

# Optimal Merchandise Testing with Limited Inventory

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The practice of “merchandise testing” refers to the deployment of fashion products to stores in limited quantities so that a retailer may learn about demand prior to the main selling season. We consider the optimal allocation of inventory to stores in a merchandise test, focusing on the tradeoff between the quantity of stores tested and the quality of observations, which can be impacted by demand censoring due to inventory stockouts. We find that the visibility into the timing of each sales transaction has a pivotal impact on the optimal allocation decisions. When such timing information is unobservable, the retailer may need to consolidate inventory in few stores to increase service levels during the test and thereby to minimize the negative impacts of demand censoring. When sales timing information is observable, the retailer is better off maximizing the number of sales during the test period without regard to stockouts in individual stores. Motivated by our analysis, we propose two heuristic allocation policies for the cases with and without sales timing information, respectively. A numerical study shows the heuristics to be near optimal for instances for which we can compare them to the optimal solution. On the other hand, inefficient inventory allocations can considerably reduce the value of the merchandise test.

*Key words:* merchandise testing; retailing; Bayesian inventory management; demand censoring

*This version:* November 5, 2014.

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## 1. Introduction

Fashion products, which are characterized by highly uncertain demand, short life cycles, and often long lead-times, pose great challenges for retailers trying to match supply with demand. With only limited replenishment opportunities and inaccurate demand forecasts, retailers often end up with significant losses in profit due to either lost sales from stockouts or heavy price discounts needed to clear excess inventory at the end of the selling season. The past two decades have seen various initiatives by retailers to streamline inventory management for fashion products: for example, quick response (Fisher and Raman 1996), backup agreements with manufacturers (Eppen and Iyer 1997), and advanced booking discount programs (Tang et al. 2004), to name a few.

An effective strategy used in practice to improve initial demand forecasts is so-called “merchandise testing,” wherein a retailer gathers demand information about new products by offering

inventory for sale in selected stores of its network during a short testing period before the primary selling season starts. The retailer then uses the information gained during the test to construct more accurate demand forecasts and thereby to make improved ordering decisions in preparation for the main selling season (Fisher and Rajaram 2000). There are three key decisions involved in such a merchandise test: (1) store selection—which store to select for the test? (2) inventory allocation—how to allocate inventory across the test stores? and (3) demand learning—how to construct demand forecasts based on the test sales data?

Fisher and Rajaram (2000) formulate the merchandise testing problem as a store clustering problem based on historical sales data of similar products and apply their methods to a large women’s apparel retailer and two major shoe retailers in the United States. Designed for practical implementation, their model requires a few simplifying assumptions. First, Fisher and Rajaram implicitly assume retailers adopt a “depth test;” that is, they assume that ample inventory is placed at each test store to meet demand during the test period. Second, they assume that store-level demand forecasting for the selling season is accomplished using a linear function of test sales that is calibrated using historical sales data. In this paper, we complement their work by relaxing these two assumptions. Our goal is to obtain insights into the interplay between limited inventory and (Bayesian) demand learning in a multi-location setting.

We consider a model in which a retailer, with multiple stores and a fixed quantity of available test inventory, manages a testing period followed by a selling period. Demands at each store in both the testing and selling periods are dependent on an unknown parameter that is common across stores and periods. Demands are independent once conditioned on this unknown parameter. Between the two periods, the retailer updates its demand forecast in a Bayesian fashion (over some prior distribution on the unknown demand parameter) based on its observations from the testing period, and it chooses inventory quantities for the selling period, during which we assume no further replenishment opportunities. The retailer’s objective is to allocate the test inventory to stores at the beginning of the testing period such that the ex-ante expected total profit in the selling period is maximized. We assume no fixed costs of testing at a store and no inventory carryover, choosing instead to focus on the statistical forces at play.

When spreading test inventory across multiple locations, a retailer must choose not only how many (and which) stores to include in the test, but also the service level to target at each of the test stores. There is a natural incentive for the retailer to test at a large number of stores, as this yields a large *quantity* of demand observations. On the other hand, spreading a limited inventory among many stores compromises the service level targeted at each store during the test, and these

service levels can impact the *quality* of information gathered in the test. How exactly service levels impact information quality depends on the retailer’s visibility into the demand process. When the retailer has access to the timing of each sales transaction within a period (e.g., Jain et al. 2014), the available inventory at a test store limits the maximum number of transactions that can be observed. However, when the sales timing information is unobservable, the retailer only observes the total sales during the test period at each store. It is known that such sales information is an imperfect demand observation that is “censored” by the amount of inventory at each store (e.g., Lariviere and Porteus 1999, Ding et al. 2002). In this case, the retailer has an incentive to concentrate inventory in fewer stores to achieve higher service levels, reduce censoring, and thereby enhance the quality of the demand observations.

We begin in §4 with the case in which sales timing information is observable. We first analyze a base case in which stores have stochastically identical demands. For general demand processes with a general prior, we prove that (in the absence of fixed costs) an optimal policy will never omit a store from a merchandise test. We further show that an “even-split” policy, which allocates the test inventory to all stores as evenly as possible, is always optimal under a Poisson demand process with a gamma prior. These results suggest the opposite of the traditional practice of a depth test which tends to avoid stockouts during the test. In fact, a high service level in the testing period is no longer a necessity when the retailer has access to sales timing information. We then extend our analysis to the non-identical-store case, in which stores may have diverse demand volumes. We characterize the monotone structure of the optimal allocation policy with respect to the relative demand volumes among stores.

A key intuition underlying the above results is that the availability of data on the timing of sales transactions largely ensures that store-level observations are of high quality, freeing the retailer to primarily consider observation quantity when allocating test inventory. Moreover, “quantity” in this context is best interpreted not in terms of the number of stores but rather in terms of the total number of sales observations—following Jain et al. (2014), each sales transaction can be viewed as an (exact) observation of an inter-arrival time in the underlying demand process. Therefore, an effective merchandise test is one that tends to maximize the quantity of sales transactions in the testing period across the store network. Motivated by this insight, we propose a “Max-Sales” heuristic allocation policy which maximizes the expected sales during the testing period.

We obtain contrasting results when the retailer does not have access to sales timing information. In this case, the form of the optimal allocation policy becomes contingent upon the amount of test inventory as well as the shape of the demand distribution, and it can be complex to characterize.

Our analysis of the case with stochastically identical stores suggests that (1) when the amount of test inventory is small, a retailer should follow a “single-store” policy which allocates all of the test inventory to only one store; (2) when the amount of test inventory is large, an even-split policy is optimal. These results are established analytically assuming a continuous gamma-Weibull demand structure (with shape parameter exceeding one), and they are corroborated numerically for the case of Poisson demand with a gamma prior. Moreover, we find examples with a moderate amount of available test inventory in which the optimal allocation policy may stock unbalanced positive quantities in each store even though stores are otherwise identical.

These findings reveal a delicate tension between the quantity of stores included in the test and the quality of observations obtained from each one. Our results show that improving the quality of each demand observation is a higher priority than seeking a large observation quantity when the total test inventory is tightly constrained. This encourages the retailer to consolidate inventory in fewer stores to increase service levels in the testing period so as to avoid the negative impact of censoring on demand learning. This may be one justification (in addition to operational fixed costs) for adopting a concentrated test at a small number of stores. Motivated by these insights, for cases with heterogeneous stores we propose a “Service-Priority” heuristic that allocates test inventory to achieve a target service level during the testing period at as many stores as the inventory budget allows.

We evaluate our heuristic allocation policies in a numerical study by comparing their performance with the optimal policies. We consider two- and three-store problems in which the optimal allocations of the test inventory can be obtained through an exhaustive enumeration. Our numerical study indicates that when timing information is observable, the Max-Sales policy yields allocations that are extremely close to the optimal solutions; in fact, the maximum optimality gap in our study is 0.01% across both two-store and three-store instances. When timing information is unobservable, the Service-Priority policy also appears near-optimal in our numerical experiments, resulting in an average gap of 0.05% and a maximum gap of 1.02% for two-store instances, and an average gap of 0.08% and a maximum gap of 1.30% for three-store instances. Furthermore, we find that using inefficient allocations—for example, using the Max-Sales and Service-Priority heuristics in the wrong settings—can result in significantly suboptimal performance.

The remainder of this paper is organized as follows. After a review of relevant literature in §2, §3 describes a general modeling framework for the merchandise testing problem with limited test inventory. §4 characterizes the structure of optimal allocation policies for test inventory when sales timing information is observable to the retailer. In §5, we analyze the case in which the retailer

does not have sales timing information. We numerically evaluate the performance of our proposed heuristics in §6. §7 concludes the paper with discussions of managerial insights and future research directions.

## 2. Literature Review

By studying the merchandise testing problem, our paper contributes to a broad literature studying strategies for retailers to learn about demand for products with short life cycles and high demand uncertainty. Other examples include the “quick response” strategy of Fisher and Raman (1996), the “advanced booking discount” program modeled by Tang et al. (2004), and models allowing for advanced demand information that is updated over time (e.g., Wang et al. 2012).

There is a well-established body of research that jointly considers demand estimation and inventory optimization when unmet demand is lost and unobservable, or in other words, when demand observations are “censored.” For a survey, we refer readers to Chen and Mersereau (2014). The majority of this literature focuses on single-location settings. Our paper belongs to a substream of this literature that uses a Bayesian framework for demand estimation. Lariviere and Porteus (1999) analyze the Bayesian inventory problem with censored demand. To achieve tractability, they assume that the underlying demand distribution is from a family of so-called “newsvendor distributions” defined by Braden and Freimer (1991) and that a gamma prior is used. The dimensionality reduction scaling technique in Scarf (1960) and Azoury (1985), both assuming fully backlogged and exactly observable demand, is extended to the censored demand case under a Weibull demand distribution with a gamma prior. The problem receives continued exploration in Ding et al. (2002), Bensoussan et al. (2007), Lu et al. (2008), Chen and Plambeck (2008), and Chen (2010). Recently, Bisi et al. (2011) closely revisit the Bayesian inventory problem with censored observations and newsvendor distribution demand and confirm that Weibull is the only member of the newsvendor family for which optimal solutions are scalable. A common insight from this stream of literature is that the retailer should stock more than myopic order-up-to levels to better learn future demand information.

The recent paper of Jain et al. (2014) extends the literature on demand learning with censored observations by incorporating the timing of individual sales transactions. (Interestingly, Jain et al. (2014) use as a motivating example the Middle Eastern cosmetics brand Mikyajy, which uses merchandise testing at a single store to make profitable purchasing decisions prior to a full product launch.) In a parsimonious Bayesian multiperiod newsvendor framework, they prove that the “stock more” result continues to hold with the additional timing information. Furthermore, their numerical

study shows that the use of timing observations significantly reduces losses in expected profit due to censoring. While their scope is again limited to a single-location setting, our paper further extends their framework to a multi-location setting, which leads to tradeoffs that are non-existent in a single-location model.

A novel aspect of our work is the focus on demand learning (for a single product) across multiple locations, which is different from Caro and Gallien (2007), who consider a dynamic assortment problem with demand learning (at a single location) for multiple products. In this regard, our model is conceptually related to Harrison and Sunar (2014), who consider a firm choosing among several modes to learn the unknown value of a project for optimizing investment timing. The cost and quality of each learning mode in Harrison and Sunar (2014) are exogenously given, while our model seeks to maximize the value of demand learning subject to a resource constraint on the amount of test inventory available.

Our paper is also related to a large number of papers concerning inventory management at a warehouse serving multiple retail locations. A detailed review can be found in Agrawal and Smith (2009). Only a small subset of this literature is applicable to fashion products with short life cycles and high demand uncertainty. Two notable examples, in addition to Fisher and Rajaram (2000), are Agrawal and Smith (2013) and Gallien et al. (2013). Agrawal and Smith (2013) consider a two-period inventory model in which the retailer has multiple non-identical stores that share a common unknown parameter and uses a Bayesian scheme to update demand forecast. Gallien et al. (2013) also develop a two-period stochastic optimization model to determine initial shipments to stores at fashion retailer Zara, accounting for the allocation of leftover and replenished stock at a central warehouse to stores in the second period. Our paper differs in several important ways. First, in the previous two contexts, the widespread rollout of a product occurs at the very beginning of the first period, which typically involves allocating a large amount of inventory to a large number of stores; however, the first period in our merchandise testing problem only involves distributing a very limited amount of inventory to a relatively small set of stores. Second, neither paper explicitly considers demand censoring when updating demand forecasts based on observations in the first period. Demand censoring is at the core of our study, as it leads to the quantity versus quality tradeoff at the heart of our research questions. Finally, neither of these papers considers using the timing of sales occurrences for demand learning.

At a high level, the current paper is related to research on the value of information and its structure in problems involving collecting information with limited resources, examples of which come from multiple disciplines including economics, simulation optimization, computer science,

and decision science. For example, Frazier and Powell (2010) consider the Bayesian ranking and selection problem in which the decision maker allocates a measurement budget to choose the best among several alternatives. They find that spreading the measurement budget equally among alternatives can be paradoxically non-optimal when the prior is identical for each alternative, due to lack of concavity of the value of information. In our merchandise testing context, we also find that the value of information is not necessarily concave, in particular, when timing information is unobservable.

### 3. Model

In this section, we describe a general framework for the merchandise testing problem. Consider a retailer that tests and sells a single product through a chain of  $N$  stores. We model two periods, labeled 1 and 2 respectively, where period 1 represents the testing period and period 2 the main selling period. At the beginning of period 1, the retailer has  $Q \in \mathbb{Z}_+$  units of inventory available in total to allocate to  $N$  stores for the merchandise test. We denote a feasible allocation of the test inventory by a vector  $\mathbf{q} \in \mathcal{Q} = \{(q_1, \dots, q_N) : \sum_{n=1}^N q_n \leq Q, q_n \in \mathbb{Z}_+\}$ .

We assume that the testing period has length  $T$  and that demand arrives at each store  $n$  according to a renewal process, denoted by  $\{D_n(t|\theta), 0 \leq t \leq T\}$ , where  $\theta \in \Theta$  is a parameter that is common to all  $N$  stores but unknown to the retailer. Let  $t_n^i$  denote the arrival time of the  $i$ -th demand at store  $n$  and  $\tau_n^i = t_n^i - t_n^{i-1}$  the inter-arrival time between the  $i$ -th and the  $(i-1)$ -th demand. We assume that  $\tau_n^i$  has probability density function (pdf)  $\psi_n(\cdot|\theta)$ , cumulative distribution function (cdf)  $\Psi_n(\cdot|\theta)$ , and complementary cdf  $\bar{\Psi}_n(\cdot|\theta)$ , and is independent of the demand processes at other stores if conditioned on  $\theta$ . Cumulative demand  $D_n(t)$  until time  $t$  has probability mass function  $f_n(\cdot|t, \theta)$  and cdf  $F_n(\cdot|t, \theta)$  (complementary cdf  $\bar{F}_n(\cdot|t, \theta)$ ). We will use  $D_n$  and  $D_n(T)$  interchangeably to denote the total demand in period 1 at store  $n$ . A Bayesian framework is employed to model demand learning and we assume that the retailer has a prior density  $\pi(\theta)$  representing its initial belief about the unknown demand parameter  $\theta$ .

In order to highlight the value of demand learning associated with the allocation of test inventory, we make the following simplifying assumptions. First, we assume no fixed costs of including a store in the merchandise test. It is intuitive that fixed costs would create an incentive to consolidate inventory; we focus instead on the statistical incentives to consolidate inventory in a few stores versus spreading it among many stores. Second, we assume that the revenue generated from sales in period 1 is negligible, as the testing period is typically short compared to the primary selling season. Finally, we do not consider inventory carryover from period 1 to period 2 for tractability

reasons; in other words, the amount of leftover inventory from the test period is assumed to be negligible compared to the substantially larger order quantities for the main selling period.

At the end of period 1, the retailer makes an observation  $X(\mathbf{q})$  which may depend on the test demand realization  $D_n$ , the allocated test inventory level  $q_n$ , and in some cases the timing of demand epochs  $\{t_n^1, t_n^2, \dots, t_n^{D_n}\}$  (or equivalently the inter-arrival times  $\{\tau_n^1, \dots, \tau_n^{D_n}\}$ ), at each store  $n$ . The retailer can obtain different types of observations during period 1 and we defer the details to §3.1. Let  $l(X(\mathbf{q})|\theta)$  denote the likelihood of observing  $X(\mathbf{q})$  for some  $\theta$  and test inventory allocation  $\mathbf{q}$ . The retailer uses Bayes rule to update its knowledge about  $\theta$  based on observation  $X(\mathbf{q})$  over prior  $\pi$  as follows:

$$\hat{\pi}(\theta) = \pi(\theta) \circ X(\mathbf{q}) = \frac{l(X(\mathbf{q})|\theta)\pi(\theta)}{\int_{\Theta} l(X(\mathbf{q})|\omega)\pi(\omega)d\omega}, \quad (1)$$

where  $\hat{\pi}(\theta)$  is the updated posterior density.

