Agenda:

1. Basic set theory
2. Sample spaces and events
3. Probability

Basic set theory

Def A set is a collection of distinct objects, which are called elements or points of the set.

- Sets are denoted by capital letters, e.g., $A, B, C, \ldots$

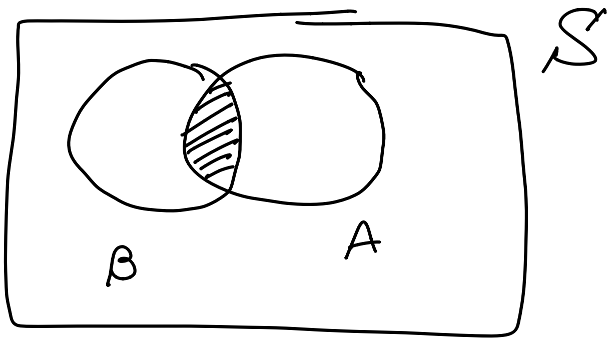
Notation

- Subset : "$A \subset B$" means A is a subset of B, i.e., all points in A are also in B.
- Null set/empty set : "$\emptyset$" denotes the set which contains no points
- Union : "$A \cup B$" denotes the collection of points which are in A or B (or both).
- Intersection : "$A \cap B$" denotes the collection of points which are in both A and B.
- Sample space : "S" denotes the universal set, i.e., the set containing all points of interest in the current situation.
- Complement : "A^c" denotes the set of

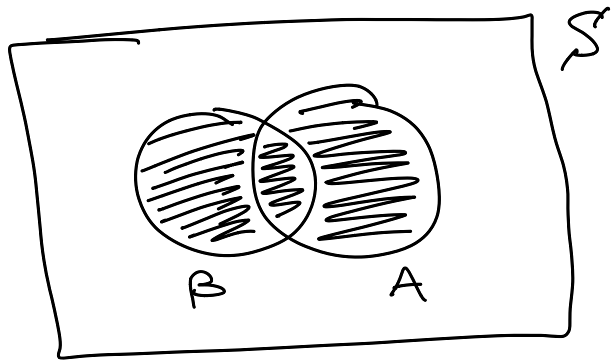
points in S not contained in A.

- <u>Mutually exclusive</u>: A and B are called mutually exclusive if $A \cap B = \emptyset$ (empty set).

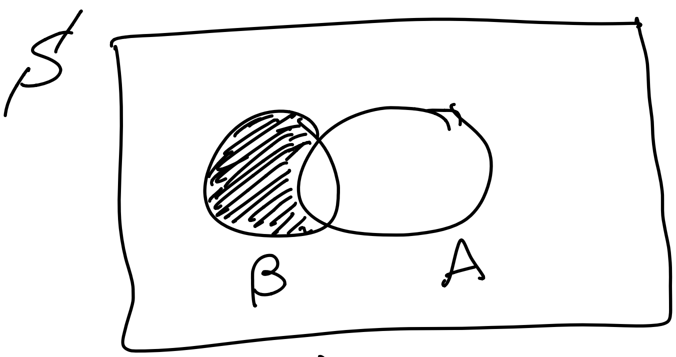
Examples



$A \cap B$



$A \cup B$



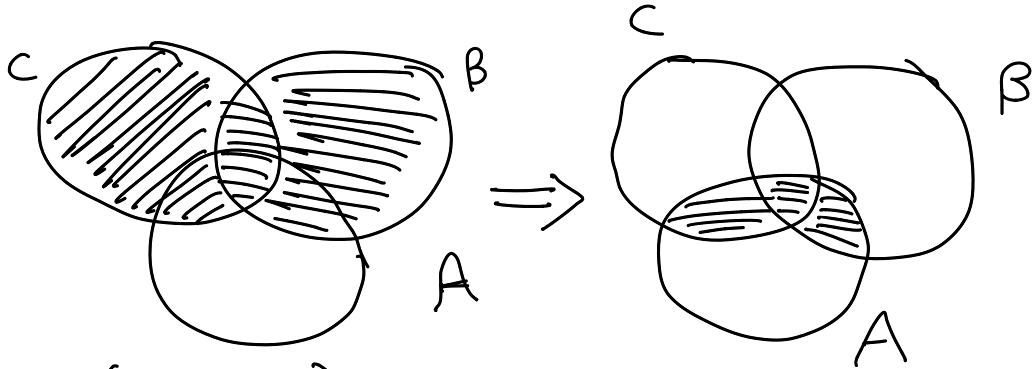
$(A^c) \cap B$

Recall A^c: all points in S not in A.

