1. Problem 5.3 in Weisberg
   I got an estimate of $\hat{\sigma}^2 = 1860.276$, so for the test statistic I got that $G = \frac{\sum_{i=1}^n (\hat{y}_i - \tilde{y}_i)^2}{\hat{\sigma}^2} = 15.296$. Then, running the bootstrap algorithm with 1000 bootstraps, I got a significance level of .002, since only $\frac{1}{1000}$ of the bootstrap samples resulted in a higher test statistic.

2. a)

$$
\begin{aligned}
X_{(i)}^T X_{(i)} &= \sum_{j \neq i} x_j^T x_j \\
X_{(i)}^T X_{(i)} &= X^T X - x_i^T x_i \\
(X_{(i)}^T X_{(i)})^{-1} &= (X^T X - x_i^T x_i)^{-1}
\end{aligned}
$$

Applying the Sherman-Morrison formula to the right hand side, we get

$$
\begin{aligned}
(X_{(i)}^T X_{(i)})^{-1} &= (X^T X)^{-1} + \frac{(X^T X)^{-1} x_i^T x_i (X^T X)^{-1}}{1 - x_i (X^T X)^{-1} x_i^T} \\
(X_{(i)}^T X_{(i)})^{-1} &= (X^T X)^{-1} + \frac{(X^T X)^{-1} x_i^T x_i (X^T X)^{-1}}{1 - h_{ii}}
\end{aligned}
$$

b)

$$
\begin{aligned}
\hat{\beta}_{(i)} &= (X_{(i)}^T X_{(i)})^{-1} X_{(i)}^T Y_{(i)} \\
\hat{\beta}_{(i)} &= \left((X^T X)^{-1} + \frac{(X^T X)^{-1} x_i^T x_i (X^T X)^{-1}}{1 - h_{ii}} \right) \left(X^T Y - x_i^T y_i \right) \\
\hat{\beta}_{(i)} &= (X^T X)^{-1} \left(X^T Y - x_i^T y_i \right) + \frac{(X^T X)^{-1} x_i^T x_i (X^T X)^{-1}}{1 - h_{ii}} \left(X^T Y - x_i^T y_i \right) \\
\hat{\beta}_{(i)} &= \hat{\beta} - (X^T X)^{-1} x_i^T y_i + \frac{(X^T X)^{-1} x_i^T x_i (X^T X)^{-1} \left(X^T Y - x_i^T y_i \right)}{1 - h_{ii}} \\
\hat{\beta}_{(i)} &= \hat{\beta} + \frac{(X^T X)^{-1} x_i^T y_i (h_{ii} - 1) + (X^T X)^{-1} x_i^T \left(x_i \hat{\beta} - h_{ii} y_i \right)}{1 - h_{ii}} \\
\hat{\beta}_{(i)} &= \hat{\beta} + \frac{(X^T X)^{-1} x_i^T \left(x_i \hat{\beta} - h_{ii} y_i + (h_{ii} - 1) y_i \right)}{1 - h_{ii}} \\
\hat{\beta}_{(i)} &= \hat{\beta} + \frac{(X^T X)^{-1} x_i^T \left(x_i \hat{\beta} - y_i \right)}{1 - h_{ii}} \\
\hat{\beta}_{(i)} &= \hat{\beta} - \frac{(X^T X)^{-1} x_i^T \hat{e}_i}{1 - h_{ii}}
\end{aligned}
$$

c)

$$
\begin{aligned}
y_i - x_i \hat{\beta}_{(i)} &= y_i - x_i \left(\hat{\beta} - \frac{(X^T X)^{-1} x_i^T \hat{e}_i}{1 - h_{ii}} \right) \\
y_i - x_i \hat{\beta}_{(i)} &= y_i - \hat{y}_i + \frac{h_{ii} \hat{e}_i}{1 - h_{ii}} \\
y_i - x_i \hat{\beta}_{(i)} &= \hat{e}_i \left(1 + \frac{h_{ii}}{1 - h_{ii}} \right) \\
y_i - x_i \hat{\beta}_{(i)} &= \hat{e}_i \frac{1}{1 - h_{ii}}
\end{aligned}
$$

d)

