FUNCTIONAL AND DIFFERENTIAL EQUATION FORMS

A function of single variable i.e., $f(x)$ the condition for an extremum is given by $df/dx = 0$.

A function of several variables $f(x_1, x_2, ... x_n)$ will have an extremum at a given $(x_1, x_2, ... x_n)$ if

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_n} = 0$$

Now consider a "functional" (i.e. function of functions) such as given below :

$$I(u) = \int_a^b F\left(u, \ \frac{du}{dx} \ , \ x \right) dx$$

We wish to find $u(x)$ which makes the functional stationary, subject to the end conditions

$u(a) = u_a$ and $u(b) = u_b$.

As an example, if $F = (1/2)\, AE\, (du/dx)^2 - (q)(u)$ and the prescribed end conditions are $u(0) = 0$; $u(L) = 0$ then this represents the problem of a uniform bar clamped at both ends and subjected to a distributed load $q(x)$.

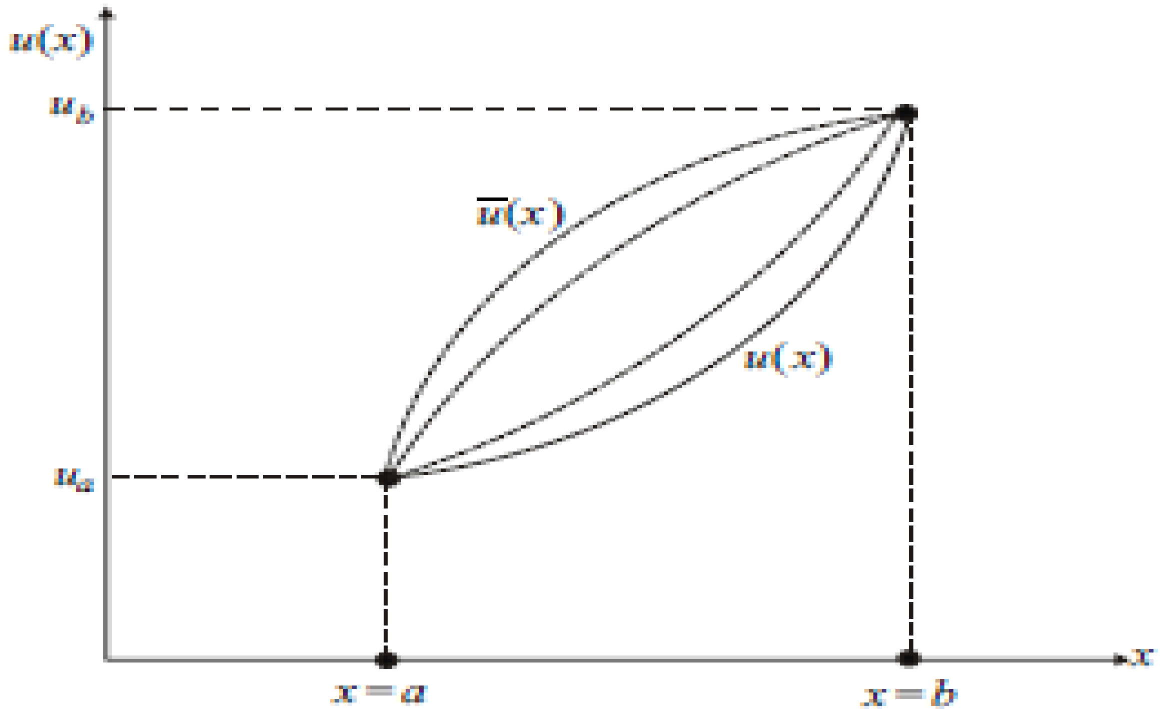

Fig. 3.1 Admissible solutions $u(x)$

all $\bar{u}(x)$ satisfy the prescribed end conditions and thus we are dealing with only "fixed end" variations.

We wish to study what happens to I, if $u(x)$ is slightly altered to $\bar{u}(x)$ i.e.,

$$\bar{u}(x) = u(x) + \varepsilon v(x)$$

where ε is a small parameter and $v(a) = 0 = v(b)$.

