1.1.DF 3 Suppose, to the contrary, $\lim_{x\to-\infty} F(x) \neq 0$.
$\Rightarrow \exists\epsilon > 0 : \lim_{x\to-\infty} F(x) = \epsilon$, since F is bounded below by 0 and nondecreasing.
$\Rightarrow$ For any sequence $x_n : \lim_{n\to\infty} x_n = -\infty$, $\lim_{n\to\infty} F(x_n) = \epsilon$.
Let x_n be such a sequence, with the additional restriction that it is strictly decreasing.
$\Rightarrow \forall n, F(x_n) > \epsilon$, since F is nondecreasing.
Let $A_n = X \leq x_n$ and $A = \bigcap_{n=1}^{\infty} A_n$. Since x_n is strictly decreasing, $\forall n > m, A_n \subset A_m$.
$\Rightarrow P[A_n] \downarrow P[A]$, since $A_n \downarrow A$.
$\Rightarrow \exists a \in A$, since $P[A] \downarrow \epsilon$ and $P[\emptyset] = 0$.
$\Rightarrow \forall n, a \leq x_n$.
This is a contradiction, since $\lim_{n\to\infty} x_n = -\infty$. Thus, we must conclude that $\lim_{x\to-\infty} F(x) = 0$

Suppose, to the contrary, $\lim_{x\to\infty} F(x) \neq 1$.
$\Rightarrow \exists\epsilon < 1 : \lim_{x\to\infty} F(x) = \epsilon$, since F is bounded above by 1 and nondecreasing.
$\Rightarrow$ For any sequence $x_n : \lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} F(x_n) = \epsilon$.
Let x_n be such a sequence, with the additional restriction that it is strictly increasing.
$\Rightarrow \forall n, F(x_n) < \epsilon$, since F is nondecreasing.
Let $A_n = X \leq x_n$ and $A = \bigcup_{n=1}^{\infty} A_n$. Since x_n is strictly increasing, $\forall n > m, A_m \subset A_n$.
$\Rightarrow P[A_n] \uparrow P[A]$, since $A_n \uparrow A$.
$\Rightarrow P[A^c] > 0$, since $P[A] \uparrow \epsilon$ and $\epsilon < 1$.
$\Rightarrow \exists a \in A^c$, since $P[\emptyset] = 0$.
$\Rightarrow \forall n, a > x_n$.
This is a contradiction, since $\lim_{n\to\infty} x_n = \infty$. Thus, we must conclude that $\lim_{x\to\infty} F(x) = 1$

DF 4 F is bounded above and nondecreasing, so we know that $F(x-)$ exists. Let x_n be a strictly increasing sequence such that $\lim_{n\to\infty} x_n = x$. Let $A_n = X \leq x_n$ and $A = \bigcup_{n=1}^{\infty} A_n$. It is clearly the case that $A = X < x$, and since $A_n \uparrow A$, $P[A_n] \uparrow P[A]$. Since $P[A_n] = F(x_n)$, we may conclude that $\lim_{n\to\infty} F(x_n) = P[X < x]$. Thus, we can conclude $F(x-) = P[X < x]$ as well.

DF 5

$$\begin{aligned} F(x) - F(x-) &= P[X \leq x] - P[X < x] \\ &= P[X = x] \end{aligned}$$

1.2. Let

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } 0 \leq x < 2 \\ 1, & \text{if } 2 \leq x \end{cases}$$

Here, $F^*(\frac{1}{2}) = 0 < 1$, yet $\frac{1}{2} = F(1)$. As well, $\frac{1}{4} < \frac{1}{2} = F(0)$, yet $F^*(\frac{1}{4}) = 0$.

1.3. $\forall v \in (0,1) : \lim_{u\downarrow 0} F^*(u) \leq F^*(v)$, since F^* nondecreasing.
$\Rightarrow \forall v \in (0,1) : \forall x \in \{x \in \mathbb{R} : v \leq F(x)\} : \lim_{u\downarrow 0} F^*(u) \leq x$, by the definition of F^*
$\Rightarrow \forall x \in \{x \in \mathbb{R} : F(x) > 0\} : \lim_{u\downarrow 0} F^*(u) \leq x$
Also, $\nexists w : \forall v \in (0,1) : \lim_{u\downarrow 0} F^*(u) < w \leq F^*(v)$.
$\Rightarrow \nexists w : \forall v \in (0,1) : \forall x \in \{x \in \mathbb{R} : v \leq F(x)\} : \lim_{u\downarrow 0} F^*(u) < w \leq F(x)$
$\Rightarrow \nexists w : \forall x \in \{x \in \mathbb{R} : F(x) > 0\} : \lim_{u\downarrow 0} F^*(u) < w \leq F(x)$
Accordingly, since $\lim_{u\downarrow 0} F^*(u)$ is the greatest upper bound of $\{x \in \mathbb{R} : F(x) > 0\}$,

