Finite Element Formulation

Procedure for FEA starting from a given DE:

- Write down the Weighted Residual statement.
- Perform integration by parts for even distribution of differentiation between the field variable and the weighting function and develop the weak form of the W–R statement.
- Re-write the weak form as a summation over "*n*" elements.
- Define finite element i.e. geometry of the element, its nodes, nodal d.o.f.

Finite Element Formulation

- Derive the shape or interpolation functions. Use these as the weighting functions also.
- Compute the element level equations by substituting these in the weak form.
- For a given topology of finite element mesh, build–up the system equations by assembling together element level equations.
- Substitute the prescribed boundary conditions and solve for the unknowns.

FE Formulation of Axial Force Element

Differential Equation and Boundary Conditions

$$AE\frac{d^2u}{dx^2} + q_0 = 0 \qquad and \qquad u(0) = u_0 \quad AE\left.\frac{du}{dx}\right|_L = P_L$$

Weighted Residual

$$\int_0^L W(X)\left[AE\frac{d^2u}{dx^2} + q_0\right]dX = 0 \quad and \quad u(0) = u_0 \quad AE\left.\frac{du}{dx}\right|_L = P_L$$

Weak Form

$$\int_0^L AE\frac{du}{dX}\frac{dW}{dX}\,dX = \int_0^L W(X)\,q_0\,dX + \left[W(X)P\right]_0^L \quad and \quad u(0) = u_0; W(0) = 0$$

Weak Form as summation over "n" elements

$$\sum\int_0^\ell AE\frac{du}{dx}\frac{dW}{dx}\,dx = \sum\int_0^\ell W(x)\,q_0\,dx + \sum\left[W(x)P\right]_0^\ell$$

For a linear element,

$$u(x) = \left(1 - \frac{x}{\ell}\right) u_k + \left(\frac{x}{\ell}\right) u_{k+1}$$

$$W(x) = \left(1 - \frac{x}{\ell}\right) \quad and \quad \left(\frac{x}{\ell}\right)$$

Element Level Equations

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_k \\ u_{k+1} \end{Bmatrix} = q_0 \begin{Bmatrix} \dfrac{\ell}{2} \\ \dfrac{\ell}{2} \end{Bmatrix} + \begin{Bmatrix} -P_0 \\ P_\ell \end{Bmatrix}$$

Force – Deflection Equations

LHS is Stiffness Matrix

RHS is Nodal Force Vector

Meaning of FE Equations

$$\frac{AE}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

First column of LHS matrix is the force vector needed to cause a deformation pattern $u_1 = 1$ and $u_2 = 0$.

Second column of LHS matrix is the force vector needed to cause a deformation pattern $u_1 = 0$ and $u_2 = 1$.

Meaning of finite element equations

• element level equations, $[K]\,\{u\} = \{F\}$

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

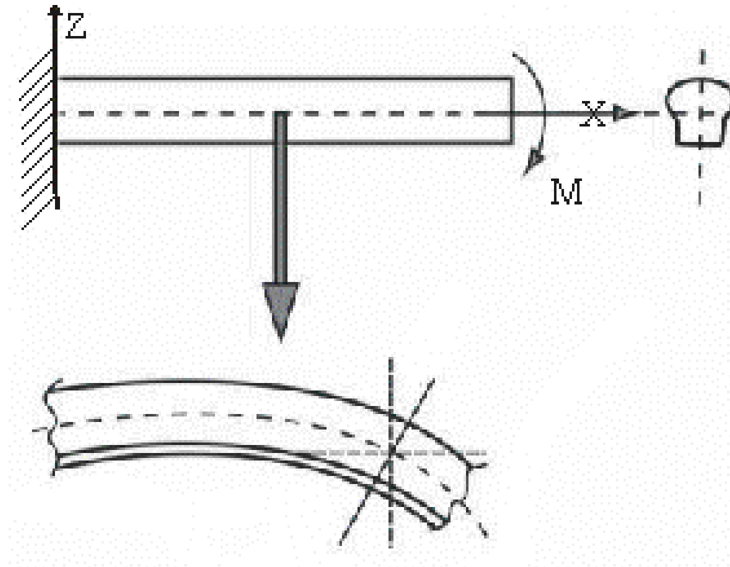
• "*elements of each column of a stiffness matrix actually represent the forces required to cause a certain deformation pattern*"