$$D_i = \frac{\|\hat{Y}_{(i)} - \hat{Y}\|^2}{p\hat{\sigma}^2}$$

$$D_i = \frac{\|X\beta_{(i)} - X\beta\|^2}{p\hat{\sigma}^2}$$

$$D_i = \frac{\|X\left(\hat{\beta} - \frac{(X^TX)^{-1}x_i^T\hat{e}_i}{1-h_{ii}}\right) - X\beta\|^2}{p\hat{\sigma}^2}$$

$$D_i = \frac{\| - X\frac{(X^TX)^{-1}x_i^T\hat{e}_i}{1-h_{ii}}\|^2}{p\hat{\sigma}^2}$$

$$D_i = \frac{\frac{\hat{e}_ix_i(X^TX)^{-1}}{1-h_{ii}}X^TX\frac{(X^TX)^{-1}x_i^T\hat{e}_i}{1-h_{ii}}}{p\hat{\sigma}^2}$$

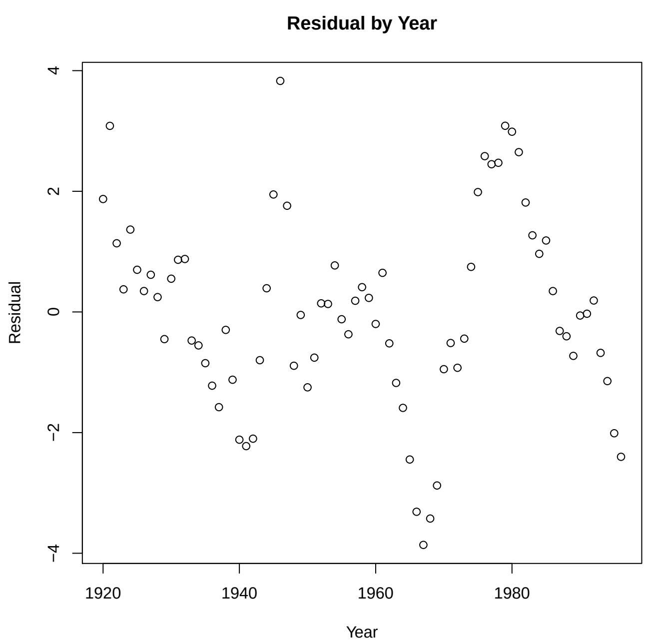
$$D_i = \frac{\frac{\hat{e}_ix_i(X^TX)^{-1}}{1-h_{ii}}X^TX\frac{(X^TX)^{-1}x_i^T\hat{e}_i}{1-h_{ii}}}{p\hat{\sigma}^2}$$

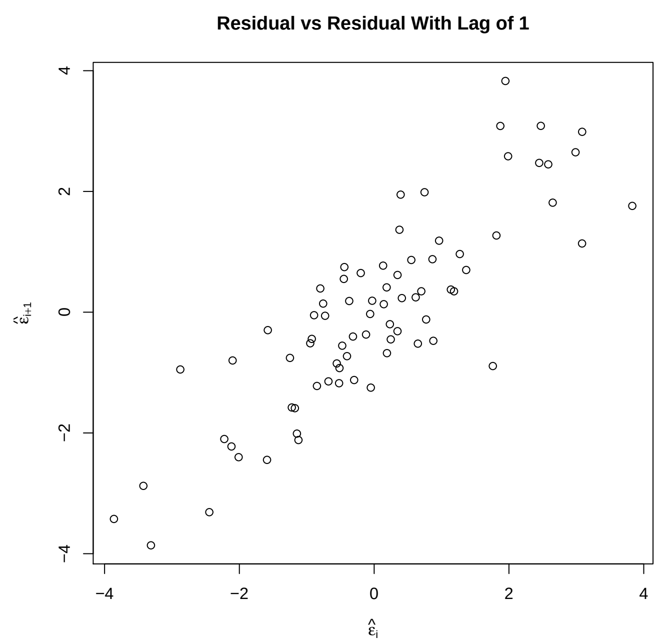
$$D_i = \frac{\frac{\hat{e}_ix_i(X^TX)^{-1}}{1-h_{ii}}\frac{x_i^T\hat{e}_i}{1-h_{ii}}}{p\hat{\sigma}^2}$$