Period 2 models the retailer's operations in the primary selling season. In essence, the retailer solves a newsvendor problem to choose the ordering quantity  $y_n$  for each store  $n$  to maximize the expected total profit generated by the entire chain based on its updated knowledge  $\hat{\pi}$  about  $\theta$ . Let  $\hat{D}_n$  be the period 2 demand at store  $n$ . Our most general model is flexible in the demand structure of period 2 in that we only assume that  $\hat{D}_n$  is distributed according to some cdf  $\hat{F}(\cdot|\theta)$  which also depends on the unknown demand parameter  $\theta$  and is independent of demand at other stores once conditioned on  $\theta$ . We assume a unit selling price  $p$  and a unit procurement cost  $c < p$ , both of which are exogenously determined and apply universally to all  $N$  stores. The expected total profit in period 2 with respect to ordering quantities  $\mathbf{y} = (y_1, \dots, y_n)$  under belief  $\hat{\pi}$  is thus given by

$$\hat{\Pi}(\mathbf{y}|\hat{\pi}) \triangleq E \left[ \sum_{n=1}^N p \min\{\hat{D}_n, y_n\} - cy_n \middle| \hat{\pi} \right].$$

Denote by  $\hat{\Phi}_n(\cdot)$  be the unconditioned cdf of demand at store  $n$  in period 2, i.e.,  $\hat{\Phi}_n(x) = \int_{\Theta} \hat{F}(x|\theta)\hat{\pi}(\theta)d\theta$ . It is straightforward to see that the optimal order quantity  $y_n^*$  for store  $n$  is given by the well-known newsvendor order quantity  $y_n^* = \hat{\Phi}_n^{-1}(\frac{p-c}{p})$ , where  $\hat{\Phi}_n^{-1}(\cdot)$  is the inverse unconditioned cdf, i.e.,  $\hat{\Phi}_n^{-1}(r) = \min\{x : \hat{\Phi}_n(x) \geq r\}$ .

Let  $\hat{V}(\hat{\pi}) = \max_{\mathbf{y}} \hat{\Pi}(\mathbf{y}|\hat{\pi})$  be the optimal expected total profit in period 2. The retailer's problem at the beginning of period 1 is to find the optimal allocation  $\mathbf{q} \in \mathcal{Q}$  of the total  $Q$  units of test inventory that maximizes its *ex-ante* expected profit  $\Pi(\mathbf{q}|\pi) = E[\hat{V}(\pi \circ X(\mathbf{q}))|\pi]$ , with the anticipation of an observation  $X(\mathbf{q})$  being made after period 1.



### 3.1. Types of Demand Observations in Period 1

We present in this subsection two types of demand observations the retailer may receive during period 1.

*Observations without Timing Information* ( $X^{NT}(\mathbf{q})$ ). This is the type of observations assumed by the majority of the literature: at the end of period 1, the retailer observes only the sales quantity at each store and whether a stockout has occurred. We denote by  $s_n = \min\{D_n, q_n\}$  the sales quantity at store  $n$  and by  $e_n = \mathbf{1}\{D_n \geq q_n\}$  a binary indicator of the store's stockout status at the end of period 1. The overall observation  $X^{NT}(\mathbf{q}) = \{\mathbf{s}, \mathbf{e}\}$  is simply a collection of two vectors where  $\mathbf{s} = (s_1, \dots, s_N)$  and  $\mathbf{e} = (e_1, \dots, e_N)$ . The superscript  $NT$  is for “No Timing.” For each store  $n$ , the likelihood of observing sales quantity  $s_n$  at time  $T$  given some demand parameter  $\theta$  is  $f(s_n|T, \theta)$  if there is excess test inventory (i.e.,  $e_n = 0$ ) and is  $\bar{F}(q_n - 1|T, \theta)$  otherwise (i.e.,  $e_n = 1$ ). Recall that we assume independent demand processes among  $N$  stores for any fixed  $\theta$ . As a result, the likelihood of observing  $X^{NT}(\mathbf{q})$  for some  $\theta$  is given by

$$l(X^{NT}(\mathbf{q})|\theta) = \prod_{i=1}^N [(1 - e_n) \cdot f(s_n|T, \theta) + e_n \cdot \bar{F}(q_n - 1|T, \theta)]. \quad (2)$$

*Observations with Timing Information* ( $X^T(\mathbf{q})$ ). This type of observation is considered by Jain et al. (2014) in a single-store setting and we extend their definition to our multi-location setting. It contains not only stores' sales quantities and stockout statuses but also the timing of all sales occurrences. Let  $\vec{\tau}_n = (\tau_n^1, \dots, \tau_n^{s_n})$  denote the observed sequence of inter-arrival times between sales at store  $n$ . Let  $X_n^T(q_n) = \{s_n, e_n, \vec{\tau}_n\}$  be the retailer's observation at store  $n$  where the superscript  $T$  stands for “Timing.” If the retailer decides not to test at store  $n$ , i.e.,  $q_n = 0$ , then it automatically stocks out (i.e.,  $e_n = 1$ ) and of course sees no sales (i.e.,  $s_n = 0$ ). Otherwise, if  $q_n > 0$ , the likelihood of it observing  $X_n^T(q_n) = (s_n, 0, \vec{\tau}_n)$  at store  $n$  is  $\prod_{i=1}^{s_n} \psi_n(\tau_n^i|\theta) \cdot \bar{\Psi}_n(T - \sum_{i=1}^{s_n} \tau_n^i|\theta)$  and of it observing  $X_n^T(q_n) = (s_n, 1, \vec{\tau}_n)$  is  $\prod_{i=1}^{s_n} \psi_n(\tau_n^i|\theta)$ . Overall, the retailer's observation,  $X^T(\mathbf{q}) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$ , is a collection of sales quantities, stockout statuses, and times between consecutive sales at all stores where we define  $\vec{\tau} = (\vec{\tau}_1, \dots, \vec{\tau}_N)$ . The likelihood of observing  $X^T(\mathbf{q})$  for some  $\theta$  is therefore given by

$$l(X^T(\mathbf{q})|\theta) = \prod_{n=1}^N \left[ e_n \cdot \prod_{i=1}^{s_n} \psi_n(\tau_n^i|\theta) + (1 - e_n) \cdot \prod_{i=1}^{s_n} \psi_n(\tau_n^i|\theta) \cdot \bar{\Psi}_n(T - \sum_{i=1}^{s_n} \tau_n^i|\theta) \right], \quad (3)$$

where we use the conventions  $\prod_{i=1}^0 \psi_n(\tau_n^i|\theta) = 1$  and  $\sum_{i=1}^0 \tau_n^i = 0$  for the case  $s_n = 0$ .

### 3.2. Marginal Value of Learning of an Additional Unit of Test Inventory

It is intuitive that the retailer would always prefer to allocate all  $Q$  units of test inventory in period 1 so as to acquire as much demand information as possible. We formalize this intuition in the following lemma which shows that the retailer's ex-ante expected profit is increasing in the test inventory quantity allocated to any store. In other words, the marginal value of learning from an additional unit of total test inventory is always nonnegative.

To facilitate our presentation throughout the rest of this paper, we introduce  $\delta_n = (0, \dots, 0, 1, 0, \dots, 0)$  as an  $N$ -dimensional vector with only the  $n$ -th element being one and all other elements zero. We use this notation mainly to describe allocation modifications. For instance, allocation  $\mathbf{q} + \delta_i - \delta_j$  modifies allocation  $\mathbf{q}$  by sending one more unit of test inventory to store  $i$  and one less to store  $j$ .

Let  $\Pi^T(\mathbf{q}|\pi)$  and  $\Pi^{NT}(\mathbf{q}|\pi)$  denote the ex-ante expected profits for the cases with and without timing information.

LEMMA 1. *The following hold for all  $\pi$ ,  $\mathbf{q} \in \mathbb{Z}_+^N$ , and  $n = 1, \dots, N$ :*

- (a)  $\Pi^T(\mathbf{q}|\pi) \leq \Pi^T(\mathbf{q} + \delta_n|\pi)$ ;
- (b)  $\Pi^{NT}(\mathbf{q}|\pi) \leq \Pi^{NT}(\mathbf{q} + \delta_n|\pi)$ ;
- (c) *There exists an optimal allocation  $\mathbf{q}$  such that  $\sum_{i=1}^N q_i = Q$ .*

All proofs can be found in the appendix. We prove Lemma 1 using results from the statistics literature on comparisons of experiments (Blackwell 1951, 1953). To prove Lemma 1(a), we define merchandise tests with allocations  $\mathbf{q} + \delta_n$  and  $\mathbf{q}$  as two statistical experiments,  $\mathcal{E}^T$  and  $\mathcal{F}^T$ , when timing information is observable. The outcomes of the two experiments are demand observations  $X^T(\mathbf{q} + \delta_n)$  and  $X^T(\mathbf{q})$ . We then establish that there exists a stochastic transformation from the distribution of  $X^T(\mathbf{q} + \delta_n)$  to that of  $X^T(\mathbf{q})$  (which is intuitive as the retailer observes more information with the additional unit of test inventory). As a result, experiment  $\mathcal{E}^T$  is said to be *sufficient* for  $\mathcal{F}^T$  and Lemma 1(a) immediately follows. The proof of part (b) uses a similar argument, and part (c) is an immediate corollary part of (a) and (b). Therefore, for the rest of the paper, we narrow our focus to the set of allocations satisfying  $\sum_{n=1}^N q_n = Q$  without loss of generality.

## 4. With Timing Information

In this section, we analyze the retailer's optimal policy for allocating test inventory when timing information is observable. We first examine in §4.1 the case in which stores have stochastically identical demand. Then we generalize our analysis to the case in which stores follow a more general demand structure.

#### 4.1. Identical Stores

We consider a base case in which all stores are identical. More specifically, we assume that stores' demand processes in period 1 share a common inter-arrival time distribution, i.e.,  $\psi_n(\tau|\theta) = \psi(\tau|\theta)$  for all  $n = 1, \dots, N$ . The identical-store case enables us to gain focused insights into the role of inventory allocation in gathering demand information from multiple locations. Practically, a group of identical stores may be interpreted as stores that have been clustered into a relatively homogeneous set in terms of demand or sales volume.

We first show in the following proposition that when the stores are identical and the retailer observes sales timing information, the retailer benefits from allocating a positive amount of test inventory to as many stores as possible under general renewal process demand with a general prior.

**PROPOSITION 1.** *Suppose that stores are identical. Then for all  $\pi$ , the following hold when timing information is observable:*

- (a) *Let  $\mathbf{q} = (q_1, \dots, q_N)$  be a test inventory allocation such that  $q_i \geq 2$  and  $q_j = 0$  for some  $i \neq j$ . Then,  $\Pi^T(\mathbf{q}|\pi) \leq \Pi^T(\mathbf{q} - \delta_i + \delta_j|\pi)$ ;*
- (b) *There exists an optimal allocation  $\mathbf{q}^* = (q_1^*, \dots, q_N^*)$  such that  $q_n^* > 0$  for  $n = 1, 2, \dots, \min\{Q, N\}$ .*

The main implication of Proposition 1 is that the retailer should cover as many stores as possible in a test without worrying about the potential to stock out at stores. This finding reveals an incentive for the retailer to deviate from a “depth test” that stocks high test inventory levels to avoid stockouts.

A formal proof appears in the appendix, but we sketch it here. We prove Proposition 1(a) by constructing two statistical experiments,  $\mathcal{E}$  and  $\mathcal{F}$ , corresponding to the two inventory allocations,  $\mathbf{q} - \delta_i + \delta_j$  and  $\mathbf{q}$ , respectively. As discussed in the sketch proof of Lemma 1, the result follows if we establish that there exists a stochastic transformation from the distribution of observation  $X^T(\mathbf{q} - \delta_i + \delta_j)$  to that of  $X^T(\mathbf{q})$ . The intuition is as follows. When timing information is observable, the retailer learns the unknown demand parameter essentially through observations of inter-arrival times. Each realized sale gives the retailer an exact observation of a single inter-arrival time. Moreover, the retailer receives a censored observation of the inter-arrival time when a store *does not* stockout, as the time until the next demand epoch is truncated at the end of period 1. Therefore, by moving one unit of test inventory from store  $i$  to store  $j$  (with no inventory), the retailer increases both the probability of selling this unit and that of getting an accurate instead of a censored observation of the inter-arrival time. Both the quantity and the quality of observations collected during the test increase (in a stochastic sense), therefore the distribution of  $X^T(\mathbf{q} - \delta_i + \delta_j)$  can

be transformed to that of  $X^T(\mathbf{q})$ . Proposition 1(b) is an immediate corollary of part (a) given we have established in Lemma 1 that it suffices to consider allocation policies that distribute all test inventory to stores.

Proposition 1 hints at the desirability of an “even-split” policy which evenly distributes test inventory to all stores, thereby maximizing the expected sales, or equivalently, the number of uncensored inter-arrival time observations during period 1. This would be true if one could generalize Proposition 1(a) to any allocation  $\mathbf{q}$  that has  $q_i - q_j \geq 2$  without requiring  $q_j = 0$ . Unfortunately, the proof generally does not extend for  $q_j > 0$ , as a stochastic transformation from  $X^T(\mathbf{q} - \delta_i + \delta_j)$  to  $X^T(\mathbf{q})$  appears no longer possible—in other words, observations under allocation  $\mathbf{q} - \delta_i + \delta_j$  do not always contain more information than that under allocation  $\mathbf{q}$ . Nevertheless, in the remainder of this subsection we present a result showing that the even-split policy is indeed optimal for an important special case.

*Poisson Demand with a Gamma Prior.* In the following, we assume that the inter-arrival times between consecutive demand epochs are exponentially distributed with an unknown rate parameter  $\lambda > 0$ , i.e.,  $\psi(\tau|\lambda) = \lambda e^{-\lambda\tau}$ . In other words, the cumulative demand up to time  $t$  at each store  $n$ ,  $\{D_n(t|\lambda), t \geq 0\}$ , is a Poisson process with unknown arrival rate  $\lambda$ , a demand process often assumed in academic research on retail inventory management. We further assume that the retailer uses a gamma prior with shape and rate parameters  $\alpha > 0$  and  $\beta > 0$ , i.e.,  $\pi(\lambda) = \pi(\lambda|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$ . When timing information is observable,  $\pi(\cdot|\alpha, \beta)$  is a conjugate prior for  $\lambda$ . More specifically, let  $X^T = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  be a realized observation in period 1 under some allocation when timing information is observable. Then the posterior, updated based on  $X^T$ , is  $\hat{\pi}(\lambda) = \pi(\lambda|\alpha, \beta) \circ X^T = \pi(\lambda|\alpha + \mathcal{S}, \beta + \mathcal{T})$ , where

$$\mathcal{S} = \sum_{n=1}^N s_n \text{ and } \mathcal{T} = \sum_{n=1}^N \left[ e_n \cdot \sum_{i=1}^{s_n} \tau_n^i + (1 - e_n)T \right]$$

constitute the sufficient statistics. Note that  $\mathcal{S}$  and  $\mathcal{T}$  are essentially the total sales quantity and the total sales duration across all stores, respectively.

The following proposition shows that the even-split policy is optimal for Poisson demand with a gamma prior when the retailer observes sales timing information.

**PROPOSITION 2.** *Suppose that the demand at each store in period 1 is a Poisson process with unknown arrival rate  $\lambda$  and that the retailer has a gamma prior  $\pi(\cdot|\alpha, \beta)$  with shape and rate parameters  $(\alpha, \beta)$ . Then, the following hold when timing information is observable:*

- (a) *Let  $\mathbf{q} = (q_1, \dots, q_N)$  be a test inventory allocation such that  $q_i - q_j \geq 2$  for some  $i \neq j$ . Then,*
- $$\Pi^T(\mathbf{q}|\alpha, \beta) \leq \Pi^T(\mathbf{q} - \delta_i + \delta_j|\alpha, \beta).$$

- (b) The “even-split” allocation  $\mathbf{q}^* = (q_1^*, \dots, q_n^*)$ , which allocates all  $Q$  units of test inventory to all  $N$  stores as evenly as possible, is optimal. In particular:
- (i) If  $Q \leq N$ ,  $q_i^* = 1$  for  $i = 1, \dots, Q$  and  $q_j^* = 0$  for  $j = Q + 1, \dots, N$ ;
  - (ii) If  $Q > N$ ,  $q_i^* = \lfloor Q/N \rfloor + 1$  for  $i = 1, \dots, (Q \bmod N)$  and  $q_j^* = \lfloor Q/N \rfloor$  for  $j = (Q \bmod N) + 1, \dots, N$ .

