

**Homework 9**

STA4322/STA5328

Introduction to Statistics Theory

Spring 2020, MWF 3:00pm

Due date: Friday, April 17th, 2020 at 5:00pm

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*. **Problems 1c and 3c is worth 1pt of extra credit.**

1. Suppose that  $Y_1, \dots, Y_n$  denote a random sample from the probability density function given by

$$f_Y(y | \theta, \lambda) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-\frac{(y-\lambda)}{\theta}} & y > \lambda \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find the likelihood ratio test statistic for testing  $H_0 : \theta = \theta_0$  versus  $H_A : \theta > \theta_0$  with  $\lambda$  unknown.
- (b) Assuming  $n$  is sufficiently large and the “regularity conditions” are satisfied, use Wilk’s theorem to find the rejection region of an asymptotic level  $\alpha$  test for the hypotheses in (a).
2. Suppose that  $X_1, \dots, X_n$  are independent and identically distributed exponential random variables with mean  $\beta_X$  and  $Y_1, \dots, Y_n$  are independent and identically distributed with mean  $\beta_Y$ . Recall that this the density for an exponentially distributed random variable  $U$  with mean  $\beta$ ,

$$f_U(u) = \frac{1}{\beta} e^{-\frac{u}{\beta}} \mathbf{1}(u > 0).$$

We wish to test the hypotheses

$$H_0 : \beta_X = \beta_Y, \quad \text{vs} \quad H_A : \beta_X \neq \beta_Y.$$

Let  $\Theta = (\beta_X, \beta_Y)$ . Let  $L(x, y | \Theta)$  be the joint likelihood of  $X_1, \dots, X_n, Y_1, \dots, Y_n$  evaluated at  $\Theta$ .

- (a) What is  $\max_{\Theta \in \Omega_0} L(x, y | \Theta)$ ?
- (b) What is  $\max_{\Theta \in \Omega_0 \cup \Omega_A} L(x, y | \Theta)$ ?
- (c) Show that the likelihood ratio test rejects  $H_0$  if

$$\frac{(\sum_{i=1}^n X_i) (\sum_{i=1}^n Y_i)}{[\sum_{i=1}^n (X_i + Y_i)]^2} < k.$$

No need to find  $k$ , you must only show that the rejection region has the form above.

3. A random sample  $W_1, \dots, W_n$  is drawn from a distribution with density

$$f_W(w) = \frac{\theta \nu^\theta}{w^{\theta+1}} \mathbf{1}(w \geq \nu)$$

where  $\theta > 0$  and  $\nu > 0$  are unknown parameters.

- (a) Find the MLE of  $\theta$  and  $\nu$ .  
(b) Give the likelihood ratio test statistic  $\lambda$  for the hypotheses

$$H_0 : \theta = 1, \nu > 0 \quad \text{vs} \quad H_A : \theta \neq 1, \nu > 0.$$

- (c) (Extra credit) Show that the LRT for

$$H_0 : \theta = 1, \nu > 0 \quad \text{vs} \quad H_A : \theta \neq 1, \nu > 0$$

has rejection region of the form

$$\{X : T(X) \leq c_1 \cup T(X) \geq c_2\}$$

where  $0 < c_1 < c_2$  are some constants (no need to find them) and

$$T(X) = \log \left[ \frac{\prod_{i=1}^n W_i}{(W_{(1)})^n} \right],$$

where  $W_{(1)} = \min(W_1, \dots, W_n)$ .