Finite Element Formulation starting from a governing differential equation

Example

$$k\frac{d^2T}{dx^2} + q = \left(\frac{P}{A_c}\right) h(T - T_\infty)$$

The Weighted Residual statement can be written as follows :

$$\int_0^L W\left(k\frac{d^2T}{dx^2} + q - \left(\frac{P}{A_c}\right) h(T - T_\infty)\right) \mathrm{dx} = 0$$

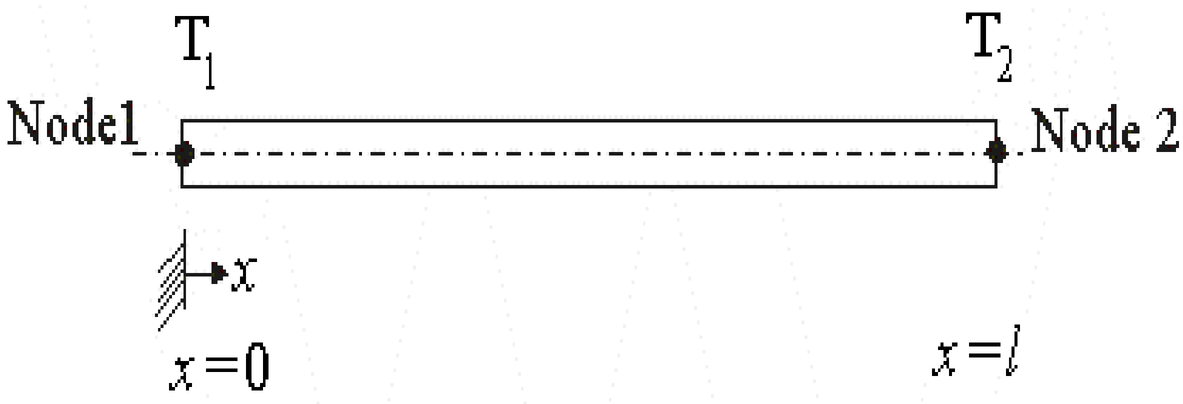
Finite Element Formulation

Weak form -- integration by parts

$$\left[Wk\frac{dT}{dx} \right]_0^L - \int_0^L k\frac{dW}{dx}\ \frac{dT}{dx}\ dx\ + \int_0^L Wq\ dx - \int_0^L W\left(\frac{P}{A_c} \right) h(T - T_\infty)\ dx = 0$$

$$\int_0^L k\frac{dW}{dx}\ \frac{dT}{dx}\ dx\ + \int_0^L W\left(\frac{P}{A_c} \right) hT\ dx = \int_0^L Wq\ dx\ + \int_0^L W\frac{P}{A_c}\ h(T_\infty)\ dx\ + \left[Wk\frac{dT}{dx} \right]_0^L$$

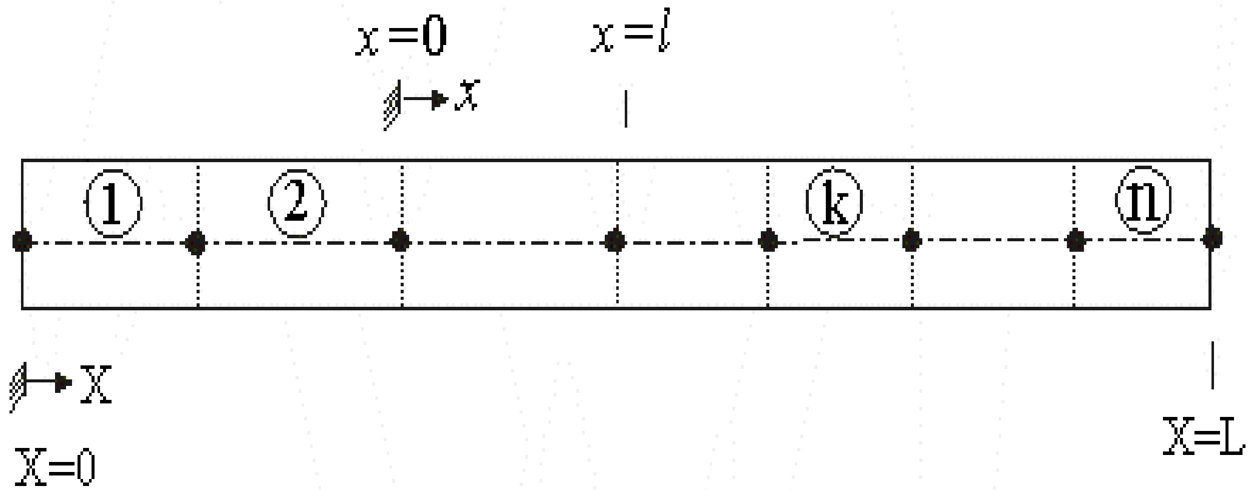
Finite Element Formulation



The weak form, for a typical mesh of "*n*" finite elements, can be written as

$$\sum_{k=1}^{n}\left[\int_0^{\ell} k\frac{dW}{dx}\frac{dT}{dx}\,dx+\int_0^{\ell} W\frac{P}{A_c}hT\,dx\right]=\sum_{k=1}^{n}\left[\int_0^{\ell} W\left(q+\frac{P}{A_c}hT_{\infty}\right)dx+\left[Wk\frac{dT}{dx}\right]_0^{\ell}\right]$$

Finite Element Formulation



$$T(x) = \left(1 - \frac{x}{\ell}\right)T_k + \left(\frac{x}{\ell}\right)T_{k+1}$$

$$W_1 = 1 - \frac{x}{\ell}, \quad \frac{dW_1}{dx} = -\frac{1}{\ell}$$

$$\frac{dT}{dx} = \frac{T_{k+1} - T_k}{\ell}$$

$$W_2 = \frac{x}{\ell}, \quad \frac{dW_2}{dx} = \frac{1}{\ell}$$

Finite Element Formulation

LHS 1st Term:

$$\frac{k}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_k \\ T_{k+1} \end{Bmatrix}$$