The proof of Proposition 2(a) builds upon Proposition 1 and extends it to any allocation  $\mathbf{q}$  that has  $q_i - q_j \geq 0$  through a two dimensional induction on  $q_i$  and  $q_j$ . The induction relies on a first-step analysis which conditions on the time until the next demand arrival at either store  $i$  or  $j$  and treats the remaining testing period as a new merchandise test with a shorter testing period and an updated prior. This first-step analysis relies on the memoryless property of Poisson processes and on the fact that  $\mathcal{S}$  and  $\mathcal{T}$  are sufficient statistics for the past demand information. Proposition 2(b) is a straightforward corollary of part (a).

The overall implication is that when sales timing information is used for demand learning in a merchandise test, the retailer need not aim for a high service level to avoid stockouts during the testing period. Instead, the retailer should allocate the limited test inventory to more stores so as to maximize the total sales, or equivalently, the quantity of exact inter-arrival time observations. The service level during the testing period is less of a concern because each sale individually reveals information about the demand distribution and has an equal value whether it is made in a store with a high or low service level.

## 4.2. Non-Identical Stores

In this subsection, we extend our analysis to the more general case where stores may be non-identical. We model non-identical demand as follows. We assume that stores’ inter-arrival times are stochastically ordered in a consistent way conditioned on any value of the unknown demand parameter. Without loss of generality, we label the stores such that their demand inter-arrival times are increasing in the sense of first-order stochastic dominance. In particular, we assume that

$$\bar{\Psi}_1(\tau|\theta) \leq \bar{\Psi}_2(\tau|\theta) \leq \dots \leq \bar{\Psi}_N(\tau|\theta) \quad (4)$$

for all  $\tau \geq 0$  and  $\theta \in \Theta$ . Recall that  $\bar{\Psi}_n(\cdot|\theta)$  is the complementary cdf of the inter-arrival times at store  $n$  given a fixed  $\theta$  and is assumed to be known to the retailer. This assumption also implies that stores’ demands are decreasing in the sense of first-stochastic dominance, i.e.,  $\bar{F}_1(x|T, \theta) \leq \bar{F}_2(x|T, \theta) \leq \dots \leq \bar{F}_N(x|T, \theta)$  for all  $x \geq 0$  and  $\theta \in \Theta$ . In this formulation, one can interpret  $\theta$  as

the overall market potential of the product. The retailer does not know  $\theta$ , but knows the market share of each store, which may be relatively more stable and predictable than the overall demand.

The following proposition extends Proposition 1 and sheds light on which stores the retailer should prefer when allocating test inventory with timing information observable.

**PROPOSITION 3.** *Suppose that stores are non-identical such that  $\bar{\Psi}_1(\tau|\theta) \leq \bar{\Psi}_2(\tau|\theta) \leq \dots \leq \bar{\Psi}_N(\tau|\theta)$  for all  $\tau \geq 0$  and  $\theta \in \Theta$ . Then for all  $\pi$ , the following hold when timing information is observable:*

- (a) *Let  $\mathbf{q} = (q_1, \dots, q_N)$  be a test inventory allocation such that  $q_i = 0$  and  $q_j \geq 1$  for some  $i < j$ . Then,  $\Pi^T(\mathbf{q}|\pi) \leq \Pi^T(\mathbf{q} + \delta_i - \delta_j|\pi)$ ;*
- (b) *There exists an optimal allocation  $\mathbf{q}^* = (q_1^*, \dots, q_N^*)$  such that  $q_n^* > 0$  for  $n = 1, \dots, m$  and  $q_n^* = 0$  for  $n > m$ , where  $m$  is some number in  $\{1, \dots, N\}$ .*

Proposition 3 indicates that the retailer should always allocate test inventory to stores with higher demand before testing at stores with lower demand. This is in line with the intuition we have gained in §4.1 that the retailer should maximize its test sales to maximize the value of the test when timing information is observable. This result is also useful if the retailer has an additional constraint on the maximum number of stores to test, say,  $M < N$  stores. In that case, instead of considering all subsets with at most  $M$  stores, the number of which is  $\sum_{m=1}^M \binom{N}{m}$  in total, the retailer need examine only  $M$  subsets, each containing the  $m$  stores with the largest relative demand,  $m = 1, 2, \dots, M$ .

Given the above proposition, it is natural to expect that the optimal quantities of test inventory allocated to stores should be ranked according to stores' relative demand volumes. That is, the retailer should send the most test inventory to store 1, the second most to store 2, and so forth. We prove that this conjecture holds for an important special case involving Poisson demand processes. Before formally stating the proposition, we first introduce our gamma-Poisson demand model for the non-identical-store setting.

*Non-Identical Poisson Demand with a Gamma Prior.* We assume that the demand inter-arrival times at store  $n$  are exponentially distributed with rate  $\gamma_n \lambda$ , i.e.,  $\psi_n(\tau|\lambda) = \gamma_n \lambda e^{-\gamma_n \lambda \tau}$ , where  $\lambda > 0$  is unknown but  $\gamma_n > 0$  is known to the retailer. In other words, the cumulative demand up to time  $t$  at each store  $n$ ,  $\{D_n(t|\lambda), t \geq 0\}$ , is a Poisson process with (partially) unknown arrival rate  $\gamma_n \lambda$ . In addition, we assume that  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_N$ . We call  $\gamma_n$  the *relative demand coefficient* of store  $n$ . The retailer still uses a gamma prior  $\pi(\lambda|\alpha, \beta)$  on  $\lambda$  with shape and rate parameters  $\alpha > 0$  and  $\beta > 0$ .

One can easily verify that the above Poisson demand model satisfies our general non-identical-store assumption in (4) by noting that the complementary cdf of inter-arrival times at store  $n$  is

given by  $\bar{\Psi}_n(\tau|\lambda) = e^{-\gamma_n \lambda \tau}$ . Furthermore, the gamma distribution is still a conjugate prior for this non-identical Poisson demand process. Let  $X^T = \{\mathbf{s}, \mathbf{e}, \bar{\tau}\}$  be a realized observation in period 1 under some allocation when timing information is observable. Then the posterior, updated based on  $X^T$ , is  $\hat{\pi}(\lambda) = \pi(\lambda|\alpha, \beta) \circ X^T = \pi(\lambda|\alpha + \mathcal{S}, \beta + \mathcal{T})$ , where

$$\mathcal{S} = \sum_{n=1}^N s_n \quad \text{and} \quad \mathcal{T} = \sum_{n=1}^N \gamma_n \cdot \left[ e_n \cdot \sum_{i=1}^{s_n} \tau_n^i + (1 - e_n)T \right].$$

The sufficient statistics still have two dimensions with one being the total sales quantity and the other the “weighted” total sales duration across all stores where the weights are the relative demand coefficients.

We prove in the following proposition that in the gamma-Poisson demand model the optimal quantities of test inventory allocated to stores are ordered by stores’ relative demand coefficients.

**PROPOSITION 4.** *Suppose that the inter-arrival times at store  $n$  are exponentially distributed with rate parameter  $\gamma_n \lambda$  where  $\lambda$  is unknown but  $\gamma_1 \geq \dots \geq \gamma_N$  are known, and that the retailer has a gamma prior on  $\lambda$  with shape and rate parameters  $(\alpha, \beta)$ . The following hold when timing information is observable:*

- (a) *Let  $\mathbf{q} = (q_1, \dots, q_N)$  be a test inventory allocation such that  $q_i < q_j$  for some  $i < j$ . Then,*  

$$\Pi^T(\mathbf{q}|\alpha, \beta) \leq \Pi^T(\mathbf{q} + \delta_i - \delta_j|\alpha, \beta);$$
- (b) *There exists an optimal allocation  $\mathbf{q}^* = (q_1^*, \dots, q_N^*)$  such that  $q_1^* \geq q_2^* \geq \dots \geq q_N^*$ .*

The structure of the optimal allocation policy characterized by Proposition 4 provides an intuitive guideline for practitioners to distribute test inventory in merchandise testing: allocate more inventory to the stores with higher demand. However, computing the exact optimal allocation quantities remains difficult due to the combinatorial nature of the problem. We propose an easy-to-implement heuristic policy in the following subsection.

### 4.3. The Max-Sales Heuristic

In this subsection, we propose a heuristic policy named “Max-Sales” based on the intuition developed in §4.1 that with timing information observable, a good policy tends to maximize the sales during the testing period.

The Max-Sales policy is a greedy heuristic which sequentially allocates  $Q$  units of test inventory to  $N$  stores such that each unit of product is sent to the store having the highest (unconditioned) probability of selling that additional unit. Let  $\phi_n(x)$  be the unconditioned probability mass function of demand being  $x$  at store  $n$  in period 1, i.e.,  $\phi_n(x) = \int_{\Theta} f_n(x|T, \theta) \pi(\theta) d\theta$ . We denote by  $\Phi_n(x)$  the

corresponding unconditioned cdf, i.e.,  $\Phi_n(x) = \sum_{u=0}^x \phi_n(u)$ , and let  $\bar{\Phi}_n(x)$  denote the unconditioned complementary cdf. The algorithm of the Max-Sales policy is as follows:

```

▷ Max-Sales Heuristic
 $q_1, \dots, q_N \leftarrow 0$ ;
for  $i \leftarrow 1$  to  $Q$ 
     $n^* \leftarrow \min\{n : \bar{\Phi}_n(q_n) \geq \bar{\Phi}_m(q_m) \text{ for all } m = 1, \dots, N\}$ ;
     $q_{n^*} \leftarrow q_{n^*} + 1$ ;
end

```

PROPOSITION 5. *Suppose that stores are non-identical such that  $\bar{\Psi}_1(\tau|\theta) \leq \bar{\Psi}_2(\tau|\theta) \leq \dots \leq \bar{\Psi}_N(\tau|\theta)$  for all  $\tau \geq 0$  and  $\theta \in \Theta$ . The following hold:*

- (a) *The Max-Sales heuristic yields an allocation  $\mathbf{q}^{MS} = (q_1^{MS}, \dots, q_N^{MS})$  that maximizes the expected total sales in period 1;*
- (b)  *$q_1^{MS} \geq q_2^{MS} \geq \dots \geq q_N^{MS}$ .*

Proposition 5(a) shows that our greedy Max-Sales heuristic indeed maximizes the expected total sales during period 1 under the general non-identical demand introduced in §4.2. Proposition 5(b) guarantees that the Max-Sales allocation is monotonic, which shares the same structure as the optimal policy under gamma-Poisson demand as we have proved in Proposition 4(b). The Max-Sales heuristic is easy-to-compute and applies to general demand processes and priors. Based on our numerical experience, this heuristic performs extremely well, coinciding with the optimal policy in almost all cases (see a detailed discussion in §6.2).

## 5. Without Timing Information

We discuss in this section the optimal test inventory allocation policy when the retailer does not observe sales timing information.

### 5.1. Identical Stores

When timing information is unobservable as is commonly assumed in the classic Bayesian inventory literature with demand censoring, analyzing the optimal allocation policy becomes particularly challenging. Even in the single-location setting, it is well-known that computing the optimal inventory policy is difficult (Bisi et al. 2011). In addition, we lose conjugacy under the discrete demand assumption, which makes it challenging even to compute the Bayes update between periods.



To achieve tractability, instead of assuming discrete, renewal process demand, we turn to a continuous Weibull demand distribution with a gamma prior, a parsimonious demand model widely adopted in the Bayesian inventory control literature with demand censoring. We will corroborate our key results numerically for gamma-Poisson demand in §4.1. For the present analysis we assume that the demand at each store in both periods is Weibull distributed with known shape parameter  $k > 0$  and unknown scale parameter  $\theta > 0$ , and that the retailer has a gamma distributed prior on  $\theta$  with shape and scale parameters  $(a, S)$  at the beginning of period 1. Note that the length  $T$  of period 1 is rendered irrelevant under this assumption. In particular,

$$f_n(x|\theta) = \hat{f}_n(x|\theta) = \theta x^{k-1} e^{-\theta x^k}, \quad \text{and } \pi(\theta) = \frac{S^a}{\Gamma(a)} \theta^{a-1} e^{-S\theta}.$$

The following proposition partially characterizes the optimal allocation policy under gamma-Weibull demand when timing information is unobservable.

**PROPOSITION 6.** *Suppose that the demand at each store in both periods is Weibull with shape parameter  $k > 0$  and unknown scale parameter  $\theta$ , and that the retailer has a gamma prior on  $\theta$  with shape and scale parameters  $(a, S)$ . The following hold for all  $a > \frac{1}{k}$  and  $S > 0$  when timing information is unobservable:*

- (a) *If  $0 < k \leq 1$ , the even-split allocation,  $q_n^* = Q/N$  for all  $n$ , is optimal;*
- (b) *If  $k > 1$ :*
  - (i) *there exists  $Q_0 > 0$  such that for  $0 \leq Q < Q_0$ , a “single-store” allocation is optimal, i.e.,  $q_i^* = Q$  for some  $i$  and  $q_n^* = 0$  for all  $n \neq i$ ;*
  - (ii) *the even-split allocation,  $q_n^* = Q/N$  for all  $n$ , becomes optimal as  $Q \rightarrow \infty$ .*

We can see from Proposition 6 that the form of the optimal allocation policy is generally complex when timing information is unobservable. It may depend on the total test inventory  $Q$  as well as the shape of the demand distribution.

When  $k \leq 1$ , the demand density is strictly decreasing with a shape similar to that of an exponential density. (The exponential distribution is itself a special case of the Weibull distribution with  $k = 1$ .) In this case, the even-split allocation is always optimal regardless of the total test inventory  $Q$ . In fact, the ex-ante expected profit  $\Pi^{NT}(\mathbf{q}|a, S)$  is jointly concave in the allocation  $\mathbf{q}$ . This resembles Bisi et al. (2011)’s result that the expected cost is convex in the inventory level under gamma-exponential demand in a single-store setting.

When  $k > 1$ , the demand density has a unimodal shape. In this case, the optimal policy is “single-store,” i.e., allocating all  $Q$  units of test inventory to only one store, when  $Q$  is sufficiently

small relative to demand. We note that  $Q_0$  is a constant that depends on demand parameters  $k$ ,  $a$ , and  $S$ . The even-split allocation becomes optimal for sufficiently large  $Q$ .

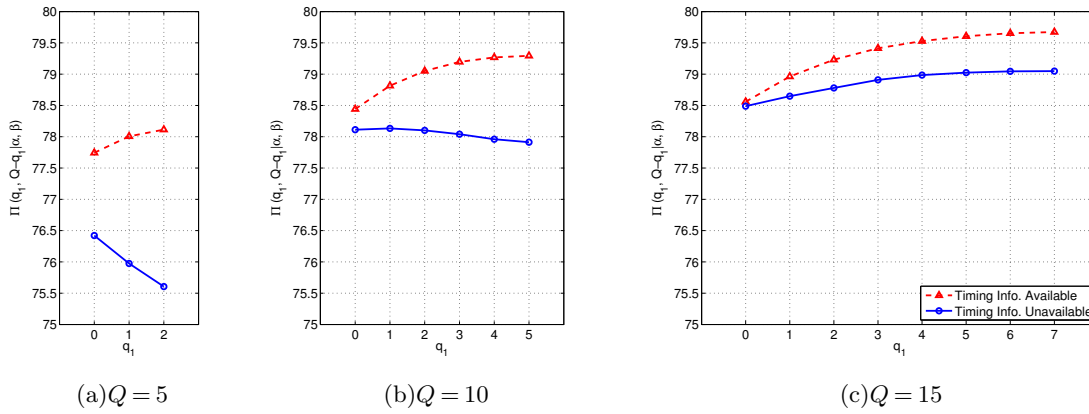
In contrast with the case in which timing information is observable, the pursuit of the quality and the quantity of demand observations need to be carefully balanced in the absence of timing information. Sending more test inventory to a store increases the service level and reduces the probability of demand being censored, thereby improving the quality of the demand observation obtained from the store. But with a fixed overall quantity of test inventory, this also means either sending less test inventory to some other store, which degrades the observation quality at that store, or excluding one or more stores from the test, which reduces the quantity of demand observations. When  $k > 1$ , Bisi et al. (2011) show in a single-store setting that the expected cost can be non-convex in the inventory level; we observe a similar phenomenon in our model, where the ex-ante expected profit  $\Pi^{NT}(\mathbf{q}|a, S)$  is non-concave in each  $q_n$ . As a result, the retailer gains little demand information from a store until it stocks sufficient test inventory at the store. Spreading the test inventory equally to all stores may not be beneficial, as the increase in the total observation quantity may not compensate for the significant loss in the quality of demand observations at each store.

Proposition 6 suggests that the retailer may want to consolidate test inventory in a few stores to achieve a sufficiently high service level during the testing period. In other words, we provide a theoretical justification, in addition to fixed costs, for the practice of avoiding stockouts in merchandise testing. We remark that this tendency to consolidate test inventory is present even though we assume zero fixed cost of conducting a test at a store.

