

Homework 7

STA 4321/5325, Fall 2019, MWF 8:30am

Professor: Aaron J. Molstad

Due date: Wednesday, October 30th, 2019

All work must be shown for complete credit. Problem #4 is worth 1pt of extra credit.

1. Suppose $X \sim N(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$ (that is, X is normally distributed with mean μ and variance σ^2). Given some realization of X , a mathematician constructs a rectangle with length $L = |X|$ and width $W = 4|X|$. What is the expected value of the area of the rectangle?

2. Let V be a random variable following the beta distribution with parameters α, β . Specifically, the density of V is

$$f_V(v) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1} (1-v)^{\beta-1} & : 0 \leq v \leq 1 \\ 0 & : \text{otherwise} \end{cases}$$

Find $E(V^k)$ for arbitrary integer k without using moment generating functions. Your answer may be left in terms of quantities involving the Γ function.

3. Derive the moment generating function of Y , a negative binomial random variable with $r = 10$ and success probability p (i.e., Y is the number of failures before the $r = 10$ th success in a sequence of independent Bernoulli trials). You may use the following facts without proof:

- (a) If X_1, X_2, \dots, X_{10} are independent geometric random variables each with success probability p , then we can write $Y = \sum_{i=1}^{10} X_i$.
- (b) If U and V are independent random variables, then for any function $g : \mathbb{R} \rightarrow \mathbb{R}$, $E[g(U)g(V)] = E[g(U)]E[g(V)]$ (assuming all relevant expected values exist).
- (c) (Lecture 12) For all constants t and r such that $|r \exp(t)| < 1$

$$\sum_{k=0}^{\infty} [r \exp(t)]^k = \frac{1}{1 - r \exp(t)}.$$

Note that when r is positive, $t < -\log(r) \implies |r \exp(t)| < 1$.

4. (Optional) It is common for engineers to work with the “error function”

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) dx$$

instead of the standard normal probability distribution function Φ , which we defined as:

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Show that the following relationship between Φ and the function erf holds for all z :

$$\Phi(z) = \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{z}{\sqrt{2}}\right).$$

5. Let W be a random variable whose probability density function is

$$f_W(w) = \begin{cases} \frac{2w}{\lambda^2} \exp\left[-\left(\frac{w}{\lambda}\right)^2\right] & : w \geq 0 \\ 0 & : \text{otherwise} \end{cases}$$

for parameter $\lambda > 0$. Note that if you cannot solve (a), you should still attempt (b) with the given MGF.

(a) Show that the moment generating function of W , $M_W(t)$ is

$$M_W(t) = \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \Gamma\left(1 + \frac{n}{2}\right)$$

(Hints: (a) recall the series expansion $\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$; (b) you will likely need to use substitution twice.)

(b) Using the moment generating function M_W , find $\text{Var}(W^2)$.