

EEE 133 Key concepts and Equations

AJ Mesa Jr.

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Differential Equations

A linear ordinary differential equation of constant coefficients follows the form

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = f(t) \quad (1)$$

A homogenous differential equation is an equation where the independent variable appears to have the same power. Equation (1) is homogenous when $f(t) = 0$.

The solution to the linear ODE in (1) has the form

$$x(t) = x_h(t) + x_p(t) \quad (2)$$

Where $x_h(t)$ is a solution to the homogenous equation while $x_p(t)$ is a solution to the nonhomogenous equation.

A homogenous equation has solutions of the form

$$x_h(t) = \exp(mt) \quad (3)$$

for m is a root of the equation

$$m^n + \frac{a_{n-1}}{a_n} m^{n-1} + \dots + \frac{a_1}{a_n} m + \frac{a_0}{a_n} = 0 \quad (4)$$

If a root m_i is repeated k times, the corresponding solutions are $\exp(m_i t)$, $t \exp(m_i t)$, \dots , $t^{k-1} \exp(m_i t)$

Some common forms (with constant coefficients) and solutions

Equation	Characteristic Equation	Determinant	Solution
$a_1 \frac{dx}{dt} + a_0 x = 0$	$a_1 m + a_0 = 0$		$x = C \exp\left(-\frac{a_0}{a_1} t\right)$
$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$	$a_2 m^2 + a_1 m + a_0 = 0$	$a_1^2 - 4a_2 a_0 > 0$	$x(t) = C_1 \exp(m_1 t) + C_2 \exp(m_2 t)$
		$a_1^2 - 4a_2 a_0 = 0$	$x(t) = (C_1 x + C_2) \exp(m_1 t)$
		$a_1^2 - 4a_2 a_0 < 0$	$x(t) = \exp(\alpha t) [C_1 \cos(\beta t) + C_2 \sin(\beta t)]$

In the last line, the characteristic equation has solutions $m_1 = \alpha + j\beta$, $m_2 = \alpha - j\beta$ where $j^2 = -1$

Laplace Transforms and Theorems

A Laplace transform of a function $f(t)$ defined for all $t \geq 0$ is the transformation

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) \exp(-st) dt$$

An inverse Laplace transform of a function $F(s)$ is defined as:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\gamma-jT}^{\gamma+jT} \exp(st) F(s) ds$$

Common Functional Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
$\delta(t)$	1	$\frac{1}{\beta - \alpha} (\exp(-\alpha t) - \exp(-\beta t)) u(t)$	$\frac{1}{(s + \alpha)(s + \beta)}$
$u(t)$	$\frac{1}{s}$	$\sin(\omega t) u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos(\omega t) u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} u(t)$	$\frac{1}{s^n}$	$\sin(\omega t + \theta) u(t)$	$\frac{s \sin(\theta) + \omega \cos(\theta)}{s^2 + \omega^2}$
$\exp(-\alpha t) u(t)$	$\frac{1}{s + \alpha}$	$\cos(\omega t + \theta) u(t)$	$\frac{s \cos(\theta) - \omega \sin(\theta)}{s^2 + \omega^2}$
$t \exp(-\alpha t) u(t)$	$\frac{1}{(s + \alpha)^2}$	$\exp(-\alpha t) \sin(\omega t) u(t)$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} \exp(-\alpha t) u(t)$	$\frac{1}{(s + \alpha)^n}$	$\exp(-\alpha t) \cos(\omega t) u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$

Common Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by constant	$cf(t)$	$cF(s)$
Addition	$f_1(t) + f_2(t) - f_3(t) + \dots$	$F_1(s) + F_2(s) - F_3(s) + \dots$
First time derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second time derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
n th time derivative	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{i=1}^n s^{n-i} \frac{d^{i-1} f(0^-)}{dt^{i-1}}$
Time integral	$\int_0^t f(x) dx$	$\frac{F(s)}{s}$

Translation in time	$f(t-a)u(t-a), a > 0$	$\exp(-as)F(s)$
Translation in frequency	$\exp(-at)f(t)$	$F(s+a)$
Scale change	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First frequency derivative	$tf(t)$	$\frac{dF(s)}{ds}$
n th frequency derivative	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
Frequency integral	$\frac{f(t)}{t}$	$\int_0^\infty F(u) du$

1 Passive components

Current and Voltage for Passive Components:

Component	Current	Voltage	Energy
Resistor R	$i = \frac{v}{R}$	$v = iR$	$w_R = \int iv dt$
Capacitor C	$i(t) = C \frac{dv}{dt}$	$v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^t i(t) dt$	$w_C(t) = \frac{1}{2} C v^2(t)$
Inductor L	$i(t) = i(t_0) + \frac{1}{L} \int_{t_0}^t v(t) dt$	$v(t) = L \frac{di}{dt}$	$w_L(t) = \frac{1}{2} L i^2(t)$

Combination of passive components:

Component	Series	Parallel
Resistor R	$R_{eq} = \sum_i R_i$	$\frac{1}{R_{eq}} = \sum_i \frac{1}{R_i}$
Capacitor C	$\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}$	$C_{eq} = \sum_i C_i$
Inductor L	$L_{eq} = \sum_i L_i$	$\frac{1}{L_{eq}} = \sum_i \frac{1}{L_i}$

1.1 Equilibrium Equations

1. Loop Current formulation:

- number of unknown currents equal number of loops
- KVL equation for each loop

2. Node Voltage formulation:

- number of unknown voltage equal number of nodes except reference
- KCL equation for each node

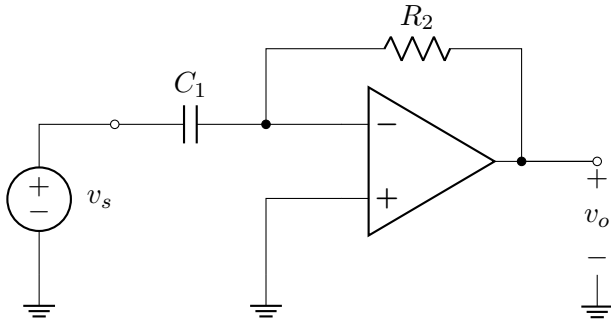


Figure 1: Op Amp Differentiator

$$v_o(t) = -R_2 C_1 \frac{dv_s}{dt}$$

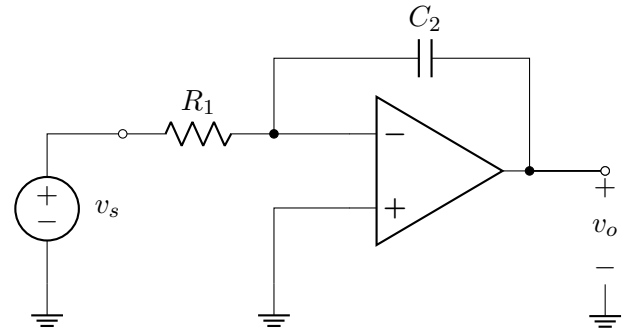


Figure 2: Op Amp Integrator

$$v_o(t) = -\frac{1}{R_1 C_2} \int_0^t v_s(t') dt'$$

1.2 First Order Circuits

1. first order circuits are any circuit with a single energy storage element, an arbitrary number of sources and resistors
2. any current or voltage in such circuit is a solution to a first order differential equation

2 First order Unforced Response

Network	Current	Voltage	Time Constant	DC Steady State
Sourcefree RL	$i_L(t) = I_0 \exp\left(-\frac{R}{L}t\right)$	$v_L = -RI_0 \exp\left(-\frac{R}{L}t\right)$	$\tau = \frac{L}{R}$	$v_L = 0$
Sourcefree RC	$i_C(t) = -\frac{V_0}{R} \exp\left(-\frac{1}{RC}t\right)$	$v_c = V_0 \exp\left(-\frac{1}{RC}t\right)$	$\tau = RC$	$i_C = 0$

Typical time constant for RL is in ms, for RC in μ s. For general RL and RC circuits, find the equivalent resistance as seen by the inductor/capacitor.

3 First order Forced Response

Consider a series RL circuit with voltage $v(t) = V_0 u(t)$. We have the following KVL equation of current

$$\begin{aligned} i(t) &= 0 & t < 0 \\ Ri + L \frac{di}{dt} &= V_0 & t > 0 \end{aligned}$$

This differential equation gives a solution of

$$i(t) = \left[\underbrace{\frac{V_0}{R}}_{\text{forced response}} - \underbrace{\left(\frac{V_0}{R} - I_0\right) \exp\left(-\frac{R}{L}t\right)}_{\text{natural response}} \right] u(t) \quad (5)$$

1. Forced response

- dependent on forcing function
- steady state response $t \gg \tau$

2. Natural response

- similar to source-free circuit
- dependent on initial values and forcing function
- transient response

Generalization:

$$i_L(t) = \left[I_f - (I_f - I_i) \exp\left(-\frac{t}{\tau}\right) \right] u(t) \quad (6)$$

$$v_c(t) = \left[V_f - (V_f - V_i) \exp\left(-\frac{t}{\tau}\right) \right] u(t) \quad (7)$$

Note that equation (7) is for the voltage across the capacitor in an RC circuit.

