EEE 135 Key concepts and Equations

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Vectors

1. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
2. $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$
3. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = |\mathbf{ABC}|$
4. $\nabla(\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$
5. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\mathbf{B} \cdot \nabla) - \mathbf{B}(\mathbf{A} \cdot \nabla)$
6. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
7. $\nabla \times (\nabla A) = \mathbf{0}$
8. $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$
9. $\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A}$
10. Divergence theorem: $\displaystyle\oint_{\text{closed surface}} \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_{\text{volume bounded}} (\nabla \cdot \mathbf{A})\,\mathrm{d}v$
11. Stokes' theorem: $\displaystyle\oint_{\text{closed loop}} \mathbf{A} \cdot \mathrm{d}\mathbf{L} = \int_{\text{surface bounded}} (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S}$

Coordinate Systems

	Rectangular	Cylindrical	Spherical
Parameters	x y z	$\rho = \sqrt{x^2 + y^2}$ $\phi = \arctan\left(\frac{y}{x}\right)$ z	$r = \sqrt{x^2 + y^2 + z^2}$ $\phi = \arctan\left(\frac{y}{x}\right)$ $\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$
Conversion		$x = \rho \cos\phi$ $y = \rho \sin\phi$ $z = z$	$x = (r\sin\theta)\cos\phi$ $y = (r\sin\theta)\sin\phi$ $z = r\cos\theta$
Unit Vectors	$\hat{\mathbf{a}}_x$ $\hat{\mathbf{a}}_y$ $\hat{\mathbf{a}}_z$	$\hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_x \cos\phi + \hat{\mathbf{a}}_y \sin\phi$ $\hat{\mathbf{a}}_\phi = -\hat{\mathbf{a}}_x \sin\phi + \hat{\mathbf{a}}_y \cos\phi$ $\hat{\mathbf{a}}_z$	$\hat{\mathbf{a}}_r = \hat{\mathbf{a}}_x \sin\theta\cos\phi + \hat{\mathbf{a}}_y \sin\theta\sin\phi + \hat{\mathbf{a}}_z \cos\theta$ $\hat{\mathbf{a}}_\phi = -\hat{\mathbf{a}}_x \sin\phi + \hat{\mathbf{a}}_y \cos\phi$ $\hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_x \cos\theta\cos\phi + \hat{\mathbf{a}}_y \cos\theta\sin\phi - \hat{\mathbf{a}}_z \sin\theta$
$\mathrm{d}v$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$	$\mathrm{d}\rho(\rho\,\mathrm{d}\phi)\,\mathrm{d}z$	$\mathrm{d}r(r\,\mathrm{d}\phi)(r\sin\theta\,\mathrm{d}\theta)$
$\mathrm{d}\mathbf{L}$	$\mathrm{d}x\hat{\mathbf{a}}_x + \mathrm{d}y\hat{\mathbf{a}}_y + \mathrm{d}z\hat{\mathbf{a}}_z$	$\mathrm{d}\rho\hat{\mathbf{a}}_\rho + \rho\,\mathrm{d}\phi\hat{\mathbf{a}}_\phi + \mathrm{d}z\hat{\mathbf{a}}_z$	$\mathrm{d}r\hat{\mathbf{a}}_r\hat{\mathbf{a}}_r + r\,\mathrm{d}\theta\hat{\mathbf{a}}_\theta + r\sin\theta\,\mathrm{d}\phi\hat{\mathbf{a}}_\phi$
Vector Field	$A_x\hat{\mathbf{a}}_x + A_y\hat{\mathbf{a}}_y + A_z\hat{\mathbf{a}}_z$	$A_\rho\hat{\mathbf{a}}_\rho + A_\phi\hat{\mathbf{a}}_\phi + A_z\hat{\mathbf{a}}_z$	$A_\rho\hat{\mathbf{a}}_\rho + A_\phi\hat{\mathbf{a}}_\phi + A_\theta\hat{\mathbf{a}}_\theta$
∇A	$\frac{\partial A}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial A}{\partial y}\hat{\mathbf{a}}_y + \frac{\partial A}{\partial z}\hat{\mathbf{a}}_z$	$\frac{\partial A}{\partial \rho}\hat{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial A}{\partial \phi}\hat{\mathbf{a}}_\phi + \frac{\partial A}{\partial z}\hat{\mathbf{a}}_z$	$\frac{\partial A}{\partial r}\hat{\mathbf{a}}_r + \frac{1}{r}\frac{\partial A}{\partial \theta}\hat{\mathbf{a}}_\theta + \frac{1}{r\sin\theta}\frac{\partial A}{\partial \phi}\hat{\mathbf{a}}_\phi$
$\nabla \cdot \mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi}$ $+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$ $+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}$
$\nabla \times \mathbf{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{a}}_x$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{a}}_y$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{a}}_z$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{a}}_\rho$ $\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\mathbf{a}}_\rho$ $\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{a}}_z$	$\frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{a}}_r$ $\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\mathbf{a}}_\theta$ $\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial(A_r)}{\partial\theta}\right)\hat{\mathbf{a}}_\phi$
$\nabla^2 A$	$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial\phi^2}$ $+\frac{\partial^2 A_z}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2}$ $+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)$