$$D_i = \frac{1}{p}\frac{\hat{e}_i^2}{\hat{\sigma}^2(1-h_{ii})}\frac{x_i(X^TX)^{-1}x_i^T}{1-h_{ii}}$$

$$D_i = \frac{1}{p}r_i^2\frac{h_ii}{1-h_{ii}}$$

3) a) The two graphs make it pretty clear there is autocorrelated errors. The first, "Residual by Year", seems to show the errors moving up and down together, rather than being the cloud centered around 0 if there was no autocorrelation. The second, "Residual vs Residual With Lag of 1" shows each residual plotted against the next residual. The evidence of autocorrelation is overwhelming here, with there being a clear positive linear relationship between one error and the next.





b) Running the AR1 model, we see an estimated autocorrelation of .972, which is extremely significant. We also see unemployed switch to being significant with a p-value of .02 in the AR1 model while just outside of of significance at .051 in the ordinary linear model. Much more importantly though, the

coefficient switches sign, speaking to the autocorrelation having a meaningful effect.
Confidence interval and estimate for autocorrelation:

	lower	est.	upper
Phi	0.653	0.972	0.998

Coefficients for autocorrelated model:

	Value	Std.Error	t-value	p-value
(Intercept)	-7.060	5.547	-1.273	0.207
unemployed	0.108	0.046	2.344	0.022
femlab	0.312	0.095	3.280	0.002
marriage	0.164	0.023	7.177	0.000
birth	-0.050	0.022	-2.267	0.026
military	0.018	0.014	1.258	0.213

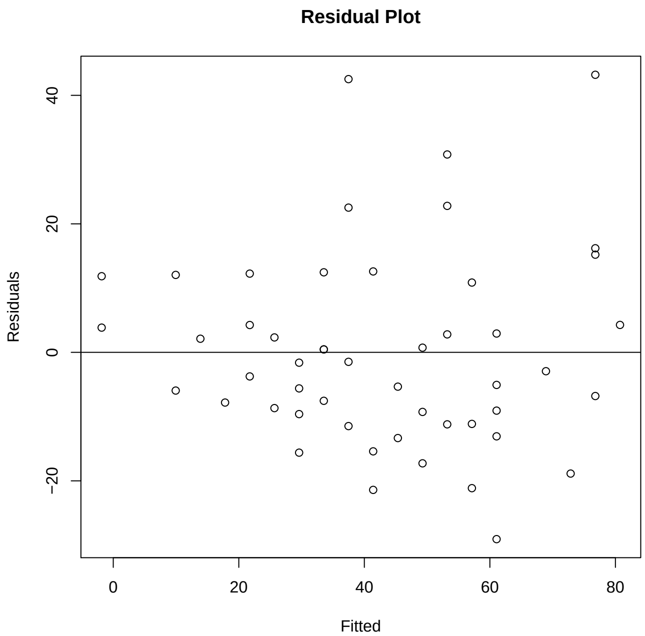
Coefficients for regular linear model:

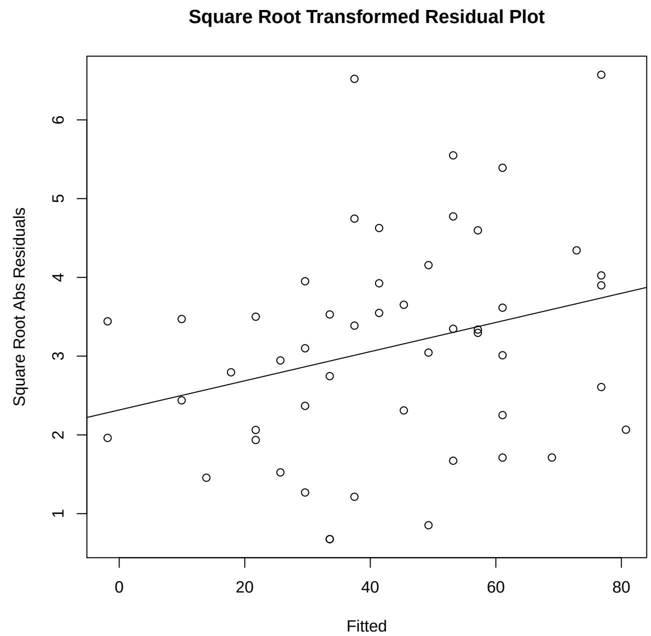
	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	2.488	3.394	0.733	0.466
unemployed	-0.111	0.056	-1.989	0.051
femlab	0.384	0.031	12.543	0.000
marriage	0.119	0.024	4.861	0.000
birth	-0.130	0.016	-8.333	0.000
military	-0.027	0.014	-1.876	0.065

