CSCI-358 Spring 2025

Midterm 01

Please note that you should type your exam using either LATEX or Word. Both templates will be provided. **Hand-written assignments will not be graded.** You need to submit a **pdf** version on Gradescope by the due date given on Canvas. You are **not** allowed to collaborate with your peers on this exam.

1. (5 points) Provide the truth table for the following formula:

$$((A\to C)\land\neg B)\lor((\neg C\lor B)\to\neg A)$$

Solution:

A	B	C	A→C	¬B	¬A	(A→C)∧¬B	¬C	¬C∨B	(A→C)∧¬B	(¬C∨B)→¬A	((A→C)∧¬B)∨((¬C∨B)→¬A)
T	T	T	T	F	F	F	F	T	F	F	F
T	T	F	F	F	F	F	T	T	F	F	F
T	F	T	T	T	F	T	F	F	T	T	T
T	F	F	F	T	F	F	T	T	F	F	F
F	T	T	T	F	T	F	F	T	F	T	T
F	T	F	T	F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	F	F	T	T	T
F	F	F	T	T	T	T	T	T	T	T	T

2. (5 points) Simplify the following formula as much as possible (resulting in an equivalent formula using 3 letters) using a series of equivalence laws. DO NOT use a truth table.

$$(\neg(R\to\neg Q)\lor(Q\land\neg R))\lor\neg(P\to R)\lor\neg(\neg Q\lor R)$$

Solution: Q ∨ (P∧¬R)

$(\neg(R\to\neg Q)\lor(Q\land\neg R))\lor\neg(P\to R)\lor\neg(\neg Q\lor R)$

$(\neg(\neg R\lor\neg Q) \quad\lor\quad (Q\land\neg R)) \quad\lor\quad \neg(\neg P\lor R) \quad\lor\quad \neg(\neg Q\lor R)$

$((R\land Q) \quad\lor\quad (Q\land\neg R)) \quad\lor\quad (P\land\neg R) \quad\lor\quad (Q\land\neg R)$

$(R\land Q) \quad\lor\quad (Q\land\neg R) \quad\lor\quad (P\land\neg R)$ (Duplicate)

$Q \quad\land\quad (R\lor\neg R) \quad\lor\quad (P\land\neg R)$

$Q \quad\lor\quad (P\land\neg R)$

$Q\lor(P\land\neg R)$

3. (10 total points) Prove the following using a series of equivalence laws.

   (a) (5 points) Prove the following is a contradiction:

$$\neg((\neg(\neg Q\to R)\land(R\land\neg P))\to(R\land P))$$

Solution: Contradiction

¬(	¬((¬Q→R))	∧	(R∧¬P)	→	(R∧P))
¬(	(¬Q∧¬R)	∧	(R∧¬P)	→	(R∧P))
	¬Q∧(¬R∧R)		∧ ¬P		
¬(	False			→	(R∧P))
¬(	True)				
False					

(b) (5 points) Prove the following is a tautology:

$$(\neg(A\rightarrow\neg B)\vee\neg(B\wedge A))\wedge((C\vee\neg B)\vee(A\vee\neg C))$$

Solution:Tautology

(¬(A→¬B)	∨	¬(B∧A))	∧	((C∨¬B)	∨	(A∨¬C))
¬(¬A∨¬B)	∨	¬(B∧A)	∧	((C∨¬B)	∨	(A∨¬C))
(A∧B)	∨	(¬B∨¬A)	∧	((C∨¬B)	∨	(A∨¬C))
True (if a or b is false, then true, if both are true then true)						
(A∨¬A) ∧ (¬B∨B)						
True						
True			∧	C∨¬B∨A∨¬C		
True			∧	true ∨ ¬B ∨ A		
True	∧	True				
True						

4. (20 total points) Prove the following arguments as valid. Use a two-column proof format and indicate the line numbers used to derive each step.
   (a) (10 points) $(C \vee \neg B) \wedge (C \rightarrow \neg A) \rightarrow (A \rightarrow (\neg B \wedge \neg C))$

Solution: Valid argument

(a) $(C \vee \neg B) \wedge (C \rightarrow \neg A) \rightarrow (A \rightarrow (\neg B \wedge \neg C))$

