Finite Elements for Dynamic Analysis

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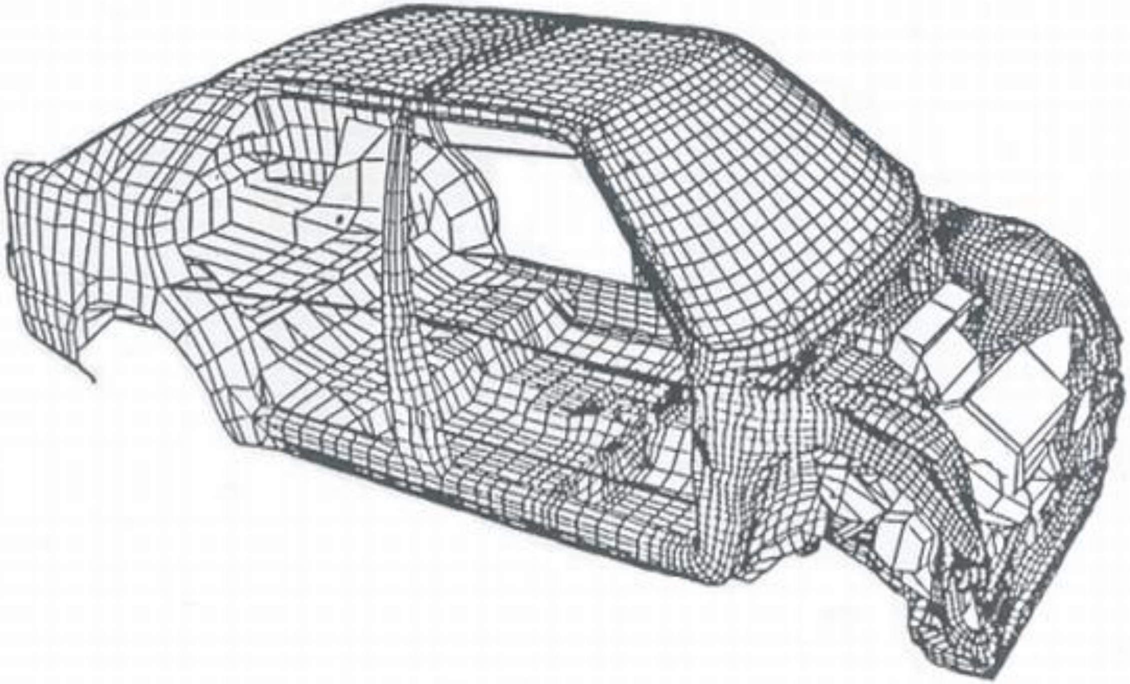
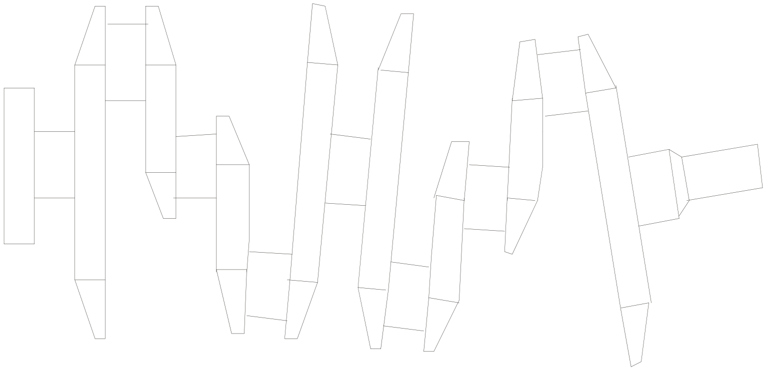
Typical Dynamic Problems

