = \sum N_i y_i \qquad i = 1,2,3,4.$

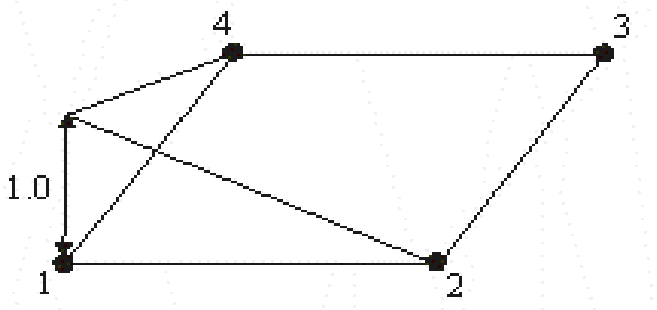


Parent element



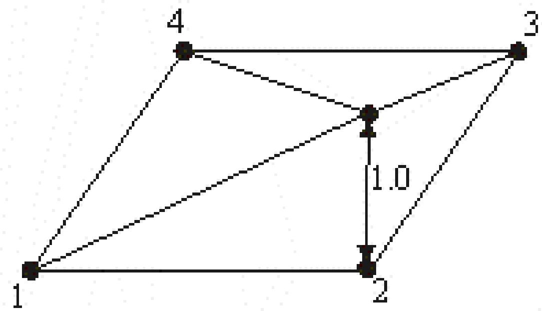
Child element

Natural co-ordinates – Quad. Elements



a) N_1

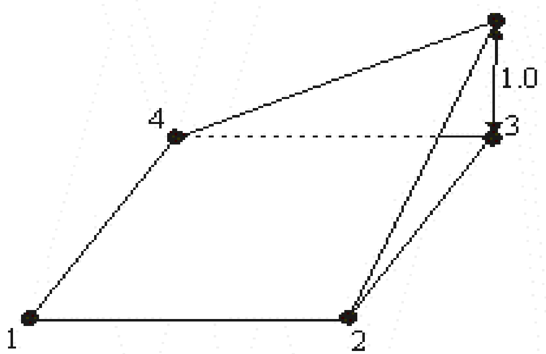
$$N_1 = \left(\frac{1}{4}\right)(1-\xi)\ (1-\eta)$$



b) N_2

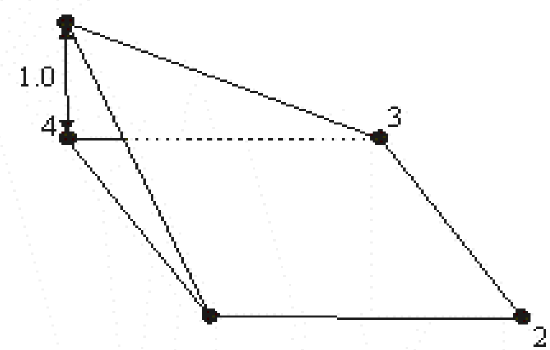
$$N_2 = \left(\frac{1}{4}\right)(1+\xi)\ (1-\eta)$$