## 5.2. Identical Stores: Numerical Illustration

In the following, we numerically test and illustrate the findings of Proposition 6 for example merchandise testing problems with identical stores, Poisson demand, and a gamma prior. To have a contrasting comparison, we also include the case with observable sales timing information in this numerical illustration. We plot in Figure 1 the ex-ante expected profit for a two-store problem as a function of  $q_1$ , the units of test inventory allocated to store 1. The dashed lines with triangle markers are for the case in which timing information is observable, whereas the solid lines with circle markers are for the case in which timing information is unobservable. Lemma 1 implies that it is sufficient to consider allocations  $\mathbf{q} = (q_1, q_2)$  that have  $q_2 = Q - q_1$ . The shape and rate parameters of the retailer's prior are  $\alpha = 2$  and  $\beta = 0.4$ , so for each store the expected arrival rate  $E[\lambda] = \alpha/\beta = 5$ . The profits are plotted for  $Q = 5, 10$ , and  $15$ , respectively. For each  $Q$  value, we only plot  $q_1$  from 0 to  $\lfloor Q/2 \rfloor$ . Given that the two stores are identical, the profits for  $q_1 \in \{\lfloor Q/2 \rfloor + 1, \dots, Q\}$  mirror those shown.

**Figure 1** Ex-ante expected profit  $\Pi(q_1, Q - q_1 | \alpha, \beta)$  as a function of  $q_1$  in a two-identical-store example under Poisson demand with a gamma prior when  $Q = 5, 10, 15$ .

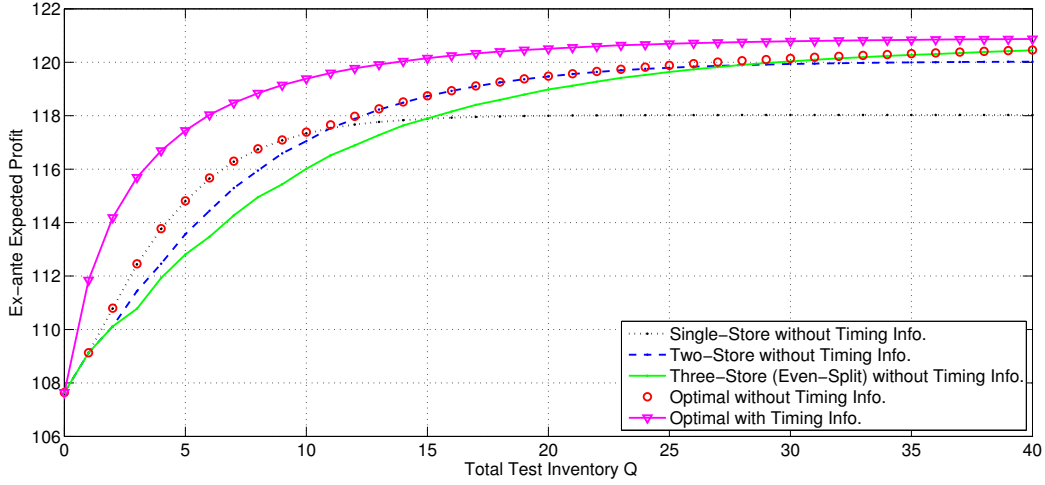


Note.  $N = 2, \alpha = 2, \beta = 0.4, p = 10, c = 1, T = 1$ .

We observe that even-split allocations (i.e., (2,3) when  $Q = 5$ , (5,5) when  $Q = 10$ , and (7,8) when  $Q = 15$ ) are always optimal when timing information is observable, consistent with Proposition 2. However, when timing information is not observable, the structure of the optimal allocation may differ as the total test inventory  $Q$  varies. The results in Figure 1(a) and 1(c) are consistent with the extreme cases of Proposition 6: if  $Q$  is small compared to the demand ( $Q = 5$ ), the single-store allocation (0,5) maximizes profits; if  $Q$  is large ( $Q = 15$ ), the even-split allocation (7,8) maximizes profits. Figure 1(b) highlights the complexity of allocating test inventory without timing information: if  $Q$  is at a moderate level ( $Q = 10$ ), an unbalanced allocation (1,9), which allocates unequal, positive quantities of test inventory to stores even though the stores are identical, can be optimal. Nonetheless, the additional benefit of using the (1,9) allocation is small compared with the single-store allocation (0,10). We find that the additional benefit of an unbalanced allocation is typically small; also, the region for an unbalanced allocation to be optimal is typically very small.

Figure 1 also yields insights into the value of using timing information for demand learning in the merchandise test. Naturally, the added value of timing information is always positive under the same allocation of test inventory, and it decreases as  $Q$  increases. An important observation is that the additional value of timing information hinges on the allocation of test inventory. Figure 1(b) gives an example in which the use of timing information may bring little extra value if the retailer employs a single-store rather than the optimal even-split allocation. Interestingly, Figure 1(c) shows that the ex-ante expected profit of using a single-store allocation with timing information is *lower* than that of using the even-split policy without timing information. In other words, a suboptimal allocation of test inventory may completely negate the advantage of observing sales timing information.

**Figure 2** Ex-ante expected profit in a three-identical-store example as a function of total test inventory  $Q$  under various allocation policies.



Note.  $N = 3, \alpha = 2, \beta = 0.4, p = 10, c = 1, T = 1$ .

Figure 2 shows the ex-ante expected profit as a function of the total test inventory  $Q$  under various allocation policies in a three-store example with identical stores, Poisson demand, and a gamma prior. The parameters are the same as those used to generate Figure 1 except that we increase the number of stores to  $N = 3$ . We consider the optimal allocation when timing information is observable (i.e., the even-split allocation) and the following four allocation policies when timing information is unobservable: (1) the single-store policy; (2) the “two-store” policy (i.e.,  $\mathbf{q} = (\frac{Q}{2}, \frac{Q}{2}, 0)$  if  $Q$  is even or  $\mathbf{q} = (\frac{Q+1}{2}, \frac{Q-1}{2}, 0)$  if  $Q$  is odd); (3) the “three-store” policy, or equivalently, the even-split policy; (4) and the optimal allocation. We obtain the optimal ex-ante expected profit for each  $Q$  value when timing information is unobservable through an exhaustive enumeration of all allocations satisfying  $q_1 + q_2 + q_3 = Q$ . Again, Figure 2 reinforces our insights from Proposition 6: the single-store allocation is optimal when  $Q$  is small ( $Q < 10$ ) while the even-split allocation is optimal when  $Q$  is large ( $Q > 30$ ). We also find that the use of timing information may increase the ex-ante expected profit, potentially by a significant margin when  $Q$  is limited. The additional value of timing information diminishes as  $Q$  increases.

We further notice in Figure 2 that when timing information is unobservable, the optimal ex-ante expected profit closely follows the envelope of the profits achieved by a class of “ $m$ -store” allocations, which allocate test inventory to  $m$  out of  $N$  stores as evenly as possible. The optimal allocation can be something other than an  $m$ -store allocation: e.g., neither the single-store nor the two-store allocation is optimal at  $Q = 11$ ; similarly, both the two-store and the three-store allocation are suboptimal at  $Q = 29$ . However, the loss in the ex-ante expected profit is negligible

in both cases if the retailer chooses either  $m$ -store allocation instead of the optimal allocation. This suggests that a retailer without access to sales timing information may start with a single-store allocation and gradually add more stores to the test as the total test inventory increases. The intuition is that the retailer need maintain a sufficient service level at test stores during the testing period to ensure the quality of the collected demand observations before seeking additional observation quantity by increasing the number of stores to test.

### 5.3. The Service-Priority Heuristic

Given the complexity of the optimal allocation policy for test inventory even in the identical-store case, computing the exact optimal allocation quantities appears to be out of reach for general demand processes when timing information is unobservable. Instead, we use the intuition uncovered in previous subsections to develop a heuristic policy for allocating the test inventory.

We have learned in §5.1 and §5.2 that the optimal allocation policy strikes a balance between observation quantity (i.e., number of stores to test) and quality (i.e., service level at each store tested). We develop our heuristic with this tradeoff in mind. The idea is to achieve a certain target service level  $r$  at as many stores as possible in period 1, where  $r$  is a tunable parameter. For this reason, we name our heuristic the “Service-Priority” policy. In particular, the heuristic allocates test inventory starting from store  $N$ , the store with the lowest relative demand. The motivating logic is that the retailer can always use less inventory to achieve the target service level  $r$  in a store with lower demand.

Let  $\Phi_n^{-1}(r)$  be the inverse unconditioned cdf of demand at store  $n$  in period 1, i.e.,  $\Phi_n^{-1}(r) = \min\{x : \Phi_n(x) \geq r\}$ . The algorithm of the Service-Priority policy is the following:

```

▷ Service-Priority Heuristic
 $n \leftarrow N$ ;
while  $Q > 0$ 
     $q_n \leftarrow \min\{Q, \Phi_n^{-1}(r)\}$ ;
     $Q \leftarrow Q - q_n$ ;
     $n \leftarrow n - 1$ ;
end

```

The question remains how to choose the target service level  $r$  for the Service-Priority policy. A naïve method would be to arbitrarily specify a relatively high  $r$ . One could also perform a search over a set of candidate  $r$  values to identify the  $r$  value that maximizes the ex-ante expected profit. In §6.3, we compare the performance of both methods for choosing  $r$ .

## 6. Performance of Heuristic Policies

In this section, we evaluate the performance of the Max-Sales policy and the Service-Priority policy proposed in §4.3 and §5.3 in an extensive numerical study. We consider a merchandise testing problem under non-identical Poisson demand with a gamma prior as introduced in §4.2. More specifically, the demand at each store  $n$  in both periods is a Poisson process with arrival rate  $\gamma_n \lambda$ , where  $\lambda$  is unknown but  $\gamma_n$ , the relative demand coefficient, is known. We normalize the  $\gamma_n$ 's such that  $\sum_{n=1}^N \gamma_n = 1$ . The prior distribution of  $\lambda$  is gamma with shape and rate parameters  $(\alpha, \beta)$ .

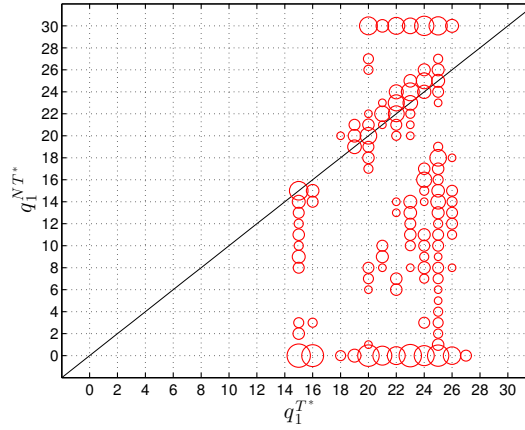
We first report results for a set of two-store instances ( $N = 2$ ). We choose values of the parameters to construct a large set of instances. We vary  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1/\gamma_2$  takes value in  $\{1, 2, 3, 4, 5\}$ . The stores are identical if  $\gamma_1 = \gamma_2 = 0.5$  and are non-identical otherwise. The shape parameter  $\alpha$  reflects the degree of uncertainty about  $\lambda$  and takes values in  $\{1, 2, 4, 8\}$ , indicating a coefficient of variation of  $\lambda$  in  $\{1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}\}$ . In order to illustrate the inventory allocations in a unified scale, we fix the total test inventory  $Q$  at 30 and vary the demand level. We choose the value of the rate parameter  $\beta$  such that the expected total arrival rate  $E[(\gamma_1 + \gamma_2)\lambda] = E[\lambda] = \alpha/\beta \in \{10, 20, 30, 40, 50, 60\}$ . The unit selling price is fixed at  $p = 20$ , and we vary the unit purchasing cost  $c$  in  $\{1, 2, \dots, 10\}$  to have a range of newsvendor ratios  $(p - c)/p$  in  $\{0.50, 0.55, \dots, 0.95\}$  targeted in period 2. This gives us a set of 1,200 instances in total. We also briefly report results for a set of  $N = 3$  instances in §6.4.

Throughout the section, we denote by  $q_n^\sigma$  the test inventory quantity allocated to store  $n$  under some policy  $\sigma$ . We use  $T^*$  and  $NT^*$  to denote the optimal allocation policy for the cases with and without timing information, respectively. We refer to the “optimality gap” (or “gap”, for short) of a policy  $\sigma$  for a problem instance as the percentage gap with respect to the ex-ante expected profit under the optimal allocation policy. For example, the optimally gap of a policy  $\sigma$  when timing information is observable is given by  $(\Pi^{T^*} - \Pi^\sigma)/\Pi^{T^*} \times 100\%$ , where  $\Pi^\sigma$  is the ex-ante expected profit of policy  $\sigma$ .

### 6.1. Optimal Test Inventory Allocations

For all the two-store instances, we compute through an exhaustive enumeration the optimal quantity of test inventory allocated to store 1 when timing information is observable,  $q_1^{T^*}$ , and that when timing information is unobservable,  $q_1^{NT^*}$ . (Recall from Lemma 1 that when  $N = 2$ , the optimal allocation quantities to store 2 are just  $Q - q_1^{T^*}$  and  $Q - q_1^{NT^*}$ , respectively.) Figure 3 shows a bubble plot of  $(q_1^{T^*}, q_1^{NT^*})$  pairs for all instances. A bubble at  $(q_1^{T^*}, q_1^{NT^*})$  means that there is at least one instance for which the optimal allocation quantity for store 1 is  $q_1^{T^*}$  if the retailer observes

**Figure 3** Comparison of the optimal test inventory allocation policies for the cases with and without timing information ( $N = 2$ ).



*Note.* The area of a bubble is linear in the number of instances at the bubble's coordinate.

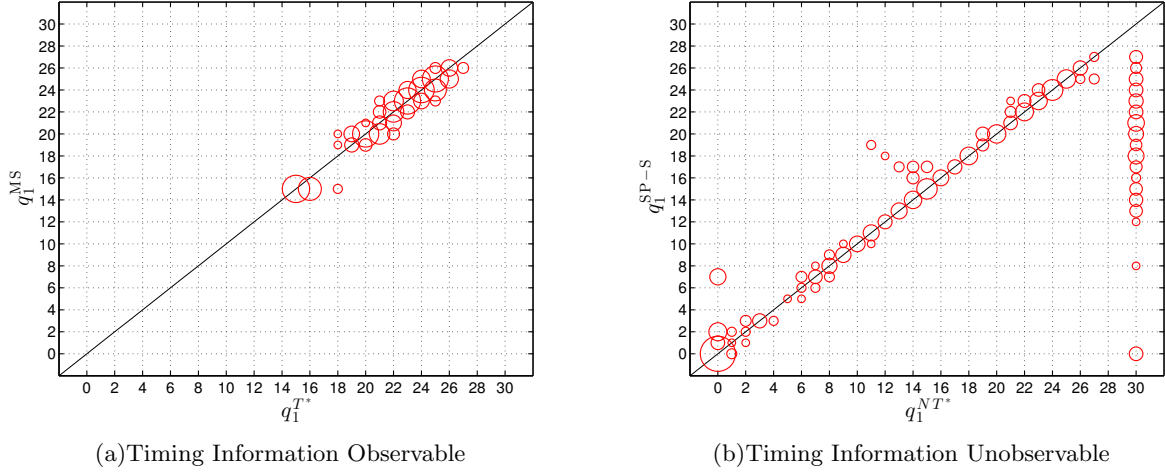
timing information and is  $q_1^{NT*}$  if the retailer does not. The size of each bubble indicates the total number of such instances out of the 1,200 total instances explored.

Figure 3 demonstrates the pronounced difference between the behavior of the optimal allocation policy when timing information is observable and that when timing information is unobservable. Also, we observe that  $q_1^{T*} \geq 15$  across all instances, which is consistent with Proposition 2 given that our instances have  $\gamma_1 \geq \gamma_2$  and  $Q = 30$ .

## 6.2. Performance of the Max-Sales Heuristic

We test the performance of the Max-Sales heuristic proposed in §4.3 against the optimal allocation policy in the case where timing information is observable. Let “MS” denote the Max-Sales policy. Figure 4(a) shows a bubble plot of  $(q_1^{T*}, q_1^{MS})$  pairs for all 1,200 instances. We observe that the bubbles are all located close to the diagonal line, implying that the Max-Sales policy closely follows the optimal allocation policy. The maximum difference between  $q_1^{T*}$  and  $q_1^{MS}$  for an instance is 3 units—at  $(q_1^{T*} = 18, q_1^{MS} = 15)$ . In addition, We compute the optimality gap of the Max-Sales policy for each instance. The average gap is 0.0007% and the maximum is only 0.01% across all 1,200 instances<sup>1</sup>. The evidence lends strong support to the near-optimality of the Max-Sales policy when timing information is observable.

<sup>1</sup> We compute all the ex-ante expected profits using Monte Carlo simulation with 1,000,000 trials. Therefore, the extremely small optimality gaps raise a natural question whether the Max-Sales heuristic is indeed optimal under Poisson demand with a gamma prior. We do not seem to have a proof (or an exact counterexample) for this claim and view it as an open question.