LHS 2nd Term: With W_1,

$$\int_0^\ell \left(1-\frac{x}{\ell}\right)\left(\frac{P}{A_c}\right) h\left[\left(1-\frac{x}{\ell}\right) T_k + \left(\frac{x}{\ell}\right) T_{k+1}\right] dx \quad = \quad \frac{Ph\,\ell}{6A_c}\left[\,2\,T_k + T_{k+1}\right]$$

Finite Element Formulation

With W_2,

$$\int_0^{\ell}\left(\frac{x}{\ell}\right)\left(\frac{P}{A_c}\right) h\left[\left(1-\frac{x}{\ell}\right) T_k + \frac{x}{\ell} T_{k+1}\right] \mathrm{dx} = \frac{Ph\,\ell}{6A_c}\left[\,T_k + 2\,T_{k+1}\right]$$

Putting together and Rearranging LHS 2nd Term:

$$\frac{Ph\,\ell}{6A_c}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_k \\ T_{k+1} \end{Bmatrix}$$

Finite Element Formulation

RHS 1st term

$$\left(q_0 + \frac{P}{A_c} \; hT_\infty \right) \begin{Bmatrix} \ell/2 \\ \ell/2 \end{Bmatrix}$$

RHS 2nd term

$$\begin{Bmatrix} -Q_0 \\ Q_\ell \end{Bmatrix}$$

Details of RHS 2nd Term

$$k\frac{dT}{dx} = Q$$

$$W_1 = 1 - \frac{x}{\ell} \qquad at \; x = 0, W_1 = 1; \qquad at \; x = l, W_1 = 0$$

$$\left[W_1 Q \right]_0^\ell = W_1 Q \big|_l - W_1 Q \big|_0 = 0 - Q_0 = -Q_0$$

$$Similarly$$

$$\left[W_2 Q \right]_0^\ell = W_2 Q \big|_l - W_2 Q \big|_0 = Q_\ell - 0 = Q_\ell$$

Finite Element Formulation

Putting together all the LHS and RHS terms,

Element level equations are obtained as:

$$\left(\frac{k}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{Ph\ell}{6A_c}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_k\\ T_{k+1}\end{Bmatrix} = \left(q_0 + \frac{Ph}{A_c}T_\infty\right)\begin{Bmatrix}\ell/2\\ \ell/2\end{Bmatrix} + \begin{Bmatrix}-Q_0\\ Q_\ell\end{Bmatrix}$$

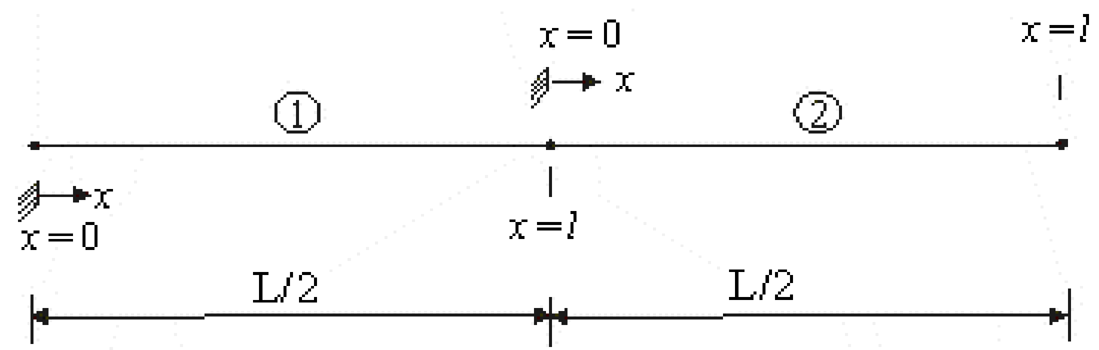
If we use only one element, this can be directly applied with boundary conditions.

If we use many elements, how do we "assemble" them together?

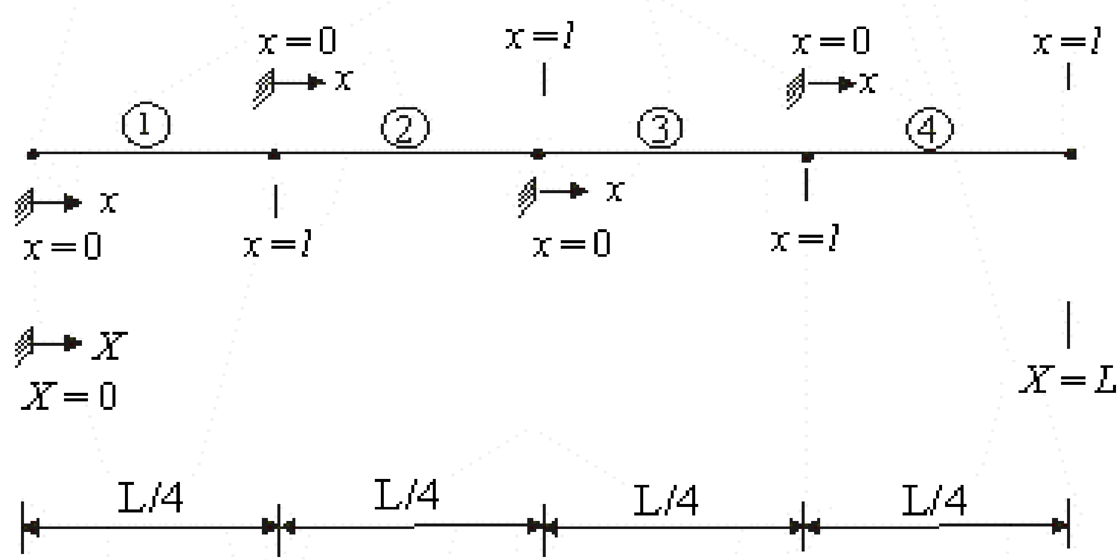
Interpretation of element level equations

- These are equations relating heat flux to temperature.
- Original differential equation is valid at each interior point i.e at every point - heat flowing in is equal to heat flowing out
- The present finite element algebraic equation is valid ONLY at the nodes ie energy balance is applied ONLY at nodes
- LHS can be called the element level conductance matrix. RHS first term is the heat flux at nodes “equivalent” to distributed heat flux over the whole element.
- RHS second term is the algebraic sum of any external heat flux applied at that node as well as internal heat flux that got “exposed” when we divided the whole domain into elements

Piecewise Approximation - Use of local coordinate frames



We define a local coordinate *x* with the origin fixed at the left end of each sub–domain.