Automobile Crash



Typical Dynamic Problems

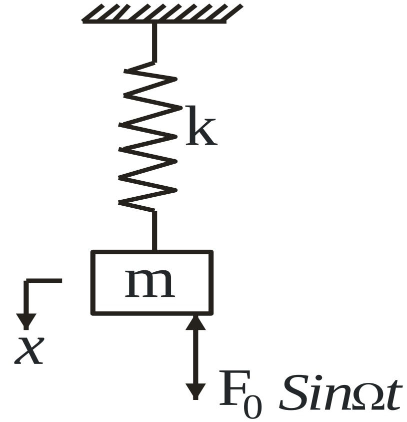
Crank Shaft Vibration



Building Vibrations

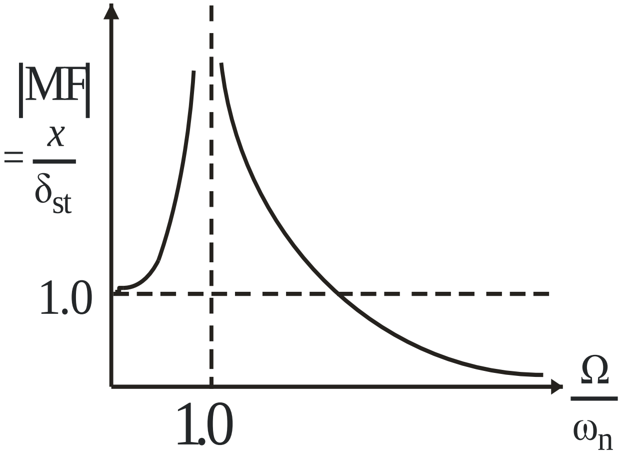
CONCEPTS IN DYNAMIC ANALYSIS

Simple spring mass system



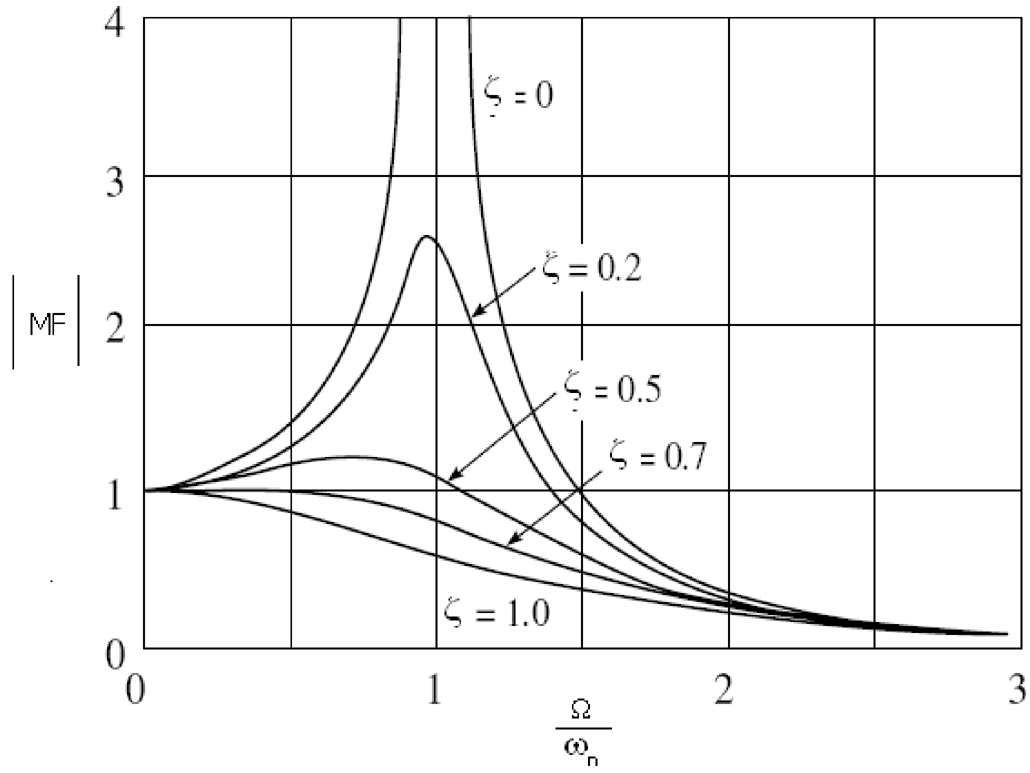
$$|MF| = \frac{1}{1-\left(\frac{\Omega}{\omega_n}\right)^2}$$

$$\Omega = \frac{1}{3}\,\omega_n, \quad |MF| = 1.125$$

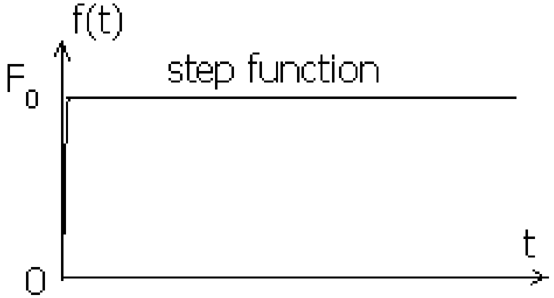


$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

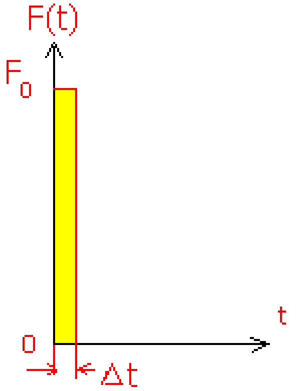
Damped Forced Vibration



Step and Impulse Response

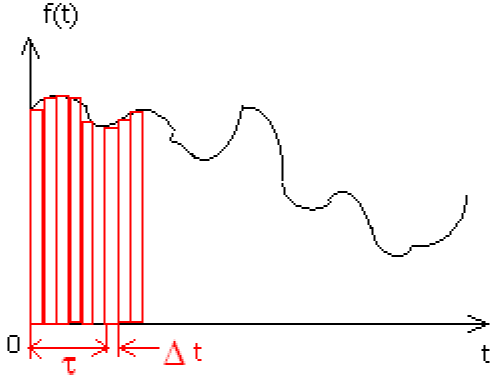


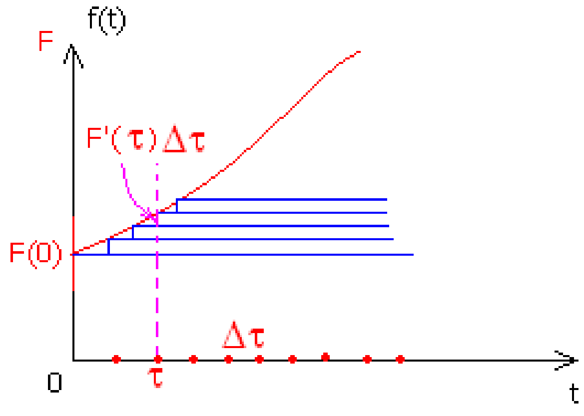
$$\frac{F_0}{k}\left[1-e^{-\zeta\omega_n t}(\cos\omega_d t+\zeta\sin\omega_d t\right]$$



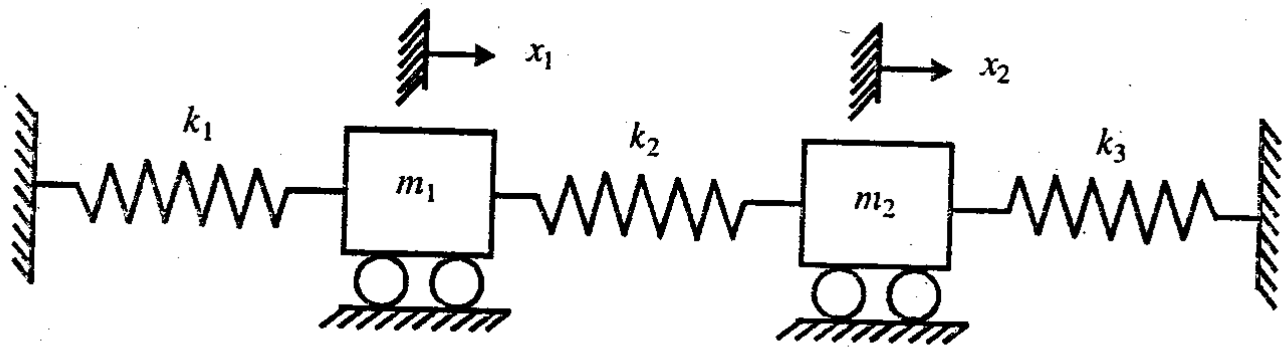
$$\frac{F_0\Delta t}{m\omega_d}\,e^{-\zeta\omega_n t}\sin\omega_d t$$

General Excitation - Response

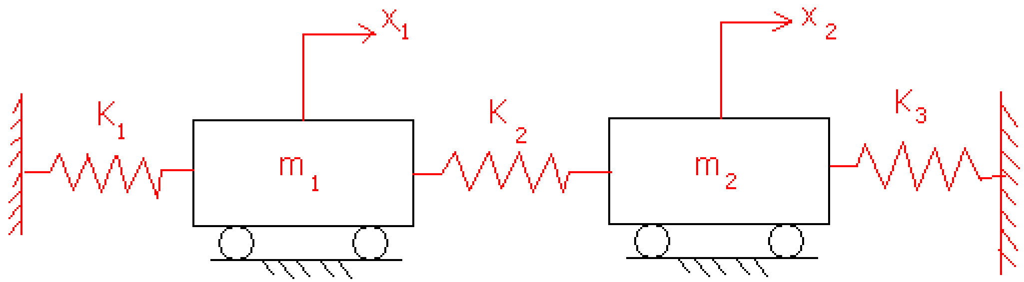


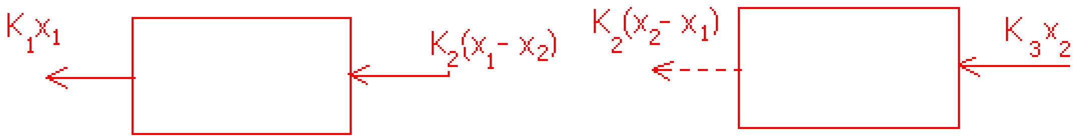


Two D.o.F. System



2-d.o.f. system





$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$[M]\left[\ddot{x}\right] + [K]\left[x\right] = 0$$

$$\begin{bmatrix} m_1\omega^2 & 0 \\ 0 & m_2\omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} K_1 + K_2 - m_1\omega^2 & -K_2 \\ -K_2 & K_2 + K_3 - m_2\omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