**Figure 4** Comparison of the heuristic and the optimal allocation policies ( $N = 2$ ).

*Note.* The area of a bubble is linear in the number of instances at the bubble's coordinate.

### 6.3. Performance of the Service-Priority Heuristic

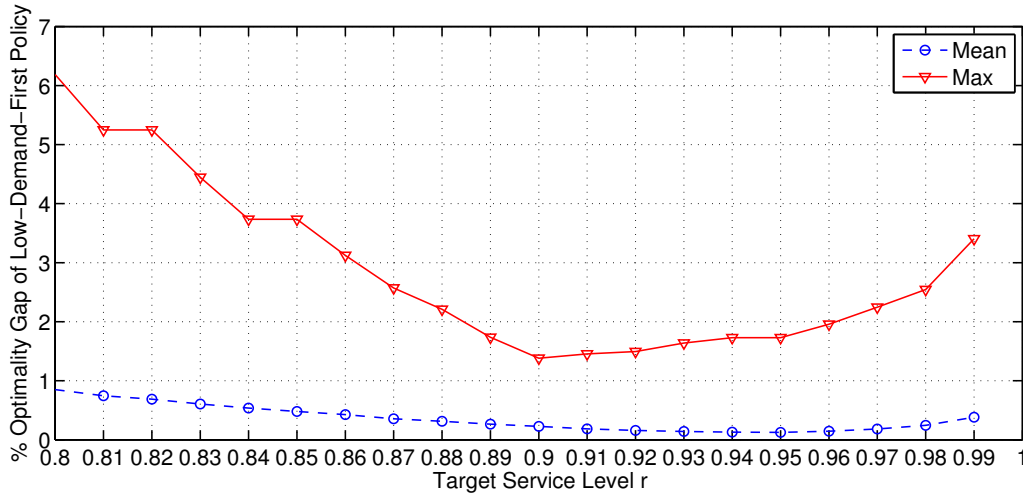
We investigate the performance of the Service-Priority heuristic proposed in §5.3 against the optimal allocation policy in the case where timing information is unobservable.

We first consider the naïve approach in which the retailer arbitrarily chooses a relatively high target service level  $r$  and applies it uniformly across all instances. Figure 5 shows the summary statistics of the optimality gaps of Service-Priority policies under various specifications for the target service level  $r$  when timing information is unobservable. We observe that as  $r$  varies from 0.80 to 0.99 in increments of 0.01, the lowest mean optimality gap is 0.13% (achieved at  $r = 0.95$ ) and the lowest maximum gap is 1.38% (achieved at  $r = 0.90$ ). These results imply that the naïve variant of the Service-Priority policy can be reasonably satisfactory as long as the retailer chooses a relatively but not extremely high  $r$ .

We then examine an alternative approach in which a search over a set of  $r$  values is performed for each instance to find the  $r$  value that maximizes the ex-ante expected profit. We numerically test the performance of this variant of the Service-Priority policy, abbreviated to SP-S, with a search over  $r \in \{0.50, 0.51, \dots, 0.99\}$  for each of the 1,200 instances. Figure 4(b) shows a bubble plot of  $(q_1^{S*}, q_1^{SP-S})$  pairs for all the instances. We observe that in most of the cases the SP-S policy closely follows the optimal policy, with a few exceptions in which the optimal policy allocates zero units of test inventory to the low demand store 2. In addition, we compute the optimality gap for each instance. The average gap of the SP-S policy is only 0.05% across all 1,200 instances and the maximum gap is 1.24%. As expected, the performance of the Service-Priority policy further improves after we include a search for a better target service level  $r$  for each instance.



**Figure 5** Optimality gaps of the Service-Priority policy with various values of target service level  $r$  when timing information is unobservable ( $N = 2$ ).

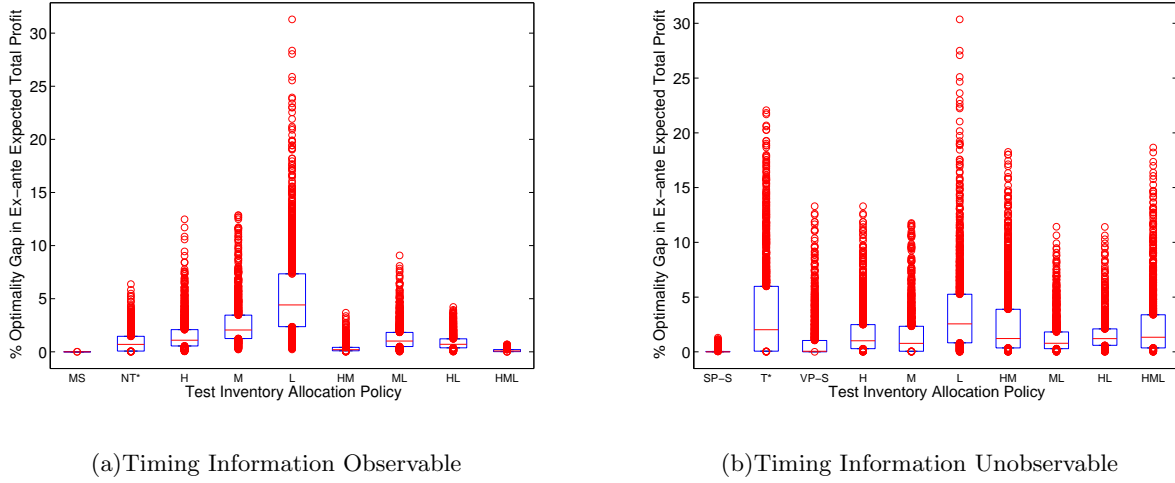


#### 6.4. Three-Store Instances

We report in this section the results for another set of instances in which  $N = 3$ . The parameter setting is the same as in the  $N = 2$  instances except for the relative demand coefficients. For ease of exposition, we label the three stores H, M, and L, respectively, which stand for relatively high, medium, and low demand. We vary their relative demand coefficients such that  $\gamma_H/\gamma_L \in \{1, 2, 3, 4, 5\}$ . The medium demand store's coefficient is set at  $\gamma_M = (\gamma_H + \gamma_L)/2$ . As in the two-store instances, we also normalize the relative demand coefficients such that  $\gamma_H + \gamma_M + \gamma_L = 1$ .

We compare the performance of the Max-Sales policy and the Service-Priority policy with a number of benchmark heuristics. We include a set of simple heuristics, each named with a set of the store labels, that always allocate the test inventory evenly to the stores in its name. For example, an H policy allocates  $Q = 30$  units of test inventory to store H, an ML policy allocates 15 units each to store M and L, and so forth. We also include a “Volume-Priority” policy (VP-S), a modification of the SP-S policy that gives priority to stores with higher, instead of lower, demand for test inventory allocation. In addition, we include the optimal policy without timing information ( $NT^*$ ) as a benchmark for the case in which timing information is observable, and also the optimal policy with timing information ( $T^*$ ) as a benchmark for the case in which timing information is unobservable.

The optimality gaps are plotted in Figure 6. When timing information is observable, we observe that the Max-Sales performs extremely well with a mean gap of only 0.0008% and a maximum gap of 0.01%. We also see that the H, HM, and HML policies dominate other simple heuristics,

**Figure 6** Optimality gaps under heuristic test inventory allocation policies ( $N = 3$ ).

*Note.* For each policy, the box shows the median and the first and the third quartiles; all the instances with an optimality gap below the first quartile or above the third quartile are plotted as an circle outside the box.

consistent with our Proposition 3. In particular, we find that the HML policy, which is the even-split policy in this  $N = 3$  case, performs reasonably well with a mean gap of 0.14% and a maximum gap of 0.71%. When timing information is unobservable, the SP-S policy significantly outperforms other heuristics with a mean gap of only 0.08% and a maximum gap of 1.30%. In addition, Figure 6 emphasizes again the pivotal impact of timing information on the inventory allocation decisions for a merchandise test. The optimal allocation policies can result in a significant loss in profit if employed in a wrong situation. The optimality gap could be as large as 6.38% if the optimal policy without timing information were used when timing information is observable, and could be as large as 22.06% if the optimal policy with timing information were used when timing information is unobservable.

## 7. Concluding Remarks

This paper uncovers new insights on the value of inventory for demand learning, in particular, on how demand censoring, demand learning across multiple locations, and the level of visibility into demand processes collectively impact inventory allocation decisions in merchandise testing.

There are fundamentally two ways to improve demand estimation given a fixed time frame to collect demand information: increasing the number, or the quantity, of the demand observations, and improving the quality of each observation. In the case where the retailer has a relatively coarse visibility into demand, i.e., the demand data contains only the sales quantities and stockout statuses, a single demand observation is made at each location if there is inventory for sale, and the quality

of the observation is negatively associated with the probability of obtaining an imperfect demand observation due to stockout. One can increase the observation quantity by stocking inventory at more locations, and can improve the observation quality at a location by raising the inventory level thereat. Our results suggest that improving the quality of each demand observation is a higher priority than seeking a large observation quantity, especially when the total inventory is tightly constrained. On the contrary, if sales timing data is observable for a reconstruction of the entire demand process, each sale transaction can be viewed as an exact demand observation. As a result, the value of inventory in demand learning is approximately maximized by simply selling as much inventory as possible so as to maximize the observation quantity. This generally involves spreading inventory among more locations.

These findings have two important managerial implications for retailers that have access to increasingly larger and richer demand data sets. First, when collecting and combining data from multiple locations for demand estimation, inventory allocation may have a significant impact on the outcome of demand learning. With the same amount of inventory, an inefficient allocation can lead to a significant loss in demand information and profit. Second, how to allocate inventory to maximizes its value in demand learning depends on the level of visibility into the demand. In particular, the use of sales timing information considerably reduces the need to maintain a high service level for demand learning as seen in reported practice.

Our work suggests several avenues for future research. In our model, demand at stores shares a common unknown parameter. Natural extensions would be to consider a hierarchical parameter structure under which each store has an unique unknown parameter in addition to the common parameter shared across stores. For example, Fisher and Rajaram (2000) cluster stores using sales histories. One could view our heuristics as solutions to the problem of inventory allocation within a store cluster; the question remains how to allocate a fixed quantity of inventory across clusters, which can be modeled by the hierarchical parameter structure described above. Another direction would be to consider a multi-product setting in which the retailer learns customer preferences in addition to demand volume through assortment experimentation. The retailer might choose to offer a full assortment with a low service level at each store, or to offer partial assortments at distinct stores with high service levels. It would be interesting to examine what would be the best inventory strategy in this setting.

## **Appendix A: Proof of Lemma 1**

The proof of Lemma 1 employs results from the statistical literature on comparison of experiments (Blackwell 1951, 1953). In the following, we introduce the definitions of statistic experiment.

DEFINITION 1 (BLACKWELL 1951, 1953, GINEBRA 2007). A statistical experiment  $\mathcal{E} = \{(X, S_X); (P_\theta, \Theta)\}$  ( $\mathcal{E}\{X; P_\theta\}$  for short) yields an observation on a random variable  $X$  defined on  $S_X$ , with an unknown probability distribution that is known to be in the family  $(P_\theta, \theta \in \Theta)$ .

The two observation types introduced in §3.1 can be viewed as outcomes of the following statistical experiments.

- (i) *Observations without timing information* ( $X^{NT}(\mathbf{q})$ ): Consider  $\mathcal{E}^{NT}(\mathbf{q}) = \{X^{NT}(\mathbf{q}); P_\theta^{NT}\}$ , where  $X^{NT}(\mathbf{q}) = \{\mathbf{s}, \mathbf{e}\}$  as defined in §3.1. The joint distribution  $P_\theta^{NT}$  of  $X^{NT}(\mathbf{q})$  under  $\theta \in \Theta$  is given by (2).  $\mathcal{E}^{NT}(\mathbf{q})$  is by definition a statistical experiment which corresponds to using an allocation  $\mathbf{q}$  when timing information is unobservable.
- (ii) *Observations with timing information* ( $X^T(\mathbf{q})$ ): Consider  $\mathcal{E}^T(\mathbf{q}) = \{X^T(\mathbf{q}); P_\theta^T\}$ , where  $X^T(\mathbf{q}) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  as defined in §3.1. The joint distribution  $P_\theta^T$  of  $X^T(\mathbf{q})$  under  $\theta \in \Theta$  is given by (3).  $\mathcal{E}^T(\mathbf{q})$  is by definition a statistical experiment which corresponds to using an allocation  $\mathbf{q}$  when timing information is observable.

Next, we introduce the sufficiency ordering between experiments.

DEFINITION 2 (GINEBRA, 2007). Experiment  $\mathcal{E} = \{X; P_\theta\}$  is *sufficient* for experiment  $\mathcal{F} = \{Y; Q_\theta\}$  if there is a stochastic transformation of  $X$  to a random variable  $W(X)$  such that  $W(X)$  and  $Y$  have identical distribution under each  $\theta \in \Theta$ .

The following lemma is a restatement of Proposition 3.2 in Ginebra (2007) by Jain et al. (2014).

LEMMA 2 (GINEBRA 2007, JAIN et al. 2014). *Experiment  $\mathcal{E}$  is sufficient for experiment  $\mathcal{F}$  if and only if, for every decision problem involving  $\theta$ , the Bayes risk for  $\mathcal{E}$  does not exceed the Bayes risk for  $\mathcal{F}$ , i.e.,  $r_\pi(\mathcal{E}) \leq r_\pi(\mathcal{F})$  for all prior  $\pi$  on  $\theta$ .*

Finally, we prove Lemma 1 by establishing sufficiency orderings between experiments with different test inventory allocations.

*Proof of Lemma 1(a).* Let  $\mathcal{E}^T = \{X^T(\mathbf{q} + \delta_n); P_\theta\}$  and  $\mathcal{F}^T = \{X^T(\mathbf{q}); Q_\theta\}$  be the experiments with timing information under allocation  $\mathbf{q} + \delta_n$  and allocation  $\mathbf{q}$ , respectively. Consider the following transformation from  $X^T(\mathbf{q} + \delta_n) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  to  $X' = \{\mathbf{s}', \mathbf{e}', \vec{\tau}'\}$ :

- (1) For all  $m \neq n$ : let  $s'_m = s_m$ ,  $e'_m = e_m$ ;
- (2) if  $s_n = q_n + 1$ ,  $e_n = 1$ : let  $s'_n = q_n$ ,  $e'_n = 1$ , and  $\vec{\tau}'_n = (\tau_n^1, \tau_n^2, \dots, \tau_n^{q_n})$ ;
- (3) if  $s_n = q_n$ ,  $e_n = 0$ : let  $s'_n = q_n$ ,  $e'_n = 1$ , and  $\vec{\tau}'_n = (\tau_n^1, \tau_n^2, \dots, \tau_n^{q_n})$ ;
- (4) if  $s_n < q_n$ ,  $e_n = 0$ : let  $s'_n = s_n$ ,  $e'_n = 0$ ,  $\vec{\tau}'_n = (\tau_n^1, \tau_n^2, \dots, \tau_n^{s_n})$ .

It can be verified that  $X'$  and  $X^T(\mathbf{q})$  have identical distributions. Therefore,  $\mathcal{E}^T$  is sufficient for  $\mathcal{F}^T$  according to Definition 2.