Unit vector dot products:

1. rectangular and cylindrical:

	$\hat{\mathbf{a}}_\rho$	$\hat{\mathbf{a}}_\phi$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{a}}_x$	$\cos\phi$	$-\sin\phi$	0
$\hat{\mathbf{a}}_y$	$\sin\phi$	$\cos\phi$	0
$\hat{\mathbf{a}}_z$	0	0	1

2. rectangular and spherical:

	$\hat{\mathbf{a}}_r$	$\hat{\mathbf{a}}_\phi$	$\hat{\mathbf{a}}_\theta$
$\hat{\mathbf{a}}_x$	$\sin\theta\cos\phi$	$-\sin\phi$	$\cos\theta\cos\phi$
$\hat{\mathbf{a}}_y$	$\sin\theta\sin\phi$	$\cos\phi$	$\cos\theta\sin\phi$
$\hat{\mathbf{a}}_z$	$\cos\theta$	0	$-\sin\theta$

1 Electrostatics

1. Coulomb's law, the force between two charged particles: $\mathbf{F}_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1 Q_2}{r^2}\hat{\mathbf{a}}_{12}$ with $\dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$
2. electric field intensity of charge Q is force per unit charge: $\mathbf{E} = \dfrac{\mathbf{F}_t}{Q_t} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1}{r^2}\hat{\mathbf{a}}_r$
3. volume charge density: $\rho_v = \lim\limits_{\Delta v \to 0}\dfrac{\Delta Q}{\Delta v} = \dfrac{\mathrm{d}Q}{\mathrm{d}v} \Longleftrightarrow Q = \displaystyle\int_{\text{volume}} \rho_v \,\mathrm{d}v$
4. field of line charge: $\mathbf{E} = \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\hat{\mathbf{a}}_\rho$
5. field of sheet charge: $\mathbf{E} = \dfrac{\rho_S}{2\varepsilon_0}\hat{\mathbf{a}}_N$
6. sketching streamlines: $\dfrac{E_y}{E_x} = \dfrac{\mathrm{d}y}{\mathrm{d}x}$
7. electric flux = induced charge of conductor on another (grounded) without contact: $\Psi = Q$
8. electric flux density = electric flux per unit area: $\mathbf{D} = \dfrac{Q}{4\pi r^2}\hat{\mathbf{a}}_r = \displaystyle\int_{\text{volume}} \dfrac{\rho_v \,\mathrm{d}v}{4\pi r^2}\hat{\mathbf{a}}_r$
9. in free space: $\mathbf{D} = \varepsilon_0\mathbf{E}$
10. Gauss's law: the electric flux passing through any closed surface is equal to the total enclosed charge by the surface. $\Psi = \displaystyle\int \mathrm{d}\Psi = \oint_{\text{closed surface}} \mathbf{D}_s \cdot \mathrm{d}\mathbf{S} = Q = \int \rho_v \,\mathrm{d}v$
11. we apply Divergence theorem: $\displaystyle\oint_{\text{closed surface}} \mathbf{D}_s \cdot \mathrm{d}\mathbf{S} = \int (\nabla \cdot \mathbf{D})\,\mathrm{d}v = \int \rho_v \,\mathrm{d}v \Longleftrightarrow \nabla \cdot \mathbf{D} = \rho_v$
12. point form of Gauss's law: $\boxed{\nabla \cdot \mathbf{D} = \rho_v}$
13. the work done by an external force to move a charge: $W = -Q\displaystyle\int_{\text{initial}}^{\text{final}} \mathbf{E} \cdot \mathrm{d}\mathbf{L}$
14. potential difference = work done by external force per unit charge: $V = -\displaystyle\int_{\text{initial}}^{\text{final}} \mathbf{E} \cdot \mathrm{d}\mathbf{L}$
15. potential difference is path independent, and $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{L} = 0$
16. relationship of potential and electric field: $\mathbf{E} = -\nabla V$
17. using the curl of divergence is 0: $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$
18. Maxwell's 2nd equation: $\boxed{\nabla \times \mathbf{E} = \mathbf{0}}$
19. electric dipole: two point charges of same magnitude, opposite charge and has a fixed distance
20. dipole moment: $\mathbf{p} = Q\mathbf{d}$ where $\mathbf{d}$ points from $-Q$ to $+Q$
21. torque on dipole by electric field: $\mathrm{d}\mathbf{T} = \mathrm{d}\mathbf{p} \times \mathbf{E}$