1.	$(C \vee \neg B) \wedge (C \rightarrow \neg A)$	Premise
2.	$(C \vee \neg B)$	Assume
3.	$(C \rightarrow \neg A)$	Assume
4.	A	Assume
5.	$A \rightarrow \neg C$	Contrapositive (3)
6.	$\neg C$	(4,5)
7.	$\neg B$	(2,6)
8.	$\neg B \wedge \neg C$	(6,7)
9.	$A \rightarrow (\neg B \wedge \neg C)$	(4,8)
10.	$(C \vee \neg B) \wedge (C \rightarrow \neg A) \rightarrow (A \rightarrow (\neg B \wedge \neg C))$	(2,3,9)

   (b) (10 points) $\neg(A \vee \neg B) \wedge (B \rightarrow (A \vee C)) \rightarrow C$

Solution: This is a valid statement

1.	$\neg(A \vee \neg B)$	Premise
2.	$(B \rightarrow (A \vee C)$	Premise
3.	$\neg A \wedge B$	1
4.	$\neg A$	Conjunction Elimination (3)
5.	B	Conjunction Elimination (3)
6.	$A \vee C$	(2,5)
7.	C	(since 4, C must be true)
8.	$\neg(A \vee \neg B) \wedge (B \rightarrow (A \vee C))$	(1-7)

5. (20 total points) Prove the following predicate arguments using a sequence of equivalence laws and inference rules. Use a two-column proof format and indicate the line numbers used to derive each step.
   (a) (10 points)

$$\exists x(L(x) \leftrightarrow R(x)) \wedge \forall x((R(x) \vee M(x)) \wedge \neg L(x)) \rightarrow \exists x(M(x))$$

Solution:This statement is valid

$$\exists x(L(x)\leftrightarrow R(x))\wedge\forall x((R(x)\vee M(x))\wedge\neg L(x))\rightarrow\exists x(M(x))$$

1.	$\exists x(L(x)\leftrightarrow R(x))$	Premise
2.	$\forall x((R(x)\vee M(x))\wedge\neg L(x))$	Premise
3.	$(R(x)\vee M(x))\wedge\neg L(x)$	2
4.	$R(x)\vee M(x)$	Conjunction Elimination (2)
5.	$\neg L(x)$	Conjunction Elimination (2)
6.	$L(x)\leftrightarrow R(x)$	1
7.	$\neg L(x)\rightarrow\neg R(x)$	6
8.	$\neg R(x)$	5,7
9.	$M(x)$	4,8
10.	$\exists x(M(x))$	9

(b) (10 points)

$$\forall x(A(x)\rightarrow B(x))\wedge\exists x(\neg B(x))\wedge\forall x(\neg C(x)\rightarrow A(x))\rightarrow\exists x(\neg A(x)\wedge C(x))$$

Solution: This statement is valid

$$\forall x(A(x)\rightarrow B(x)) \wedge \exists x(\neg B(x)) \wedge \forall x(\neg C(x)\rightarrow A(x))\rightarrow\exists x(\neg A(x)\wedge C(x))$$

1.	$\forall x(A(x)\rightarrow B(x))$	Premise
2.	$\exists x(\neg B(x))$	Premise
3.	$\forall x(\neg C(x)\rightarrow A(x))$	Premise
4.	$A(x)\rightarrow B(x)$	1
5.	$\neg B(x)$	2
6.	$\neg C(x)\rightarrow A(x)$	3
7.	$\neg A(x)$	4,5
8.	$\neg A(x)\rightarrow C(x)$	6
9.	$C(x)$	7,8
10.	$\neg A(x)\wedge C(x)$	7,9
11.	$\exists x(\neg A(x)\wedge C(x))$	10
12.	$\forall x(A(x)\rightarrow B(x)) \wedge \exists x(\neg B(x)) \wedge \forall x(\neg C(x)\rightarrow A(x))\rightarrow\exists x(\neg A(x)\wedge C(x))$	

6. (10 total points) For the following questions, translate the English statement into a logical expression using predicates, quantifiers, and logical connectives, or translate the logical expression to English.