$$\lim_{u\downarrow 0} F^*(u) = \inf\{x \in \mathbb{R} : F(x) > 0\}$$

Let $x \in \mathbb{R} : F(x) \in (0,1)$.
$\Rightarrow \exists v \in (0,1) : F(x) < v < 1$

$\Rightarrow F^*(v) > x$ by the switching formula.
$\Rightarrow \lim_{u\uparrow 1} F^*(u) \geq F^*(v) > x$, since F^* is nondecreasing.
$\Rightarrow \lim_{u\uparrow 1} F^*(u)$ is an upper bound of $\{x \in \mathbb{R} : F(x) < 1\}$
Suppose, to the contrary, $\exists w : \forall x \in \{x \in \mathbb{R} : F(x) < 1\} : x \leq w < \lim_{u\uparrow 1} F^*(u)$.
$\Rightarrow \forall v \in (0,1) : F^*(v) \leq w < \lim_{u\uparrow 1} F^*(u)$.
$\Rightarrow \lim_{u\uparrow 1} F^*(u) \leq w < \lim_{u\uparrow 1} F^*(u)$
Since the above is a contradiction, we must conclude that $\lim_{u\uparrow 1} F^*(u)$ is the least upper bound of $\{x \in \mathbb{R} : F(x) < 1\}$, and therefor

$$\lim_{u\uparrow 0} F^*(u) = \sup\{x \in \mathbb{R} : F(x) > 0\}$$

1.4. $F(x) \leq F(x)$
$\Rightarrow F^*(F(x)) \leq x$ by the switching formula.
Let $\delta > 0$.
$\Rightarrow F(x) < F(x) + \delta$
$\Rightarrow x < F^*(F(x+\delta))$ by the switching formula.
$\Rightarrow x \leq F^*(F(x)+)$
Accordingly,

$$F^*(F(x)) \leq x \leq F^*(F(x)+)$$

1.5. We can show that

$$\begin{aligned} F^*(u) &\leq F^*(u) & \\ u &\leq F(F^*(u)) & \text{by the switching formula} \\ F^*(u) &\leq F^*(F(F^*(u))) & \text{since } F^* \text{ is nondecreasing} \end{aligned}$$

Now suppose, to the contrary, that $F^*(u) < F^*(F(F^*(u)))$.
$\Rightarrow F(F^*(u)) < F(F^*(u))$ by the switching formula.
Since this is an obvious contradiction, we may conclude that

$$F^*(u) = F^*(F(F^*(u)))$$

We can also show that

$$\begin{aligned} F(x) &\leq F(x) & \\ x &\leq F^*(F(x)) & \text{by the switching formula} \\ F(x) &\leq F(F^*(F(x))) & \text{since } F \text{ is nondecreasing} \end{aligned}$$

If $F(x) < F(F^*(F(x)))$, this would imply by the switching formula that $F^*(F(x)) > F^*(F(x))$, which an obvious contradiction. Thus, we can conclude that

$$F(x) = F(F^*(F(x)))$$

1.6. Let F not be continious at x.
$\Rightarrow F(x-) \neq F(x)$, since $F(x) = F(x)$ by DF2.
$\Rightarrow F(x-) < F(x)$ since F is nondecreasing.
$\Rightarrow \exists \epsilon > 0 : \forall \delta > 0 : \exists x_1 \in (x-\delta, x) : F(x) - F(x_1) > \epsilon$
$\Rightarrow \forall w < x, F(x) - \epsilon > F(w)$, since F is nondecreasing, and $\exists x_1 \in (w,x) : F(w) \leq F(x_1) < F(x) - \epsilon$
$\Rightarrow \forall u \in (F(x)-\epsilon, F(x)) : F^*(u) \geq x$, since x is a lower bound of $\{x \in \mathbb{R} : u \leq F(x)\}$.
$\Rightarrow F(F^*(u)) \geq F(x) > u$, since F is nondecreasing.
$\Rightarrow$ It is not the case that $\forall u \in (0,1) : F(F^*(u)) = u$
Now, going the other direction, let F be continious, and, to the contrary, let $F(F^*(u)) > u$ for some $u \in (0,1)$

