EEE 137 Key concepts and Equations

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September 30, 2020

Contents

1 Sets and Probabilities

Some set operations and properties:

1. If $A \subseteq B$ and $B \subseteq A$, then $A = B$
2. The set difference $A - B$ or $A \setminus B$ contains the elements in A that are not in
3. The set $A', \overline{A},$ orA^C contains all the elements in the universal set but not in A
4. Commutative:
   (a) $A \cap B = B \cap A$
   (b) $A \cup B = B \cup A$
5. Distributive:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. Associative:
   (a) $A \cup (B \cup C) = (A \cup B) \cup C$

   (b) $A \cap (B \cap C) = (A \cap B) \cap C$

7. De Morgan's Law:
   (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

8. Duality principle: the following replacements preserves the identity
   (a) $\cup \longleftrightarrow \cap$
   (b) $S \longleftrightarrow \varnothing$
   (c) $A \longleftrightarrow \overline{A}$

Stochastic experiment

1. Experiment: process that gives information
2. Outcome: information recorded as result of experiment
3. Stochastic (random) experiment: process with different outcomes
4. Replicate: single performance of random experiment
5. Sample space: set which every outcome is represented by one and only one element
   (a) Discrete: countable elements
   (b) Continuous: elements are uncountable
6. Event: any occurence that result from performance of experiment:
   (a) Elementary: single outcome
   (b) Compound: more than 1 outcome
7. Probability:
   (a) Classical: $P(E) = \dfrac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$
   (b) Empirical: $P(E) = \dfrac{\text{frequency of occurence of favorable outcome}}{\text{total frequency}}$
8. Axioms of probability:
   (a) For any event $E \subseteq S,\ 0 \leq P(E) \leq 1$
   (b) $P(S) = 1$
   (c) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (d) $P(\overline{E}) = 1 - P(E)$
   (e) Principle of Inclusion and Exclusion: $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}\left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} P(A_I)\right)$
   (f) where $A_I := \bigcap_{i \in I} A_i$

2 Conditional Probability

1. probability of event A given B: $P\left(A|B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)}$

2. special cases:

   (a) A is subset of B : $P\left(A|B\right) = \dfrac{P\left(A\right)}{P\left(B\right)}$

   (b) B is subset of A : $P\left(A|B\right) = 1$

   (c) A and B are disjoint: $P\left(A|B\right) = 0$

3. Independent events: $P\left(A|B\right) = P\left(A\right)$

   (a) $P\left(A \cap B\right) = P\left(A\right) P\left(B\right)$

4. Bayes' Theorem: If a sample space S is partitioned into a set of events A_i for $i = 1,\ 2, \ldots,\ n$ such that $\displaystyle\bigcup_{i=1}^{n} A_i = S$, then $P\left(B\right) = \displaystyle\sum_{i=1}^{n} P\left(B \cap A_i\right) = \sum_{i=1}^{n} P\left(B|A_i\right) P\left(A_i\right)$

5. $P\left(A_j|B\right) = \dfrac{P\left(B|A_j\right) P\left(A_j\right)}{\sum_{i=1}^{n} P\left(B|A_i\right) P\left(A_i\right)}$

6. Conditional expectation: $E\left[X|A\right] = \displaystyle\sum_{x} x p_{X|A}\left(x\right)$

3 Random Variables

A random variable is a function that associates a unique numerical values with every outcome of an experiment

1. Discrete RV: has a countable number of distinct values

2. Continuous RV: has infinite number of possible values

3. Probability Distribution Function $(p_X(x))$: list of probabilities associated with each possible values of RV.

4. $p(x_i) = P(X = x_i)$

5. Cumulative Distribution Function $F_X(x)$: probability that a funmber is less than or equal to x

6. $F_X(x) = P(X \leq x)$

7. properties of $p_X(x)$:

   (a) follows that axioms of probability

   (b) $\displaystyle\sum_{i} P_X(x_i) = 1$

8. properties of $F_X(x)$:

   (a) if $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$

   (b) $F_X(+\infty) = 1,\ F_X(-\infty) = 0$

   (c) $P(X > x) = 1 - F_X(x)$

3.1 Discrete Random Variables

1. for disjoint events A that occurs m ways and B that occurs n ways, the event A or B occurs in $m+n$ ways
2. suppose events C can be decomposed of steps A and B (unrelated events). C occurs mn ways
3. Bernoulli trial: only 2 possible outcomes. has probability mass function
$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$
4. Expectation: $E[X] = \overline{x} = \frac{\sum_i N_i x_i}{\sum_i N_i} = \sum_i x_i p_X(x_i)$
5. Properties of expectation:
   (a) $E[c] = c$
   (b) $E[cg(x)] = cE[g(x)]$
   (c) $E[g_1(x) + g_2(x) = E[g_1(x)] + E[g_2(x)]$
6. Variance: $\sigma_X^2 = E\left[\left(X - \overline{X}\right)^2\right] = \overline{\left(X - \overline{X}\right)^2} = E\left[X^2\right] - \left(E[X]\right)^2$
7. Standard deviation: σ_X
8. Mean square value: $E[X^2] = \overline{X^2}$
9. nth moment: $m_n = E[X^n] = \overline{X^n}$
10. nth central moment: $\mu_n = E\left[\left(X - \overline{X}\right)^n\right] = \overline{\left(X - \overline{X}\right)^n}$
11. for Bernoullie trial:
    (a) $E[X] = 0(1-p) + 1(p) = p$
    (b) $E\left[X^2\right] = 0^2(1-p) + 1^2 p = p$
    (c) $\sigma_X^2 = p - p^2$
    (d) number of successes on n trials: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$
    (e) mean: $E[X] = (np) \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-(y+1)} = np$
    (f) mean square: $E\left[X^2\right] = np(1-p) + n^2 p^2$
    (g) variance: $\sigma_X^2 = np(1-p)$

3.2 Continuous Random Variables

1. Expectation: $E[X] = \overline{x} = \int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x$
2. Uniform Random Variable: sub-intervals of the same length are equally likely
3. Gaussian: model for natural random phenomena:
   (a) Probability Distribution Function:
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \overline{x})^2}{2\sigma^2}\right) \tag{2}$$

   (b) mean: $\overline{x}$
   (c) variance: σ^2
4. any Gaussian RV has the standard form: $f_X(x) = G\left(\frac{x-\overline{x}}{\sigma}\right)$
5. Exponential Random Variable : model for waiting time
   (a) Probability Distribution Function:

$$f_X(t) = \lambda \exp\left(-\lambda t\right);\ \lambda \geq 0 \tag{3}$$

   (b) mean: $\frac{1}{\lambda}$
   (c) variancec: $\frac{1}{\lambda^2}$

3.3 Bivariate Random Variable

1. Joint distribution: determined by 2 independent random variables
   (a) $F_{X,Y} = \sum_{n=1}^{N}\sum_{m=1}^{M} P\left(x_n, y_m\right) u\left(x - x_n\right) u\left(y - y_m\right)$
   (b) $P\left(x_n, y_m\right)$ is the probability of the join event $\{X = x_n, Y = y_m\}$
2. Properties of Joint Distribution:
   (a) $F_{X,Y}\left(-\infty,-\infty\right) = F_{X,Y}\left(x,-\infty\right) = F_{X,Y}\left(-\infty,y\right) = 0$
   (b) $F_{X,Y}\left(+\infty,+\infty\right) = 1$
   (c) $0 \leq F_{X,Y}\left(x,y\right) \leq 1$
   (d) $P\{x_1 < X \leq x_2.y_1 < Y \leq y_2\} = F_{X,Y}\left(x_2,y_2\right) + F_{X,Y}\left(x_1,y_1\right) - F_{X,Y}\left(x_1,y_2\right) - F_{X,Y}\left(x_2,y_1\right)$
   (e) Marginal Distribution Functions:
      i. $F_{X,Y}\left(x,+\infty\right) = F_X(y)$
      ii. $F_{X,Y}\left(+\infty,y\right) = F_y(y)$
3. Joint Probability Density Function

$$f_{X,Y}\left(x,y\right) = \frac{\partial^2 F_{X,Y}\left(x,y\right)}{\partial x \partial y} \tag{4}$$

$$f_{X,Y}\left(x,y\right) = \sum_{n=1}^{N}\sum_{m=1}^{M} P\left(x_n, y_m\right) \delta\left(x - x_n\right) \delta\left(y - y_m\right) \tag{5}$$

4. Properties of Joint PDF:
   (a) $f_{X,Y}\left(x,y\right) \geq 0$
   (b) $\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} f_{X,Y}\left(x,y\right) \mathrm{d}x\,\mathrm{d}y = 1$
   (c) $F_{X,Y}\left(x,y\right) \int\limits_{-\infty}^{y}\int\limits_{-\infty}^{x} f_{X,Y}\left(x',y'\right) \mathrm{d}x'\,\mathrm{d}y'$
   (d) $F_X\left(x\right) \int\limits_{-\infty}^{x}\int\limits_{-\infty}^{+\infty} f_{X,Y}\left(x',y'\right) \mathrm{d}y'\,\mathrm{d}x'$

(e) $F_Y(y) \int\limits_{-\infty}^{y} \int\limits_{-\infty}^{+\infty} f_{X,Y}\left(x', y'\right) \mathrm{d}x' \, \mathrm{d}y'$