Piecewise Approximation

- Two line segment approximation (with $\ell = L/2 = 1/2$)

$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0) + [\, x/\ell\,]\, f(0.5) \qquad (0 < x < \ell)$

$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0.5) + [\, x/\ell\,]\, f(1) \qquad (0 < x < \ell)$

- Four line segment approximation (with $\ell = L/4 = 1/4$)

$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0) + [\, x/\ell\,]\, f(0.25) \qquad (0 < x < \ell)$

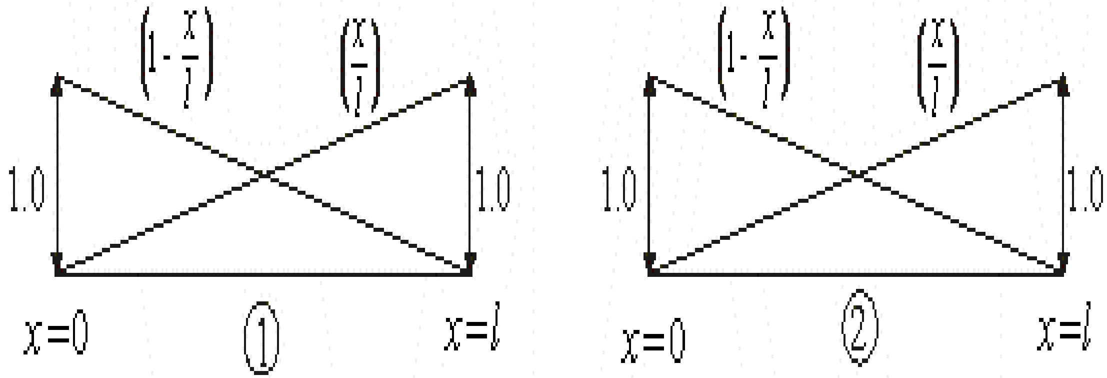
$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0.25) + [\, x/\ell\,]\, f(0.5) \qquad (0 < x < \ell)$

$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0.5) + [\, x/\ell\,]\, f(0.75) \qquad (0 < x < \ell)$

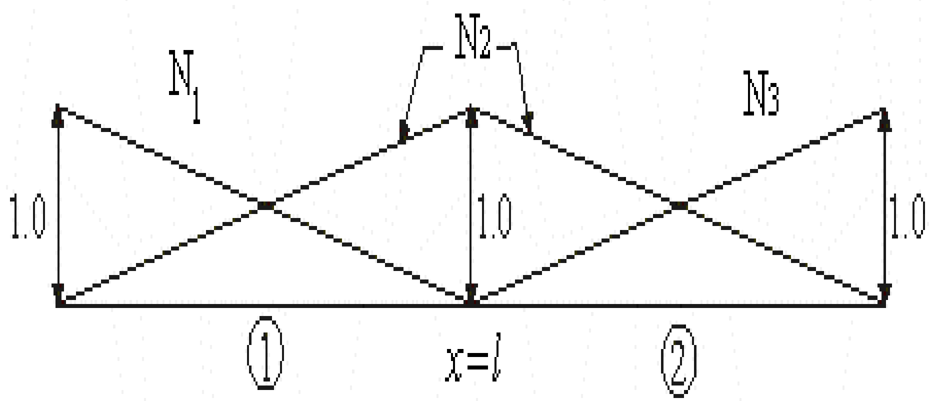
$f(x) \approx [\, 1 - (x/\ell)\,]\, f(0.75) + [\, x/\ell\,]\, f(1) \qquad (0 < x < \ell)$

$$\boxed{f(x) \approx [\, 1 - (x/\ell)\,]\, f_{k-1} + [\, x/\ell\,]\, \mathrm{f}_{k} \qquad (0 < x < \ell = L/n)}$$

Piecewise Approximation – Shape Functions



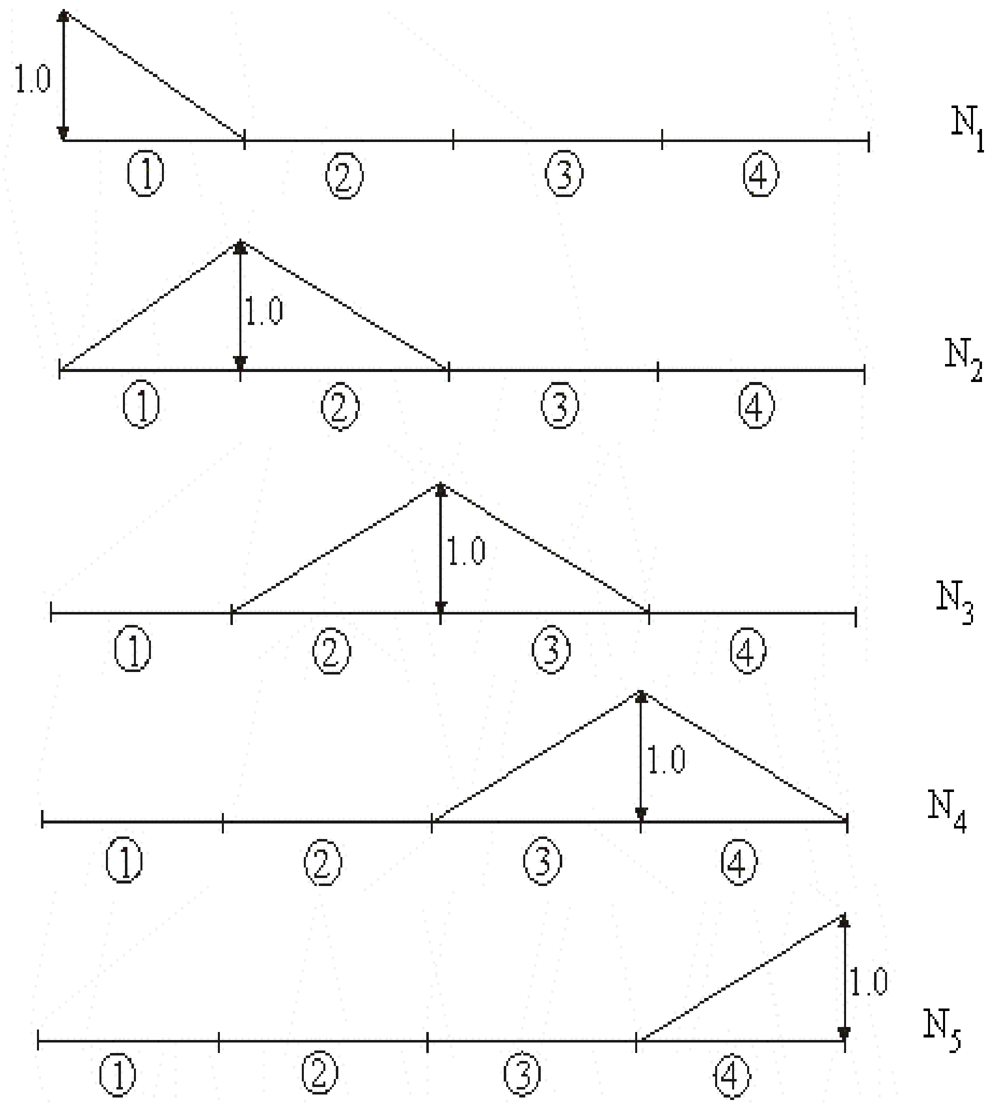
(a) Interpolation function within each sub-domain



