Weighted Residual Formulation

What is FEM

- General technique for solving partial differential equations
- The partial differential equations are transformed into algebraic equations which can be solved using digital computers
- Algebraic equations are cast into matrix form to allow ease of computer programming.
- The essence of the finite element method is to break large, complex geometry into smaller interconnected components called "elements".

Contd…

- Each element has a function which is assumed to satisfy the required differential equations over the volume of the element

- The more elements you have the better the answers (usually)

- The idea is similar to what we discussed in interpolation – instead of using a single, high degree polynomial curve fit; better to use several lower degree (spline) curve fits

Finite Element Formulation

- Finite element analysis is versatile because of the common features in the mathematical formulation of seemingly different physical problems.

- Many problems can be represented by partial differential equations.

- In some cases, the same *type* of partial differential equation (e.g. the two dimensional Laplace /Poisson equation) can represent a large number of physical problems e.g. ground water seepage, torsion of bars, heat flow etc.

Finite Element Formulation Starting from a Governing Differential Equation

WEIGHTED RESIDUAL METHOD

General problem is described in the form of

- **a differential equation (DE) in domain Ω**
- **prescribed boundary conditions (BC) on the boundary Γ.**

Steps for finding approximate solution to DE.

Step-1 **Assume a trial solution to the problem. For a one dimensional problem, trial solution may be as follows:**

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots.$$

WEIGHTED RESIDUAL METHOD

Assumed function may not satisfy DE within the domain Ω and/or the BCs on Γ.

In the context of finite element method, trial solutions are such that they satisfy the applicable boundary conditions

Step-2 **Substitute the assumed function in the DE , applying the BCs of problem & find the "domain residual"**

Step-3 **Determine the unknown parameters (c_0, c_1, c_2 ….) in the assumed trial function so as to make these residuals as low as possible.**

Example 1

$$\frac{d^2 f}{dx^2} - p = 0 \qquad f(0) = 0 \qquad \frac{df}{dx}(L) = 0$$

$$Let \;\; f(x) \approx c_0 + c_1 x + c_2 x^2$$

$$\frac{df}{dx} = c_1 + 2c_2 x \qquad \frac{d^2 f}{dx^2} = 2c_2$$

$$Substitute$$

$$\mathrm{Re}\,sidual,\; R = 2c_2 - p \neq 0$$

If c_2 = p/2, the assumed solution satisfies the d.e and b.c

Example 2

Governing DE and BCs are given by,

$$EI\ \frac{d^4v}{dx^4} - q_0 = 0$$

v **(0) = 0,** $\qquad \dfrac{d^2v}{dx^2}(0) = 0$

v **(*L*) = 0,** $\qquad \dfrac{d^2v}{dx^2}(L) = 0$

WEIGHTED RESIDUAL METHOD

Step 1 – Assume a trial solution:

$$\text{Let} \quad v(x) \approx c_1 \sin(\pi x / L)$$

This one parameter trial solution satisfies all b.c.

Step 2 – Find the domain residual:

$$R_d = c_1 (\pi/L)^4 (EI) \sin(\pi x / L) - q_0$$

Step 3 – Minimise the residual:

Domain residual is varying from point to point within the domain! Only one coefficient has to be determined !!

WR Method - Point Collocation Technique

Residuals can be set to zero at chosen points within the domain.

Number of points being equal to the number of coefficients in the trial function that need to be determined.

Residual might be unduly large at some other points within the domain. Therefore residual must be “minimised” over the entire domain rather than setting it identically zero at only selected few points.

Point Collocation Method

Make $R_d = 0$ at $x = L/4$, i.e.

$$c_1 (\pi/L)^4 (EI) \sin(\pi/4) - q_0 = 0$$

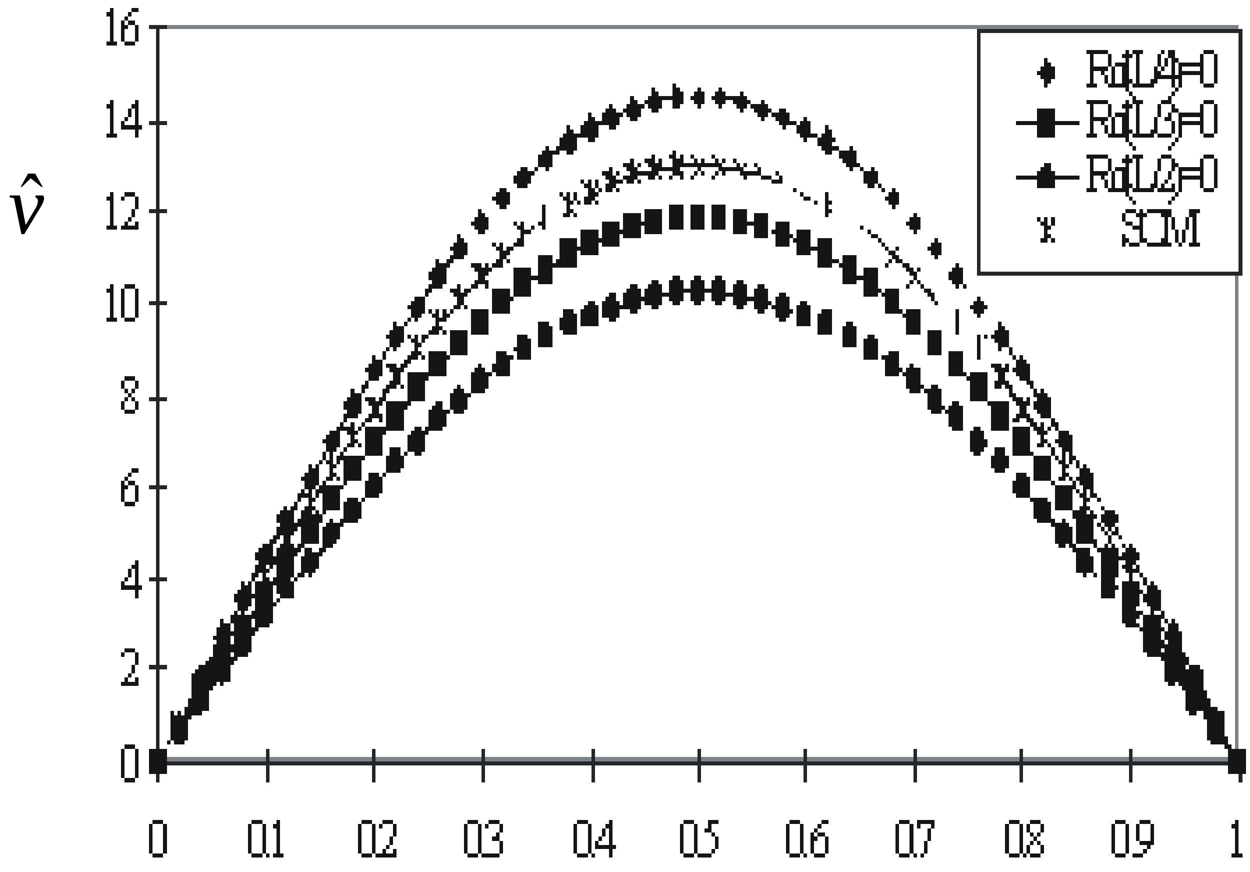
Therefore

$$c_1 = \frac{\sqrt{2}}{\pi^4} \frac{q_0 L^4}{EI}$$

Resulting trial solution is -

$$\hat{v}(x) = \frac{\sqrt{2}}{\pi^4} \frac{q_0 L^4}{EI} \sin \frac{\pi x}{L}$$

Point Collocation Technique



Two – term trial solution

WE CAN IMPROVE OUR TRIAL SOLUTION BY ADDING ONE MORE TERM TO THE SINE SERIES I.E.,

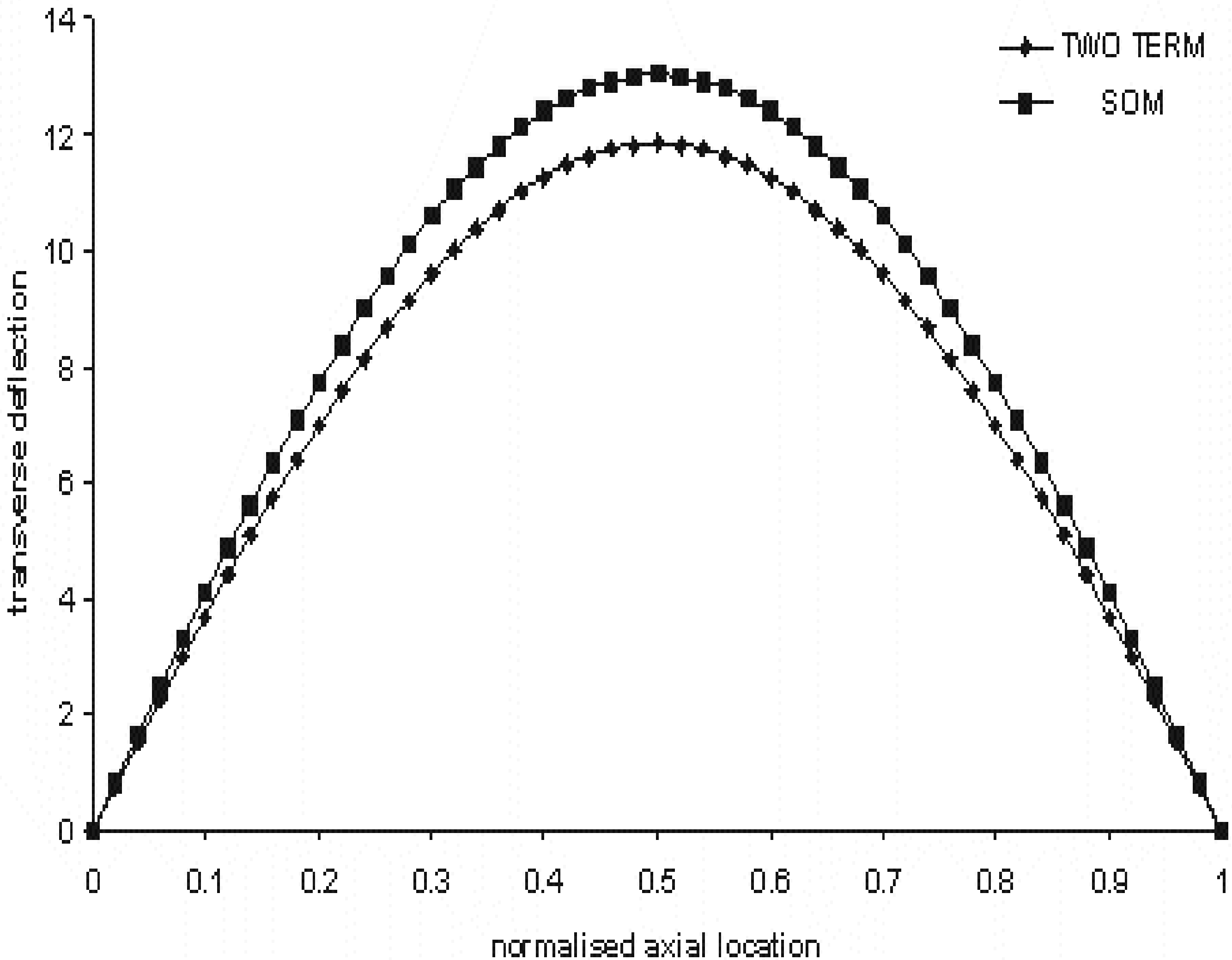
$$v(X) \approx v(X) = C_1 \,\mathrm{SIN}\,(\pi X / L) + C_3 \,\mathrm{SIN}\,(3\pi X / L)$$

DOMAIN RESIDUAL

$$R_D = C_1 (\pi/L)^4 (EI)\,\mathrm{SIN}\,(\pi X / L) + C_3 (3\pi/L)^4 (EI)\,\mathrm{SIN}\,(3\pi X / L) - q_0$$

TWO CONSTANTS TO BE FOUND (C_1 and C_3)

SET THE RESIDUAL ZERO AT TWO SELECTED POINTS


14
12
10
8
6
4
2
0
transverse deflection
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
normalised axial location
TWO TERM
SOM

Key Idea – 1: Solution by Weighted Residual Technique

minimise R_d in an overall sense

For the problem at hand, we will formulate this technique as:

$$\int_0^L W_i(x)\, R_d(x)\, dx = 0$$

where $W_i(x)$ are weighting functions.