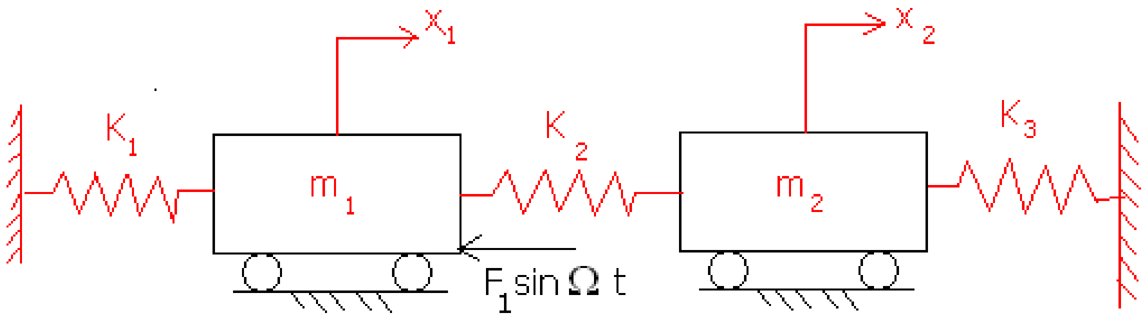
$$\begin{vmatrix} K_1 + K_2 - m_1\omega^2 & -K_2 \\ -K_2 & K_2 + K_3 - m_2\omega^2 \end{vmatrix} = 0$$

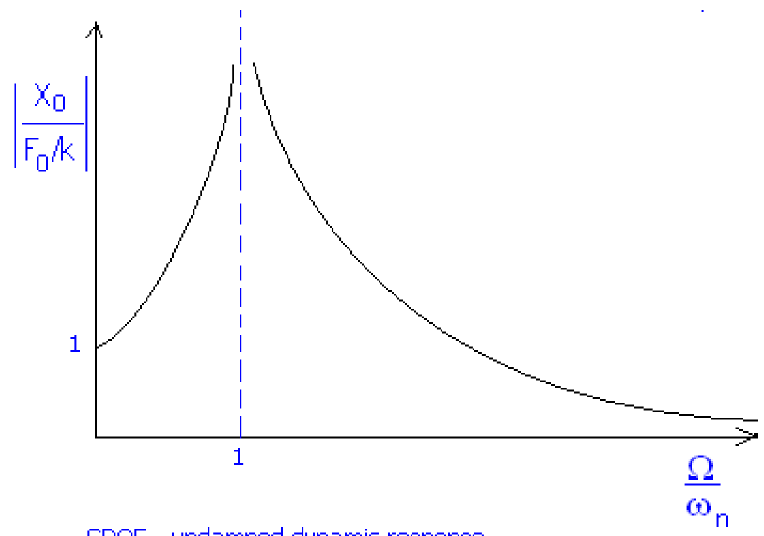
Orthogonality of mode shapes

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}$$

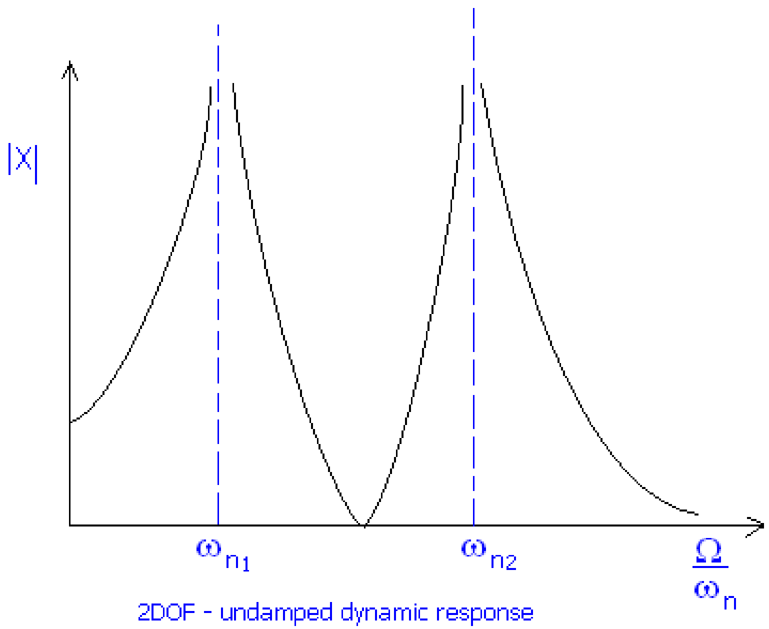
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$$