22. potential energy of dipole on electric field: $U = -\,\mathrm{d}\mathbf{p} \cdot \mathbf{E}$

23. total energy of system of N charges: $W_E = \frac{1}{2} \sum_{n=1}^{n=N} Q_n V_n$

24. for continuous charge distribution: $W_E = \frac{1}{2} \int_{\text{volume}} \rho_v V \, \mathrm{d}v = \frac{1}{2} \int_{\text{volume}} \mathbf{D} \cdot \mathbf{E} \, \mathrm{d}v = \frac{1}{2} \int_{\text{volume}} \varepsilon E^2 \, \mathrm{d}v$

25. energy density: $\frac{\mathrm{d}W_E}{\mathrm{d}v} = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}$

2 Conductors and Dielectrics, Capacitance

1. current = charges passing through an area per unit time: $I = \frac{\mathrm{d}q}{\mathrm{d}t}$

2. current density = current per area: $I = \oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$

3. let Q_i be the charge flowing OUT a surface. the continuity equation: $\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}Q_i}{\mathrm{d}t}$

4. point form of continuity equation: $\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t}$

5. electron drift speed: $\mathbf{v}_d = -\mu_e \mathbf{E}$ with μ = mobility

6. conductivity: $\sigma_e = -\rho_e \mu_e$

7. vector form of Ohm's law: $\mathbf{J} = \sigma \mathbf{E}$

8. Ohm's law: $V = IR$

9. resistance $R = \frac{L}{\sigma S} = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot \mathrm{d}\mathbf{L}}{\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}}$

10. boundary conditions for conductors:
    - $D_t = E_t = 0$
    - $D_N = \varepsilon \mathbf{E}_N = \rho_s$
    - there is no field inside
    - the field on the surface is normal to the surface
    - the surface is an equipotential surface

11. Method of Images: the effect of an infinite conducting sheet is the same as the effect of the sheet removed and all other charges reflected with respect to the sheet.

12. semiconductors: there is contribution of both electrons ansd holes $\sigma = -\rho_e \mu_e + \rho_h \mu_h$

13. polarization = dipole moment per unit volume: $\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta b} \mathbf{p}_i$

14. total bounded charges: $Q_b = -\oint_S \mathbf{P} \cdot \mathrm{d}\mathbf{S}$

15. total free charges: $Q = Q_T - Q_b = \oint_S (\varepsilon_0 \mathbf{E} + \mathbf{P}) \cdot \mathrm{d}\mathbf{S}$
16. linear relationship of $\mathbf{P}$ and $\mathbf{E}$: $\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$ where χ_e is the electric susceptibility of the material
17. relative permittivity: $\varepsilon_r = \chi_e + 1$
18. permitivity: $\varepsilon = \varepsilon_0 \varepsilon_r$
19. in polarizable material: $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E}(1 + \chi_e) = \varepsilon \mathbf{E}$
20. boundary conditions for perfect dielectric materials:
    - $E_{t,\,1} = E_{t,\,2} \iff (\mathbf{E}_1 - \mathbf{E}_2) \times \hat{\mathbf{n}} = 0$
    - $\dfrac{D_{t,\,1}}{D_{t,\,2}} = \dfrac{\varepsilon_1}{\varepsilon_2}$
    - $D_{N,\,1} = D_{N,\,2} \iff (\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_S$
    - $\varepsilon_1 E_{N,\,1} = \varepsilon_2 E_{N,\,2}$
    - $D_2 = D_1 \sqrt{\cos^2 \theta_1 + \left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 \sin^2 \theta_1}$
    - $E_2 = E_1 \sqrt{\sin^2 \theta_1 + \left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2 \theta_1}$
21. capacitance: dependent only on the geometry of the capacitor: $C = \dfrac{\oint_S \varepsilon \mathbf{E} \cdot \mathrm{d}\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot \mathrm{d}\mathbf{L}}$
22. parallel plate: $C = \dfrac{\varepsilon S}{d}$
23. cylindrical capacitor: $C = \dfrac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}$
24. spherical capacitor: $C = 4\pi\varepsilon \left(\dfrac{1}{a} - \dfrac{1}{b}\right)^{-1}$
25. Poisson equation: $\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$
26. Laplace equation: $\nabla^2 V = 0$

