

Homework 9

STA 4321/5325, Fall 2019, MWF 8:30am

Professor: Aaron J. Molstad

Due date: Monday, November 25th, 2019

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*. Problem #8 is worth 2pts of extra credit.

1. Let the joint density function of random pair (U, V) be

$$f_{U,V}(u, v) = \begin{cases} \frac{1}{4}(u + 2v) & 0 < v < 1, 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

In Homework 8, we showed

$$f_U(u) = \begin{cases} \frac{(u+1)}{4} & 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the density function of the random variable $Z = 9/(U + 1)^2$?

2. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

(a) What are $E(Y_1)$ and $E(Y_2)$?

(b) What is $E(Y_1 Y_2)$?

(Hint: $E(Y_1 Y_2) = E(Z^3)$ and the MGF of the standard normal is $M_Z(t) = e^{\frac{t^2}{2}}$.)

(c) What is $\text{Cov}(Y_1, Y_2)$?

(d) Are Y_1 and Y_2 independent? If not, provide some information which contradicts their independence.

3. A worker leaves for work between 9:00am and 9:30am and takes between 45 and 55 minutes to arrive. Let the random variable Y denote this worker's time of departure, and the random variable X the travel time. Assuming that Y and X are independent and uniformly distributed, find the probability that the worker arrives at work before 10:00am.

4. Suppose the distribution of X , conditional on $U = u$, is $N(u, u^2)$ where the marginal distribution of U is uniform $(0, \theta)$.

(a) Find $E(X)$, $\text{Var}(X)$, and $\text{Cov}(X, U)$.

(b) Prove that X/U and U are independent.

5. (W.M.S 6.29) The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f_V(v) = av^2 e^{-bv^2}, \quad v > 0$$

where $b = m/2kT$ with k, T , and m denoting Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively.

- (a) Derive the distribution of $W = mV^2/2$, the *kinetic energy* of the molecule.
(b) Find $E(W)$.

6. Let X have a distribution function given by

$$F_X(x) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y^2} & y \geq 0 \end{cases}$$

Find a transformation $g(U)$ such that if U has a uniform distribution on the interval $(0, 1)$, $g(U)$ has the same distribution as Y .

7. Let X_1 and X_2 be two random variables with joint density

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} e^{x_1} & 0 \leq x_2 \leq x_1 < \infty \\ 0 & \text{otherwise} \end{cases}$$

- (a) Compute $P(X_1 < 2, X_2 > 1)$.
(b) Compute $P(X_1 \geq 2X_2)$.
(c) Compute $P(X_1 - X_2 \geq 1)$
8. (Optional) (W.M.S, 6.52) Let V_1 and V_2 be independent Poisson random variables with means λ_1 and λ_2 , respectively.
- (a) What is the probability distribution function of $V_1 + V_2$?
(b) What is the conditional probability function of Y_1 given that $Y_1 + Y_2 = m$?