Only one coefficient, that of unemployed, switched to the other side of the .05 threshold with the auto correlated term, but the estimate and p-value of all the other variables changes noticeably, speaking to how not dealing with autocorrelation can invalidate a model.

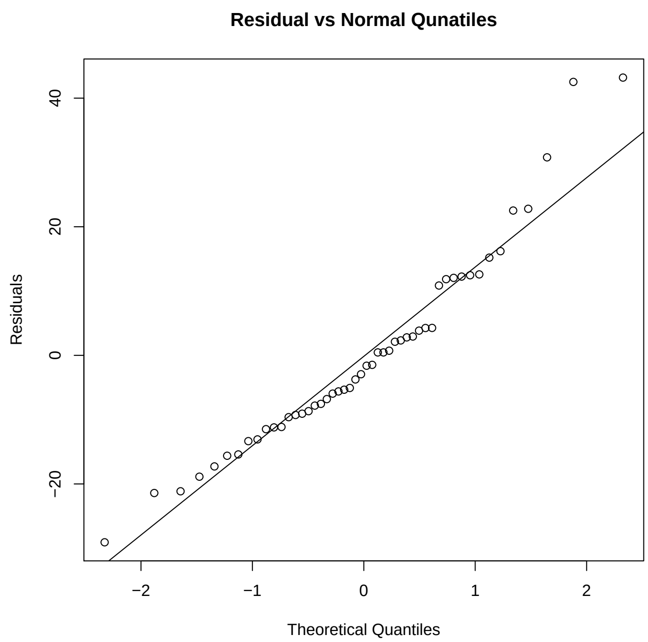
c) My suspicion is that the autocorrelation largely has to do with the relative size of different generations. The portion of the US population in different age groups varies a lot over time, and divorce rates are likely quite different between different age groups. Since the relative portion of people in each age group is highly auto-correlated (since everyone either dies or gets just one year older), our divorce rate is also auto-correlated.

4) It looks like we have a lack of fit since the variance isn't constant. Looking at the first plot, the residual plot, nothing is glaringly wrong, but it looks like the points farther to the right are somewhat more spread out. Plotting the square root transformed residuals against the predicted values, and adding a trend line, we see a much more obvious increase in variance over the range, with a clear upwards trend. Testing this formally with the variance test, where we split the residuals into those with speed above and below 14, we get a ratio of .206, with an associated p-value of .0009, which seals it. We also appear to have a mildly non-linear response.

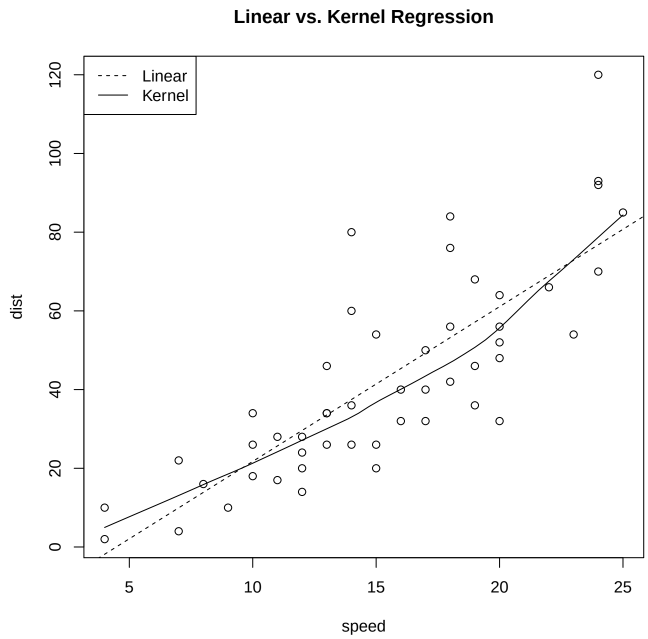




As far as other issues, none jump out. From the residual plot, no residual looks extreme enough to be an outlier. Running a QQ plot, the residuals on the whole look roughly normal.