• *i*th column of the stiffness matrix shows a deformation pattern wherein the *i*th d.o.f. is given unit displacement (translational or rotational) and all other d.o.f. are held zero

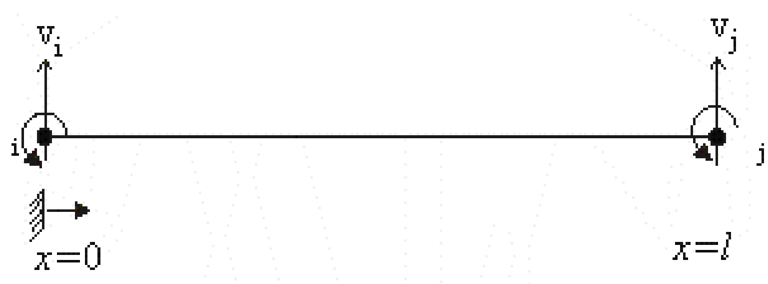
• This is called the direct method of formulation for FE equations

Beam Element

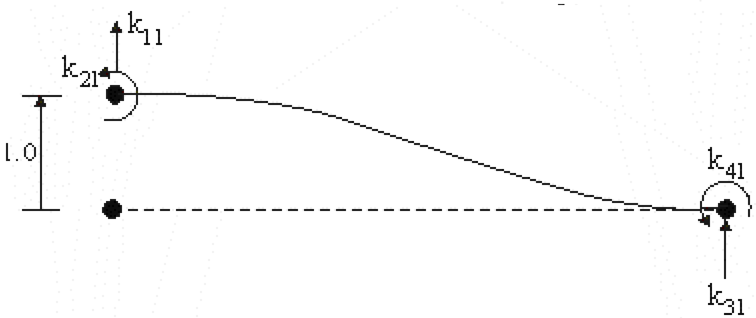
- Euler-Bernoulli beam theory
- c/s has the same transverse deflection as the neutral axis
- sections perpendicular to the neutral axis remain so after bending
- axial deformation, $u_P = -(z)\ (dv/dx)$



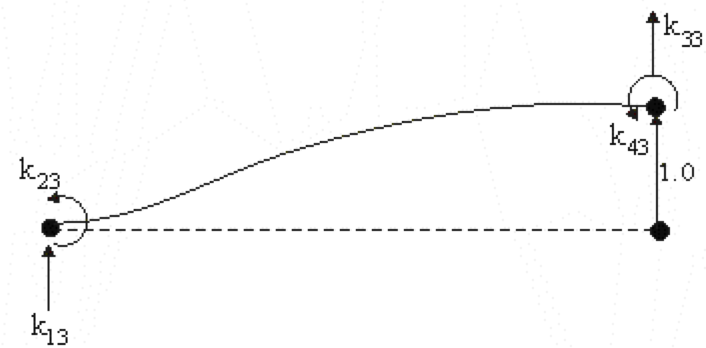
- Beam element – line element representing neutral axis
- To ensure continuity of deformation at any point – use v and dv/dx as the nodal dof