Using Serendipity approach



c) N_3

$$N_3 = \left(\frac{1}{4}\right)(1+\xi)\ (1+\eta)$$

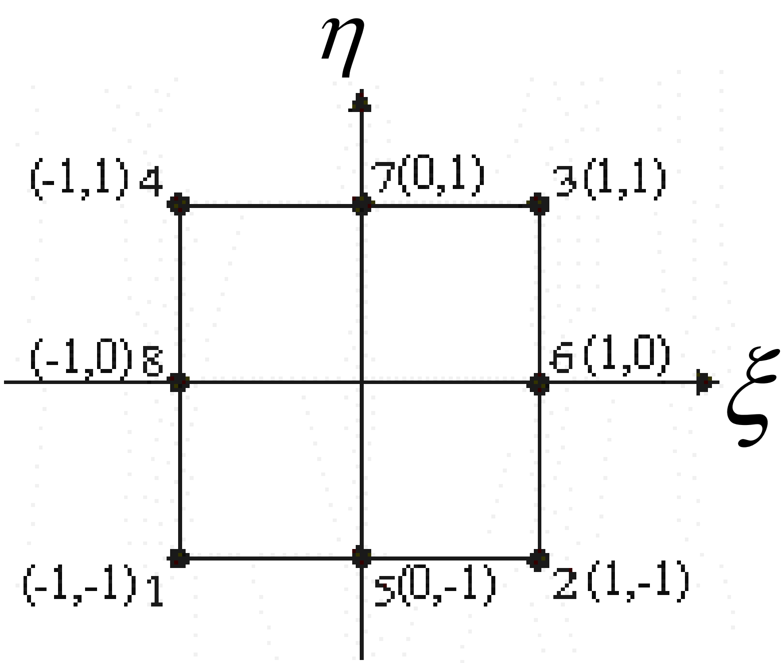


d) N_4

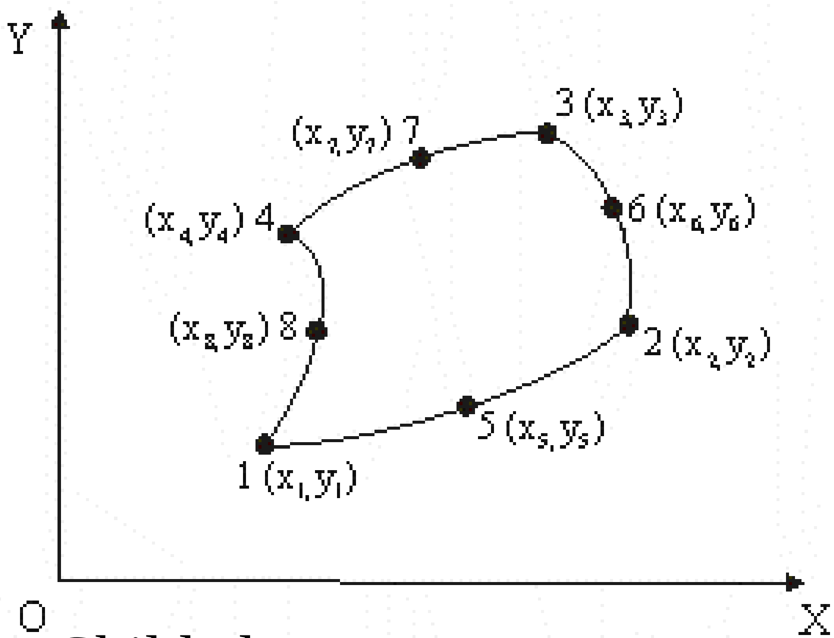
$$N_4 = \left(\frac{1}{4}\right)(1-\xi)\ (1+\eta)$$

Natural co-ordinates – Quad. Elements (Quad 8)

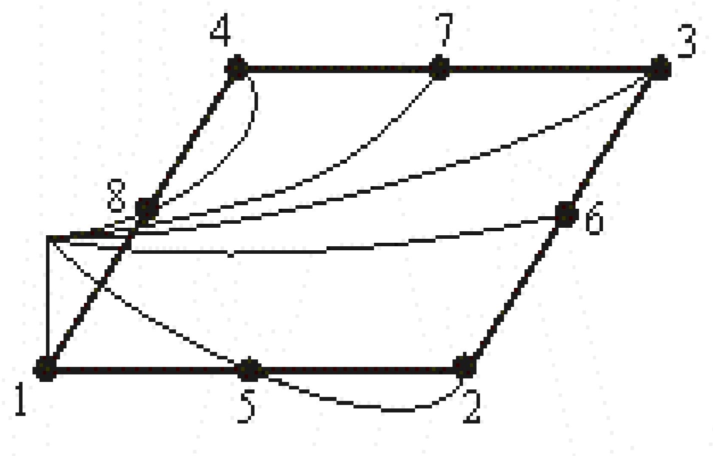
- Quadratic variation of shape functions
- $x_P = \sum N_i x_i$ $\qquad y_P = \sum N_i y_i$ $\qquad i = 1,\ldots.8$



Parent element

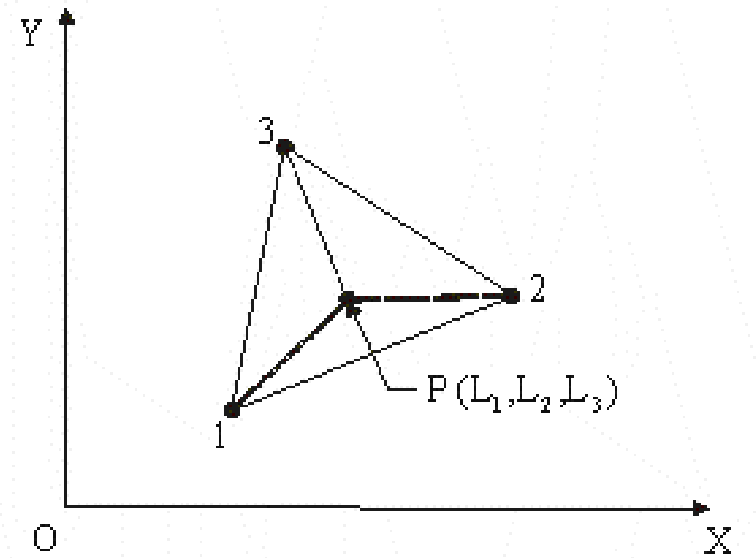


Child element



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta)$$

Natural co-ordinates – Triangular Elements



- Linear variation of shape functions
- $x_P = \sum L_i x_i \qquad y_P = \sum L_i y_i \qquad i = 1, 2, 3.$

$$L_1 = \frac{A_1}{A} = \frac{Area\ of\ \Delta P23}{Area\ of\ \Delta 123}$$

$$L_2 = \frac{A_2}{A} = \frac{Area\ of\ \Delta 1P3}{Area\ of\ \Delta 123}$$

$$L_3 = \frac{A_3}{A} = \frac{Area\ of\ \Delta P12}{Area\ of\ \Delta 123}$$

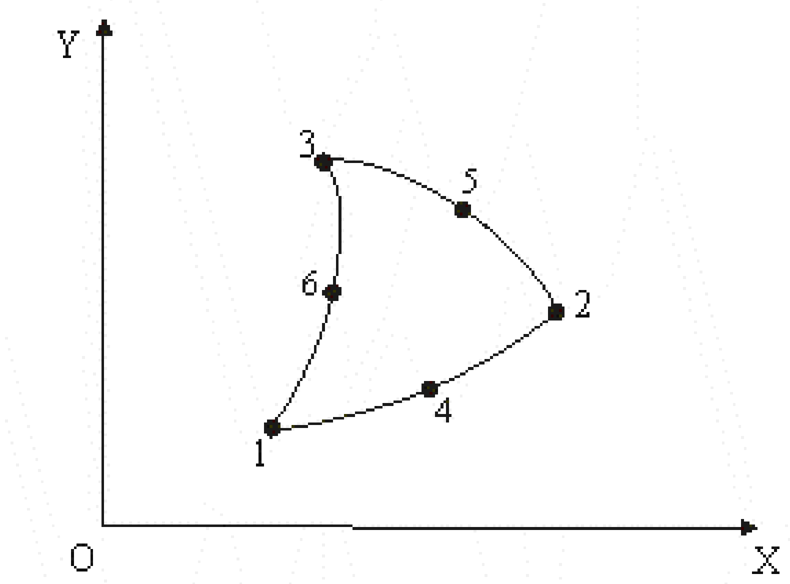
$$L_1 + L_2 + L_3 = 1$$

$$N_1 = L_1,\ N_2 = L_2,\ N_3 = L_3$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{Bmatrix}$$

Natural co-ordinates – Triangular Elements (6-noded)

- Quadratic variation of shape functions
- $x_P = \sum N_i x_i$
- $y_P = \sum N_i y_i \quad$ i = 1, ..6.