3.1 Square Waves and Sequentially Switched Circuits

Now consider the series RL circuit with voltage $v(t) = V_0 u(t) - V_0 u(t - t_0)$ and $I_0 = 0$. We can apply superposition to find the current:

$$i(t) = i_1(t) + i_2(t)$$

$$i(t) = \underbrace{\left[\frac{V_0}{R} \left(1 - \exp\left(-\frac{R}{L}t\right) \right) \right] u(t)}_{\text{caused by } V_0 u(t) \text{ alone}} - \underbrace{\left[\frac{V_0}{R} \left(1 - \exp\left(-\frac{R}{L}(t - t_0)\right) \right) \right] u(t - t_0)}_{\text{caused by } V_0 u(t - t_0) \text{ alone}}$$

For sequentially switched circuits, we consider the pulse width (PW) and period (T) of the pulsing.

Condition	Output	Condition	Output
PW $\gg \tau$	time enough to fully charge	T - PW $\gg \tau$	time enough to fully discharge
PW $\ll \tau$	time NOT enough to fully charge	T - PW $\ll \tau$	time NOT enough to fully discharge

3.2 RC Oscillator

1. For a low pass RC circuit, the output (voltage at capacitor) is simply the input for low frequencies

- (a) at $\omega = 0$, $v_o = v_i$
- (b) at $\omega \rightarrow \infty$, $v_o \rightarrow 0$

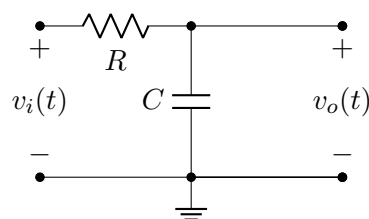
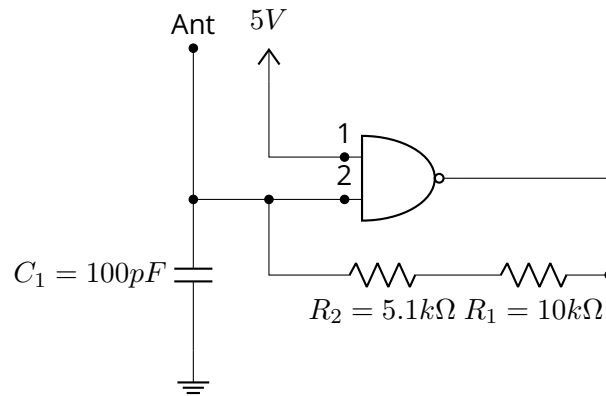


Figure 3: Low Pass Circuit



2. Schmitt Trigger RC

- (a) $V_p = 2.9V \approx 3.0V, V_n = 1.9V \approx 2.0V$
- (b) As the voltage on C_1 reaches V_p , the output will become voltage low. Then the voltage across the capacitor decays
- (c) When the voltage across C_1 decays to V_n , output will become voltage high. Then the voltage across the capacitor increases
- (d) The behavior oscillates.
- (e) the voltage across the charging capacitor is

$$v = V_{cc} \left[1 - \exp \left(-\frac{t}{\tau} \right) \right]$$

- (f) the time to charge to V_p is

$$t_1 = -\tau \ln \left(1 - \frac{V_p}{V_{cc}} \right)$$

- (g) the voltage across the discharging capacitor is

$$v = V_p \exp \left(-\frac{t}{\tau} \right)$$

- (h) the time to discharge to V_n is

$$t_2 = -\tau \ln \left(\frac{V_n}{V_p} \right)$$

- (i) the voltage across the charging capacitor from V_n to V_p is

$$v = V_{cc} - (V_{cc} - V_n) \exp \left(-\frac{t}{\tau} \right)$$

- (j) the time to charge to from V_n to V_p is

$$t_3 = -\tau \ln \left(\frac{V_{cc} - V_p}{V_{cc} - V_n} \right)$$

3. For a high pass RC circuit, the output (voltage at capacitor) is simply the input for high frequencies

- (a) at $\omega = 0, v_o = 0$
- (b) at $\omega \rightarrow \infty, v_o \rightarrow v_i$