   1. Let $M(x)$ denote "x is a musician"
   2. Let $T(x)$ denote "x plays the trumpet"
   3. Let $P(x)$ denote "x plays the piano"
   4. Let $I(x,\ y)$ denote "x inspired y"
   5. Let l denote Loius
   6. The domain on x is **all people**

   Do not translate each statement exactly literally. For example do not translate $\forall x(P(x) \rightarrow T(x))$ as *"For all people x, if x plays the piano, then x plays the trumpet."* Instead, a correct translation could be *"All piano players play the trumpet." or "Every piano player can also play trumpet"*.

   (a) (3 points) Translate to a logical expression: Anyone who plays the trumpet is a musician.

   Solution: $\forall x(T(x) \rightarrow M(x))$

   (b) (3 points) Translate to a logical expression: All musicians have another musician who inspired them.

   Solution: $\forall x(M(x) \rightarrow \exists y(M(y) \wedge I(y,x))$

   (c) (2 points) Translate to English: $T(l) \wedge \forall x(T(x) \rightarrow I(l,\ x))$

   Solution: Louis is a trumpet player and has inspired every trumpet player

   (d) (2 points) Translate to English: $\exists x(\neg T(x) \wedge P(x))$

   Solution: Someone does not play the trumpet but plays the piano

7. (10 total points) Prove the following statements.
   (a) (5 points) For any two integers x and y, if $x \times y \geq 30$ then $|x| \geq 6$ or $|y| \geq 6$.

Solution:

Proof by Contrapositive $\forall x,y \in Z, (x \cdot y \geq 30) \rightarrow (x \geq 6 \vee |y| \geq 6)$

Contrapositive: $(x<6 \wedge |y|<6) \rightarrow (x \cdot y<30)$

x<6 means that $x \leq 5$

$|y|<6$ means that $-5 \leq y \leq 5$

Maximum occurs at x = 5 and y = 5

Minimum occurs at x = 5 and y = -5

In Max: $x \cdot y = 25$

In Min: $x \cdot y = -25$

25 is less than 30, and 25 is the maximum value

Since the contrapositive is true, the original statement is also true

   (b) (5 points) Let x and y be integers. Prove that if x and y satisfy the equation $16x + 6y = 110$, then at least one of x or y must be odd.

Solution: $16x+6y=110$

Proof. Prove that both x and y cannot be positive

$16(2a) + 6(2b) = 110$ *x=2a and y=2b guarantees that x and y are even*

$2(16a + 6b) = 110$

$(16a + 6b) = 55$

Both 16a and 6b are guaranteed to be even, and we cannot add two even numbers to get an odd number, this equation is impossible assuming both numbers are even.

Therefor, at least one of the numbers, either x or y, must be odd.

8. (20 points) Use mathematical induction to prove the following:
   (a) (10 points) $7^{2n} + 16n - 1$ is divisible by 64 for all integers $n \geq 1$.

Solution: $7^{2n}+16n-1$ is divisible by 64 for all integers $n \geq 1$.

Proof.

Base case n = 1 49 + 16 - 1 = 64 which is divisible by 64

Formula $7^{2n}+16n - 1 = 64a$ a is some integer providing multiples of 64

n+1 $7^{2(n+1)}+16(n+1) - 1 = 7^{2n+2}+16n+16 - 1$

$(7^{2n}\cdot 7^2)+16n+16-1 = 49(7^{2n})+16n+15$

$49(64a -16n +1) + 16n + 15 = 49*64a - 48*16n + 64$

$64(49a - 12n + 1)$ Since any value of a and n is multiplied by 64, the answer must also be divisible by 64. As both the base case n=1 and the case n+1 are both true, the statement must be true.

(b) (10 points) Prove that $\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

Solution:

Proof.

Base case n = 1 LEFT = 1/2 RIGHT = 1 - 1/2 = 1/2 Equal

Case k + 1 Sum (i=1 through k+1) for 1/((i+1)!) = **Sum (i=1 through k) for 1/((i+1)!)**+1/((k+2)!)

Replace bolded part as inductive hypothesis Highlighted left side to read easier

Sum (i=1 through k+1) for 1/((i+1)!) = (1 - 1/((k+1)!)) + (1/((k+2)!)

Sum (i=1 through k+1) for 1/((i+1)!) = (1 - 1/((k+2)!))

(1 - 1/((k+2)!)) = (1 - 1/((k+2)!))

Proving that k+1 holds true as well as the base case proves that this is true.