Companion to "Demand Estimation from Censored Observations with Inventory Record Inaccuracy"

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This document includes appendices for "Demand Estimation from Censored Observations with Inventory Record Inaccuracy." References herein point to that document, and we use notation specified there.

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A. A Corollary to Proposition 1

The following corollary to Proposition 1 holds for the general model of Section 4.

Corollary 4. Suppose that

a) the discrepancies ϵ_t are independent over time and the distribution H_t is stationary and such that $\Pr\{\epsilon_t > 0\} \geq \Pr\{\epsilon_t < 0\}$ (and $\Pr\{\epsilon_t = 0\} < 1$),

b) demand follows a newsvendor distribution with known $g()$ and unknown parameter $\theta > 0$, on which the sellers share a non-degenerate prior distribution with well-defined density, and

c) the naive and tracking sellers stock the same fixed, bounded sequence of quantities $0 < j_t < M$ and experience the same realization of demands and discrepancies.

Then with probability approaching one as $t \to \infty$, the tracking seller's predictive distribution of demand $\hat{F}_t(\cdot|\vec{s}_t, \vec{c}_t^r)$ first-order stochastically dominates the naive seller's predictive distribution of demand $\hat{F}_t(\cdot|\vec{s}_t, \vec{c}_t^n)$.

Corollary 4 is proven with the help of two lemmas.

Lemma 5. Under the assumptions of Corollary 4, $\Pr\{C_\tau^n = 1\} - \Pr\{C_\tau^r = 1\} = \mathsf{E}\left[C_\tau^n - C_\tau^r\right] \leq \eta$ where $\eta < 0$ is a constant with respect to j_τ.

Proof. We consider two cases. First, suppose $\Pr\{\epsilon < 0\} = 0$ so that $\Pr\{j_\tau \geq I_\tau\} = 1$ and case (C) of Table 1 occurs with probability zero. Under this assumption, Table 1 implies that we can write

$$\begin{aligned}
\Pr\{C_\tau^n = 1\} - \Pr\{C_\tau^r = 1\} &= -\Pr\{j_\tau > D_\tau \geq I_\tau\} - \Pr\{D_\tau \geq j_\tau > I_\tau\} \\
&= -\Pr\{D_\tau \geq j_\tau > I_\tau\} \leq -\Pr\{D_\tau \geq j_\tau\}\Pr\{\epsilon_\tau > 0\} \\
&\leq -\Pr\{D_\tau \geq M\}\Pr\{\epsilon_\tau > 0\} = -\Pr\{\epsilon_\tau > 0\}e^{-\theta g(M)},
\end{aligned}$$

where the first inequality relies on the independence of D_τ and I_τ given j_τ.

Second, suppose $\Pr\{\epsilon < 0\} > 0$. The first inequality below is in the proof of Proposition 1.

$$\begin{aligned}
\Pr\{C^n_\tau = 1\} - \Pr\{C^r_\tau = 1\} &\leq \Pr\{D_\tau \geq j_\tau \geq I_\tau\} - \Pr\{D_\tau \geq I_\tau\} \\
&= \Pr\{D_\tau > j_\tau \geq I_\tau\} - \Pr\{D_\tau \geq I_\tau > j_\tau\} - \Pr\{j_\tau > D_\tau \geq I_\tau\} - \Pr\{D_\tau \geq j_\tau \geq I_\tau\} \\
&\leq -\Pr\{D_\tau \geq I_\tau > j_\tau\} \\
&= -\Pr\{D_\tau \geq I_\tau | I_\tau > j_\tau\} \Pr\{\epsilon_\tau < 0\} \\
&\leq -\Pr\{D_\tau \geq M - \epsilon_\tau | I_\tau > j_\tau\} \Pr\{\epsilon_\tau < 0\} \\
&\leq -(1/2) \Pr\{D_\tau \geq M - m | I_\tau > j_\tau\} \Pr\{\epsilon_\tau < 0\} \\
&= -(1/2) \Pr\{\epsilon_\tau < 0\} e^{-\theta g(M-m)},
\end{aligned}$$

where m is the median of the distribution of $\epsilon | \epsilon < 0$. ■

Lemma 6. Under the assumptions of Corollary 4, $\lim_{t\to\infty} \Pr\left\{\sum_{\tau=1}^t (C^n_\tau - C^r_\tau) < 0\right\} = 1$