(b) Compact representation of shape function

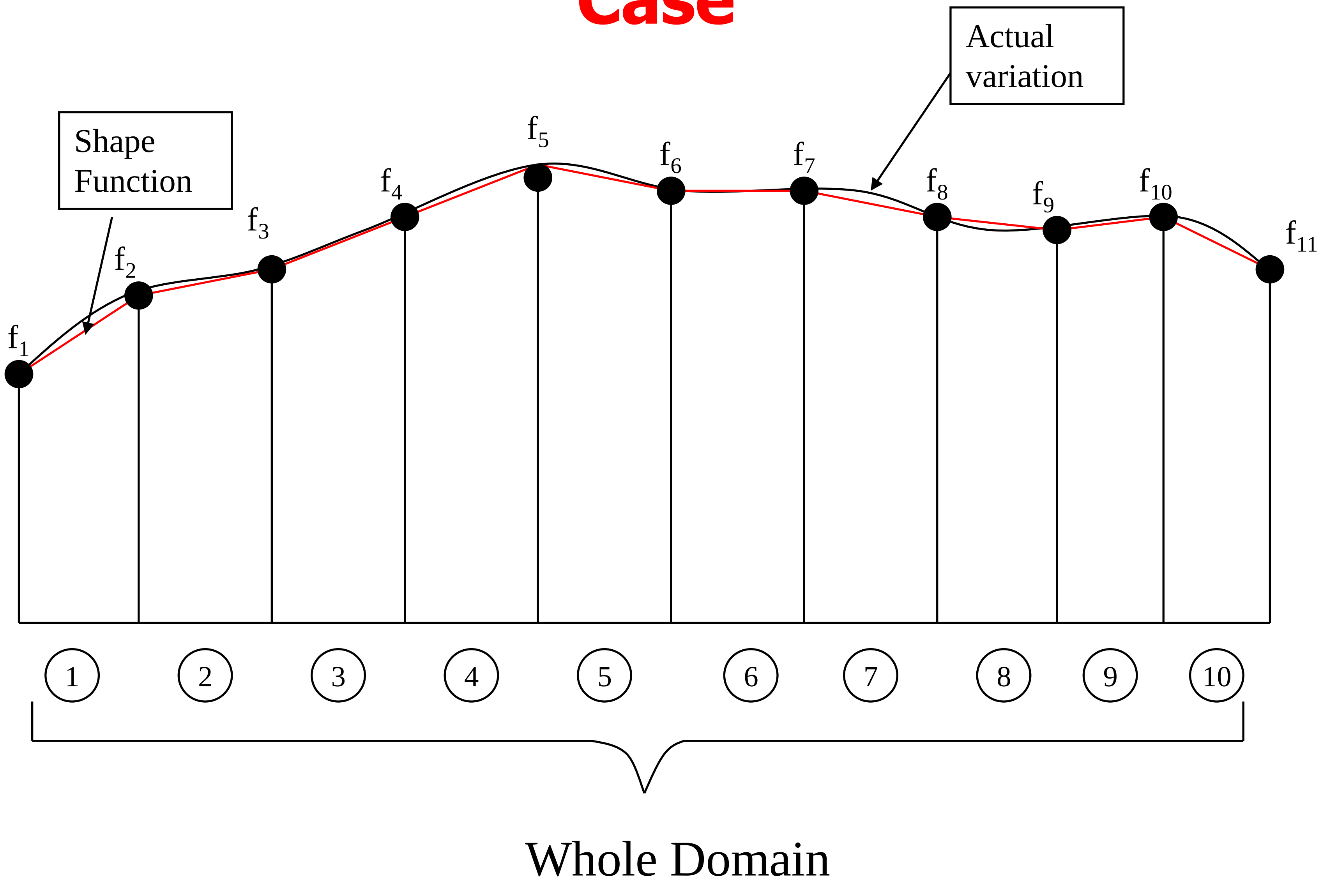
[1 – (x/ℓ)] and [x/ℓ] used in our interpolation are called "*interpolation functions*" *or* "*shape functions*"

Piecewise Approximation – Shape Functions



Contribution of a given *fk* to
the value of the function
at any point *P* within the
domain 0<*X*<*L*.

