Lecture 13
STA4321 and STA5325, Spring 2022
Introduction to Probability
Department of Statistics, University of Florida

Agenda

1. Memoryless property of geometric random variables
2. Negative binomial distribution
3. Poisson distribution

1 Geometric random variables continued

Recall, a Geometric random variable arises out of a series of independent Bernoulli trials which are repeated until the first success is achieved. The number of variables before the first success, X, is said to be a Geometric random variable. Last lecture, we showed (or argued)

$$p_X(x) = p(1-p)^x, \;\; x = 0, 1, 2, \ldots$$

and

$$\mathrm{E}(X) = \frac{1-p}{p}, \;\; \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Example: A firm has a new position which needs fluency in both English and Spanish. Applications are selected randomly from (a very large) pool, and interviewed until the first applicant who is fluent in both English and Spanish is found. If 20% of applicants in the pool are fluent in both English and Spanish, what is the expected number of unqualified applicants who will be interviewed before a qualified candidate is found?

Since each applicant is either qualified or unqualified, each interview is a Bernoulli trial.

However, each Bernoulli trial is not independent and identical, but if the population is very large, we can approximately assume this to be true!

Let $X =$ # of unqualified applicants interviewed before the first qualified applicant. Then X is a geometric random variable where the success probability of the Bernoulli experiment is 0.2. Hence,

$$\mathrm{E}(X) = \frac{1-0.20}{0.20} = 4.$$

Memoryless property of the geometric distribution

Let j, k be positive integers and let X be a geometric random variable. Then,

$$P(X \geq j + k \mid X \geq j) = P(X \geq k).$$

Proof:

$$\begin{aligned}
P(X \geq j+k \mid X \geq j) &= \frac{P(\{X \geq j+k\} \cap \{X \geq j\})}{P(\{X \geq j\})} \\
&= \frac{P(\{X \geq j+k\}}{P(\{X \geq j\})} \\
&= \frac{\sum_{x=j+k}^{\infty} p(1-p)^x}{\sum_{x=j}^{\infty} p(1-p)^x} \\
&= \frac{p(1-p)^{j+k} \sum_{x=0}^{\infty} (1-p)^x}{p(1-p)^j \sum_{x=0}^{\infty} (1-p)^x} \\
&= (1-p)^k
\end{aligned}$$

and furthermore,

$$\begin{aligned}
P(X \geq k) &= \sum_{x=k}^{\infty} p(1-p)^x \\
&= p(1-p)^k \sum_{x=0}^{\infty} (1-p)^x = \frac{p(1-p)^k}{1-(1-p)} = (1-p)^k.
\end{aligned}$$

In words, this property means that given that there have been j failures, the chance of at least k more failures before the first success is exactly the same as if we are just beginning the experiment and want to know the probability of having at least k failures before the first success.

2 Negative binomial distribution

A geometric random variable corresponds to the number of failures before the first success in a sequence of independent Bernoulli trials. But what if we were interested in the number of failures before the rth success for some positive integer r?

Let $X =$ # the number of failures observed before the rth success in a sequence of Bernoulli trials.

Then X is said to follow a negative binomial distribution, i.e., X is a negative binomial random variable.

Clearly, the random variable X can take any non-negative integers as a value, i.e.,

$$\mathfrak{X} = \text{Range}(X) = \{0, 1, 2, \ldots, \}.$$

Moreover,

$$\begin{aligned} P(X = x) =& P(\text{first } x + r - 1 \text{ trials contain } x \text{ failures and } r - 1 \text{ successes,} \\ & \quad \text{with the } (x + r)\text{th trial being a success}) \\ =& P(\text{first } x + r - 1 \text{ contain } x \text{ failures and } r - 1 \text{ successes})\, P((x + r)\text{th trial is a success}) \\ =& \left\{ \underbrace{\binom{x + r - 1}{x}}_{A_1} p^{r-1}(1 - p)^x \right\} p = \binom{x + r - 1}{x} p^r (1 - p)^x \end{aligned}$$

where $A_1 =$ # of ways of choosing the x trials with failures.

Hence, the probability mass function of a negative binomial random variable is

$$p_X(x) = \binom{x + r - 1}{x} p^r (1 - p)^x, \quad x = 0, 1, 2, \ldots$$

Negative binomial distribution

Let X be random variable following a negative binomial distribution arising from independent Bernoulli trials each with success probability p and continuing until the rth success. Then,

$$\text{E}(X) = \frac{r(1 - p)}{p}, \quad \text{Var}(X) = \frac{r(1 - p)}{p^2}.$$

Example: Amongst a large lot of tires, 5% of tires are defective. Four tires need to be chosen from the lot and placed on a car.

1. Find the probability that 2 defective tires are found before four good ones.
2. Find the expected value and the variance of the number of defective tires chosen before finding four good tires.

(Assume that the number of tires is large enough so that choosing tires successively can be treated as independent and indentical Bernoulli experiments.)

As given to us, choosing each tire can be thought of as a Bernoulli trial with a sucess if the tire is good (which occurs with probability 0.95). Hence,

$$X = \text{ \# of bad tires chosen before finding 4 good tires},$$

so that X is a negative binomial random variable with $r = 4$ and $p = 0.95$. Hence,

$$\begin{aligned} P(X = 2) &= \binom{4+2-1}{2}(0.95)^4(0.05)^2 \\ &= 10(0.95)^4(0.05)^2 \\ &= 0.02036 \end{aligned}$$

Moreover,

$$\begin{aligned} \mathrm{E}(X) &= 4\left(\frac{0.05}{.095}\right) = 0.211 \\ \mathrm{Var}(X) &= 4\left(\frac{0.05}{.095^2}\right) = 0.222 \end{aligned}$$