Forced vibration – 2 dof

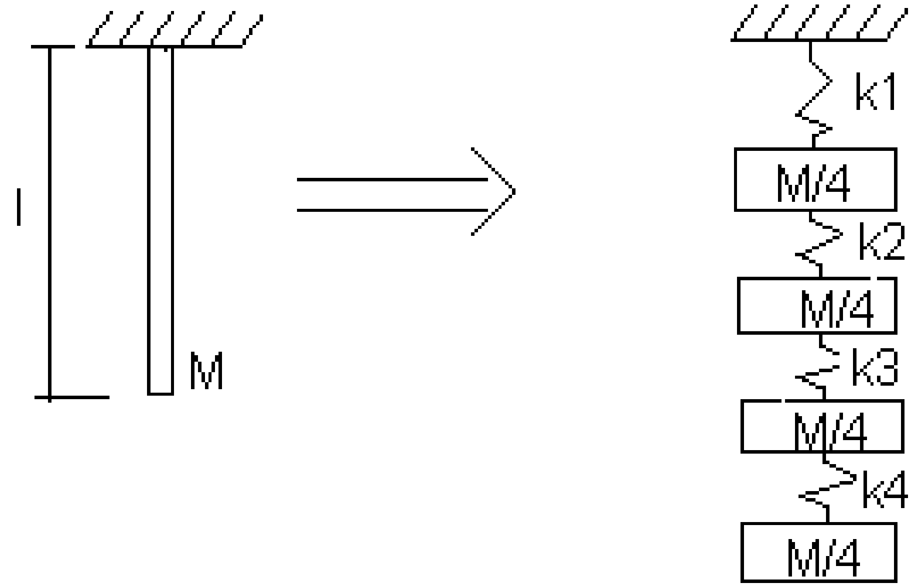




SDOF - undamped dynamic response

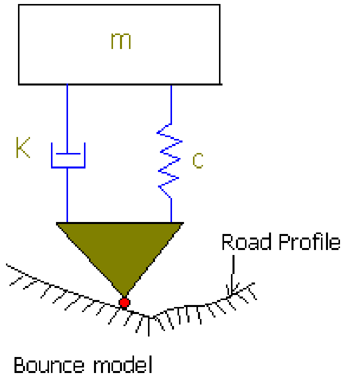


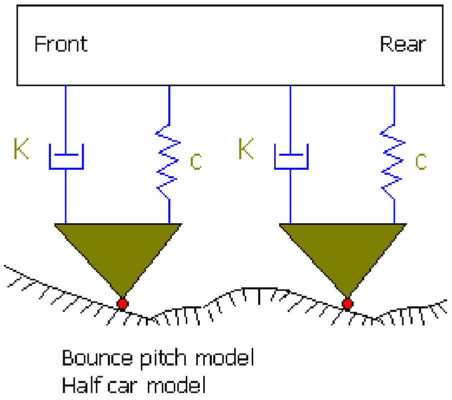
2DOF - undamped dynamic response

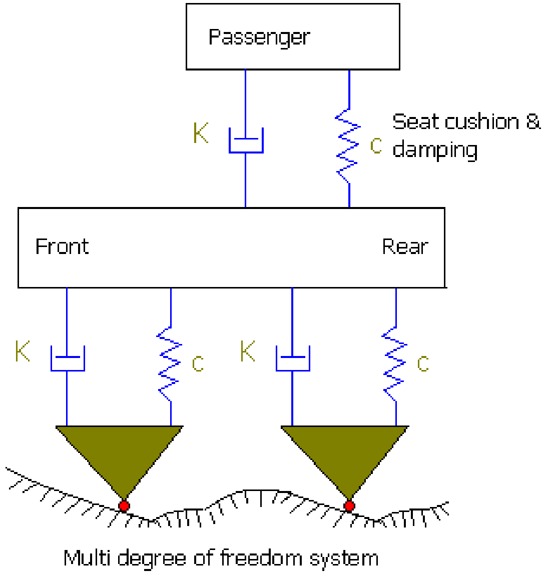


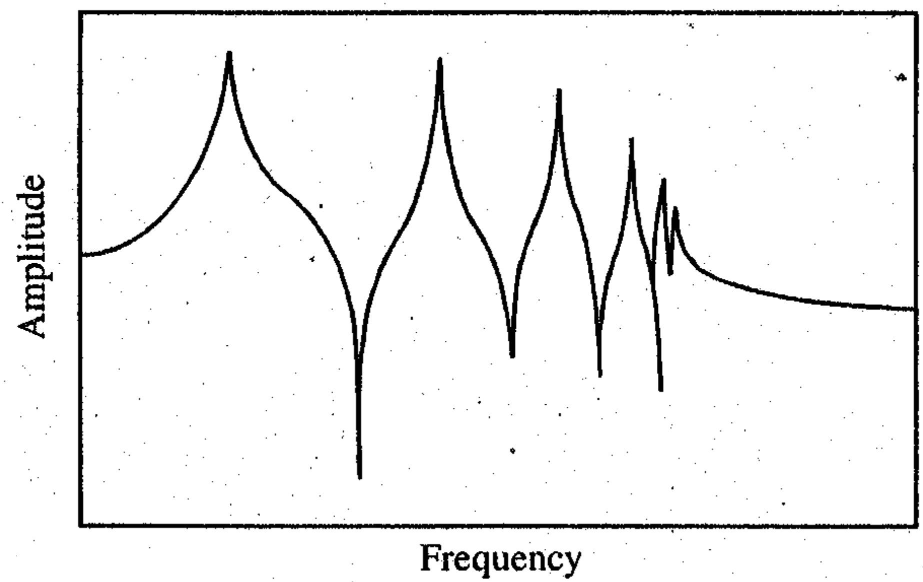
Distributed Parameter System

Model of a Vehicle



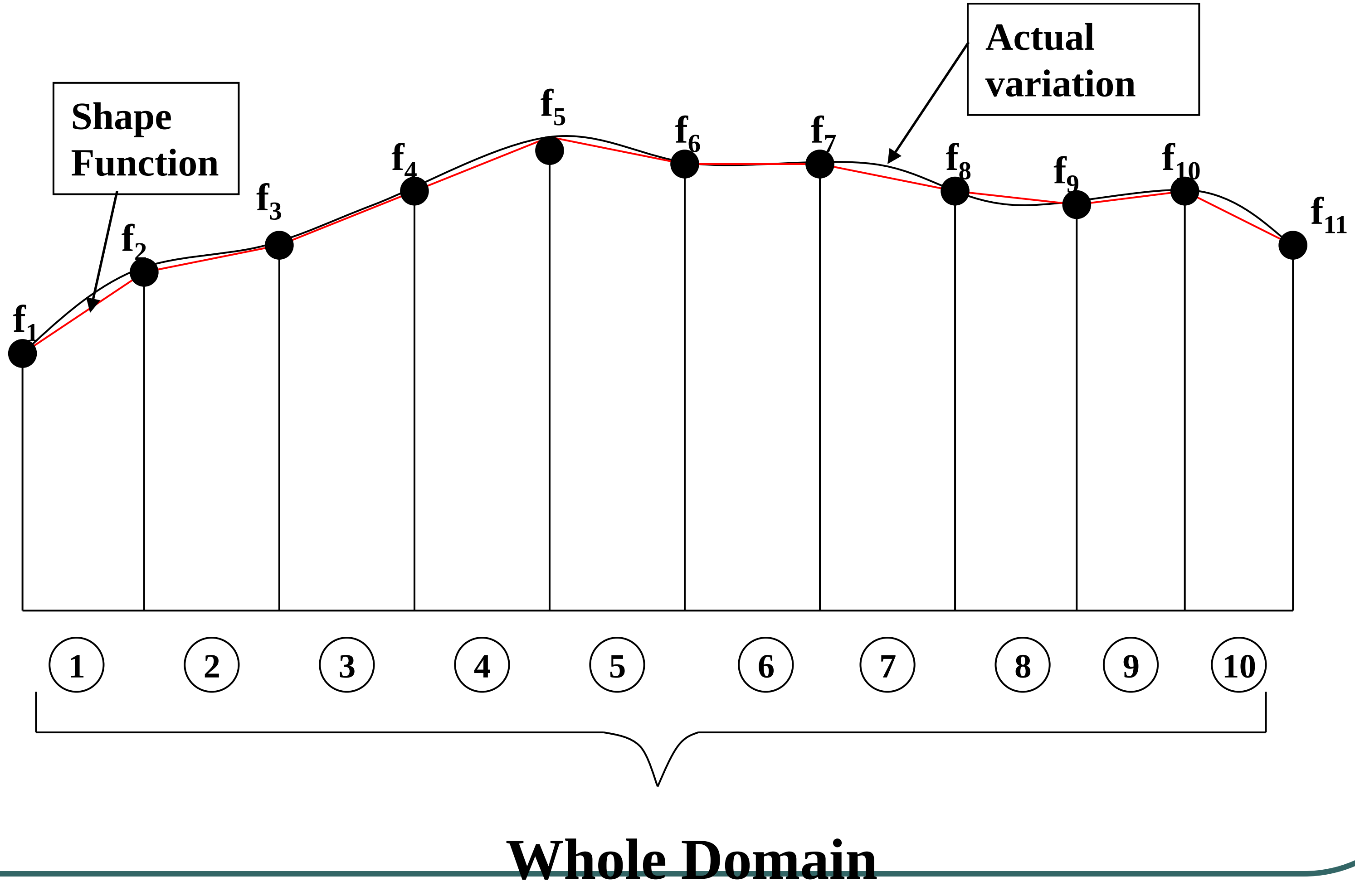




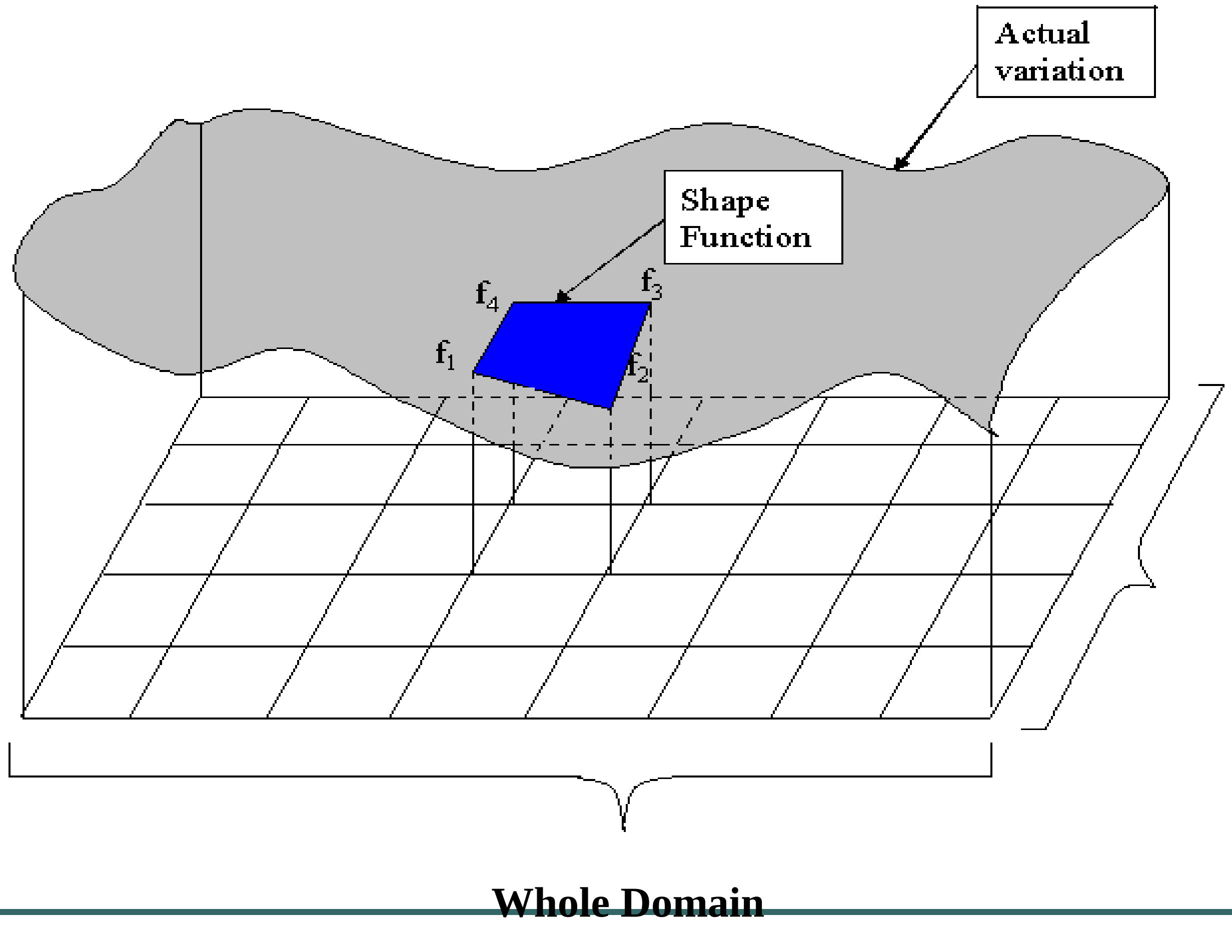


Dynamic response of a multi d.o.f. system.

Piecewise Curve Fit – One Dimensional Case



Piecewise Curve Fit – Two Dimensional Case



Evaluation of Weighted Residual

$$\int_0^L W_i(X)\, R_d(X)\, dX \;=\; \sum_1^{n} \int_0^l W_i(x)\, R_d(x)\, dx$$

Where, n == the number of segments/pieces

Thus we evaluate over each segment and then sum up

Shape functions over each segment are same/similar and hence calculations easy

Essence of Finite element method

- Evaluate the sub-domain level contributions to the weighted residual by merely computing the integral $\int W(x)R_d(x)dx$ or its weak form, just once for the kth sub–domain

- Build–up the entire coefficient matrices [A] & {b} by appropriately placing these sub–domain level contributions in the appropriate rows and columns.

- Solve the (n+1) algebraic equations to determine the unknowns viz., function values f_k at the ends of the sub–domains.

Piecewise Approximation

- Each of the sub–domains is called a **"finite element"** – to be distinguished from the **"differential element"** used in continuum mechanics.
- The ends of the sub–domain are referred to as the **"nodes"** of the element.
- Later on we see elements with nodes not necessarily located at only the ends e.g. an element can have mid–side nodes, internal nodes etc.
- The unknown function values *fk* at the ends of the sub–domains are known as the **"nodal degrees of freedom (d.o.f) ".**

Finite Element Formulation

- A general finite element can admit the function values as well as its derivatives as nodal d.o.f.