Piecewise Curve Fit – One Dimensional Case



Evaluation of Weighted Residual

$$\int_0^L W_i(X)\, R_d(X)\, dX \;=\; \sum_1^{n} \int_0^l W_i(x)\, R_d(x)\, dx$$

n == the number of segments/pieces

Thus we evaluate over each segment and then sum up

Shape functions over each segment are same/similar and hence calculations become repetitive – easily programmed

Segments can be of different length

Shape function need not be same for all segments

Illustration of Assembly for 2-elements

Galerkin FEM -> weighting functions same as shape functions

$$W_1(x) = 1 - \frac{x}{\ell} \quad 0 < x < \ell, 1^{st}\ element$$

$$= 0 \qquad 2^{nd}\ element$$

$$W_2(x) = \frac{x}{\ell} \quad 0 < x < \ell, 1^{st}\ element$$

$$= 1 - \frac{x}{\ell} \quad 0 < x < \ell, 2^{nd}\ element$$

$$W_3(x) = 0 \qquad 1^{st}\ element$$

$$= \frac{x}{\ell} \quad 0 < x < \ell, \quad 2^{nd}\ element$$

$$\sum_{k=1}^{n}\left[\int_0^\ell k\frac{dW}{dx}\;\frac{dT}{dx}\;dx\;+\int_0^\ell W\frac{P}{A_c}hT\;dx\right]=\sum_{k=1}^{n}\left[\int_0^\ell W\left(q+\frac{P}{A_c}hT_\infty\right)\;dx+\left[Wk\frac{dT}{dx}\right]_0^\ell\right]$$

LHS 1st Term

$$\int_0^L k\frac{dW_1}{dX}\frac{dT}{dX}dX=\sum_1^2\int_0^\ell k\frac{dW_1}{dx}\frac{dT}{dx}dx=\int_0^\ell k\left(-\frac{1}{\ell}\right)\left(\frac{T_2-T_1}{\ell}\right)dx+0$$

$$=\frac{k}{\ell}\left(T_1-T_2\right)$$

$$\int_0^L k\frac{dW_2}{dX}\frac{dT}{dX}dX=\sum_1^2\int_0^\ell k\frac{dW_2}{dx}\frac{dT}{dx}dx=\int_0^\ell k\left(\frac{1}{\ell}\right)\left(\frac{T_2-T_1}{\ell}\right)dx+\int_0^\ell k\left(-\frac{1}{\ell}\right)\left(\frac{T_3-T_2}{\ell}\right)dx$$

$$=\frac{k}{\ell}\left[\left(-T_1+T_2\right)+\left(T_2-T_3\right)\right]$$

$$\int_0^L k\frac{dW_3}{dX}\frac{dT}{dX}dX=\sum_1^2\int_0^\ell k\frac{dW_3}{dx}\frac{dT}{dx}dx=0+\int_0^\ell k\left(\frac{1}{\ell}\right)\left(\frac{T_3-T_2}{\ell}\right)dx$$

$$=\frac{k}{\ell}\left(T_3-T_2\right)$$

LHS 2nd Term

$$\int_0^L W_1 \frac{Ph}{A_c} T dX = \sum_1^2 \int_0^\ell W_1 \frac{Ph}{A_c} T dx = \int_0^\ell \left(1-\frac{x}{\ell}\right)\frac{Ph}{A_c}\left[\left(1-\frac{x}{\ell}\right)T_1 + \left(\frac{x}{\ell}\right)T_2\right]dx + 0$$

$$= \frac{Ph\ell}{6A_c}\left[2T_1 + T_2\right]$$

$$\int_0^L W_2 \frac{Ph}{A_c} T dX = \sum_1^2 \int_0^\ell W_2 \frac{Ph}{A_c} T dx$$

$$= \int_0^\ell \left(\frac{x}{\ell}\right)\frac{Ph}{A_c}\left[\left(1-\frac{x}{\ell}\right)T_1 + \left(\frac{x}{\ell}\right)T_2\right]dx + \int_0^\ell \left(1-\frac{x}{\ell}\right)\frac{Ph}{A_c}\left[\left(1-\frac{x}{\ell}\right)T_2 + \left(\frac{x}{\ell}\right)T_3\right]dx$$

$$= \frac{Ph\ell}{6A_c}\left[\left(T_1 + 2T_2\right) + \left(2T_2 + T_3\right)\right]$$

$$\int_0^L W_3 \frac{Ph}{A_c} T dX = \sum_1^2 \int_0^\ell W_3 \frac{Ph}{A_c} T dx = 0 + \int_0^\ell \left(\frac{x}{\ell}\right)\frac{Ph}{A_c}\left[\left(1-\frac{x}{\ell}\right)T_2 + \left(\frac{x}{\ell}\right)T_3\right]dx$$

$$= \frac{Ph\ell}{6A_c}\left[T_2 + 2T_3\right]$$

RHS 1st Term

$$\int_0^L W_1\left(q_0+\frac{Ph}{A_c}T_\infty\right)dX=\sum_1^2\int_0^\ell W_1\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx=\int_0^\ell\left(1-\frac{x}{\ell}\right)\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx+0$$

$$=\left(q_0+\frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2}\right)$$

$$\int_0^L W_2\left(q_0+\frac{Ph}{A_c}T_\infty\right)dX=\sum_1^2\int_0^\ell W_2\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx$$

$$=\int_0^\ell\left(\frac{x}{\ell}\right)\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx+\int_0^\ell\left(1-\frac{x}{\ell}\right)\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx$$

$$=\left(q_0+\frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2}+\frac{\ell}{2}\right)$$

$$\int_0^L W_3\left(q_0+\frac{Ph}{A_c}T_\infty\right)dX=\sum_1^2\int_0^\ell W_3\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx=0+\int_0^\ell\left(\frac{x}{\ell}\right)\left(q_0+\frac{Ph}{A_c}T_\infty\right)dx$$

$$=\left(q_0+\frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2}\right)$$

RHS 2nd Term

$$\left[W_1 k \frac{dT}{dX}\right]_0^L = \sum_1^2 \left[W_1 k \frac{dT}{dx}\right]_0^\ell = \left[\left(1-\frac{x}{\ell}\right)Q\right]_0^\ell + 0 = -Q_0^1$$

$$\left[W_2 k \frac{dT}{dX}\right]_0^L = \sum_1^2 \left[W_2 k \frac{dT}{dx}\right]_0^\ell = \left[\left(\frac{x}{\ell}\right)Q\right]_0^\ell + \left[\left(1-\frac{x}{\ell}\right)Q\right]_0^\ell = Q_\ell^1 - Q_0^2$$

$$\left[W_3 k \frac{dT}{dX}\right]_0^L = \sum_1^2 \left[W_3 k \frac{dT}{dx}\right]_0^\ell = 0 + \left[\left(\frac{x}{\ell}\right)Q\right]_0^\ell = Q_\ell^2$$

