EEE 133 Key concepts and Equations

AJ Mesa Jr.

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Differential Equations

A linear ordinary differential equation of constant coefficients follows the form

$$a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \ldots + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = f(t) \tag{1}$$

A homogenous differential equation is an equation where the independent variable appears to have the same power. Equation (1) is homogenous when $f(t) = 0$.
The solution to the linear ODE in (1) has the form

$$x(t) = x_h(t) + x_p(t) \tag{2}$$

Where $x_h(t)$ is a solution to the homogenous equation while $x_p(t)$ is a solution to the nonhomogenous equation.
A homogenous equation has solutions of the form

$$x_h(t) = \exp(mt) \tag{3}$$

for m is a root of the equation

$$m^n + \frac{a_{n-1}}{a_n} m^{n-1} + \ldots + \frac{a_1}{a_n} m + \frac{a_0}{a_n} = 0 \tag{4}$$

If a root m_i is repeated k times, the corresponding solutions are $\exp(m_i t),\ t\exp(m_i t),\ \ldots,\ t^{k-1}\exp(m_i t)$

Some common forms (with constant coefficients) and solutions

Equation	Characteristic Equation	Determinant	Solution
$a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = 0$	$a_1 m + a_0 = 0$		$x = C\exp\left(-\frac{a_0}{a_1}t\right)$
$a_2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = 0$	$a_2 m^2 + a_1 m + a_0 = 0$	$a_1^2 - 4a_2a_0 > 0$	$x(t) = C_1\exp(m_1 t) + C_2\exp(m_2 t)$
		$a_1^2 - 4a_2a_0 = 0$	$x(t) = (C_1 x + C_2)\exp(m_1 t)$
		$a_1^2 - 4a_2a_0 < 0$	$x(t) = \exp(\alpha t)\left[C_1\cos(\beta t) + C_2\sin(\beta t)\right]$

In the last line, the characteristic equation has solutions $m_1 = \alpha + j\beta,\ m_2 = \alpha - j\beta$ where $j^2 = -1$

Laplace Transforms and Theorems

A Laplace transform of a function $f(t)$ defined for all $t \geq 0$ is the transformation

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)\exp(-st)\,\mathrm{d}t$$

An inverse Laplace transform of a function $F(s)$ is defined as:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{\gamma-jT}^{\gamma+jT}\exp(st)F(s)\,\mathrm{d}s$$

Common Functional Tranforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[F(s)]$	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[F(s)]$
$\delta(t)$	1	$\frac{1}{\beta-\alpha}\left(\exp(-\alpha t) - \exp(-\beta t)\right)u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\sin(\omega t+\theta)u(t)$	$\frac{s\sin(\theta)+\omega\cos(\theta)}{s^2+\omega^2}$
$\exp(-\alpha t)u(t)$	$\frac{1}{s+\alpha}$	$\cos(\omega t+\theta)u(t)$	$\frac{s\cos(\theta)-\omega\sin(\theta)}{s^2+\omega^2}$
$t\exp(-\alpha t)u(t)$	$\frac{1}{(s+\alpha)^2}$	$\exp(-\alpha t)\sin(\omega t)u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}\exp(-\alpha t)u(t)$	$\frac{1}{(s+\alpha)^n}$	$\exp(-\alpha t)\cos(\omega t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Common Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by constant	$cf(t)$	$cF(s)$
Addition	$f_1(t)+f_2(t)-f_3(t)+\dots$	$F_1(s)+F_2(s)-F_3(s)+\dots$
First time derivative	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$	$sF(s)-f(0^-)$
Second time derivative	$\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}$	$s^2F(s)-sf(0^-)-\frac{\mathrm{d}f(0^-)}{\mathrm{d}t}$
nth time derivative	$\frac{\mathrm{d}^nf(t)}{\mathrm{d}t^n}$	$s^nF(s)-\sum_{i=1}^{n}s^{n-1}\frac{\mathrm{d}^{i-1}f}{\mathrm{d}t^{i-1}}$
Time integral	$\int_0^t f(x)\,\mathrm{d}x$	$\frac{F(s)}{s}$