$N_1 = (SF)$ (Eqn of Line 2–5–3) (Eqn of Line 4–6) $= (L_1)\,(2L_1 - 1)$

Similarly, $N_2 = (L_2)\,(2L_2 - 1)$, $N_3 = (L_3)\,(2L_3 - 1)$

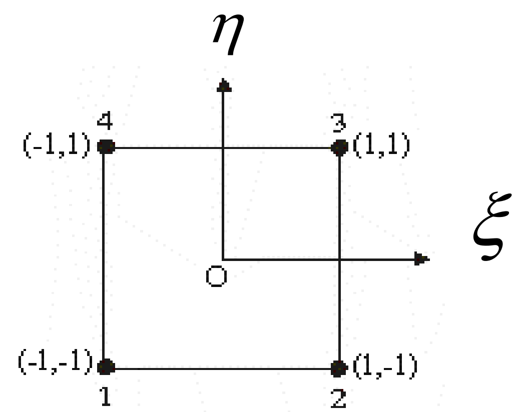
$N_4 = 4\,L_1 L_2, \quad N_5 = 4\,L_2 L_3, \quad N_6 = 4\,L_3 L_1$

4-node Quadrilateral element (Quad 4)

$$(x, y) \Leftrightarrow f(\xi, \eta)$$

$$x_P = \sum \mathrm{N_i x_i} \qquad y_P = \sum N_i y_i$$

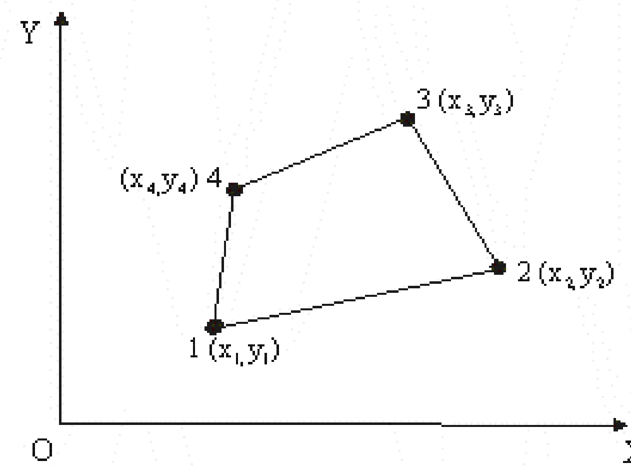
$$\mathrm{i} = 1,2,3,4$$



Parent element

Using iso-parametric formulation

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$



Child element

4-node Quadrilateral element (Quad 4)

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

- u,v in terms of Ni's – Ni's in terms of ξ, η
- Express $(x, y) \Leftrightarrow f(\xi, \eta)$
- Jacobian matrix → relates partial derivatives in x, y to that in ξ, η co-ordinates
- When element is badly distorted, Jacobian could pose problems

AVOID ELEMENT DISTORTION

Jacobian

Consider f = f(x,y) → x = x(ξ, η) & y = y(ξ, η)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

Writing other way round

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \quad \& \quad \frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

→

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = [\,J\,] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \qquad [\,J\,] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Jacobian

- In other words, $$[\,J\,] = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

- For Quad 4,

$$[\,J\,] = \begin{bmatrix} \left(\frac{1-\eta}{4}\right)(x_2 - x_1) + \left(\frac{1+\eta}{4}\right)(x_3 - x_4) & \left(\frac{1-\eta}{4}\right)(y_2 - y_1) + \left(\frac{1+\eta}{4}\right)(y_3 - y_4) \\ \left(\frac{1-\xi}{4}\right)(x_4 - x_1) + \left(\frac{1+\xi}{4}\right)(x_3 - x_2) & \left(\frac{1-\xi}{4}\right)(y_4 - y_1) + \left(\frac{1+\xi}{4}\right)(y_3 - y_2) \end{bmatrix}$$

Jacobian

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}$$

|J| appears in the denominator

When elements are badly distorted, this could become zero

AVOID ELEMENT DISTORTION

HEED WARNINGS OF ANSYS

Jacobian Matrix

- relates partial derivatives in the physical co-ordinates to that in natural co-ordinates
- the sum of the determinants of the Jacobian matrix = area of the element
- As the shape of the element distorts from rectangle - accuracy decreases. Indicators of element distortion:
  - Included angles at element vertices
  - variation of |J| from one Gauss point to another
  - element aspect ratio
- Software issues an error for unacceptable element distortion

Computational expense

- Stiffness matrix,

$$[k]^e = \int_v [B]^T_{6\times3}[D]_{3\times3}[B]_{3\times6}\ dv$$

- Stiffness matrix – 6 X 6
- considering symmetry – need to evaluate 21 integrals
- since [B] & [D] are constant

$$[k]^e = [B]^T [D] [B] (t)(A)$$

- For any other element, [B] matrix is not constant
- Explicit evaluation of integral becomes tedious
- Numerical integration – facilitated by natural co-ordinates