The difference between $\overline{u}(x)$ and $u(x)$ is termed the "variation" in $u(x)$ and we denote this by $\delta u(x)$.

$$\delta u(x) = \overline{u}(x) - u(x)$$
$$= \varepsilon v(x)$$

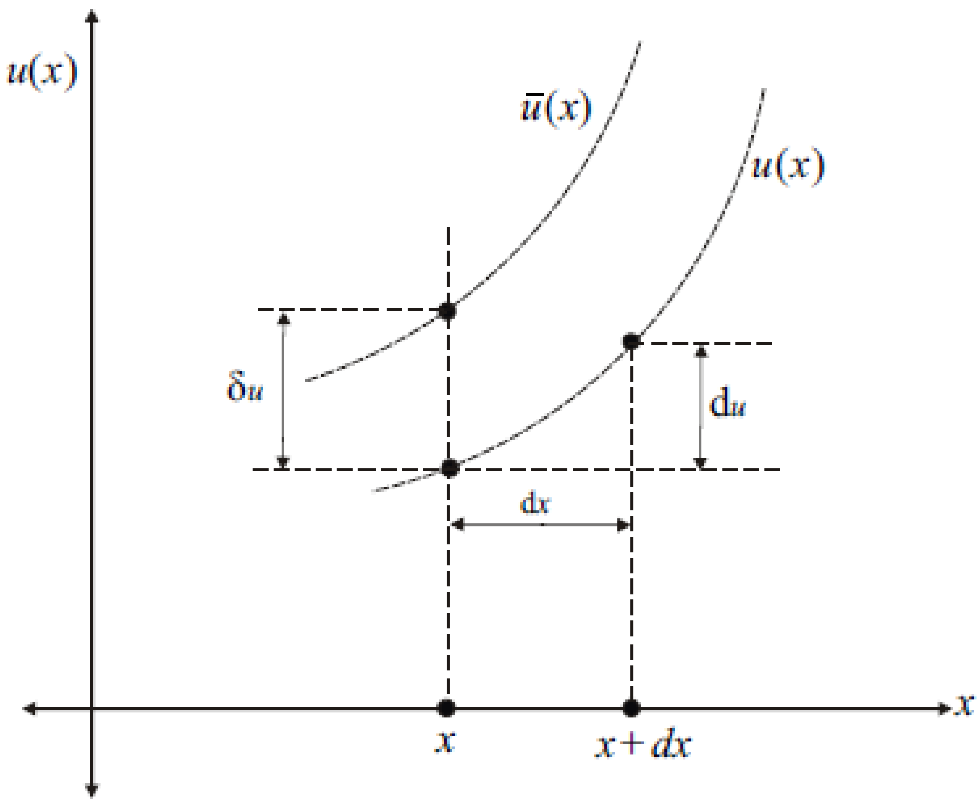

Fig. 3.2 Distinction between δu, $\mathrm{d}u$

Variation in u' and F

$\delta(u')$ = difference in slope of $\overline{u}(x)$ and $u(x)$

$= \overline{u}'(x) - u'(x) \quad = u'(x) + \varepsilon v'(x) - u'(x)$

$= \varepsilon v'(x) = [\delta u]'$

$$\Delta F = F(\overline{u}, \overline{u}', x) - F(u, u', x)$$

$$= F(u + \delta u, u' + \delta u', x) - F(u, u', x)$$

$$F(u+\delta u,\ u'+\ \delta u',x)\ =\ F(u,\ u',x)\ +\ \frac{\partial F}{\partial u}\ \delta u+\ \frac{\partial F}{\partial u}\ \delta u'$$

$$+\ \frac{1}{2!}\left[\frac{\partial^2 F}{\partial u^2}\ \delta u^2+\frac{2\partial^2 F}{\partial u \partial u'}\ \delta u \delta u'\ +\ \frac{\partial^2 F}{\partial u'^2}\ \delta u'^2\right]+\ldots$$

$$\Delta F=\left(\frac{\partial F}{\partial u}\ \delta u+\frac{\partial F}{\partial u'}\ \delta u'\right)+\frac{1}{2!}\left(\frac{\partial^2 F}{\partial u^2}\ \delta u^2+\frac{2\partial^2 F}{\partial u \partial u'}\ \delta u \delta u'+\frac{\partial^2 F}{\partial u'^2}\ \delta u'^2\right)+\ldots$$