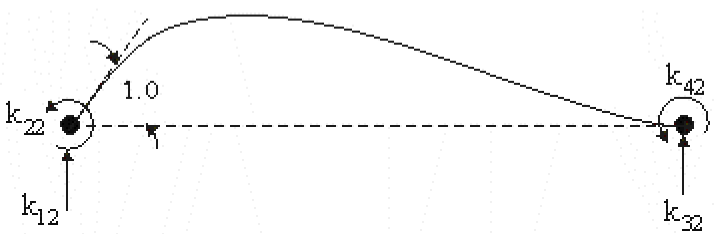
Direct method for beam element



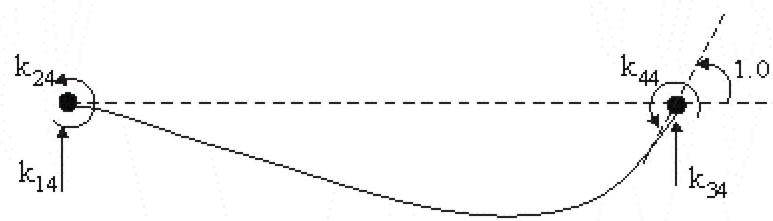
v_1=1, $v_2 = \theta_1 = \theta_2 = 0$



v_1=0, v_2 =1, $\theta_1 = \theta_2 = 0$



$v_1 = v_2$=0, θ_1 = 1, θ_2 = 0



$v_1 = v_2$=0, θ_1 = 0, θ_2 = 1

Direct method for beam element

- element level equations,

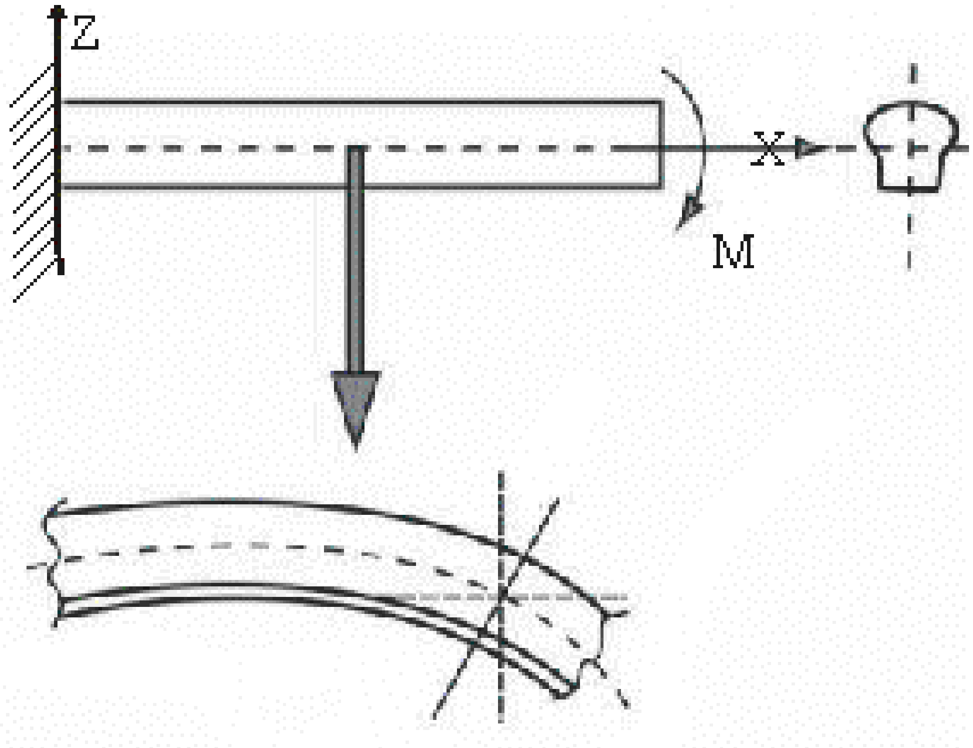
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

