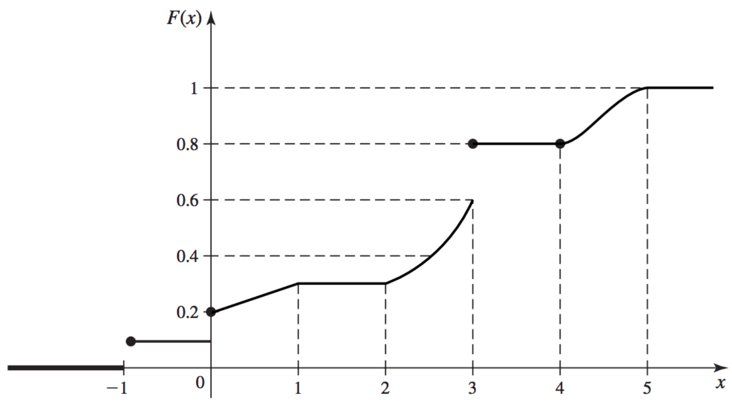
Homework 5
STA 4321/5325, Fall 2019, MWF 8:30am
Professor: Aaron J. Molstad
Due date: Wednesday, October 16th, 2019

All work must be shown for complete credit. W.M.S. denotes the course textbook (*Mathematical Statistics with Applications*). For clarity, we use $\exp(x) \equiv e^x$ interchangably.

1. Let the following be the probability distribution function for the random variable X.



   Compute the following probabilities. (Notice that there are points of discontinuity, so that for this random variable, it is not true that $P(X \leq a) = P(X < a)$ for all a.)

   (a) $P(X = -1)$ (b) $P(X < 0)$ (c) $P(X \leq 0)$ (d) $P(X = 1)$
   (e) $P(0 \leq X \leq 3)$ (f) $P(0 < X < 3)$ (g) $P(1 < X \leq 2)$ (h) $P(1 \leq X \leq 2)$
   (i) $P(X > 5)$ (j) $P(X \geq 5)$ (k) $P(3 \leq X \leq 4)$

2. Let Y be a random variable following the binomial distribution based on n independent Bernoulli trials, each with success probability p.

   (a) Compute $\mathrm{E}[Y(Y-1)(Y-2)]$
   (b) Compute $\mathrm{E}\left[(Y+1)^{-1}\right]$

3. Suppose Y is a negative binomial random variable with parameters r and p (i.e, Y is the number of failures before the rth success in a sequence of Bernoulli trials, each with success probability p). Define $\tilde{Y} = Y + r$. Then, $\tilde{Y}$ can be interpreted as the number of the trial on which the rth success is observed.

   (a) What is the probability mass function of $\tilde{Y}$?
   (b) Using your answer to (a), compute $\mathrm{E}(\tilde{Y})$.

4. Let Z be a random variable with probability density function

$$f_Z(z) = c \cdot \exp\left(-\frac{z^2}{2}\right), \quad z \in (-\infty, \infty).$$

Determine c so that f_Z is a valid probability density function.

5. Suppose that the random variable X has probability density function

$$f_X(x) = \begin{cases} \frac{4}{3}(1 - x^3) & : 0 < x < 1 \\ 0 & : \text{otherwise} \end{cases}$$

   (a) Compute the probability distribution function F_X.
   (b) Sketch the probability density function f_X.
   (c) Evaluate the following probabilities: $P\left(X < 1/2\right), P\left(1/4 < X < 3/4\right), P\left(X > 1/3\right)$

6. Suppose that the probability distribution function of a random variable X is as follows:

$$F_X(x) = \begin{cases} \exp(x - 3) & : x \leq 3 \\ 1 & : x > 3 \end{cases}$$

Find and sketch the probability density function f_X.

7. Suppose that a continuous random variable X takes values over the interval $[a, b]$ (i.e., $\mathfrak{X} = \mathrm{Range}(X) = [a, b]$). Furthermore, suppose X takes values such that every interval of the same length has equal probability (assuming the interval is entirely contained in $[a, b]$). That is,

$$P(c < X < c + h) = P(d < X < d + h)$$

for all $c, d,$ and $h \in \mathbb{R}$ such that $[c, c + h] \subseteq [a, b]$ and $[d, d + h] \subseteq [a, b]$. What is the probability density function for X?