The one potential issue is that the true relationship between the predictor and outcome may not quite be linear. Looking at the linear fit versus a kernel regression fit, it seems like the function may actual be somewhat convex.



Attempting to approximate this with a linear spline, we allow for a different slope above and below a speed of 20, where we seem to have a mild kink. When we do this, and run we ANOVA, we just get significant decrease in variance, with a p-value of .0378.

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	48.000	11353.521				
2	47.000	10347.643	1.000	1005.878	4.569	0.038

5) In general, it seems to be the case that without dealing with outliers, one can get meaningfully different results between robust regression and non-robust regression. However, when outliers are taken out, they give much more similar results. We see this most strongly with the coefficient for air flow. With the outlier in, we get coefficients of about .8 with robust regression and a smaller, though still significant, coefficient of .7 with regular linear regression. When an outlier is removed, all the methods get about .8.

With the outliers in, we got the results below. In addition to the difference with air flow noted above, the coefficient with water temp seems to vary a lot between methods, though is always significant, while acid concentration is significant in every one except the least absolute deviation, where its just barely significant.

a) Regular least squares:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-39.920	11.896	-3.356	0.004
Acid.Conc.	-0.152	0.156	-0.973	0.344
Water.Temp	1.295	0.368	3.520	0.003
Air.Flow	0.716	0.135	5.307	0.000

b) Least absolute deviation:

	coefficients	lower bd	upper bd
(Intercept)	-39.690	-41.620	-29.678
Acid.Conc.	-0.061	-0.213	-0.029
Water.Temp	0.574	0.322	1.411
Air.Flow	0.832	0.523	1.141

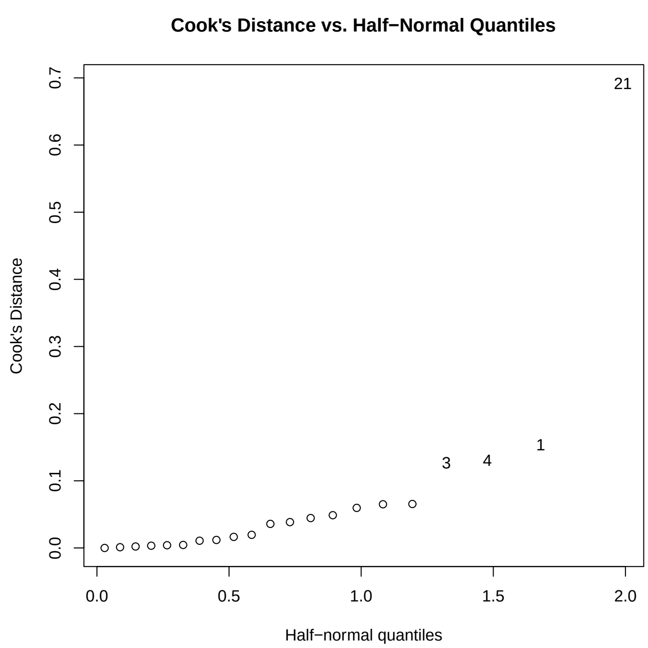
c) Hubber method:

	Value	Std. Error	t value
(Intercept)	-41.027	9.807	-4.183
Acid.Conc.	-0.128	0.129	-0.992
Water.Temp	0.926	0.303	3.052
Air.Flow	0.829	0.111	7.460