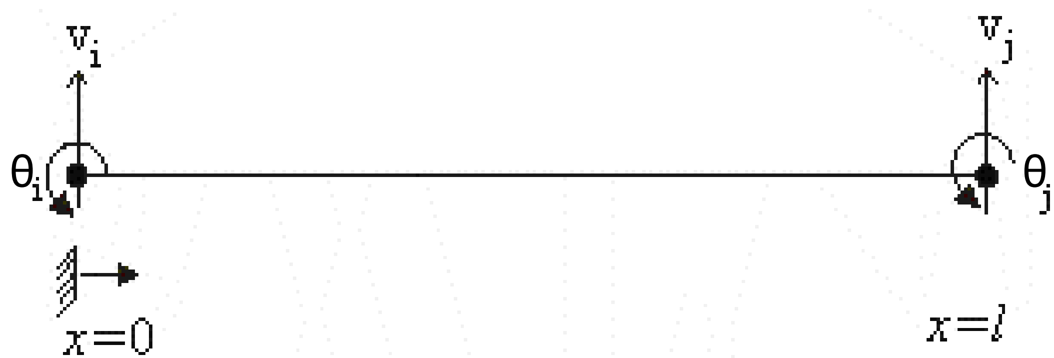
- Using direct method,

$$[\mathrm{k}] = \begin{bmatrix} \frac{12EI}{\ell^3} & & & Sym. \\ \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & & \\ \frac{-12EI}{\ell^3} & \frac{-6EI}{\ell^2} & \frac{12EI}{\ell^3} & \\ \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & \frac{-6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix}$$

- displacement field, $v(x) = N_1 v_i + N_2 \theta_i + N_3 v_j + N_4 \theta_j$

BEAM BENDING ELEMENT





BEAM ELEMENT – Fourth Order D.E

$$EI\frac{d^4v}{dX^4} - q(X) = 0$$

$$v(0) = 0 \qquad \frac{d^2v}{dX^2}(0) = 0$$

$$v(L) = 0 \qquad \frac{d^2v}{dX^2}(L) = 0$$

Weighted residual FE formulation

WEIGHTED RESIDUAL STATEMENT

$$\int_0^L W(X)\left[EI\,\frac{d^4v}{dX^4}-q(X)\right]dX=0$$

$Develop\ Weak\ Form$

$Define\ and\ Develop\ "Beam"\ Element$

Weak Form – Integration by Parts

$$\int_0^L W(X)\, EI\,\frac{d^4v}{dX^4}dX = \int_0^L W(X)\, d\left[EI\,\frac{d^3v}{dX^3}\right]$$

$$= \left[W(X)\left[EI\,\frac{d^3v}{dX^3}\right]\right]_0^L - \int_0^L \left[EI\,\frac{d^3v}{dX^3}\right]\frac{dW}{dX}\,dX$$

$$= \left[W(X)\left[EI\,\frac{d^3v}{dX^3}\right]\right]_0^L - \int_0^L \frac{dW}{dX}\,d\left[EI\,\frac{d^2v}{dX^2}\right]$$

$$= \left[W(X)\left[EI\,\frac{d^3v}{dX^3}\right]\right]_0^L - \left[\left[\frac{dW}{dX}\left[EI\,\frac{d^2v}{dX^2}\right]\right]_0^L - \int_0^L \frac{d^2W}{dX^2}\left[EI\,\frac{d^2v}{dX^2}\right]dX\right]$$

From the boundary conditions,

v (0) = v(L) = 0 and also,

$$\frac{d^2v}{dX^2} = 0 \; at \;\; X = 0, L$$

Weak form reduces to:

$$\left[\int_0^L \frac{d^2W}{dX^2} \left[EI \frac{d^2v}{dX^2} \right] dX \right] = \int_0^L W(X)\, q(X)\, dX$$

Weak form as a summation over "n" elements

$$\sum_{1}^{n} \left[\int_{0}^{l} \frac{d^2W}{dx^2} \left[EI \frac{d^2v}{dx^2} \right] dx \right] = \sum_{1}^{n} \int_{0}^{l} W(x)\, q(x)\, dx$$

