All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. Find the maximum likelihood estimator of θ based on the random sample of size n from a Uniform$(0, 2\theta)$ distribution.

2. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with density
$$f_X(x \mid \theta) = \begin{cases} e^{-(x-\theta)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
where $\theta > 0$ is an unknown constant.

   (a) Find an estimator $\hat{\theta}_1$ for θ using the method of moments.

   (b) Find the maximum likelihood estimator (MLE) of θ, $\hat{\theta}_2$.

   (c) Modify both $\hat{\theta}_1$ and $\hat{\theta}_2$ so that they are unbiased for θ.

   (d) Using the modified versions from part (c), compute the efficiency of the modified version of $\hat{\theta}_2$ relative to the modified version of $\hat{\theta}_1$.

3. Suppose that m integers are randomly drawn *with replacement* from the integers $1, 2, \ldots, M$. That is, each integer has probability $1/M$ of taking on any values $1, 2, \ldots, M$, and the sampled values are independent.

   (a) Find the method of moments estimator of $\hat{M}$ of M.

   (b) Compute $\mathrm{E}(\hat{M})$ and $\mathrm{Var}(\hat{M})$.

4. The geometric distribution has probability mass function
$$P(Y = y) = p(1 - p)^y.$$

   Suppose we have observed a random sample of size n from the geometric distribution with unknown parameter p.

   (a) Find the method of moments estimator of p.

   (b) Find the MLE of p.

   (c) Using a sufficient statistic, find an unbiased estimator of p.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density

$$f_X(x \mid \theta) = \begin{cases} \frac{3x^2}{\theta^3} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $X_{(n)} = \max(X_1, \ldots, X_n)$ is a sufficient statistic.

(b) Find the MLE of θ.

(c) Find a function of the MLE which is a pivotal quantity.

(d) Use the pivotal quantity to construct a $100(1 - \alpha)\%$ confidence interval for θ.