Consistent v/s Lumped load vector

- For distributed loads, 2 schemes to evaluate load vector
  - ➢ Consistent load vector
    - equivalent nodal loads which will do the same amount of work as distributed load
    - based on equilibrium conditions
  - ➢ Lumped load vector
    - Lump total force on particular nodes
    - based on judgement

4-node Quadrilateral element (Quad 4)

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \left[J \right]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{|\, J \,|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}$$

- Using this in constitutive equations

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{[B_1]} \begin{Bmatrix} \dfrac{\partial \mathrm{u}}{\partial \mathrm{x}} \\ \dfrac{\partial u}{\partial \mathrm{y}} \\ \dfrac{\partial \mathrm{v}}{\partial \mathrm{x}} \\ \dfrac{\partial \mathrm{v}}{\partial y} \end{Bmatrix}$$

4-node Quadrilateral element (Quad 4)

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \underbrace{\begin{bmatrix} J_{22}/|J| & -J_{12}/|J| & 0 & 0 \\ -J_{21}/|J| & J_{11}/|J| & 0 & 0 \\ 0 & 0 & J_{22}/|J| & -J_{12}/|J| \\ 0 & 0 & -J_{21}/|J| & J_{11}/|J| \end{bmatrix}}_{[B_2]} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \partial N_1/\partial\xi & 0 & \partial N_2/\partial\xi & 0 & \partial N_3/\partial\xi & 0 & \partial N_4/\partial\xi & 0 \\ \partial N_1/\partial\eta & 0 & \partial N_2/\partial\eta & 0 & \partial N_3/\partial\eta & 0 & \partial N_4/\partial\eta & 0 \\ 0 & \partial N_1/\partial\xi & 0 & \partial N_2/\partial\xi & 0 & \partial N_3/\partial\xi & 0 & \partial N_4/\partial\xi \\ 0 & \partial N_1/\partial\eta & 0 & \partial N_2/\partial\eta & 0 & \partial N_3/\partial\eta & 0 & \partial N_4/\partial\eta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$= [\, B_3 \,] \{\delta\}^e$$

4-node Quadrilateral elemenet (Quad 4)

- For Quad 4 elements,

$$N_1 = \left(\frac{1}{4}\right)(1-\xi)\ (1-\eta) \qquad \frac{\partial N_1}{\partial \xi} = \frac{-(1-\eta)}{4} \qquad \frac{\partial N_1}{\partial \eta} = \frac{-(1-\xi)}{4}$$

$$N_2 = \left(\frac{1}{4}\right)(1+\xi)\ (1-\eta) \qquad \frac{\partial N_2}{\partial \xi} = \frac{1-\eta}{4} \qquad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4}$$

$$N_3 = \left(\frac{1}{4}\right)(1+\xi)\ (1+\eta) \qquad \frac{\partial N_3}{\partial \xi} = \frac{1+\eta}{4} \qquad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4}$$

$$N_4 = \left(\frac{1}{4}\right)(1-\xi)\ (1+\eta) \qquad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4} \qquad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

- Thus, $\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [B]\{\delta\}^e$ $\qquad [B] = [B1]\ [B2]\ [B3]$

4-node Quadrilateral element (Quad 4)

- element stiffness matrix

$$[k]^e_{8\times 8} = \int_v [B]^T \; [D][B]\, dv = \iint [B]^T [D]\, [B]\, t\; dx\, dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B]\, t\, \left| J \right| d\xi\, d\eta$$

- need to evaluate 36 integrals for each element
- elements o [B] vary from point to point within the element
- explicit evaluation of stiffness matrix – too tediou

8-node Quadrilateral elemenet (Quad 8)

- **Three possibilities**

Coordinate Interpolation	**Displacement Interpolation**	**Formulation**
Linear (i.e. four vertex nodes)	Quadratic (i.e. all eight nodes)	Sub-parametric
Quadratic	Quadratic	Iso-parametric
Quadratic	Linear	Super-parametric

- Sub-parametric → structural geometry – simple polynomial
  field variable – sharp
- Iso-parametric → curved edge modeling
  quadratic displacement variation
- Super-parametric → curved features in a low stress region

8-node Quadrilateral element (Quad 8)

- For iso-parametric formulation

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{2\times1} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{bmatrix}_{2\times16} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \end{Bmatrix}_{16\times1}$$

- Strain-displacement relation

$$\begin{aligned} \{\varepsilon\}_{3\times1} &= \underbrace{[B_1]_{3\times4}\ [B_2]_{4\times4}\ [B_3]_{4\times16}}\ \{\delta\}^e_{16\times1} \\ &= [B]_{3\times16}\{\delta\}^e_{16\times1} \end{aligned}$$

- Jacobian

$$[\,J\,] = \begin{bmatrix} \displaystyle\sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi}\, x_i & \displaystyle\sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi}\, y_i \\ \displaystyle\sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta}\, x_i & \displaystyle\sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta}\, y_i \end{bmatrix}$$

8-node Quadrilateral element (Quad 8)

- For Quad 8 elements,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)\ (1-\eta)\ (-1+\xi-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)\ (1+\eta)\ (-1+\xi+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)\ (1+\eta)\ (-1-\xi+\eta)$$

$$N_5 = \left(\frac{1}{2}\right)\left(1-\xi^2\right)\left(1-\eta\right)$$

$$N_6 = \left(\frac{1}{2}\right)\left(1+\xi\right)\left(1-\eta^2\right)$$

$$N_7 = \left(\frac{1}{2}\right)\left(1-\xi^2\right)\left(1+\eta\right)$$

$$N_8 = \left(\frac{1}{2}\right)\left(1-\xi\right)\left(1-\eta^2\right)$$

- Element stiffness matrix

$$\left[k\right]^e = \int_v [B]^T [D][B]\, dv = \iint [B]^T [D][B]\, t\, dx\, dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B]\, t\, \left| J \right| d\xi\, d\eta$$