$\Rightarrow \forall x \in \mathbb{R} : F(x-) = F(x)$.
$\Rightarrow \forall \epsilon > 0 : \exists \delta > 0 : \forall x_1 \in (x - \delta, x), F(x) - F(x_1) < \delta$
$\Rightarrow \exists \delta > 0 : \forall w \in (F^*(u) - \delta, F^*(u)) : F(F^*(u)) - F(w) < F(F^*(u)) - u$
$\Rightarrow F(w) > u$ and $F^*(u) > w$
However, by definition, $F^*(u)$ is less than or equal to all x such that $F(x) \geq u$, so we have reached a contradiction. Thus, we may conclude that if F is continious, it must also be the case that $F(F^*(u)) \leq u$. We can reject that $F(F^*(u)) < u$, since that would mean $F^*(u) < F^*(u)$ by the switching formula, so we are left with $F(F^*(u)) = u$.
Taking both direction together, we get the result

$$\forall u \in (0,1) : F(F^*(u)) = u \Leftrightarrow F \text{is continious}$$

Let F be strictly increasing.
$\Rightarrow$ if $x_2 < x_1$, then $F(x_2) < F(x_1)$.
$\Rightarrow x_2 < F^*(F(x_1))$ by the switching formula.
$\Rightarrow x_2 < F^*(F(x_1)) \leq x_1$ by IDF4.
$\Rightarrow \forall \epsilon > 0 : x_1 - \epsilon < F^*(F(x_1)) \leq x_1$.
$\Rightarrow x_1 = F^*(F(x_1))$.
Now, let F not be strictly increasing.
$\Rightarrow \exists x_1, x_2 \in \mathbb{R} : x_2 < x_1$ and $F(x_2) \geq F(x_1)$
$\Rightarrow F(x_2) = F(x_1)$, since F is nondecreasing.
$\Rightarrow F^*(F(x_1)) \leq x_2 < x_1$
$\Rightarrow$ It is not the case that $\forall x \in \{x \in \mathbb{R} : 0 < F(x) < 1\} : F^*(F(x)) = x$.
Taking both directions together, we get the result that

$$\forall x \in A := \{x \in \mathbb{R} : 0 < F(x) < 1\} : F^*(F(x)) = x \ \Leftrightarrow F \text{ is strictly increasing over } A$$

1.7. To acheive a desired p-value, it is sufficient that, for a test statistic x, $F(x) \leq p$.

1.8. a) Let $u \geq F(x-)$
$\Leftrightarrow \forall \delta_1 > 0 : u \geq F(x - \delta_1)$, since F is nondecreasing.
$\Leftrightarrow \forall \delta_1, \delta_2 > 0 : u + \delta_2 > F(x - \delta_1)$
$\Leftrightarrow \forall \delta_1, \delta_2 > 0 : F^*(u + \delta_2) > x - \delta_1$ by the switching formula.
$\Leftrightarrow \forall \delta_2 > 0 : F^*(u + \delta_2) \geq x$
$\Leftrightarrow F^*(u+) \geq x$

b) Let $A = \{w : u \geq F(w-)\}$
$\Rightarrow A = \{w : F^*(u+) \geq w\}$ by part a)
$\Rightarrow \forall x \in A : F^*(u+) \geq x$
$\Rightarrow F^*(u+) \geq \sup(A)$
Now suppose, to the contrary, that $F^*(u+) > \sup(A)$
$\Rightarrow \exists x \in \mathbb{R} : F^*(u+) > x > \sup(A)$
$\Rightarrow x \in A$ and $x > \sup A$
This is a contradiction, so we must conclude that it cannot be the case that $\sup A < F^*(u+)$, and rather it must be that $\sup A = F^*(u+)$.