Choose as many weighting functions as necessary to generate the required number of equations for the solution of the undetermined coefficients in the trial function.

Galerkin W-R Method

Galerkin (1915) $W(x)$ same as the trial functions themselves.

Let $v(x) \approx c_1 \sin(\pi x / L)$

The domain residual,

$$R_d = c_1 (\pi/L)^4 (EI) \sin(\pi x / L) - q_0$$

Using Galerkin procedure,

$$\int_0^L \underbrace{\left[\sin\left(\frac{\pi x}{L}\right)\right]}_{W(x)} \underbrace{\left[c_1 \left(\frac{\pi}{L}\right)^4 EI \sin \frac{\pi x}{L} - q\right]}_{R_d(x)} dx = 0$$

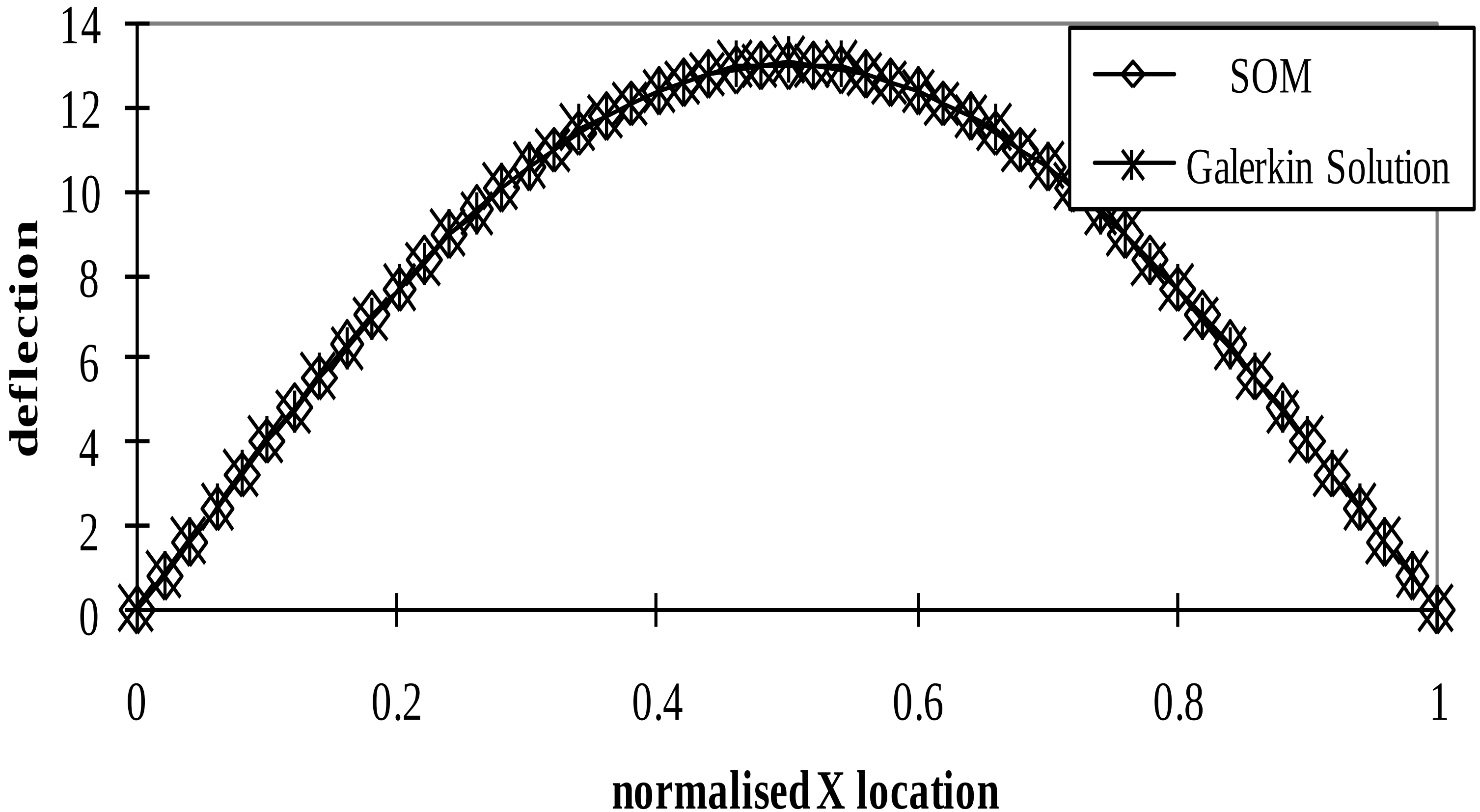
Galerkin W-R Method

$$\left(\frac{\pi}{L}\right)^4 EIc_1 \int_0^L \sin^2\frac{\pi x}{L}\, dx = \int_0^{\mathrm{L}} q\ \sin\frac{\pi x}{L}\, dx$$

Therefore, c_1 = 0.013071q_0L4 / *EI*.

$$\hat{v}(x) = 0.013071\ \frac{q_0 L^4}{EI}\ \sin\frac{\pi x}{L}$$

Galerkin Weighted residual solution vs exact solution



The error in v(x) at the mid–span is just 0.38%.

General Weighted Residual Technique

- ✓ Trial function should satisfy the BCs of the problem.
- ✓ Trial solution is assumed as follows:

$$f(x) = \phi(x) + \Sigma c_i N_i(x)$$

where $\phi(x)$ and $N_i(x)$ are fully known functions of x, pre–selected such that the boundary conditions are satisfied by $f(x)$,

- ✓ Weighting functions to be $W_i(x) = \partial f/\partial c_i = N_i(x)$

Key Idea – 2: Weak Form of W-R statement

- **By carrying out integration by parts of general WR statement, reduce the continuity requirement on the trial function assumed in the solution.**
- **Example : Let Governing DE be:** $K_1 \dfrac{d^2u}{dx^2} + \mathrm{K}_2 x = 0$

  with BCs $u(0) = 0$, $\left.\dfrac{du}{dx}\right|_{x=L} = 0$
- **Let** $\hat{u}$ **be the trial solution assumed. Substituting in DE,**

$$R_d = K_1 \frac{d^2\hat{u}}{dx^2} + \mathrm{K}_2 x$$

Weak Form of W-R statement

With *W*(*x*) as the weighting function we have, WR statement,

$$\int_0^L W(x)\left[K_1\frac{d^2\hat{u}}{dx^2} + K_2 x\right]dx = 0$$

$$\int_0^L W(x)\,K_1\frac{d^2\hat{u}}{dx^2}\,dx + \int_0^L W(x)\,K_2 x\,dx = 0$$

$$\int_0^L W(x)\;d\left\{K_1\frac{d\hat{u}}{dx}\right\} + \int_0^L W(x)\;K_2 x\,dx = 0$$

We Know that, integration by parts,

$$\int_\alpha^\beta u\;dv = [uv]_\alpha^\beta - \int_\alpha^\beta v\;du$$

$$= (uv)\big|_\beta - (uv)\big|_\alpha + \int_\alpha^\beta v\;du$$

Weak Form of W-R statement

- **By performing integration by parts for our problem,**

$$\left[W(x)\ \mathrm{K}_1 \frac{d\hat{u}}{dx} \right]_0^L - \int_0^L \left(K_1 \frac{d\hat{u}}{dx} \right) \frac{dW}{dx}\ dx + \int_0^L W(x)\ \mathrm{K}_2 x\ dx = 0$$

- **Writing** $K_1 \dfrac{d\hat{u}}{dx} = \mathrm{P}$ **above equation becomes**

$$W(L)P_L - W(0)P_0 - \int_0^L K_1 \frac{d\hat{u}}{dx} \frac{dW}{dx}\ dx + \int_0^L W(x)\ \mathrm{K}_2 x\ dx = 0$$

- **Continuity requirement on u(x) has gone down.**
- **Natural BCs are explicitly defined in WR statement.**
- **Trial functions have to satisfy only essential BCs.**

Weak Form of W-R statement

Equivalent Representation of a Problem Statement

Differential equation	Weighted Residual Statement	Weak form
$\mathrm{K}_1 \dfrac{d^2u}{dx^2} + \mathrm{K}_2 x = 0$ **subject to** ***u*(0) = 0** $\mathrm{K}_1 \dfrac{du}{dx}(L) = 0$	$\int_0^L W \left(\mathrm{K}_1 \dfrac{d^2\hat{u}}{dx^2} + \mathrm{K}_2 x \right) dx = 0$ **subject to *û*(0) = 0** $\mathrm{K}_1 \dfrac{d\hat{u}}{dx}(L) = 0$	$\int_0^L K_1 \dfrac{d\hat{u}}{dx} \dfrac{dW}{dx}\, dx = \int_0^L W\ \mathrm{K}_2 x\, dx$ **subject to** ***û*(0) = 0** ***W*(0) = 0**

Weak Form of W-R statement

- **Back to Example Problem, let**

$$u(x) \approx \hat{u}(x) = c_1 x + c_2 x^2$$

- **Then we have**

$$d\hat{u}/dx = c_1 + 2c_2\, x\,,$$

$$W_1 = x\,,\quad W_2 = x^2\,,$$

$$dW_1/dx = 1,\quad dW_2/dx = 2x$$

- **We observe that $\hat{u}(0) = 0$, $W_1(0) = 0$ and $W_2(0) = 0$ as required.**

Weak Form of W-R statement

- **Weak form w.r.t W_1 ,**

$$\int_0^L (K_1)\,(c_1 + 2c_2\, x)\,(1)\, dx = \int_0^L (x)\,(K_2 x)\, dx$$

$$K_1\,(c_1 L + c_2 L^2) = K_2 L^3 / 3$$

Weak Form of W-R statement

- **Weak form w.r.t W_2 ,**

$$\int_0^L (\mathrm{K}_1)\,(c_1 + 2c_2 x)(2x)\,dx = \int_0^L (x^2)(K_2 x)\,dx$$

$$\mathrm{K}_1\left(\frac{c_1 L^2 + 4c_2 L^3}{3}\right) = \frac{K_2 L^4}{4}$$

Weak Form of W-R statement

- **Solving the two equations, we have**

$$c_1 = \frac{7K_2L^2}{12\ \mathrm{K}_1}\ , \qquad c_2 = -\frac{K_2L}{4K_1}$$

- **Thus the $\hat{u}(x)$ is given as:**

$$\hat{u}(x) = \frac{K_2L}{12\ K_1}\,(7xL - 3x^2)$$

Comparison of DE, WR & Weak forms

- **If $\hat{u}(x)$ is the exact solution, then the weighted residual form is implicitly satisfied for any and every weighting function $W(x)$.**

- **WR statement is equivalent only to the DE and does not account for BCs. Trial solution should satisfy all the BCs and it should be differentiable as many times as required in the original DE.**

Comparison of DE, WR & Weak forms

- The weighting functions $W(x)$ can in principle be any non-zero, integrable function. By choosing "n" different weighting functions, generate necessary "n" equations for determination of coefficients c_i $(i = 1,2,\cdots,n)$ in the trial solution field assumed.

