Homework 1
STA 4321/5325, Fall 2019, MWF 8:30am
Professor: Aaron J. Molstad
Due date: Wednesday, September 4th, 2019

All work must be shown for complete credit. W.M.S. denotes the course textbook (*Mathematical Statistics with Applications*).

1. A point (x, y) is to be selected from square S containing all points (x, y) such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that the probability that the selected point will belong to each specified subsets of S is equal to the area of that subset. Find the probability of the following subsets:
   (a) the subset of points such that $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \geq 1/4$ (Hint: the equation for a circke with radius r centered at (c, d) is $(x - c)^2 + (y - d)^2 = r^2$.)
   (b) the subset of points such that $\frac{1}{2} \leq x + y \leq \frac{3}{2}$
   (c) the subset of points such that $y \leq 1 - x^2$
   (d) the subset of points such that $x = y$

2. Using the laws and axioms from lecture, prove that for arbitrary events A and B,
$$P(A) = P(A \cap B) + P(A \cap B^c).$$
You may use $P(\varnothing) = 0$ without proof (since we proved this in lecture as a consequence of the axioms).

3. Suppose a bad contains 50 balls: 20 red and 30 blue. If we select 10 balls at random without replacement, what is the probability at exactly 6 red balls will be selected?

4. (2.59 of W.M.S.) Fives cards are drawn at random from a standard 52-card deck. What is the probability we draw:
   (a) one ace, one two, one three, one four, one five (this is one way to get a "straight", which occurs when we draw five cards of sequential rank)?
   (b) any kind of straight (where here, we do not consider 10-jack-queen-king-ace a valid straight, i.e., ace only counts as a low card sequentially)?

5. Two people each toss a fair coin n times. Find the probability that they will toss the same number of tails. Note that you may simplify your answer using the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

6. If k students are seated at random in a row containing $2k$ seats, what is the probability that no two students will occupy two contiguous seats?

7. Suppose n letters are placed into n mailboxes at random, with no preference for any mailbox. Find the probability that exactly one mailbox remains empty.

8. (Similar to 2.71 of W.M.S.) If two events, A and B, are such that $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$, find the following:

   (a) $P(A \mid B)$

   (b) $P(B \mid A)$

   (c) $P(A \mid A \cup B)$

   (d) $P(A \mid A \cap B)$

   (e) $P(A \cap B \mid A \cup B)$

   It may be helpful to draw Venn diagrams to solve these problems.

9. A pair of events A and B cannot be mutually exclusive and independent. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

   (a) If A and B are mutually exclusive, they cannot be independent.

   (b) If A and B are independent, they cannot be mutually exclusive.