We can recast our merchandise testing problem as a statistical decision problem with period 2 ordering quantities  $\mathbf{y}$  as the decision, and a loss function  $L(\mathbf{y}, \theta) = \hat{\Pi}^*(\theta) - \hat{\Pi}(\mathbf{y}|\theta)$ , where  $\hat{\Pi}^*(\theta)$  is the optimal expected profit in period 2 under  $\theta$ , and  $\hat{\Pi}(\mathbf{y}|\theta)$  the expected profit if the retailer's ordering decision is  $\mathbf{y}$ . The Bayes risk is thus  $r_\pi(\mathcal{E}^T) = E_\pi[E_{P_\theta}[L(\mathbf{y}(\pi \circ X^T(\mathbf{q} + \boldsymbol{\delta}_n)), \theta)]] = E_\pi[E_{P_\theta}[\hat{\Pi}^*(\theta)]] - E_\pi[E_{P_\theta}[\hat{V}(\pi \circ X^T(\mathbf{q} + \boldsymbol{\delta}_n))]] = E_\pi[E_{P_\theta}[\hat{\Pi}^*(\theta)]] - \Pi^T(\mathbf{q} + \boldsymbol{\delta}_n|\pi)$  for  $\mathcal{E}^T$ , and is  $r_\pi(\mathcal{F}^T) = E_\pi[E_{P_\theta}[L(\mathbf{y}(\pi \circ X^T(\mathbf{q})), \theta)]] = E_\pi[E_{P_\theta}[\hat{\Pi}^*(\theta)]] - E_\pi[E_{Q_\theta}[\hat{V}(\pi \circ X^T(\mathbf{q}))]] = E_\pi[E_{Q_\theta}[\hat{\Pi}^*(\theta)]] - \Pi^T(\mathbf{q}|\pi)$  for  $\mathcal{F}^T$ . Note that  $E_\pi[E_{P_\theta}[\hat{\Pi}^*(\theta)]] = E_\pi[E_{Q_\theta}[\hat{\Pi}^*(\theta)]] = E_\pi[\hat{\Pi}^*(\theta)]$ . Lemma 1(a) thus follows from  $r_\pi(\mathcal{E}^{NT}) \leq r_\pi(\mathcal{F}^{NT})$  as a result of Lemma 2.  $\square$

*Proof of Lemma 1(b).* Let  $\mathcal{E}^{NT} = \{X^{NT}(\mathbf{q} + \boldsymbol{\delta}_n); P_\theta\}$  and  $\mathcal{F}^{NT} = \{X^{NT}(\mathbf{q}); Q_\theta\}$  be the experiments without timing information under allocation  $\mathbf{q} + \boldsymbol{\delta}_n$  and allocation  $\mathbf{q}$ , respectively. Consider the following transformation from  $X^{NT}(\mathbf{q} + \boldsymbol{\delta}_n) = \{\mathbf{s}, \mathbf{e}\}$  to  $X' = \{\mathbf{s}', \mathbf{e}'\}$ :

- (1) For all  $m \neq n$ : let  $s'_m = s_m$ ,  $e'_m = e_m$ ;
- (2) if  $s_n = q_n + 1$ ,  $e_n = 1$ : let  $s'_n = q_n$ ,  $e'_n = 1$ ;
- (3) if  $s_n = q_n$ ,  $e_n = 0$ : let  $s'_n = q_n$ ,  $e'_n = 1$ ;
- (4) if  $s_n < q_n$ ,  $e_n = 0$ : let  $s'_n = s_n$ ,  $e'_n = 0$ .

It can be verified that  $X'$  and  $X^{NT}(\mathbf{q})$  have identical distributions. Therefore,  $\mathcal{E}^{NT}$  is sufficient for  $\mathcal{F}^{NT}$  according to Definition 2. We show  $\Pi^{NT}(\mathbf{q}) \leq \Pi^{NT}(\mathbf{q} + \boldsymbol{\delta}_n)$  by an argument similar to that in proving part (a).  $\square$

*Proof of Lemma 1(c).* Suppose that allocation  $\mathbf{q}^*$  is optimal with timing information, i.e.,  $\Pi^T(\mathbf{q}^*) \geq \Pi^T(\mathbf{q})$  for any allocation  $\mathbf{q}$ . If  $\sum_{n=1}^N q_n^* < Q$ , consider allocation  $\mathbf{q}' = \mathbf{q}^* + (Q - \sum_{n=1}^N q_n^*)\boldsymbol{\delta}_1$ , which has  $\sum_{n=1}^N q'_n = Q$ . It follows from part (a) that  $\Pi^T(\mathbf{q}') \geq \Pi^T(\mathbf{q}^*)$ , which indicates that  $\mathbf{q}'$  is also optimal. A similar argument applies to the case without timing information.  $\square$

## Appendix B: Proof of Proposition 1

*Proof of Proposition 1(a).* Let  $\mathcal{E} = \{X^T(\mathbf{q} - \boldsymbol{\delta}_i + \boldsymbol{\delta}_j); P_\theta\}$  and  $\mathcal{F} = \{X^T(\mathbf{q}); Q_\theta\}$  be the experiments with timing information under allocation  $\mathbf{q} - \boldsymbol{\delta}_i + \boldsymbol{\delta}_j$  and allocation  $\mathbf{q}$ , respectively. Consider the following transformation from  $X^T(\mathbf{q} - \boldsymbol{\delta}_i + \boldsymbol{\delta}_j) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  to  $X' = \{\mathbf{s}', \mathbf{e}', \vec{\tau}'\}$ :

- (1) For  $m \neq i, j$ , let  $s'_m = s_m$ ,  $\vec{\tau}'_m = \vec{\tau}_m$ , and  $e'_m = e_m$ ;
- (2) Let  $s'_j = 0$ ,  $\vec{\tau}'_j = \emptyset$ ,  $e'_j = 1$ ;
- (3) if  $s_i < q_i - 1$ : let  $s'_i = s_i$ ,  $\vec{\tau}'_i = \vec{\tau}_i$ , and  $e'_i = 0$ ;
- (4) if  $s_i = q_i - 1$ ,  $s_j = 0$ : let  $s'_i = q_i - 1$ ,  $\vec{\tau}'_i = \vec{\tau}_i$ ,  $e'_i = 0$ ;

- (5) if  $s_i = q_i - 1$ ,  $s_j = 1$ , and  $\tau_j^1 > T - \sum_{k=1}^{q_i-1} \tau_i^k$ : let  $s'_i = q_i - 1$ ,  $\vec{\tau}'_i = \vec{\tau}_i$ , and  $e'_i = 0$ ;  
 (6) if  $s_i = q_i - 1$ ,  $s_j = 1$ , and  $\tau_j^1 \leq T - \sum_{k=1}^{q_i-1} \tau_i^k$ : let  $s'_i = q_i$ ,  $\vec{\tau}'_i = \{\tau_i^1, \dots, \tau_i^{q_i-1}, \tau_j^1\}$ ,  $e'_i = 1$ .

It can be verified that  $X'$  and  $X^T(\mathbf{q})$  have identical distributions. Therefore,  $\mathcal{E}$  is sufficient for  $\mathcal{F}$ .

We show  $\Pi^T(\mathbf{q}) \leq \Pi^T(\mathbf{q} - \boldsymbol{\delta}_i + \boldsymbol{\delta}_j)$  by an argument similar to that in the proof of Lemma 1.  $\square$

*Proof of Proposition 1(b).* Immediately follows from part (a).  $\square$

## Appendix C: Proof of Proposition 2

*Proof of Proposition 2(a).* We show the proof only for  $N = 2$  and let  $i = 1$  and  $j = 2$  without loss of generality, namely,  $\Pi^T(q_1, q_2 | \alpha, \beta) \leq \Pi^T(q_1 - 1, q_2 + 1 | \alpha, \beta)$  for all  $q_2 \geq 0$  and  $q_1 \geq q_2 + 2$ . The proof extends to the  $N > 2$  cases by conditioning on the demand processes at stores other than  $i$  and  $j$ .

We prove by two inductions—an inner induction nested inside an outer induction. Proposition 1 guarantees that part (a) holds for  $q_2 = 0$  and all  $q_1 \geq 2$ . Suppose that it holds for some  $q_2 = q \geq 0$  and all  $q_1 \geq q + 2$ , that is,

ASSUMPTION 1.  $\Pi_T^T(q_1, q | \alpha, \beta) \geq \Pi_T^T(q_1 - 1, q + 1 | \alpha, \beta)$  for some  $q \geq 0$  and all  $q_1 \geq q + 2$ .

Assumption 1 is the assumption for the outer induction. The subscript  $T$  in  $\Pi_T^T$  makes explicit the length  $T$  of period 1. We first show it holds for  $q_2 = q + 1$  and  $q_1 = q_2 + 2 = q + 3$ . By conditioning on the time  $s \geq 0$  until the first demand arrival at either store, which is exponential with rate  $2\lambda$ , we have

$$\begin{aligned}
 & \Pi_T^T(q + 3, q + 1 | \alpha, \beta) \\
 &= E_{\lambda | \alpha, \beta} E_{s | \lambda} \left[ \left\{ \frac{1}{2} \Pi_{T-s}^T(q + 2, q + 1 | \alpha + 1, \beta + 2s) + \frac{1}{2} \Pi_{T-s}^T(q + 3, q | \alpha + 1, \beta + 2s) \right\} \mathbf{1}_{s \leq T} \right. \\
 & \quad \left. + \Pi_0^T(q + 3, q + 1 | \alpha, \beta + 2T) \mathbf{1}_{s > T} \right], \\
 & \Pi_T^T(q + 2, q + 2 | \alpha, \beta) \\
 &= E_{\lambda | \alpha, \beta} E_{s | \lambda} \left[ \left\{ \frac{1}{2} \Pi_{T-s}^T(q + 1, q + 2 | \alpha + 1, \beta + 2s) + \frac{1}{2} \Pi_{T-s}^T(q + 2, q + 1 | \alpha + 1, \beta + 2s) \right\} \mathbf{1}_{s \leq T} \right. \\
 & \quad \left. + \Pi_0^T(q + 2, q + 2 | \alpha, \beta + 2T) \mathbf{1}_{s > T} \right].
 \end{aligned}$$

Note that

- (i)  $\Pi_{T-s}^T(q + 2, q + 1 | \alpha + 1, \beta + 2s) = \Pi_{T-s}^T(q + 1, q + 2 | \alpha + 1, \beta + 2s)$  because stores are identical;
- (ii)  $\Pi_{T-s}^T(q + 3, q | \alpha + 1, \beta + 2s) \leq \Pi_{T-s}^T(q + 2, q + 1 | \alpha + 1, \beta + 2s)$  by Assumption 1;
- (iii)  $\Pi_0^T(q + 3, q + 1 | \alpha, \beta + 2T) = \Pi_0^T(q + 2, q + 2 | \alpha, \beta + 2T)$  since period 1 has zero length.

As a result,  $\Pi_T^T(q+3, q+1|\alpha, \beta) \leq \Pi_T^T(q+2, q+2|\alpha, \beta)$ ; i.e., Assumption 1 holds for  $q+1$  and  $q_1 = q+3$ .

We still need to show that Assumption 1 holds for  $q+1$  and all  $q_1 > q+3$  to complete the outer induction. We prove that by an inner induction on  $q_1$  which makes the following induction assumption.

ASSUMPTION 2.  $\Pi_T^T(q + \Delta_q + 3, q+1|\alpha, \beta) \geq \Pi_T^T(q + \Delta_q + 2, q+2|\alpha, \beta)$  for  $q$  and some  $\Delta_q \geq 0$ .

We have shown that Assumption 2 is true for  $q$  and  $\Delta_q = 0$ . To show it holds for  $\Delta_q = 1$ , we again condition on the time  $s \geq 0$  until the first demand arrival at either store and get

$$\begin{aligned} & \Pi_T^T(q + \Delta_q + 1 + 3, q+1|\alpha, \beta) \\ &= E_{\lambda|\alpha, \beta} E_{s|\lambda} \left[ \left\{ \frac{1}{2} \Pi_{T-s}^T(q + \Delta_q + 3, q+1|\alpha+1, \beta+2s) + \frac{1}{2} \Pi_{T-s}^T(q + \Delta_q + 4, q|\alpha+1, \beta+2s) \right\} \mathbb{1}_{s \leq T} \right. \\ & \quad \left. + \Pi_0^T(q + \Delta_q + 4, q+1|\alpha, \beta+2T) \mathbb{1}_{s > T} \right], \\ & \Pi_T^T(q + \Delta_q + 1 + 2, q+2|\alpha, \beta) \\ &= E_{\lambda|\alpha, \beta} E_{s|\lambda} \left[ \left\{ \frac{1}{2} \Pi_{T-s}^T(q + \Delta_q + 2, q+2|\alpha+1, \beta+2s) + \frac{1}{2} \Pi_{T-s}^T(q + \Delta_q + 3, q+1|\alpha+1, \beta+2s) \right\} \mathbb{1}_{s \leq T} \right. \\ & \quad \left. + \Pi_0^T(q + \Delta_q + 3, q+2|\alpha, \beta+2T) \mathbb{1}_{s > T} \right]. \end{aligned}$$

Note that

- (i)  $\Pi_{T-s}^T(q + \Delta_q + 3, q+1|\alpha+1, \beta+2s) \leq \Pi_{T-s}^T(q + \Delta_q + 2, q+2|\alpha+1, \beta+2s)$  by Assumption 2;
- (ii)  $\Pi_{T-s}^T(q + \Delta_q + 4, q|\alpha+1, \beta+2s) \leq \Pi_{T-s}^T(q + \Delta_q + 3, q+1|\alpha+1, \beta+2s)$  by Assumption 1;
- (iii)  $\Pi_0^T(q + \Delta_q + 4, q+1|\alpha, \beta+2T) = \Pi_0^T(q + \Delta_q + 3, q+2|\alpha, \beta+2T)$  since period 1 has zero length.

Consequently we have  $\Pi_T^T(q + \Delta_q + 1 + 3, q+1|\alpha, \beta) \leq \Pi_T^T(q + \Delta_q + 1 + 2, q+2|\alpha, \beta)$  which completes the inner induction. This also completes the outer induction in showing that Assumption 1 holds for  $q+1$  and all  $q_1 \geq (q+1) + 2 = q+3$ .  $\square$

*Proof of Proposition 2(b).* Immediately follows from part (a).  $\square$

## Appendix D: Proof of Proposition 3

*Proof of Proposition 3(a).* Let  $\mathcal{E} = \{X^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j); P_\theta\}$  and  $\mathcal{F} = \{X^T(\mathbf{q}); Q_\theta\}$  be the experiments with timing information under allocation  $\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j$  and allocation  $\mathbf{q}$ , respectively. Consider the following transformation from  $X^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  to  $X' = \{\mathbf{s}', \mathbf{e}', \vec{\tau}'\}$  for every  $\theta \in \Theta$ :

- (1) For  $m \neq i, j$ , let  $s'_m = s_m$ ,  $\vec{\tau}'_m = \vec{\tau}_m$ , and  $e'_m = e_m$ ;
- (2) let  $s'_i = 0$ ,  $\vec{\tau}'_i = \emptyset$ ,  $e'_i = 1$ ;
- (3) if  $s_j < q_j - 1$  ( $q_j \geq 2$ ): let  $s'_j = s_j$ ,  $\vec{\tau}'_j = \vec{\tau}_j$ ,  $e'_j = 0$ ;

- (4) if  $s_i = 0$ ,  $s_j = q_j - 1$ : let  $s'_j = s_j = q_j - 1$ ,  $\bar{\tau}'_j = \bar{\tau}_j$ ,  $e'_j = 0$ ;  
 (5) if  $s_i = 1$ ,  $s_j = q_j - 1$ , and  $\Psi_i(\tau_i^1|\theta) > \Psi_j(T - \sum_{k=1}^{q_j-1} \tau_j^k|\theta)$ : let  $s'_j = q_j - 1$ ,  $\bar{\tau}'_j = \bar{\tau}_j$ ,  $e'_j = 0$ ;  
 (6) if  $s_i = 1$ ,  $s_j = q_j - 1$ , and  $\Psi_i(\tau_i^1|\theta) \leq \Psi_j(T - \sum_{k=1}^{q_j-1} \tau_j^k|\theta)$ : let  $s'_j = q_j$ ,  $\bar{\tau}'_j = \{\tau_j^1, \dots, \tau_j^{q_j-1}, \Psi_j^{-1}(\Psi_i(\tau_i^1|\theta)|\theta)\}$ ,  $e'_j = 1$ ,

where  $\Psi_j^{-1}(\cdot|\theta)$  is the inverse cdf of the corresponding inter-arrival time cdf  $\Psi_j(\cdot|\theta)$ . It can be verified that  $X'$  and  $X^T(\mathbf{q})$  have identical distributions under every  $\theta \in \Theta$ . Therefore,  $\mathcal{E}$  is sufficient for  $\mathcal{F}$ . We show  $\Pi^T(\mathbf{q}) \leq \Pi^T(\mathbf{q} + \delta_i - \delta_j)$  by an argument similar to that in the proof of Lemma 1.  $\square$

*Proof of Proposition 3(b).* Immediately follows from part (a).  $\square$

## Appendix E: Proof of Proposition 4

The proof involves an equivalent transformation from a problem in which stores have non-identical arrival rates and identical lengths of period 1, to one in which stores have identical arrival rates and non-identical lengths of period 1. We use  $\Pi^T(\mathbf{q}|\gamma_1\lambda_0, \dots, \gamma_N\lambda_0; T_1, \dots, T_N; \alpha, \beta)$  to denote the ex-ante expected profit of an allocation  $\mathbf{q}$  for a merchandise testing problem with observable timing information, unknown arrival rate parameter  $\lambda_0$ , relative demand coefficients  $\gamma_1, \dots, \gamma_N$ , and lengths of period 1,  $T_1, \dots, T_N$ . The retailer has a gamma prior with parameters  $(\alpha, \beta)$  on  $\lambda_0$ . The following lemma shows that under gamma-Poisson demand, a store  $n$  with arrival rate  $\gamma_n\lambda_n$  and length  $T_n$  in period 1 is equivalent in terms of ex-ante expected profit to one with arrival rate  $\lambda_n$  and length  $\gamma_n T_n$ .