Comments on Weak Form (WF)

- In WF, continuity demand on trial function *u* has gone down and that on the weighting function *W* has increased.
- Even order DE can be transformed into WF, having equal continuity demand on trial solution function and the weighting function.

Comments on Weak Form (WF)

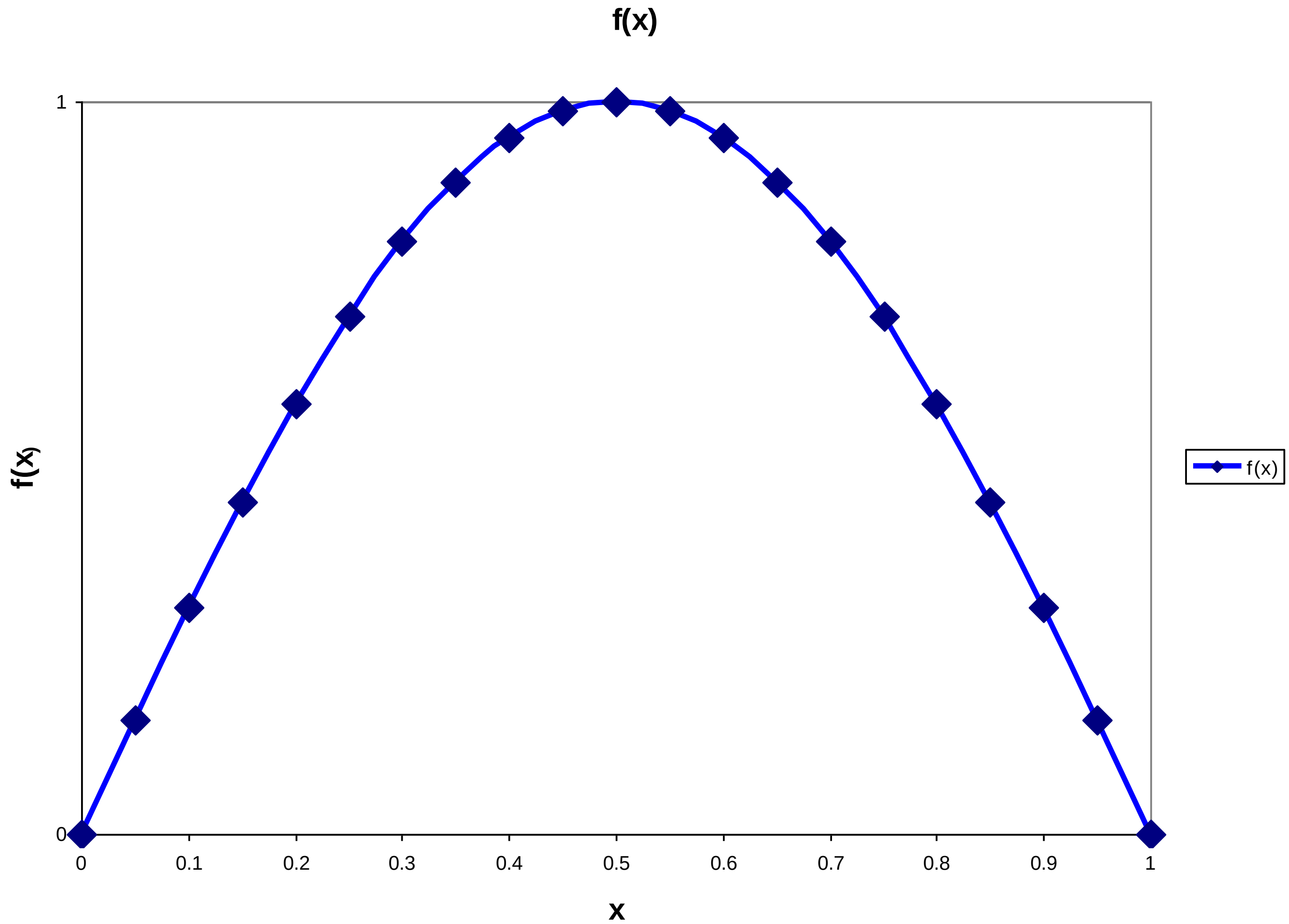
- **Only essential BCs have to be satisfied by the trial solution of the WF equation as natural BCs are taken into account while forming WF.**
- **Weighting function should satisfy the homogeneous part (i.e., $u = 0$) of the prescribed essential BCs.**
- **Much wider choice of trial solution functions is possible with weak form as continuity demanded has gone down**

Key Idea 3: Piecewise Approximation

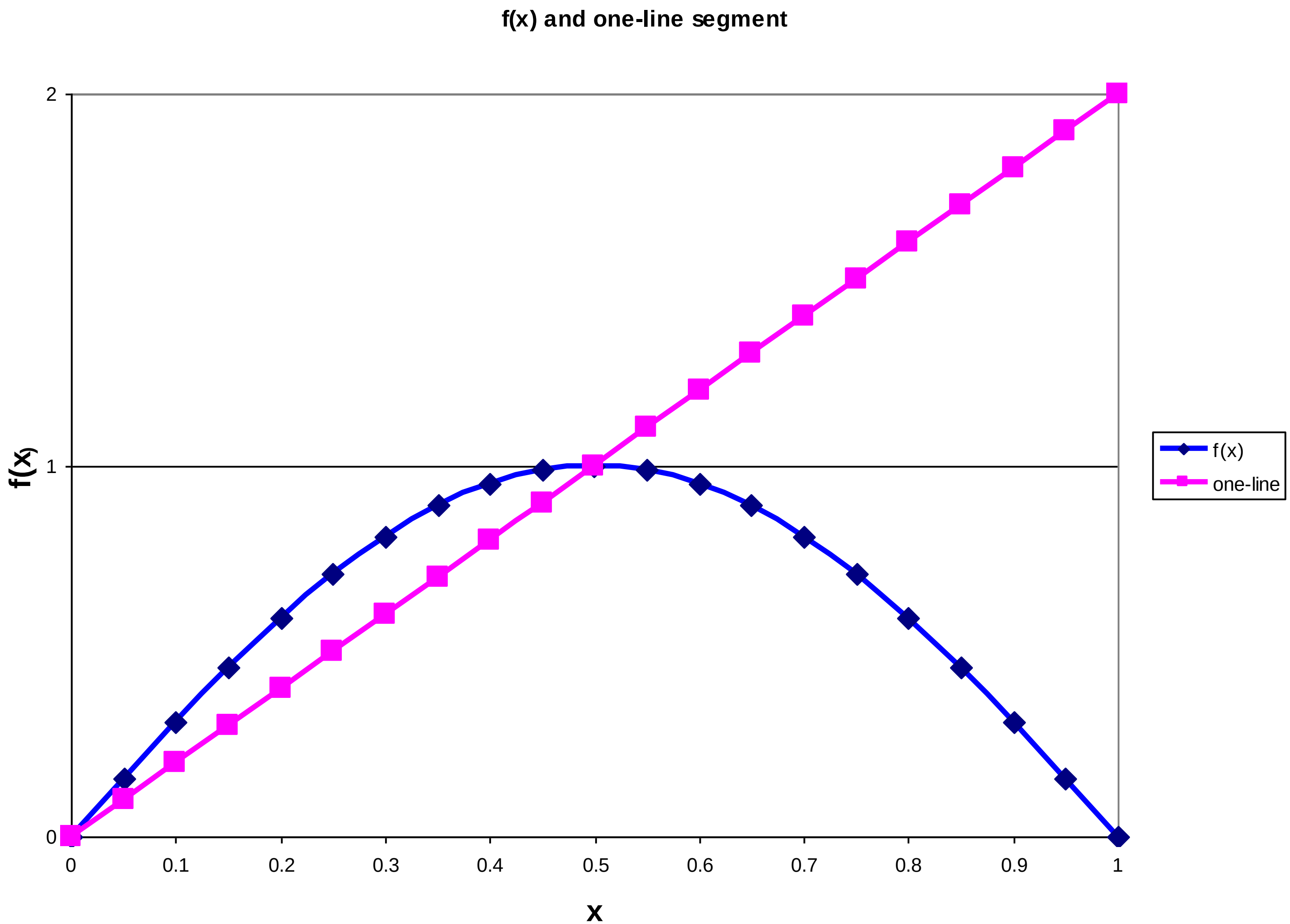
Piece-wise Continuous Trial Function Solution of Weak Form

- Trial function chosen was a single composite function, valid over the entire solution domain.