1.9. a) Let A be a closed and bounded supbinterval of B.
$\Rightarrow f(A) \subseteq [f(\min(A)), f(\max(A))]$, since f nondecreasing.
$\Rightarrow \forall \epsilon > 0 :$ Let $J_{A,\epsilon} = \{x \in A : f(x+) - f(x-) > \epsilon\}$
$\Rightarrow |J_{A,\epsilon}| \in \mathbb{N}$, since, due to f being nondecreasing,

$$\sum_{j \in J_{A,\epsilon}} \epsilon \leq \sum_{j \in J_{A,\epsilon}} f(j+) - f(j-) \leq f(\max(A)) - f(\min(A))$$

Now, let, where $K = \{z \in \mathbb{Z} : B \cap [z-1, z] \neq \emptyset\}$

$$J = \bigcup_{z \in K} \bigcup_{n=1}^{\infty} J_{[z-1,z], \frac{1}{n}}$$

$\Rightarrow D_f \subseteq J$, since if $j \in D_f$, $\exists z \in \mathbb{Z} : j \in [z-1, z]$ and $\exists n \in \mathbb{N}_0 : f(j+) - f(j-) > \frac{1}{n}$.
Let $h : \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0 \to J$ be defined as such: The triple (z, n_1, n_2) gets mapped to the n_2th largest j in $J_{[z-1,z], \frac{1}{n_1}}$. We may choose the n_2th largest since we have already shown $J_{[z-1,z], \frac{1}{n_1}}$ is finite. This is clearly a surjective function, where $\forall j \in J : \exists (z, n_1, n_2) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0 : h(z, n_1, n_2) = j$. Therefor, since such a mapping exists, we may conclude that

$$|\mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0| \geq |J| \geq |D_f|$$

Since $\mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0$ is countable, this means that D_f is also at most countable.

b) Let $x \in B$ and $\epsilon > 0$. $(x - \epsilon, x + \epsilon)$ is an uncountable set, so there can't possibly be a surjective function $h : D_f \to (x - \epsilon, x_\epsilon)$, since D_f is at most countable. Thus, it must be the case that $\exists c \in (x - \epsilon, x_\epsilon) : c \in D_f^c = C_f$. We can thus conclude C_f is dense in B.

1.10. a) Let w be a continuity point of F and $u \in (0,1)$ with $F^*(u) > w$
$\Rightarrow u > F(w)$ by the switching formula
Let $\epsilon = u - F(w)$. $\Rightarrow \exists m : \forall m \geq n : |F_n(w) - F(w)| < \epsilon$, since $\lim_{n\to\infty} F_n(w) = F(w)$
$\Rightarrow \forall n \geq m$

$$\begin{aligned} u &= F(w) + (u - F(w) \\ &> F(w) + |F_n(w) - F(w)| \\ &> F(w) + F_n(w) - F(w) \\ &> F_n(w) \end{aligned}$$

b) Let y be a continuity point of F and $u \in (0,1)$ with $F^*(u+) < y$
$\Rightarrow \exists \delta > 0 : \forall \epsilon \in (0, \delta) : F^*(u + \epsilon) < y$
$\Rightarrow u + \epsilon \leq F(y)$ by the switching formula
$\Rightarrow u \leq F(y) - \epsilon$
Since $\lim_{n\to\infty} F_n(y) = F(y), \exists m : \forall n > m : |F_n(y) - F(y)| < \epsilon$
$\Rightarrow \forall n > m$

$$\begin{aligned} u &\leq F(y) - \epsilon \\ &< F(y) - |F_n(y) - F(y)| \\ &< F(y) - (F(y) - F_n(y)) \\ &< F_n(y) \\ F^*(u) &< y && \text{By the switching formula} \end{aligned}$$

c) Suppose, to the contrary, $F^*(u) > \liminf_n F_n^*(u)$
$\Rightarrow \exists c \in C_f : F^*(u) > c > \liminf_n F_n^*(u)$, since C_f is dense in $\mathbb{R}$.
$\Rightarrow \exists m \in \mathbb{N} : \forall n > m : F_n^*(u) > c$, by part a.
$\Rightarrow \forall n > m, \inf_n F_n^*(u) \geq c$
$\Rightarrow \liminf_n F_n^*(u) \geq c$
This is a contradiction, so we must conclude $\liminf_n F_n^*(u) \geq F^*(u)$

$\inf_n F^*(u) \leq \sup_n F^*(u)$ by definition.
$\Rightarrow \liminf_n F_n^*(u) \leq \limsup_n F_n^*(u)$

Suppose, to the contrary, $\limsup_n F_n^*(u) > F^*(u+)$
$\Rightarrow \exists c \in C_f : \limsup_n F_n^*(u) > c > F^*(u+)$, since C_f is dense in $\mathbb{R}$a
$\Rightarrow \exists m \in \mathbb{N} : \forall n > m : F_n^*(u) \leq c$, by part b.