**LEMMA 3.**  $\Pi^T(\mathbf{q}|\gamma_1\lambda_0, \dots, \gamma_n\lambda_0, \dots, \gamma_N\lambda_0; T_1, \dots, T_N; \alpha, \beta) = \Pi^T(\mathbf{q}|\gamma_1\lambda_0, \dots, \lambda_0, \dots, \gamma_N\lambda_0; T_1, \dots, \gamma_n T_n, \dots, T_N)$  for all  $\mathbf{q}$ ,  $\gamma_n > 0$ ,  $\alpha > 0$ , and  $\beta > 0$ .

*Proof.* We show the proof only for  $N = 1$ , i.e.,  $\Pi^T(q|\gamma\lambda_0; T_0; \alpha, \beta) = \Pi^T(q|\lambda_0; \gamma T_0; \alpha, \beta)$ . The result extends to the  $N > 1$  cases by conditioning on the demand processes at other stores.

Observations during period 1 can be summarized by a pair of sufficient statistics  $(s, t)$ , where  $s$  is the sales quantity and  $t$  is the effective sales duration. Let  $\hat{\alpha}$  and  $\hat{\beta}$  denote the posterior parameters. Let  $P(D(T|\lambda) = s)$  be the probability of total arrivals being  $s$  during time  $[0, T]$  in a Poisson process with arrival rate  $\lambda$ , and  $g(t|q, \lambda)$  be the probability density of the  $q$ -th arrival time. Then,

- (i) if  $\lambda = \gamma\lambda_0$ ,  $T = T_0$ :  $\hat{\alpha} = \alpha + s$ ,  $\hat{\beta} = \beta + \gamma t$ . By conditioning on unknown parameter  $\lambda_0$  and observation  $(s, t)$ , one obtains

$$\begin{aligned} & \Pi^T(q|\gamma\lambda_0; T_0; \alpha, \beta) \\ &= E_{\lambda_0|\alpha, \beta} E_{(s, t)|\gamma\lambda_0} [\hat{V}(\alpha + s, \beta + \gamma t)] \\ &= E_{\lambda_0|\alpha, \beta} \left[ \sum_{s=0}^{q-1} \hat{V}(\alpha + s, \beta + \gamma T_0) \cdot P(D(T_0|\gamma\lambda_0) = s) + \int_0^{T_0} \hat{V}(\alpha + q, \beta + \gamma t) \cdot g(t|q, \gamma\lambda_0) dt \right] \end{aligned}$$



$$= E_{\lambda_0|\alpha,\beta} \left[ \sum_{s=0}^{q-1} \hat{V}(\alpha+s, \beta+\gamma T_0) \cdot \frac{(\gamma\lambda_0 T_0)^s e^{-\gamma\lambda_0 T_0}}{s!} + \int_0^{T_0} \hat{V}(\alpha+q, \beta+\gamma t) \cdot \frac{(\gamma\lambda_0)^q t^{q-1} e^{-\gamma\lambda_0 t}}{\Gamma(q)} dt \right]; \quad (5)$$

(ii) if  $\lambda = \lambda_0$ ,  $T = \gamma T_0$ :  $\hat{\alpha} = \alpha + s$ ,  $\hat{\beta} = \beta + t$ . Similarly, by conditioning on  $\lambda_0$  and observation  $(s', t')$  we have

$$\begin{aligned} & \Pi^T(q|\lambda_0; \gamma T_0; \alpha, \beta) \\ &= E_{\lambda_0|\alpha,\beta} E_{(s',t')|\lambda_0} [\hat{V}(\alpha+s', \beta+t')] \\ &= E_{\lambda_0|\alpha,\beta} \left[ \sum_{s'=0}^{q-1} \hat{V}(\alpha+s', \beta+\gamma T_0) \cdot P(D(\gamma T_0|\lambda_0) = s') + \int_0^{\gamma T_0} \hat{V}(\alpha+q, \beta+t') \cdot g(t'|q, \lambda_0) dt \right] \\ &= E_{\lambda_0|\alpha,\beta} \left[ \sum_{s'=0}^{q-1} \hat{V}(\alpha+s', \beta+\gamma T_0) \cdot \frac{(\gamma\lambda_0 T_0)^{s'} e^{-\gamma\lambda_0 T_0}}{s'!} + \int_0^{\gamma T_0} \hat{V}(\alpha+q, \beta+t') \cdot \frac{\lambda_0^q (t')^{q-1} e^{-\gamma\lambda_0 t'}}{\Gamma(q)} dt' \right]. \end{aligned}$$

Let  $s' = s$ ,  $t' = \gamma t$ , then

$$\begin{aligned} & \Pi^T(q|\lambda_0; \gamma T_0; \alpha, \beta) \\ &= E_{\lambda_0|\alpha,\beta} \left[ \sum_{s=0}^{q-1} \hat{V}(\alpha+s, \beta+\gamma T_0) \cdot \frac{(\gamma\lambda_0 T_0)^s e^{-\gamma\lambda_0 T_0}}{s!} + \int_0^{T_0} \hat{V}(\alpha+q, \beta+\gamma t) \cdot \frac{\lambda_0^q (\gamma t)^{q-1} e^{-\gamma\lambda_0 \gamma t}}{\Gamma(q)} d\gamma t \right] \\ &= \Pi^T(q|\gamma\lambda_0; T_0; \alpha, \beta). \end{aligned} \quad (6)$$

The last equality follows from a comparison between (5) and (6).  $\square$

The following lemma is a counterpart to Proposition 3 in which stores have identical inter-arrival time distributions but have different lengths of period 1.

LEMMA 4. Suppose that stores have identical inter-arrival time distributions  $\Psi(\tau|\theta)$ . Then,  $\Pi^T(\mathbf{q}|\Psi, \dots, \Psi; T_1, \dots, T_N; \pi) \leq \Pi^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j | \Psi, \dots, \Psi; T_1, \dots, T_N; \pi)$  for all  $\mathbf{q}$  that has  $q_i = 0$  and  $q_j > 0$  for some  $i < j$ ,  $T_1 \geq \dots \geq T_N$ , and all  $\pi$ .

*Proof.* Let  $\mathcal{E} = \{X^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j); P_\theta\}$  and  $\mathcal{F} = \{X^T(\mathbf{q}); Q_\theta\}$  be the experiments with timing information under allocation  $\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j$  and allocation  $\mathbf{q}$ , respectively. Consider the following transformation from  $X^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j) = \{\mathbf{s}, \mathbf{e}, \vec{\tau}\}$  to  $X' = \{\mathbf{s}', \mathbf{e}', \vec{\tau}'\}$  for every  $\theta \in \Theta$ :

- (1) For  $m \neq i, j$ , let  $s'_m = s_m$ ,  $\tau'_m = \tau_m$ , and  $e'_m = e_m$ ;
- (2) let  $s'_i = 0$ ,  $\tau'_i = \emptyset$ ,  $e'_i = 1$ ;
- (3) if  $s_j < q_j - 1$ : let  $s'_j = s_j$ ,  $\tau'_j = \tau_j$ ,  $e'_j = 0$ ;
- (4) if  $s_i = 0$ ,  $s_j = q_j - 1$ : let  $s'_j = q$ ,  $\tau'_j = \tau_j$ ,  $e'_j = 0$ ;
- (5) if  $s_i = 1$ ,  $s_j = q_j - 1$ , and  $\tau_i^1 > T_j - \sum_{k=1}^{q_j-1} \tau_j^k$ : let  $s'_j = q_j - 1$ ,  $\tau'_j = \tau_j$ ,  $e'_j = 0$ ;
- (6) if  $s_i = 1$ ,  $s_j = q_j - 1$ , and  $\tau_i^1 \leq T_j - \sum_{k=1}^{q_j-1} \tau_j^k$ : let  $s'_j = q_j$ ,  $\tau'_j = \{\tau_j^1, \dots, \tau_j^{q_j-1}, \tau_i^1\}$ ,  $e'_j = 1$ .

It can be verified that  $X'$  and  $X^T(\mathbf{q})$  have identical distributions. (Note that  $T_i \geq T_j$  guarantees that  $\sum_{k=1}^{q_j} \tau_j'^k = \sum_{k=1}^{q_j-1} \tau_j^k + \tau_i^1$  in (6) covers the entire  $[0, T_j]$  interval.) Therefore,  $\mathcal{E}$  is sufficient for  $\mathcal{F}$ . The lemma follows from an argument similar to that in the proof of Lemma 1.  $\square$

The following corollary applies Lemma 4 to gamma-Poisson demand.

**COROLLARY 1.** *Suppose that demand is gamma-Poisson and that stores have identical arrival rates  $\lambda$ , lengths  $T_1 \geq \dots \geq T_N \geq 0$  of period 1. Then,  $\Pi^T(\mathbf{q}|\lambda, \dots, \lambda; T_1, \dots, T_N; \pi) \leq \Pi(\mathbf{q} + \delta_i - \delta_j|\lambda, \dots, \lambda; T_1, \dots, T_N; \pi)$  for all  $\mathbf{q}$  that has  $q_i = 0$  and  $q_j > 0$  for some  $i < j$ ,  $\alpha > 0$ , and  $\beta > 0$ .*

The following corollary applies Proposition 3 to gamma-Poisson demand.

**COROLLARY 2.** *Suppose that demand is gamma-Poisson and that stores have identical length  $T$  of period 1 and relative demand coefficients  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_N$ , respectively. Then,  $\Pi^T(\mathbf{q}|\gamma_1\lambda, \dots, \gamma_N\lambda; T, \dots, T; \pi) \leq \Pi^T(\mathbf{q} + \delta_i - \delta_j|\gamma_1\lambda, \dots, \gamma_N\lambda; T, \dots, T; \pi)$  for all  $\mathbf{q}$  that has  $q_i = 0$  and  $q_j > 0$  for some  $i < j$ ,  $\alpha > 0$ , and  $\beta > 0$ .*

*Proof.* Follows from Lemma 3 and Proposition 3.  $\square$

The following lemma shows that under gamma-Poisson demand, the retailer prefers allocation  $\mathbf{q} + \delta_i - \delta_j$  to allocation  $\mathbf{q}$  that has  $q_j = q_i + 1$  for some  $i > j$ , if stores have identical arrival rates but store  $i$  has a longer length of period 1 than store  $j$  does.

**LEMMA 5.** *Suppose that demand is gamma-Poisson and that stores have identical arrival rates  $\lambda$  and lengths  $T_1, \dots, T_N$  of period 1, where  $T_i \geq T_j$  for some  $i \neq j$ . Then,  $\Pi^T(\mathbf{q}|\lambda, \dots, \lambda; T_1, \dots, T_N; \alpha, \beta) \leq \Pi^T(\mathbf{q} + \delta_i - \delta_j|\lambda, \dots, \lambda; T_1, \dots, T_N; \alpha, \beta)$  for all  $\mathbf{q}$  that has  $q_j = q_i + 1$ ,  $\alpha > 0$ , and  $\beta > 0$ .*

*Proof.* We show the proof only for  $N = 2$  and let  $i = 1$  and  $j = 2$  without loss of generality, i.e.,  $\Pi^T(q, q+1|\lambda, \lambda; T_1, T_2; \alpha, \beta) \leq \Pi^T(q+1, q|\lambda, \lambda; T_1, T_2; \alpha, \beta)$  for all  $q = 0, 1, \dots, \alpha$ , and  $\beta$ . The proof extends to the  $N > 2$  cases by conditioning on the demand processes at stores other than  $i$  and  $j$ .

We write  $T = T_2 \geq 0$  and  $\Delta T = T_1 - T_2 \geq 0$ . Consider a modification of the problem where period 1 at each store always *ends*, instead of starts, at the same time. In this case, after the modification, period 1 at store 1 covers time interval  $[0, T + \Delta T]$ , whereas period 1 at store 2 covers time interval  $[\Delta T, T + \Delta T]$ . We use a tilde as an identifier for corresponding notation in the modified problem. Note that the lengths of period 1 for both stores remain the same:  $\tilde{T}_1 = T + \Delta T = T_1$ ,  $\tilde{T}_2 = T = T_2$ . Since the stores are independent conditional on  $\lambda$  and period 1 is purely for information learning purpose, such a modification in the start time of testing at store 2 does not affect the ex-ante expected profit, i.e.,

$$\tilde{\Pi}^T(q, q+1|\lambda, \lambda; T_1, T_2; \alpha, \beta) = \Pi^T(q, q+1|\lambda, \lambda; T_1, T_2; \alpha, \beta),$$

$$\tilde{\Pi}^T(q+1, q|\lambda, \lambda; T_1, T_2; \alpha, \beta) = \Pi^T(q+1, q|\lambda, \lambda; T_1, T_2; \alpha, \beta).$$

Let  $\bar{T} = \min\{t_1^q, \Delta T\}$ , where  $t_1^q$  is the time of  $q$ -th arrival at store 1. By conditioning on  $\lambda, \bar{T}$ , and observation  $(s, t)$  (sales quantity and effective selling duration) during  $[0, \bar{T}]$ , we obtain

$$\begin{aligned} \tilde{\Pi}^T(q, q+1|\lambda, \lambda; T + \Delta T, T; \alpha, \beta) &= E_{\lambda|\alpha, \beta} \left[ \sum_{s=0}^{q-1} \Pi^T(q-s, q+1|\lambda, \lambda; T, T; \alpha+s, \beta+\Delta T) P(D(\Delta T|\lambda) = s) \right. \\ &\quad \left. + \int_0^{\Delta T} \tilde{\Pi}^T(0, q+1|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t) g(t|q, \lambda) dt \right], \\ \tilde{\Pi}^T(q+1, q|\lambda, \lambda; T + \Delta T, T; \alpha, \beta) &= E_{\lambda|\alpha, \beta} \left[ \sum_{s=0}^{q-1} \Pi^T(q+1-s, q|\lambda, \lambda; T, T; \alpha+s, \beta+\Delta T) P(D(\Delta T|\lambda) = s) \right. \\ &\quad \left. + \int_0^{\Delta T} \tilde{\Pi}^T(1, q|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t) g(t|q, \lambda) dt \right], \end{aligned}$$

where  $g(t|q, \lambda)$  is the pdf of  $t_1^q$  conditional on  $\lambda$ . Note that

- (i)  $\Pi^T(q-s, q+1|\lambda, \lambda; T, T; \alpha+s, \beta+\Delta T) \leq \Pi^T(q+1-s, q|\lambda, \lambda; T, T; \alpha+s, \beta+\Delta T)$  for all  $s = 0, \dots, q-1$ , following from Proposition 2;
- (ii)  $\tilde{\Pi}^T(0, q+1|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t) = \Pi^T(0, q+1|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t) \leq \Pi^T(1, q|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t) = \tilde{\Pi}^T(1, q|\lambda, \lambda; T + \Delta T - t, T; \alpha+q, \beta+t)$ , where the inequality follows from Corollary 1.

As a result,  $\tilde{\Pi}^T(q, q+1|\lambda, \lambda; T + \Delta T, T; \alpha, \beta) \leq \tilde{\Pi}^T(q+1, q|\lambda, \lambda; T + \Delta T, T; \alpha, \beta)$ , or,  $\Pi^T(q, q+1|\lambda, \lambda; T_1, T_2; \alpha, \beta) \leq \Pi^T(q+1, q|\lambda, \lambda; T_1, T_2; \alpha, \beta)$ .  $\square$

The following corollary is a counterpart to Lemma 5 in which stores have identical lengths of period 1 but different arrival rates.

**COROLLARY 3.** *Suppose that demand is gamma-Poisson and that stores have identical lengths  $T$  of period 1, and relative demand coefficients  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_N$ . Then,  $\Pi^T(\mathbf{q}|\gamma_1\lambda, \dots, \gamma_N\lambda; T, \dots, T; \alpha, \beta) \leq \Pi^T(\mathbf{q} + \boldsymbol{\delta}_i - \boldsymbol{\delta}_j|\gamma_1\lambda, \dots, \gamma_N\lambda; T, \dots, T; \alpha, \beta)$  for all  $\mathbf{q}$  that has  $q_j = q_i + 1$  for some  $i < j$ ,  $\alpha > 0$ , and  $\beta > 0$ .*

*Proof.* Follows from Lemma 3 and Lemma 5.  $\square$

*Proof of Proposition 4(a).* We show the proof only for  $N = 2$  and let  $i = 1$  and  $j = 2$  without loss of generality, i.e.,  $\Pi^T(q, q+n|\gamma_1\lambda, \gamma_2\lambda; T, T; \alpha, \beta) \leq \Pi^T(q+1, q+n-1|\gamma_1\lambda, \gamma_2\lambda; T, T; \alpha, \beta)$  for  $q = 0, 1, \dots, n = 1, 2, \dots, \gamma_1 \geq \gamma_2, T > 0, \alpha > 0$ , and  $\beta > 0$ . The proof extends to the  $N > 2$  cases by conditioning on the demand processes at stores other than  $i$  and  $j$ .

The proof is by induction. The proposition holds for  $q = 0$  and all  $n \geq 1$  according to Corollary 2. It also holds for all  $q \geq 0$  and  $n = 1$  according to Corollary 3.