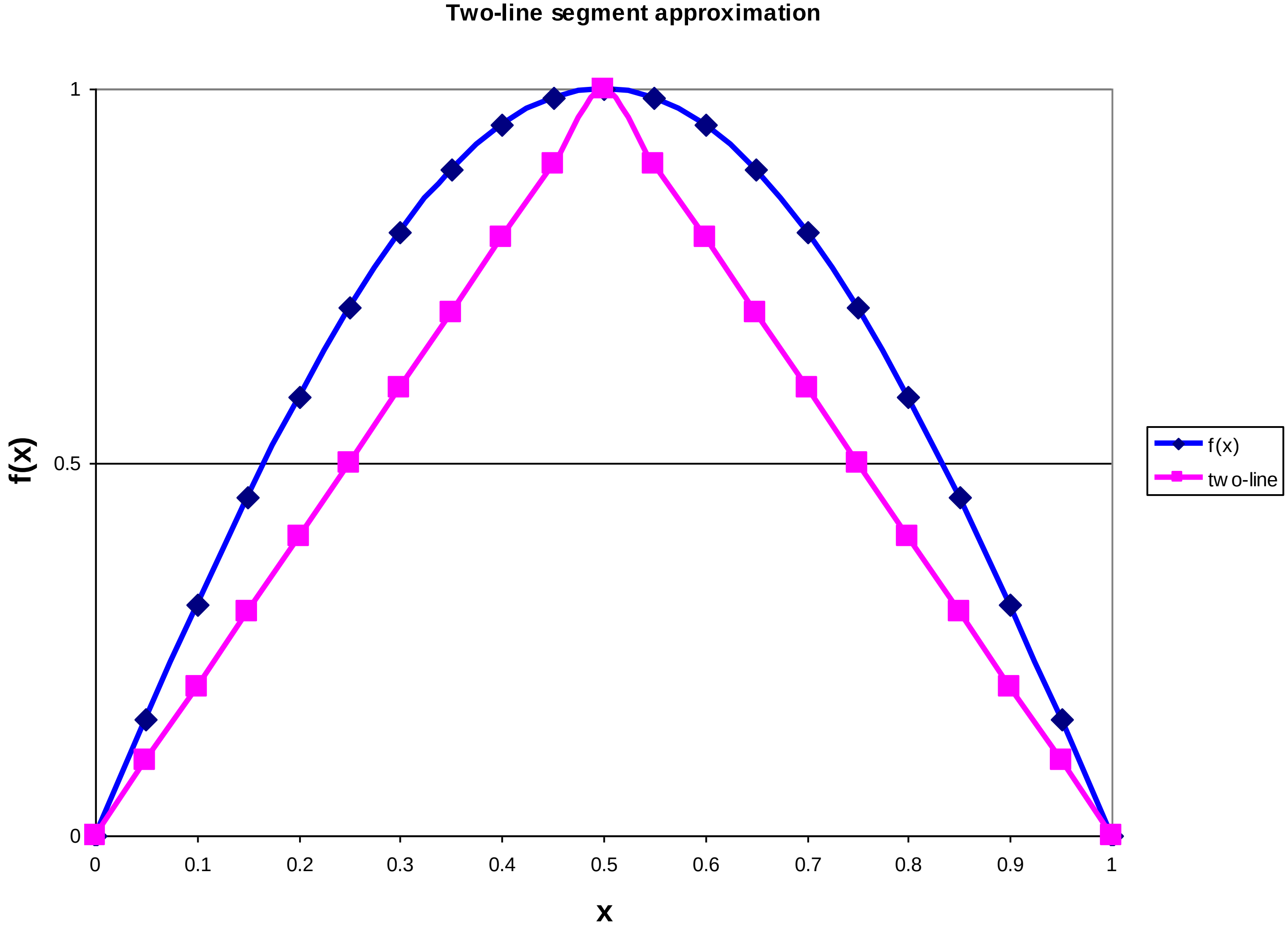
- W–R method is essentially a process of “curve fitting” and is best done "piece–wise" == the more the number of pieces, the better the fit


f(x)
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
f(x)
f(x)

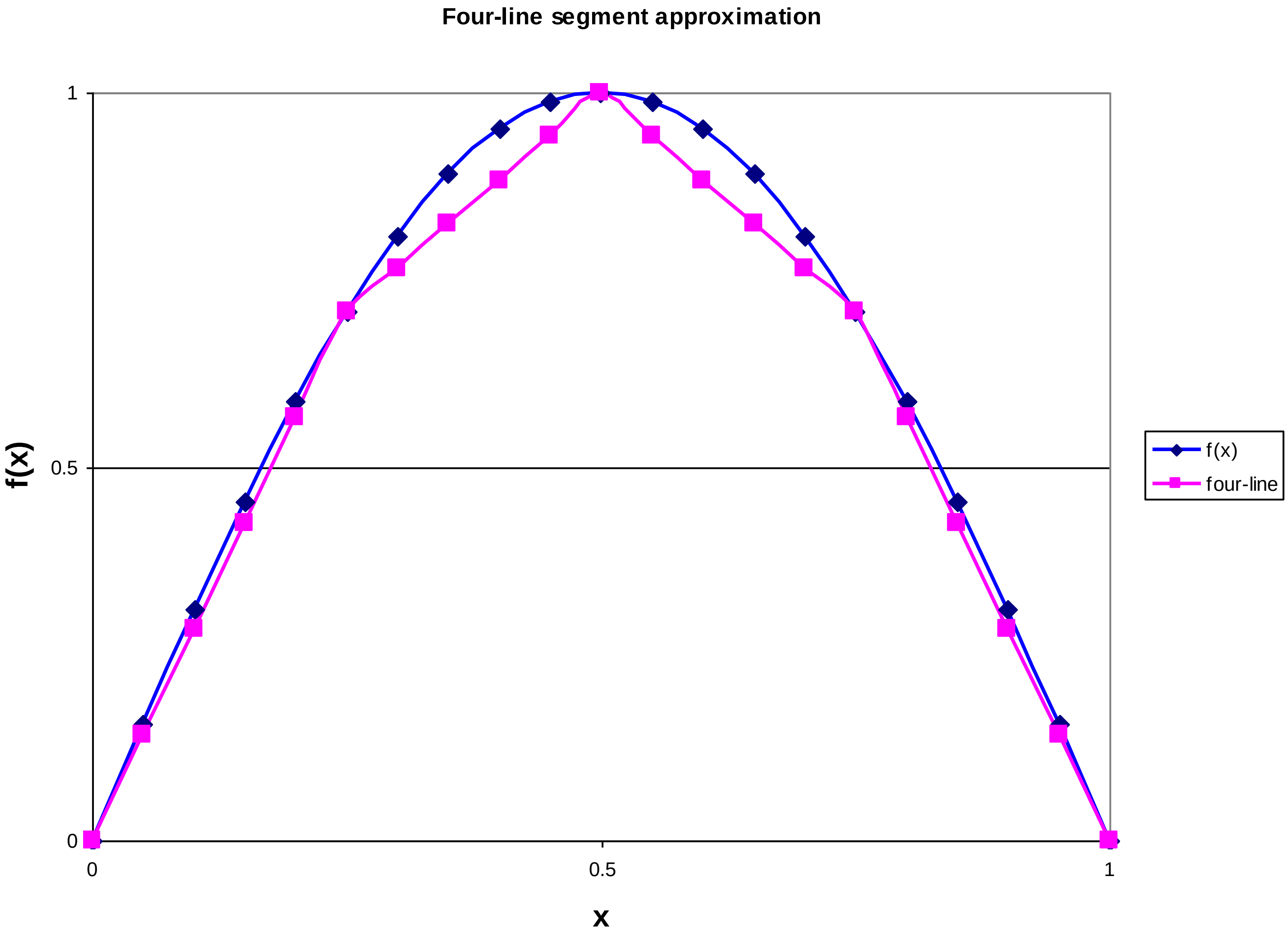
Approximation Using One Straight Line Segment