Beam Element

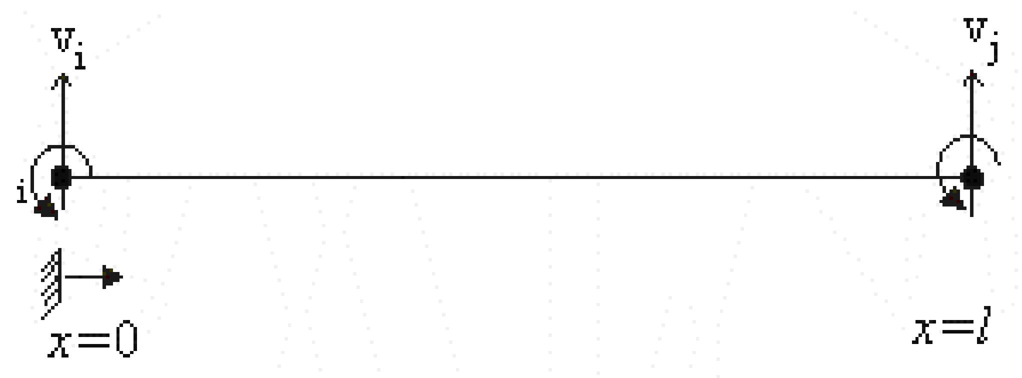
$v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$dv/dx = c_1 + 2\ c_2 x + 3\ c_3 x^2$

At $x = 0$ and l, we have,

$v_i = c_0$, $\theta_i = c_1$,

$v_j = c_0 + c_1 l + c_2 l^2 + c_3 l^3$, $\theta_j = c_1 + 2c_2 l + 3c_3 l^2$



$$v(x) = N_1 v_i + N_2 \theta_i + N_3 v_j + N_4 \theta_j$$

$N_1 = 1 - 3x^2/l^2 + 2x^3/l^3$, $N_2 = x - 2x^2/l + x^3/l^2$

$N_3 = 3x^2/l^2 - 2x^3/l^3$, $N_4 = -x^2/l + x^3/l^2$

$W_1 = N_1$

$W_2 = N_2$

$W_3 = N_3$

$W_4 = N_4$

Beam element equations

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

$$[\mathrm{k}] = \begin{bmatrix} \frac{12EI}{\ell^3} & & & Sym. \\ \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & & \\ \frac{-12EI}{\ell^3} & \frac{-6EI}{\ell^2} & \frac{12EI}{\ell^3} & \\ \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & \frac{-6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix}$$

Consistent vs lumped load

- uniform load q_0 on beam element,

$$\{f\}^e = \int_0^{\ell} [N]^T q_0 \; dx = \begin{Bmatrix} q_0\ell/2 \\ q_0\ell^2/12 \\ q_0\ell/2 \\ -\; q_0\ell^2/12 \end{Bmatrix}$$

- **consistent load** – equivalent to the distributed force

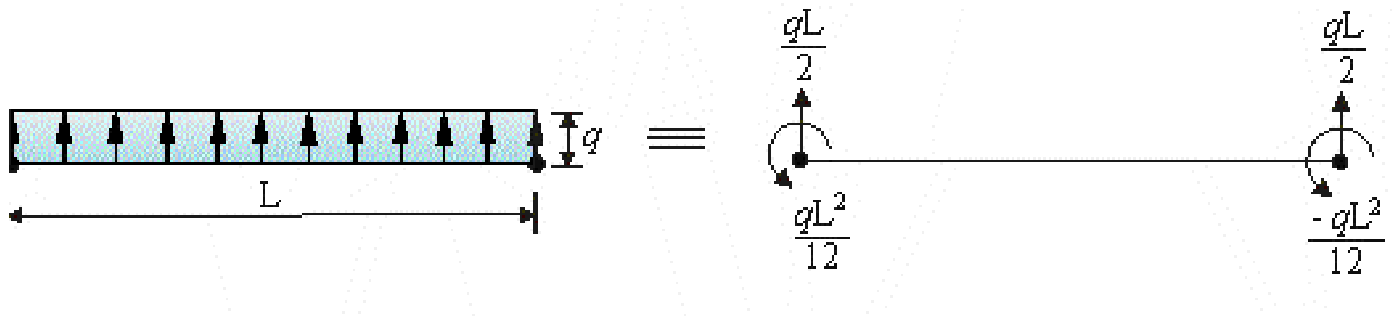
  – nodal forces acting through the nodal displacements do the same amount of work as the distributed force

