Project 3: Differential equations and the solar system, Fys 4150

Ask J. Markestad, Thorbjørn V. Larsen

Abstract

In this paper we discuss the basic ideas behind numerical solutions to ordinary differential equations, and their limitations by applying two well known methods; the forward Euler and velocity Verlet methods, to the many-body problem of our solar system. We then use these methods to do some numerical experiments, namely escape velocity of earth from the sun, and testing a general relativity correction to the Newtonian gravitational force.

Introduction

Solving differential equations is one of the core problems that physicists encounter. It is therefore important to have a proper understanding of how to solve them, and since most cannot be solved analytically, numerical methods are a necessity in a physicists toolbox. We will briefly introduce two methods for solving ordinary differential equations; the forward Euler and velocity Verlet methods. We will then talk about their uses and limitations, conservation of energy and stability. To learn their applications we will use them to model our solar system and then use our model to do some numerical experiments. We will figure our the escape velocity of earth from the sun by running though a range of initial velocities. We will then look at a correction to the Newtonian gravitational force from General Relativity and test its validity by comparing our experiment to the observed perihelion precession of Mercury's orbit.

Theory and Algorithms

We will state the forward Euler and velocity Verlet methods, discuss their properties, and look at the solar system in more detail in this section. For a detailed derivation of these methods see Computational Physics Lecture Notes Fall 2015 [1] by Morten Hjort-Jensen.

The forward Euler method can be thought of geometrically. If you have an initial point, then you can find the next point by estimating the derivative in the initial point and using this gradient as the path to travel from the initial point to the next one by moving along it for a standard distance/step length.

$$y_{i+1} = y_i + hy_i^{(1)} + O(h^2), \quad h = \frac{y_{final} - y_{initial}}{Number\ of\ steps} \tag{1}$$

$$y_{i+1}^{(1)} = y_i^{(1)} + hy_i^{(2)} + O(h^2)$$

Here we have set up the forward Euler method for a second order differential equation. For a higher order differential equation one would have several equations each using the next one to estimate the gradient of that order to be used. This method has a local error of $O(h^2)$ but will wave a global error of $O(h)$ when it is applied N times to find the complete solution. Further, the forward Euler method is unstable in that not all step lengths give a finite solution for a differential equations. In other words, for exact solutions that converge, the numerical solution diverges if the step length is not chosen with care.

The velocity Verlet method is like most of our methods based on Taylor expansions. However, unlike the forward Euler method we also expand $y^{(2)}(t+h)0v^{(1)}(t+h)$ and use it to get the relation $hv_i^{(2)} \approx v_{i+1}^{(1)} - v_i^{(1)}$. This allows us to go to a higher order in the Taylor expansion of y and v.

$$y_{i+1} = y_i + hv_i + \frac{h^2}{2} v_i^{(1)} + O(h^3) \tag{2}$$

$$v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} - v_i^{(1)}\right) + O(h^3)$$

Note that we need to know the acceleration for both the point we are at and for the next position to calculate the next velocity step, meaning we need to do the position calculation first. This method is also stable, i.e. for every step length the computed and exact solution will have the same characteristics, but will have a local error of $O(h^3)$ and the solution will have a global error of $O(h^2)$. Another characteristic of the velocity Verlet method is that is preserves energy, which we will explicitly test and compare with the forward Euler method.

Let us look more specifically at the solar system at hand. The system is described by Newtons law of gravitation and Newtons second law, which means we know the forces and thus the accelerations involved as functions of the position. The gravitational law for a two body system, f.eks. the sun earth system is given by:

$$F_G = \frac{GM_{sun}M_{earth}}{r^2} \tag{3}$$

Further we want to do a variable change into some more fitting units for simulating the solar system, namely astronomical units (1 AU $= 1.5 \times 10^{11}$m), years and solar masses($M_{sun} = 2 \times 10^{30} kg$). To achieve this we rescale using a circular orbit of 1 AU. Then, from Newtons second law:

$$v^2 r = GM_{sun} = 4\pi^2 \frac{AU^3}{yr^2} \tag{4}$$

Where we used that the period for the orbit should be one year. This allows us to get rid of the GM_{sun} factor in our calculations. We then divide the force by the mass of the moving object to find the acceleration and we can then apply the numerical methods previously mentioned. The next step is then to introduce more bodies into our calculations and thus taking us away for situations that are analytically solvable. Let us look are the inclusion of a third planet, Venus. Each body will now have a force acting on them that is the sum of the forces for each two body combination that they are a part of. So, the earth will have a force term that is identical with the previous earth sun term, but will also have a term:

$$F_G = \frac{GM_{venus}M_{earth}}{r^2_{earth-venus}} \tag{5}$$

Again, we are interested in the acceleration that this provides earth with so we will dived by the earth mass here but we are now left with a GM_{venus} factor. Here we multiply and divide by M_{sun} so we can write M_{venus} in terms of solar masses and we get the GM_{sun} factor that we have rescaled to be $(2\pi)^2$. The forces for Venus will be similar but with the earth mass having to be rescaled. This leaves us with the coupled differential equations for earth in the x direction:

$$\frac{dx}{dt} = v(t) \qquad \frac{dv_x}{dt} = -\frac{(2\pi)^2 M_{venus}}{r^2_{earth-venus}} - \frac{(2\pi)^2}{r^2_{earth-sun}} \tag{6}$$

With similar equations for the y and z directions, and for Venus and the Sun. Adding more planets will then equate to adding more of these acceleration terms, and we can then add arbitrarily many bodies to our simulation.

The perihelion precession of Mercury's orbit has been known for a very long time, and was a known problem to Einstein when he developed General relativity, and it was the very first success of General Relativity that it predicted and explained Mercury's perihelion precession. We will attempt recreate this effect in our simulation by adding a correction term to the force, from General Relativity:

$$F_G = \frac{GM_{sun}M_{mercury}}{r^2_{sun-mercury}}\left[1 + \frac{3l^2}{r^2_{sun-mercury}c^2}\right] \tag{7}$$

Where $l = |\vec{r} \times \vec{v}|$ is the magnitude of mercury's orbital angular momentum. We should note here that this correction will be very small given that c, the speed of light, is a very large number and it therefore only gives an effect for planets sufficiently close to each other, explaining why only Mercury observes this effect. We can measure the precession by checking the perihelion angle $\tan(\theta_p) = \frac{y_p}{x_p}$, which changes by an observed 43 arcseconds per century.

Memory handling and algorithms

Results

Conclusion

References

[1] Morten Hjort-Jensen *Computational Physics Lecture Notes Fall 2015* Department of Physics, University of Oslo 2015 https://github.com/CompPhysics/ComputationalPhysics/blob/master/doc/Lectures/lectures2015.pdf