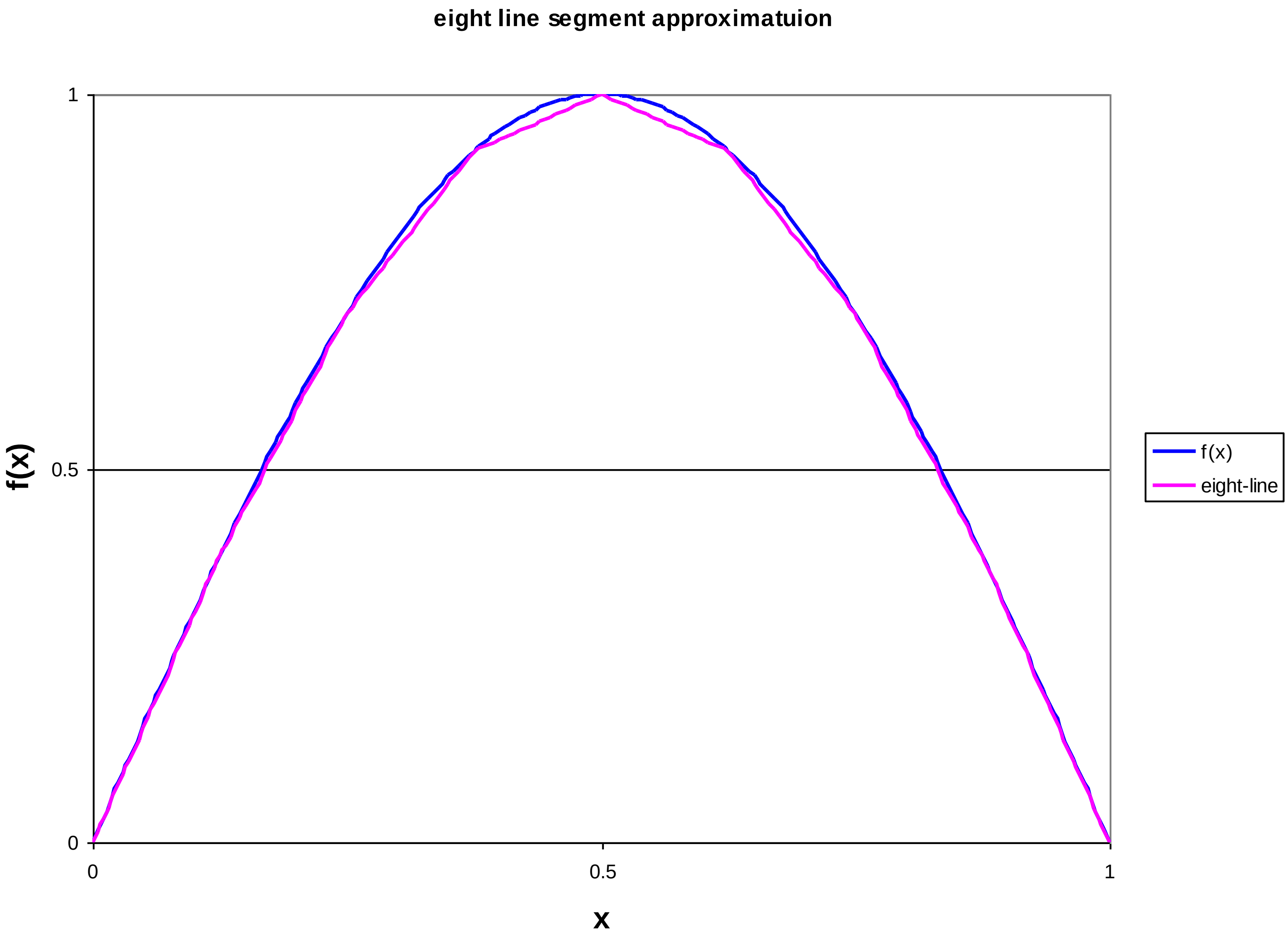
Approximation Using Two Line Segments



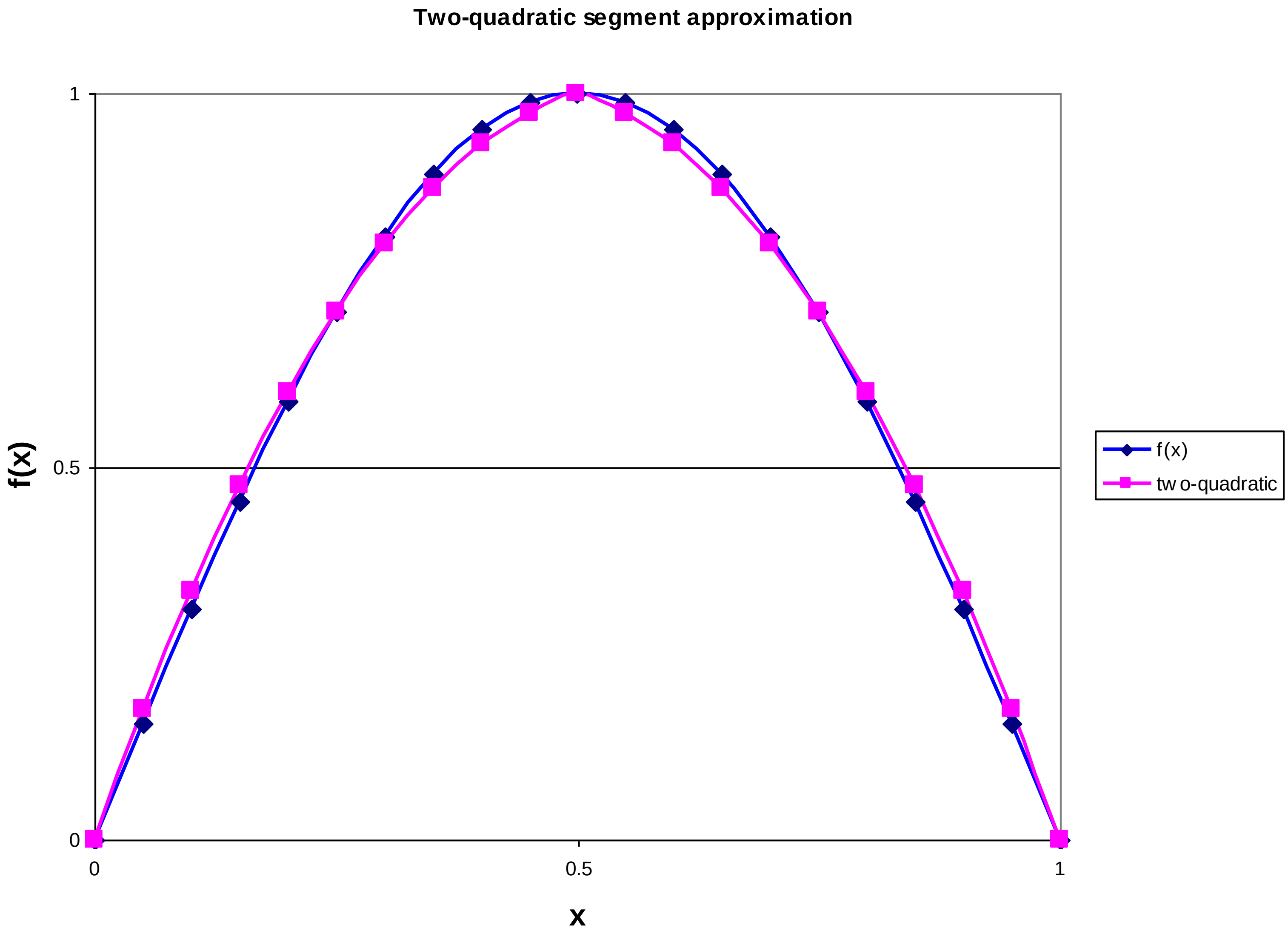
Approximation Using Four Line Segments



Approximation Using Eight Line Segments



Two Quadratic Segment Approximation



Piecewise Approximation

- **One–line segment approximation**

$f(X) \approx X = f(0.5)*2X$ $\quad (0 < X < 1)$

- **Two–line segment approximation**

$f(X) \approx f(0.5)*2X$ $\quad (0 < X < 0.5)$

$f(X) \approx f(0.5) + (X - 0.5)$ $\quad (0.5 < X < 1)$

- **Four–line segment approximation**

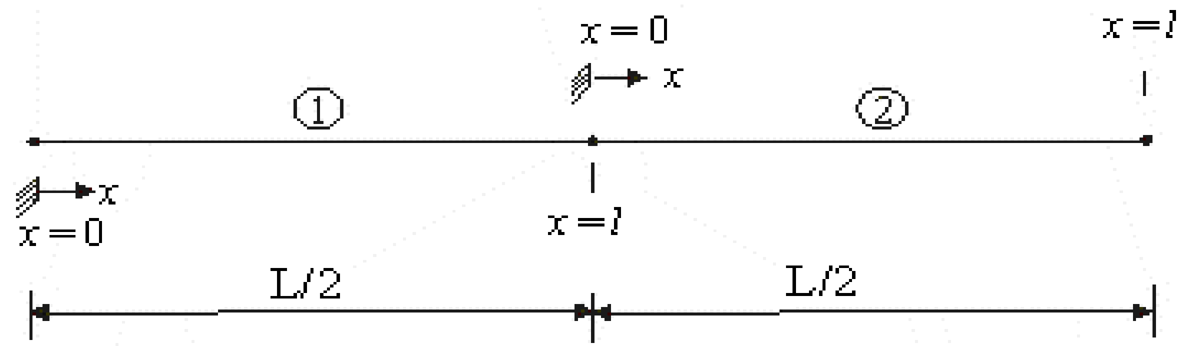
$f(X) \approx f(0.25)*4X$ $\quad (0 < X < 0.25)$

$f(X) \approx f(0.25) + (X - 0.25)$ $\quad (0.25 < X < 0.5)$

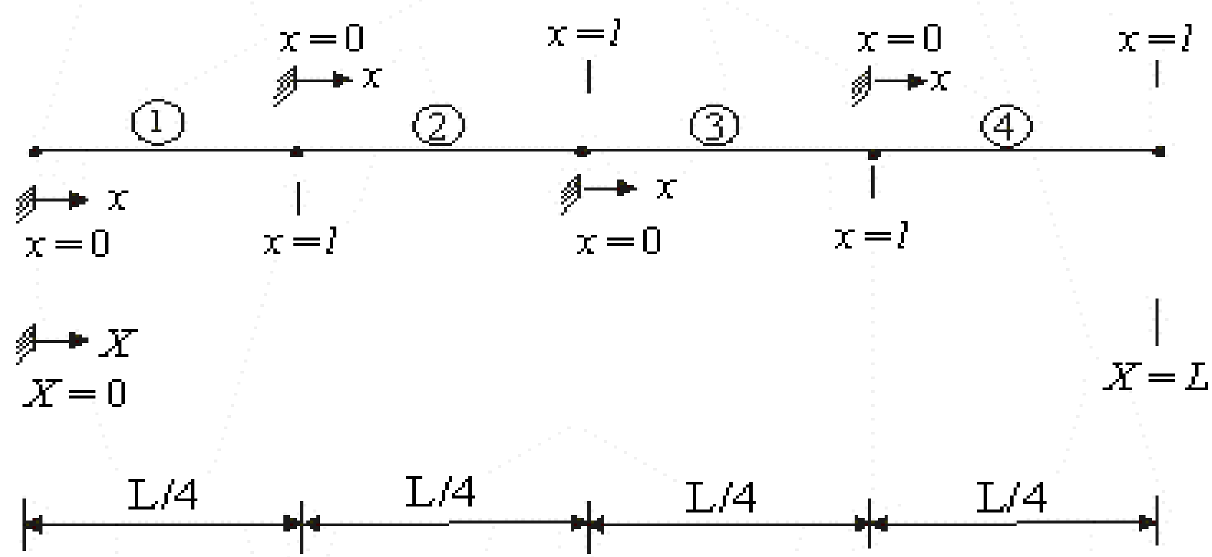
$f(X) \approx f(0.5) + (X - 0.5)$ $\quad (0.5 < X < 0.75)$

$f(X) \approx f(0.75) + (X - 0.75)$ $\quad (0.75 < X < 1)$

Piecewise Approximation - Use of local coordinate frames



We define a local coordinate *x* with the origin fixed at the left end of each sub–domain.



Piecewise Approximation

- **Two line segment approximation (with $\ell = L/2 = 1/2$)**

$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0) + [\,x/\ell\,]\, f(0.5) \qquad (0 < x < \ell)$$

$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0.5) + [\,x/\ell\,]\, f(1) \qquad (0 < x < \ell)$$

- **Four line segment approximation (with $\ell = L/4 = 1/4$)**

$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0) + [\,x/\ell\,]\, f(0.25) \qquad (0 < x < \ell)$$

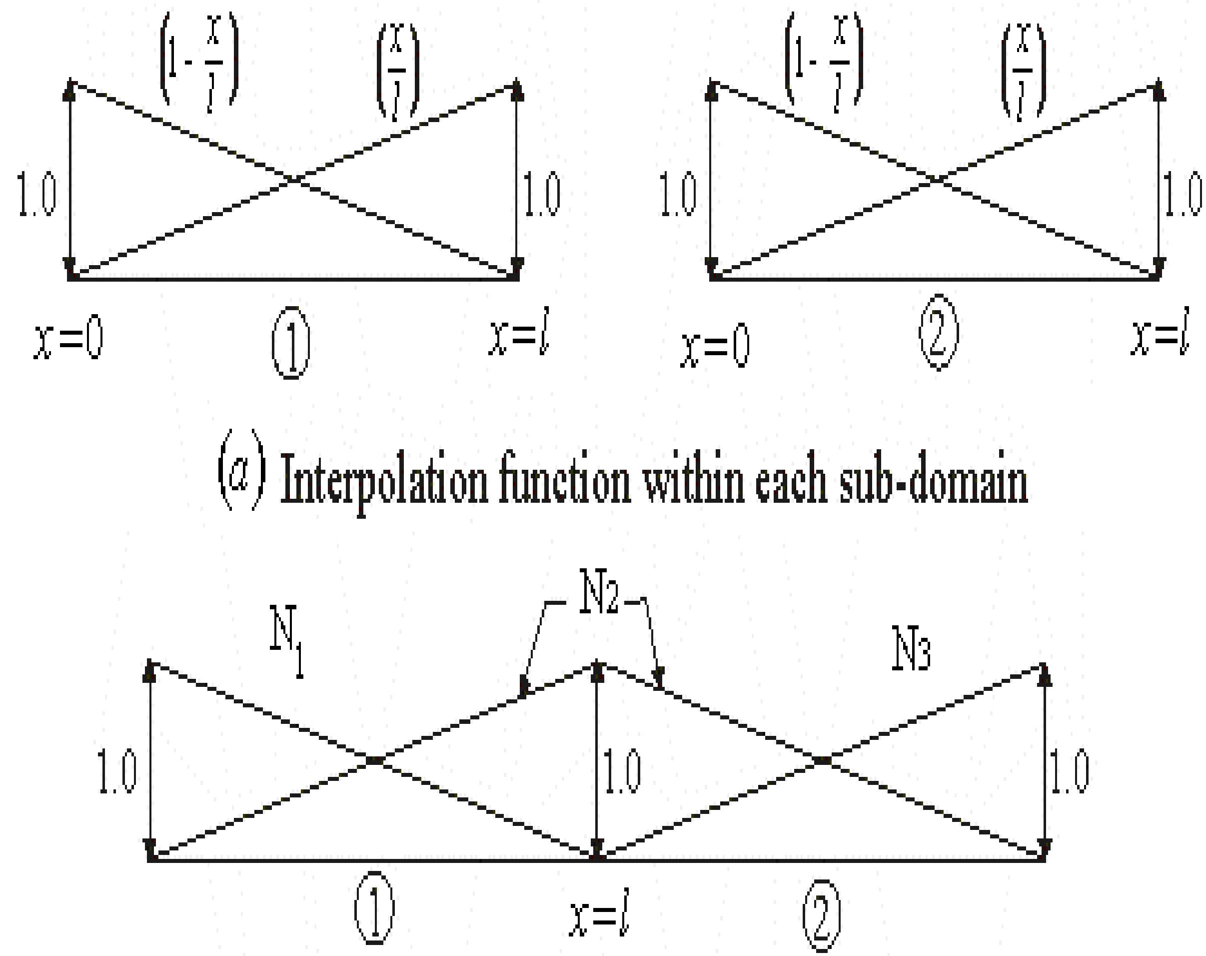
$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0.25) + [\,x/\ell\,]\, f(0.5) \qquad (0 < x < \ell)$$