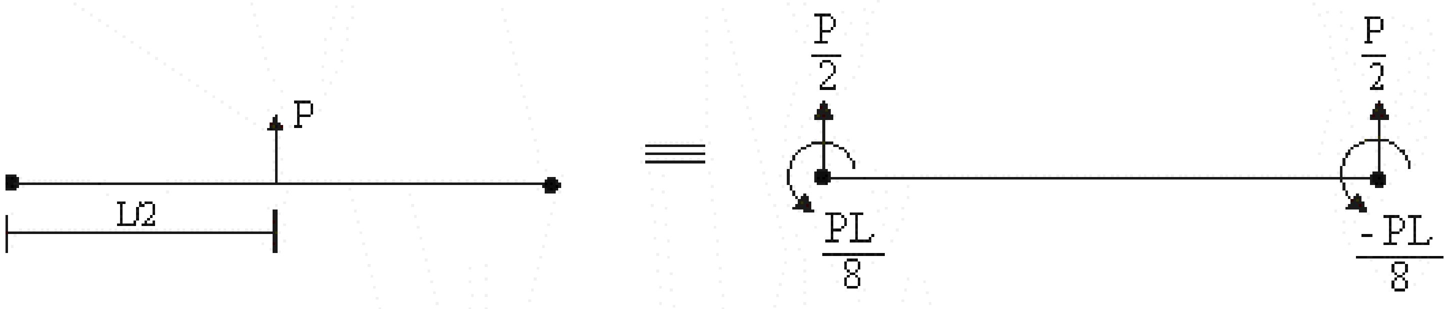
- **lumped load** – lump half of the total load on each node