- The sub-domain level contributions to the weak form are typically referred to as **"element level equations".**

- The process of building-up the entire coefficient matrices [A] & {b} is known as the process of **"assembly"** i.e. assembling or appropriately placing the individual element equations to generate the system level equations.

Three Key Ideas in FEM

- Weighted Residual Method – assume a solution and minimise residual
- Weak form of WR Method – to reduce continuity demand so that lower order trial solution can be used
- Piecewise curve fit – divide and assemble

EQUATIONS OF MOTION BASED ON WEAK FORM

- AXIAL VIBRATION OF A ROD

$$AE\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2}$$

$$\boxed{AE\frac{d^2U}{dx^2} + \rho A\omega^2 U = 0}$$

- The Weighted–Residual statement

$$\int_0^L W(x)\left(AE\frac{d^2U}{dx^2} + \rho A\omega^2 U\right) dx = 0$$

$$\left[W(x)\, AE\frac{dU}{dx}\right]_0^L - \int_0^L AE\frac{dU}{dx}\frac{dW}{dx}\, dx + \int_0^L W(x)\,\rho A\omega^2 U(x)\, dx = 0$$

Contd..

$$U(x) = \left(1 - \frac{x}{\ell}\right) U_1 + \left(\frac{x}{\ell}\right) U_2$$

$$W_1(x) = 1 - \frac{x}{\ell} \qquad W_2(x) = \frac{x}{\ell}$$

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} -P_0 \\ P_\ell \end{Bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{\rho \mathrm{AL}\omega^2}{6}\begin{Bmatrix} \mathrm{U}_1 \\ \mathrm{U}_2 \end{Bmatrix}$$

$$\boxed{[K]^e = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \qquad \boxed{[M]^e = \frac{\rho A\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

TRANSVERSE VIBRATION OF A BEAM

The governing equation for free transverse vibration of a beam

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0 \qquad V(x,t) = V(x)\, e^{i\omega t}$$

$$\boxed{EI\frac{d^4 V}{dx^4} - \rho A\omega^2 V = 0}$$

Weighted–Residual statement

$$\int_0^L W(x)\left[EI\frac{d^4 V}{dx^4} - \rho A\omega^2 V\right]dx = 0$$

Contd..

Performing integration by parts

$$\left[W(x)\, EI \frac{d^3V}{dx^3} \right]_0^L - \left[\frac{dW}{dx}\; EI \frac{d^2V}{dx^2} \right]_0^L + \int_0^L EI \frac{d^2V}{dx^2}\; \frac{d^2W}{dx^2}\, dx$$

$$- \int_0^L \rho A \omega^2\; W(x) V(x)\, dx \;=\; 0$$

$$\boxed{V(x) \;=\; N_1 V_1 \;+\; N_2 \theta_2 \;+\; N_3 V_3 \;+\; N_4 \theta_4}$$

$$N_1 \;=\; 1 - 3x^2/L^2 \;+\; 2x^3/L^3$$

$$N_2 \;=\; x - 2x^2/L \;+\; x^3/L^2$$

$$N_3 \;=\; 3x^2/L^2 \;-\; 2x^3/L^3$$

$$N_4 \;=\; -x^2/L \;+\; x^3/L^2$$

Contd...

$$[m]^e = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & & & sym. \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

CONSISTENT MASS MATRICES FOR VARIOUS ELEMENTS

Bar element

$$
\begin{aligned}
[m]^e &= \int_v \rho [N]^T [N]\, dv \\
&= \rho A \int_0^{\ell} \begin{bmatrix} 1-\frac{x}{\ell} \\ \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} 1-\frac{x}{\ell} & \frac{x}{\ell} \end{bmatrix} dx \\
&= \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\end{aligned}
$$

Beam element

$$
\begin{aligned}
[\mathrm{m}]^{\mathrm{e}} &= \int_{\mathrm{v}} \rho [\mathrm{N}]^{\mathrm{T}} [\mathrm{N}]\, \mathrm{dv} \\
&= \rho A \int_0^{\ell} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \mathrm{dx}
\end{aligned}
$$

Contd..

$$[m]^e = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & & & sym. \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

Lumped mass matrices:

$$[m^e]_{lumped} = \begin{bmatrix} \frac{\rho A \ell}{2} & 0 \\ 0 & \frac{\rho A \ell}{2} \end{bmatrix}$$

$$[m^e]_{lumped} = \begin{bmatrix} \frac{\rho A \ell}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho A \ell}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example : natural freq. of uniform cross section bar

- One element solution – lumped & cons. mass

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega^2 \rho A \ell \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\frac{AE}{L} u_2 = \omega_{lump}^2 \frac{\rho A L}{2} u_2 \qquad \omega_{lump} = \sqrt{\frac{2E}{\rho L^2}} = \frac{1.414}{L}\sqrt{\frac{E}{\rho}}$$

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega^2 \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\omega_{cons.} = \sqrt{\frac{3E}{\rho L^2}} = \frac{1.732}{L}\sqrt{\frac{E}{\rho}}$$

- Two element solution – lumped mass

$$\frac{AE}{\left(\frac{L}{2}\right)}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \rho A\left(\frac{L}{2}\right)\omega^2\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2+1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-(\lambda/2) \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-(\lambda/2) \end{vmatrix} = 0$$

Natural frequencies of a fixed–free bar

($L = 1m$, $E = 2 \times 10^{11}\ N/m^2$, $\rho = 7800$ kg/m3, $A = 30 \times 10^{-6}\ m^2$)

No.of element / Mode	1	2	3	4	8	16	Exact
1	1140.0	1234.0	1252.0	1258.0	1264.0	1265.0	1265.9
	1396.0	1299.0	1280.0	1274.0	1268.0	1266.0	
2	----	2978.0	3420.0	3582.0	3743.0	3784.0	3797.8
		4537.0	4188.0	4019.0	3853.0	3812.0	
3	----	----	4670.0	5366.0	6078.0	6266.0	6329.6
			7597.0	7301.0	6586.0	6393.0	
4	----	----	----	6319.0	8180.0	8688.0	8861.5
				10560.0	9563.0	9037.0	
5	----	----	----	----	10000.0	11030.0	11393.3
					12850.0	11770.0	