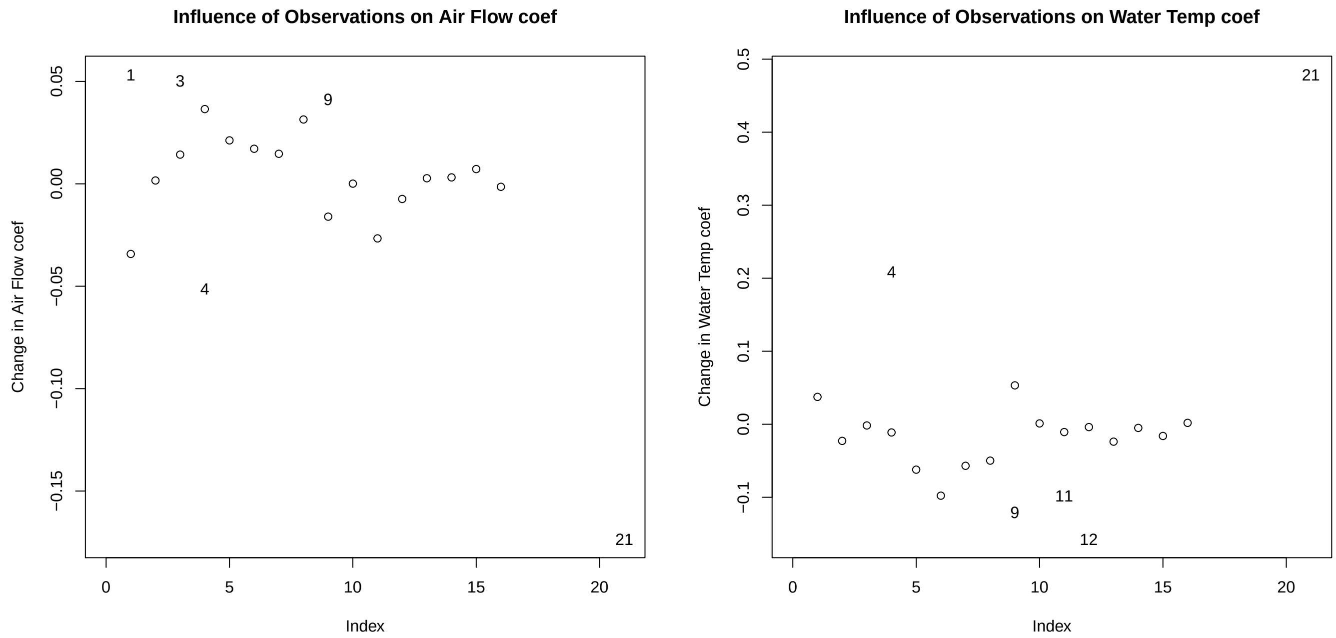
d) Least trimmed squares:

	coef(sl.lts)
(Intercept)	-35.806
Acid.Conc.	-0.000
Water.Temp	0.333
Air.Flow	0.750

Looking at the Cook's Distances, its clear that the 21st observation is immensely influential. 3, 4, and 1 also have relatively high Cook's Distances, but they don't seem that out of line with the rest of them. Double checking the studentized residuals, 21 a value of 3.330, while the next highest is 2.052. Since the Bonferroni critical value at a p-value of .05 is 3.256, I concluded that 21 is worth removing, while the rest can remain.



Looking over the effects of the inclusion of each observation on the coefficients, we see that 21 has a much bigger effect over air flow and water temp than other variables, further supporting that it should be removed as an outlier.



Rerunning each of the regressions with 21 removed, we get the results below. The air flow coefficient is now fairly consistent between the robust and non-robust regressions. The water temp coefficient is still meaningfully different between the robust and non-robust regressions, though much smaller than before. It may be that observation 4, as seen above, is still having undue influence. Finally, acid concentration continues to be insignificant, or just barely significant.

a) Regular least squares:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-43.704	9.492	-4.605	0.000
Acid.Conc.	-0.107	0.125	-0.860	0.402
Water.Temp	0.817	0.325	2.512	0.023
Air.Flow	0.889	0.119	7.481	0.000

b) Least absolute deviation:

	coefficients	lower bd	upper bd
(Intercept)	-39.986	-54.137	-30.213
Acid.Conc.	-0.057	-0.398	-0.035
Water.Temp	0.564	0.267	1.107
Air.Flow	0.835	0.825	1.167

c) Hubber method:

	Value	Std. Error	t value
(Intercept)	-42.841	8.619	-4.970
Acid.Conc.	-0.108	0.113	-0.953
Water.Temp	0.685	0.295	2.322
Air.Flow	0.918	0.108	8.509

d) Least trimmed squares:

	coef(sl.lts.no)
(Intercept)	-35.806
Acid.Conc.	-0.000
Water.Temp	0.333
Air.Flow	0.750