  – do not consider moment load

$$\{f\}^e = \begin{Bmatrix} q_0\ell/2 \\ 0 \\ q_0\ell/2 \\ 0 \end{Bmatrix}$$

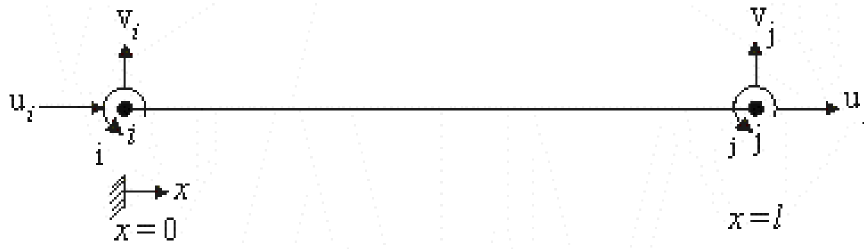
Consistent loads





Frame element

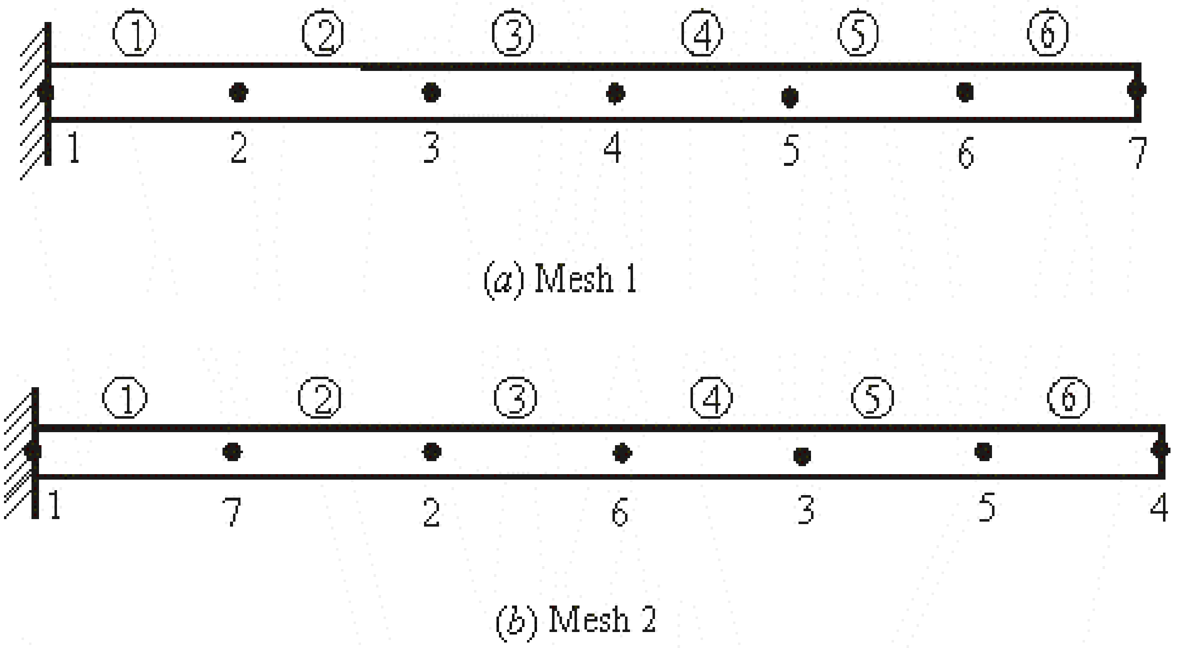
• combination of bar and beam element



$$\{\delta\}^e = \begin{Bmatrix} u_i & v_i & \theta_i & u_j & v_i & \theta_j \end{Bmatrix}^T$$

$$[k]^e = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12AE/L^3 & 6AE/L^2 & 0 & -12AE/L^3 & 6AE/L^2 \\ 0 & 6AE/L^2 & 4AE/L & 0 & -6AE/L^2 & 2AE/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12AE/L^3 & -6AE/L^2 & 0 & 12AE/L^3 & -6AE/L^2 \\ 0 & 6AE/L^2 & 2AE/L & 0 & -6AE/L^2 & 4AE/L \end{bmatrix}$$

Effect of node numbering on assembled matrix equation



- Element stiffness matrix

$$[k]^e = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembly of Element Matrices

$$[k]^e = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{array}{lcc} \text{Global} & [1] & [2] \\ \text{Local} & [1] & [2] \end{array}$$

$$\begin{array}{cccc} 1 & -1 & [1] & [1] \\ -1 & 1 & [2] & [2] \end{array}$$

$$\begin{array}{lcc} \text{Global} & [2] & [3] \\ \text{Local} & [1] & [2] \end{array}$$

$$\begin{array}{cccc} 1 & -1 & [1] & [2] \\ -1 & 1 & [2] & [3] \end{array}$$

$$\begin{array}{cc} [1] & [7] \\ [1] & [2] \end{array}$$

$$\begin{array}{cccc} 1 & -1 & [1] & [1] \\ -1 & 1 & [2] & [7] \end{array}$$

$$\begin{array}{cc} [2] & [6] \\ [1] & [2] \end{array}$$

$$\begin{array}{cccc} 1 & -1 & [1] & [2] \\ -1 & 1 & [2] & [6] \end{array}$$

Effect of node numbering on assembled matrix equation

- Assembled stiffness matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Mesh 1 (banded matrix) Mesh 1 (scattered matrix)

- Stiffness matrix is symmetric
- Efficient node numbering scheme reduces computational expense
- No need to store non-zero matrix coefficients
- Commercial softwares automatically apply efficient node numbering scheme and matrix storage schemes