Natural frequencies (Hz) of a simply supported beam

No.of element / Mode	2	3	4	8	Exact
1	14.42	14.52	14.52	14.52	14.52
	14.21	14.46	14.51	14.52	
	14.58	14.53	14.52	14.52	
2	----	57.67	58.07	58.09	58.11
	104.3	56.84	57.84	58.03	
	64.47	58.32	58.11	58.09	
3	----	122.4	130.5	130.7	130.75
	148.2	120.2	129.3	130.4	
	162.1	133.1	130.9	130.7	
4	----	----	230.7	232.3	232.45
	180.0	416.2	227.4	231.4	
	295.5	257.9	233.3	232.4	
5	----	----	354.7	362.8	363.20
	----	481.3	348.0	360.6	
	----	408.9	366.4	363.3	

FORM OF FINITE ELEMENT EQUATIONS FOR VIBRATION PROBLEMS

Governing equation

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\}$$

For un damped free vibration problems

$$[M]\{\ddot{X}\} + [K]\{X\} = 0$$

Assuming harmonic vibration at a frequency ω_i

$$\{X_i\} = \{U_i\} \sin \omega_i t$$

$$[K]_{n\times n}\{U_i\}_{n\times 1} = \omega_i^2 [M]_{n\times n} \{U_i\}_{n\times 1}$$

Eigenvalue problem

$$[K]_{n\times n}\{U_i\}_{n\times 1} = \omega_i^2 [M]_{n\times n}\{U_i\}_{n\times 1}$$

re-writing $[M]^{-1}[K]\{U_i\} = \omega_i^2\{U_i\}$ **or** $\frac{1}{\omega_i^2}\{U_i\} = [K]^{-1}[M]\{U_i\}$

$$\boxed{[A]\{U_i\} = \lambda_i\{U_i\}}$$

where $[A] = [M]^{-1}[K]$ $\lambda_i = \omega_i^2$

or $[A] = [K]^{-1}[M]$ $\lambda_i = \frac{1}{\omega_i^2}$

SOLUTION OF EIGENVALUE PROBLEMS

Methods of solution **:**

- Determinant based methods
- Transformation based methods
- Vector iteration based method

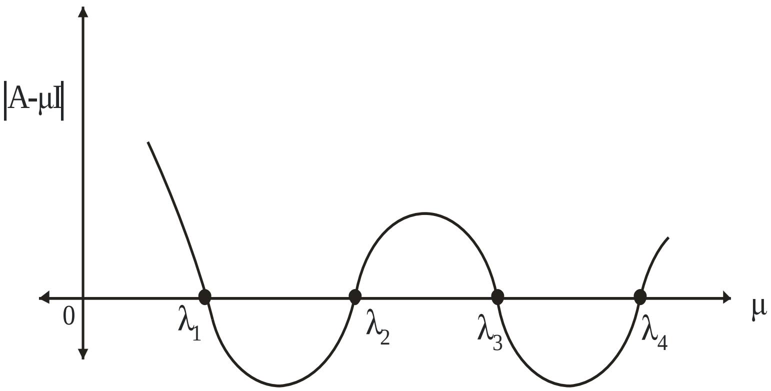
Determinant based methods

- Primarily based on

$$[A]\{U\} = \lambda\{U\}$$

$$[\,[A] - \lambda[I]\,]\{U\} = \{0\}$$

$$\big|\,[A] - \lambda[I]\,\big| = 0 \text{ for a non–trivial } \{U\}$$



- Take trial values of λ
- Compute determinant $|[A]-\lambda[I]|$.
- Not useful for practical implementation
- Heavy computational cost
- Evaluation of each determinant of size ($n\times n$) requires of the order of n^3 floating point operations

Transformation based methods

- Given $[A]\{U\} = \lambda\{U\}$
- Transform *[A]* into a diagonal matrix using
  a series of matrix transformations of the type $= [T]^T\ [A][T]$
  where [*T*] is an orthogonal matrix i.e. $[\mathrm{T}]^{\mathrm{T}} = [\mathrm{T}]^{-1}$
- Well known methods-
  - Givens method.
  - Householders method.
  - Jacobi method.
  - Lanczos method.

$$[\Phi] = [\{U_1\} \cdots \{U_n\}]$$

$$[\Phi]^T \; [K] \; [\Phi] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \mathrm{O} & \\ & 0 & & \lambda_n \end{bmatrix}$$

$$[\Phi]^T \; [M] \; [\Phi] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}$$

$$[A_1] = [A]$$

$$[A_2] = [T_1]^T [A_1] [T_1]$$

$$[A_3] = [T_2]^T [A_2] [T_2] = ([T_1] [T_2])^T [A] ([T_1] [T_2])$$

Jacobi Method

$$[T]=\begin{array}{cl}
\left[\begin{array}{cccccccccc}
1 & & 0 & 0 & & & & & & \\
 & 1 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & \cos\theta & 0 & -\sin\theta & & & \\
 & & & & 0 & 1 & \vdots & & & \\
 & & & & \vdots & & \vdots & & & \\
 & & & & \sin\theta & & \cos\theta & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1
\end{array}\right]
&
\begin{array}{l}
\\ \\ \\ \\ \leftarrow\ i^{th}\ \text{row} \\ \\ \\ \leftarrow\ j^{th}\ \text{row} \\ \\ \\ \\
\end{array}
\\
\begin{array}{cccc}
 & \begin{array}{c}\uparrow \\ i^{th}\ \text{col,}\end{array} & \begin{array}{c}\uparrow \\ j^{th}\ \text{col.}\end{array} &
\end{array}
&
\end{array}$$

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} \qquad \mathbf{a_{ii} \neq a_{jj},}$$

$$\theta = \frac{\pi}{4} \qquad \textbf{if } \mathbf{a_{ii} = a_{jj}}$$

$$[K_1] = \begin{bmatrix} 0.360\times10^8 & -0.180\times10^8 & 0 \\ -0.180\times10^8 & 0.360\times10^8 & -0.180\times10^8 \\ 0 & -0.180\times10^8 & 0.180\times10^8 \end{bmatrix}$$

$$[M_1] = \begin{bmatrix} 0.052 & 0.013 & 0 \\ 0.013 & 0.052 & 0.013 \\ 0 & 0.013 & 0.026 \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K_2] = [T_1]^T\ [K_1]\ [T_1] = 10^8 \begin{bmatrix} 1.08 & 0 & 0.180 \\ 0 & 0.360 & -0.18 \\ 0.18 & -0.18 & 0.18 \end{bmatrix}$$

$$[M_2] = [T_1]^T\ [M_1]\ [T_1] = \begin{bmatrix} 0.078 & 0 & -0.013 \\ 0 & 0.13 & 0.013 \\ -0.013 & 0.013 & 0.026 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 1 & 0 & -0.309 \\ 1 & 0 & 0 \\ 1.24 & 1 & 0 \end{bmatrix}$$

$$[K_3] = [T_2]^T\ [K_2]\ [T_2] = \ 10^8 \begin{bmatrix} 1.80 & -0.222 & 0 \\ -0.222 & 0.36 & -0.18 \\ 0 & -0.18 & 0.172 \end{bmatrix}$$

$$[M_3] = [T_2]^T\ [M_2]\ [T_2] = \begin{bmatrix} 0.0856 & 0.0161 & -0.694\times10^{-17} \\ 0.0161 & 0.130 & 0.013 \\ -0.694\times10^{-17} & 0.013 & 0.0415 \end{bmatrix}$$