$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0.5) + [\,x/\ell\,]\, f(0.75) \qquad (0 < x < \ell)$$

$$f(x) \approx [\,1 - (x/\ell)\,]\, f(0.75) + [\,x/\ell\,]\, f(1) \qquad (0 < x < \ell)$$

$$\boxed{f(x) \approx [\,1 - (x/\ell)\,]\, f_{k-1} + [\,x/\ell\,]\, f_k \qquad (0 < x < \ell = L/n)}$$

Piecewise Approximation – Shape Functions

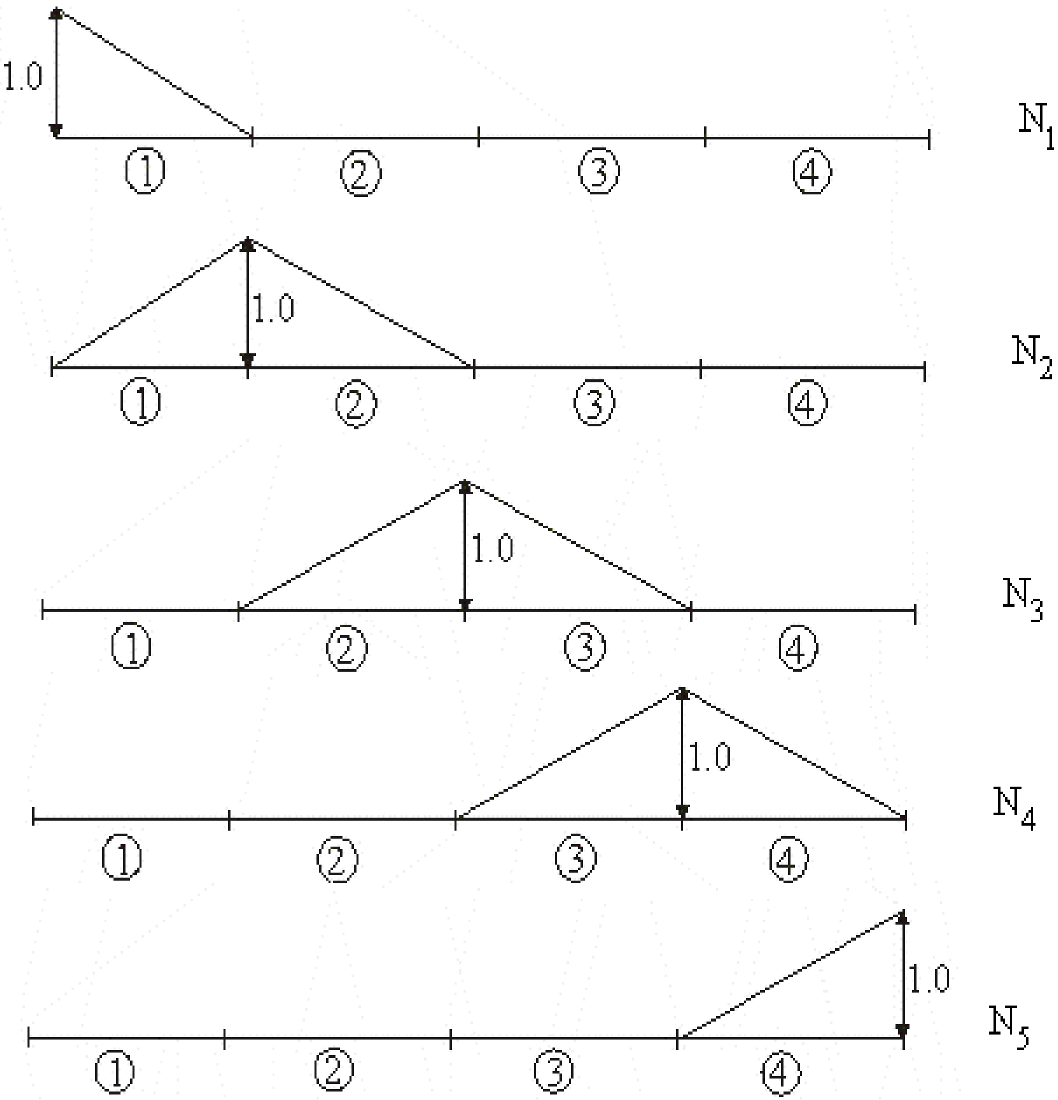


(a) Interpolation function within each sub-domain

(b) Compact representation of shape function

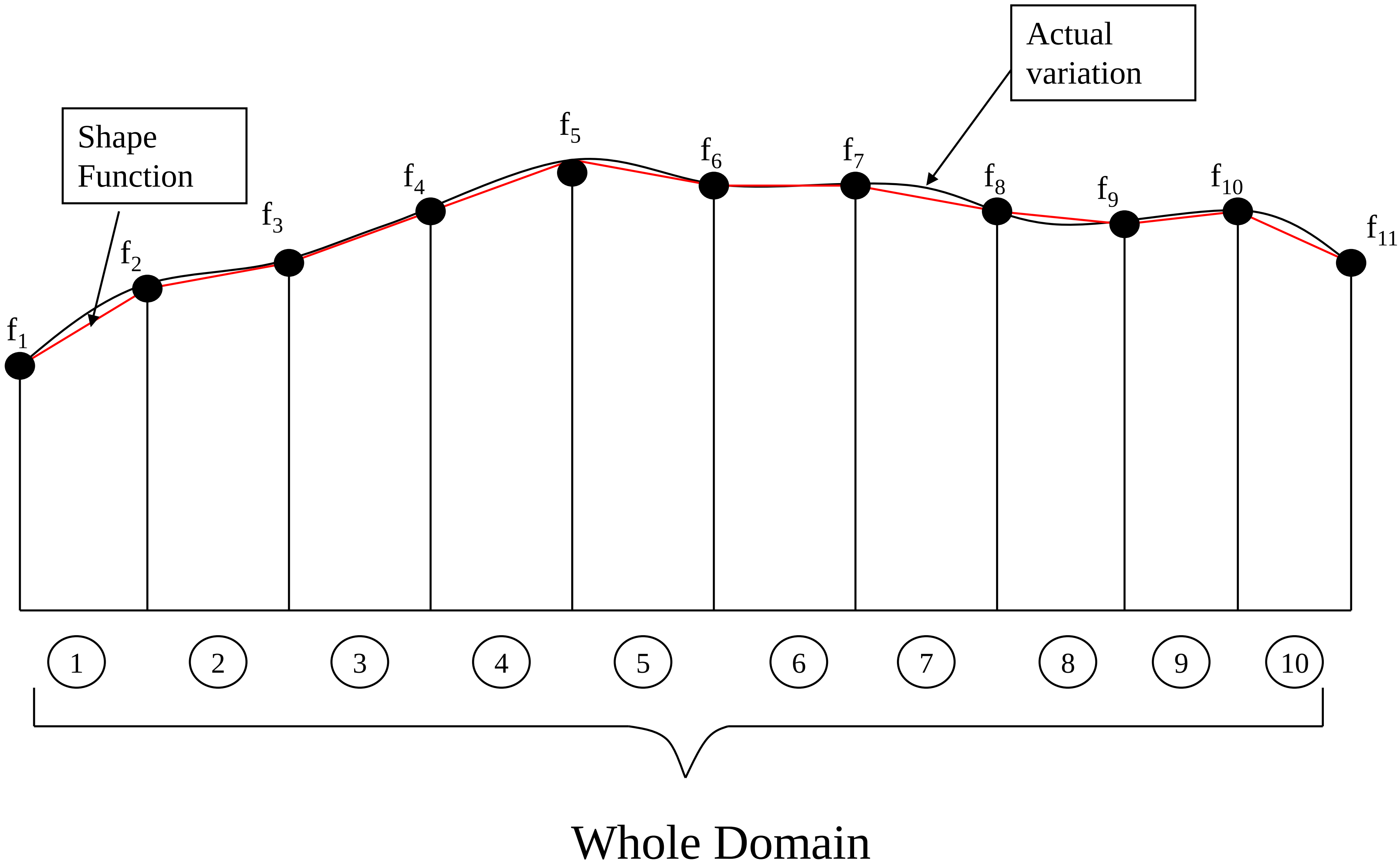
[1 – (x/ℓ)] and [x/ℓ] used in our interpolation are called "***interpolation functions***" *or* "***shape functions***"

Piecewise Approximation – Shape Functions

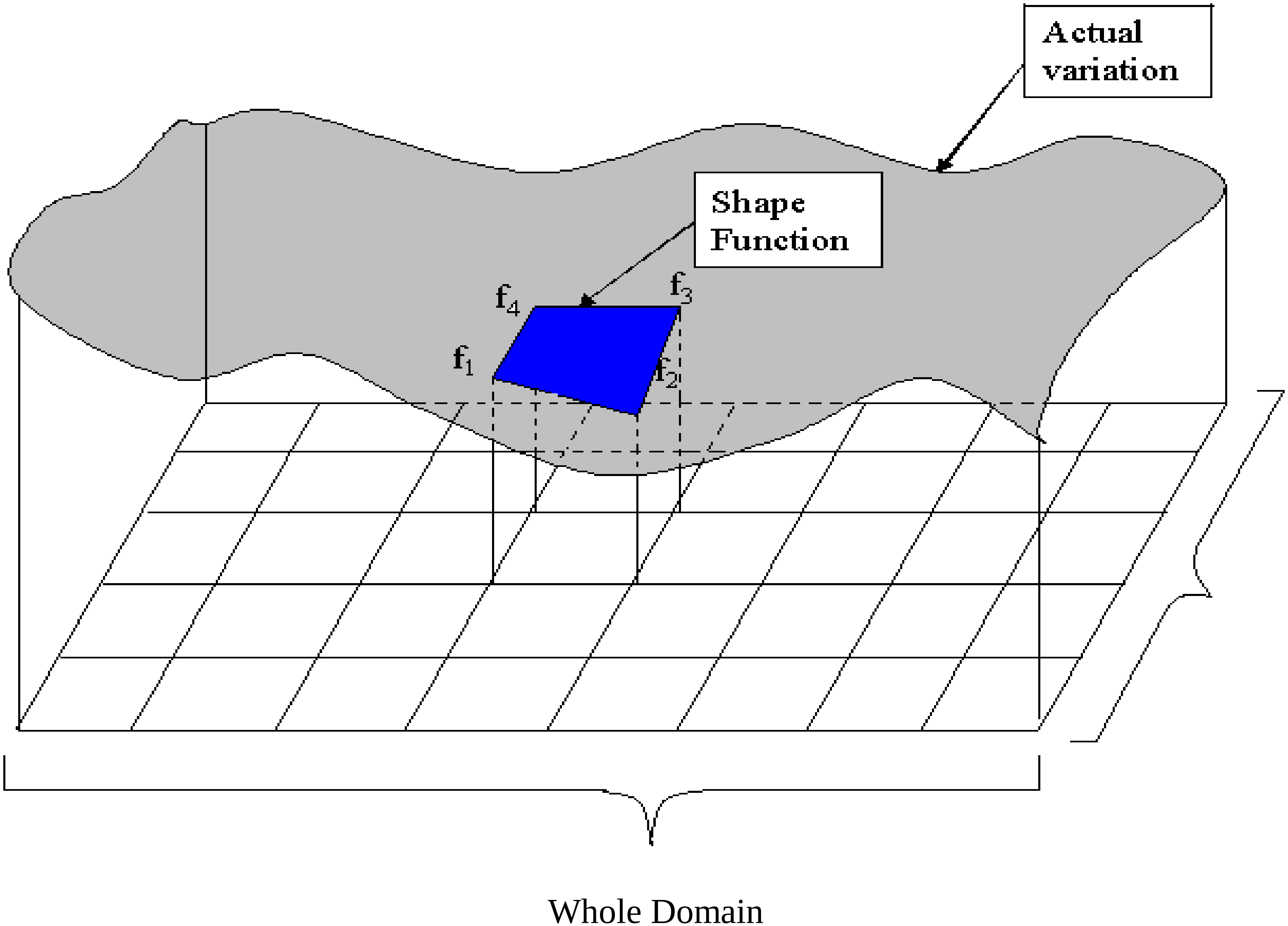


Contribution of a given *fk* to the value of the function at any point *P* within the domain $0<X<L$.