Final Equations for 2 elements

$$\frac{k}{\ell}\left(T_1 - T_2\right) + \frac{Ph\ell}{6A_c}\left[2T_1 + T_2\right] = \left(q_0 + \frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2}\right) + \left(-Q_0^1\right)$$

$$\frac{k}{\ell}\left[\left(-T_1 + T_2\right) + \left(T_2 - T_3\right)\right] + \frac{Ph\ell}{6A_c}\left[\left(T_1 + 2T_2\right) + \left(2T_2 + T_3\right)\right] = \left(q_0 + \frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2} + \frac{\ell}{2}\right) + \left(Q_\ell^1 - Q_0^2\right)$$

$$\frac{k}{\ell}\left(T_3 - T_2\right) + \frac{Ph\ell}{6A_c}\left[T_2 + 2T_3\right] = \left(q_0 + \frac{Ph}{A_c}T_\infty\right)\left(\frac{\ell}{2}\right) + \left(Q_\ell^2\right)$$

In matrix form, the equations are given by:

LHS

$$\left(\frac{k}{\ell}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \left(\frac{Ph\ell}{6A_c}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2+2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} =$$

Final Equations for 2 elements

Completing the equation by:

RHS

$$= \left(q_0 + \frac{Ph}{A_c} T_\infty \right) \begin{Bmatrix} \dfrac{\ell}{2} \\ \dfrac{\ell}{2} + \dfrac{\ell}{2} \\ \dfrac{\ell}{2} \end{Bmatrix} + \begin{Bmatrix} -Q_o^1 \\ Q_o^1 - Q_o^2 \\ Q_\ell^2 \end{Bmatrix}$$

Essence of Finite element method

- Evaluate the sub-domain level contributions to the weighted residual by merely computing the integral $\int W_i(x)\, R_d(x)\, dx$ or its weak form, just once for the kth sub–domain
- Build–up the entire coefficient matrices on LHS and RHS by appropriately placing these sub–domain level contributions in the appropriate rows and columns.
- Solve the $(n+1)$ algebraic equations to determine the unknowns viz., function values f_k at the ends of the sub–domains.
- This is the essence of the Finite element method.

Finite Element Method

- Each of the sub–domains is called a "finite element" – to be distinguished from the "differential element" used in continuum mechanics.

- The ends of the sub–domain are referred to as the "nodes" of the element.

- Later on we see elements with nodes not necessarily located at only the ends e.g. an element can have mid–side nodes, internal nodes etc.

- The unknown function values *fk* at the ends of the sub–domains are known as the "nodal degrees of freedom (d.o.f) ".

Finite Element Formulation

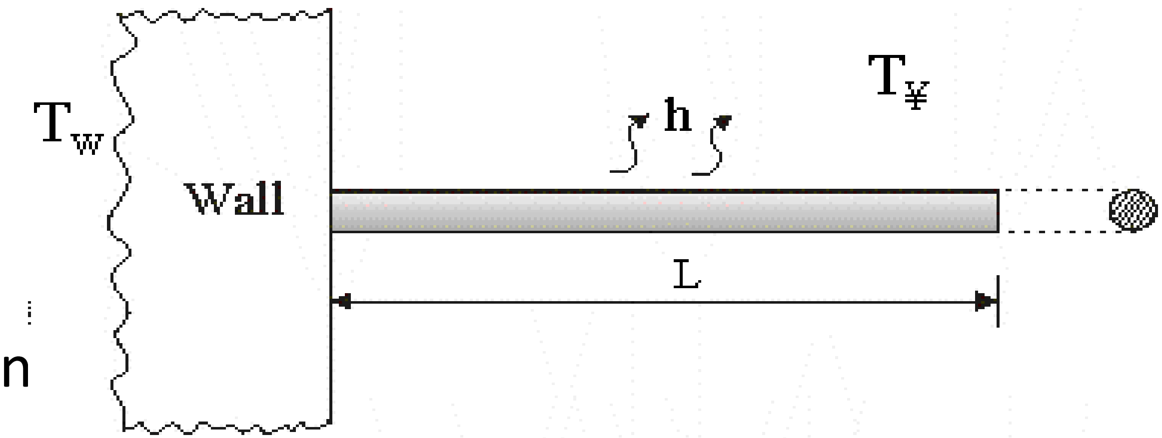
- A general finite element can admit the function values as well as its derivatives as nodal d.o.f.

- A general finite element can have quadratic or cubic etc shape functions

- The sub-domain level contributions to the weak form are typically referred to as "element level equations".

- The process of building-up the entire coefficient matrices on LHS and RHS is known as the process of "assembly" i.e. assembling or appropriately placing the individual element equations to generate the system level equations.

Example: Temperature distribution in a pin–fin



One element solution

$$\left(\frac{200}{0.05}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{(\pi)\,(0.001)\,(20)\,(0.05)}{(6)\,(\pi)\,(0.0005)^2}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}$$

From the boundary conditions, we have, $T1$ = 300^{0}C, $Qtip$ = 0. Thus we get, $T2$ = 198.75 ^{o}C.

$$=\frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30)\begin{Bmatrix}0.025\\ 0.025\end{Bmatrix}+\begin{Bmatrix}Q_{wall}\\ Q_{tip}\end{Bmatrix}$$

Pin Fin Example

Two element solution

$$\left(\frac{200}{0.025} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{(\pi)\,(0.001)\,(20)\,(0.025)}{(6)\,(\pi)\,(0.0005)^2} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$= \frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30) \begin{Bmatrix} 0.0125 \\ 0.025 \\ 0.0125 \end{Bmatrix} + \begin{Bmatrix} Q_{wall} \\ 0 \\ Q_{tip} \end{Bmatrix}$$