3 Magnetostatics

1. Biot-Savart law: $\mathrm{d}\mathbf{H} = \dfrac{I\,\mathrm{d}\mathbf{L} \times \hat{\mathbf{a}}_r}{4\pi r^2}$
2. magnetic field intensity $\mathbf{H}$ is analogous to electric field
3. only the integral form has experimental basis: $\mathbf{H} = \oint \dfrac{I\,\mathrm{d}\mathbf{L} \times \hat{\mathbf{a}}_r}{4\pi r^2} = \oint \dfrac{K \times \hat{\mathbf{a}}_r\,\mathrm{d}S}{4\pi r^2} = \oint \dfrac{J \times \hat{\mathbf{a}}_r\,\mathrm{d}v}{4\pi r^2}$
4. surface current density: $I\,\mathrm{d}\mathbf{L} = \mathbf{K}\,\mathrm{d}S = \mathbf{J}\,\mathrm{d}v$
5. Ampere's circuital law: $\oint \mathbf{H} \cdot \mathrm{d}\mathbf{L} = I$
6. point form of Ampere's circuital law: $\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$

7. magnetic flux density = magnetic flux per area (in free space): $\mathbf{B} = \mu_0 \mathbf{H}$
8. magnetic flux: $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$
9. for a closed surface: $\Phi = \oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$
10. using divergence theoren: $\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_{\text{volume}} (\nabla \cdot \mathbf{B})\,\mathrm{d}v = 0$
11. point form of magnetic flux: $\boxed{\nabla \cdot \mathbf{B} = 0}$
12. Maxwell's equations for static fields:

Differential	Integral
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q$
$\nabla \times \mathbf{E} = \mathbf{0}$	$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{L} = 0$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{L} = I$
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$

13. Scalar magnetic potential: we designate a scalar magnetic potential similar to electric potential such that $\mathbf{H} = -\nabla V_m$(when$\mathbf{J} = \mathbf{0}$) because the curl of a gradient is 0.
14. this scalar magnetic potential obey's Laplace's equation: $\nabla^2 V_m = 0$
15. however, this potential is NOT CONSERVATIVE. $V_{m,\,ab} = -\int_b^a \mathbf{H} \cdot \mathrm{d}\mathbf{L}, \quad \oint \mathbf{H} \cdot \mathrm{d}\mathbf{L} = I \neq 0$
16. vector potential: we choose a vecor potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$
17. from differential current elements $\mathbf{A} = \oint \frac{\mu_0 I\,\mathrm{d}\mathbf{L}}{4\pi r} = \oint_S \frac{\mu_0 \mathbf{K}\,\mathrm{d}S}{4\pi r} = \oint_{\text{volume}} \frac{\mu_0 \mathbf{J}\,\mathrm{d}v}{4\pi r}$
18. magnetic force: $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ (does no work on the object)
19. Lorentz force equation: $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
20. differential force on a current element: $\mathrm{d}\mathbf{F} = \mathbf{J} \times \mathbf{B}\,\mathrm{d}v = \mathbf{K} \times \mathbf{B}\,\mathrm{d}S = I\,\mathrm{d}\mathbf{L} \times \mathbf{B}$
21. force between differential current elements: $\mathrm{d}(\mathrm{d}\mathbf{F})2) = \mu_0 \frac{I_1 I_2}{4\pi r_{12}^2}\,\mathrm{d}\mathbf{L}_2 \times (\mathrm{d}\mathbf{L}_2 \times \hat{\mathbf{a}}_{r12})$
22. in a space of uniform magnetic flux density $F = -I \oint \mathbf{B} \times \mathrm{d}\mathbf{L} = -I\mathbf{B} \times \oint \mathrm{d}\mathbf{L} = \mathbf{0}$
23. define the torque with respect to an origin as $\mathbf{T} = \mathbf{R} \times \mathbf{F}$
24. torque on a loop: $\mathrm{d}\mathbf{T} = I\,\mathrm{d}\mathbf{S} \times \mathbf{B} = \mathrm{d}\mathbf{m} \times \mathbf{B}$
25. define magnetic dipole moment $\mathrm{d}\mathbf{m} = I\,\mathrm{d}\mathbf{S}$
26. magnetization: magnetic dipole moment per unit volume: $\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i$

27. the bound current in a contour $I_B = \oint \mathbf{M} \cdot \mathrm{d}\mathbf{L}$

28. the free current: $I = I_T - I_B = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot \mathrm{d}\mathbf{L}$

29. hence we have $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \Longleftrightarrow \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$

30. magnetization is linear in linear isotropic media: $\mathbf{M} = \chi_m \mathbf{H}$ with magnetic susceptibility χ_m

31. relative permeability: $\mu_r = 1 + \chi_m$

32. permeability $\mu = \mu_r \mu_0$

33. relationship of $\mathbf{B}$ and $\mathbf{H}$: $\mathbf{B} = \mu \mathbf{H}$

34. boundary conditions of magnetic materials:
    - $B_{N2} = B_{N1}$
    - $H_{N2} = \frac{\mu_1}{\mu_2} H_{N1}$
    - $H_{t1} - H_{t2} = K$
    - $\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K$