Numerical Integration

- **element stiffness matrix**

$$[k]^e = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\,t\,|J|\,d\xi\,d\eta$$

- **need to evaluate many integrals for each element**
- **elements of [B] vary from point to point within the element. explicit evaluation of stiffness matrix – too tedious**
- **Resort to Numerical integration**
- **Methods available: → Trapezoidal rule**

  → Simpson's 1/3 rd rule

  → Newton-Cotes formula

  → Gauss Quadrature Formula

Gauss Quadrature

- Required integral, $I(x) = \int_{-1}^{1} f(x)\, dx$
- Approximation $I \approx W_1 f_1 + W_2 f_2 + \ldots + W_n f_n$
- n- no of Gauss points, W_i – weight functions
- f_i – function value at i^{th} Gauss point

No of Gauss points (n)	***Location of sampling point x_i***	***Weight factor W_i***
1	0	2
2	$1/\sqrt{3}$ $-1/\sqrt{3}$	1 1
3	$\sqrt{0.6}$ 0 $-\sqrt{0.6}$	5/9 8/9 5/9

Gauss Quadrature

- method gives exact solution if "f" is a polynomial of degree,

$$p \leq (2n - 1)$$

- 1 point rule for linear polynomial
- 2 point rule for (upto) cubic polynomial

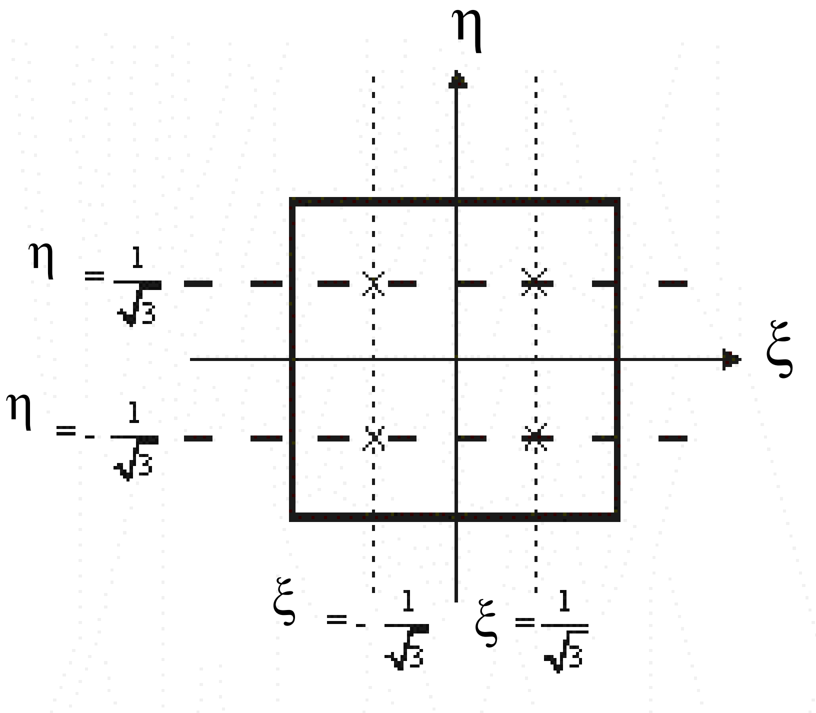
For 2D case

$$[k]^e = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\,t\,\left|J\right|\,d\xi\;d\eta$$

$$= \sum_i \sum_j \left([B]^T[D][B]\,t\,\left|J\right|\right) W_i W_j$$

Gauss Quadrature Implementation

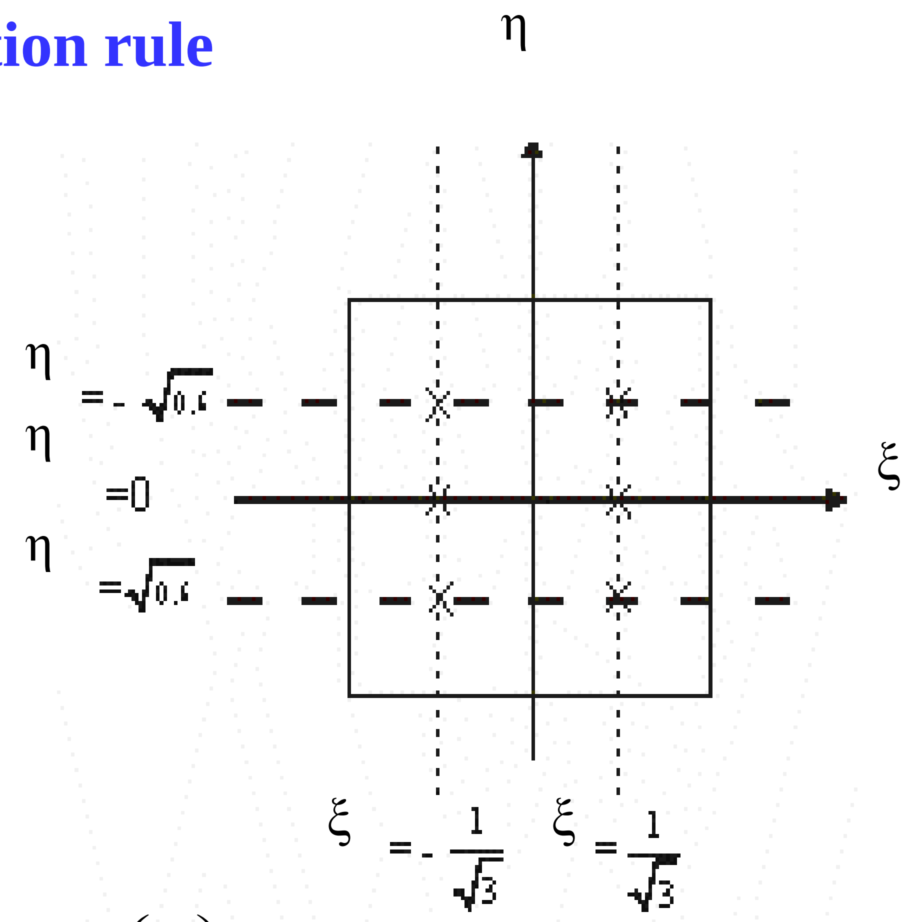
2 x 2 rule for integration:



$$I \approx (1)(1)\ f\left(\frac{-1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}}\right) + (1)(1)\ f\left(\frac{1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}}\right)$$

$$+ (1)(1)\ f\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + (1)(1)\ f\left(\frac{-1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

2 x 3 integration rule



$$I \approx (1)\left(\frac{5}{9}\right) f_1 + (1)\left(\frac{5}{9}\right) f_2 + (1)\left(\frac{8}{9}\right) f_3$$

$$+ (1)\left(\frac{8}{9}\right) f_4 + (1)\left(\frac{5}{9}\right) f_5 + (1)\left(\frac{5}{9}\right) f_6$$

Gauss Quadrature method

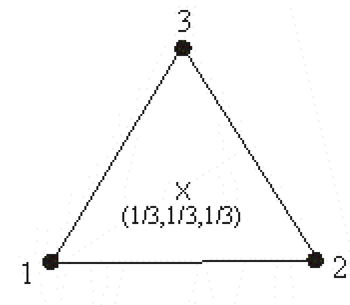
No of Guass points needed for different element types

Element type	No of Gauss points needed	
	$[K]^e$	**f^e**
Linear (Quad4)	1	1
Quadratic (Quad 8 & Quad 9)	2	2
Cubic	3	3

Gauss Quadrature for triangular element

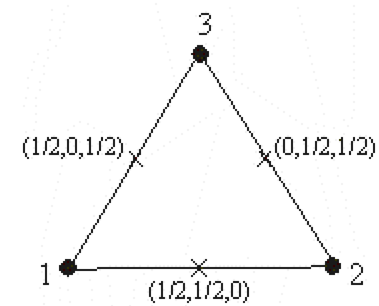
- Weight functions are not taken in 2 directions

$$I \; = \int_0^1 \int_{-1}^{1-L_1} f(L_1,\ L_2)\, dL_1 dL_2 \;\approx\; \sum W_i\, f_i$$

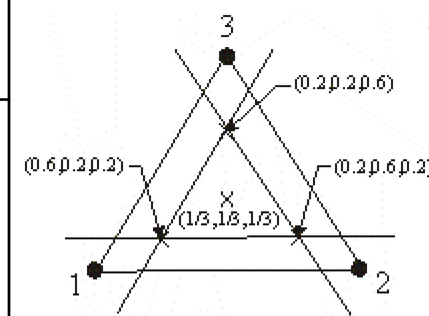


a) One point rule

n, no. of points	Location of sampling points	Weight factor W
1	$L_1 = 1/3$ $L_2 = 1/3$ $L_3 = 1/3$	1
3	(1/2, 1/2, 0) (1/2, 0, 1/2) (0, 1/2, 1/2)	1/3 1/3 1/3
4	(1/3, 1/3, 1/3) (0.6, 0.2, 0.2) (0.2, 0.6, 0.2) (0.2, 0.2, 0.6)	–27/48 25/48 25/48 25/48



b) 3 point rule



c) 4 point rule

Solution of Static Equilibrium Equations

General form:

$$[K]_{n\times n}\{\delta\}_{n\times 1} = \{F\}_{n\times 1}$$

Methods of Solution:

- Gauss elimination
- Gauss-Siedel
- Conjugate gradient method, etc.

Significant computational effort !!

Boundary conditions

- Let the system equations be: $[A]_{n\times n}\{x\}_{n\times 1} = \{b\}_{n\times 1}$

- Direct method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- If $x_1 = \underline{x}_1$ then

$$x_1 = \bar{x}_1$$
$$a_{22}x_2 + a_{23}x_3 = b_2 - a_{21}\bar{x}_1$$
$$a_{32}x_2 + a_{33}x_3 = b_3 - a_{31}\bar{x}_1$$

- In other words,
  - put aii = 1
  - aij = 0 in that row and aji = 0 in that coloumn
  - put RHS = $\{(b_j - a_{ji}\bar{x}_i)\}$ $\quad j = 1, 2, \ldots, n$

Boundary conditions

- Penalty function approach:

use -

$$\bar{a}_{ii} = a_{ii} + a_p$$

$$\bar{b}_i = a_p \bar{x}_i$$

- a_p- penalty number (very high ≈ 10^3-10^6)
- If $a_p >> a_{ij}$ → $x_i = \underline{x}_i$ where $\underline{x}_i$ is the prescribed boundary condition
- Very high values of a_p may lead to numerical difficulties

- Note:

- until the proper bc's are substituted, there is a possibility of rigid body motion →matrix A is singular
- software may show errors → hence ensure proper bc's

