

### Homework 7

STA 4321/5325, Fall 2019, MWF 8:30am

Professor: Aaron J. Molstad

Due date: Wednesday, October 30th, 2019

All work must be shown for complete credit. Problem #4 is worth 1pt of extra credit.

1. Suppose  $X \sim N(\mu, \sigma^2)$  for some  $\mu \in \mathbb{R}$  and  $\sigma > 0$  (that is,  $X$  is normally distributed with mean  $\mu$  and variance  $\sigma^2$ ). Given some realization of  $X$ , a mathematician constructs a rectangle with length  $L = |X|$  and width  $W = 4|X|$ . What is the expected value of the area of the rectangle?

2. Let  $V$  be a random variable following the beta distribution with parameters  $\alpha, \beta$ . Specifically, the density of  $V$  is

$$f_V(v) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1} (1-v)^{\beta-1} & : 0 \leq v \leq 1 \\ 0 & : \text{otherwise} \end{cases}$$

Find  $E(V^k)$  for arbitrary integer  $k$  without using moment generating functions. Your answer may be left in terms of quantities involving the  $\Gamma$  function. (Note: when does

3. Derive the moment generating function of  $Y$ , a negative binomial random variable with  $r = 10$  and success probability  $p$  (i.e.,  $Y$  is the number of failures before the  $r = 10$ th success in a sequence of independent Bernoulli trials). You may use the following facts without proof:

- (a) If  $X_1, X_2, \dots, X_{10}$  are independent geometric random variables each with success probability  $p$ , then we can write  $Y = \sum_{i=1}^{10} X_i$ .
- (b) If  $U$  and  $V$  are independent random variables, then for any function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $E[g(U)g(V)] = E[g(U)]E[g(V)]$  (assuming all relevant expected values exist).

4. (Optional) It is common for engineers to work with the “error function”

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) dx$$

instead of the standard normal probability distribution function  $\Phi$ , which we defined as:

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Show that the following relationship between  $\Phi$  and the function  $\operatorname{erf}$  holds for all  $z$ :

$$\Phi(z) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right).$$

5. Let  $W$  be a random variable whose probability density function is

$$f_W(w) = \begin{cases} \frac{2w}{\lambda^2} \exp\left[-\left(\frac{w}{\lambda}\right)^2\right] & : w \geq 0 \\ 0 & : \text{otherwise} \end{cases}$$

for parameter  $\lambda > 0$ . Note that if you cannot solve (a), you should still attempt (b) with the given MGF.

(a) Show that the moment generating function of  $W$ ,  $M_W(t)$  is

$$M_W(t) = \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \Gamma\left(1 + \frac{n}{2}\right)$$

(Hints: (a) recall the series expansion  $\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ ; (b) you will likely need to use substitution twice.)

(b) Using the moment generating function  $M_W$ , find  $\text{Var}(W^2)$ .