35. magnetic circuit potential: $V_{m,\ ab} = \int_a^b \mathbf{H} \cdot \mathrm{d}\mathbf{L}$

36. vectorOhm's law analog: $\mathbf{B} = \mu \mathbf{H}$

37. current analog: $\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$

38. Ohm's law analog: $V_m = \Phi \mathfrak{R}$

39. reluctance is dependent on geometry: $\mathfrak{R} = \frac{d}{\mu S}$

40. potential energy in magnetic field $W_H = \frac{1}{2} \int_{\text{volume}} \mathbf{B} \cdot \mathbf{H} \, \mathrm{d}v$

41. inductance = ratio of total flux to the current they link: $L = \frac{N\Phi}{I} = \frac{2W_H}{I^2} = \frac{1}{I^2} \int_{\text{volume}} \mathbf{B} \cdot \mathbf{H} \, \mathrm{d}v = \frac{1}{I^2} \int_{\text{volume}} \mathbf{A} \cdot \mathbf{J} \, \mathrm{d}v$

42. mutual inductance = depends on magnetic interaction between two currents: $M_{12} = \frac{1}{I_1 I_2} \int_{\text{volume}} (\mu \mathbf{H}_2 \cdot \mathbf{H}_2 \, \mathrm{d}v) = M_{21}$

4 Time-Varying Fields

1. Faraday's law = changing magnetic flux creates an emf: $\mathcal{E} = -\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$
2. Lenz's law = the induced voltage acts to produce an opposing flux
3. Faraday's law: $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{L} = -\dfrac{\mathrm{d}}{\mathrm{d}t} \int_s \mathbf{B} \cdot \mathrm{d}\mathbf{S} \Longleftrightarrow \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
4. motional emf (adds contribution of magnetic field is not constant): $\mathbf{E}_m = \dfrac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B}$
5. displacement current density = due to changing electric flux: $\mathbf{J}_d = \dfrac{\partial \mathbf{D}}{\partial t} \Longleftrightarrow \nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$
6. when $\mathbf{J} = 0$, we have: $\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t}$ and $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
7. integrating the equation on displacement current with respect to a surface: $\int_S \nabla \times \mathbf{H} \cdot \mathrm{d} = \int_S \mathbf{J} \cdot \mathrm{d} + \int_S \mathbf{J}_d \cdot \mathrm{d} \Longleftrightarrow \oint \mathbf{H} \cdot \mathrm{d}\mathbf{L} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S}$
8. Maxwell's equations for time-varying fields:

Differential	Integral
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q$
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{L} = -\int_S \dfrac{\partial B}{\partial t} \cdot \mathrm{d}\mathbf{S}$
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{L} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S}$
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$

9. retarted potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$
10. EM waves travel at speed $v = \dfrac{1}{\sqrt{\mu\varepsilon}}$
11. replace time with $t' = t - \dfrac{R}{v}$ where R is the distance between the differential charge element and point where potential will be determined
12. $V = \displaystyle\int_{\text{volume}} \dfrac{[\rho_v]}{4\pi\varepsilon R} \,\mathrm{d}v$
13. $\mathbf{A} = \displaystyle\int_{\text{volume}} \dfrac{\mu[\mathbf{J}]}{4\pi R} \,\mathrm{d}v$

5 Plane Waves

1. In free space, the medium is sourceless $\rho_v = 0,\ \mathbf{J} = \mathbf{0}$, hence Maxwells' equations becomes:

Maxwell's equations
$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$ $\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$ $\nabla \cdot \mathbf{E} = 0$ $\nabla \cdot \mathbf{H} = 0$

2. we consider the case such that $\mathbf{E} = E_x \hat{\mathbf{a}}_x$ and $\mathbf{H} = H_y \hat{\mathbf{a}}_y$ and they vary only in the z component in a sinusoidal manner with angular frequency ω

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} = -\mu \frac{\partial \left(\mathbf{H} \exp\left(j\omega t \right) \right)}{\partial t} = -j\omega\mu \mathbf{H}$$
$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \varepsilon \frac{\partial \left(\mathbf{E} \exp\left(j\omega t \right) \right)}{\partial t} = j\omega\varepsilon \mathbf{E}$$

Getting the curl of these two equations in the leftmost side gives

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-j\omega\mu \nabla \times \mathbf{H}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla(0) - \nabla^2 \mathbf{E}$$
$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (j\omega\mu \nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \nabla(0) - \nabla^2 \mathbf{H}$$
$$\Updownarrow$$
$$\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \mathbf{E} = 0 \tag{1}$$
$$\nabla^2 \mathbf{H} + \omega^2 \mu\varepsilon \mathbf{H} = 0 \tag{2}$$