Solution of static equilibrium equations

- Let the system equations be: $[A]_{n\times n}\{x\}_{n\times 1} = \{b\}_{n\times 1}$
- Gauss elimination method – widely used
- Reduce A into an upper-triangular matrix

$$\begin{bmatrix} \bar{a}_{11} & a_{11} & \dots & \dots & \dots & a_{1n} \\ 0 & \bar{a}_{22} & \dots & \dots & \dots & \bar{a}_{2n} \\ 0 & 0 & \bar{a}_{33} & \dots & \dots & \bar{a}_{3n} \\ & & & \ddots & & \\ & & & & \bar{a}_{n-1,n-1} & \bar{a}_{n-1,n} \\ 0 & 0 & \dots & \dots & \dots & \bar{a}_{nn} \end{bmatrix} \begin{Bmatrix} x \\ \\ \\ \\ \\ \\ \end{Bmatrix} = \begin{Bmatrix} \bar{b} \\ \\ \\ \\ \\ \\ \end{Bmatrix}$$

Solution of static equilibrium equations

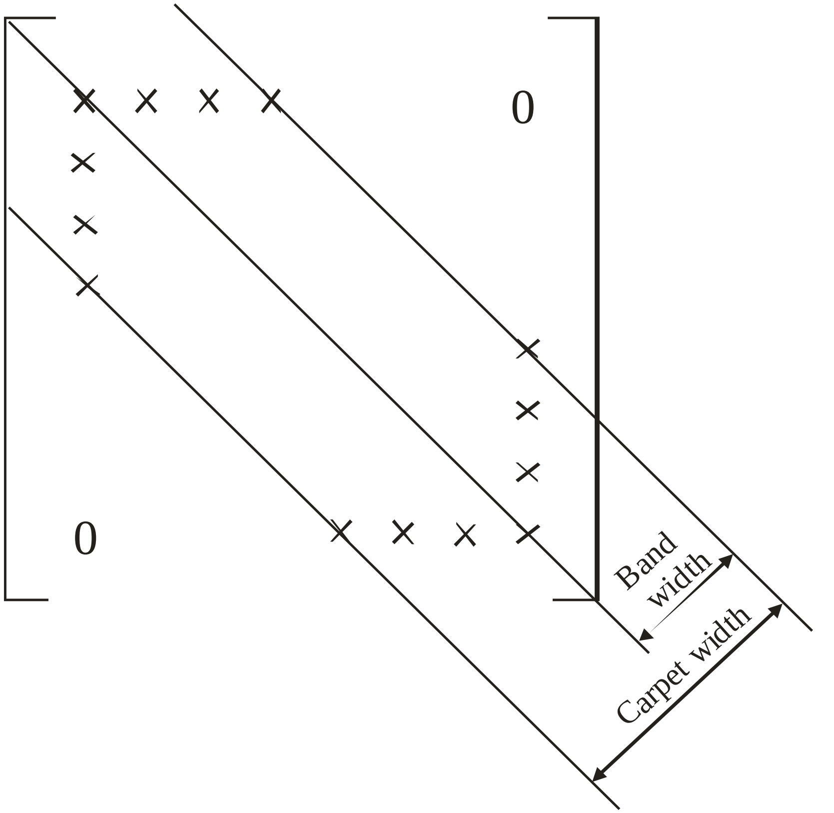
- Back substitution:

$$\bar{a}_{nn} x_n = \bar{b}_n \qquad \rightarrow x_n = \frac{\bar{b}_n}{\bar{a}_{nn}}$$

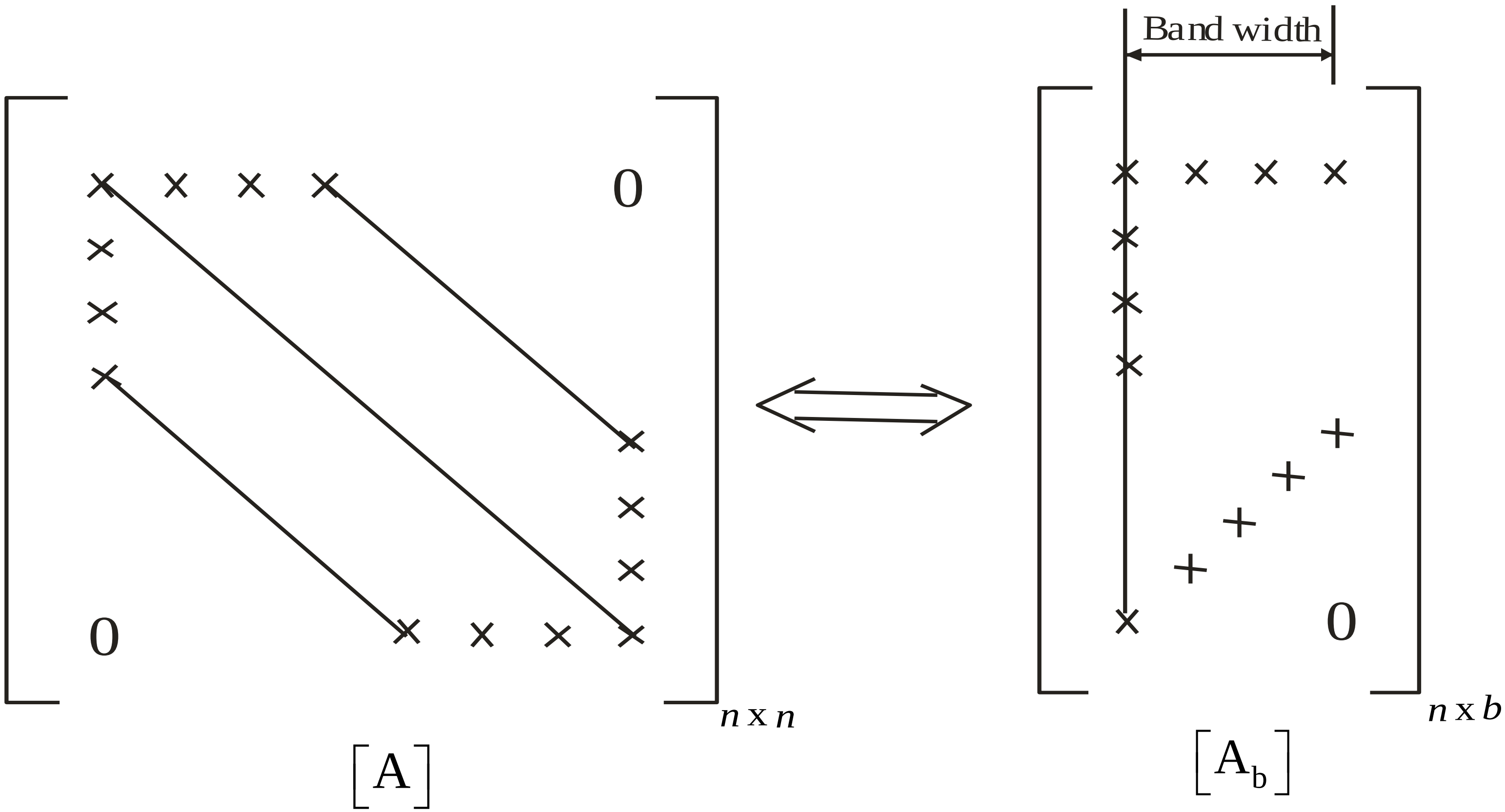
- penultimate row:

$$\bar{a}_{n-1,n-1} x_{n-1} + \bar{a}_{n-1,n} x_n = \bar{b}_{n-1}$$

- <u>Computer Storage required</u>:
- Stiffness matrix is banded, symmetric and positive definite
- no need to store elements beyond carpet width
- store only non-zero elements in the band-width



Solution of static equilibrium equations

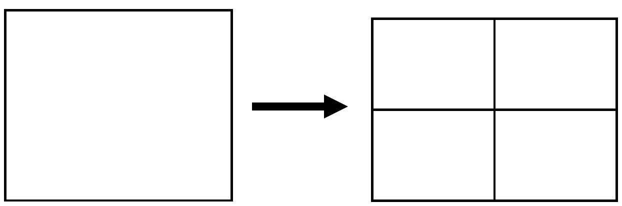


• Several thousands of elements - present in a typical problem

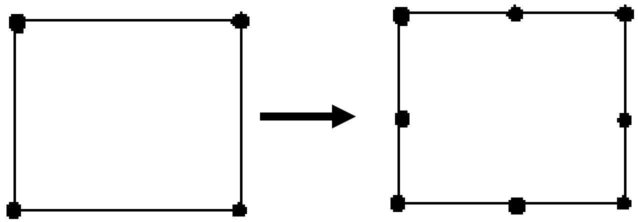
• The above method translates into a considerable reduction of computer storage space required

Modeling considerations

- |J| should not be negative anywhere in the element
    - ➢ Shape of the element should not be too distorted
- Initially start with a course mesh, obtain results and check with analytical results
- Refine the mesh to get more accurate results →
    - h – convergence (use more elements of lower order)
    - p – convergence (increase the order of the existing elements)



h-refinement

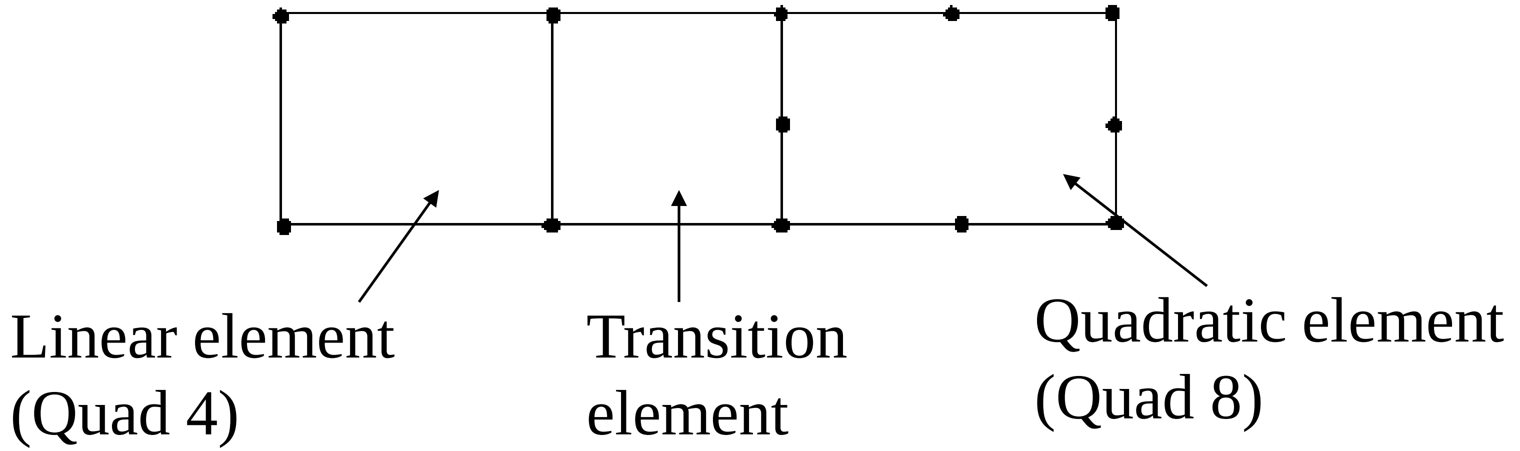


p-refinement

Modeling considerations

• To combine different types of elements - use transition element

• eg combining linear element to quadratic element



• Use local mesh of high density in the region where stresse stress gradients are high (eg. at a notch)

• Make use of the symmetry/ anti-symmetry of the problem