Translation in time	$f(t-a)u(t-a),\ a>0$	$\exp(-as)F(s)$
Translation in frequency	$\exp(-at)f(t)$	$F(s+a)$
Scale change	$f(at),\ a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First frequency derivative	$tf(t)$	$\frac{\mathrm{d}F(s)}{\mathrm{d}s}$
nth frequency derivative	$t^n f(t)$	$(-1)^n\frac{\mathrm{d}^nF(s)}{\mathrm{d}s^n}$
Frequency integral	$\frac{f(t)}{t}$	$\int\limits_0^\infty F(u)\,\mathrm{d}u$

1 Passive components

Current and Voltage for Passive Components:

Component	Current	Voltage	Energy
Resistor R	$i=\frac{v}{R}$	$v=iR$	$w_R=\int iv\,\mathrm{d}t$
Capacitor C	$i(t)=C\frac{\mathrm{d}v}{\mathrm{d}t}$	$v(t)=v(t_0)+\frac{1}{C}\int_{t_0}^t i(t)\,\mathrm{d}t$	$w_C(t)=\frac{1}{2}Cv^2(t)$
Inductor L	$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^t v(t)\,\mathrm{d}t$	$v(t)=L\frac{\mathrm{d}i}{\mathrm{d}t}$	$w_L(t)=\frac{1}{2}Li^2(t)$

Combination of passive components:

Component	Series	Parallel
Resistor R	$R_{eq}=\sum_i R_i$	$\frac{1}{R_{eq}}=\sum_i\frac{1}{R_i}$
Capacitor C	$\frac{1}{C_{eq}}=\sum_i\frac{1}{C_i}$	$C_{eq}=\sum_i C_i$
Inductor L	$L_{eq}=\sum_i L_i$	$\frac{1}{L_{eq}}=\sum_i\frac{1}{L_i}$

1.1 Equilibrium Equations

1. **Loop Current formulation**:
   - number of unknown currents equal number of loops
   - KVL equation for each loop
2. **Node Voltage formulation**:
   - number of unknown voltage equal number of nodes except reference
   - KCL equation for each node

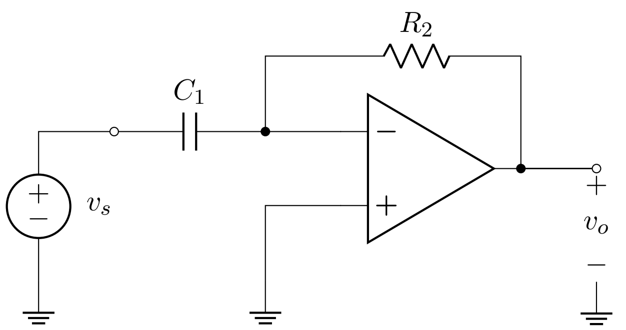


Figure 1: Op Amp Differentiator

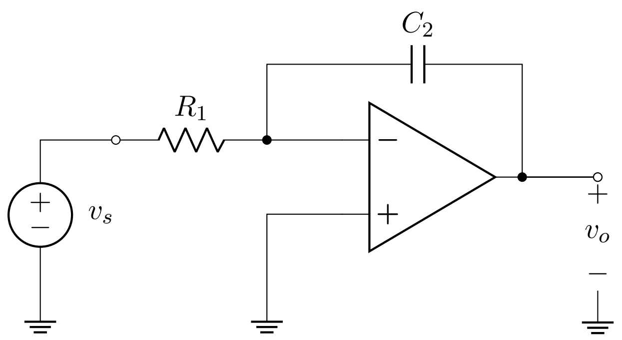


Figure 2: Op Amp Integrator

$$v_o(t) = -R_2 C_1 \frac{\mathrm{d}v_s}{\mathrm{d}t}$$

$$v_o(t) = -\frac{1}{R_1 C_2} \int_0^t v_s(t')\,\mathrm{d}t'$$

1.2 First Order Circuits

1. first order circuits are any circuit with a single energy storage element, an arbitrary number of sources and resistors
2. any current or voltage in such circuit is a solution to a first order differential equation