Observe that k(1,2) and m(1,2) have again become nonzero!

$$[K_{10}] = [T_9]^T[K_9][T_9] = 10^8\begin{bmatrix} 1.96 & -0.216\times10^{-10} & 0.284\times10^{-17} \\ -0.216\times10^{-10} & 0.161 & 0.222\times10^{-22} \\ 0.416\times10^{-16} & 0.421\times10^{-16} & 0.424 \end{bmatrix}$$

$$[\mathrm{M}_{10}] = [\mathrm{T}_9]^{\mathrm{T}}\ [\mathrm{M}_9]\ [\mathrm{T}_9] = \begin{bmatrix} 0.0859 & 0.156\times10^{-11} & -0.204\times10^{-18} \\ 0.156\times10^{-11} & 0.248 & 0 \\ -0.195\times10^{-18} & 0.694\times10^{-17} & 0.0612 \end{bmatrix}$$

$$\mathrm{f}_1 = \frac{1}{2\pi}\sqrt{\frac{\mathrm{k}_{11}}{\mathrm{m}_{11}}} = \frac{1}{2\pi}\sqrt{\frac{1.96\times10^8}{0.0859}} = 7597\ \mathrm{Hz}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k_{22}}{m_{22}}} = \frac{1}{2\pi}\sqrt{\frac{0.161\times10^8}{0.248}} = 1280.43\ Hz$$

$$f_3 = \frac{1}{2\pi}\sqrt{\frac{k_{33}}{m_{33}}} = \frac{1}{2\pi}\sqrt{\frac{0.424\times10^8}{0.0612}} = 4187.64\ Hz$$

$$[\bar{\Phi}] = [T_1][T_2]\cdots[T_9]$$

$$[\bar{\Phi}] = \begin{bmatrix} 0.697 & 0.745 & -0.886 \\ -1.21 & 1.29 & 0 \\ 1.39 & 1.49 & 0.886 \end{bmatrix}$$

Vector iteration based methods

- Assume a trial eigen vector
- Perform repeated matrix manipulations—

  to converge to the desired eigen vector

- Available in many commercial finite element software packages.

Basis of Vector Iteration Methods

$$\{X^1\} = c_1\{U_1\} + c_2\{U_2\} + c_3\{U_3\} + \cdots + c_n\{U_n\}$$

$$\begin{aligned}\{X^2\} &= [A]\{X^1\} = c_1 [A]\{U_1\} + c_2 [A]\{U_2\} + \cdots + c_n [A]\{U_n\} \\ &= c_1 \lambda_1 \{U_1\} + c_2 \lambda_2 \{U_2\} + \cdots + c_n \lambda_n \{U_n\}\end{aligned}$$

$$\begin{aligned}\{X^{m+1}\} &= c_1 \lambda_1^m \{U_1\} + c_2 \lambda_2^m \{U_2\} + \cdots + c_n \lambda_n^m \{U_n\} \\ &= c_1 \lambda_1^m \left[\{U_1\} + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^m \{U_2\} + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^m \{U_n\} \right]\end{aligned}$$

If $\lambda_1 > \lambda_2 > \lambda_3 \cdots > \lambda_n$ **i.e.** $\left(\frac{\lambda_2}{\lambda_1} < 1, \ \frac{\lambda_n}{\lambda_1} < 1\right)$

$$\{X^{m+1}\} \approx c_1 \lambda_1^m \{U_1\}$$

Inverse Iteration Scheme

- **Step 1:** Formulate the global [K] and [M] for the structure
- **Step 2:** Assume a trial vector $\{X^1\}$
- **Step 3:** Compute $\{R\} = [M]\{X^1\}$
- **Step 4:** Solve $[K]\{\bar{X}\} = \{R\}$
- **Step 5:** Obtain $\{X^2\}$ from $\{\bar{X}\}$ such that $\{X^2\}^T [M] \{X^2\} = 1$

$$\{X^2\} = \frac{1}{\sqrt{\{\bar{X}\}^T [M] \{\bar{X}\}}} \{\bar{X}\}$$

- **Step 6:** Compute $\lambda = \{X^2\}^T [K] \{X^2\}$

Repeat steps (3) – (6) till λ converges to within a pre–set tolerance.

$$[K] = 10^8 \begin{bmatrix} 0.360 & -0.180 & 0 \\ -0.180 & 0.360 & -0.180 \\ 0 & -0.180 & 0.180 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 0.052 & 0.013 & 0 \\ 0.013 & 0.052 & 0.013 \\ 0 & 0.013 & 0.026 \end{bmatrix}$$

Sub-space Iteration

Choose trial vector $\{\bar{X}_1\}_{n\times m}$

extract an orthonormal set of vectors $\{X_1\}_{n\times m}$ from $\{\bar{X}_1\}_{n\times m}$ i.e. $\{X_1\}^T[M]\{X_1\}=[I]$

$$\{R\}_{n\times m} = [M]_{n\times n}\{X_1\}_{n\times m}$$

$$[K]_{n\times n}\{\bar{X}_2\}_{n\times m} = \{R\}_{n\times m}$$

$$[\lambda]_{m\times m} = \{X_1\}^T[K]\,\{X_1\}$$

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \text{O} & \\ 0 & & & \lambda_n \end{bmatrix}$$

Subspace

$$[\bar{K}]_{m\times m} = \{\bar{X}\}^{T}_{m\times n}[K]_{n\times n}\{\bar{X}\}_{n\times m}$$

$$[\bar{M}]_{m\times m} = \{\bar{X}\}^{T}_{m\times n}[M]_{n\times n}\{\bar{X}\}_{n\times m}$$

$$[\bar{K}]_{m\times m}\{\phi\}_{m\times m} = [\bar{\lambda}]_{m\times m}[\bar{M}]_{m\times m}\{\phi\}_{m\times m}$$

$$\{X\}_{n\times m} = \{\bar{X}\}_{n\times m}\{\phi\}_{m\times m}$$

$$[K] = 10^8 \begin{bmatrix} 0.360 & -0.180 & 0 \\ -0.180 & 0.360 & -0.180 \\ 0 & -0.180 & 0.180 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 0.052 & 0.013 & 0 \\ 0.013 & 0.052 & 0.013 \\ 0 & 0.013 & 0.026 \end{bmatrix}$$