These two equations (Helmholtz Equation) have these solutions (define the wave number $k = \omega\sqrt{\mu\varepsilon}$):

$$E_x(z,\ t) = E_f \exp\left(-jkz \right) + E_r \exp\left(jkz \right)$$
$$H_y(z,\ t) = H_f \exp\left(-jkz \right) + H_r \exp\left(jkz \right)$$

3. the relationship between E and H: $H_y(z,\ t) = \frac{1}{\eta}\left[E_f \exp\left(-jkz \right) - E_r \exp\left(jkz \right) \right]$

4. the speed of the wave propagation is $v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$

5. the wavelength is the distance between two consecutive reference points: $\lambda = \frac{2\pi}{k} = \frac{v_p}{f}$

6. the perfect medium has intrinsic impedance $\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$

7. in a lossy medium, there is conductivity and Maxwell's curl equations become

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \sigma \mathbf{E}$$

Helmholtz equation becomes:

$$\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon} \right) \mathbf{E} = 0$$

The propagation constant(γ = attenuation constant + imaginary wave number) is complex:

$$\gamma = jk = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \Longleftrightarrow \nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0$$

The solution to this equation is

$$\begin{aligned} E_x(z,\ t) &= E_f \exp(-\gamma z) + E_r \exp(\gamma z) \\ E_x(z,\ t) &= E_f \exp(-\alpha z)\cos(\omega t - \beta z) + E_r \exp(\alpha z)\cos(\omega t + \beta z) \end{aligned}$$

8. the intrinsic impedance is $\eta = \dfrac{j\omega\mu}{\gamma}$
9. the magnetic field intensity is $H_y(z) = \dfrac{1}{\eta}\left[E_f \exp(-\gamma z) - E_r \exp(\gamma z)\right]$
10. define the skin depth (depth of penetration) as $\delta_s = \dfrac{1}{\alpha} = \sqrt{\dfrac{2}{\omega\mu\varepsilon}}$
11. the Poynting vector gives the power output of an EM wave: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$
12. the average value of the poynting vector is $\langle\mathbf{S}\rangle = \dfrac{1}{2}\mathfrak{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\}$ with $\mathbf{E}_s = E_0 \exp(-\beta z)\,\hat{\mathbf{a}}_x,\ \mathbf{H}_s^* = H_0 \exp(+\beta z)\,\hat{\mathbf{a}}_y$
13. the boundary conditions for lossless medium to lossless medium:
    - $D_{N1} = D_{N2}$
    - $E_{t1} = E_{t2}$
    - $B_{N1} = B_{N2}$
    - $H_{t1} = H_{t2}$
14. let the boundary be $z = 0$ and $\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$. Then at the left side of the boundary is:

$$\begin{aligned} \mathbf{E}_1(0) &= (E_1^f + E_1^r)\hat{\mathbf{a}}_x \\ \mathbf{H}_2(0) &= \frac{1}{\eta_1}(E_1^f + E_1^r)\hat{\mathbf{a}}_y \end{aligned}$$

at the right side of the boundary:

$$\begin{aligned} \mathbf{E}_2(0) &= E_2^f\hat{\mathbf{a}}_x \\ \mathbf{H}_2(0) &= \frac{1}{\eta_2}E_2^f\hat{\mathbf{a}}_y \end{aligned}$$

these, along with the boundary conditions, give:

$$\frac{E_1^r}{E_1^f} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma$$

15. the reflection coefficient Γ is the ratio of the amplitudes of the REFLECTED wave to the INCIDENT wave
16. the transmission coefficient $T = 1 + \Gamma = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ is the ratio of the amplitudes of the TRANSMITTED wave to the INCIDENT wave
17. the boundary conditions from lossless medium to perfect electric conductor (0 electric fields inside)

- $D_{N1} = 0$
- $E_{t1} = 0$
- $B_{N1} = B_{N2}$
- $H_{t1} = J$

18. at the left side:

$$\mathbf{E}_1(0) = (E_1^f + E_1^r)\hat{\mathbf{a}}_x$$
$$\mathbf{H}_2(0) = \frac{1}{\eta_1}(E_1^f + E_1^r)\hat{\mathbf{a}}_y$$

using the boundary conditions:

$$E_1^f + E_1^r = 0$$
$$H_1^f + H_1^r = J$$

this means the electric field is reflected completety

19. the boundary conditions from lossless medium to good conductor: we can use the definition of the reflection coefficient with

$$\eta_2 = j\frac{\omega\mu_2}{\gamma_2} = \sqrt{\frac{j\omega\mu_2}{\sigma + j\omega\varepsilon_2}}$$