(f) $P\left(x_1 < X \leq x_2, y_1 < Y \leq y_2\right) = \int\limits_{y_1}^{y_2} \int\limits_{x_1}^{x_2} f_{X,Y}\left(x', y'\right) \mathrm{d}x' \, \mathrm{d}y'$

(g) marginal distribution of X : $f_X(x) = \int\limits_{-\infty}^{+\infty} f_{X,Y}\left(x', y'\right) \mathrm{d}y' = \dfrac{\mathrm{d}F_X(x)}{\mathrm{d}x}$

(h) marginal distribution of Y : $f_Y(y) = \int\limits_{-\infty}^{+\infty} f_{X,Y}\left(x', y'\right) \mathrm{d}x' = \dfrac{\mathrm{d}F_Y(y)}{\mathrm{d}y}$

(i) X and Y are statistically independent if $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

5. Conditional distribution:

   (a) $f_{X,Y}(y|x) = \dfrac{f_{X,Y}(x, y)}{f_X(x)}$

   (b) $f_{X,Y}(x|y) = \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$

6. Independence: Two events $\{X = x\}$ and A are independent if $\mathbf{P}(X = x \text{ and } A) = \mathbf{P}(X = x) \mathbf{P}(A) = p_X(x) \mathbf{P}(A)$

   (a) if $P(A) > 0$, then $p_{X|A}(x) = p_X(x)$

7. Joint moment about the origin:

$$m_{n,k} = E\left[X^n Y^k\right] = \iint\limits_{-\infty}^{+\infty} x^n y^k f_{X,Y}(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{6}$$

8. Correlation: $R_{XY} = m_{1,1} = E[XY]$

   (a) if X and Y are statistically independent then $R_{X,Y} = E[X] E[Y]$ and X and Y are *uncorrelated*

   (b) if $R_{XY} = 0$, then X and Y are *orthogonal*

9. Joint Central moments: $\mu_{n,k} = E\left[\left(X - \overline{X}\right)^n \left(Y - \overline{Y}\right)^k\right]$

   (a) Covariance: $C_{XY} = \mu_{1,1} = E\left[\left(X - \overline{X}\right)\left(Y - \overline{Y}\right)\right] = R_{XY} - E[X] E[Y]$

   (b) Normalized covariance: $\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y} = \dfrac{R_{XY} - E[X] E[Y]}{\sigma_X \sigma_Y}$

4 Functions of Random Variables

4.1 Transformation of RV

1. Suppose X is a discrete RV with probability distribution $f(x)$. Let $Y = g(X)$ be a one-to-one transformation such that $x = h(y)$. The probability distribution of Y is

$$f_Y(y) = f[h(y)] \tag{7}$$

2. the process for joint probability is the same (but you hve two substitutions)

3. Suppose X is a continuous RV with probability distribution $f_X(x)$. Let $Y = g(X)$ be a one-to-one transformation such that $x = h(y)$. The probability distribution of Y is

$$f_Y(y) = f[h(y)]\,|J| \tag{8}$$

where $J = \dfrac{\mathrm{d}h}{\mathrm{d}y}$ and is called the Jacobian of the transformation

4. for joint probability, the Jacobian is $J = \det \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix}$

4.2 Linear Combination of RV

1. Suppose $Z = X + Y$ where X and Y are independent
   (a) Discrete case:
      i. $P(Z = z) = \displaystyle\sum_{k=-\infty}^{+\infty} P(X = k)\,P(Y = z - k)$
   (b) Continuous case:
      i. cumulative function is $F_Z(z) = P(Z \le z) = P(X + Y \le z) = \displaystyle\int_R f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y$
      ii. probability distribution function: $f_Z(z) = \displaystyle\int_{-\infty}^{+\infty} f_Y(z - x)\,f_X(x)\,\mathrm{d}x \equiv f_Y(y) * f_X(x)$

5 Central Limit Theorem

1. Population: totality of concerned observations
2. Sample: subset of the population
   (a) Sample mean: $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$
   (b) Sample median: $\tilde{x} = \begin{cases} x_{(n+1)/2}, & \text{if } n \text{ is odd} \\ \dfrac{1}{2}\left(x_{n/2} + x_{(n+2)/2}\right), & \text{if } n \text{ is even} \end{cases}$
   (c) Sample mode: most frequent
   (d) Sample variance: $S^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \dfrac{1}{n(n-1)}\left[n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2\right]$
   (e) Sample standard deviation: $S = \sqrt{S}$
   (f) Sample range: $R = X_{\text{max}} - X_{\text{min}}$
3. Central Limit Theorem: If $\overline{X}$ is the mean of a random sample of size n taked from a population with mean μ and finite variance σ^2, then the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{9}$$