2 First order Unforced Response

Network	Current	Voltage	Time Constant	DC Steady State
Sourcefree RL	$i_L(t) = I_0 \exp\left(-\frac{R}{L}t\right)$	$v_L = -RI_0 \exp\left(-\frac{R}{L}t\right)$	$\tau = \frac{L}{R}$	$v_L = 0$
Sourcefree RC	$i_C(t) = -\frac{V_0}{R} \exp\left(-\frac{1}{RC}t\right)$	$v_c = V_0 \exp\left(-\frac{1}{RC}t\right)$	$\tau = RC$	$i_C = 0$

Typical time constant for RL is in ms, for RC in μs. For general RL and RC circuits, find the equivalent resistance as seen by the inductor/capacitor.

3 First order Forced Response

Consider a series RL circuit with voltage $v(t) = V_0 u(t)$. We have the following KVL equation of current

$$\begin{aligned} i(t) &= 0 & t &< 0 \\ Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} &= V_0 & t &> 0 \end{aligned}$$

This differential equation gives a solution of

$$i(t) = \left[\underbrace{\frac{V_0}{R}}_{\text{forced response}} - \underbrace{\left(\frac{V_0}{R} - I_0\right) \exp\left(-\frac{R}{L}t\right)}_{\text{natural response}} \right] u(t) \tag{5}$$

1. Forced response

- dependent on forcing function
- steady state response $t \gg \tau$

2. Natural response

- similar to source-free circuit
- dependent on initial values and forcing function
- transient response

Generalization:

$$i_L(t) = \left[I_f - (I_f - I_i)\exp\left(-\frac{t}{\tau}\right)\right]u(t) \quad (6)$$

$$v_c(t) = \left[V_f - (V_f - V_i)\exp\left(-\frac{t}{\tau}\right)\right]u(t) \quad (7)$$

Note that equation (7) is for the voltage across the capacitor in an RC circuit.

3.1 Square Waves and Sequentially Switched Circuits

Now consider the series RL circuit with voltage $v(t) = V_0u(t) - V_0u(t-t_0)$ and $I_0 = 0$. We can apply superposition to find the current:

$$i(t) = i_1(t) + i_2(t)$$

$$i(t) = \underbrace{\left[\frac{V_0}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\right]u(t)}_{\text{caused by } V_0u(t) \text{ alone}} - \underbrace{\left[\frac{V_0}{R}\left(1 - \exp\left(-\frac{R}{L}(t-t_0)\right)\right)\right]u(t-t_0)}_{\text{caused by } V_0u(t-t_0) \text{ alone}}$$

For sequentially switched circuits, we consider the pulse width (PW) and period (T) of the pulsing.

Condition	Output	Condition	Output
PW $\gg \tau$	time enough to fully charge	T - PW $\gg \tau$	time enough to fully discharge
PW $\ll \tau$	time NOT enough to fully charge	T - PW $\ll \tau$	time NOT enough to fully discharge

3.2 RC Oscillator

1. For a low pass RC circuit, the output (voltage at capacitor) is simply the input for low frequencies
   (a) at $\omega = 0$, $v_o = v_i$
   (b) at $\omega \to \infty$, $v_o \to 0$