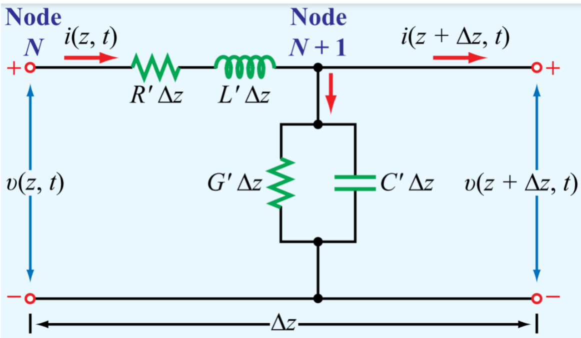
note that as $\sigma \to \infty \implies \eta_2 \to 0,\ \Gamma \to -1$

6 Transmission Lines

1. transmission line = structure or media that transder information or energy between two points

2. transmission line theory:

- physical dimensions are a fraction or multiple of wavelengths
- has a distributed parameter network
- voltages and currents vary in magnitude and phase over the length



3. lumped element model = transmission line is represented with L-network of R', L', G', and C' of length Δz. these are in per unit length elements (eg. Ω/m)
KVL analysis of the big loop:

$$v(z,\ t) = R'\Delta z i(z,\ t) + L'\Delta z\frac{\partial i(z,\ t)}{\partial t} + v(z + \Delta z,\ t) \implies -\frac{\partial v(z,\ t)}{\partial z} = R' i(z,\ t) + L'\frac{\partial i(z,\ t)}{\partial t}$$

KCL at node $N+1$

$$i(z,\ t) = G'\Delta v(z+\Delta z,\ t) + C'\Delta z\frac{\partial v(z+\Delta z,\ t)}{\partial t} + i(z+\Delta z,\ t) \Longrightarrow -\frac{\partial i(z,\ t)}{\partial z} = G'v(z,\ t) + C'\frac{\partial v(z,\ t)}{\partial t}$$

4. Telegrapher's equations:

$$-\frac{\partial v(z,\ t)}{\partial z} = R'i(z,\ t) + L'\frac{\partial i(z,\ t)}{\partial t}$$
$$-\frac{\partial i(z,\ t)}{\partial z} = G'v(z,\ t) + C'\frac{\partial v(z,\ t)}{\partial t}$$

we define the following (as sinusoidal steady state transmission):

$$v(z,\ t) = \mathfrak{Re}\left\{V(z)\exp\left(j\omega t\right)\right\}$$
$$i(z,\ t) = \mathfrak{Re}\left\{I(z)\exp\left(j\omega t\right)\right\}$$

the telegrapher's equations become:

$$-\frac{\partial v(z)}{\partial z} = \left(R' + j\omega L'\right) i(z) \tag{3}$$
$$-\frac{\partial i(z)}{\partial z} = \left(G' + j\omega C'\right) v(z) \tag{4}$$

differentiating (3) w. r. t. z then combine with (4), and differentiate (4) w. r. t z then combine with (3):

$$\frac{\partial^2 v(z)}{\partial z^2} - \left(R' + j\omega L'\right)\left(G' + j\omega C'\right) v(z) = 0$$
$$\frac{\partial^2 i(z)}{\partial z^2} - \left(R' + j\omega L'\right)\left(G' + j\omega C'\right) i(z) = 0$$

let $\gamma = \sqrt{\left(R' + j\omega L'\right)\left(G' + j\omega C'\right)} = \alpha + j\beta$
the solutions are

$$v(z) = V_0^f\exp\left(-\gamma z\right) + V_0^r\exp\left(\gamma z\right)$$
$$i(z) = I_0^f\exp\left(-\gamma z\right) + I_0^r\exp\left(\gamma z\right)$$

5. the relationship of $i(z)$ and $v(z)$: $i(z) = \dfrac{\gamma}{R' + j\omega L'}\left[V_0^f\exp\left(-\gamma z\right) - V_0^r\exp\left(-\gamma z\right)\right]$

6. the characteristic impedance is $Z_0 = \dfrac{V_0^f}{I_0^f} = -\dfrac{V_0^r}{I_0^r} = \dfrac{R' + j\omega L'}{\gamma} = \sqrt{\dfrac{R' + j\omega L'}{G' + j\omega C'}}$

7. the time domain expressions are:

$$v(z,\ t) = V_0^f\exp\left(-\alpha z\right)\cos\left(\omega t - \beta z\right) + V_0^r\exp\left(\alpha z\right)\cos\left(\omega t + \beta z\right)$$
$$i(z,\ t) = I_0^f\exp\left(-\alpha z\right)\cos\left(\omega t - \beta z\right) + I_0^r\exp\left(\alpha z\right)\cos\left(\omega t + \beta z\right)$$