Piecewise Curve Fit – One Dimensional Case



Piecewise Curve Fit – Two Dimensional Case



Evaluation of Weighted Residual

$$\int_0^L W_i(X)\, R_d(X)\, dX = \sum_1^n \int_0^l W_i(x)\, R_d(x)\, dx$$

n == the number of segments/pieces

Thus we evaluate over each segment and then sum up

Shape functions over each segment are same/similar and hence calculations become repetitive – easily programmed

Segments can be of different length

Shape function need not be same for all segments

Essence of Finite element method

- Evaluate the sub-domain level contributions to the weighted residual by merely computing the integral $\int W_i(x)\, R_d(x)\, dx$ or its weak form, just once for the kth sub–domain

- Build–up the entire coefficient matrices $[A]$ & $\{b\}$ by appropriately placing these sub–domain level contributions in the appropriate rows and columns.

- Solve the $(n+1)$ algebraic equations to determine the unknowns viz., function values f_k at the ends of the sub–domains.

- This is the essence of the Finite element method.

Finite Element Method

- **Each of the sub–domains is called a** “finite element” **– to be distinguished from the** “differential element” **used in continuum mechanics.**

- **The ends of the sub–domain are referred to as the** “nodes” **of the element.**

- **Later on we see elements with nodes not necessarily located at only the ends e.g. an element can have mid–side nodes, internal nodes etc.**

- **The unknown function values *fk* at the ends of the sub–domains are known as the** “nodal degrees of freedom (d.o.f) ”.

Finite Element Formulation

- A general finite element can admit the function values as well as its derivatives as nodal d.o.f.

- The sub-domain level contributions to the weak form are typically referred to as "element level equations".

- The process of building-up the entire coefficient matrices [A] & {b} is known as the process of "assembly" i.e. assembling or appropriately placing the individual element equations to generate the system level equations.

Finite Element Formulation

Example

$$k\frac{d^2T}{dx^2} + q = \left(\frac{P}{A_c}\right) h(T - T_\infty)$$

The Weighted Residual statement can be written as follows :

$$\int_0^L W\left(k\frac{d^2T}{dx^2} + q - \left(\frac{P}{A_c}\right) h(T - T_\infty)\right) \mathrm{dx} = 0$$

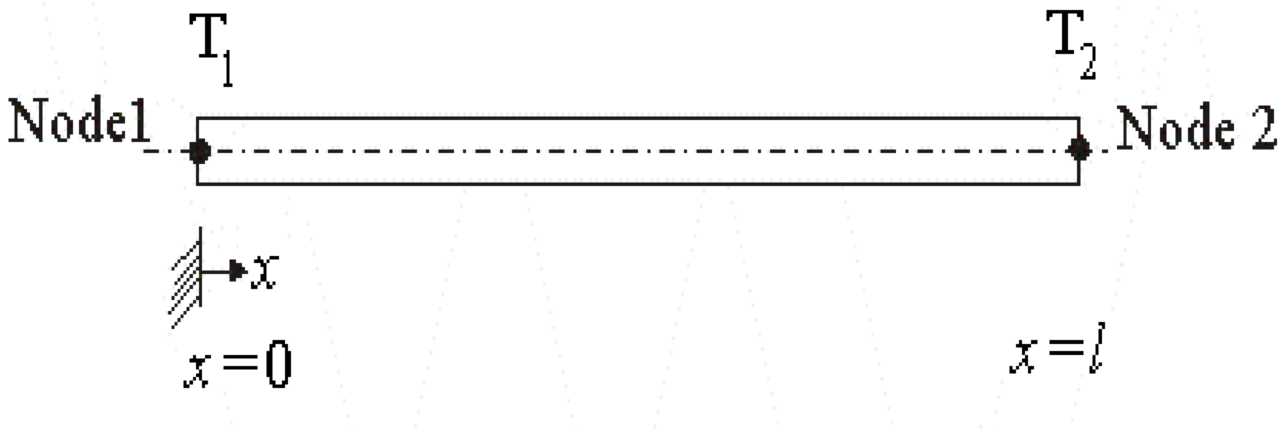
Finite Element Formulation

Weak form -- integration by parts

$$\left[Wk\frac{dT}{dx} \right]_0^L - \int_0^L k\frac{dW}{dx}\ \frac{dT}{dx}\ dx\ + \int_0^L Wq\ dx - \int_0^L W\left(\frac{P}{A_c}\right)\ h(T - T_\infty)\ dx = 0$$

$$\int_0^L k\frac{dW}{dx}\ \frac{dT}{dx}\ dx\ + \int_0^L W\left(\frac{P}{A_c}\right) hT\ dx = \int_0^L Wq\ dx\ + \int_0^L W\frac{P}{A_c}\ h(T_\infty)\ dx\ + \left[Wk\frac{dT}{dx} \right]_0^L$$

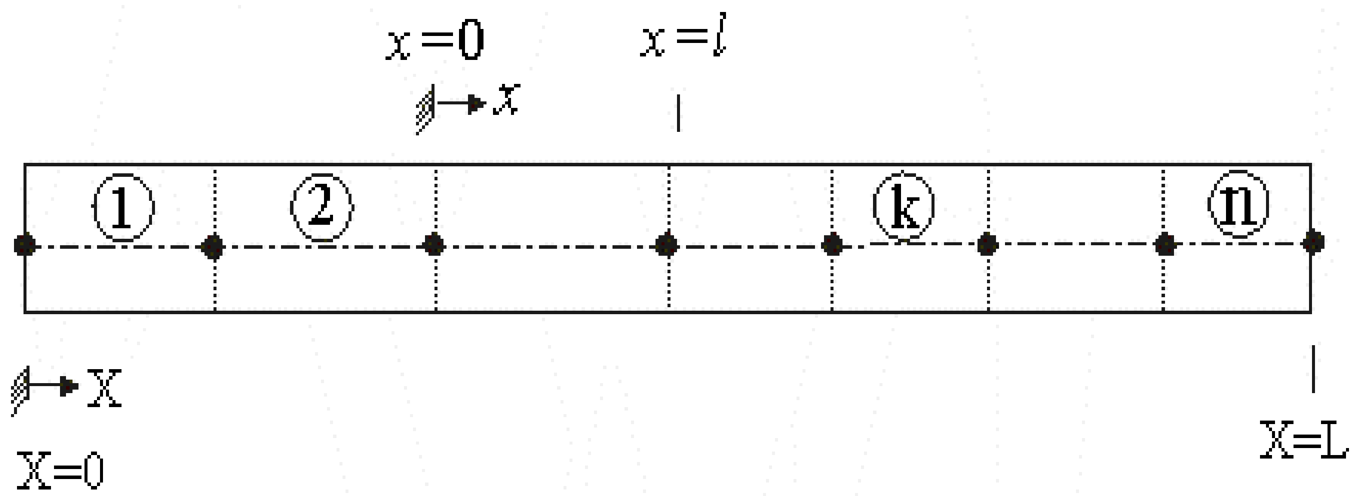
Finite Element Formulation



The weak form, for a typical mesh of "*n*" finite elements, can be written as

$$\sum_{k=1}^{n}\left[\int_0^{\ell} k\frac{dW}{dx}\;\frac{dT}{dx}\;dx\;+\int_0^{\ell} W\frac{P}{A_c}hT\;dx\right]=\sum_{k=1}^{n}\left[\int_0^{\ell} W\left(q+\frac{P}{A_c}hT_{\infty}\right)dx+\left[Wk\frac{dT}{dx}\right]_0^{\ell}\right]$$

Finite Element Formulation



$$T(x) = \left(1 - \frac{x}{\ell}\right) T_k + \left(\frac{x}{\ell}\right) T_{k+1}$$

$$W_1 = 1 - \frac{x}{\ell}, \quad \frac{dW_1}{dx} = -\frac{1}{\ell}$$

$$\frac{dT}{dx} = \frac{T_{k+1} - T_k}{\ell}$$

$$W_2 = \frac{x}{\ell}, \quad \frac{dW_2}{dx} = \frac{1}{\ell}$$

Finite Element Formulation

LHS 1st Term:

$$\frac{k}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_k \\ T_{k+1} \end{Bmatrix}$$

LHS 2nd Term: With ***W_1***,

$$\int_0^{\ell}\left(1-\frac{x}{\ell}\right)\left(\frac{P}{A_c}\right) h\left[\left(1-\frac{x}{\ell}\right) T_k + \left(\frac{x}{\ell}\right) T_{k+1}\right] dx = \frac{Ph\,\ell}{6A_c}\left[\,2\,T_k + T_{k+1}\right]$$

Finite Element Formulation

With W_2,

$$\int_0^{\ell}\left(\frac{x}{\ell}\right)\left(\frac{P}{A_c}\right) h\left[\left(1-\frac{x}{\ell}\right) T_k + \frac{x}{\ell} T_{k+1}\right] \mathrm{dx} = \frac{Ph\,\ell}{6A_c}\left[\; T_k + 2\,T_{k+1}\right]$$

Putting together and Rearranging LHS 2nd Term:

$$\frac{Ph\,\ell}{6A_c}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_k \\ T_{k+1} \end{Bmatrix}$$

Finite Element Formulation

RHS 1st term

$$\left(q_0 + \frac{P}{A_c} \; hT_\infty \right) \begin{Bmatrix} \ell/2 \\ \ell/2 \end{Bmatrix}$$