as $n \to \infty$ is the standard normal distribution $n(z; 0, 1)$

4. Consequences:
   (a) let $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{x} X_i$ and $S_n = \sum_{i=1}^{x} X_i$
   (b) $\overline{X}$ and S_n have approximately normal distribution
   (c) $\sigma^2_{\overline{X}} = \sigma^2/n$
   (d) $\mu_{\overline{X}} = \mu$
   (e) $\mu_s = n\mu$
   (f) $\sigma^2_S = n\sigma^2$
5. for two independent samples of size n_1 and n_2:
$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - (\mu_1 - \mu_2)}{\sqrt{\left(\sigma_1^2/n_1\right)^2 + \left(\sigma_2^2/n_2\right)^2}} \tag{10}$$

6 Random Process

1. a random (stochastic) process assigns a time function $X(t, s)$ for every outcome s
2. for a fixed s, the graph of $X(t, s)$ is a realization, sample path, or sample function of the random process
3. kth order distribution function of time samples:
$$F_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) = P\{X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_k) \le x_k\} \tag{11}$$
4. for k RVs, the joint density function is:
$$f_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) = \frac{\partial^k (x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k)}{\partial x_1 \partial x_2 \ldots \partial x_k} \tag{12}$$
5. Univariate central moment: $\overline{X} = E[X(t, s)] = \int\limits_{-\infty}^{+\infty} X(t, s) f_s(s)\,\mathrm{d}s$
6. Variance: $VAR = E\left[\left(X - \overline{X}\right)^2\right] = \int\limits_{-\infty}^{+\infty} \left(X - \overline{X}\right)^2 f_s(s)\,\mathrm{d}s$
7. For independent vectors of RV: the joint density is the product of the individual density functions of each vector
8. Autocorrelation:
$$R_{XX}(t_1, t_2) \equiv E[X(t_1) X(t_2)] = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} x_1 x_2 f_{x(t_1)x(t_2)}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{13}$$
9. Autocovariance:
$$\begin{aligned} C_{XX}(t_1, t_2) &\equiv E\{[X(t_1 - \mu_x(t_1))][X(t_2 - \mu_x(t_2))]\} \\ &= R_{XX}(t_1, t_2) - \mu_x(t_1)\mu_x(t_2) \end{aligned}$$
10. Variance: $VAR = C_{XX}(t, t)$
11. Correlation coefficient:
$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)}\sqrt{C_{XX}(t_2, t_2)}} \tag{14}$$
12. Cross-correlation: $R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)]$
   (a) product of two expectations if X and Y are statistically independent

6.1 Stationary Random Processes

1. Definition: if the distribution is independent of the time origin
2. Wide-sense Stationary Process (WSS) satisfies the following:
   (a) $E\left[X\left(t\right)\right]=\overline{X}$
   (b) $E\left[X\left(t\right)X\left(t+\tau\right)\right]$ is a function of τ only
3. Jointly WSS: both satisfy the above condition and $E\left[X\left(t\right)Y\left(t+\tau\right)\right]$ is a function of τ only the correlation of WSS and jointly WSS are also functions of τ only
4. Properties of WSS:
   (a) $\left|R_{XX}\left(\tau\right)\right|\leq R_{XX}\left(0\right)$
   (b) $R_{XX}\left(\tau\right)=R_{XX}\left(-\tau\right)$
   (c) $R_{XX}\left(0\right)=E\left[\left(X\left(t\right)\right)^{2}\right]$
   (d) $\lim\limits_{|\tau|\to\infty}R_{XX}\left(\tau\right)=\overline{X}^{2}$ if $\overline{X}\neq 0$ and $X\left(t\right)$ has no periodic part
   (e) if $X\left(t\right)$ has periodic component, then $R_{XX}\left(\tau\right)$ will have a periodic component with the same period
   (f) if $X\left(t\right)$ is ergodic, zero-mean and has no periodic component, the $\lim\limits_{|\tau|\to\infty}R_{XX}\left(\tau\right)=0$
   (g) $R_{XX}\left(\tau\right)$ can't have an arbitrary shape
5. time average:
$$A\left[x\left(t\right)\right]=\lim_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}x\left(t\right)\mathrm{d}t \tag{15}$$
6. time autocorrelation: $\mathfrak{R}_{XX}\left(\tau\right)=A\left[x\left(t\right)x\left(t+\tau\right)\right]$
7. Ergodicity: must satisfy the following conditions
   (a) $E\left[x\right]=A\left[x\left(t\right)\right]$
   (b) $\mathfrak{R}_{XX}\left(\tau\right)=R_{XX}\left(\tau\right)$

7 Bernoulli Processes

1.

8