8. the wavelength $\lambda = \dfrac{2\pi}{\beta}$

9. the wavespeed $v_p = \dfrac{\omega}{\beta} = \lambda f$

10. Lossless transmission lines: $R' = 0,\ G' = 0$

- $\gamma = \sqrt{((0) + j\omega L')((0) + j\omega C')} = (0) + j\beta = j\omega\sqrt{L'C'}$
- $Z_0 = \sqrt{\dfrac{(0) + j\omega L'}{(0) + j\omega C'}} = \sqrt{\dfrac{L'}{C'}}$
- $v(z) = V_0^f \exp(-j\beta z) + V_0^r \exp(j\beta z)$
- $i(z) = I_0^f \exp(-j\beta z) + I_0^r \exp(j\beta z)$
- $\lambda = \dfrac{2\pi}{\omega\sqrt{L'C'}}$
- $v_p = \dfrac{1}{\sqrt{L'C'}}$

11. in a lossy transmission line with permeability μ, surface resistance $R_S = \dfrac{1}{\sigma\delta_S}$, complex permittivity $\varepsilon = \varepsilon' - j\varepsilon''$:

$$\begin{aligned}
W_m = \frac{\mu}{4}\int_S \mathbf{H}_S \cdot \mathbf{H}_S^* \,\mathrm{d}S &\Longrightarrow L' = \frac{\mu}{|I_0|^2}\int_S \mathbf{H}_S \cdot \mathbf{H}_S^* \,\mathrm{d}S \\
W_e = \frac{\varepsilon}{4}\int_S \mathbf{E}_S \cdot \mathbf{E}_S^* \,\mathrm{d}S &\Longrightarrow C' = \frac{\varepsilon'}{|V_0|^2}\int_S \mathbf{E}_S \cdot \mathbf{E}_S^* \,\mathrm{d}S \\
R' &= \frac{R_S}{|I_0|^2}\int_{C1+C2} \mathbf{H}_S \cdot \mathbf{H}_S^* \,\mathrm{d}L \\
G' &= \frac{\omega\varepsilon''}{|V_0|^2}\int_S \mathbf{E}_S \cdot \mathbf{E}_S^* \,\mathrm{d}S
\end{aligned}$$

12. terminanted transmission line. consider the case at which a lossless line is terminated by an impedance Z_L at the receiving port and extends infinitely from one end. let the point where the load side connects with the transmission line be $z = 0$.

$$\begin{aligned}
V(0) &= V_0 = V_0^f + V_0^r \\
I(0) &= \frac{V_0}{Z_L} = \frac{V_0^f}{Z_0} - \frac{V_0^r}{Z_0} \\
&\Updownarrow \\
V_0^f &= \frac{V_0}{2}\left(\frac{1}{Z_0} + \frac{1}{Z_l}\right) \\
V_0^r &= \frac{V_0}{2}\left(\frac{1}{Z_0} - \frac{1}{Z_l}\right)
\end{aligned}$$

The current and voltage is:

$$\begin{aligned}
V(z) &= \frac{V_0}{2}\left(\frac{1}{Z_0} + \frac{1}{Z_l}\right)\exp(-jkz) + \frac{V_0}{2}\left(\frac{1}{Z_0} - \frac{1}{Z_l}\right)\exp(jkz) \\
I(z) &= \frac{V_0}{2Z_0}\left(\frac{1}{Z_0} + \frac{1}{Z_l}\right)\exp(-jkz) + \frac{V_0}{2Z_0}\left(\frac{1}{Z_0} - \frac{1}{Z_l}\right)\exp(jkz)
\end{aligned}$$

the relationship of the forward and reverse wave is $\dfrac{V_0^r}{V_0^f} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

13. special cases:

	Reflected Voltage	Remarks
Open circuit ($Z_L \to \infty$)	$\frac{V_0^r}{V_0^f} = 1$	same phase
Short circuit ($Z_L \to 0$)	$\frac{V_0^r}{V_0^f} = -1$	π out of phase
Matched load ($Z_L = Z_0$)	$\frac{V_0^r}{V_0^f} = 0$	no reflection, maximum power transfer

14. Impedance transformation. using Ohm's law: $Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{Z_L + jZ_0 \tan(\beta z)}{Z_0 + jZ_L \tan(\beta z)}$

15. at the quarter wavelength line (quarter wave transform), $z = \frac{\lambda}{4}, \beta = \frac{2\pi}{\lambda} \implies \beta z = \frac{\pi}{2}$

$$Z_{eq} = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{\pi}{2}\right)}{Z_0 + jZ_L \tan\left(\frac{\pi}{2}\right)}$$

$$\Updownarrow$$

$$Z_{eq} Z_L = Z_0^2$$

special cases:

	Z_L	Z_{eq}
Open circuit	∞	0
Short circuit	0	∞
Matched load	Z_0	Z_0