Distributive law

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*)



$(B \cup C) \equiv D$ $\quad$ $A \cap D$

$\boxed{\text{De Morgan's law:}}$

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

$\longrightarrow$ Both laws will be useful for probability computations.

$\boxed{\text{Example}}$ $S = \{1, 2, 3, 4\}$

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{4\}$,

- $A \cup B : \{1, 2, 3, 4\}$
- $A \cap B : \{2, 3\}$
- $A \cap C : \emptyset$
- $C \subset B \Longrightarrow$ ("implies")

  $C \cup B = B$, $C \cap B = C$
- $A^c = \{4\}$

Sample spaces and events

- Perform a random experiment/observe a random phenomenon. For example, consider the tossing of a coin 4 times or the proportion of people in a population affected by an epidemic.

$\boxed{\text{Def}}$ The <u>sample space</u> S of a random experiment is the set of all possible outcomes of the experiment listed in a mutually exclusive and exhaustive way.

Example

- Toss a coin 4 times:

  $S = \{HHHH, THHH, \dots, TTTT\}$

  In total, $2^4 = 16$ possible outcomes

- Percentage of people in a population affected by an epidemic

  $S = [0, 100]$, i.e., all real numbers from 0 to 100. This is an example of a continuous or "uncountable" sample space.

Def An event is an collection of sample points. In other words, any subset of the sample space S is called an event.

Example

- Toss a coin three times:

$S = \{HHH, THH, HTH, HHT, TTH, HTT, THT, TTT\}$

A = event that there is at least one heads

$= HHH, THH, HTH, HHT, TTH, HTT, THT\}$

B = event that there is at most one heads

$= \{TTH, HTT, THT\}$

Probability

Formal def of probability:

Intuitively, the "probability" of an event is a number between 0 and 1 expressing our belief in the occurence of the event in a single performance of an experiment.

$\mathcal{S}$ = sample space of random experiment

$\mathcal{A}$ = Collection of all possible events

Def A probability assignment P for a random experiment is a numerically valued function that assigns a value $P(A)$ to every event A so that the following axioms hold:

1. $P(A) \geq 0$ for every event A
2. $P(\mathcal{S}) = 1$
3. If $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Consequences of 1 - 3

- $P(\emptyset) = 0$: Set $A_1 = \mathcal{S}$, all other $A_j = \emptyset$ and apply 3.
- $A \cap B = \emptyset \Rightarrow P(A \cap B) = P(A) + P(B)$

  Set $A_1 = A$, $A_2 = B$, all others to $\emptyset$, apply 3.
- $A \subset B \Rightarrow P(A) \leq P(B)$

Defining and calculating the probability of an event by the sample point method
(* discrete sample space)

1. Define the experiment
2. Construct the sample space
3. Assign probabilities to each of the sample points (ensuring they sum to one)
4. Express event of interest as a collection of sample points
5. Find $P(A)$ by summing the probabilities of all the sample points in A.

$\boxed{\text{Proof of } P(\emptyset)=0}$ For axiom 3), choose $A_1 = S$, $A_i = \emptyset$ for all $i \geq 2$. It is easy to verify that these events are mutually exclusive. Hence, we get that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

$$\Rightarrow P(S) = P(S) + \sum_{i=2}^{\infty} P(A_i) \quad \text{since } \bigcup_{i=1}^{\infty} A_i = S$$

$$\Rightarrow \sum_{i=2}^{\infty} P(\emptyset) = 0.$$

Since $P(\emptyset) \geq 0$ by axiom 1), it follows that $P(\emptyset) = 0$. $\square$

Proof of $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

For axiom 3), choose $A_1 = A$, $A_2 = B$, $A_i = \emptyset$ for all $i \geq 3$. It is easy to verify that these events are mutually exclusive. Hence, we get that,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\Longrightarrow P(A \cup B) = P(A) + P(B) + \sum_{i=3}^{\infty} P(\emptyset)$$

$$\Longrightarrow P(A \cup B) = P(A) + P(B)$$

since $P(\emptyset) = 0$.