Proof. Consider the random variables

$$X(\tau) = C^n_\tau - C^r_\tau, \qquad Y(t) = \sum_{\tau=1}^t (C^n_\tau - C^r_\tau) = \sum_{\tau=1}^t X(\tau)$$

Note that $X(\tau)$ is defined on $\{-1, 0, 1\}$ and therefore its variance $v(\tau)$ is upper bounded by 4. Because the $X(\tau)$'s are independent of each other conditional on $j_1, \ldots, j_t$, therefore $V(t) \equiv \mathsf{Var}\,(Y(t)) = \sum_{\tau=1}^t v(\tau) \leq 4t$. By Lemma 5 we have that $\overline{Y}(t) \equiv \text{Mean}(Y(t)) \leq \eta t < 0$. The one-tailed Chebyshev inequality applied to $Y(t)$, yields

$$\Pr\left\{Y(t) - \overline{Y}(t) \geq k\sqrt{V(t)}\right\} \leq \frac{1}{1+k^2} \text{ for any } k > 0.$$

Specifically, set $k = -\overline{Y}(t)/\sqrt{V(t)}$, then

$$\Pr\{Y(t) \geq 0\} = \Pr\{Y(t) - \overline{Y}(t) \geq -\overline{Y}(t)\} \leq \left(1 + \frac{(\overline{Y}(t))^2}{V(t)}\right)^{-1} \leq \left(1 + \frac{\eta^2 t^2}{4t}\right)^{-1} = \left(1 + \frac{\eta^2 t}{4}\right)^{-1} \xrightarrow{t\to\infty} 0.$$

■

To complete the proof of Corollary 4, it remains to show that $\sum_{\tau=1}^t (c^n_\tau - c^r_\tau) = \sum_{\tau=1}^t (1 - c^r_\tau) - \sum_\tau^t (1 - c^n_\tau) < 0$ implies that the tracking seller's predictive distribution stochastically dominates that of the naive seller. Because the naive and tracking sellers stock the same quantities and are subject to the same sequences of demands and errors, we have $s^r_\tau = s^n_\tau$ for all τ and therefore $\sum_{\tau=1}^t g(s^r_t) = \sum_{\tau=1}^t g(s^n_t)$. The desired result then follows directly from Proposition 6.3 of Braden and Freimer (1991).

B. Numerical Solutions to Equation (12)

In this appendix we present data from a numerical investigation of equation (12) and associated simulations. Data from this appendix is plotted in Figures 2 and 3 and discussed in Section 5.4. Tables 3 and 4 show solutions to equation (12) for $\theta = 1/20$ and $\theta = 1/2$, respectively. Each row of the tables represents a separate instance. For each instance, we report the following values:

- Discrepancy distribution: We assume stationary distributions for the discrepancies ϵ_t.
- γ: Critical fractile parameter to naive seller's stocking policy.
- Adjusted γ: The (type 1) service level obtained by the critical fractile policy if θ were known. There can be a small difference between γ and Adjusted γ because discrepancies interfere with service even if θ is known.
- Actual SL: The actual (type 1) service level as seem by customers when $\tilde{j}$ is stocked. This is determined analytically by comparing the distribution of D_t with the distribution of $I_t = \tilde{j} - \epsilon$.
- $1 - \mathsf{E}[\tilde{C}^n]$: The service level implied by the stockout indicators observed by the naive seller.
- $1/\tilde{\theta}$: The naive seller's estimate of mean demand at the fixed point.
- Simulated 95% CI: We simulated the estimate-order-estimate process for 100 replications begining from a gamma$(3, \theta/3)$ prior. We report 95% confidence intervals for the posterior means after 10000 periods. (We conservatively chose 10000 periods so that our estimates stabilize and are not impacted by initial conditions. We have found that fewer periods are often sufficient for this purpose.)

For five instances in Table 4 we found no solution to equation (12). For these instances, we calculate "Actual SL" and $1 - \mathsf{E}[\tilde{C}^n]$ based on the stocking quantity implied by the simulated average posterior mean.