$\Rightarrow \forall n > m, \sup_n F_n^*(u) \leq c$
$\Rightarrow \limsup_n F_n^*(u) \leq c$
This is a contradiction, so we must conclude $\limsup_n F_n^*(u) \leq F^*(u+)$

Combining the three above results, we have that

$$F^*(u) \leq \liminf_n F_n^*(u) \leq \limsup_n F_n^*(u) \leq F^*(u+)$$

d) If $u \in C_f$, by definition $F^*(u) = F^*(u+)$
$\Rightarrow F^*(u) = \liminf_n F_n^*(u) = \limsup_n F_n^*(u)$ by part c.
$\Rightarrow \forall \epsilon > 0 : \exists m \in \mathbb{N} : \forall n > m :$

$$F^*(u) - \epsilon < \inf_n F_n^*(u) \leq \sup_n F_n^*(u) < F^*(u) + \epsilon$$

$\Rightarrow \forall n > m : F^*(u) - \epsilon < F_n^*(u) < F^*(u) + \epsilon$
$\Rightarrow \lim_{n\to\infty} F_n^*(u) = F^*(u)$

e) Let

$$F_n^*(u) = (2u-1)^{-n} \text{ and } F^*(u) = \begin{cases} -1, & \text{if } 0 < x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

arising from

$$F_n(u) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{x^n+1}{2}, & \text{if } -1 \leq x < 1 \\ 1, & \text{if } 1 \leq x \end{cases} \text{ and } F(u) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{2}, & \text{if } -1 \leq x < 1 \\ 1, & \text{if } 1 \leq x \end{cases}$$

$\lim_{n\to\infty} F_n^*(u) = F^*(u)$ for all continuity points of F^*. However, where F^* isn't continious at $\frac{1}{2}$, $\lim_{n\to\infty} F_n^*(\frac{1}{2}) = 0 \neq -1 = F^*(\frac{1}{2})$.

2.1. Let $F(x)$ not be strictly increasing over $A = \{x \in \mathbb{R} : 0 < F(x) < 1\}$.
$\Rightarrow \exists x_1, x_2 \in A : x_1 < x_2$ and $F(x_1) = F(x_2)$
$\Rightarrow F^*(F(x_1)) \leq x_1$, since $\inf\{x : F(x_1) \leq F(x)\} \leq x_2$
Let $B = \{x \in A : F(x) = F(x_1)\}$
$\Rightarrow \forall x \in \{x \in A : F(x) > F(x_1)\}, x \geq \sup B$, since the fact that F is nondecreasing makes x and upper bound on B
$\Rightarrow \forall v > F(x_1), F^*(v) \geq \sup B$
$\Rightarrow \forall \delta > 0$

$$\begin{aligned} F^*(F(x) + \delta) &\geq \sup B - F^*(F(x_1) + F^*(F(x_1)) \\ F^*(F(x) + \delta) - F^*(F(x_1)) &\geq \sup B - F^*(F(x_1) \\ &\geq x_2 - x_1 \end{aligned}$$

$\Rightarrow F^*(u+) \neq F^*(u)$

Now, let F be strictly increasing over A and $x \in A$ such that $x = F^*(u)$ and $F(x) < 1$. Let $y \in (x, 1)$.
$\Rightarrow \exists w \in (x, y)$
$\Rightarrow u \leq F(x) < F(w) < F(y)$ since F is strictly increasing.
$\Rightarrow u + F(w) - F(x) \leq F(w)$
$\Rightarrow F^*(u + F(w) - F(x)) \leq w < y$ by the switching formula.
$\Rightarrow F^*(u+) < y$
$\Rightarrow F^*(u+) = x = F^*(u)$

Between these two results, we get that F^* is continious if and only if F is strictly increasing.

2.2. a)

$$\begin{aligned} P[T(Y) \in B] &= P[Y \in x : T(x) \in B] \\ &= P[Z \in x : T(x) \in B] \\ &= P[T(Z) \in B] \end{aligned}$$

$\Rightarrow Y \ Z$

b) Let $P[Y = Z] = 1$

$$\begin{aligned} P[Y \in B] &= P[Y = Z \cap Y \in B] \\ &= P[Y = Z \cap Z \in B] \\ &= P[Z \in B] \end{aligned}$$

$\Rightarrow Y \ Z$