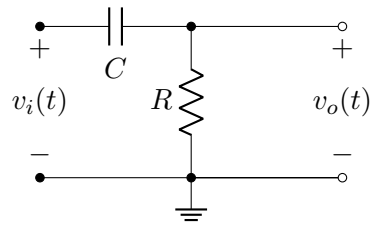


Figure 4: High Pass Circuit

4 Diode and Transistor Switching: Half Wave Rectifier

1. The half wave rectifier has the following circuit

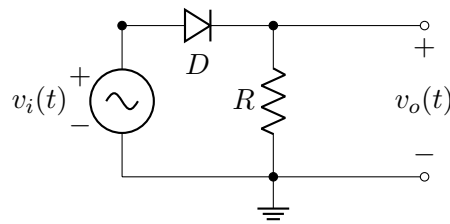


Figure 5: Half Wave Rectifier

(a) the diode is open only when $v_i(t) < v_{on}$, hence only the positive voltage are seen by $v_o(t)$

i. Average Value $\frac{V_m}{\pi}$

ii. $V_{rms} = \frac{V_m}{2}$

iii. $I_m = \frac{V_m}{R_L}$

iv. Ripple Factor $\frac{I_{rms}}{I_{DC}}$

v. Efficiency $e = \frac{\text{DC Output Power}}{\text{AC Output Power}}$

(b) a smoothing capacitor is added (capacitor filter) so that the output waveform does not have a 0 value:

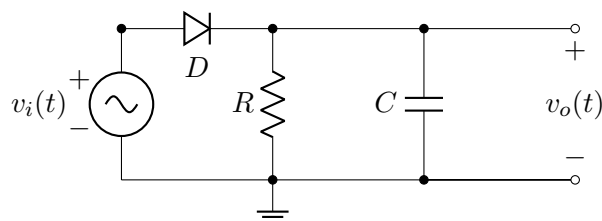


Figure 6: Half Wave Rectifier with Capacitor Filter

i. C is chosen such that $RC \gg T$ so the exponential seems linear

5 LC Oscillations

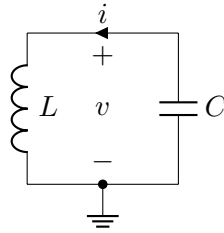


Figure 7: LC Oscillator

1. most of the time, either $v(0)$ or $i(0)$ are given.
2. by conservation of energy: $\frac{1}{2}Li^2 = \frac{1}{2}Cv^2$
3. the differential equations are $v = L\frac{di}{dt}$ and $i = -C\frac{dv}{dt} \Rightarrow i = \frac{1}{LC}\frac{d^2i}{dt^2}$
4. we have $\omega = \frac{1}{\sqrt{LC}}$
5. the solutions have the form

$$i = I_0 \cos\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

$$v = -I_0\sqrt{\frac{L}{C}} \sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

or

$$v = V_0 \cos\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

$$i = -V_0\sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

6 Source-free RLC Circuits

	Characteristic Equation	(α)	(ω_0)	Damping Factor (ζ)
Series $i(t)$	$s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$	$\alpha = \frac{R}{2L}$	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$
Parallel $v(t)$	$s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC} = 0$	$\alpha = \frac{1}{2RC}$	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad (8)$$

Case	Condition	Characteristic	Roots
Overdamped	$\zeta > 1$	Does not oscillate about the steady state value	Real, Distinct
Underdamped	$\zeta < 1$	Oscillation with decay envelope	Complex
Critically damped	$\zeta = 1$	Decays fastest to steady state without oscillation	Real, equal

7 Resonance

1. resonance - the fixed amplitude forcing function produces a response of maximum amplitude

2. Series Resonant Conditions:

(a) $\omega_0 = \frac{1}{\sqrt{LC}}$

(b) $Z_{in} = R$

(c) voltage and current of source in phase

(d) current magnitude is maximum

(e) electric filter

(f) the half-power-point frequency is where the power is half:

$$\omega_{1,2} = \mp \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \quad (9)$$

(g) bandwidth: $BW = \omega_2 - \omega_1 = \frac{R}{L}$

(h) Quality factor: $Q = \frac{\omega_0}{BW} = \frac{1}{R} \sqrt{\frac{L}{C}}$

(i) $\omega_0^2 = \omega_1 \omega_2$

3. Parallel Resonant Conditions:

(a) $\omega_0 = \frac{1}{\sqrt{LC}}$

(b) $Y_{in} = \frac{1}{R}$

(c) voltage and current of source in phase

(d) voltage magnitude is maximum

(e) electric filter

(f) the half-power-point frequency is where the power is half:

$$\omega_{1,2} = \mp \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad (10)$$