Pin Fin Example

Four element solution

$$\left(\frac{200}{0.0125} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} + \frac{(\pi)\,(0.001)\,(20)\,(0.0125)}{(6)\,(\pi)\,(0.0005)^2} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

$$= \frac{(\pi)\,(0.001)\,(20)}{(\pi)\,(0.0005)^2}\,(30) \begin{Bmatrix} 0.00625 \\ 0.0125 \\ 0.0125 \\ 0.0125 \\ 0.00625 \end{Bmatrix} + \begin{Bmatrix} Q_{wall} \\ 0 \\ 0 \\ 0 \\ Q_{tip} \end{Bmatrix}$$

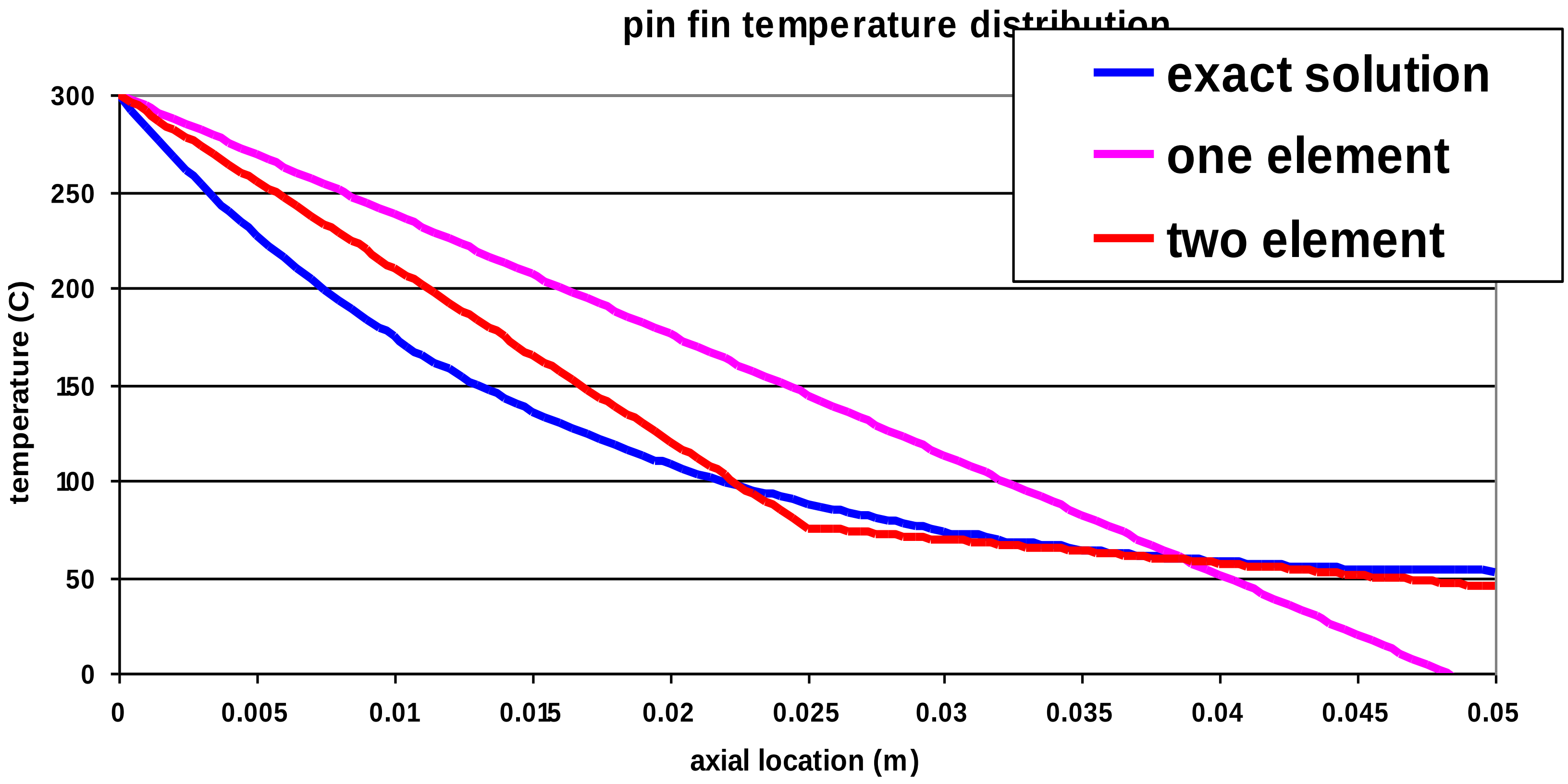
Pin Fin Example

Exact solution

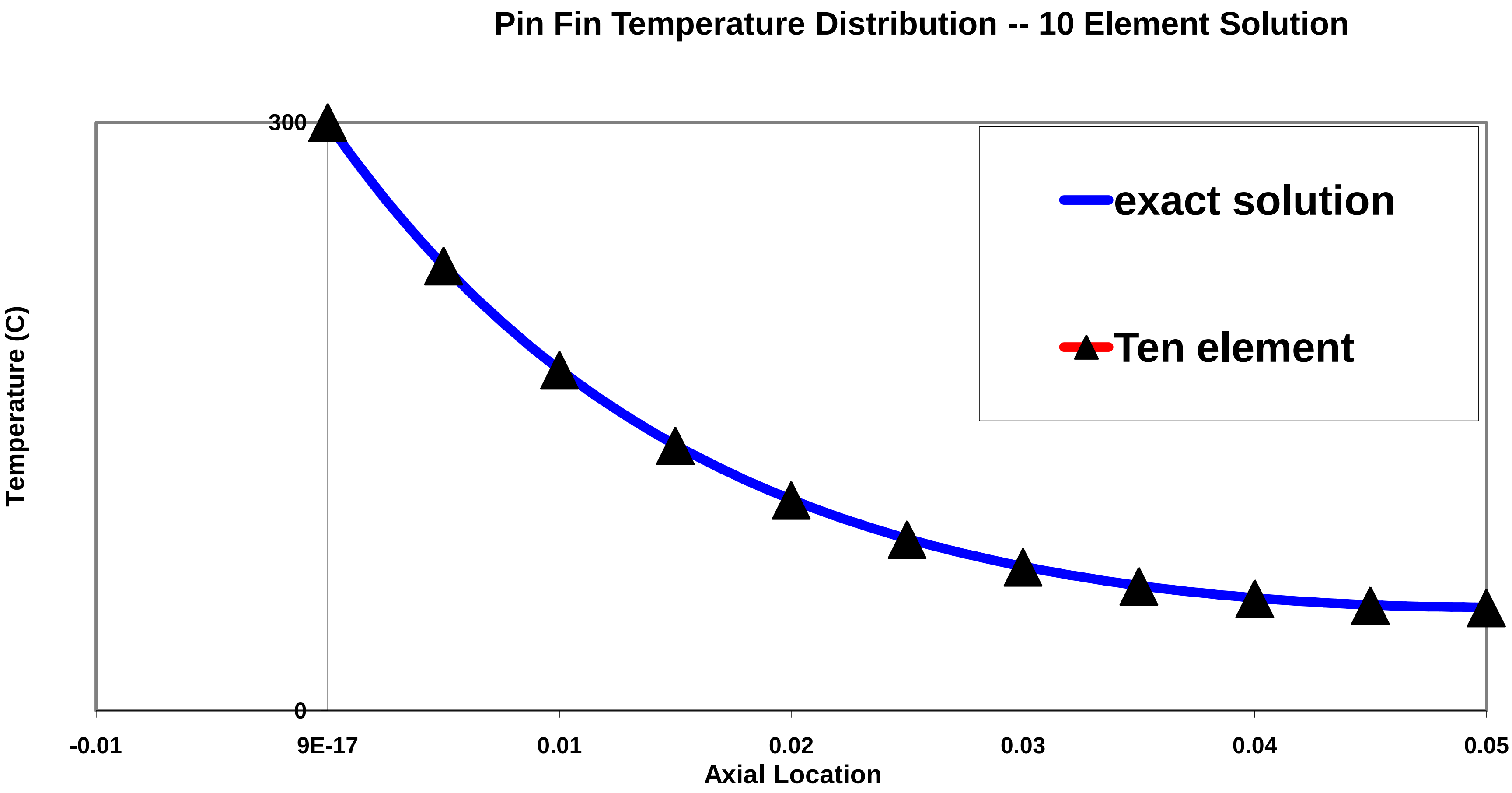
$$T(x) = T_\infty + (T_{wall} - T_\infty)\left[\frac{\cosh\, m\,(L-x)}{\cosh\, mL}\right]$$

$$m = \sqrt{\frac{hP}{k\,A_c}}$$

Pin Fin Example - Comparison of exact and Finite Element solution



Comparison of exact and 10 element solution



Finite Element Formulation

Procedure for FEA starting from a given DE:

- Write down the Weighted Residual statement.
- Perform integration by parts for even distribution of differentiation between the field variable and the weighting function and develop the weak form of the W–R statement.
- Re-write the weak form as a summation over "*n*" elements.
- Define finite element i.e. geometry of the element, its nodes, nodal d.o.f.

Finite Element Formulation

- Derive the shape or interpolation functions. Use these as the weighting functions also.
- Compute the element level equations by substituting these in the weak form.
- For a given topology of finite element mesh, build–up the system equations by assembling together element level equations.
- Substitute the prescribed boundary conditions and solve for the unknowns.

Exercise 1

- Derive shape functions for a quadratic element
- Using these shape functions, derive element matrices
- Find solution using one and two quadratic elements

Exercise 2

- In the heat transfer example, the solutions for temperature obtained are as follows:
- One element – T_1 = 300^{0}C, T_2 = 198.75 ^{o}C.
- Two elements - ℓ = 0.025 m;

  T_1 = 300^{0}C , T_2 = 226.185^{o}C , T_3 = 203.548^{o}C
- Four elements - ℓ = 0.0125 m;

  T_1 = 300^{0}C ,

  T_2 = 256.37^{o}C , T_3 = 227.03^{o}C

  T_4 = 210.14^{o}C , T_5 = 204.63^{0}C
- Find Temperature at x=0.015; Compare the FEM solution with 1/2/4 elements and exact solution
- Discuss the behavior of first derivative of temperature ie heat flux

FE Formulation of Axial Force Element

Differential Equation and Boundary Conditions

$$AE\frac{d^2u}{dx^2}+q_0=0 \qquad \textit{and} \qquad u(0)=u_0 \quad AE\left.\frac{du}{dx}\right|_L=P_L$$

Weighted Residual

$$\int_0^L W(X)\left[AE\frac{d^2u}{dx^2}+q_0\right]dX=0 \quad \textit{and} \quad u(0)=u_0 \quad AE\left.\frac{du}{dx}\right|_L=P_L$$

Weak Form

$$\int_0^L AE\frac{du}{dX}\frac{dW}{dX}\,dX=\int_0^L W(X)\,q_0\,dX+\left[W(X)P\right]_0^L \quad \textit{and} \quad u(0)=u_0;W(0)=0$$

Weak Form as summation over "n" elements

$$\sum\int_0^\ell AE\frac{du}{dx}\frac{dW}{dx}\,dx=\sum\int_0^\ell W(x)\,q_0\,dx+\sum\left[W(x)P\right]_0^\ell$$

For a linear element,

$$u(x) = \left(1 - \frac{x}{\ell}\right) u_k + \left(\frac{x}{\ell}\right) u_{k+1}$$

$$W(x) = \left(1 - \frac{x}{\ell}\right) \quad and \quad \left(\frac{x}{\ell}\right)$$

Element Level Equations

$$\frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_k \\ u_{k+1} \end{Bmatrix} = q_0 \begin{Bmatrix} \dfrac{\ell}{2} \\ \dfrac{\ell}{2} \end{Bmatrix} + \begin{Bmatrix} -P_0 \\ P_\ell \end{Bmatrix}$$

Force – Deflection Equations

LHS is Stiffness Matrix

RHS is Nodal Force Vector

Meaning of FE Equations

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

First column of LHS matrix is the force vector needed to cause a deformation pattern u1 = 1 and u2 = 0.

Second column of LHS matrix is the force vector needed to cause a deformation pattern u1 = 0 and u2 = 1.

Practice Exercise - Assembly