First "variation of *F*" is defined as

$$\delta F\ =\ \frac{\partial F}{\partial u}\ \delta u+\frac{\partial F}{\partial u'}\ \delta u'$$

Second "variation of *F*" is defined as

$$\delta^2 F\ =\ \delta(\delta F)\ =\ \frac{\partial^2 F}{\partial u^2}\ \delta u^2+\ \frac{2\partial^2 F}{\partial u \partial u'}\ \delta u\ \delta \mathrm{u}'\ +\ \frac{\partial^2 \mathrm{F}}{\partial \mathrm{u}'^2}\ \delta \mathrm{u}'^2$$

Variation in I

$$\Delta I = I(\bar{u}, \bar{u}', x) - I(u, u', x)$$

$$= \int_a^b F(\bar{u}, \bar{u}', x)\, dx - \int_a^b F(u, u', x)\, dx$$

$$= \int_a^b \Delta F \; dx$$

$$= \int_a^b \left(\delta F + \frac{1}{2!}\, \delta^2 F \; + \dots \right) dx$$

$$\delta I \;=\; \int_a^b \delta F \; dx \qquad\qquad \delta^2 I \;=\; \int_a^b \delta^2 F \; dx$$

$\Delta I = \delta I + \delta^2 I + \ldots\ldots.$

- Since $u(x)$ minimises I, $\Delta I \geq 0$. As ε is reduced, ΔI approaches zero and when $\bar{u}(x) = u(x)$, I attains a minimum and $\Delta I \equiv 0$.
- By studying the relative orders of magnitude of the various terms, it is possible to show that δI is of the order $O(\varepsilon)$ while $\delta^2 I$ is $O(\varepsilon^2)$, etc.
- When ε is sufficiently small, $\delta^2 I$ and higher variations become negligible compared to δI and thus the condition for I to be stationary becomes, $\delta I = 0$.

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right) dx$$

$$\delta(u') = (\delta u)' = \frac{d}{dx} (\delta u)$$

$$\int_a^b \frac{\partial F}{\partial u'} \delta u' \, dx = \int_a^b \frac{\partial F}{\partial u'} d(\delta u)$$

$$\int_a^b \frac{\partial F}{\partial u'} d(\delta u) = \left[\frac{\partial F}{\partial u'} \delta u \right]_a^b - \int_a^b (\delta u) \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) dx$$

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) \right] \delta u \, dx + \left[\frac{\partial F}{\partial u'} \delta u \right]_a^b = 0$$

Since all the trial functions $u(x)$ satisfy the end conditions at $x = a$ and b, we have

$\delta u(a) = 0 = \delta \mathrm{u}(b)$

For arbitrary δu, we therefore have

$$\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] = 0$$

This is known as the Euler – Lagrange equation.

$$I = \int \left\{ \left(\frac{1}{2}\right) AE \left(\frac{du}{dx}\right)^2 - (q)(u) \right\} dx$$

$$F = \left(\frac{1}{2}\right) AE \left(\frac{du}{dx}\right)^2 - (q)(u)$$

$$\frac{\partial F}{\partial u} = -q$$

$$\frac{\partial F}{\partial u'} = AE \frac{du}{dx}$$

$$\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) \right] = 0$$

$$AE \frac{d^2 u}{dx^2} + q = 0$$

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) \right] \delta u \, dx + \left[\frac{\partial F}{\partial u'} \, \delta u \right]_a^b = 0$$

$\delta u(0) = 0$; $\delta u(L) = 0$ in view of prescribed boundary conditions at x=0, L

Recapitulating the Weighted Residual statement for this problem we have

$$\int_a^b \left[q + AE \frac{d^2 u}{dx^2} \right] \mathrm{W} \, dx = 0$$

While developing the weak form, we demanded that $W(x)$ be zero at those points where u is prescribed. Thus $W(0) = 0 = W(L)$