

Homework 3

STA4322/STA5328

Introduction to Statistics Theory

Spring 2020, MWF 3:00pm

Due date: Friday, February 7th, 2020

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. Let Y_1, Y_2, \dots, Y_n denote a random sample from a Poisson distribution with mean λ . Consider $\hat{\lambda}_1 = (Y_1 + Y_2)/2$ and $\hat{\lambda}_2 = \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$. Derive the relative efficiency of $\hat{\lambda}_1$ relative to $\hat{\lambda}_2$.

2. Let W_1, W_2, \dots, W_n denote a random sample from the probability density function

$$f_W(w) = \begin{cases} \theta w^{\theta-1} & : 0 < w < 1 \\ 0 & : \text{otherwise} \end{cases}$$

where $\theta > 0$. Show that $\bar{W} = \sum_{i=1}^n W_i$ is a consistent estimator of $\theta/(\theta + 1)$.

3. Suppose that Y_1, Y_2, \dots, Y_n is a random sample where $Y_i \sim \text{Poisson}(\lambda)$ for all $i = 1, \dots, n$. Assume that $n = 2k$ for some integer k (i.e., assume that n is even). Let

$$\hat{\lambda} = \frac{1}{2k} \sum_{i=1}^k (Y_{2i} - Y_{2i-1})^2.$$

- (a) Show that $\hat{\lambda}$ is an unbiased estimator of λ .
(b) Show that $\hat{\lambda}$ is a consistent estimator of λ .

4. Let Z_1, Z_2, \dots, Z_n be independent standard normal random variables. In the previous homework, we used the fact that $Z_i^2 \sim \chi_1^2$ for $i = 1, \dots, n$ and because the Z_i 's are independent $Z_1^2 + Z_2^2 + \dots + Z_n^2 \sim \chi_n^2$. Now, let us define

$$W_n = \frac{1}{n} \sum_{i=1}^n Y_i^2.$$

Does W_n converge in probability to some constant c ? If so, what is the value of c ?

5. Let Y_1, Y_2, \dots, Y_n be independent random variables with density

$$f_Y(y) = \begin{cases} 3y^2 & : 0 \leq y \leq 1 \\ 0 & : \text{otherwise} \end{cases}$$

Show that $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$ converges in probability to some constant and find the constant.

6. Suppose that X_1, X_2, \dots, X_n is a random sample from an exponential distribution with mean θ .

(a) Show that $\hat{\theta}_1 = nX_{(1)} = n \min(X_1, \dots, X_n)$ is an unbiased estimator of θ .

(b) Show that $\text{MSE}(\hat{\theta}_1) = \theta^2$.

(c) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2 = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$.