C. Numerical Results for Non-Stationary Model

In this appendix we present a set of numerical solutions to equation (8). We solve the equation for a set of instances with true exponential demand parameter $\theta = 1/20$, cycle count frequency $T = 10$, and error processes roughly matched to those used in generating Figure 2. Specifically, we use normal error distributions with variances 0.182 and 0.727, which correspond with average discrepancy variances of 1 and 4 over a 10-period cycle, respectively. We try a ZeroNormal error distribution in which no error occurs with probability 0.947 each period and a normal(0,4) error occurs otherwise; this yields an average $\Pr\{\epsilon > 0\} = 0.125$ as in the ZeroNormal distribution used in Figure 2. We try a ZeroExponential error distribution in which no error occurs with probability 0.869 each period and an exponential(1) error occurs otherwise; this yields an average $\Pr\{\epsilon > 0\} = 0.5$ as in the ZeroExponential distribution used in Figure 2.

Discrepancy Distribution	γ	Adj. γ	Actual SL	1-$\mathsf{E}[\tilde{C}^n]$	$1/\tilde{\theta}$	Simulated 95% CI
$\epsilon \sim \text{normal}(0,1)$	0.70	0.700	0.590	0.795	14.84	[14.82 14.91]
	0.80	0.800	0.747	0.874	17.10	[17.08 17.16]
	0.85	0.850	0.819	0.909	18.00	[17.99 18.07]
	0.90	0.900	0.885	0.942	18.78	[18.75 18.84]
	0.95	0.950	0.946	0.973	19.44	[19.37 19.44]
	0.99	0.990	0.990	0.995	19.90	[19.88 19.96]
$\epsilon \sim \text{normal}(0,4)$	0.70	0.699	0.587	0.795	14.78	[14.76 14.85]
	0.80	0.799	0.746	0.873	17.07	[17.05 17.14]
	0.85	0.849	0.817	0.909	17.98	[17.97 18.05]
	0.90	0.900	0.884	0.942	18.76	[18.74 18.83]
	0.95	0.950	0.945	0.973	19.44	[19.36 19.44]
	0.99	0.990	0.990	0.995	19.90	[19.88 19.96]
$\epsilon \sim \begin{cases} \delta_0 & \text{w.p. } 3/4 \\ \text{normal}(0,4) & \text{w.p. } 1/4 \end{cases}$	0.70	0.700	0.679	0.719	18.87	[18.80 18.89]
	0.80	0.800	0.789	0.816	19.35	[19.31 19.39]
	0.85	0.850	0.843	0.863	19.54	[19.51 19.59]
	0.90	0.900	0.897	0.910	19.71	[19.68 19.77]
	0.95	0.950	0.949	0.955	19.87	[19.82 19.90]
	0.99	0.990	0.990	0.991	19.97	[19.96 20.05]
$\epsilon \sim \begin{cases} \delta_0 & \text{w.p. } 1/2 \\ \text{exponential}(1) & \text{w.p. } 1/2 \end{cases}$	0.70	0.692	0.569	0.895	14.40	[14.38 14.48]
	0.80	0.795	0.737	0.936	16.90	[16.85 16.94]
	0.85	0.846	0.812	0.954	17.87	[17.83 17.92]
	0.90	0.897	0.881	0.971	18.70	[18.67 18.75]
	0.95	0.949	0.944	0.986	19.41	[19.38 19.46]
	0.99	0.990	0.989	0.997	19.89	[19.90 19.98]

Table 3: Solutions to equation (12) for $1/\theta = 20$.