Effect of Node Numbering on System Matrices

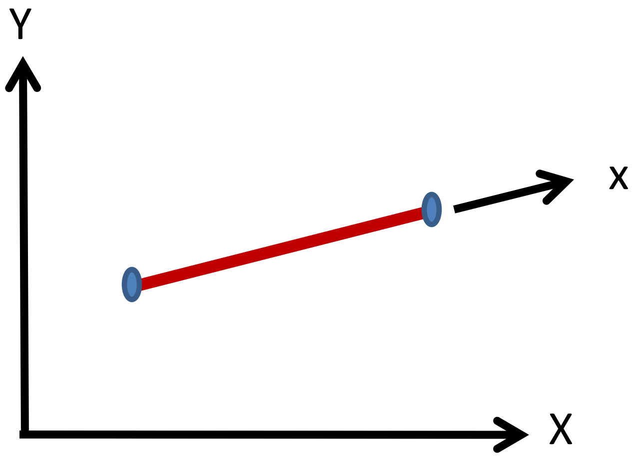
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Assembly – nodal dof are vectors!

- Temperatures are scalar quantities
- So direction independent
- Element matrices or equations wont change if coordinate frame is changed
- Such scalar problems – element matrices can be directly assembled
- In some problems eg. Structural displacements are vectors and hence direction dependent
- Need to transform all of the dof into one common global reference coordinate frame



$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

$$u_i = u_{iX} \cos\,\theta + u_{iY} \sin\,\theta$$

$$u_j = u_{jX} \cos\,\theta + u_{jY} \sin\,\theta$$

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_{iX} \\ u_{iY} \\ u_{jX} \\ u_{jY} \end{Bmatrix}$$

$$\{\delta\}_\ell^e = [T]\{\delta\}_g^e$$

$$[k]_\ell^e \{\delta\}_\ell^e = \{F\}_\ell^e \qquad [k]_\ell^e [T]\{\delta\}_g^e = \{F\}_\ell^e$$

$$\left([T]^T [k]_\ell^e [T] \right) \{\delta\}_g^e = [T]^T \{F\}_\ell^e$$

$$[T]^T \{F\}_\ell^e = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \begin{Bmatrix} F_i \cos\theta \\ F_i \sin\theta \\ F_j \cos\theta \\ F_j \sin\theta \end{Bmatrix}$$

$$[k]_g^e \{\delta\}_g^e = \{F\}_g^e$$

$$[k]_g^e = [T]^T [k]_\ell^e [T] \qquad [T]^T \{F\}_\ell^e = \{F\}_g^e$$

$$[k]_\ell^e = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{array}{lcccc} Global & (1X) & (1Y) & (3X) & (3Y) \\ Local & (iX) & (iY) & (jX) & (jY) \end{array}$$

$$[k]_g^{(1)} = \frac{AE\cos\alpha}{L}\begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha & -\cos^2\alpha & -\sin\alpha\cos\alpha \\ & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ & & \cos^2\alpha & \sin\alpha\cos\alpha \\ Sym. & & & \sin^2\alpha \end{bmatrix}\begin{array}{cc} Local & Global \\ (iX) & (1X) \\ (iY) & (1Y) \\ (jX) & (3X) \\ (jY) & (3Y) \end{array}$$

Summary : General Method of Assembly

Step 1: Identify the local (i.e. element level) and global (i.e. structure level) coordinate frames of reference.

Step 2: Obtain the element matrices in the local reference frame.

Step 3: Obtain the coordinate transformation matrix $[T]$ between the local and global frames of reference.

Step 4: Transform the element matrices into the common global reference frame.

Step 5: Identify the global locations of the individual coefficients of the element matrices based on local and global node numbers.

Step 6: Assemble the element matrices by placing the coefficients of the element matrices in their appropriate places as identified in Step 5.