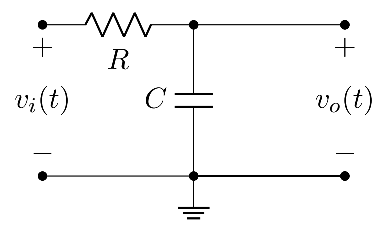
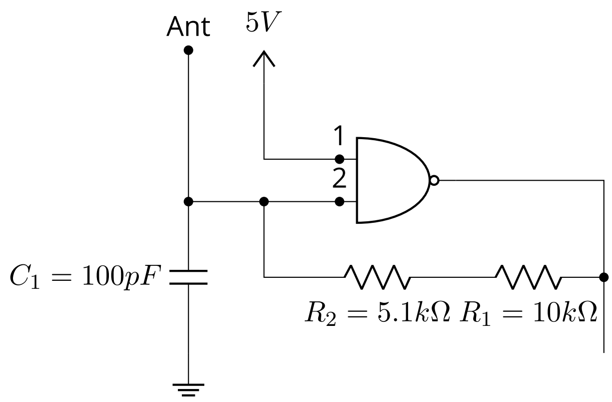


Figure 3: Low Pass Circuit



2. Schmitt Trigger RC
   (a) $V_p = 2.9V \approx 3.0V, V_n = 1.9V \approx 2.0V$
   (b) As the voltage on C_1 reaches V_p, the output will become voltage low. Then the voltage across the capacitor decays
   (c) When the voltage across C_1 decays to V_n, output will become voltage high. Then the voltage across the capacitor increases
   (d) The behavior oscillates.
   (e) the voltage across the charging capacitor is
   $$v = V_{cc}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$
   (f) the time to charge to V_p is
   $$t_1 = -\tau \ln\left(1 - \frac{V_p}{V_{cc}}\right)$$
   (g) the voltage across the discharging capacitor is
   $$v = V_p \exp\left(-\frac{t}{\tau}\right)$$
   (h) the time to discharge to V_n is
   $$t_2 = -\tau \ln\left(\frac{V_n}{V_p}\right)$$
   (i) the voltage across the charging capacitor from V_n to V_p is
   $$v = V_{cc} - (V_{cc} - V_n)\exp\left(-\frac{t}{\tau}\right)$$
   (j) the time to charge to from V_n to V_p is
   $$t_3 = -\tau \ln\left(\frac{V_{cc} - V_p}{V_{cc} - V_n}\right)$$
3. For a high pass RC circuit, the output (voltage at capacitor) is simply the input for high frequencies
   (a) at $\omega = 0$, $v_o = 0$
   (b) at $\omega \to \infty$, $v_o \to v_i$

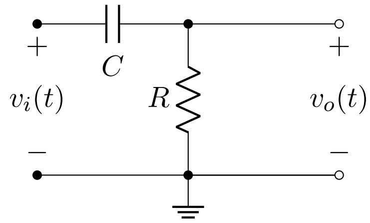


Figure 4: High Pass Circuit

4 Diode and Transistor Switching: Half Wave Rectifier

1. The half wave rectifier has the following circuit

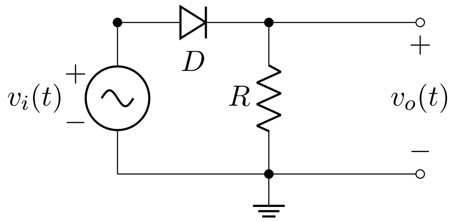


Figure 5: Half Wave Rectifier

   (a) the diode is open only when $v_i(t) > v_{\text{on}}$, hence only the positive voltage are seen by $v_o(t)$
      i. Average Value $\frac{V_{\text{m}}}{\pi}$
      ii. $V_{\text{rms}} = \frac{V_{\text{m}}}{2}$
      iii. $I_{\text{m}} = \frac{V_{\text{m}}}{R_{\text{L}}}$
      iv. Ripple Factor $\frac{I_{\text{rms}}}{I_{\text{DC}}}$
      v. Efficiency $e = \frac{\text{DC Output Power}}{\text{AC Output Power}}$
   (b) a smoothing capacitor is added (capacitor filter) so that the output waveform does not have a 0 value:

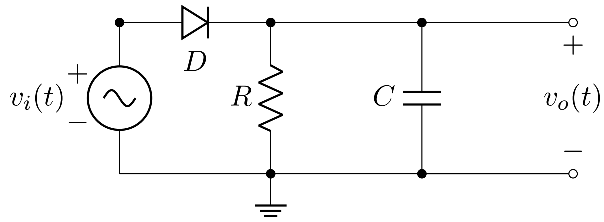


Figure 6: Half Wave Rectifier with Capacitor Filter

      i. C is chosen such that $RC \gg T$ so the exponential seems linear

5 LC Oscillations

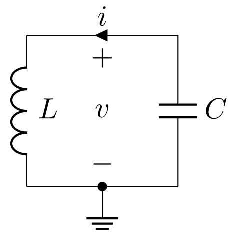


Figure 7: LC Oscillator

1. most of the time, either $v(0)$ or $i(0)$ are given.
2. by conservation of energy: $\frac{1}{2}Li^2 = \frac{1}{2}Cv^2$
3. the differential equations are $v = L\frac{\mathrm{d}i}{\mathrm{d}t}$ and $i = -C\frac{\mathrm{d}v}{\mathrm{d}t} \implies i = \frac{1}{LC}\frac{\mathrm{d}^2 i}{\mathrm{d}t^2}$
4. we have $\omega = \frac{1}{\sqrt{LC}}$
5. the solutions have the form

$$i = I_0 \cos\left(\frac{t}{\sqrt{LC}} + \phi\right)$$
$$v = -I_0\sqrt{\frac{L}{C}} \sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

or

$$v = V_0 \cos\left(\frac{t}{\sqrt{LC}} + \phi\right)$$
$$i = -V_0\sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

6 Source-free RLC Circuits

	Characteristic Equation	Neper Frequency (α)	Resonant Frequency (ω_0)	Damping Factor (ζ)
Series	$s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$	$\alpha = \frac{R}{2L}$	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$
Parallel	$s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC} = 0$	$\alpha = \frac{1}{2RC}$	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$

Case	Condition	Characteristic
Overdamped	$\zeta > 1$	Does not oscillate about the steady state value but takes time
Underdamped	$\zeta < 1$	Oscillation with decay envelope
Critacally damped	$\zeta = 1$	Decays fastes to steady state without oscillation

7 Laplace Transform Applications

7.1 Method 1

1. Write the differential equations for the unknown function $x(t)$
2. Apply Laplace transform on the equation
3. Use algebraic manipulation for $X(s)$
4. Apply inverse Laplace transform to get $x(t)$

7.2 Method 2

1. Apply Laplace transform on each element
2. Use DC circuit analysis techniques to write the s-domain equations involving $X(s)$
3. Apply inverse Laplace transform to get $x(t)$

Examples

1. Resistors: $v = iR \iff V = IR$, where $V = \mathcal{L}\{v\},\ I = \mathcal{L}\{i\}$
2. Inductors: $v = L\dfrac{\mathrm{d}i}{\mathrm{d}t} \iff V = L\left[sI - i(0^-)\right] = sLI - LI_0$
3. Capacitors: $i = C\dfrac{\mathrm{d}v}{\mathrm{d}t} \iff I = C\left[sV - v(0^-)\right] = sCV - CV_0$

7.3 $v(t) = V_0 u(t)$

1. Find the equivalent circuit using the Laplace equivalent of the circuit
2. Solve for $V(s)$. May involve partial fraction decomposition. This is a nice guide
3. Use Inverse Laplace transform to get $v(t)$

Transfer function. The s-domain ratio of the output to input signal.

$$H(s) = \frac{Y(s)}{X(s)} \tag{8}$$

		$H(s)$ of series RL
Input is $V(t)$	Output is $i(t)$	$H(s) = \dfrac{1}{R + sL}$
	Output is $v(t)$	$H(s) = \dfrac{sL}{R + sL}$

7.3.1 $v(t) = V_0 \cos(\omega t)$

The response has the form $i(t) = I_1 \cos(\omega t) + I_2 \sin(\omega t)$. Exploit the transfer function of the circuit.

8 Modeling of Electromechanical Systems

9 Second order Unforced Responses

10 Second order Forced Responses