Discrepancy Distribution	γ	Adj. γ	Actual SL	1-$\mathsf{E}[\tilde{C}^n]$	$1/\tilde{\theta}$	Simulated 95% CI
$\epsilon \sim \text{normal}(0,1)$	0.70	0.662	0.461	0.752	1.165	[1.283 1.298]
	0.80	0.774	0.686	0.861	1.591	[1.592 1.604]
	0.85	0.830	0.780	0.903	1.728	[1.725 1.735]
	0.90	0.887	0.863	0.940	1.838	[1.829 1.838]
	0.95	0.943	0.937	0.972	1.927	[1.922 1.930]
	0.99	0.989	0.988	0.995	1.987	[1.980 1.988]
$\epsilon \sim \text{normal}(0,4)$	0.70	0.598	0.452†	0.757†	NA	[1.194 1.204]
	0.80	0.706	0.568†	0.833†	NA	[1.361 1.374]
	0.85	0.770	0.660†	0.880†	NA	[1.498 1.510]
	0.90	0.840	0.760	0.922	1.609	[1.662 1.674]
	0.95	0.918	0.898	0.969	1.852	[1.849 1.858]
	0.99	0.984	0.983	0.995	1.976	[1.968 1.977]
$\epsilon \sim \begin{cases} \delta_0 & \text{w.p. } 3/4 \\ \text{normal}(0,4) & \text{w.p. } 1/4 \end{cases}$	0.70	0.675	0.616	0.685	1.699	[1.807 1.818]
	0.80	0.777	0.747	0.800	1.836	[1.869 1.879]
	0.85	0.830	0.812	0.854	1.888	[1.898 1.907]
	0.90	0.885	0.876	0.905	1.931	[1.932 1.940]
	0.95	0.942	0.939	0.954	1.968	[1.966 1.974]
	0.99	0.988	0.988	0.991	1.994	[1.987 1.995]
$\epsilon \sim \begin{cases} \delta_0 & \text{w.p. } 1/2 \\ \text{exponential}(1) & \text{w.p. } 1/2 \end{cases}$	0.70	0.595	0.262†	0.690†	NA	[0.776 0.810]
	0.80	0.720	0.541†	0.827†	NA	[1.311 1.330]
	0.85	0.786	0.627	0.863	1.365	[1.533 1.546]
	0.90	0.855	0.797	0.929	1.694	[1.719 1.730]
	0.95	0.926	0.911	0.970	1.876	[1.874 1.883]
	0.99	0.985	0.984	0.995	1.979	[1.973 1.981]

Table 4: Solutions to equation (12) for $1/\theta = 2$. In cases marked with †, in which the equation has no solution, we estimate Actual SL and $1 - \mathsf{E}[\tilde{C}^n]$ based the simulated average posterior mean demand.

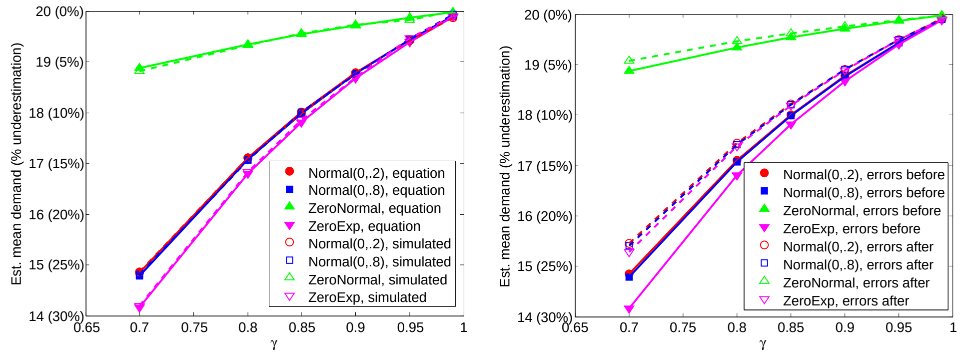

Figure 8: Numerical solutions of equation (8) compared with simulated results (left) and model with inventory errors occurring at the end of periods (right).

The left pane of Figure 8 shows solutions to equation (8). A comparison with Figure 2 reveals that the biases are similar in the two figures. This is expected given the similarity of equations (12) and (8) and given that we have matched average error parameters to those used in Figure 2.

The left pane also reports results of a simulation based on the model in Section 5.1, but in which we relax a modeling assumption. Specifically, we do not allow the seller to return articles to her supplier at inspection epochs. Here we assume that the naive seller can update her demand estimates each period and that she may not return articles to her supplier. We report demand estimates after 10000 periods, averaged over 100 simulation runs. We observe that the simulation results are hardly distinguishable from the equation solutions, suggesting that the impact of this assumption is negligible for these instances.

The right pane tests another assumption of our model, that inventory errors occur at the beginning of each period. If errors instead occurred at the end of each period, the analysis of Section 5.1 holds with only minor modifications. We would set discrepancies to zero immediately following a cycle count, and we would start accumulating errors in the period thereafter. The net impact would be $T-1$ error accumulations per count cycle rather than T. In the right pane, we show fixed points for this modified model. We see, not surprisingly, that the bias is smaller. However, the differences are modest and the insights are the same. We expect the differences to be even smaller for longer cycles. Given that the errors-before-demand and the errors-after-demand models can be viewed as extremes of a model in which demand and errors are interleaved, we believe our model is a reasonable approximation to a model with interleaved demand and errors.

References

Braden, D. J., M. Freimer. 1991. Informational dynamics of censored observations. *Management Sci.* **37**(11) 1390–1405.