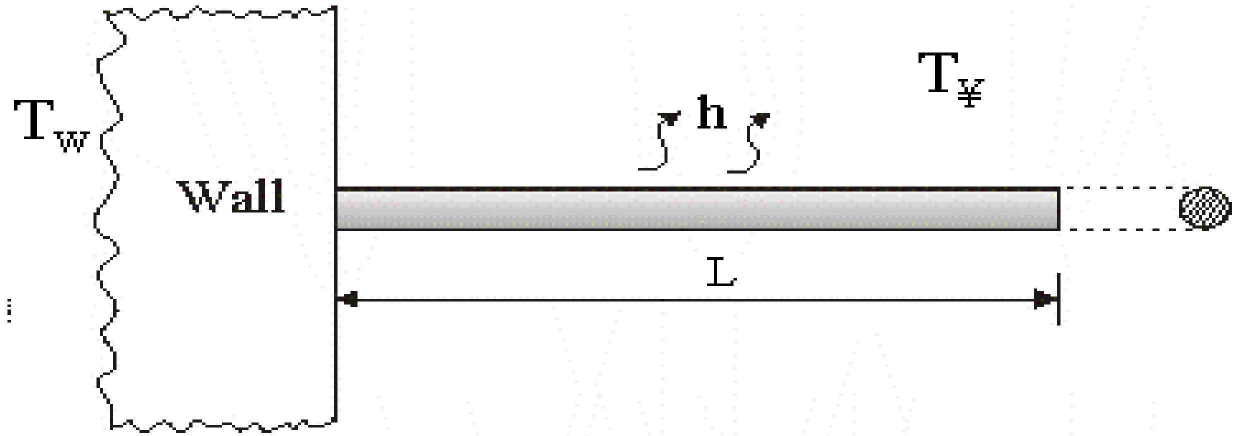
RHS 2nd term

$$\begin{Bmatrix} -Q_0 \\ Q_\ell \end{Bmatrix}$$

Element level equations

$$\left(\frac{k}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{Ph\ell}{6A_c} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_k \\ T_{k+1} \end{Bmatrix} = \left(q_0 + \frac{Ph}{A_c} \; T_\infty \right) \begin{Bmatrix} \ell/2 \\ \ell/2 \end{Bmatrix} + \begin{Bmatrix} -Q_0 \\ Q_\ell \end{Bmatrix}$$

Example: Temperature distribution in a pin–fin



One element solution

$$\left(\frac{200}{0.05}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{(\pi)\,(0.001)\,(20)\,(0.05)}{(6)\,(\pi)\,(0.0005)^2}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}$$

$$=\frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30)\begin{Bmatrix}0.025\\ 0.025\end{Bmatrix}+\begin{Bmatrix}Q_{wall}\\ Q_{tip}\end{Bmatrix}$$

From the boundary conditions, we have, $T1$ = 300⁰C, $Qtip$ = 0. Thus we get, $T2$ = 198.75 ⁰C.

Pin Fin Example

Two element solution

$$\left(\frac{200}{0.025} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{(\pi)\,(0.001)\,(20)\,(0.025)}{(6)\,(\pi)\,(0.0005)^2} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$= \frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30) \begin{Bmatrix} 0.0125 \\ 0.025 \\ 0.0125 \end{Bmatrix} + \begin{Bmatrix} Q_{wall} \\ 0 \\ Q_{tip} \end{Bmatrix}$$

Pin Fin Example

Four element solution

$$\left(\frac{200}{0.0125} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} + \frac{(\pi)\,(0.001)\,(20)\,(0.0125)}{(6)\,(\pi)\,(0.0005)^2} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

$$= \frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30) \begin{Bmatrix} 0.00625 \\ 0.0125 \\ 0.0125 \\ 0.0125 \\ 0.00625 \end{Bmatrix} + \begin{Bmatrix} Q_{wall} \\ 0 \\ 0 \\ 0 \\ Q_{tip} \end{Bmatrix}$$

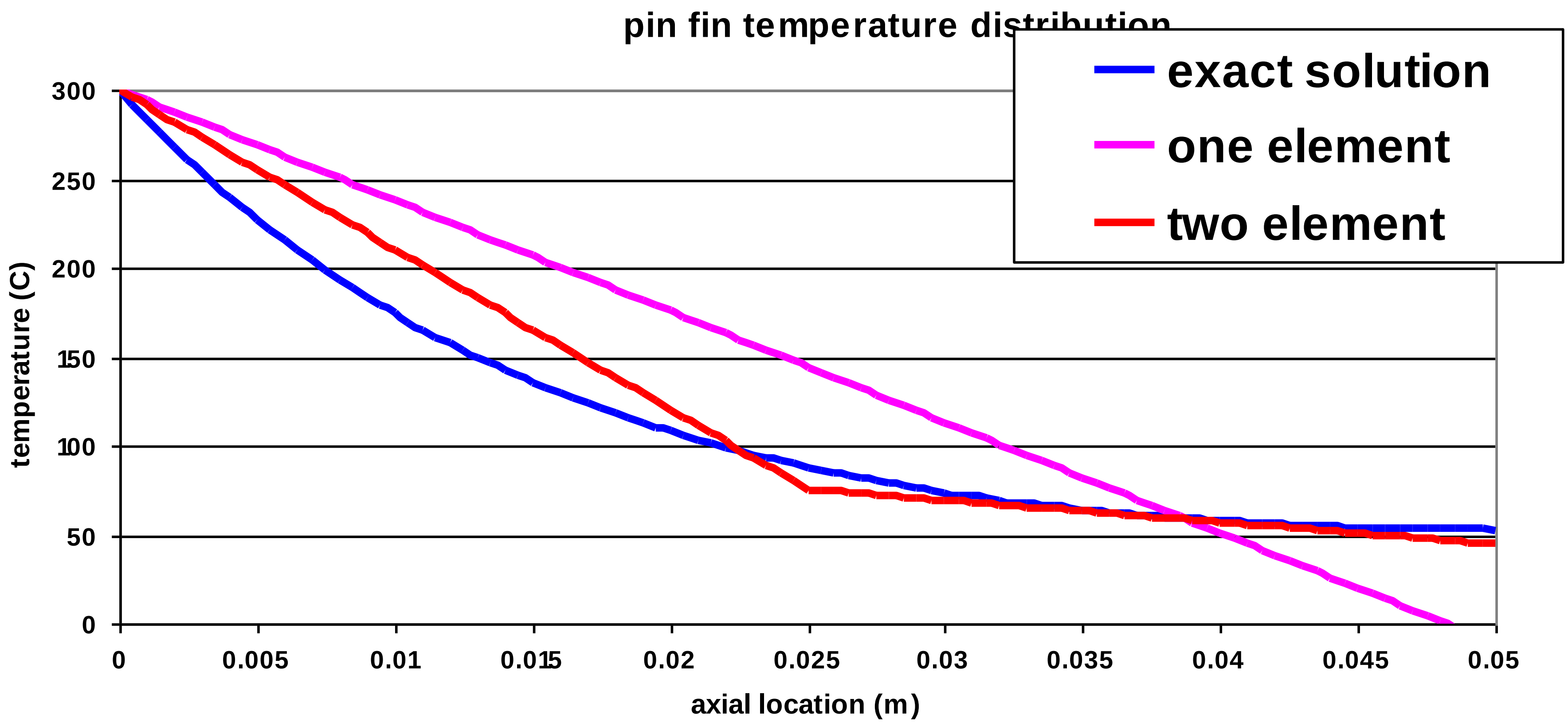
Pin Fin Example

Exact solution

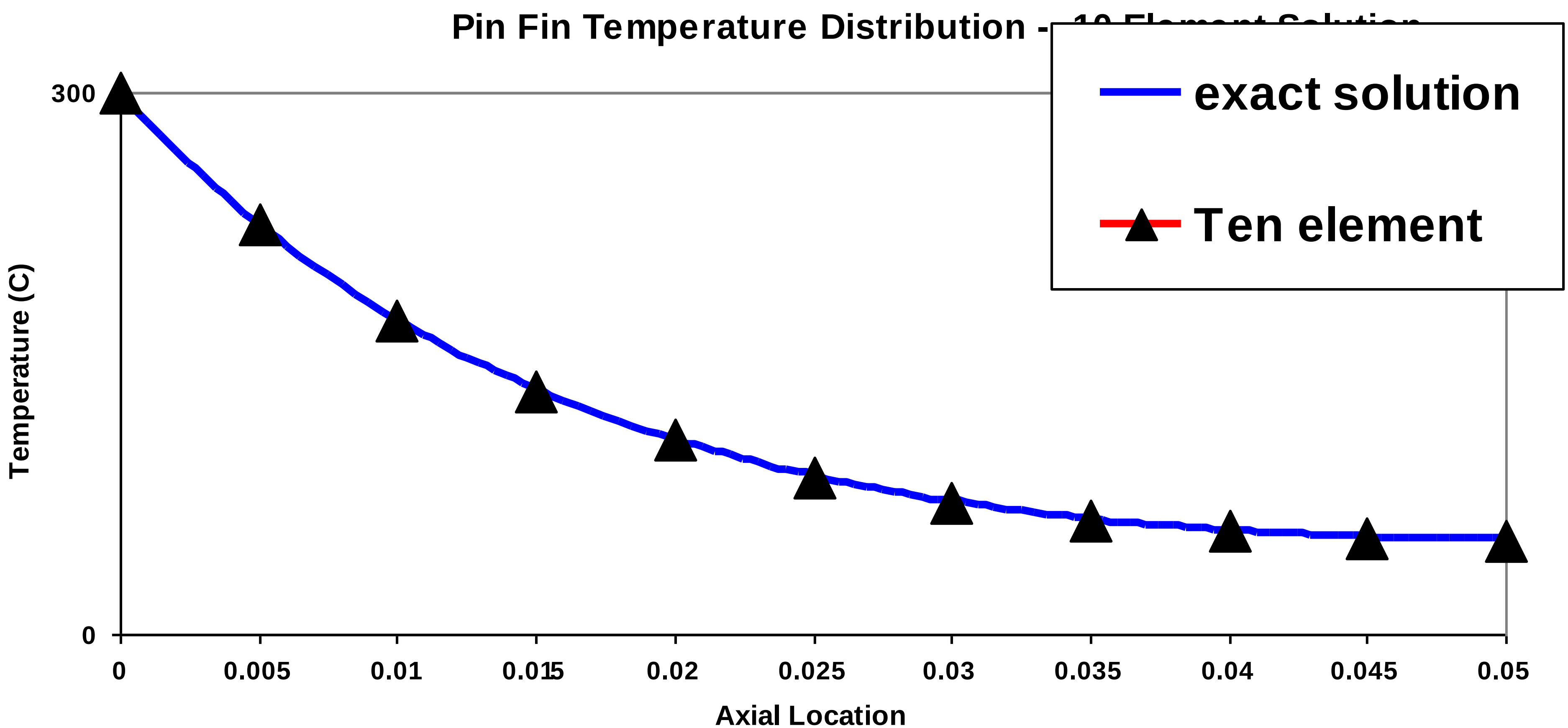
$$T(x) = T_\infty + (T_{wall} - T_\infty)\left[\frac{\cosh m(L-x)}{\cosh mL}\right]$$

$$m = \sqrt{\frac{hP}{k A_c}}$$

Pin Fin Example - Comparison of exact and Finite Element solution



Comparison of exact and 10 element solution



Finite Element Formulation

Procedure for FEA starting from a given DE:

- **Write down the Weighted Residual statement.**
- **Perform integration by parts for even distribution of differentiation between the field variable and the weighting function and develop the weak form of the W–R statement.**
- **Re-write the weak form as a summation over "*n*" elements.**
- **Define finite element i.e. geometry of the element, its nodes, nodal d.o.f.**

Finite Element Formulation

- Derive the shape or interpolation functions. Use these as the weighting functions also.
- Compute the element level equations by substituting these in the weak form.
- For a given topology of finite element mesh, build–up the system equations by assembling together element level equations.
- Substitute the prescribed boundary conditions and solve for the unknowns.