(g) bandwidth: $BW = \omega_2 - \omega_1 = \frac{1}{RC}$

(h) Quality factor: $Q = \frac{\omega_0}{BW} = R \sqrt{\frac{C}{L}}$

(i) $\omega_0^2 = \omega_1 \omega_2$

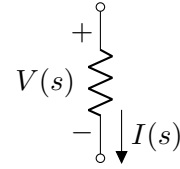
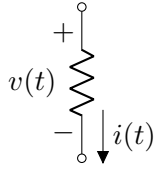
4. Physical model: R_1 series to L , C parallel to that series and to R_2

(a) $\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R_1}{L}\right)^2}$

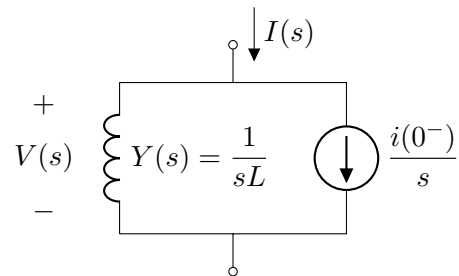
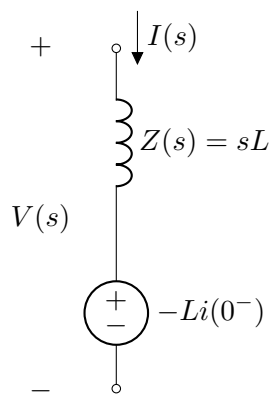
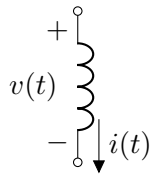
8 Laplace Transform Applications

Transform of circuit elements:

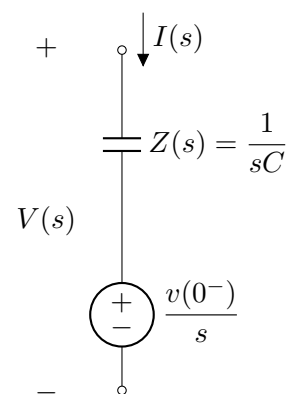
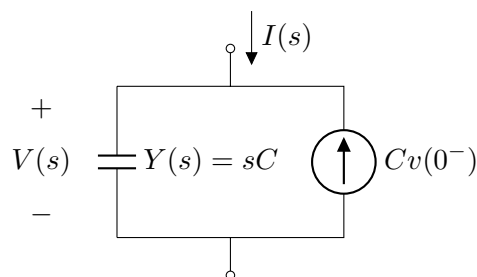
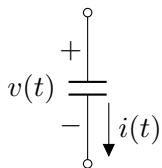
1. Resistor:



2. Inductor:



3. Capacitor:



8.1 Method 1

1. Write the differential equations for the unknown function $x(t)$
2. Apply Laplace transform on the equation

3. Use algebraic manipulation for $X(s)$
4. Apply inverse Laplace transform to get $x(t)$

8.2 Method 2

1. Apply Laplace transform on each element
2. Use DC circuit analysis techniques to write the s -domain equations involving $X(s)$
3. Apply inverse Laplace transform to get $x(t)$

Examples

1. Resistors: $v = iR \iff V = IR$, where $V = \mathcal{L}\{v\}$, $I = \mathcal{L}\{i\}$
2. Inductors: $v = L \frac{di}{dt} \iff V = L [sI - i(0^-)] = sLI - LI_0$
3. Capacitors: $i = C \frac{dv}{dt} \iff I = C [sV - v(0^-)] = sCV - CV_0$

8.3 $v(t) = V_0 u(t)$

1. Find the equivalent circuit using the Laplace equivalent of the circuit
2. Solve for $V(s)$. May involve partial fraction decomposition. This is a nice guide
3. Use Inverse Laplace transform to get $v(t)$

Transfer function. The s -domain ratio of the output to input signal.

$$H(s) = \frac{Y(s)}{X(s)} \quad (11)$$

		$H(s)$ of series RL
Input is $V(t)$	Output is $i(t)$	$H(s) = \frac{1}{R + sL}$
	Output is $v(t)$	$H(s) = \frac{sL}{R + sL}$

8.3.1 $v(t) = V_0 \cos(\omega t)$

The response has the form $i(t) = I_1 \cos(\omega t) + I_2 \sin(\omega t)$. Exploit the transfer function of the circuit.

9 Ramp and Sine Response

10 Second Order System

11 Transients in Power Systems

12 Switching Power Supply

13 Filter Design