Assume that the proposition holds for some  $q$  and all  $n \geq 1$ . In addition, assume that it holds for  $q+1$  and some  $n \geq 1$ . We show that it continues to hold for  $q+1$  and  $n+1$  by conditioning on the time  $t \geq 0$  until the next arrival at either store, which is exponential with rate  $(\gamma_1 + \gamma_2)\lambda$ . Once arrives, the next arrival occurs at store 1 with probability  $\frac{\gamma_1}{\gamma_1 + \gamma_2}$  and at store 2 with probability  $\frac{\gamma_2}{\gamma_1 + \gamma_2}$ . We have

$$\begin{aligned}
& \Pi^T(q+1, q+1+n+1 | \gamma_1\lambda, \gamma_2\lambda; T, T; \alpha, \beta) \\
&= E_{\lambda|\alpha, \beta} E_{t|\lambda} \left[ \left\{ \frac{\gamma_1}{\gamma_1 + \gamma_2} \Pi^T(q, q+1+n+1 | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \right. \right. \\
&\quad \left. \left. + \frac{\gamma_2}{\gamma_1 + \gamma_2} \Pi^T(q+1, q+1+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \right\} \mathbb{1}\{t \leq T\} \right. \\
&\quad \left. + \Pi^T(q+1, q+1+n+1 | \gamma_1\lambda, \gamma_2\lambda; 0, 0; \alpha, \beta + (\gamma_1 + \gamma_2)T) \mathbb{1}\{t > T\} \right], \\
& \Pi^T(q+2, q+1+n | \gamma_1\lambda, \gamma_2\lambda; T, T; \alpha, \beta) \\
&= E_{\lambda|\alpha, \beta} E_{t|\lambda} \left[ \left\{ \frac{\gamma_1}{\gamma_1 + \gamma_2} \Pi^T(q+1, q+1+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \right. \right. \\
&\quad \left. \left. + \frac{\gamma_2}{\gamma_1 + \gamma_2} \Pi^T(q+2, q+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \right\} \mathbb{1}\{t \leq T\} \right. \\
&\quad \left. + \Pi^T(q+2, q+1+n | \gamma_1\lambda, \gamma_2\lambda; 0, 0; \alpha, \beta + (\gamma_1 + \gamma_2)T) \mathbb{1}\{t > T\} \right].
\end{aligned}$$

Note that

- (i)  $\Pi^T(q, q+1+n+1 | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \leq \Pi^T(q+1, q+1+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t)$ , following from the first induction assumption;
- (ii)  $\Pi^T(q+1, q+1+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t) \leq \Pi^T(q+2, q+n | \gamma_1\lambda, \gamma_2\lambda; T-t, T-t; \alpha+1, \beta + \gamma_1 t + \gamma_2 t)$ , following from the second induction assumption;
- (iii) and  $\Pi^T(q+1, q+1+n+1 | \gamma_1\lambda, \gamma_2\lambda; 0, 0; \alpha, \beta + (\gamma_1 + \gamma_2)T) = \Pi^T(q+2, q+1+n | \gamma_1\lambda, \gamma_2\lambda; 0, 0; \alpha, \beta + (\gamma_1 + \gamma_2)T)$  by definition.

Therefore, the induction is complete.  $\square$

*Proof of Proposition 4(b).* Immediately follows from part (a).  $\square$

## Appendix F: Proof of Proposition 5

*Proof of Proposition 5(a).* We formulate the problem of maximizing the expected sales during period 1 as a dynamic program. Let  $S_n(q)$  denote the sales at store  $n$  with test inventory level  $q$ . Then

$$E[S_n(q)] = \sum_{x=0}^q x \phi_n(x) + q \sum_{x=q+1}^{\infty} \phi_n(x),$$

where  $\phi_n(\cdot)$  is the unconditioned pmf of demand at store  $n$ . We thus have

$$\Delta E[S_n(q)] = E[S_n(q+1)] - E[S_n(q)] = \bar{\Phi}_n(q), \quad (7)$$

where  $\bar{\Phi}_n(q) = \sum_{x=q+1}^{\infty} \phi_n(x)$ .

Let  $S(\mathbf{q})$  be the expected total test sales under allocation  $\mathbf{q} = (q_1, \dots, q_N)$ , i.e.,  $S(\mathbf{q}) = \sum_{n=1}^N S_n(q_n)$ . Let  $V_q(\mathbf{q})$  denote the maximum additional expected total test sales with  $q$  units of test inventory left to allocate given an allocation  $\mathbf{q}$ . The problem of allocating  $Q$  units of test inventory to maximize test sales can be formulated as a longest path problem with the following Bellman equations:

$$\begin{aligned} V_q(\mathbf{q}) &= \max_{n \in \{1, \dots, N\}} \{ \Delta E[S_n(q_n)] + V_{q-1}(\mathbf{q} + \delta_n) \} \\ &= \max_{n \in \{1, \dots, N\}} \{ \bar{\Phi}_n(q_n) + V_{q-1}(\mathbf{q} + \delta_n) \}, \end{aligned}$$

with  $V_0(\mathbf{q}) = 0$  for all  $\mathbf{q}$ . The maximum expected total test sales with  $Q$  units of test inventory is given by  $V_Q(\mathbf{0})$ .

The proof of part (a) is by backward induction. For any allocation  $\mathbf{q} = (q_1, \dots, q_N)$ , relabel the stores by  $n_1, n_2, \dots, n_N$ , a permutation of  $1, 2, \dots, N$ , such that  $\bar{\Phi}_{n_1}(q_{n_1}) \geq \bar{\Phi}_{n_2}(q_{n_2}) \geq \dots \geq \bar{\Phi}_{n_N}(q_{n_N})$ . Note that

$$V_1(\mathbf{q}) = \max_{i=1, \dots, N} \{ \bar{\Phi}_{n_i}(q_{n_i}) + V_0(\mathbf{q} + \delta_{n_i}) \} = \bar{\Phi}_{n_1}(q_{n_1}) + V_0(\mathbf{q} + \delta_{n_1}),$$

as  $V_0(\mathbf{q}) = 0$  for all allocation  $\mathbf{q}$ . Suppose that for some  $q \geq 1$  and allocation  $\mathbf{q}$ ,

$$V_q(\mathbf{q}) = \max_{i=1, \dots, N} \{ \bar{\Phi}_{n_i}(q_{n_i}) + V_{q-1}(\mathbf{q} + \delta_{n_i}) \} = \bar{\Phi}_{n_1}(q_{n_1}) + V_{q-1}(\mathbf{q} + \delta_{n_1})$$

where stores are relabeled according to  $\mathbf{q}$  such that  $\bar{\Phi}_{n_1}(q_{n_1}) \geq \bar{\Phi}_{n_2}(q_{n_2}) \geq \dots \geq \bar{\Phi}_{n_N}(q_{n_N})$ . Consider

$$V_{q+1}(\mathbf{q}) = \max_{i=1, \dots, N} \{ \bar{\Phi}_{n_i}(q_{n_i}) + V_q(\mathbf{q} + \delta_{n_i}) \}.$$

Note that  $\bar{\Phi}_{n_1}(q_{n_1}) \geq \bar{\Phi}_{n_i}(q_{n_i}) \geq \bar{\Phi}_{n_i}(q_{n_i} + 1)$  for  $i = 2, \dots, N$ . Therefore,

$$V_q(\mathbf{q} + \delta_{n_i}) = \bar{\Phi}_{n_1}(q_{n_1}) + V_{q-1}(\mathbf{q} + \delta_{n_i} + \delta_{n_1})$$

for  $i = 2, \dots, N$  by the induction assumption. It follows that

$$\begin{aligned} \bar{\Phi}_{n_i}(q_{n_i}) + V_q(\mathbf{q} + \delta_{n_i}) &= \bar{\Phi}_{n_i}(q_{n_i}) + \bar{\Phi}_{n_1}(q_{n_1}) + V_{q-1}(\mathbf{q} + \delta_{n_i} + \delta_{n_1}) \\ &= \bar{\Phi}_{n_1}(q_{n_1}) + \bar{\Phi}_{n_i}(q_{n_i}) + V_{q-1}(\mathbf{q} + \delta_{n_1} + \delta_{n_i}) \\ &\leq \bar{\Phi}_{n_1}(q_{n_1}) + V_q(\mathbf{q} + \delta_{n_1}), \end{aligned}$$

where the inequality follows from the definition of  $V_q(\mathbf{q} + \boldsymbol{\delta}_{n_1})$ . As a result, we have

$$V_{q+1}(\mathbf{q}) = \bar{\Phi}_{n_1}(q_{n_1}) + V_q(\mathbf{q} + \boldsymbol{\delta}_{n_1}).$$

This completes the induction.  $\square$

*Proof of Proposition 5(b).* We prove part (b) by showing that the ordering  $q_1 \geq q_2 \geq \dots \geq q_N$  is preserved before each step of the Max-Sales algorithm. The ordering holds trivially at the beginning of the algorithm as  $q_1 = \dots = q_N = 0$ . Suppose that the ordering holds before some step  $i$ . By definition  $n^* = \min\{n : \bar{\Phi}_n(q_n) \leq \bar{\Phi}_m(q_m), \forall m \neq n\}$ . The ordering is preserved before step  $i + 1$  if  $n^* = 1$ . When  $n^* > 1$ , assume that  $q_{n^*-1} < q_{n^*} + 1$ , then  $\bar{\Phi}_{n^*}(q_{n^*}) > \bar{\Phi}_{n^*-1}(q_{n^*-1}) \geq \bar{\Phi}_{n^*}(q_{n^*-1}) \geq \bar{\Phi}_{n^*}(q_{n^*})$ , where the first inequality follows from the definition of  $n^*$  and the last from the fact that  $q_{n^*-1} \geq q_{n^*}$ . This leads to contradiction. Hence we must have  $q_{n^*-1} \geq q_{n^*} + 1$ , i.e., the ordering holds before step  $i + 1$ .  $\square$

## Appendix G: Proof of Proposition 6

We show the proof only for  $N = 2$ . The proof extends to the  $N > 2$  cases by conditioning on demand at other stores and noting the fact that stores are identical.

Let  $\hat{V}(\hat{a}, \hat{S})$  denote the optimal expected profit in period 2 at a single store under some gamma posterior with parameters  $(\hat{a}, \hat{S})$ . The ex-ante expected profit of allocation  $(q_1, q_2)$  under a gamma prior with parameters  $(a, S)$  is given by

$$\begin{aligned} \Pi^{NT}(q_1, q_2 | a, S) = \Pi(q_1, q_2) = & \int_0^\infty \left[ \int_0^{q_1} \int_0^{q_2} \hat{V}(a + 2, S + x_1^k + x_2^k) f(x_1 | \theta) f(x_2 | \theta) dx_1 dx_2 \right. \\ & + \int_{q_1}^\infty \int_0^{q_2} \hat{V}(a + 1, S + q_1^k + x_2^k) f(x_1 | \theta) f(x_2 | \theta) dx_1 dx_2 \\ & + \int_0^{q_1} \int_{q_2}^\infty \hat{V}(a + 1, S + x_1^k + q_2^k) f(x_1 | \theta) f(x_2 | \theta) dx_1 dx_2 \\ & \left. + \int_{q_1}^\infty \int_{q_2}^\infty \hat{V}(a, S + q_1^k + q_2^k) f(x_1 | \theta) f(x_2 | \theta) dx_1 dx_2 \right] \pi(\theta | a, S) d\theta. \end{aligned}$$

Following Bisi et al. (2011), we can write  $\hat{V}(a, S) = \hat{S}^{\frac{1}{k}} \hat{v}(\hat{a})$ , where  $\hat{v}(\hat{a}) = \hat{V}(\hat{a}, 1)$ . Taking the derivatives, we have

$$\begin{aligned} \frac{\partial \Pi(q_1, q_2)}{\partial q_1} &= q_1^{k-1} \left[ A(S + q_1^k)^{\frac{1}{k}-a-1} + (B - A)(S + q_1^k + q_2^k)^{\frac{1}{k}-a-1} \right], \\ \frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1 \partial q_2} &= k \left( \frac{1}{k} - a - 1 \right) (B - A) q_1^{k-1} q_2^{k-1} (S + q_1^k + q_2^k)^{\frac{1}{k}-a-2}, \\ \frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2} &= A q_1^{k-2} (S + q_1^k)^{\frac{1}{k}-a-2} [(k-1)S - k a q_1^k] \\ &\quad + (B - A) q_1^{k-2} (S + q_1^k + q_2^k)^{\frac{1}{k}-a-2} [(k-1)(S + q_2^k) - k a q_1^k], \end{aligned} \quad (8)$$

where  $A$  and  $B$  are constants given by  $A = S^a k a \left[ \frac{a+1}{a+1-1/k} \hat{v}(a+2) - \hat{v}(a+1) \right]$  and  $B = S^a k \left( a - \frac{1}{k} \right) \left[ \frac{a}{a-1/k} \hat{v}(a+1) - \hat{v}(a) \right]$  that have  $B > A > 0$ . We also have  $\frac{\partial \Pi(q_1, q_2)}{\partial q_2} = \frac{\partial \Pi(q_2, q_1)}{\partial q_2}$  and  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2} = \frac{\partial^2 \Pi(q_2, q_1)}{\partial q_2^2}$  as  $\Pi(q_1, q_2)$  is symmetric with respect to  $q_1$  and  $q_2$ .

It can be verified that  $\frac{\partial \Pi(q_1, q_2)}{\partial q_1} > 0$  for all  $q_1 > 0, q_2 \geq 0$ , and that  $\frac{\partial \Pi(q_1, q_2)}{\partial q_2} > 0$  for all  $q_1 \geq 0, q_2 > 0$ .

Therefore, it suffices to consider allocations that satisfy  $q_1 + q_2 = Q$ .

*Proof of Proposition 6(a).* When  $0 < k \leq 1$ , from (8) we have  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2} < 0$  and  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2} < 0$  for all  $q_1 > 0, q_2 > 0$ . Furthermore, we have  $(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2})(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2}) - (\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1 \partial q_2})^2 > 0$  for all  $q_1 > 0, q_2 > 0$ . Hence,  $\Pi(q_1, q_2)$  is jointly concave in  $q_1$  and  $q_2$  for  $q_1 > 0, q_2 > 0$ . Also,  $\Pi(q_1, q_2)$  is continuous at points with  $q_1 = 0$  and/or  $q_2 = 0$ . Therefore, allocation  $\mathbf{q}^* = (q_1^*, q_2^*)$  with  $q_1^* = q_2^* = Q/2$  maximizes  $\Pi(q_1, q_2)$ .  $\square$

*Proof of Proposition 6(b).* When  $k > 1$ ,

- (i)  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2} > 0$  and  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2} > 0$  for all  $q_1 < Q_0, q_2 < Q_0$  where constant  $Q_0 = \left[ \frac{(k-1)S}{ka} \right]^{\frac{1}{k}}$ . Also,  $(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2})(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2}) - (\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1 \partial q_2})^2 > 0$  for all  $q_1 < Q_0, q_2 < Q_0$ . Hence,  $\Pi(q_1, q_2)$  is jointly convex on  $q_1 < Q_0, q_2 < Q_0$ . As a result, for all  $Q < Q_0$ ,  $\Pi(q_1, q_2)$  is jointly convex on  $\{(q_1, q_2) : q_1 + q_2 = Q\}$ , and a single-store allocation  $(Q, 0)$  or  $(0, Q)$  maximizes  $\Pi(q_1, q_2)$ .
- (ii)  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2} < 0$  for all  $q_1 > \bar{Q}(q_2)$  and  $\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2} < 0$  for all  $q_2 > \bar{Q}(q_1)$  where function  $\bar{Q}(q) = \left[ \frac{(k-1)(S+q^k)}{ka} \right]^{\frac{1}{k}}$ . Also,  $(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1^2})(\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_2^2}) - (\frac{\partial^2 \Pi(q_1, q_2)}{\partial q_1 \partial q_2})^2 > 0$  for all  $q_1 > \bar{Q}(q_2), q_2 > \bar{Q}(q_1)$ . Hence, for  $q_1 > \bar{Q}(q_2), q_2 > \bar{Q}(q_1)$ ,  $\Pi(q_1, q_2)$  is jointly concave in  $q_1$  and  $q_2$ , thus  $(Q/2, Q/2)$  is a local maximal. Also, as  $Q \rightarrow \infty$ ,  $\Pi(Q/2, Q/2|a, S) \rightarrow \int_0^\infty \int_0^\infty \int_0^\infty \hat{V}(a+2, S+x_1^k+x_2^k) f(x_1|\theta) f(x_2|\theta) dx_1 dx_2 \pi(\theta) d\theta \geq \Pi(q_1, q_2)$  for all  $q_1 \geq 0$  and  $q_2 \geq 0$ .  $\square$

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